August 2016

Finite Generation of Ext-Algebras of Finite Dimensional Algebras and Associated Monomial Algebras

Melissa Margaret DiMarco
Syracuse University

ABSTRACT

In this thesis, we investigate the Ext-algebra of a basic, finite dimensional K-algebra $A = K\mathcal{Q}/I$, where K is an algebraically closed field and $\mathcal{Q}$ is a finite quiver. We denote the Ext-algebra of A by $E(A)$. We denote $\bar{A} = A/A^+$ to be the direct sum of all simple modules over A.

In the first part, we use the work of Green, Solberg, and Zacharia to construct a family of elements in $K\mathcal{Q}$, which we call $\{f_i^j\}$. These elements yield a minimal projective resolution of $\bar{A}$ over A. Consequently, $\{f_i^j\}$ form a dual basis of $E(A)$. In Chapter 2, we see that the subalgebra of $E(A)$ generated in degrees 0 and 1 is of the form $K\mathcal{Q}^*/I^!$ and prove the relations in $I^!$ can be directly computed using $\{f_i^j\}$. In the case A is graded, we provide an alternate proof to the result of Löfwall and Priddy, namely that $A^!$ is quadratic. Then we proceed to compute the relations which generate $I^!$. In the case A is monomial, we prove that the family $\{f_i^m\}$ is exactly the set of m-chains used by Green and Zacharia.

In the second part, we use a construction by Anick, Green, and Solberg to form a family $\{x_i^j\}$ which yields a projective resolution of $\bar{A}$, called the AGS resolution. If A is a monomial algebra, we prove there are easily checked conditions for $E(A)$ to be generated in degrees 0,1, and 2. If A is not necessarily monomial, we consider the case where the AGS resolution is minimal. In that situation, we look to the associated monomial algebra of A, found in [8] and [9], which we denote A_{MON}. We prove that if the AGS resolution is minimal and $E(A_{\text{MON}})$ is finitely generated, then $E(A)$ is finitely generated.

Finite Generation of Ext-Algebras of Finite Dimensional Algebras and Associated Monomial Algebras.

by

Melissa M. DiMarco

B.A., Binghamton University, 2009

M.S., Syracuse University, 2011

Dissertation

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

Syracuse University

August 2016

Contents

Chapter 1

Background and Notation

1.1 Introduction

Let A be a finite dimensional K-algebra where K is an algebraically closed field. Let $\bar{A}$ be the sum of all the simple modules over A. We may now define the following:

Definition 1.1. The *Ext-algebra* of A is denoted $E(A)$ and

$$E(A) = \bigoplus_{n \geq 0} \operatorname{Ext}_A^n(\bar{A}, \bar{A})$$

$E(A)$ is a graded K-algebra with the usual addition and with multiplication the Yoneda product. It is well known that $E(A)$ is a finite dimensional K-algebra if and only if A has finite global dimension. However, the following question arises: Under what conditions is $E(A)$ finitely generated as a K-algebra? Although this question is currently open, there have been many partial solutions. In the case where A is a monomial algebra, [?D] has presented solutions for cycle algebras and [?CKKM] has presented solutions in the local case.

When A is a finite dimensional K-algebra, not necessarily monomial, the finite generation of the Ext-algebra has also been investigated. It is well known that for a Koszul algebra A, $E(A)$ is always generated in degrees 0 and 1. As a generalization of Koszul algebras, [?GMMZ] determines when $E(A)$ is finitely generated for a D-Koszul algebra A. In [?GrS], first A is assumed to be a graded algebra such that the associated monomial algebra is δ-resolution determined. Then it can be determined when $E(A)$ is finitely generated. In [7], [?CPS], they first consider when $E(A)$ is generated in degrees 0,1, and 2. Then if A is an algebra with that property, A is called a $\mathcal{K}_2$ algebra. In [12], 2-d-determined algebras are investigated, leaving the open problem of when 2-d-determined algebras are K_2.

In this thesis, we continue generalizing the main problem to more algebras. First we explore the shriek algebra of A, that is, the subalgebra of $E(A)$ generated in degrees 0 and 1. This algebra is denoted by $A^!$. We are able to write $A^!$ in the form $K\mathcal{Q}^*/I^!$ where $\mathcal{Q}^*$ is a finite quiver and $I^!$ is given by an explicit set of generateors. Then we look at monomial algebras and find easily checked conditions for a monomial algebra to be $\mathcal{K}_2$. We then use the associated monomial algebra of A to state our main theorem:

Theorem 1.2. *Suppose the AGS resolution is minimal. Then if $E(A_{MON})$ is generated in degrees $0, 1, ..., m$ for some m, then $E(A)$ is also generated in degrees $0, 1, ..., m$. In particular, if $E(A_{MON})$ is finitely generated, then $E(A)$ is finitely generated.*

We then look to 2-d-determined algebras and prove the following result:

Theorem 1.3. *Suppose A is a 2-d-determined algebra such that the AGS resolution of $\bar{A}$ over A is minimal. Then $E(A)$ is generated in degrees 0,1,2.*

Then we find a example of a finite dimensional 2-d-determined algebra A so that $E(A)$ is not K_2.

Throughout the thesis, we always view A as a quotient of a path algebra, that is, $A = K\mathcal{Q}/I$ for some finite quiver $\mathcal{Q}$ and some admissible ideal I. We then use the construction found in [15], [14]: We start with a projective resolution of $\bar{A}$,

$$... \to P^n \xrightarrow{d^n} P^{n-1} \to ... \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \bar{A} \to 0$$

where P^n is a projective right A-module. For each $n \geq 0$, $P^n = \bigoplus_{i=1}^{l} f_i A$ where f_i is an element in $K\mathcal{Q}$. Although the projective resolution is unique in some sense, there are many ways to choose $\{f_i\}$. In this thesis, we focus on two ways to choose $\{f_i\}$. In Chapter 2, we use [15] to choose a family $\{f_i^n\}_{i=1}^{l^n}$ so that when $P^n = \bigoplus_{i=1}^{l^n} f_i^n A$, the corresponding projective resolution is minimal. In Chapter 4, we use [1], [14] to choose a family $\{x_i^n\}$ so that when $P^n = \bigoplus_{i=1}^{t^n} x_i^n A$ we still have a projective resolution of $\bar{A}$ over A, only it need not be minimal. We call this particular resolution the AGS resolution. When the AGS resolution is minimal, it is essentially the same projective resolution as in [15].This construction also yields a basis of $\operatorname{Ext}_A^n(\bar{A}, \bar{A})$, which allows us to prove our results.

In this Chapter we review the basic results pertaining to finite dimensional K-algebras. We do so by introducing quivers and path algebras, graded algebras, ext-algebras, quadratic algebras, and Koszul algebras.

1.2 Quivers and Path Algebras

Throughout this thesis we are concerned with finite dimensional, basic $K-$algebras over an algebraically closed field K of characteristic 0. Due to the nature of our investigation, we only concern ourselves with algebras which are isomorphic to a quotient of a path algebra over a finite quiver $\mathcal{Q}$. Here, we review the basic definitions of quivers and path algebras. For additional information, we cite [3], [2].

Definition 1.4. Let K be a field. A K-algebra is a ring A with an identity element (denoted by 1) such that A has a K-vector space structure compatible with the multiplication structure of the ring. That is, for all $\lambda \in K$ and $a, b \in A$, then $\lambda(ab) = (a\lambda)b = (ab)\lambda$. A is finite dimensional as a K algebra if it is finite dimensional as a K-vector space.

Definition 1.5. [2] A quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ is a quadruple consisting of two sets: $\mathcal{Q}_0$ (whose elements are called vertices) and $\mathcal{Q}_1$ (whose elements are called arrows), and two maps $s, t : \mathcal{Q}_1 \to \mathcal{Q}_0$ which associate to each arrow $a \in \mathcal{Q}_1$ its source $s(a) \in \mathcal{Q}_0$ and its target $t(a) \in \mathcal{Q}_0$ respectively.

We abbreviate $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ by $\mathcal{Q}$. We say that $\mathcal{Q}$ is finite if $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are finite sets.

Example 1.6. Consider the quiver $\mathcal{Q}$

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Here, $\mathcal{Q}_0 = \{1, 2, 3\}$ and $\mathcal{Q}_1 = \{a, b\}$. We have $s(a) = 1, t(a) = 2$, and $s(b) = 2, t(b) = 3$.

Definition 1.7. A path of length $l \geq 1$ is a sequence of l arrows $a_1 a_2 ... a_l$ such that for $1 \leq i \leq l - 1$, $t(a_i) = s(a_{i+1})$. We will also consider paths of length 0 at a vertex v, which consists of that vertex v. Such a path is called the *trivial path* at vertex v.

Sometimes we denote a path of length m by p and write $p = a_1 ... a_m$ for $a_i \in \mathcal{Q}_1$ or write $l(p) = m$. In 1.6 the only path of length 2 in $\mathcal{Q}$ is ab. The paths of length 1 are a and b, and the paths of length 0 are 1, 2, and 3.

Definition 1.8. [2, 1.2] Let $\mathcal{Q}$ be a quiver. The path algebra $K\mathcal{Q}$ of $\mathcal{Q}$ is both a ring with a copy of K in its center as well as a K-vector space that has as its basis the set of all paths $a_1 a_2 ... a_l$ of length $l \geq 0$ in $\mathcal{Q}$. The product of two basis vectors $a_1 ... a_l$ and $b_1 ... b_k$ of $K\mathcal{Q}$ is defined by

$$a_1 ... a_l b_1 ... b_k$$

if $t(a_1) = s(b_1)$ and 0 otherwise. Then the product of basis elements is extended to arbitrary elements of $K\mathcal{Q}$ by linearity.

By abuse of notation, if $x \in K\mathcal{Q}$ is a linear combination of paths of length m, we say the *degree* of x is m and write $l(x) = m$.

Recall an algebra is connected if it is not the direct product of two algebras. $K\mathcal{Q}$ has the following properties

Lemma 1.9. [2, lemma 1.4,1.7] *Let $\mathcal{Q}$ be a quiver and $K\mathcal{Q}$ its path algebra. Then*

1. *$K\mathcal{Q}$ is an associative algebra,*
2. *$K\mathcal{Q}$ has an identity element if and only if $\mathcal{Q}_0$ is finite, and*
3. *$K\mathcal{Q}$ is finite dimensional if and only if $\mathcal{Q}$ is finite and acyclic.*
4. *Assuming $\mathcal{Q}$ is a finite quiver, $K\mathcal{Q}$ is connected if and only if $\mathcal{Q}$ is a connected quiver.*

We now consider some special two-sided ideals of $K\mathcal{Q}$

Definition 1.10. A two-sided ideal I of $K\mathcal{Q}$ is called *admissible* if there exists some $m \geq 2$ such that

$$J^m \subset I \subset J^2$$

where J is the ideal of $K\mathcal{Q}$ generated by all arrows.

Thus for an admissible ideal I, it follows that $K\mathcal{Q}/I$ is a finite dimensional algebra.

We now discuss representations of quivers.

Definition 1.11. [2] Let $\mathcal{Q}$ be a finite quiver. A *representation* M, denoted (M_v, Φ_a), of $\mathcal{Q}$ satisfies two properties:

1. To each vertex v in $\mathcal{Q}_0$ we associate to it a K-vector space M_v.

2. To each arrow a in $\mathcal{Q}_1$ such that $s(a) = v$ and $t(a) = w$, we associate to it a K-linear map $\Phi_a : M_v \to M_w$.

(M_v, Φ_a) is called *finite dimensional* if each M_v is a finite dimensional K-vector space.

If $M = (M_v, \Phi_a)$ and $M' = (M'_v, \Phi'_a)$ are two different representations of a finite quiver $\mathcal{Q}$, we may define a *morphism* $f : M \to M'$ as a family of K-linear maps $(f_v)_{v \in \mathcal{Q}_0}$ such that the following diagram commutes:

$$\begin{array}{ccc} M_v & \xrightarrow{\Phi_a} & M_w \\ \downarrow{\scriptstyle f_v} & & \downarrow{\scriptstyle f_w} \\ M'_v & \xrightarrow{\Phi'_a} & M'_w \end{array}$$

Definition 1.12. [2] Let $\mathcal{Q}$ be a finite quiver and $M = (M_v, \Phi_a)$ a representation. Also, let $p = a_1 ... a_l$ be a path in $\mathcal{Q}$ where $s(p) = v$ and $t(p) = w$. We may define the *evaluation* of M on the path p to be the K-linear map from M_v to M_w

$$\Phi_p = \Phi_{a_l} \Phi_{a_{l-1}} ... \Phi_{a_1}$$

We may extend by linearity to define Φ_q for any element $q \in K\mathcal{Q}$. We may now define a category of finite dimensional representations of a finite quiver $\mathcal{Q}$, which we denote $\text{rep}(\mathcal{Q})$.

Now suppose $I = \langle \rho_1, ... \rho_m \rangle$ is an admissible ideal of $K\mathcal{Q}$. We say a representation M is *bound* by I if $\Phi_{\rho_i} = 0$ for every i. We may denote by $\text{rep}(\mathcal{Q}, I)$ the full subcategory of $\text{rep}(\mathcal{Q})$ containing the representations of $\mathcal{Q}$ bound by I.

Theorem 1.13. [2] *Let $A = K\mathcal{Q}/I$ for I an admissible ideal and $\mathcal{Q}$ a finite quiver. Then there is a K-linear equivalence of categories*

$$F : modA \to rep(\mathcal{Q}, I)$$

So we can assoicate to each simple module of A a representation of $\mathcal{Q}$ bound by I. For any vertex v, we may consider the representation $S_v = (S(v)_w, \Phi_a)$ which is defined as follows:

$$S(v)_w = \begin{cases} 0 & w \neq v \\ K & w = v \end{cases}$$

and

$$\Phi_a = 0$$

for all $a \in \mathcal{Q}_1$. We call S_v the *simple module of A at vertex v.* By 1.13, we see the indecomposable simple modules of A are in 1-1 correspondance to $\{S_v \mid v \in \mathcal{Q}_0\}$.

1.3 Graded Algebras

A ring A is called *graded* if we can write $A = A_0 \bigoplus A_1 \bigoplus ....$ where each A_i is an abelian group and $A_iA_j \subseteq A_{i+j}$ for all i, j. If $A_iA_j = A_{i+j}$ for all i, j, we say A is generated in degree 1. In other words, if A is generated in degree 1, then $A_i = (A_1)^i$ for each $i \geq 1$.

Definition 1.14. Let $A = A_0 \bigoplus A_1 \bigoplus ...$ be a graded ring and $x \in A$. We say the degree of x is i if $x \in A_i$. In this case we also say that x is homogeneous of degree i.

Example 1.15. Let $A = K[x_1, ..., x_n]$ where the degree of each x_i is 1. Then we may write

$$A = K \bigoplus \text{Span}\{x_1, ..., x_n\} \bigoplus \text{Span}\{x_i x_j\}_{i,j \geq 1} \bigoplus ...$$

Remark 1.16. Suppose A is graded. Then $A_0 \subseteq A$ is a subring of A called the initial subring of A. Note that by [10], we may assume $1 \in A_0$.

If A is a graded ring as well as a K-algebra, we say A is a *graded K-algebra*. In addition to the properties of a graded ring, we will also require that A satisfies the following properites:

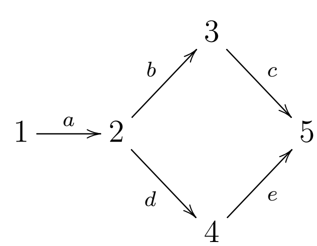
1. For all i, A_i is a finite dimensional K-vector space.

2. A is generated in degree 1

3. $A_0 = K \times K \times ... \times K$.

If $A_0 = K$, we say that A is connected.

Example 1.17. Suppose $A = K\mathcal{Q}$ where $\mathcal{Q}$ is given by the following quiver



$$A_0 = K^5$$

$$A_1 = Span(a, b, c, d, e)$$

$$A_2 = Span(ab, ad, bc, de)$$

$$A_3 = Span(abc, ade)$$

and $A_i = 0$ for all $i \geq 4$ because there are no paths in A of length greater than or equal to 4. Note that $K\mathcal{Q}$ is connected as a K-algebra because $\mathcal{Q}$ is a connected quiver; however, it is not connected as a graded K algebra because $A_0 = K^5$.

Definition 1.18. Let A be graded. An ideal I of A is called *graded* or *homogeneous* if

$$I = \bigoplus_{i \geq 0} A_i \cap I$$

In other words, I is graded if and only if $I = I_0 \bigoplus I_1 \bigoplus I_2 \bigoplus ...$ where $I_i \subset A_i$ is an A_0 submodule and $A_i I_j \subset I_{i+j}$.

If $\mathcal{Q}$ is a finite quiver and $I \subset K\mathcal{Q}$ is a homogeneous ideal, then $A = K\mathcal{Q}/I$ is a "length graded" algebra generated in degree 1. Conversely, if A is a graded K-algebra generated in degree 1, then $A = K\mathcal{Q}/I$ for some homogeneous ideal I.

Example 1.19. If I is an ideal of a path algebra $K\mathcal{Q}$ and I is generated by homogeneous elements, then I is a graded ideal.

Definition 1.20. Let A^+ be the homogeneous ideal of A, $A^+ = A_1 \bigoplus A_2 \bigoplus ....$ We call A^+ the *radical* of A (also called the graded radical).

It is important to note that the graded radical of A need not equal the Jacobson radical of A. Also, if $A_0 = K$, then A^+ is a maximal ideal of A. Otherwise, it is not.

Example 1.21. Let $A = K[x]$. Then $A^+ = \text{Span}\{x, x^2, ...\} = \langle x \rangle$. However, the Jacobson radical is 0.

Definition 1.22. A right A-module M is said to be *graded* if $M = \bigoplus_{i \geq 0} M_i$ where for all i, j, $M_i A_j \subseteq M_{i+j}$.

Suppose $M = \bigoplus_{i \geq 0} M_i$ and $N = \bigoplus_{i \geq 0} N_i$ are two graded right A-modules. Then a homomorphism $f : M \to N$ is *graded of degree d* if $f(M_i) \subseteq N_{i+d}$ for all i. By convention, we say f is *graded* if it is graded in degree 0.

Let M be a graded, right A-module. We say that a graded projective resolution $\mathcal{P}$ of M

$$... \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

is *minimal* if $\text{im} d^i \subset P^{i-1} A^+$ for all i.

1.4 The Ext Algebra of A

Suppose $A = K\mathcal{Q}/I$ is a finite dimensional K-algebra and I an admissible ideal. Let $(K\mathcal{Q})^+ = J$ and $A^+ = \mathfrak{r}$. Let $\bar{A} = K\mathcal{Q}/J = A/\mathfrak{r}$. Notice that $\bar{A}$ is also a right A-module because $I \subset J$. We often will refer to $\bar{A}$ as the *top of A*.

Definition 1.23. The *Ext Algebra* of A is denoted $E(A)$ and

$$E(A) = \bigoplus_{n \geq 0} \text{Ext}^n_A(\bar{A}, \bar{A})$$

where multiplication is given by the Yoneda product. It is easy to see that $E(A)$ is a graded algebra. Its internal degree is given by the homological degree over A. Moreover, if A is a graded algebra, then $E(A)$ is *bigraded*. In particular, for all $n, \mathrm{Ext}_A^n(\bar{A}, \bar{A})$ is a graded K-vector space. If $\mu \in \mathrm{Ext}_A^n(\bar{A}, \bar{A})$, we may write $\mu \in \mathrm{Ext}_A^n(\bar{A}, \bar{A})_{-p}$ to denote that μ is of degree p in $\mathrm{Ext}_A^n(\bar{A}, \bar{A})$.

Let M be a right A-module. If $\mathcal{P}$ is a minimal projective resolution of M,

$$... \xrightarrow{d^{n+1}} P^n \xrightarrow{d^n} ... \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \to M \to 0$$

then $\mathrm{Ext}_A^n(M, \bar{A})$ is the cohomology of the complex

$$0 \to \mathrm{Hom}_A(P^1, \bar{A}) \xrightarrow{d^{1*}} \mathrm{Hom}_A(P^2, \bar{A}) \xrightarrow{d^{2*}} ... \tag{1.1}$$

and, since $\bar{A}$ is semisimple, the boundary maps of this complex are all zero. It follows that, for each $n \geq 0$, we have

$$\mathrm{Ext}_A^n(M, \bar{A}) = \mathrm{Hom}_A(P^n, \bar{A})$$

As mentioned earlier, the multiplication in $E(A)$ is given by the Yoneda product. Here is a way of defining the product:

If $\epsilon \in \mathrm{Ext}_A^i(\bar{A}, \bar{A})$ and $\nu \in \mathrm{Ext}_A^j(\bar{A}, \bar{A})$, we may think of $\epsilon \in \mathrm{Hom}(P^i, \bar{A})$ and $\nu \in \mathrm{Hom}(P^j, \bar{A})$ where

$$... \to P^n \to .... \to P^0 \to \bar{A} \to 0$$

is a fixed minimal projective resolution of $\bar{A}$. We construct the following commutative diagram where $l_0, l_1, ...$ denote consecutive liftings of ν.

$$
\begin{array}{ccccccccccccc}
... & \longrightarrow & P^{i+j} & \longrightarrow & P^{i+j-1} & \longrightarrow & ... & \longrightarrow & P^{j+1} & \longrightarrow & P^{j} & & \\
 & & \downarrow{\scriptstyle l_j} & & \downarrow{\scriptstyle l_{j-1}} & & & & \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l_0} & \searrow{\scriptstyle \nu} & \\
... & \longrightarrow & P^{i} & \longrightarrow & P^{i-1} & \longrightarrow & ... & \longrightarrow & P^{1} & \longrightarrow & P^{0} & \longrightarrow & \bar{A} \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \epsilon} & & & & & & & & & & \\
 & & \bar{A} & & & & & & & & & &
\end{array}
$$

Then we put $\nu\epsilon = \epsilon \circ l_j$. It can be shown that this definition is independent of choice of lifting. If A is a graded algebra and $\epsilon \in \mathrm{Ext}^i_A(\bar{A}, \bar{A})_{-p}$ and $\nu \in \mathrm{Ext}^j_A(\bar{A}, \bar{A})_{-q}$, then it follows that $\nu\epsilon \in \mathrm{Ext}^{i+j}_A(\bar{A}, \bar{A})_{-(p+q)}$.

Suppose now that S_i is a simple right A-module corresponding to vertex i in $\mathcal{Q}_0$ and P_i is the indecomposable projective cover of S_i. Then for each i we define

$$\mathrm{Ext}^*_A(P_i, \bar{A}) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(P_i, \bar{A})$$

which is a simple right module of $E(A)$, because for each i, $\mathrm{Ext}^*_A(P_i, \bar{A}) = \mathrm{Hom}_A(P_i, \bar{A})$, so it is a one-dimensional $E(A)$ module. Moreover, for each i,

$$\mathrm{Ext}^*_A(S_i, \bar{A}) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(S_i, \bar{A})$$

is an indecomposable projective right $E(A)$-module. In fact, it is the projective cover of $\mathrm{Ext}^*_A(P_i, \bar{A})$. Consequently,

$$E(A) = \bigoplus_{i=1}^{n} \mathrm{Ext}^*_A(S_i, \bar{A})$$

1.5 Quadratic Algebras

Suppose $A = K\mathcal{Q}/I$ where $I = \langle \rho_1, \rho_2, ..., \rho_m \rangle$. We call A a *quadratic algebra* if for all i, $\rho_i = \sum \lambda_{i,k} q_{i,k}$ so that $\lambda_{i,k} \in K$ and $q_{i,k}$ is a path in $\mathcal{Q}$ of length 2. It is easy to see that I is a graded ideal of $K\mathcal{Q}$, so A is also a graded algebra.

For every quadratic algebra $A = K\mathcal{Q}/I$, we assoiciate its *quadratic dual*, which we denote $A^{\perp}$. To construct $A^{\perp}$, we first define

$$V = \text{Span}\,(\text{paths in } \mathcal{Q} \text{ of length 2 })$$

We now define the bilinear form

$$\langle\ ,\ \rangle : V \times V \to K$$

as follows: for any two basis elements p_i, p_j, the product is

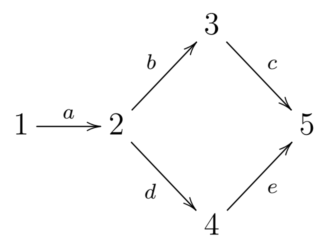
$$\langle p_i, p_j \rangle = \begin{cases} 1 & p_i = p_j \\ 0 & p_i \neq p_j \end{cases}$$

and we extend linearly to define a bilinear form on all V. Now we define

$$I^{\perp} = \langle v \in V \mid \langle u, v \rangle = 0\ \forall u \in I \rangle$$

Then $A^{\perp} = K\mathcal{Q}/I^{\perp}$.

Example 1.24. Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the following quiver



and $I = \langle ad, bc - de \rangle$. To compute $I^\perp$, we let $\alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de \in V$ for $\alpha_i \in K$. We compute

$$\langle \alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de, ad \rangle = \alpha_2$$

and

$$\langle \alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de, bc - de \rangle = \alpha_3 + \alpha_4$$

Thus

$$I^\perp = \langle \alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de \mid \alpha_2 = 0 \text{ and } \alpha_3 - \alpha_4 = 0 \rangle$$
$$= \langle \alpha_1 ab + \alpha_3 (bc + de) \rangle$$
$$= \langle ab, bc + de \rangle$$

and $A^\perp = K\mathcal{Q}/\langle ab, bc + de \rangle$.

1.6 Koszul Algebras and Linear Modules

We now wish to describe an important class of algebras called *Koszul Algebras*. To do so, we require the following terminology. Let A be a finitely generated graded K-algebra as in the previous section, but not necessarily quadratic. In this section we assume all modules are graded and finitely generated in degree $j \geq i_0$, as defined below. We also assume all modules are right A-modules and the homomorphisms are degree 0 homomorphisms.

Definition 1.25. A right A-module $M = \bigoplus_{k \geq i_0} M_k$ is *generated in degree* i_0 if for all $j > i_0$, $M_j = M_{i_0} A_{j-i_0}$. A module M has a *linear resolution* if there exists a graded projective resolution

$$... \to P^i \xrightarrow{d^i} P^{i-1} \to .... \to P^0 \to M \to 0$$

where, for each i, P^i is generated in degree i. We say M is a *linear module* if it has a linear resolution.

It is important to note that if M has a linear resolution, then M is generated in degree 0. Also, since $\text{Im} d^i \subset (P^{i-1})_{\geq i-2} = P^{i-1}A^+$, we see that a linear resolution is always minimal.

Example 1.26. Let $A = K[x]/(x^2)$ and let $M = K$. Consider the following resolution

$$... \to A \xrightarrow{\bar{x}} A \xrightarrow{\bar{x}} A \xrightarrow{\bar{x}} A \xrightarrow{\bar{x}} A \to K \to 0$$

Where $P^0 = A$ is generated in degree 0, $P^1 = A[-1]$ is generated in degree 1, $P^2 = A[-2]$ is generated in degree 2, and P^n is generated in degree n for all n. Thus K is linear.

Example 1.27. Let $A = K[x]/(x^4)$ and let $M = K$. Consider the following resolution

$$... \to A(-10) \xrightarrow{\bar{x}^3} A(-7) \xrightarrow{\bar{x}} A(-4)) \xrightarrow{\bar{x}^3} A(-1) \xrightarrow{\bar{x}} A \to K \to 0$$

Where $P^0 = A$ is generated in degree 0, $P^1 = A(-1)$ is generated in degree 1, but $P^2 = A(-4)$ is generated in degree 4. Thus K is not linear.

Remark 1.28. Suppose $A = K\mathcal{Q}/I$ where I is a graded two-sided ideal of $K\mathcal{Q}$. Also suppose $\mathcal{Q}_1 = \{a_1, ..., a_m\}$ and M is a right A-module. If

$$...P^i \xrightarrow{d^i} P^{i-1} \to ... \to P^0 \to M \to 0 \tag{1.2}$$

is a minimal projective resolution of M, we can describe the maps d^i with more detail. To do so, we write $P^i = \bigoplus P_j$ and $P^{i-1} = \bigoplus Q_k$ where for every i and k, P_j, Q_k are indecomposable projective modules. We may write d^i as a matrix with entries in $\text{Hom}_A(P_j, Q_k)$. However, as indecomposable projectives, for every j, k, we may write $P_j = eA$ and $Q_k = fA$ where e, f are primitive idempotents (vertices) in A. Recall $\text{Hom}_A(eA, fA) \cong fAe$ as a K-vector space. Consequently, we may think of the entries of d^i as multiplication by elements in A.

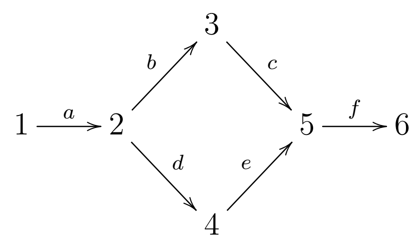
Proposition 1.29. *The resolution (1.2) is linear if and only if each d^i is a matrix whose nonzero entries are homogeneous elements of degree 1 in $K\mathcal{Q}$.*

Definition 1.30. An algebra A is *Koszul* if $\bar{A}$ is a linear A-module.

We now explore the relationship between Koszul and quadratic algebras. The following proposition is well known.

Proposition 1.31. [4, proposition 1.2.3] *If A is a Koszul algebra, then it is quadratic.*

The converse is not true. Consider the following example, due to Zacharia in unpublished notes. Let $\mathcal{Q}$ be the following quiver



and the algebra $A = K\mathcal{Q}/I$ where $I = \langle ab, ef, bc - de \rangle$. Clearly, A is quadratic. However, it is not Koszul. If we compute the projective resolution of S_1, we see that P^3 is generated in degree 4.

However, there are situations in which the converse does hold.

Proposition 1.32. [16] *Suppose $A = K\mathcal{Q}/I$ where I is generated by paths. Then A is quadratic if and only if it is Koszul.*

Proposition 1.33. [13] *Suppose A is an algebra of global dimension 2. Then A is quadratic if and only if it is Kozsul.*

Theorem 1.34. [13] *Let A be a graded K-algebra. The following are equivalent:*

1. *A is a Koszul algebra.*

2. *A is quadratic and $E(A) \cong A^{\perp}$.*

Moreover,

Proposition 1.35. [13],[4] *Let A be a Koszul algebra. Then $E(A)$ is also a Koszul algebra and $E(E(A)) \cong A$ as graded K-algebras.*

Chapter 2

The Shriek Algebra

2.1 Introduction

In this Chapter we investigate the Shriek algebra of a finite dimensional K-algebra A, which has been studied in [17], [19], and [13]. To do so, we first use [15] to form a family $\{f_i^j\}$ of elements in $K\mathcal{Q}$. We may use those elements to construct a minimial projective resolution of $\bar{A}$ over A. In doing so, we show that $\{f_i^j\}$ form a dual basis of $E(A)$. Consequently, we may denote the basis of $E(A)$ by $\{(f_i^j)^*\}$ and investigate the product of two basis elements. Then we consider the subalgebra of A generated in degrees 0 and 1, called the *shriek algebra* of A and denoted $A^!$. We see $A^!$ is of the form $K\mathcal{Q}^*/I^!$ and prove the relations of $I^!$ can be directly computed using $\{f_i^j\}$. In the case A is graded, we provide an alternate proof to the result found in [17] and [19], namely that $A^!$ is quadratic. Then we proceed to compute the relations that generate $I^!$.

2.2 Background and Notation

Let $A = K\mathcal{Q}/I$, where A is a basic finite dimensional K-algebra, I is a two sided ideal, and $\mathcal{Q}$ is a finite quiver with n vertices. Let J be the ideal of $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$ and denote by $\bar{A} = K\mathcal{Q}/J$ the top of A. In other words, $\bar{A} = S_1 \bigoplus S_2 \bigoplus .... \bigoplus S_n$ where S_i is the simple module corresponding to vertex i. By abuse of notation, for an element $q \in K\mathcal{Q}$, denote $qK\mathcal{Q}/qI$ as qA.

As defined in [15], an element in $K\mathcal{Q}$ is *uniform* (respectively *left uniform, right uniform*) if it is a linear combination of paths in $K\mathcal{Q}$, all starting at the same vertex and all ending at the same vertex (respectively, all starting at the same vertex, all ending at the same vertex).

Following [15], a minimal projective resolution of a finitely generated, right A-module M

can be given in terms of a family $\{f_i^j\}_{i,j\geq 0}$ where each $f_i^j \in K\mathcal{Q}$ is a right uniform element. Moreover, for each fixed j, the family $\{f_i^j\}$ is finite. This resolution is of the form

$$... \longrightarrow \bigoplus f_i^2 K\mathcal{Q}/f_i^2 I \longrightarrow \bigoplus f_i^1 K\mathcal{Q}/f_i^1 I \longrightarrow \bigoplus f_i^0 K\mathcal{Q}/f_i^0 I \longrightarrow M \longrightarrow 0 \tag{2.1}$$

where the maps are induced by inclusion maps in $K\mathcal{Q}$, see below. We recall how this resolution is constructed. First, the family $\{f_i^0\}_{i\geq 1}$ consists of vertices of $K\mathcal{Q}$, which yields the short exact sequence of $K\mathcal{Q}$ modules

$$0 \longrightarrow \Omega^1_{K\mathcal{Q}}(M) \longrightarrow \bigoplus_{i\geq 1} f_i^0 K\mathcal{Q} \longrightarrow M \longrightarrow 0.$$

Second, the family $\{f_i^1\}$ consists of arrows of $K\mathcal{Q}$. We may write $\Omega^1_{K\mathcal{Q}}(M) = \bigoplus_i f_i^1 K\mathcal{Q}$ since $K\mathcal{Q}$ is hereditary, [3]. Now we can inductively construct the f_i^js as follows. To construct the family $\{f_i^{n+1}\}$, consider the intersection $(\bigoplus_i f_i^n K\mathcal{Q}) \cap (\bigoplus_k f_k^{n-1} I)$. If this intersection is 0, then we stop. If it is nonzero, we may apply [8, Theorem 5.4], and we set $(\bigoplus_i f_i^n K\mathcal{Q}) \cap (\bigoplus_k f_k^{n-1} I) = \bigoplus_l f_l^{n+1'} K\mathcal{Q}$ for a set of elements $\{f_l^{n+1'}\}$. If we want the set to yield a minimal projective resolution, we may need to discard a subset of $\{f_i^{n+1'}\}$. We apply Theorems 2.2 and 2.4 of [15], which ensures us a minimal projective resolution by instructing us to discard the minimal number of elements $f^{n+1'}$ such that the remaining subset, which we denote $\{f_i^{n+1}\}$, is such that no proper K-linear combination of a subset of it is in $\bigoplus f_i^n I + \bigoplus f^{n+1'} J$. If $\{f_i^{n+1}\}$ is empty, we stop. Let l^j be the cardinality of the set $\{f_i^j\}$, in other words, $\{f_i^j\} = \{f_1^j, f_2^j, ..., f_{l^j}^j\}$. In [15], it is also determined that $f_r^j = \sum_i f_i^{j-1} h_{i,r}^{j-1,j}$ where we can assume each h is a uniform element in $K\mathcal{Q}$. We expand this notation to write f_k^j as an element in $\bigoplus f_i^{j-s} K\mathcal{Q}$ (where $0 \le s \le j$) as follows: $f_k^j = \sum_i f_i^{j-s} h_{i,k}^{j-s,j}$ for uniform elements $h_{i,k}^{j-s,j}$ in $K\mathcal{Q}$. Using this construction, we have the following filtration of $\bigoplus_{i=1}^{n} f_i^0 K\mathcal{Q}$ viewed as a right $K\mathcal{Q}$ module:

$$... \subset \bigoplus f_i^n K\mathcal{Q} \subset \bigoplus f_i^{n-1} K\mathcal{Q} \subset ... \subset \bigoplus f_i^2 K\mathcal{Q} \subset \bigoplus f_i^1 K\mathcal{Q} \subset \bigoplus f_i^1 K\mathcal{Q} \subset \bigoplus f_i^0 K\mathcal{Q}.$$

For each $n \ge 0$, if we let $\mathcal{P}^n = \bigoplus f_i^n K\mathcal{Q}/f_i^n I$ and $D^n : \mathcal{P}^n \to \mathcal{P}^{n-1}$ be the homomorphism induced by the inclusion $\bigoplus f_i^n K\mathcal{Q} \subset \bigoplus f_j^{n-1} K\mathcal{Q}$, then Theorems 1.3 and 2.4 of [15] prove that

$$(\mathcal{P}, D)... \longrightarrow \mathcal{P}^n \xrightarrow{D^n} \mathcal{P}^{n-1} ... \longrightarrow \mathcal{P}^1 \xrightarrow{D^1} \mathcal{P}^0 \longrightarrow M \longrightarrow 0 \tag{2.2}$$

is a minimal projective resolution of M over A.

Let S_i denote the simple module corresponding to vertex i, and $\epsilon_i^\bullet$ be the minimal projective resolution of S_i constructed as above. Since top$S_i = S_i$, all the f_k^js appearing in ϵ_i can be chosen to be uniform with $s(f_k^j) = v_i$. If we take the direct sum of $\epsilon_i^\bullet$ for all i, then we have a minimal projective resolution of $\bar{A}$ constructed from a family $\{f_k^j\}$ of uniform elements.

We would like to rewrite this minimal projective resolution of $\bar{A}$ using modules of the form $v_k A$ where $v_k \in \mathcal{Q}_0$ and maps of the form $(\bar{h}_{i,r}^{j-1,j})$, where $\bar{h}_{i,r}^{j-1,j}$ is the image of $h_{i,r}^{j-1,j}$ in A. We start with a minimal projective resolution of $\bar{A}$ as in (2.2) where the f_i^js are chosen to be uniform. Then we note that for a vertex v such that $f_i^j v \neq 0$, we get an isomorphism of $K\mathcal{Q}$-modules, $\Phi : f_i^j v K\mathcal{Q} \longrightarrow vK\mathcal{Q}$ where, for $\lambda \in K$ and $q \in KQ$, $\Phi(\lambda f_i^j v_k q) = \lambda v_k q$. Reducing modulo I, we still get an isomorphism, but this time, of A-modules.

For all k, j, let $v_k^j = t(f_k^j)$, the target of f_k^j. Then the set $\{v_k^j\}_{k=1}^{l^j}$ is in a $1-1$ correspondence with the set $\{f_k^j\}_{k=1}^{l^j}$ Note that if we ignore the k-indexing, the set $\{v_k^j\}$ may contain multiple copies of the same vertex. This set determines the terms in our resolution, $\mathcal{P}^j = \bigoplus_{i=1}^{l^j} v_i^j A$.

Since $f_k^j = \sum_{i=1}^{l^{j-1}} f_i^{j-1} h_{i,k}^{j-1,j}$, for each k, we have the following commutative diagram of $K\mathcal{Q}$ modules.

$$\begin{array}{ccc}
f_k^j K\mathcal{Q} & \overset{\iota}{\hookrightarrow} & \bigoplus_{i=1}^{l^{j-1}} f_i^{j-1} K\mathcal{Q} \\
\downarrow \Phi & & \downarrow \hat{\Phi} \\
v_k^j K\mathcal{Q} & \overset{(h_{i,k}^{j-1,j})}{\hookrightarrow} & \bigoplus_{i=1}^{l^{j-1}} v_i^{j-1} K\mathcal{Q}
\end{array}$$

where for each path q, $\Phi(f_k^j v_k^j q) = v_k^j q$ is an isomorphism, ι is the inclusion map, and $(h_{i,k}^{j-1,j})$ is the column matrix where the i^{th} row is multiplication by $h_{i,k}^{j-1,j}$. Also, $\hat{\Phi}$ is the diagonal $l^{j-1} \times l^{j-1}$ matrix

$$\begin{pmatrix} \Phi & 0 & \dots & 0 \\ 0 & \Phi & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Phi \end{pmatrix}$$

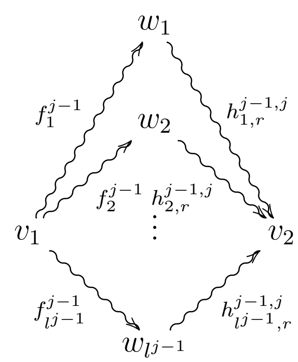
So modulo I, for each k, j, the following diagram also commutes:

$$\begin{array}{ccc}
f_k^j A & \overset{\bar{\iota}}{\longrightarrow} & \bigoplus_{i=1}^{l^{j-1}} f_i^{j-1} A \\
\downarrow \bar{\Phi} & & \downarrow \bar{\hat{\Phi}} \\
v_k^j A & \overset{(\bar{h}_{i,k}^{j-1,j})}{\longrightarrow} & \bigoplus_{i=1}^{l^{j-1}} v_i^{j-1} A
\end{array}$$

and $\bar{\Phi}$, $\bar{\bar{\Phi}}$ are isomorphisms. We may now rewrite (2.2) with $\mathcal{P}^n = \bigoplus_{k=1}^{l^n} v_k^n A$ and $D^n = (\bar{h}_{i,r}^{n-1,n})$. Consequently, we have the following projective resolution of $\bar{A}$ which is isomorphic to (2.2).

$$\ldots \longrightarrow \bigoplus_{k=1}^{l^2} v_k^2 A \xrightarrow{(\bar{h}_{i,k}^{1,2})} \bigoplus_{k=1}^{l^1} v_k^1 A \xrightarrow{(\bar{h}_{i,k}^{0,1})} \bigoplus_{1}^{l^0} v_k^0 A \xrightarrow{(\pi_i)_1^n} \bar{A} \longrightarrow 0$$

The map $(\bar{h}_{i,k}^{j-1,j})$ can be pictured with the following diagram. Suppose v_1 and v_2 are two vertices of $\mathcal{Q}$ and $s(f_r^j) = v_1$ and $t(f_r^j) = v_2$ for some f_r^j. Suppose also that $f_r^j = \sum_i f_i^{j-1} h_{i,r}^{j-1,j}$ where $\mathrm{t}(f_i^{j-1}) = w_i$.



Note that $\rightsquigarrow$ denotes a uniform element in $K\mathcal{Q}$, not necessarily a path. In this case the map $v_2 A \to \bigoplus w_i A$ would send v_2 to $\sum_{i=1}^{l^{j-1}} w_i \bar{h}_{i,r}^{j-1,j}$.

Remark 2.1. We may recover $\{f_i^j\}$ by composing the maps $(h_{i,k}^{j,j+1})$. Inductively, we can see the composition $(h_{i,k}^{0,1})(h_{i,k}^{1,2})\ldots(h_{i,k}^{j-1,j})$, is an $l^0 \times l^j$ matrix with nonzero entries in the set $\{f_i^j\}_{i=1}^{l^j}$. More specifically, each column contains exactly one non-zero entry and each row t contains only the f_i^js which start at vertex t.

In the case where $j = 0$, $(h_{i,k}^{0,1})$ is the $l^0 \times l^1$ matrix with exactly 1 arrow in each column. Because the arrows are the f_i^1s, we see that the nonzero entries in the matrix are the f_i^1s. Each column has exactly one nonzero entry, and each row t contains the f_i^1s which start at vertex t.

For the composition $(h_{i,k}^{0,1})...(h_{i,k}^{j-1,j})$, the inductive step is as follows.

Consider the composition $B = (h_{i,k}^{0,1})...(h_{i,k}^{j-2,j-1})$, which is an $l^0 \times l^{j-1}$ matrix whose nonzero entries are in the set $\{f_1^{j-1}, f_2^{j-1}, ..., f_{l^{j-1}}^{j-1}\}$ and each column has exactly one nonzero entry and each row j contains exactly the f_i^{j-1}s which start at vertex j. Writing $B = (x_{ij})$ and $H = (h_{i,k}^{j-1,j})$, then

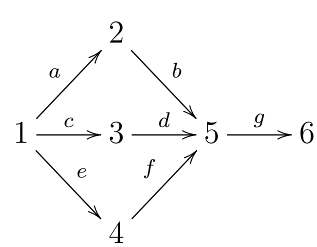
$$BH = \begin{pmatrix} x_{11} & ... & x_{1l^{j-1}} \\ \vdots & \vdots & \vdots \\ x_{l^0 1} & ... & x_{l^0 l^{j-1}} \end{pmatrix} \begin{pmatrix} h_{1,1}^{j-1,j} & ... & h_{1,l^j}^{j-1,j} \\ \vdots & \vdots & \vdots \\ h_{l^{j-1}1}^{j-1,j} & ... & h_{l^{j-1}l^j}^{j-1,j} \end{pmatrix}$$

The (s,k) entry of BH is $\sum\limits_t x_{s,t} h_{t,k}^{j-1,j}$ where $x_{s,t} = f_t^{j-1}$ if f_t^{j-1} starts at vertex s and 0 otherwise. Now suppose for some index k $f_k^j = \sum\limits_i^{l^{j-1}} f_i^{j-1} h_{i,k}^{j-1,j}$ and suppose $s(f_k^{j-1}) = m$. Then for all i such that $h_{i,k}^{n-1,n} \neq 0$, we must have $s(f_i^{j-1}) = m$. Then the (m,k) entry of BH, $\sum\limits_t x_{m,t} h_{t,k}^{j-1,j} = f_k^j$

A consequence of this remark is that $(h_{i,k}^{0,1})(h_{i,k}^{1,2})...(h_{i,k}^{j-1,j})(0,0,...,v_i^m,0,...,0)^t$ is the $l^0 \times 1$ matrix with the only nonzero entry being f_i^m.

Here is an example which illustrates all the notation discussed thus far:

Example 2.2. Let A be given by the quiver $\mathcal{Q}$:



bound by relations $I = \langle ab - cd, cd - ef, fg \rangle$. The f_i^js are as follows:

$$\begin{array}{l|l|l} f_1^0 = 1 & f_1^1 = a & f_1^2 = ab - cd \\ f_2^0 = 2 & f_2^1 = b & f_2^2 = cd - ef \\ f_3^0 = 3 & f_3^1 = c & f_3^2 = fg \\ f_4^0 = 4 & f_4^1 = d & \\ f_5^0 = 5 & f_5^1 = e & \\ f_6^0 = 6 & f_6^1 = f & \\ & f_7^1 = g & \end{array}$$

These are all the f_i^js because our algebra is of global dimension 2.

First, $f_i^0 = v_i^0$ because $\{f_i^0\}$ are the vertices of $\mathcal{Q}$. Also,

$$a = f_1^1 = f_1^0 a \Rightarrow h_{1,1}^{0,1} = a \text{ and } v_1^1 = 2$$

$$b = f_2^1 = f_2^0 b \Rightarrow h_{2,2}^{0,1} = b \text{ and } v_2^1 = 5$$

$$c = f_3^1 = f_1^0 c \Rightarrow h_{1,3}^{0,1} = c \text{ and } v_3^1 = 3$$

$$d = f_4^1 = f_3^0 d \Rightarrow h_{3,2}^{0,1} = d \text{ and } v_4^1 = 5$$

$$e = f_5^1 = f_1^0 e \Rightarrow h_{1,5}^{0,1} = e \text{ and } v_5^1 = 4$$

$$f = f_6^1 = f_4^0 f \Rightarrow h_{4,6}^{0,1} = f \text{ and } v_6^1 = 5$$

$$g = f_7^1 = f_5^0 g \Rightarrow h_{5,7}^{0,1} = g \text{ and } v_7^1 = 6$$

$$ab - cd = f_1^2 = f_1^1 b - f_3^1 c \Rightarrow h_{1,1}^{1,2} = b,\ h_{3,1}^{1,2} = -c \text{ and } v_1^2 = 5$$

$$cd - ef = f_2^2 = f_3^1 d - f_5^1 f \Rightarrow h_{3,2}^{1,2} = d,\ h_{5,2}^{1,2} = -f \text{ and } v_2^2 = 5$$

$$fg = f_3^2 = f_6^1 g \Rightarrow h_{6,3}^{1,2} = g \text{ and } v_3^2 = 6$$

Using the notation found in [15], we could write a minimal projective resolution of $\bar{A}$ as follows

$$0 \to \begin{pmatrix} (ab-cd)A \\ \oplus(cd-ef)A \\ \oplus fgA \end{pmatrix} \to \begin{pmatrix} aA \\ \oplus cA \\ \oplus eA \\ \oplus bA \\ \oplus dA \\ \oplus fA \\ \oplus gA \end{pmatrix} \to \begin{pmatrix} v_1A \\ \oplus v_2A \\ \oplus v_3A \\ \oplus v_4A \\ \oplus v_5A \\ \oplus v_6A \end{pmatrix} \to \bar{A} \to 0$$

Using our notation, we rewrite the resolution as

$$0 \to \begin{pmatrix} v_5A \\ \oplus v_5A \\ \oplus v_6A \end{pmatrix} \xrightarrow{D^2} \begin{pmatrix} v_2A \\ \oplus v_3A \\ \oplus v_4A \\ \oplus v_5A \\ \oplus v_5A \\ \oplus v_5A \\ \oplus v_6A \end{pmatrix} \xrightarrow{D^1} \begin{pmatrix} v_1A \\ \oplus v_2A \\ \oplus v_3A \\ \oplus v_4A \\ \oplus v_5A \\ \oplus v_6A \end{pmatrix} \xrightarrow{D^0} \bar{A} \to 0$$

where

$$D^1 = \begin{pmatrix} \bar{a} & \bar{c} & \bar{e} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{g} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$D^2 = \begin{pmatrix} \bar{b} & 0 & 0 \\ -\bar{d} & \bar{d} & 0 \\ 0 & -\bar{f} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{g} \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix $D^1 = (\bar{h}^{0,1}_{i,k}) = (h^{0,1}_{i,k})$, is a matrix with exactly one nonzero entry in each column, and the nonzero entries in the i^{th} row are the arrows which start at vertex v^0_i. Also,

$$D^1D^2 = \begin{pmatrix} \overline{ab-cd} & \overline{cd-ef} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{f}g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the 6×3 matrix whose nonzero entries are the images of f^2_i in A. Each column has exactly one nonzero entry, and each row t contains the images of the f^2_i that starts at vertex t. Also, $\mathcal{P}^2 = \bigoplus_{k=1}^{l^2} v^2_k A = v^2_1 A \bigoplus v^2_2 A \bigoplus v^2_3 A = P_3 \bigoplus P_3 \bigoplus P_6$ where P_i is the indecomposable projective module at vertex i. Note P_3 appears twice as a summand of $\mathcal{P}^2$ because there are exactly two f^2_is which terminate at vertex 2.

2.3 The Extensions of $\bar{A}$

The minimal projective resolution constructed in the previous subsection gives information about the Ext-algebra, $E(A)$, of A.

Let us recall from [15] the structure of $E(A)$ using the f^j_is. As a graded algebra, $E(A) = \bigoplus_{m \geq 0} \mathrm{Ext}^m_A(\bar{A}, \bar{A})$ with the usual addition, and multiplication given by the Yoneda product. In order to understand how $E(A)$ looks, we start with our minimal projective resolution of $\bar{A}$.

$$\ldots \longrightarrow \bigoplus_{k=1}^{l^2} v^2_k A \xrightarrow{(\bar{h}^{1,2}_{i,k})} \bigoplus_{k=1}^{l^1} v^1_k A \xrightarrow{(\bar{h}^{0,1}_{i,k})} \bigoplus_1^{l^0} v^0_k A \xrightarrow{(\pi_i)^n_1} \bar{A} \longrightarrow 0$$

where $(\pi_i)_1^n$ is defined below. From 1.1, we have

$$\begin{aligned}\operatorname{Ext}_A^m(\bar{A},\bar{A}) &= \operatorname{Hom}_A(\bigoplus_{k=1}^{l^m} v_k^m A, \bar{A}) \\ &= \bigoplus_{k=1}^{l^m} \operatorname{Hom}_A(v_k^m A, \bar{A}) \\ &= \bigoplus_{k=1}^{l^m} \operatorname{Hom}_A(v_k^m A, S_k^m)\end{aligned}$$

where S_j^m is the simple module corresponding to vertex v_j^m. Recall it is possible for $S_j^m \cong S_i$ for some i as simple A-modules. Note that $\operatorname{Hom}_A(v_j^m A, S_j^m)$ is a one-dimensional vector space with basis element π_j^m, where for each j, $\pi_j^m : v_j^m A \to \bar{A}$ is the map taking v_j^m to $(0,0,...,1,...,0) \in K^{l^0}$, where 1 is in the entry corresponding to the top of $v_j^m A$. So the set $\{(\pi_1^m, 0, ..., 0), (0, \pi_2^m, 0, ..., 0), ..., (0, ..., 0, \pi_{l^m}^m)\}$ is an ordered basis of $\operatorname{Ext}_A^m(\bar{A},\bar{A})$. For notational purposes, we have the following:

Definition 2.3. For each i, m, define $(f_i^m)^* = (0, ..., 0, \pi_i^m, 0, .., 0)$ to be the matrix with nonzero i^{th} entry π_i^m.

With this notation, it follows that $\{(f_i^m)^*\}$ is an ordered basis of $\operatorname{Ext}_A^m(\bar{A},\bar{A})$. This means that any element μ in $\operatorname{Ext}_A^m(\bar{A},\bar{A})$ can be written as $\mu = a_1(f_1^m)^* + a_2(f_2^m)^* + ... + a_{l^m}(f_{l^m}^m)^*$ for $a_i \in K$. Let's look at an example.

Example 2.4. Let A be as in example 2.2. For the vertex set $\{1,2,3,4,5,6\}$, denote the simple modules corresponding to the vertices by $S_1, S_2, ..., S_6$ respectively, and maps $\pi_1 : P_1 \to S_1, ..., \pi_6 : P_6 \to S_6$ their projective covers. Because $\mathcal{P}^2 = \bigoplus_{i=1}^{l^2} v_i^2 A$, we see that $\operatorname{Ext}_A^2(\bar{A},\bar{A}) = \operatorname{Span}\{\pi_1^2, \pi_2^2, \pi_3^2\}$. where $\pi_1^2 = \pi_5$, $\pi_2^2 = \pi_5$, and $\pi_3^2 = \pi_6$.

2.4 Multiplication of Elements in $E(A)$

Recall $\{f_i^1\}$ is the set of arrows in $\mathcal{Q}$. For any path $p = f_{k_1}^1 ... f_{k_t}^1$ in $\mathcal{Q}$, we would ultimately like to compute the product $\prod_{i=1}^{t} (f_{k_i}^1)^*$ in $E(A)$. Doing so would give us the multiplication structure of the subalgebra of $E(A)$ generated in degrees 0 and 1. However, to compute that product, we need the following definitions.

Definition 2.5. Suppose $f_i^j = \sum_1^m \lambda_k p_k$ is a uniform element of $K\mathcal{Q}$ where each λ_k is a nonzero field element. We define the *support* of f_i^j to be the set $\{\lambda_k p_k \mid 1 \leq k \leq m\}$. For example, in 2.2, we can see that $\{cd, -ef\}$ is the support of $cd - ef$. We will denote the support of f_i^j as $\text{supp}(f_i^j)$.

Since the $\{(f_i^j)^*\}_{i,j}$ form an ordered basis of $E(A)$, we want to express the product $(f_r^t)^*(f_t^1)^*$ as a linear combination of elements in $\{(f_i^j)^*\}$ and determine the multiplication constants. Because $f_k^{j+1} = \sum_{i=1}^{l^j} f_i^j h_{i,k}^{j,j+1}$, we can express any f_k^{j+1} as an element of $\bigoplus_{i=1}^{l^j} f_i^j K\mathcal{Q}$.

Lemma 2.6. $(f_r^j)^*(f_t^1)^* \neq 0$ *if and only if there exists some* f_k^{j+1} *and a* $1 \leq t \leq l^1$ *such that a scalar multiple of* f_t^1 *is a nonzero term of* $h_{r,k}^{j,j+1}$.

Proof. Without loss of generality, assume $r = 1$ and $j \geq 2$. Thus we are computing $(f_1^j)^*(f_t^1)^*$. Also assume $\text{t}(f_1^j) = v_1^0$ and $\text{s}(f_t^1) = v_s^0$ for some s. We consider the following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \bigoplus_{k=1}^{l^{j+1}} v_k^{j+1}A & \xrightarrow{(\bar{h}_{i,k}^{j,j+1})} & \bigoplus_{k=1}^{l^{j}} v_k^{j}A & \xrightarrow{(\bar{h}_{i,k}^{j-1,j})} \cdots & & \bigoplus_{k=1}^{l^{0}} v_k^{0}A & \longrightarrow & \bar{A} & \longrightarrow & 0 \\
& \downarrow L^1 & & \downarrow L^0 & \searrow (f_1^j)^* & & & & & & \\
\cdots \longrightarrow & \bigoplus_{k=1}^{l^{1}} v_k^{1}A & \xrightarrow{(\bar{h}_{i,k}^{0,1})} & \bigoplus_{k=1}^{l^{0}} v_k^{0}A & \longrightarrow & \bar{A} & \longrightarrow & 0 & & & \\
& \downarrow (f_1^1)^* & & & & & & & & & \\
& \bar{A} & & & & & & & & &
\end{array} \tag{2.3}$$

where $(f_1^j)^* = \begin{pmatrix} \pi_1^j & 0 & ... & 0 \end{pmatrix}$, $f_t^1 = \begin{pmatrix} 0 & \cdots & \pi_t^1 & \cdots & 0 \end{pmatrix}$ and $(\bar{h}_{i,k}^{0,1})$ is a matrix such that the nonzero entries in the 1^{st} row are the arrows starting at vertex v_1^0.

$$L^0 = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Notice that

$$L^0(\bar{h}_{i,k}^{j,j+1}) = \begin{pmatrix} \bar{h}_{1,1}^{j,j+1} & \bar{h}_{1,2}^{j,j+1} & ... & \bar{h}_{1,l^2}^{j,j+1} \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}$$

and let

$$L^1 = \begin{pmatrix} a_{1,1} & ... & a_{1,l^{j+1}} \\ a_{2,1} & ... & a_{2,l^{j+1}} \\ \vdots & \vdots & \vdots \\ a_{l^1,1} & ... & a_{l^1,l^{j+1}} \end{pmatrix}$$

where $a_{i,j}$ is right multiplication by an element $a_{i,j}$ in A.

($\Rightarrow$) Assume $(f_1^j)^*(f_t^1)^* \neq 0$. Then $(f_1^j)^*(f_t^1)^* = (f_t^1)^* L^1 = \begin{pmatrix} \pi_t^1 a_{t,1} & ... & \pi_t^1 a_{t,l^{j+1}} \end{pmatrix} \neq 0$ implies there exists some index m and vertex v_m^{j+1} such that $v_m^{j+1} = v_t^1$ as vertices and $a_{t,m} : v_m^{j+1}A \to v_t^1 A$ is an isomorphism. Thus $a_{t,m}$ is multiplication by an invertible element of $v_t^1 A v_t^1$. So we assume this element is of the form $\lambda v_t^1 + v_t^1 r v_t^1$ where $\lambda \in K^*$ and $r \in \mathfrak{r}$. Thus $f_t^1 a_{t,m} \neq 0$ because $f_t^1 \lambda v_t^1 \neq 0$.

Now consider $\mathrm{s}(f_t^1) = v_s^0$ for some index s. Then $h_{s,1}^{0,1} = f_t^1$. The (s,m) entry of the product $(\bar{h}_{i,k}^{0,1})L^1$ is

$$\sum_{k=1}^{l^0} h_{s,k}^{0,1} a_{k,m} = h_{s,t}^{0,1} a_{t,m} + \sum_{k \neq t} h_{s,k}^{0,1} a_{k,m}$$

$$= f_t^1 a_{t,m} + \sum_{k \neq t} h_{s,k}^{0,1} a_{k,m}.$$

Because $\{h_{i,k}^{0,1}\} = \{f_k^1\}$ is the set of arrows in $\mathcal{Q}$, we know that if $k \neq t$, then $h_{s,k}^{0,1} \neq f_t^1$ (See Remark 2.1). Thus the (s,m) entry of $(\bar{h}_{i,k}^{0,1})L^1$ contains $\bar{f}_t^1 a_{1,m}$ as a nonzero term. However, by the commutativity of (2.3), the (s,m) entry of $L^0(\bar{h}_{i,k}^{j,j+1})$ must be nonzero. That is the case if and only if $s = 1$. Consequently, $f_t^1 = h_{1,m}^{0,1}$. Moreover, again by the commutativity of (2.3), $\bar{f}_t^1 a_{1,m} + \sum_{k \neq t} \bar{h}_{1,k}^{0,1} a_{k,m} = \bar{h}_{1,m}^{j,j+1}$, which proves the claim.

($\Leftarrow$). Suppose $\bar{h}_{1,m}^{j,j+1} = \lambda \bar{f}_t^1 + \bar{q}$ for some element q in $K\mathcal{Q}$ and $\lambda \in K^*$. By diagram chasing, $\lambda \bar{f}_t^1 + \bar{q} = \bar{f}_t^1 a_{1,m} + \sum_{k \neq t} \bar{h}_{1,k}^{0,1} a_{k,m}$. Note for all $k \neq t, \bar{h}_{i,k}^{0,1} \neq \bar{f}_t^1$, thus $\bar{h}_{i,k} a_{k,m} \neq \alpha \bar{f}_t^1$ for any $\alpha \in K^*$. Thus multiplication by $\lambda \bar{f}_t^1$ must be a term of $\bar{f}_t^1 a_{t,m}$, which implies that $a_{t,m}$ is an isomorphism, and the claim holds. □

Recall that for every j, k, $f_k^{j+1} = \sum f_r^j h_{r,k}^{j,j+1}$ where $\{f_i^j\}$ yield a minimal projective resolution of $\bar{A}$.

Definition 2.7. Let $Z_{r,t}^{j,j+1} = \{f_k^{j+1} \mid$ a scalar multiple of f_t^1 is a nonzero term of $h_{r,k}^{j,j+1}\}$

Proposition 2.8. $(f_r^j)^* (f_t^1)^* = \sum_{f_k^{j+1} \in Z_{r,t}^{j,j+1}} \lambda_k (f_k^{j+1})^*$ *where* $\lambda_k \in K^*$.

Proof. Without loss of generality, we may suppose $r = 1$. We want to compute $(f_1^j)^* (f_t^1)^*$. Let $Z_{1,t}^{j,j+1} = \{f_{i_1}^{j+1}, f_{i_2}^{j+1}, \ , \ldots \ , f_{i_m}^{j+1}\}$. Clearly, we may rearrange the ordering of the f_i^{j+1}s such that $f_1^{j+1} = f_{i_1}^{j+1}, f_2^{j+1} = f_{i_2}^{j+1}, ..., f_m^{j+1} = f_{i_m}^{j+1}$. Consider the following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \bigoplus_{k=1}^{l^{j+1}} v_k^{j+1}A & \xrightarrow{(\bar{h}_{i,k}^{j,j+1})} & \bigoplus_{k=1}^{l^{j}} v_k^{j}A & \xrightarrow{(\bar{h}_{i,k}^{j-1,j})} & \cdots & & \bigoplus_{k=1}^{l^{0}} v_k^{0}A & \longrightarrow & \bar{A} \longrightarrow 0 \\
& \downarrow L^1 & & \downarrow L^0 & \searrow (f_1^j)^* & & & & & \\
\cdots \longrightarrow & \bigoplus_{k=1}^{l^{1}} v_k^{1}A & \xrightarrow{(\bar{h}_{i,k}^{0,1})} & \bigoplus_{k=1}^{l^{0}} v_k^{0}A & \longrightarrow & \bar{A} & \longrightarrow & 0 & & \\
& \downarrow (f_1^1)^* & & & & & & & & \\
& \bar{A} & & & & & & & &
\end{array} \tag{2.4}$$

in which all maps are as in the proof of 2.6. Notice $f_s^{j+1} \in Z_{1,t}^{j,j+1}$ if and only if $1 \leq s \leq m$ by construction. Moreover, $f_s^{j+1} \in Z_{1,t}^{j,j+1}$ if and only if $\lambda_k \bar{f}_t^1$ is a term in $\bar{h}_{1,s}^{j,j+1}$ for some $\lambda_k \in K^*$. By applying proposition 2.8, we see that $f_s^{j+1} \in Z_{1,t}^{j,j+1}$ if and only if $a_{t,s}$ is an isomorphism; equivalently, if and only if $a_{t,s} = \lambda_k v_t^1 + v_t^1 q v_t^1$ where $q \in \mathfrak{r}$. Note

$$\begin{aligned}
\pi_t^1 a_{t,s} &= \pi_t^1(\lambda_k v_t^1 + v_t^1 q v_t^1) \\
&= \lambda_k \pi_t^1
\end{aligned}$$

So, by the Yoneda product,

$$\begin{aligned}
(f_1^j)^*(f_t^1)^* &= \begin{pmatrix} 0 & 0 & \cdots & \pi_t^1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{1,1} & \ldots & a_{1,l^2} \\ a_{2,1} & \ldots & a_{2,l^2} \\ \vdots & \vdots & \vdots \\ a_{l^1,1} & \ldots & a_{l^1,l^2} \end{pmatrix} \\
&= \begin{pmatrix} \pi_t^1 a_{t,1} & \pi_t^1 a_{t,2} & \ldots & \pi_t^1 a_{t,l^2} \end{pmatrix} \\
&= \begin{pmatrix} \pi_t^1 a_{t,1} & \pi_t^1 a_{t,2} & \ldots & \pi_t^1 a_{t,m} & 0 & \ldots & 0 \end{pmatrix} \\
&= \sum_{f_k^{j+1} \in Z_{1,t}^{j,j+1}} \lambda_k (f_k^{j+1})^*
\end{aligned}$$

which proves the proposition. $\square$

The following corollary restates proposition 2.8 using the matrix $(h^{j,j+1})$. Sometimes computations are easier this way.

Corollary 2.9. $(f_r^j)^*(f_t^1)^* = \sum \lambda_k (f_k^{j+1})^*$ where $\lambda_k \in K^*$ and the sum is taken over all k such that the (r,k) entry of $(h^{j,j+1})$ contains $\lambda_k f_t^1$ as a nonzero term.

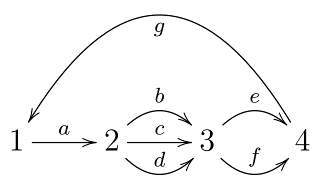
We briefly explore the case when $j = 1$. Recall for any k we may write $f_k^2 = \sum_{i=1}^{l^1} f_i^1 h_{i,k}^{1,2}$ where $\{f_i^1\} = \mathcal{Q}_1$. Consequently, for any index r, if $f_r^1 h_{r,k}^{1,2} \neq 0$, then $f_r^1 h_{r,k}^{1,2}$ is a nonzero summand of f_k^2. But

$$Z_{r,t}^{1,2} = \{f_k^2 \mid h_{r,k}^{1,2} \text{ contains a scalar multiple of } f_t^1 \text{ as a nonzero term }\}$$

and so we have the following corollary:

Corollary 2.10. $f_k^2 \in Z_{r,t}^{1,2}$ if and only if $\lambda f_r^1 f_t^1$ is a nonzero term of f_k^2.

Example 2.11. Let us work with an example from [15]. Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the following quiver:



and $I = \langle ab + ac + ad, be + df, be + ce, de + df, eg, ga \rangle$. We now compute the following:

$f_1^0 = 1$	$f_1^1 = a$	$f_1^2 = ab + ac + ad$	$f_1^3 = ceg + deg$	$f_1^4 = f_1^3 a$	$f_1^5 = f_1^4(b + c + d)$
$f_2^0 = 2$	$f_2^1 = b$	$f_2^2 = be + df$	$f_2^3 = (be + ce)g$	$f_2^4 = f_2^3 a$	$f_2^5 = f_2^4(b + c + d)$
$f_3^0 = 3$	$f_3^1 = c$	$f_3^2 = be + ce$	$f_3^3 = ega$	$f_3^4 = f_3^3(b + c + d)$	
$f_4^0 = 4$	$f_4^1 = d$	$f_4^2 = de + df$	$f_4^3 = ga(b + c + d)$		
	$f_5^1 = e$	$f_5^2 = eg$			
	$f_6^1 = f$	$f_6^2 = ga$			
	$f_7^1 = g$				

We can construct the following minimal projective resolution of $\bar{A}$

$$0 \to \begin{pmatrix} v_3A \\ \oplus v_3A \end{pmatrix} \xrightarrow{D^5} \begin{pmatrix} v_2A \\ \oplus v_2A \\ \oplus v_3A \end{pmatrix} \xrightarrow{D^4} \begin{pmatrix} v_1A \\ \oplus v_1A \\ \oplus v_2A \\ \oplus v_3A \end{pmatrix} \xrightarrow{D^3} \begin{pmatrix} v_3A \\ \oplus v_4A \\ \oplus v_4A \\ \oplus v_4A \\ \oplus v_1A \\ \oplus v_2A \end{pmatrix} \xrightarrow{D^2} \begin{pmatrix} v_2A \\ \oplus v_3A \\ \oplus v_3A \\ \oplus v_3A \\ \oplus v_4A \\ \oplus v_4A \\ \oplus v_1A \end{pmatrix} \xrightarrow{D^1} \begin{pmatrix} v_1A \\ \oplus v_2A \\ \oplus v_3A \\ \oplus v_4A \end{pmatrix} \xrightarrow{D^0} \bar{A} \to 0$$

where

$$D^1 = \begin{pmatrix} \bar{a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{b} & \bar{c} & \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{e} & \bar{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{g} \end{pmatrix}$$

and

$$D^2 = \begin{pmatrix} \overline{b+c+d} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{e} & \bar{e} & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \\ 0 & \bar{f} & 0 & \overline{e+f} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{g} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{a} \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\bar{g} & 0 & 0 & 0 \\ \bar{g} & \bar{g} & 0 & 0 \\ \bar{g} & 0 & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & \overline{b+c+d} \end{pmatrix}$$

$$D^4 = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & \overline{b+c+d} \\ 0 & 0 & 0 \end{pmatrix}$$

$$D^5 = \begin{pmatrix} \overline{b+c+d} & 0 \\ 0 & \overline{b+c+d} \\ 0 & 0 \end{pmatrix}$$

Here are some example computations:

1. $(f_1^3)^*(f_1^1)^*$: Note $Z_{1,1}^{3,4} = \{f_k^4 \mid$ for some $\lambda_k \in K^*, \lambda_k f_1^1$ is a nonzero term in $h_{1,k}^{3,4}\}$. By inspection of D^4, we see the $(1,k)$ entry is $h_{1,k}^{3,4}$. Thus $Z_{1,1}^{3,4} = \{f_1^4\}$. We compute $(f_1^3)^*(f_1^1)^* = (f_1^4)^*$.

2. $(f_3^2)^*(f_7^1)^*$:

   Note $Z_{3,7}^{2,3} = \{f_k^3 \mid$ For some $\lambda_k \in K^*, \lambda_k f_7^1$ is a nonzero term in $h_{3,k}^{2,3}\}$. By inspection of D^3, we see the $(3,k)$ entry is $h_{3,k}^{2,3}$. Thus $Z_{3,7}^{2,3} = \{f_1^3, f_2^3\}$. We compute $(f_3^2)^*(f_7^1)^* = (f_1^3)^* + (f_2^3)^*$.

3. $(f_2^1)^*(f_5^1)^*(f_7^1)^*$: First we compute $(f_2^1)^*(f_5^1)^*$. By 2.10 we see $(f_2^1)^*(f_5^1)^* = (f_2^2)^*+(f_3^2)^*$. Thus

   $$\begin{aligned}(f_2^1)^*(f_5^1)^*(f_7^1)^* &= ((f_2^2)^* + (f_3^2)^*)(f_7^1)^* \\ &= (f_2^2)^*(f_7^1)^* + (f_3^2)^*(f_7^1)^*\end{aligned}$$

   Note $Z_{2,7}^{2,3} = \{f_k^3 \mid$ For some $\lambda_k \in K^*, \lambda_k f_7^1$ is a term in$h_{2,k}^{2,3}\}$. By inspection of D^3, we see the $(2,k)$ entry is $h_{2,k}^{2,3}$. Thus $Z_{2,7}^{2,3} = \{f_1^3\}$. We compute $(f_3^2)^*(f_7^1)^* = -(f_1^3)^*$. Note $(f_3^2)^*(f_7^1)^* = (f_1^3)^* + (f_2^3)^*$ by the previous example. Thus

   $$\begin{aligned}(f_2^1)^*(f_5^1)^*(f_7^1)^* &= ((f_2^2)^* + (f_3^2)^*)(f_7^1)^* \\ &= (f_2^2)^*(f_7^1)^* + (f_3^2)^*(f_7^1)^* \\ &= -(f_1^3)^* + (f_1^3)^* + (f_2^3)^* \\ &= (f_2^3)^*\end{aligned}$$

We now present two lemmas.

Lemma 2.12. *Let p be a path in $\mathcal{Q}$ and $\lambda \in K^*$. If λp is in the support of some f_i^j with $j \geq 0$, then $l(p) \geq j$.*

Proof. Proceed by induction on j. For $j = 0, 1$ the result is clear. Let $f_i^j = \sum_{k=1}^{l^{j-2}} f_k^{j-2} h_{k,i}^{j-2,j}$ where $h_{k,i}^{j-2,j} \in I$ and assume $p \in \text{supp}(f_i^j)$. Then $p = qr$ where $q \in \text{supp}(f_k^{j-2})$ for some k, and $r \in I$. By the inductive hypothesis, $l(q) \geq j - 2$. Because I is admissible, we have $l(r) \geq 2$. The result follows. □

Definition 2.13. Let $p = f_{k_1}^1 ... f_{k_m}^1$ be a path in $\mathcal{Q}$. Define

$$V_p = \{f_k^m \mid \lambda p \in \text{supp}(f_k^m) \text{ for some } \lambda \in K^*\}$$

Lemma 2.14. *For every path $p = f_{k_1}^1 ... f_{k_m}^1$ in $\mathcal{Q}$, we have $\prod_{i=1}^{m} (f_{k_i}^1)^* = \sum_{f_k^m \in V_p} \lambda_k^m (f_k^m)^*$ for $\lambda_k^m \in K$ such that $\lambda_k^m p \in \text{supp}(f_k^m)$.*

Observe that $\{(f_k^m)^*\}$ form a basis of $E(A)$ so such a unique respresentation always exists. However, 2.14 determines the multiplication constants λ_k^m. In particular, if $\lambda_k^m p \notin \text{supp}(f_k^m)$, by the definition, we know that $\lambda_k^m = 0$. This will simplify some computations found later in the chapter.

Proof. Proceed by induction on m. For $m = 2$, the claim holds by 2.10. So assume the claim holds for all the paths of length less than m and consider $p = \prod_{i=1}^{m} f_{k_i}^1$. Let $q = \prod_{i=1}^{m-1} f_{k_i}^1$, then

$$\begin{aligned}
\prod_{i=1}^{m} (f_{k_i}^1)^* &= \left(\prod_{i=1}^{m-1} (f_{k_i}^1)^* \right) (f_{k_m})^* \\
&= \sum_{f_i^{m-1} \in V_q} \lambda_i^{m-1} (f_i^{m-1})^* (f_{k_m})^* \\
&= \sum_{f_i^{m-1} \in V_q} \lambda_i^{m-1} \left(\sum_{f_k^m \in Z_{i,m}^{m-1,m}} \lambda_{ik} (f_k^m)^* \right) \\
&= \sum_{f_i^{m-1} \in V_q} \sum_{f_k^m \in Z_{i,m}^{m-1,m}} \lambda_i^{m-1} \lambda_{ik} (f_k^m)^*
\end{aligned}$$

where λ_{ik} is the constant such that $\lambda_{ik} f_{k_m}^1 \in \text{supp}(h_{i,k}^{m-1,m})$. However, for all i in the sum, f_i^{m-1} contains $\lambda_i^{m-1} f_{k_1}^1 ... f_{k_{m-1}}^1$ in its support. Thus we may rewrite the sum in the form $\sum_{k=1}^{l^m} \beta_k (f_k^m)^*$ where $\beta_k = 0$ if either the following 2 conditions hold:

1. $f_t^m = \sum f_i^{m-1} h_{i,t}^{m-1,m}$ and there does not exist an index j and nonzero constant λ such that $\lambda f_{k_1}^1 ... f_{k_m}^1$ is a term in $(f_j^{m-1} h_{j,k}^{m-1,m})$. This is because in order for f_k^m to be included in the above sum, we require that in the decomposition $f_k^m = \sum f_i^{m-1} h_{i,k}^{m-1,m}$ there exists and index j such that $\lambda_{i,k} f_{k_m}^1$ is a term of $h_{j,k}^{m-1,m}$ and $\lambda_j^{m-1} f_{k_1}^1 ... f_{k_{m-1}}^1$ is in the support of f_i^{m-1} Thus if there does not exist such a j and nonzero constant λ such that $\lambda f_{k_1}^1 ... f_{k_m}^1$ is a term in $(f_j^{m-1} h_{j,k}^{m-1,m})$, then f_k^m does not appear in the above sum.

2. $\sum_i \lambda_i^{m-1} \lambda_{ik} = 0$.

Otherwise, $\beta_k = \sum_S \lambda_i^{m-1} \lambda_{i,k}$ where $S = \{i \mid f_i^{m-1} \in V_q \text{ and } \lambda_{ik} \in \text{supp}(h_{i,k}^{m-1,m})\}$. We claim that β_k is chosen so that $\beta_k f^1_{k_1}...f^1_{k_m} \in \text{supp}(f^m_k)$. Note

$$\begin{aligned} f^m_k &= \sum_{i=1}^{l^{m-1}} f^{m-1}_i h^{m-1,m}_{i,t} \\ &= \sum_{f^{m-1}_i \in V_q} f^{m-1}_i h^{m-1,m}_{i,t} + \sum_{f^{m-1}_i \notin V_q} f^{m-1}_i h^{m-1,m}_{i,t} \\ &= \sum_{S_1} f^{m-1}_i h^{m-1,m}_{i,t} + \sum_{S_2} f^{m-1}_i h^{m-1,m}_{i,k} + \sum_{f^{m-1}_i \notin V_q} f^{m-1}_i h^{m-1,m}_{i,t} \end{aligned}$$

where

$$S_1 = \{i \mid f^{m-1}_i \in V_q \text{ and } \lambda_{ik} f^1_{k_m} \in \text{supp}(h^{m-1,m}_{i,k})\}$$

and

$$S_2 = \{i \mid f^{m-1}_i \in V_q \text{ and } \lambda_{ik} f^1_{k_m} \notin \text{supp}(h^{m-1,m}_{i,k})\}$$

If γ_i is a nonzero constant such that $\gamma_i f^1_{k_1}...f^1_{k_m} \in \text{supp}(f^{m-1}_i h^{m-1,m}_{i,t})$, then $i \in S_1$ and $\gamma_i = \lambda^{m-1}_i \lambda_{ik}$. Moreover, every nonzero term of the sum $\sum_{i \in S_1} \gamma_i f^1_{k_1}...f^1_{k_m}$ is an element of $\text{supp}(f^m_k)$. Thus

$$\beta_k f^1_{k_1}...f^1_{k_m} \in \text{supp}(f^1_k) \qquad \square$$

It is important to note that if $\lambda^m_k p \notin \text{supp}(f^m_k)$ for any k, then $\prod_{i=1}^{m} (f^1_{k_i})^* = 0$

2.5 The Shriek Algebra

The *shriek algebra* $A^!$ is the subalgebra of $E(A)$ generated by $E(A)_0$ and $E(A)_1$. It is well known how to compute $A^!$ in the case where A is quadratic.

Theorem 2.15. *([20],[17], [19]) Given a quadratic algebra $A = K\mathcal{Q}/I$, $A^! = A^\perp$.*

In this section we show how to compute $A^!$ for any finite dimensional K-algebra A using $\{f_i^j\}$.

Let $\mathcal{Q}^*$ denote the quiver of $A^!$. $\mathcal{Q}_0^* = \{(f_i^0)^*\}_{i=1}^{l^0}$, and $\mathcal{Q}_1^* = \{(f_i^1)^*\}_{i=1}^{l^1}$ where $(f_i^0)^* \xrightarrow{(f_k^1)^*} (f_j^0)^*$ if and only if $f_i^0 \xrightarrow{f_k^1} f_j^0$. Because $\{(f_i^0)^*\}$ is in 1-1 correspondence with the vertices of $\mathcal{Q}$ and $\{(f_i^1)^*\}$ is in 1-1 correspondence with the arrows of $\mathcal{Q}$, we see that $\mathcal{Q} \cong \mathcal{Q}^*$ as quivers.

Definition 2.16. $Z_i^j := \text{supp}(f_i^j) \cap V^j$ where V^j is the subspace of $K\mathcal{Q}$ generated by the paths of length j. Then define $\text{supp}_j(f_i^j) := \sum_{z \in Z_i^j} z$

Notice that $\text{supp}_j(f_i^j) = 0$ if and only if f_i^j has no homogeneous terms of degree $-j$. We may think of $\text{supp}_j(f_i^j)$ as the "part" of f_i^j which is homogenous of degree $-j$.

We may generalize the definition of the bilinear form to the spaces V^j when $j \geq 2$. For $j \geq 2$, let $\{p_1^j, p_2^j, ..., p_{n_j}^j\}$ be the set of all paths of length j.

Definition 2.17. Let p and q be paths of length j. Define

$$\langle p, q \rangle_j = \begin{cases} 0 & p \neq q \\ 1 & p = q \end{cases}$$

and extend by linearity to define a bilinear form $\langle\ ,\ \rangle_j : V^j \times V^j \longrightarrow K$.

Because $\mathcal{Q} \cong \mathcal{Q}^*$ as quivers, we may define the subspaces $V^{j*} \subset K\mathcal{Q}^*$ to be the subspace spanned by the set $\{p_i^{j*}\}$. We may now define a family of subspaces of $K\mathcal{Q}$.

Definition 2.18. Let $I^!_{-j}$ be the subspace of $K\mathcal{Q}$ spanned by the set

$$\{x^* \in V^{j^*} \mid \langle x, \text{supp}_j(f_i^j)\rangle_j = 0\}$$

and define $I^! = \langle \bigoplus_{j\geq 0} I^!_{-j} \rangle$ to be a graded ideal of $K\mathcal{Q}^*$.

Theorem 2.19. *Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is a finite quiver. Fix the family $\{f_i^j\}$. Then $A^! \cong K\mathcal{Q}^*/I^!$.*

Proof. Let x^* be a homogeneous element in $K\mathcal{Q}^*$ of degree $-j$ and $\bar{x^*}$ its image in $A^!$. Because $\mathcal{Q}^* \cong \mathcal{Q}$ as quivers, we let x be the corresponding homogenous element in $K\mathcal{Q}$ of degree $-j$. We claim $\bar{x^*} = 0 \iff x^* \in I^!_{-j}$. First, we look at the case where x^* is a path in $K\mathcal{Q}^*$. We may write $x^* = \alpha \prod_{t=1}^{j} (f^1_{k_t})^*$ with $\alpha \in K^*$, and apply 2.14 to see

$$\begin{aligned}
\bar{x^*} = 0 &\iff \prod_{t=1}^{j} (f^1_{k_t})^* = 0 \\
&\iff \beta \prod_{t=1}^{j} (f^1_{k_t}) \notin \operatorname{supp}(f^j_k) \ \forall \text{ indices } k, \text{ and } \beta \in K^* \\
&\iff \langle x, \operatorname{supp}_j(f^j_k)\rangle_j = 0 \ \forall k \\
&\iff x^* \in I^!_{-j}
\end{aligned}$$

Second, we consider the case where $x^* = \sum_{i=1}^{m} \alpha_i z_i^*$ for $\alpha_i \in K^*$, $z_i^* = \prod_{t=1}^{j} (f^1_{i,k_t})^*$ is a path of length j in $\mathcal{Q}^*$, and $f^1_{i,k_t} \in \{f^1_i\}$. Then $x = \sum_{i=1}^{m} \alpha_i z_i$ where for $\alpha_i \in K^*$, $z_i = \prod_{t=1}^{j} (f^1_{i,k_t})$. Note $\bar{x^*} = \sum \alpha_i \bar{z_i^*}$ so we may assume $\alpha \bar{z_i^*} \neq 0$ for all i. Apply 2.14 to $\bar{z_i^*}$ and we see

$$\begin{aligned}
\bar{z_i^*} &= \prod_{t=1}^{j} (f^1_{i,k_t})^* \\
&= \sum_{f^j_k \in V_{z_i}} \lambda^j_{ik} (f^j_k)^*
\end{aligned}$$

where $\lambda_{i_k}^j$ is the scalar such that $\lambda_{i_k}^j z_i \in \text{supp}(f_k^j)$. Thus

$$\begin{aligned}
\bar{x^*} &= \sum_{i=1}^{m} \alpha_i \bar{z_i^*} \\
&= \sum_{i=1}^{m} \alpha_i (\sum_{f_k^j \in V_{z_i}} \lambda_{i_k}^j (f_k^j)^*) \\
&= \sum_{i=1}^{m} \sum_{f_k^j \in V_{z_i}} \alpha_i \lambda_{i_k}^j (f_k^j)^* \\
&= \sum_{f_k^j \in V_{z_i}} \left(\sum_{i=1}^{m} \alpha_i \lambda_{i_k}^j \right) (f_k^j)^*
\end{aligned}$$

Because the $\{f_i^j\}$ yield a minimal projective resolution of $\bar{A}$, we know $\{(f_1^j)^*, ..., (f_{l^j}^j)^*\}$ form a basis of $\text{Ext}_A^j(\bar{A}, \bar{A})$, so they must be linearly independent. Consequently,

$$\bar{x^*} = 0 \iff \sum_{\{i \ | \ f_k^j \in V_{z_i}\}} \alpha_i \lambda_{i_k}^j = 0$$

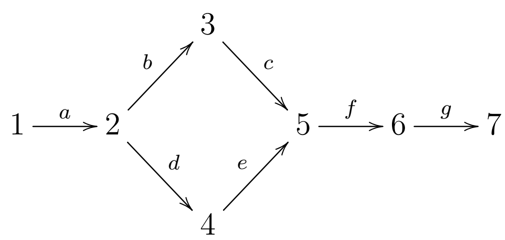
for all i. Now for each f_k^j we have

$$\begin{aligned}
\langle x, \text{supp}_j(f_k^j) \rangle_j &= \langle \sum_{i=1}^{m} \alpha_i z_i, \text{supp}_j(f_k^j) \rangle_j \\
&= \sum_{i=1}^{m} \langle \alpha_i z_i, \text{supp}_j(f_k^j) \rangle_j \\
&= \sum_{\{i \ | \ f_k^j \in V_{z_i}\}} \langle \alpha_i z_i, \text{supp}_j(f_k^j) \rangle_j \\
&= \sum_{\{i \ | \ f_k^j \in V_{z_i}\}} \alpha_i \lambda_{i_k}^j
\end{aligned}$$

Thus $\langle x, \text{supp}_j(f_k^j) \rangle = 0$ if and only if $\sum_{\{i \ | \ f_k^j \in V_{z_i}\}} \alpha_i \lambda_{i_k}^j = 0$ for all i. By the remark above, $\langle x, \text{supp}_j(f_k^j) \rangle_j = 0$ if and only if $\bar{x^*} = 0$.

$\square$

Example 2.20. Let $A = K\mathcal{Q}/I$ and let $\mathcal{Q}$ be the quiver



and $I = \langle ad, bc - de, cf, efg \rangle$. If we compute the f_i^js we get

$f_1^0 = 1$	$f_1^1 = a$	$f_1^2 = ad$	$f_1^3 = adef$	$f_1^4 = adefg$
$f_2^0 = 2$	$f_2^1 = b$	$f_2^2 = bc - de$	$f_2^3 = bcfg - defg$	
$f_3^0 = 3$	$f_3^1 = c$	$f_3^2 = cf$		
$f_4^0 = 4$	$f_4^1 = d$	$f_4^2 = efg$		
$f_5^0 = 5$	$f_5^1 = e$			
$f_6^0 = 6$	$f_6^1 = f$			
$f_7^0 = 7$	$f_7^1 = g$			

To compute $A^!$ using (2.19), we first find I_2, I_3, and I_4. To find I_2, note

$$\mathrm{supp}_2(ad) = ad$$
$$\mathrm{supp}_2(bc - de) = bc - de$$
$$\mathrm{supp}_2(cf) = cf$$

and

$$\mathrm{supp}_2(efg) = 0$$

Then for $x = \alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de + \alpha_5 cf + \alpha_6 ef + \alpha_7 fg$ where $\alpha_i \in K$,

$$\begin{aligned}
I_2 &= \{x \in V^2 \mid \langle x, \mathrm{supp}_2(f_i^2)\rangle_2 = 0\} \\
&= \{x \mid \langle x, ad\rangle_2 = 0 \text{ and } \langle x, bc - de\rangle_2 = 0 \text{ and } \langle x, cf\rangle_2 = 0\} \\
&= \{x \mid \alpha_2 = 0, \alpha_3 = \alpha_4, \text{ and } \alpha_5 = 0\} \\
&= \{x = \alpha_1 ab + \alpha_3(bc + de) + \alpha_4 de + \alpha_6 ef + \alpha_7 fg \mid \alpha_i \in K^*\} \\
&= \langle ab, bc + de, ef, fg\rangle
\end{aligned}$$

To find I_3, note $\mathrm{supp}_3(f_i^3) = 0$ for all i. Thus

$$I_3 = \langle x \in V^3 \mid \langle x, 0\rangle_3 = 0\} = V^3$$

Similarly,

$$I_4 = V^4$$

However, note V^{3^*}, and consequently V^{4^*} are annihilated by the ideal $\langle a^*b^*, b^*c^*+d^*e^*, e^*f^*, f^*g^*\rangle$ in $K\mathcal{Q}$. In other words, $V^{3^*} \subset I_2^*$. Thus $A^! = K\mathcal{Q}^*/I_2^*$.

2.5.1 The Graded Case

Let $\mathcal{Q}$ be a finite quiver and $A = K\mathcal{Q}/I$. Recall $K\mathcal{Q}$ is a graded algebra if we assign to all arrows degree 1. Let I be a 2-sided ideal in $K\mathcal{Q}$ generated by a set of homogeneous elements.

Lemma 2.21. *If f_i^2 is a homogeneous element for $i = 1, ..., l^2$, then f_i^n can be chosen to be a homogeneous element for all i, n.*

Proof. Proceed by induction on n. The case where $n = 2$ is assumed, so inductively assume f_i^j

is homogeneous for all i and for all $j \leq n$. Then consider $\bigoplus f_i^{n+1'} K\mathcal{Q} = \bigoplus f_i^n K\mathcal{Q} \cap \bigoplus f_i^{n-1} I$. Notice both I and $K\mathcal{Q}$ are graded. Also, by the induction hypothesis, for all i, j, f_i^n and f_j^{n-1} are homogeneous elements. Thus both ideals $f_i^n K\mathcal{Q}$ and $f_j^{n-1} I$ are generated by homogeneous elements. Consequently, $f_i^n K\mathcal{Q} \cap f_i^{n-1} I$ can be genereated by homogeneous elements $\{f_k^{n+1'}\}$. By construction, $f_i^{n+1} = f_k^{n+1'}$ for some k, thus f_i^{n+1} must be homogeneous. □

In the above proof, we did not need the fact that $\{f_i^n\}$ yields a minimal projective resolution of $\bar{A}$.

Recall how the set $\{f_i^n\}$ was formed. First a set $\{f_i^{n'}\}$ was constructed so that $\bigoplus f_i^{n'} K\mathcal{Q} = \bigoplus f_k^{n-1} K\mathcal{Q} \cap \bigoplus f_k^{n-2} I$. Then a subset of $\{f_i^{n'}\}$ was discarded so that the remaining elements, $\{f_i^n\}$, are such that no proper linear combination of them is in the set $\bigoplus f_i^{n-1} I + \bigoplus f_i^{n'} J$.

Lemma 2.22. *If $f_i^{n'}$ is homogeneous of degree n, then $f_i^{n'}$ cannot be discarded in the construction of $\{f_k^n\}$.*

Proof. Suppose that $x \in K\mathcal{Q}$ and x is homogeneous of degree n. If $x \in \bigoplus f_i^{n-1} I + \bigoplus f_i^{n'} J$ then any term in x has length at least $n+1$, which contradicts our choice of x. If we choose $x = f_i^{n'}$, then x is not removed in the construction of $\{f_k^n\}$. □

Remark 2.23. Let $f_i^{j+1} = \sum\limits_{i=1}^{j} f_i^j h_{i,j}^{j,j+1}$. For all j, i, k, $h_{i,k}^{j,j+1}$ has terms of length at least 1.

These next two remarks are quite technical, but provide insight into $\text{supp}_j(f_i^j)$ for any indices i, j.

Remark 2.24. $I = \bigoplus f_i^1 K\mathcal{Q} \cap \bigoplus f_i^0 I = \bigoplus f_i^{2'} K\mathcal{Q}$. We will show that $I = \langle f_i^2 \rangle$: If $f_t^{2'}$ is discarded in the construction of the set $\{f_i^2\}$, we must have

$$f_t^{2'} + \sum_{i \in T_1} \lambda_i f_i^{2'} = \sum_{j \in T_2} f_j^{2'} r_j + \sum_{k \in T_3} f_k^1 s_k$$

where T_1 is an indexing set where $t \notin T_1$ and $\lambda_i \in K^*$. T_2, T_3 are also indexing sets, $r_i \in J$, and $s_i \in I$. Because A is graded, we may assume $f_r^{2'}$ is homogeneous for all r, thus can assume $l(f_i^{2'}) = l(f_j^{2'} r_j) = l(f_k^1 s_k) = l(f_t^{2'})$ for all i, j, k in the above equation (Recall that if y is a homogeneous element of $K\mathcal{Q}$, we denote by $l(y)$ the length of any term in y). Solving for $f_t^{2'}$ we have

$$f_t^{2'} = -\sum_{i \in T_1} \lambda_i f_i^{2'} + \sum_{j \in T_2} f_j^{2'} r_j + \sum_{k \in T_3} f_k^1 s_k$$

Recall $l(r_j) \geq 1$, so we must have $l(f_j^{2'}) < l(f_t^{2'})$, which implies $t \notin T_2$. Also, $s_k \in I$ implies that $s_k \in \bigoplus f_r^{2'} K\mathcal{Q}$. However, because $l(f_k^1 s_k) = l(f_t^{2'})$, $s_k \in \bigoplus_{i \neq t} f_i^{2*} K\mathcal{Q}$. We now have that $f_t^{2'} \in \bigoplus_{i \neq t} f_i^{2'} K\mathcal{Q}$ which implies $I = \langle f_i^{2'} \rangle_{i \neq t}$. If we repeat this process for all t such that $f_t^{2'}$ is discarded, we can, without loss of generality, say $I = \langle f_1^2, f_2^2, ..., f_{l^2}^2 \rangle$ where for $1 \leq i \leq m$, $l(f_i^2) = 2$ and for $m+1 \leq i \leq l^2$, $l(f_i^2) > 2$. Notice in this case, either $\operatorname{supp}_2(f_i^2) = f_i^2$ or $\operatorname{supp}_j(f_i^2) = 0$.

Remark 2.25. Using (2.21), we see for every j, either $\operatorname{supp}_j(f_i^j) = f_i^j$ or $\operatorname{supp}_j(f_i^j) = 0$. We now use (2.22) to see that for every j, there is an index $1 \leq m_j \leq l^j$ such that $\{f_1^j, ..., f_{m_j}^j\}$ consists of all the elements of length j. That particular set $\{(f_1^j)^*, ..., (f_m^j)^*\}$ forms a basis of $\operatorname{Ext}_A^j(\bar{A}, \bar{A})_{-j}$.

For the next lemma we introduce a new definition which will simplify notation.

Definition 2.26. Let p be a path of length j where $p = \prod_{t=1}^{j} (f_{k_t}^1)$ for a sequence of arrows $(f_{k_t}^1)$. Define $p^* = \prod_{t=1}^{j} (f_{k_t}^1)^*$

In the graded case, the following lemma was essentially proved in [20], and [17], and can be found completely in [19]. The proof recognizes that $\operatorname{Ext}_A^i(\bar{A}, \bar{A})$ is the i^{th} cohomology of the cobar complex of $\bar{A}$. Here, we provide an alternate proof using explicit computations of $\{f_i^j\}$ and basic linear algebra. We also generalize the result to all finite dimensional K-algebras.

Lemma 2.27. *Let $A = K\mathcal{Q}/I$ be a finite dimensional K-algebra. Then $(Ext_A^1(\bar{A}, \bar{A}))^j$ is the subspace of $Ext_A^j(\bar{A}, \bar{A})$ spanned by the set $\{(f_i^j)^* \mid \mathrm{supp}_j(f_i^j) \neq 0\}$. If A is a graded algebra, then $\bigoplus_{j \geq 0}(Ext_A^1(\bar{A}, \bar{A}))^j = \bigoplus_{j \geq 0} Ext_A^j(\bar{A}, \bar{A})_{-j}$.*

Proof. Let $j \geq 0$. We will use the following fact from linear algebra: If V and W are vector spaces where $V \subseteq W$, W is m-dimensional, and V contains m linearly independent elements, then $V = W$. For us, we let $W = Span((f_i^j)^* \mid \mathrm{supp}_j(f_i^j) \neq 0)$ which, without loss of generality, we may assume has basis $\{(f_1^j)^*, ..., (f_m^j)^*\}$ for some $1 \leq m \leq l^j$. Also, we let $V = (\mathrm{Ext}_A^1(\bar{A}, \bar{A}))^j$. We will show that V contains m linearly independent elements. If $\{p_1, p_2, ..., p_n\}$ is the set of all paths in $\mathcal{Q}$ of length j, then $\{p_1^*, p_2^*, ..., p_n^*\}$ is a spanning set of V. By (2.14), we can write $p_k^* = \sum_{i=1}^{m} \lambda_{k,i}(f_i^j)^*$ where $\lambda_{k,i}p_k \in \mathrm{supp}(f_i^j)$ and $\lambda_{k,i} \in K$. For $\lambda_{k,i} \neq 0$, we must have $\lambda_{k,i}p_k \in \mathrm{supp}(f_i^j)$, which implies $\mathrm{supp}_j(f_i^j) \neq 0$ and $1 \leq i \leq m$. Consequently, $V \subseteq W$. Now we show that m of the p_k^*s are linearly independent. To do so, recall that $f_i^j = \sum_{k=1}^{n} \lambda_{k,i}p_k$ and we can construct the following matrix

$$M = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \dots & \lambda_{1,m} \\ \lambda_{2,1} & \lambda_{2,2} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n,1} & \lambda_{n,2} & \dots & \lambda_{n,m} \end{pmatrix}$$

We claim that the columns of M are linearly independent: Assume we have a linear combination of these columns equal to zero,

$$c_1(\lambda_{1,1}, \lambda_{2,1}, ..., \lambda_{n,1})^t + c_2(\lambda_{1,2}, \lambda_{2,2}, ..., \lambda_{n,2})^t + ... + c_m(\lambda_{1,m}, \lambda_{2,m}, ..., \lambda_{n,m})^t = 0$$

and the resulting system of linear equations,

$$\begin{gathered} c_1\lambda_{1,1} + c_2\lambda_{1,2} + ... + c_m\lambda_{1,m} = 0 \\ c_1\lambda_{2,1} + c_2\lambda_{2,2} + ... + c_m\lambda_{2,m} = 0 \\ \vdots \\ c_1\lambda_{n,1} + c_2\lambda_{n,2} + ... + c_m\lambda_{n,m} = 0 \end{gathered}$$

which implies

$$\begin{aligned} 0 &= (c_1\lambda_{1,1} + ... + c_m\lambda_{1,m})p_1 + (c_1\lambda_{2,1} + ... + c_m\lambda_{2,m})p_2 + ... + (c_1\lambda_{n,1} + ... + c_m\lambda_{n,m})p_m \\ &= \sum_{i=1}^{m} c_i(\lambda_{1,i}p_1 + \lambda_{2,i}p_2 + ... + \lambda_{n,i}p_m) \\ &= \sum_{i=1}^{m} c_i(\sum_{k=1}^{n} \lambda_{k,i}p_k) \\ &= \sum_{i=1}^{m} c_i f_i^j \end{aligned}$$

As the f_i^js are linearly independent, we have that $c_i = 0$ for all i, and this proves our claim. Consequently, the row rank of the matrix M must also be m, which means there are m linearly independent rows. Without loss of generality, suppose the first m rows are linearly independent. We will now show that $\{p_k^*\}_1^m$ is a linearly independent set of V. First we look at the first m rows of M. Because they are linearly independent, for all $c_k \in K$, we have

$$c_1(\lambda_{1,1}, \lambda_{1,2}..., \lambda_{1,m}) + ... + c_m(\lambda_{m,1}, \lambda_{m,2}, ..., \lambda_{m,m}) = 0 \tag{2.5}$$

implies $c_k = 0$ for all k. Second, we consider

$$\begin{aligned}
0 &= \sum_{k=1}^{m} c_k p_k^* \\
&= \sum_{k=1}^{m} c_k \left(\sum_{i=1}^{m} \lambda_{k,i} (f_i^j)^* \right) \\
&= \sum_{k=1}^{m} c_k (\lambda_{k,1}(f_1^j)^* + \lambda_{k,2}(f_2^j)^* + ... + \lambda_{k,m}(f_m^j)^*) \\
&= c_1(\lambda_{1,1}(f_1^j)^* + ... + \lambda_{1,m}(f_m^j)^*) + ... + c_m(\lambda_{m,1}(f_1^j)^* + ... + \lambda_{m,m}(f_m^j)^*) \\
&= (c_1\lambda_{1,1} + c_2\lambda_{2,1} + ... + c_m\lambda_{m,1})(f_1^j)^* + ... + (c_1\lambda_{1,m} + c_2\lambda_{2,m} + ... + c_m\lambda_{m,m})(f_m^j)^*
\end{aligned}$$

Because the $\{f_i^j\}_{i=1}^m$ form a basis of W, we must have the coefficients are equal to 0. In other words,

$$\begin{gathered}
c_1\lambda_{1,1} + c_2\lambda_{2,1} + ... + c_m\lambda_{m,1} = 0 \\
\vdots \\
c_1\lambda_{1,m} + c_2\lambda_{2,m} + ... + c_m\lambda_{m,m} = 0
\end{gathered}$$

However, by (2.5), we see this implies $c_k = 0$ for all k. Thus $\{p_k^*\}_{k=1}^m$ must be linearly independent elements of V. Thus, $V = W$.

Because A is length graded, $\text{supp}_j(f_i^j) \neq 0$ if and only if f_i^j is homogeneous of degree j. However, $\{(f_i^j)^* \mid l(f_i^j) = j\}$ is a basis for $W = \text{Ext}_A^j(\bar{A}, \bar{A})_{-j}$. This holds for every $j \geq 0$, we see the claim follows. □

We want to show that $A^!$ is a quadratic algebra. This was first shown in [17], but we provide a proof using $\{f_i^j\}$. To do so, we require the following construction. Write

$I = \langle f_1^1, ... f_{l^2}^2 \rangle$. There exists unique $1 \leq 1 \leq m \leq l^2$ such that $l(f_i^2) = 2$ for $1 \leq i \leq m$, and for $m \leq i \leq l^2$, $l(f_i^2) > 2$.

Definition 2.28. Let $I_Q = \langle f_1^2, ..., f_m^2 \rangle$ and $A_Q = K\mathcal{Q}/I_Q$.

Clearly, A_Q is a quadratic algebra. Because $\bigoplus_{i \geq 1} f_i^{n'} K\mathcal{Q}$ is graded, its degree n component is the subspace $Span(f_i^{n'} \mid l(f_i^{n'}) = n)$. However, by 2.22, we see this is the same subspace spanned by $\{f_i^n \mid l(f_i^n) = n\}$. Denote this subspace by C^n. Also, let D^{n-1} be the subspace $Span(f_i^{n-1} r_i \mid 1 \leq i \leq l^{n-1}, l(f_i^{n-1}) = n-1, r_i \in J$ and $r_i \notin J^2)$ and E^{n-2} be the subspace $Span(f_i^{n-2} q_i \mid 1 \leq i \leq l^{n-2}, l(f_i^{n-2}) = n-2, q_i \in I_Q$ and $q_i \notin I_Q J + J I_Q)$.

Lemma 2.29. *For every n, $C^n = D^{n-1} \cap E^{n-2}$.*

Proof. Suppose $l(f_k^n) = n$ and $f_k^n = \sum f_i^{n-1} h_{i,k}^{n-1,n}$. By (2.23), $l(h_{i,k}^{n-1,n}) \geq 1$. By (2.12), if $p \in \operatorname{supp}(f_i^{n-1})$, then $l(p) \geq n-1$. Consequently, for all i such that $h_{i,k}^{n-1,n} \neq 0$, we must have f_i^{n-1} homogeneous of degree $n-1$ and $l(h_{i,k}^{n-1,n}) = 1$. Thus $f_k^n \in D^{n-1}$. We show in a similar manner that $f_k^n \in E^{n-2}$. To do so, we know $f_k^n = \sum f_i^{n-1} h_{i,k}^{n-2,n}$ where $h_{i,k}^{n-2,n} \in I$. I is admissible, so $l(h_{i,k}^{n-2,n}) \geq 2$. By (2.12), if $p \in \operatorname{supp}(f_i^{n-2})$, then $l(p) \geq n-2$. Consequently, for all i such that $h_{i,k}^{n-2,n} \neq 0$, f_i^{n-2} is homogeneous of degree $n-2$ and $l(h_{i,k}^{n-2,n}) = 2$. By construction, $h_{i,k}^{n-2,n}$ is an element of length 2 in I if and only if $h_{i,k}^{n-2,n} = \sum_{i=1}^{m} \gamma_i f_i^2$ for $\gamma_i \in K$. Notice $\sum_{i=1}^{m} \gamma_i f_i^2 \in E^{n-2}$.

For the reverse containment, let $x \in D^{n-1} \cap E^{n-2}$. We also know that $x \in \bigoplus f_i^{n'} K\mathcal{Q}$. Thus there are two different ways to express the element x.

1. $x = \sum_{l(f_i^{n-1}) = n-1} f_i^{n-1} r_i$ where $r_i \in J$ but $r_i \notin J^2$.
2. $x = \sum_i f_i^{n'} u_i$ where $u_i \in K\mathcal{Q}$.

Clearly x is homogeneous of degree n, so if $l(u_i) \neq 0$, then $l(f_i^{n'}) \leq n$. However, by (2.12), $l(f_i^{n'}) \geq n$, which implies $l(f_i^{n'}) = n$ and $l(u_i) = 0$, which implies $x \in C^n$. □

Now we focus on the algebra A_Q. We construct uniform elements $\{t_i^j\}$ to yield a minimal projective resolution of $\bar{A}_Q$ using the methods found in [15]. We want to compare $\{t_i^j\}$ to the set $\{f_i^j\}$, where $\{f_i^j\}$ is constructed to produce a minimal projective resolution of $\bar{A}$ over A.

Lemma 2.30. *There exists a set $\{t_i^n\}$ of uniform elements of $K\mathcal{Q}$ that can be constructed so that $\{t_i^n \mid l(t_i^n) = n\} = \{f_i^n \mid l(f_i^n) = n\}$ for all $n \geq 0$ and $\{t_i^j\}$ yield a minimal projective resolution of $\bar{A}_Q$.*

Proof. Proceed by induction on n. For $n = 1$, we set $\{t_i^1\} = \{f_i^1\}$, the set of all the arrows in $\mathcal{Q}$. Inductively assume the claim holds for all $j \leq n$. The degree n part of $\bigoplus t_i^{n-1} K\mathcal{Q} \cap \bigoplus t_i^{n-2} I_Q$ is the subspace generated by the set $\{t_i^n \mid l(t_i^n) = n\}$ and we denote this subspace C_Q^n. We may use the fact that $(I_Q)_Q = I_Q$ and apply 2.29 to see $C_Q^n = D_Q^{n-1} \cap E_Q^{n-2}$ where D_Q^{n-1} is the subspace $Span(f_i^{n-1} r_i \mid 1 \leq i \leq l^{n-1}, l(f_i^{n-1}) = n-1, r_i \in J$ and $r_i \notin J^2)$ and E_Q^{n-2} is the subspace $Span(f_i^{n-2} q_i \mid 1 \leq i \leq l^{n-2}, l(f_i^{n-2}) = n-2, q_i \in I_Q$ and $q_i \notin I_Q J + J I_Q)$. Here we apply the inductive hypothesis. If D^{n-1} is the subspace $Span(f_i^{n-1} r_i \mid 1 \leq i \leq l^{n-1}, l(f_i^{n-1}) = n-1, r_i \in J$ and $r_i \notin J^2)$ and E^{n-2} is the subspace $Span(f_i^{n-2} q_i \mid 1 \leq i \leq l^{n-2}, l(f_i^{n-2}) = n-2, q_i \in I_Q$ and $q_i \notin I_Q J + J I_Q)$ we see $C_Q^n = D^{n-1} \cap E^{n-2}$ as K-vector spaces and $K\mathcal{Q}$ modules. By 2.29, we see $C^n = D^{n-1} \cap E^{n-2}$. So, $C^n = C_Q^n$, which proves our claim. □

Lemma 2.31. *Let $v_i^j \in \mathcal{Q}_0$. Then $\operatorname{Hom}_{A_Q}(v_i^j A_Q, \bar{A}_Q) \cong \operatorname{Hom}_A(v_i^j A, \bar{A})$ as K-vector spaces.*

Proof. We use the following fact from module theory: Let B be a ring, let I be a two-sided ideal in B, and let M be a right B-module. Then $\mathrm{Hom}_B(M, B/I) \cong \mathrm{Hom}_{B/I}(M/MI, B/I)$. First we set $B = A, I = J$, and $M = v_i^j A$. Then

$$\mathrm{Hom}_A(v_i^j A, \bar{A}) = \mathrm{Hom}_{\bar{A}}(v_i^j \bar{A}, \bar{A}) \cong v_i^j \bar{A}$$

If we set $B = A_Q, I = J$, and $M = v_i^j A_Q$, then

$$\mathrm{Hom}_{A_Q}(v_i^j A_Q, \bar{A}_Q) = \mathrm{Hom}_{\bar{A}_Q}(v_i^j \bar{A}_Q, \bar{A}_Q) \cong v_i^j \bar{A}_Q$$

But $\bar{A} = K\mathcal{Q}/J = \bar{A}_Q$, so the claim holds. □

We are now ready to present an alternative proof to the result in [17], [20], [19].

Theorem 2.32. *Let A be a graded algebra. Then $A^!$ is a quadratic algebra.*

Proof. We have $I_Q = \langle f_1^2, ..., f_m^2 \rangle \subseteq I$. Also, let $A_Q = K\mathcal{Q}/I_Q$. and a family of sets $\{t_i^j\}$ yield a minimal projective resolution of $\bar{A}_Q$ over $\bar{A}$. We have seen that if $\{f_i^j\}$ is constructed to create a minimal projective resolution of $\bar{A}$ over A, then the $\{t_i^j \mid l(t_i^j) = j\}$ and $\{f_i^j \mid l(f_i^j) =$

$j\}$ are equal for every j. As vector spaces,

$$\begin{aligned}
A^! &= \bigoplus (\mathrm{Ext}^1_A(\bar{A},\bar{A}))^j \\
&= \bigoplus \mathrm{Ext}^j_A(\bar{A},\bar{A})_{-j} \\
&= \bigoplus_{l(f_i^j)=j} \mathrm{Hom}_A(v_i^j A, \bar{A}) \\
&\cong \bigoplus_{l(t_i^j)=j} \mathrm{Hom}_{A_Q}(v_i^j A_Q, \bar{A}_Q) \\
&= \bigoplus \mathrm{Ext}^j_{A_Q}(\bar{A}_Q,\bar{A}_Q)_{-j} \\
&= A^!_Q
\end{aligned}$$

Consequently, $I^! = I^!_Q$ as vector spaces, which implies that $I^!/I^!_Q = 0$. $I^!_Q \subseteq I^!$ as two-sided ideals in $K\mathcal{Q}$, so $I^! = I^!_Q$ as two-sided ideals of $K\mathcal{Q}$. Since $A^! \cong A^!_Q/\left(I^!/I^!_Q\right)$ as graded algebras, we see $A^! \cong A^!_Q$ as graded K-algebras. So, $A^!$ is quadratic. □

Corollary 2.33. Let $A = K\mathcal{Q}$ be a graded algebra. Then $A^! = K\mathcal{Q}/I^!$ where $I^!$ is generated by $\{x \in V^2 \mid \forall\, i, \langle x, \mathrm{supp}_2(f_i^2)\rangle_2 = 0\}$.

Proof. For each i, the element f_i^2 is homogeneous, and the sets $\{f_i^2 \mid 1 \le i \le m\}$, $\{f_i^2 \mid \mathrm{supp}_2(f_i^2) \neq 0\}$ are equal. So,

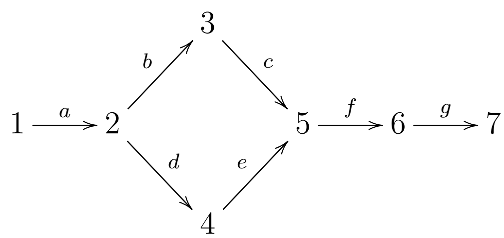
$$\begin{aligned}
I^! &\cong I^!_Q \\
&= Span(x \in V^2 \mid \langle x, f_i^2\rangle_{1\le i\le m} = 0)\ \forall\, i \\
&= Span(x \in V^2 \mid \langle x, \mathrm{supp}_2(f_i^2)\rangle_2 = 0)\ \forall\, i
\end{aligned}$$

and the corollary is proved. □

Corollary 2.34. Let $A = K\mathcal{Q}/I$ be a graded algebra where $I_Q = 0$. Then $A^! = K\mathcal{Q}/J^2$

Proof. If $I_Q = 0$, then $\mathrm{supp}_2(f_i^2) = 0$ for all i. Then $I_2 = \{x \in V^2 \mid \langle x, 0\rangle_2 = 0\} = V^2$. Note $\langle V^2 \rangle = J^2$ and the result follows. $\square$

Example 2.35. Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver



and $I = \langle ad, bc - de, cf, efg\rangle$. Then $I_Q = \langle ad, bc - de, cf\rangle$. Recall we compute

$f_1^0 = 1$	$f_1^1 = a$	$f_1^2 = ad$	$f_1^3 = adef$	$f_1^4 = adefg$
$f_2^0 = 2$	$f_2^1 = b$	$f_2^2 = bc - de$	$f_2^3 = bcfg - defg$	
$f_3^0 = 3$	$f_3^1 = c$	$f_3^2 = cf$		
$f_4^0 = 4$	$f_4^1 = d$	$f_4^2 = efg$		
$f_5^0 = 5$	$f_5^1 = e$			
$f_6^0 = 6$	$f_6^1 = f$			
$f_7^0 = 7$	$f_7^1 = g$			

If we compute $A^!$ using (2.33), we note

$$\mathrm{supp}_2(ad) = ad$$
$$\mathrm{supp}_2(bc - de) = bc - de$$
$$\mathrm{supp}_2(cf) = cf$$

and

$$\mathrm{supp}_2(efg) = 0$$

Then if $x = \alpha_1 ab + \alpha_2 ad + \alpha_3 bc + \alpha_4 de + \alpha_5 cf + \alpha_6 ef + \alpha_7 fg$ where $\alpha_i \in K$, then $I^!$ is generated by the set

$$= \{x \in V^2 \mid \langle x, \mathrm{supp}_2(f_i^2)\rangle_2 = 0\}$$

$$= \{x \mid \langle x, ad\rangle_2 = 0 \text{ and } \langle x, bc - de\rangle_2 = 0 \text{ and } \langle x, cf\rangle_2 = 0\}$$

$$= \{x \mid \alpha_2 = 0, \alpha_3 = \alpha_4, \text{ and } \alpha_5 = 0\}$$

$$= \{x = \alpha_1 ab + \alpha_3(bc + de) + \alpha_4 de + \alpha_6 ef + \alpha_7 fg \mid \alpha_i \in K^*\}$$

$$= \langle ab, bc + de, ef, fg\rangle$$

Thus $A^! = K\mathcal{Q}/\langle ab, bc + de, ef, fg\rangle$.

2.5.2 If A is a Quadratic Algebra

Let $A = K\mathcal{Q}/I$ be a quadratic algebra where $I = \langle \rho_1, ..., \rho_k\rangle$ where $\{\rho_i\}$ is a minimal set of generators and each ρ_i is a quadratic relation. By remark (2.24), we know $\{f_i^2\} = \{\rho_i\}$. Moreover, for every i, $\rho_i = \mathrm{supp}_2(f_i^2)$. We can apply theorem 2.33 to see $I^!$ is generated by the set

$$\{x \in V^2 \mid \langle x, \mathrm{supp}_2(f_i^2)\rangle = 0\}$$

$$= \{x \in V^2 \mid \langle x, \rho_i\rangle = 0\}$$

$$= I_2^{\perp}$$

Thus in the quadratic case, we recapture the well-known result, namely that $A^!$ is the quadratic dual of A.

Chapter 3

The Ext-Algebra of a Monomial Algebra

3.1 Introduction

Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is a finite quiver and I is an admissible ideal generated by paths. We call A a *monomial algebra* and each generator a *monomial relation*. Let $\{f_i^j\}$ be chosen as in Chapter 2 such that a minimal projective resolution of $\bar{A}$ can be constructed using them.

We wish to compare the f_i^js to the m-chains found in [16], as both use paths in $\mathcal{Q}$ to compute a minimal projective resolution of $\bar{A}$. To do so, we will first review how the m-chains are constructed and how they relate to the Ext-algebra of A. Then we use the m-chains to construct $A^!$ and show that $A^!$ is always a Koszul algebra. Lastly, we show that for every m, $\{f_i^m\}_{i\geq 0}$ is the set of m-chains.

3.2 m-Chains

Let us review how the m-chains are constructed and how they relate to the Ext-algebra of A. Let $I = \langle p_1, p_2, ..., p_n\rangle$ where each p_i is a path in $\mathcal{Q}$ of length at least 2 and no p_i is a subpath of another p_j. We construct three sets, Γ_0 = vertices of $\mathcal{Q}$, Γ_1 = arrows of $\mathcal{Q}$, and $\Gamma_2 = \{p_1, p_2, ..., p_n\}$. We now use the construction used in [11], [16] to define Γ_{m+1} inductively for $m \geq 2$. Let $\mathcal{M} = \{p \text{ path in } \mathcal{Q} \mid \text{ image of } p \text{ in } A \text{ is nonzero }\}$. A path p in $\mathcal{Q}$ is called an *m-prechain* if $p = qrs$ where $q \in \Gamma_{m-1}, qr \in \Gamma_m, s$ is a nontrivial path in $\mathcal{M}$, and rs contains a subpath which is in Γ_2. Then Γ_{m+1} is the set of all the m-prechains which have the property that no proper initial subpath is an m-prechain. We say $p \in \Gamma_m$ is a *mimimal* m-chain if $p \neq qr$ where $q \in \Gamma_{m-k}$ and $r \in \Gamma_k$. For example, Γ_3 is the paths $p = qrs$ where q is an arrow, qr is a relation, $s \notin I$, and rs contains a relation. Moreover,

$\{\Gamma_i\}_{i\geq 0}$ are mutually disjoint sets.

We recall from [15],[16] that $(\mathcal{P})$ is a minimal projective resolution of $\bar{A}$ over A

$$\to \mathcal{P}^i \xrightarrow{\delta^i} ... \to \mathcal{P}^2 \xrightarrow{\delta^2} \mathcal{P}^1 \xrightarrow{\delta^1} \mathcal{P}^0 \to \bar{A} \to 0$$

where $\mathcal{P}_m = \bigoplus_{p\in\Gamma_m} v_pA$ and v_p is the vertex corresponding to the terminus of the path p. Note that we can choose these vertices in such a way that if p and q are distinct paths in $\mathcal{Q}$, then v_p is distinct from v_q (although it is possible $v_pA \cong v_qA$). Consider the pairs (p, v_p) where p runs over the set of m-chains. Let $\hat{\Gamma}_m = \{(p, v_p) \mid p \text{ is an } m-\text{chain}\}$. Then

$$\mathcal{P}^m = \bigoplus_{(p,v_p)\in\hat{\Gamma}_m} v_pA$$

which induces a basis $\{\pi_i^m\}$ of $\mathrm{Ext}_A^m(\bar{A}, \bar{A})$ in the following way. We can linearly order each set $\hat{\Gamma}_m := \{(p_1^m, v_1^m), ..., (p_{m_i}^m, v_{m_i}^m)\}$. Then for each m, v_k^mA is a direct summand of $\mathcal{P}^m$ and there is a map $\pi_k^m : \mathcal{P}^i \to \bar{A}$ where

$$\pi_k^m(v_j^m) = \begin{cases} 0 & \text{if } k \neq j \\ (0, ..., 1, 0, ..., 0) & \text{if } k = j \end{cases}$$

Let $S = \{\pi_k^m \mid p_k^m \in \Gamma_m\}$ and let $\bar{S} = \{\pi_k^m \mid p_k^m \in \Gamma_m \text{ is a minimal } m\text{-chain }\}$. Note $\mathcal{Q}_0 = \Gamma_0$ and $\mathcal{Q}_1 = \Gamma_1$ are both subsets of $\bar{S}$. Moreover, S is identified with a spanning set of $E(A)$ with K-basis $\bar{S}$. We may define $S^m = \{\pi_k^m \mid 1 \leq k \leq m_j \text{ and } p_k^m \in \Gamma_m\}$ as a spanning set of $\mathrm{Ext}_A^m(\bar{A}, \bar{A})$ with K-basis $\bar{S}^m$ where $\bar{S}^m = \{\pi_k^m \mid 1 \leq k \leq t_m \text{ and } p_k^m \text{ is a minimal } m\text{-chain }\}$.

3.3 The Quiver of $A^!$ Using m-Chains

The quiver of $E(A)$ is determined in [16]. We review it below. Let Δ be a finite quiver. Let $\Delta_0 = \{\pi_k^0 \mid p_k^0 \in \mathcal{Q}_0\}$. If x and y are two vertices in Δ, then there is an arrow $x \xrightarrow{\hat{p}_k^j} y$ if and only if there exists a $p_k^j \in \bar{S}$ such that $s(p_k^j) = x$ and $t(p_k^j) = y$. Thus there is a quiver embedding $\mathcal{Q} \xrightarrow{\nu} \Delta$. We also denote the degree of an arrow in Δ as follows: Let $\hat{p} \in \Delta_1$. Then $\deg(\hat{p}) = i$ if and only if $p \in \Gamma_i$. Note if p is an arrow in $\mathcal{Q}$, then $\nu(p) = \hat{p}$ and $\deg(\hat{p})$=1. By abuse of notation, if $p \in \mathcal{Q}_1$, by considering $\nu(p)$, we can also consider p as an arrow in Δ.

This general construction holds in even more general situations. Suppose $R = R_0 \oplus R_1 \oplus ...$ is a graded ring where $R_0 = K \times K \times ... \times K$, $R_i = R_1^i$ and $\dim R_i < \infty$. Let $R^+ = R_1 \oplus R_2 \oplus ...$ and $(R^+)^2 = R_2 \oplus R_3 \oplus ....$ Then the number of arrows in the quiver of R is equal to $\dim R^+/(R^+)^2 = \dim R_1$. Letting $e_i = (0, ..., 0, 1, 0, ..., 0)$ where 1 is in the i^{th} entry, we see the number of arrows in the quiver of R from i to j is equal to $\dim(e_i(R^+/(R^+)^2)e_j) = \dim(e_i R_1 e_j)$.

Theorem 3.1. [16, theorem B] *Suppose $A = K\mathcal{Q}/I$ is a monomial algebra. Then $E(A) = K\Delta/I_\Delta$ where for $a_i, q_i \in \Delta$, I_Δ is generated by the following relations:*

1. *$a_1a_2...a_m$ such that $a_1...a_m \notin \Gamma_{deg(a_1...a_m)}$*
2. *$a_1a_2...a_n - q_1...q_m$ such that as paths in $\mathcal{Q}$, $a_1...a_m = q_1...q_n$ and $deg(a_1...a_m) = deg(q_1...q_n)$.*

The relations mentioned in the second part of the theorem are sometimes called *binomial relations.*

There are a few consequences of this theorem. Let $\{r_i\}$ be a set of relations which generate I_Δ. Note for each i, we may assume r_i is either monomial or binomial. We now explore the degrees of terms of the binomial relations in $E(A)$. More specifically, in any binomial relation, each term contains an arrow of degree ≥ 2. To see why, suppose there exists a generating relation r such that $r = x - y$ where $x = x_1x_2...x_k$ and $y = y_1y_2...y_j$ are distinct paths in Δ where x_i, y_i are arrows in Δ. Suppose $\deg(x_i) = 1$ for all i, so x_i is an arrow in $\mathcal{Q}$. Then $\deg(x) = k$ in Δ, which implies $\deg(y) = k$. If $\deg(y_i) = 1$ for all i, then y_i is an arrow in $\mathcal{Q}$. Then by 3.1*part 2, $x_1...x_k = y_1...y_k$ as paths in $\mathcal{Q}$, which implies they are exactly the same path. Thus the two paths are not distinct in Δ. This contradiction implies that there must exist some i such that $\deg(y_i) > 1$. Then there is some value b, w such that $x_1...x_b...x_{b+w}...x_k = y_1...y_i...y_j$ where $x_b...x_{b+w} = y_i$ as paths in $\mathcal{Q}$ and $\deg(x_b...x_{b+w})$ $=\deg(y_i)$. Thus $y_i \notin \bar{S}$, a contradiction. Thus there exists a z such that $\deg(x_z) \geq 2$. This leads to the following remark.

Remark 3.2. Let r be a binomial relation in I_Δ. If x is a term in r of length k, then $\deg(x) = \deg\,(r) \geq k+1$.

We now turn our focus to $A^!$. $A^!$ is generated by $\mathrm{Ext}^0_A(\bar{A}, \bar{A})$ and $\mathrm{Ext}^1_A(\bar{A}, \bar{A})$, which have bases $\bar{S}^0$ and $\bar{S}^1$. We see that the quiver of $A^!$ is given by $\nu(\mathcal{Q})$. Let $\mathcal{Q}^*$ denote the quiver $\nu(\mathcal{Q})$. We have $\mathcal{Q}^* \cong \mathcal{Q}$. Now we determine the ideal $I^!$ such that $A^! \cong K\mathcal{Q}^*/I^!$. Denote by p^* the image in $A^!$ of an arrow p in the quiver $\mathcal{Q}$. By abuse of notation, we will also use p^* to denote the image of an arrow p in $E(A)$.

Lemma 3.3. *Let A be a monomial algebra. Then $A^!$ is a monomial algebra.*

Proof. Suppose $p_1^*p_2^*...p_m^* = \sum_i q_{i_1}^*q_{i_2}^*...q_{i_{n_i}}^*$ in $A^!$ where $p_j, q_{i_j} \in \mathcal{Q}_1$, $\deg(p_1...p_m) = m$, and

$\deg(q_{i_1}...q_{i_{n_i}}) = n_i$. If this equality holds in in $A^!$, then it also holds in $E(A)$, which implies $p_1^*...p_n^* - \sum_i q_{i_1}^*...q_{i_{n_i}}^* = 0$ in $E(A)$. Using 3.1, we have that $p_1^*...p_m^* - \sum_i q_{i_1}^*...q_{i_{n_i}}^* \in I_\Delta$. If $I_\Delta = \langle \{r_i, m_j\}_{i,j\geq 0}\rangle$ where r_i is a binomial relation and m_i is a monomial, then write $p_1^*...p_m^* - \sum_i q_{i_1}^*...q_{i_{n_i}}^* = \sum \alpha_i r_i + \sum \beta_i m_i$. Suppose $p_1^*...p_m^*$ is a term of r_i for some i. Then $\deg(r_i) \geq m$ by 3.2, a contradiction. Thus $p_1^*...p_m^*$ appears as a term of m_k for some k, which implies $p_1^*...p_m^* = m_k$ A similar argument yields that for every i, $q_{i_1}^*...q_{i_{n_i}}^* = m_{j_i}$ for some j_i. Thus $p_1^*...p_m^* = 0$ and $q_{i_1}^*...q_{i_{n_i}}^* = 0$, proving our claim. □

We now introduce an ideal C of $K\mathcal{Q}^*$. We will ultimately prove that $A^! \cong K\mathcal{Q}^*/C$.

Definition 3.4. Let $C =: \langle \{a_1^* a_2^* \mid a_i \in \mathcal{Q}_1 \text{ and } a_1 a_2 \notin \Gamma_2\}\rangle$.

Theorem 3.5. *$A^! \cong K\mathcal{Q}^*/C$.*

Proof. Suppose $a_i \in \mathcal{Q}_1$ and $\nu(a_i) = a_i^*$ is the image of a_i in $A^!$, and in turn, $E(A)$. Because $A^!$ is a monomial algebra, it suffices to show that whenever $a_1^*...a_m^* = 0$ in $A^!$, there exists some $1 \leq i \leq m-1$ such that $a_i^* a_{i+1}^* = 0$. To do this, we proceed by induction on m.

Suppose $m = 2$. If $a_1^* a_2^* = 0$ in $A^!$, then $a_1^* a_2^* = 0$ in $E(A)$, which implies $a_1 a_2 \notin \Gamma_2$, so the claim holds.

Now assume that for $b_i \in \mathcal{Q}_1$ and $k \leq m-1$, that if $b_1^*...b_k^* = 0$ in $A^!$, then there exists some $1 \leq i \leq k-1$ such that $b_i^* b_{i+1}^* = 0$. Consider the subpath $a_1..a_{m-2}$ of $a_1...a_m$. There are two possibilities: either $a_1...a_{m-2} \in \Gamma_{m-2}$ or $a_1...a_{m-2} \notin \Gamma_{m-2}$.

1. Case I: If $a_1...a_{m-2} \notin \Gamma_{m-2}$, then by the inductive hypothesis, $a_1...a_{m-2} \in I_\Delta$, which implies $a_1^*...a_{m-2}^* = 0$ and so there exists an $1 \leq i \leq m-3$ such that $a_i^* a_{i+1}^* = 0$, proving our claim.

2. Case II: Assume $a_1...a_{m-2} \in \Gamma_{m-2}$ and consider $a_1...a_{m-1}$. If $a_1...a_{m-1} \notin \Gamma_{m-1}$, then the claim holds (by an argument similar to Case I). So, assume $a_1...a_{m-1} \in \Gamma_{m-1}$. Now consider the subpath $a_{m-1}a_m$. Either $a_{m-1}a_m \in \Gamma_2$ or $a_{m-1}a_m \notin \Gamma_2$. If $a_{m-1}a_m \notin \Gamma_2$, then $a^*_{m-1}a^*_m = 0$, proving the claim. So, assume $a_{m-1}a_m \in \Gamma_2$. Then $a_1...a_m$ is an $m-1$ prechain, thus an element of Γ_m. This is a contradiction because if $a_1...a_m \in \Gamma_m$, then $a^*_1...a^*_m \neq 0$ in $E(A)$, which implies it is nonzero in $A^!$.

We note that if $C = 0$, then $A^! \cong K\mathcal{Q}^*$. The proof is complete. □

We have the following consequence:

Corollary 3.6. If A is a monomial algebra, then $A^!$ is Koszul.

Proof. $A^!$ is monomial by 3.3 and $I^!$ is quadratic by 3.5. The result follows by 1.32. □

Example 3.7. Let $A = K\mathcal{Q}/I$ be given by the following quiver:

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

where $I = \langle abc, bcd \rangle$. Here, $\Gamma_0 = \{1,2,3,4,5\}, \Gamma_1 = \{a,b,c,d\}, \Gamma_2 = \{abc, bcd\}$, and $\Gamma_3 = \{abcd\}$. For $i \geq 4$, $\Gamma_i = \emptyset$. We can use these sets to form a minimal projective resolution of $\bar{A}$. Let v_i be the vertex i. Then $\mathcal{P}^i = \bigoplus_{p_i \in \Gamma_i} \mathrm{t}(p_i)A$ and we see

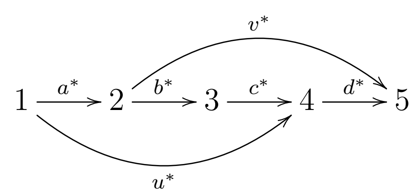
$$\mathcal{P}^0 = v_1A \oplus v_2A \oplus v_3A \oplus v_4A \oplus v_5A$$
$$\mathcal{P}^1 = v_2A \oplus v_3A \oplus v_4A \oplus v_5A$$
$$\mathcal{P}^2 = v_4A \oplus v_5A$$
$$\mathcal{P}^3 = v_5A$$
$$\mathcal{P}^4 = 0$$

and

$$0 \to \mathcal{P}^3 \to \mathcal{P}^2 \to \mathcal{P}^1 \to \mathcal{P}^0 \to \bar{A}$$

is a minimal projective resolution. Because there are 12 elements among the Γ_is, we know $E(A)$ is a 12-dimensional algebra. To construct the quiver of $E(A)$, we must determine the minimal m-chains. Note $abcd = (abc)d$ where $abc \in \Gamma_2$ and $d \in \Gamma_1$, so $abcd$ is not a minimal m-chain. However, note abc and bcd are minimal m-chains. For example, $abc = (a)(b)(c)$ where $a, b, c \in \Gamma_1$. Because deg(a)+deg(b)+deg(c)=3 and deg(abc)=2, we know abc is minimal.

Thus, the minimal m-chains are $1, 2, 3, 4, 5, a, b, c, d, abc$, and bcd. Letting $u = abc$ and $v = bcd$, the quiver of $E(A)$ is given by the following quiver Δ:



As for the relations, notice $a^*b^* = 0$ because $ab \notin \langle \Gamma_2 \rangle$. Similarly, $b^*c^* = 0$, and $c^*d^* = 0$. As for the nonzero relations, consider the path $abcd = (abc)d = a(bcd)$. Thus $u^*d^* = a^*v^*$, and $E(A) = K\Delta/\langle a^*b^*, b^*c^*, c^*d^*, u^*d^* - a^*v^* \rangle$.

Now we want to construct $A^!$. To do so, we use (3.5) and we obtain $A^! \cong K\mathcal{Q}/\langle ab, bc, cd \rangle$.

3.4 The Connection to $\{f_i^j\}$

Consider the quiver in 3.7. We repeat the above example using the f_i^js instead of the m-chains.

Example 3.8. $A = K\mathcal{Q}/I$ is given by the following quiver:

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

where $I = \langle abc, bcd \rangle$. We compute the f_i^js:

$f_1^0 = 1$	$f_1^1 = a$	$f_1^2 = abc$	$f_1^5 = abcd$
$f_2^0 = 2$	$f_2^1 = b$	$f_2^2 = bcd$	
$f_3^0 = 3$	$f_3^1 = c$		
$f_4^0 = 4$	$f_4^1 = d$		
$f_5^0 = 5$			

By setting $\mathcal{P}^n = \bigoplus_{k=1}^{l^n} v_k^n A$, we get the same minimal projective resolution of $\bar{A}$ as in (3.7).

$$0 \longrightarrow v_1^5 A \longrightarrow \bigoplus_{k=1}^{2} v_k^2 A \longrightarrow \bigoplus_{k=1}^{4} v_k^1 A \longrightarrow \bigoplus_{k=1}^{5} v_k^0 A \longrightarrow \bar{A} \to 0$$

In this example, notice $\{f_i^j\} = \Gamma_j$. In fact, this is the case for all monomial algebras. We will prove this by the end of the section.

Lemma 3.9. *Let $A = K\mathcal{Q}/I$ be a monomial algebra where $\mathcal{Q}$ is a finite quiver and I is an admissible monomial ideal. Then a family $\{f_i^j\}$ of monomials can be chosen to yield a minimal projective resolution of $\bar{A}$.*

Proof. We proceed by induction on j. Suppose $I = \langle \rho_1, \rho_2, ...\rho_m \rangle$ where $\{\rho_i\}$ is a minimal set of paths generating I. For $j = 2$, we show that $\{f_i^2\} = \{\rho_i\}$. To do so, consider $\bigoplus f_i^1 K\mathcal{Q} \cap \bigoplus f_i^0 I = \bigoplus f_i^0 I$ because $\bigoplus f_i^0 I \subset \bigoplus f_i^1 K\mathcal{Q}$. Then let $\{\rho_i'\}$ be chosen such that

1. Each ρ_i' is a path.

2. $\{\rho_i\} \subset \{\rho_i'\}$

3. $\bigoplus f_i^0 I = I = \bigoplus \rho_i' K\mathcal{Q}$

We may set $\{\rho_i'\} = \{f_i^{2'}\}$. Now consider $\rho_i' = a_i\rho_i b_i$ where a_i, b_i are paths in $\mathcal{Q}$. Suppose $l(a_i) \geq 1$. Then $a_i \in f_k^1 K\mathcal{Q}$ because $\{f_k^1\}$ is the set consisting of arrows of $\mathcal{Q}$. Thus $a_i\rho_i b_i \in \oplus f_i^1 I$, so $\rho_i' \neq f_k^2$ for any k.

Now suppose a_i is a vertex and $l(b_i) \geq 1$. Then $a_i\rho_i b_i \in \oplus f_i^{2'} J$, so $\rho_i' \neq f_k^2$ for any k. Thus for any path ρ_i', if $\rho_i' \neq \rho_i$, then $\rho_i' \neq f_r^2$ for any index r. Now consider ρ_k. Note $\rho_k \notin \bigoplus f_i^1 I + \bigoplus \rho_s' J$ because otherwise either $\rho_k = f_i^1 r$ for some arrow f_i^1 and some element $r \in I$, or $\rho_k = \rho_s' x$ for some $x \in J$, because ρ_k is monomial. Both of these cases lead to a contradiction of our choice of $\{\rho_i\}$. Thus the family $\{f_i^2\} = \{\rho_i\}$.

Now assume for $t < j$ that $\{f_i^t\}$ is chosen to be monomial and no linear combination of a subset of $\{f_i^t\}$ lies in $\bigoplus f^{j-1} I + \bigoplus f^{j'} J$. Consider the intersection

$$S = \bigoplus f_i^{j-1} K\mathcal{Q} \cap \bigoplus f_i^{j-2} I$$

By the induction hypothesis, $f_i^{j-1} K\mathcal{Q}$ is generated by monomials and $f_i^{j-2} I$ is also generated by monomials. The intersection is also generated by monomials and we can write $S = \bigoplus f_i^{j*}$ where f_i^{j*} are monomials and construct $\{f_i^j\}$ by removing the elements $f_i^{j*} \in \bigoplus f_i^{j-1} I + \bigoplus f_i^{j*} J$. Now we apply theorem 2.4 of [15] to conclude that $\{f_i^j\}$ yields a minimal projective resolution of $\bar{A}$. □

Corollary 3.10. $f_k^j = f_i^{j-1} h_{i,k}^{j-1,j}$ for some path $h_{i,k}^{j-1,j}$ where $1 \leq i \leq l^j$.

Proof. Recall $f_k^j = \sum_{i=1}^{j-1} f_i^{j-1} h_{i,k}^{j-1,j}$. By the previous lemma, the f_i^{j-1} are paths for each i. Note $f_i^{j-1} h_{i,k}^{j-1,j} \neq \lambda f_t^{j-1} h_{t,k}^{j-1,j}$ for any constant λ for any $t \neq i$. Because f_k^j is also a path, we must have $h_{i,k}^{j-1,j}$ is nonzero for exactly one value of i. Moreover, if $f_k^j = f_i^{j-1}h_{i,k}^{j-1,j}$ and f_k^j and f_i^{j-1} are paths, then so is $h_{i,k}^{j-1,j}$. □

Theorem 3.11. *Let A be a monomial algebra and let the f_i^js be chosen as in the preceding chapter to yield a minimal projectve resolution of $\bar{A}$. Then $\{f_i^j\} = \Gamma_j$ for all j where Γ_j is the set of j-chains.*

Proof. Proceed by induction on j. Note for $j = 0$ $\{f_i^0\} = \{\text{vertices in } K\mathcal{Q}\} = \Gamma_0$. Assume the claim holds for $n \leq j$ and consider $f_k^{j+1} \in \oplus f_i^j K\mathcal{Q} \cap \oplus f_i^{j-1} I$. By (3.10),

$$f_k^{j+1} = f_i^j h_{i,k}^{j,j+1} = f_t^{j-1} h_{t,i}^{j-1,j} h_{i,k}^{j,j+1}$$

where $h_{t,i}^{j-1,j} h_{i,k}^{j,j+1} \in I$, thus contains a subpath in Γ_2 by 3.9. By the induction hypothesis, each $f_t^{j-1} \in \Gamma_{j-1}$ and $f_t^{j-1} h_{t,i}^{j-1,j} = f_i^j \in \Gamma_j$. Thus each f_k^{j+1} is a j-chain, so it is an element of Γ_{j+1}.

For the reverse containment, suppose that $p = qrs$ is a j-chain in Γ_{j+1}. Then $q \in \Gamma_{j-1}$ implies that $q = f_t^{j-1}$ by the inductive hypothesis. Similarly, $qr \in \Gamma_j$ implies that $qr = f_i^j$ for some i. Thus $f_i^j = f_t^{j-1} r$ implies $h_{t,i}^{j-1,j} = r$. Because p is a j-chain, rs contains a subpath in Γ_2, which implies that $rs \in I$. Thus

$$p = f_t^{j-1} rs = f_i^j s \in \oplus f_i^j K\mathcal{Q} \cap \oplus f_i^{j-1} I$$

Thus $p = f_k^{j+1*}$ for some value of k. If $p \notin \oplus f_i^j I \bigoplus \oplus f_i^{j+1'} J$, then $p = f_k^j$ and we are done. So suppose $p \in \oplus f_i^j I \bigoplus \oplus f_i^{j+1'} J$. Because p is a path, then $p = f_i^j z$ for some path $z \in I$ or $p = f_i^{j+1'} z$ for some path z.

Case I: $p = f_i^j z$ for some path $z \in I$. Then $p = f_i^j s$ implies $z = s$. Because $s \notin I$, we get a contradiction.

Case II: $p = f_i^{j+1'} z$ for some path z. Then $f_i^{j+1'} \in \oplus f_i^j K\mathcal{Q} \cap \oplus f_i^{j-1} I$ implies $f_i^{j+1'}$ is a j-chain. However, $p = f_k^{j+1'}$ is also a j-chain. This is a contradiction because no j-chain can left divide another by construction.

Thus, $p = f_k^{j+1}$ for some k. □

We can now cite a theorems found in [16] regarding j-chains. However, because the set of j-chains is $\{f_i^j\}$, we rewrite the result using $\{f_i^j\}$.

Theorem 3.12. [16, Proposition 1.2] *Let A be a monomial algebra. Then $(f_i^j)^*(f_r^s)^* \neq 0$ in $E(A)$ if and only if $(f_i^j)^*(f_r^s)^* = \lambda(f_k^{j+s})^*$ where $\lambda f_i^j f_r^s = f_k^{j+s}$ for some nonzero constant λ.*

Chapter 4

The Associated Monomial Algebra

4.1 Introduction

The following is an open question: If $A = K\mathcal{Q}/I$ and I is admissible, when is $E(A)$ finitely generated? In general, $E(A)$ need not be finitely generated, even in the monomial case, as shown in [?GMH]. However, this chapter offers some partial solutions. It is well known that if $E(A)$ is generated in degrees 0 and 1, then A must be a Koszul algebra. We seek to generalize this notion. First we look to monomial algebras: If A is monomial, we find easily checked conditions for $E(A)$ to be generated in degrees 0,1, and 2. To do so, we use a construction from [14] to form a family $\{x_i^j\}$ which yields a projective resolution of $\bar{A}$, called the AGS resolution. This construction is also useful if A is not necessarily monomial. In that situation, we consider the case where the AGS resolution is minimal and look to the associated monomial algebra of A, found in [8] and [9], which we denote A_{MON}. We prove that if the AGS resolution is minimal and $E(A_{\text{MON}})$ is finitely generated, then $E(A)$ is finitely generated. Next we look at 2-d-determined algebras, as defined in [12]. Let A be a 2-d-determined algebra. We prove that if the AGS resolution is minimal, then $E(A)$ is generated in degrees 0,1, and 2. However, if the AGS resolution is not minimal, we prove that $E(A)$ need not be generated in degrees 0,1, and 2.

4.2 Noncommutative Gröbner Bases

Let $A = K\mathcal{Q}/I$ be a finite dimensional K-algebra. In this section we explore how to construct a noncommutative Gröbner basis of A.

Definition 4.1. $K\mathcal{Q}$ as a K-algebra with K-basis

$$\mathcal{B} := \{\text{ all the directed paths in } \mathcal{Q}\}$$

Throughout this section we use terminology and results from [8], [9].

In order to construct a Gröbner basis we need to impose a well-order $>$ on $\mathcal{B}$. Recall $>$ is a well-order if it is a total order and every nonempty subset of $\mathcal{B}$ has a minimal element. To find a well-order that works well with the structure of a path algebra, we require an *admissible ordering* on $\mathcal{B}$.

Definition 4.2. A well-order on $\mathcal{B}$ is admissible if it satisfies the following three conditions for any $p, q, r, s \in \mathcal{B}$

1. If $p < q$, then $pr < qr$ if both pr and qr are nonzero.

2. If $p < q$, then $sp < sq$ if both sp and sq are nonzero.

3. If $p = qr$, then $p > q$ and $p > r$.

The admissible order used in this chapter will be the (left) length-lexicographic order. We review it now. Suppose $\mathcal{Q}_0 = \{v_1, ..., v_n\}$ and $\mathcal{Q}_1 = \{a_1, ..., a_m\}$. We order first the vertices and the arrows in an arbitrary way and set the vertices smaller than the arrows.

$$v_1 < v_2.... < v_n < a_1 < a_2.... < a_m$$

Recall that we define the *length* of a path $p = a_1 a_2 ... a_m$ is m, and we denote it $l(p) = m$. An ordering on all the paths now follows: if p and q are two paths, then $p < q$ if one of the following cases hold:

1. $l(p) < l(q)$,

2. $l(p) = l(q)$, $p = a_1 ... a_m$ and $q = b_1 ... b_m$ for $a_i, b_i \in \mathcal{Q}_1$, and $a_1 < b_1$.

3. $l(p) = l(q)$, $p = a_1 ... a_m$ and $q = b_1 ... b_m$ for $a_i, b_i \in \mathcal{Q}_1$, and there exists some $1 \leq i \leq m$, such that $a_j = b_j$ for $j < i$ and $a_i < b_i$.

From now on, we will always assume that $<$ is an admissible ordering on B.

Definition 4.3. Let $v = \sum_i \alpha_i p_i$ be a nonzero element of $K\mathcal{Q}$ where each $\alpha_i \in K^*$ and $p_i \in \mathcal{B}$. We say the *tip* of v is p_i if $p_i > p_j$ for all $j \neq i$. We denote the tip of v by $\text{tip}(v)$. We denote the coefficient of the tip of v by $C\,\text{tip}$ and write $C\,\text{tip}(v) = \alpha_i$ if $\text{tip}(v) = p_i$. If X is a subset of $K\mathcal{Q}$, we let

$$\text{tip}(X) = \{b \in \mathcal{B} \mid b = \text{tip}(x) \text{ for some nonzero } x \in X\}$$

We also define $\text{Nontip}(v) = v - \text{tip}(v)$ and for a set X of $K\mathcal{Q}$, we define

$$\text{Nontip}(X) = \{\text{nontip}(v) \mid v \in X\}$$

We will now define a noncommutative Gröbner basis for I.

Definition 4.4. If I is a two-sided ideal of $K\mathcal{Q}$, define $I_{\text{MON}} = \langle \text{tip}(I) \rangle$.

I_{MON} is an important ideal because it is used in defining the following:

Definition 4.5. Let $A = K\mathcal{Q}/I$ where I is an admissible ideal. Then *the associated monomial algebra of* A, denoted A_{MON} is $A_{\text{MON}} = K\mathcal{Q}/I_{\text{MON}}$.

Definition 4.6. [8, 2.4] Let I be a two sided ideal of $K\mathcal{Q}$. We say that a subset $\mathcal{G} \subset I$ is a *Gröbner basis* for I with respect to $<$ if

$$\langle \text{tip}(\mathcal{G}) \rangle = I_{\text{MON}}$$

There are a few things to note. First, $\mathcal{G}$ need not necessarily be finite. However, if I_{MON} can be generated finitely many elements, we have the following.

Theorem 4.7. [8, Proposition 2.8] *If I is an admissible ideal of a path algebra $K\mathcal{Q}$, then I will always have a finite Gröbner basis.*

Because I_{MON} is a monomial ideal, the following proposition applies.

Proposition 4.8. [8, 2.5] *Suppose $\mathcal{Q}$ is a finite quiver and I an admissible ideal of $K\mathcal{Q}$. Then I_{MON} has a unique minimal monomial generating set.*

By the above proposition, I_{MON} has a minimal monomial generating set, which we will denote $\mathcal{T}$. If $\mathcal{G}$ is a Gröbner basis of I, then tip$(\mathcal{G})$ contains $\mathcal{T}$. We may now define a *reduced Gröbner basis of I*.

Definition 4.9. [8, 2.5] Let I be an admissible ideal in $K\mathcal{Q}$. Then $\mathcal{G}$ is a *reduced Gröbner basis* for I with respect to $>$ if $\mathcal{G}$ is a Gröbner basis for I and for every $g \in \mathcal{G}$, $C\,\text{tip}(g) = 1$ and if a path p is a term of g, and p contains a subpath t, where $t = \text{tip}(g')$ for some $g' \in \mathcal{G}$, then $g = g'$.

We also have the following

Proposition 4.10. [8] *Let I be an admissible ideal in $K\mathcal{Q}$. A set $\mathcal{G}$ of elements in I is a reduced Gröbner basis for I (with respect to $<$) if the following conditions hold:*

1. *If* $g \in \mathcal{G}$*, then* $C\,\mathrm{tip}(g) = 1$*.*

2. *If* $g \in \mathcal{G}$*, then* $g - \mathrm{tip}(g) \in \mathit{Span}(\mathit{NonTip}(I))$*.*

3. $\mathrm{tip}(\mathcal{G})$ *is the minimal monomial generating set for* I_{MON}

Definition 4.11. We say a set $S \subset K\mathcal{Q}$ is *tip-reduced* if $s, s' \in S$ and tip(s)=tip(s')p for some path $p \in \mathcal{Q}$, then $s = s'$.

It follows that every reduced Gröbner basis is also a tip-reduced Gröbner basis. We now paraphrase a proposition of Green specific to our situation:

Proposition 4.12. [8, 2.2,2.4.1] *Let* I *be an admissible ideal of* $K\mathcal{Q}$ *generated by uniform elements. Then* I *has a finite reduced Gröbner basis* $\mathcal{G} = \{G^2_1, ..., G^2_{l^2}\}$ *with respect to* $>$ *where each* G^2_i *is a uniform element of* $K\mathcal{Q}$*.*

Assume now that I is an admissible ideal generated by uniform elements. Because the minimal generating set $\mathcal{T}$ of $I_{\mathrm{MON}} \subset \mathrm{tip}(\mathcal{G})$, we see that I_{MON} is generated by elements $\{\mathrm{tip}(G^2_1), \mathrm{tip}(G^2_2), ..., \mathrm{tip}(G^2_{l^2})\}$. In view of the above proposition, since I_{MON} is monomial and $\mathcal{G}$ is tip-reduced, and $\{\mathrm{tip}(G^2_1), \mathrm{tip}(G^2_2), ..., \mathrm{tip}(G^2_{l^2})\}$ is a minimal set of generators for I_{MON}.

The following notion will be needed later in the chapter:

Definition 4.13. We say two paths p and q *overlap* if there are three paths t, r, s such that such that

1. $l(r) \geq 1$

2. $trs = ps = tq$

3. $tr = p$

4.3 The AGS resolution

As seen in Chapter 2, a sequence of finite families $\{f_i^j\}$ were constructed in an algorithmic way such that

$$... \longrightarrow \bigoplus f_i^2 K\mathcal{Q}/f_i^2 I \longrightarrow \bigoplus f_i^1 K\mathcal{Q}/f_i^1 I \longrightarrow \bigoplus f_i^0 K\mathcal{Q}/f_i^0 I \longrightarrow \bar{A} \longrightarrow 0 \tag{4.1}$$

is a minimal projective resolution of $\bar{A}$, where the differentials are induced by inclusion maps in $K\mathcal{Q}$. We attained this minimality by applying [15, theorem 2.4]. That is, after forming the set $\{f_i^{n+1'}\}$ such that $\bigoplus f_i^{n+1'} K\mathcal{Q} = \bigoplus f_i^n K\mathcal{Q} \cap \bigoplus f_j^{n-1} I$, we discard enough elements of the form $f^{n+1'}$ to obtain a set $\{f_i^{n+1}\}$ such that no proper K-linear combination of a subset of $\{f_j^{n+1}\}$ is in $\bigoplus f_i^n I + \bigoplus f^{n+1'} J$. However, there is no unique way to choose the f_i^js. In this section we construct families $\{x_i^n\}, \{x_i^{n'}\}$ which satisfy the following properties. Recall, by abuse of notation, that $\bigoplus x_i^n A$ denotes $\bigoplus (x_i^n K\mathcal{Q}/x_i^n I)$:

1. $\{x_i^0\} = \mathcal{Q}_0$

2. $\{x_i^1\} = \mathcal{Q}_1$

3. The set $\mathcal{G} = \{x_i^2\}$, where $\mathcal{G}$ is a reduced Gröbner basis for I.

4. $\{x_i^{n+1}\}$ can be chosen so that if $P^n = \bigoplus_i x_i^{n+1} A$, then

$$... \to P^{n+1} \to P^n \to ... \to P^0 \to \bar{A} \to 0$$

   is a projective resolution of $\bar{A}$ (but not necessarily minimal).

5. $x_i^{n+1} \notin \bigoplus x_k^n I$

6. For every i and n, $(\bigoplus x_i^n A) \bigoplus \left(\bigoplus x_i^{n'} A\right) = \bigoplus x_i^{n-1} A \cap x_i^{n-2} I$

These families, $\{x_i^n\}$, and $\{x_i^{n'}\}$ are not unique. In this chapter, we employ an algorithm found in [14], [1] to construct these sets. We call this algorithm the *AGS algorithm.* However, we reemphasize that $\{x_i^n\}$ is constructed in a way where it is possible that some K-linear combination of a subset of the $\{x_i^n\}$ is in $\bigoplus x_i^{n-1} I + \bigoplus x^{n'} J$, so the minimality of the resolution is not assured. For the convenience of the reader, we will review the algorithm. First, for the case $n = 2$, and then for the general case. To do so, we need the following definition.

Definition 4.14. [14] Let I be an admissible ideal of $K\mathcal{Q}$ and $\mathcal{G}$ be a fixed tip-reduced Gröbner basis of I. Let p be a path in $\mathcal{Q}$ of length ≥ 1 and let $X(p)$ be the set of all paths q which satisfying the following conditions:

1. p left divides q and $p \neq q$, that is $q = pr$ for some path r of length $l(r) \geq 1$.

2. There is some element $G_i^2 \in \mathcal{G}$ such that $\text{tip}(G_i^2)$ right divides q and $q \neq \text{tip}(G_i^2)$, that is $q = r\,\text{tip}(G_i^2)$ for some path r with $l(r) \geq 1$.

3. If $q = q's$ and q' satisfies (1) and (2), then s is a vertex.

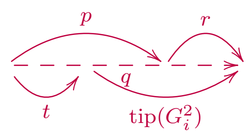
Note this definition is not symmetric and is only useful when studying right A-modules. We also note that $X(p)$ equals the disjoint union $X(p) = W(p) \sqcup O(p)$ where

$$W(p) = \{q \in X(p) \mid q = pz\,\text{tip}(G_i^2),\ \text{for some path } z \text{ such that } \ l(z) \geq 1\}$$

and

$$O(p) = X(p) - W(p)$$

$O(p)$ is a set which will be used repeatedly throughout the chapter for various paths p. We may use the following diagram to visualize the set $O(p)$: it is the set of shortest paths q such that the following diagram can be constructed.



We review now how this can be used to construct two families, $\{x^2_{(i,q)}\}_{i=1}^{l^1}$ and $\{x^{2'}_{(i,q)}\}_{i=1}^{l^1}$. We will ultimately show $\{x^2_{(i,q)}\} = \{G^2_i\}$ after certain properties of $\{x^2_{(i,q)}\}$ are established. The construction of $\{x^2_{(i,q)}\}$ and $\{x^{2'}_{(i,q)}\}$ is outlined in the following steps:

Step 1: Set $\mathcal{Q}_0 = \{x^0_i\}_{i\in\hat{T}_0}$ and $\mathcal{Q}_1 = \{x^1_i\}_{i\in\hat{T}_1}$ where $\hat{T}_0, \hat{T}_1$ are indexing sets. Consequently, we have the following $K\mathcal{Q}$-presentation of $\bar{A}$

$$0 \to \bigoplus_{i\in\hat{T}_1} x^1_i K\mathcal{Q} \xrightarrow{H^1} \bigoplus_{i\in\hat{T}_0} x^0_i K\mathcal{Q} \xrightarrow{\pi} \bar{A} \to 0$$

where H^1 is the inclusion map. Note that x^0_i and x^1_j are uniform elements of $K\mathcal{Q}$ for all i, j and the set $\{x^1_i\}_{i\in\hat{T}_1}$ is tip-reduced.

Step 2: Define the following sets

$$T_2 = \{(i,q) \mid i \in \hat{T}_1 \text{ and } q \in O(x^1_i)\}$$

and

$$U_2 = \{(i,q) \mid i \in \hat{T}_1 \text{ and } q \in W(x^1_i)\}$$

where the sets $W(x^1_i)$ and $O(x^1_i)$ are defined above. Note that we can have two different paths, q_1 and q_2, which both correspond to the same arrow i.

Remark 4.15. For each $(i,q) \in T_2$, we have $q = x_i^1 p_q = q' \operatorname{tip}(G_j^2)$ for some paths p_q and q' and some index j. We consider $x_i^1 p_q - q' G_j^2$, which has no terms of length 0 because neither $x_i^1 p_q$ nor $q' G_j^2$ does. Thus $\pi(x_i^1 p_q - q' G_j^2) = 0$, which implies $x_i^1 p_q - q' G_j^2 \in \bigoplus_{i \in \hat{T}_1} x_i^1 K\mathcal{Q}$. Hence, $x_i^1 p_q - q' G_j^2 = \sum_{k \in \hat{T}_1} x_k^1 r_k$ for some r_k in $K\mathcal{Q}$. It is important to note that $x_i^1 p_q = \operatorname{tip}(q' G_j^2)$, which implies $\operatorname{tip}(\sum_{k \in \hat{T}_1} x_k^1 r_k) < x_i^1 p_q$. Consequently, $x_i^1 p_q$ is not a term in $\sum_{k \in \hat{T}_1} x_k^1 r_k$. However it is possible to have $x_i^1 r_i$ appear as a summand in $\sum_{k \in \hat{T}_1} x_k^1 r_k$. If that is the case, then $\operatorname{tip}(r_i) < p_q$.

If $G_j^2 = \operatorname{tip}(G_j^2)$, then $x_i^1 p_q - q' G_j^2 = 0$ as both $x_i^1 p_q$ and $q' G_j^2$ will be paths.

Step 3: For each $(i,q) \in T_2$, we set $x_{(i,q)}^2 = x_i^1 p_q - \sum_{k \in T_1} x_k^1 r_k$. It is clear $x_{(i,q)}^2 \in \bigoplus x_i^1 K\mathcal{Q}$. Because $q' G_j^2 \in \bigoplus x_i^0 I$,

$$x_{(i,q)}^2 \in \bigoplus_{i \in T_1} x_i^1 K\mathcal{Q} \cap \bigoplus_{i \in T_0} x_i^0 I$$

Moreover, $\operatorname{tip}(x_{(i,q)}^2) = x_i^1 p_q$ by construction.

Step 4: If $(i,q) \in U_2$ then there exists a path z_q and G_j^2 such that $q = x_i^1 z_q \operatorname{tip}(G_j^2)$. Define $x_{(i,q)}^{2'} = x_i^1 z_q G_j^2$. It is clear that $x_{(i,q)}^{2'} \in \bigoplus x_i^1 I$ and $\operatorname{tip}(x_{(i,q)}^{2'}) = x_i^1 z_q \operatorname{tip}(G_j^2) = q$.

We show now that if $(i,q) \in T_2$, then $x_{(i,q)}^2 = G_j^2$ for some j. Because $q = x_i^1 p_q = q' \operatorname{tip}(G_j^2) \in O(x_i^1)$, we must have $l(q') = 0$ (Otherwise, $(i,q) \notin T_2$). This implies that

$$x_i^1 p_q - G_j^2 = -\operatorname{nontip}(G_j^2)$$

which can be written as an element in $\bigoplus x_k^1 K\mathcal{Q}$. Suppose $\text{nontip}(G_j^2) = \sum_{k\in\hat{T}_1} x_k^1 r_k$. Then

$$x_{(i,q)}^2 = x_i^1 p_q + \sum_{k\in\hat{T}_1} x_k^1 r_k = \text{tip}(G_j^2) + \text{nontip}(G_j^2) = G_j^2$$

We have shown that every $x_{(i,q)}^2 = G_j^2$ for some j. Now suppose $(i_1, q_1) \neq (i_2, q_2)$ are two pairs corresponding to $x_{(i_1,q_1)}^2$ and $x_{(i_2,q_2)}^2$, respectively. By construction, $\text{tip}(x_{(i_j,q_j)}^2) = x_{i_j}^1 p_{q_j}$ for $j = 1, 2$. Because $(i_1, q_1) \neq (i_2, q_2)$ either $i_1 \neq i_2$ or $q_1 \neq q_2$. If $i_1 \neq i_2$, then $x_{i_1}^1 \neq x_{i_2}^1$, which implies $x_{(i_1,q_1)}^2 \neq x_{(i_2,q_2)}^2$. If $q_1 \neq q_2$ then the same conclusion can be reached. Thus each pair (i, q) corresponds uniquely to a G_j^2.

Now we show that for every j, $G_j^2 = x_{(i,q)}^2$ for some pair (i, q). Notice that $\text{tip}(G_j^2) = x_i^1 p$ for some path p and some arrow x_i^1. $\{G_i^2\}$ is tip-reduced, so there is a unique pair (i, q) such that $q = x_i^1 p$. By the above argument, we know this pair (i, q) yields $x_{(i,q)}^2 = G_j^2$. Then, if necessary, reindex the set $\{x_{(i,q)}^2\}$ with a single index as $\{x_j^2\}$ in a way so that $x_j^2 = G_j^2$.

We are now ready to review the algorithm found in [14] to construct the sets $\{x_{(i,q)}^n\}, \{x_{(i,q)}^{n'}\}$ for all $n \geq 3$. We do this inductively, using the case $n = 2$ as our base case. Assume $\hat{T}_j = \{x_{(i,q)}^j\}$ and $\hat{U}_j = \{x_{(i,q)}^{j'}\}$ has been constructed for all $j \leq n$. We may reindex and write

$$\hat{T}_j = \{x_i^j\}_{\{1\leq j\leq l^j\}}$$

and

$$\hat{U}_j = \{x_i^{j'}\}_{\{1\leq j\leq u^j\}}$$

we then construct an element $x_{(i,q)}^{n+1}$ by the following steps:

<u>Step 1</u>: For every i, we may assume, by the inductive hypothesis, that $\text{tip}(x_i^n) = \text{tip}(x_k^{n-1})t_{k,i}^{n-1,n}$

where $t_{k,i}^{n-1,n}$ is a path in $\mathcal{Q}$.

Step 2: Compute $X(t_{k,i}^{n-1,n})$ and construct

$$T_{n+1} = \{(i,q) \mid x_i^n \in \hat{T}_n \text{ and } q \in O(t_{k,i}^{n-1,n})\}$$

and

$$U_{n+1} = \{(i,q) \mid x_i^n \in \hat{T}_n \text{ and } q \in N(t_{k,i}^{n-1,n})\}$$

Step 3: For $(i,q) \in T_{n+1}$, write $q = t_{k,i}^{n-1,n} p_q = z_q \operatorname{tip}(x_t^2)$ for some x_t^2, where z_q and p_q are paths in $\mathcal{Q}$. Then

$$\begin{aligned}\operatorname{tip}(x_i^n)p_q &= \operatorname{tip}(x_k^{n-1})t_{k,i}^{n-1,n}p_q \\ &= \operatorname{tip}(x_k^{n-1})q \\ &= \operatorname{tip}(x_k^{n-1})z_q \operatorname{tip}(x_t^2)\end{aligned}$$

and let

$$d = x_i^n p_q - x_k^{n-1} z_q x_t^2$$

Notice that

$$x_i^n p_q \in \bigoplus_{\hat{T}_n} x_j^n K\mathcal{Q}$$

so $x_k^{n-1} \in \bigoplus_{\hat{T}_{n-2}} x_j^{n-2} K\mathcal{Q}$ by the induction hypothesis. We may assume $x_k^{n-1} z_q x_t^2 \in \bigoplus_{\hat{T}_{n-2}} x_j^{n-2} I$ as $x_t^2 \in I$. Thus

$$x_k^{n-1} z_q x_t^2 \in \bigoplus_{\hat{T}_{n-1}} x_j^{n-1} K\mathcal{Q} \cap \bigoplus_{\hat{T}_{n-2}} x_j^{n-2} I$$

By the inductive hypothosis,

$$\bigoplus_{\hat{T}_{n-1}} x_j^{n-1} K\mathcal{Q} \cap \bigoplus_{\hat{T}_{n-2}} x_j^{n-2} I = \left(\bigoplus_{\hat{T}_n} x_j^n K\mathcal{Q} \right) \bigoplus \left(\bigoplus_{\hat{U}_n} x_j^{n'} K\mathcal{Q} \right)$$

Thus

$$d \in \left(\bigoplus_{\hat{T}_n} x_j^n K\mathcal{Q} \right) \bigoplus \left(\bigoplus_{\hat{U}_n} x_j^{n'} K\mathcal{Q} \right)$$

and so we can write

$$d = x_i^n p_q - \lambda x_k^{n-1} z_q x_t^2 = \sum_{\hat{T}_n} x_j^n r_j + \sum_{\hat{U}_n} x_j^{n'} w_j$$

Step 3: Set $x_{(i,q)}^{n+1} = x_i^n p_q - \sum_{\hat{T}_n} x_j^n r_j$.

Step 4: For (i,q) in U_{n+1}, $\operatorname{tip}(x_k^n) z_q \operatorname{tip}(x_t^2) = q$ for some index t and some path z_q. Set $x_{(i,q)}^{n+1'} = x_k^n z_q x_t^2$.

Remark 4.16. Clearly $x_{(i,q)}^{n+1} \in \bigoplus_{\hat{T}_n} x_j^n K\mathcal{Q}$, and

$$x_{(i,q)}^{n+1} = x_k^{n-1} z_q x_t^2 + \sum_{\hat{U}_n} x_j^{n'} w_j \in \bigoplus_{\hat{T}_{n-1}} x_j^{n-1} I$$

Also, $x_{(i,q)}^{n+1'} \in \bigoplus_{\hat{T}_n} x_j^n I$, which implies

$$x_{(i,q)}^{n+1'} \in \bigoplus_{\hat{T}_n} x_j^n K\mathcal{Q} \cap \bigoplus_{\hat{T}_{n-1}} x_j^{n-1} I$$

We must have

$$\left(\bigoplus_{(j,q)\in T_{n+1}} x^{n+1}_{(i,q)} K\mathcal{Q}\right) \bigoplus \left(\bigoplus_{(i,q)\in U_{n+1}} x^{n+1'}_{(i,q)} K\mathcal{Q}\right) \subset \bigoplus_{\hat{T}_n} x^n_j K\mathcal{Q} \cap \bigoplus_{\hat{T}_{n-1}} x^{n-1}_j I$$

In fact, equality holds. For the reverse containment, we repeat the proof from [14] for the convenience of the reader. Suppose

$$\bigoplus_{\hat{T}_{n+1}} x^{n+1}_i A + \bigoplus_{\hat{U}_{n+1}} x^{n+1'}_j A$$

does not generate $\bigoplus x^n_i K\mathcal{Q} \cap \bigoplus x^{n-1}_i I$. Let $y \in \bigoplus x^n_i K\mathcal{Q} \cap \bigoplus x^{n-1}_i I$ be chosen such that $\text{tip}(y)$ is minimal with respect to the property that

$$y \notin \left(\bigoplus_{\hat{T}_{n+1}} x^{n+1}_i A\right) \bigoplus \left(\bigoplus_{\hat{U}_{n+1}} x^{n+1'}_i A\right)$$

However, because $y \in \bigoplus_{\hat{T}_n} x^n_i K\mathcal{Q}$, $\text{tip}(y) = \text{tip}(x^n_i)p$ for some $x^n_i \in \hat{T}_n$ and some path p in $\mathcal{Q}$. But because $y \in \bigoplus_{\hat{T}_{n-1}} x^{n-1}_j I$, we have $\text{tip}(y) = \text{tip}(x^{n-1}_k)w\,\text{tip}(x^2_r)z$ for some index k and some paths w, z. We may choose the index r such that w has minimal length. In other words, if $w_1\,\text{tip}(x^2_i)z_1 = w_2\,\text{tip}(x^2_r)z_2$, we know $l(w_2) \leq l(w_1)$. Thus $\text{tip}(x^{n-1}_k)w\,\text{tip}(x^2_r)z = \text{tip}(x^n_i)p$ and either $\text{tip}(x^2_r)$ overlaps $\text{tip}(x^n_i)$ or it does not. If they do overlap, then there exists some l such that $\text{tip}(x^{n+1}_l)z = \text{tip}(y)$. Then, for $\lambda \in K^*$,

$$y - \lambda x^{n+1}_l z \notin \bigoplus_{\hat{T}_{n+1}} x^{n+1}_i A + \bigoplus_{\hat{U}_{n+1}} x^{n+1'}_j A$$

However $\text{tip}(y - \lambda x^{n+1}_l z) < \text{tip}(y)$, which contradicts our choice of y. Thus $\text{tip}(x^n_r)$ and $\text{tip}(x^n_i)$

do not overlap. Consequently, there exists some $x_l^{n+1'} \in \hat{U}_{n+1}$ such that $\text{tip}(x_l^{n+1'})z = \text{tip}(y)$ for some path $z \in \mathcal{Q}$. This leads to a similar contradiction. Thus no such y exists and

$$\bigoplus x_i^n K\mathcal{Q} \cap \bigoplus x_i^{n-1} I = \left(\bigoplus_{\hat{T}_{n+1}} x_i^{n+1} K\mathcal{Q}\right) \bigoplus \left(\bigoplus_{\hat{U}_{n+1}} x_j^{n+1'} K\mathcal{Q}\right)$$

We now have a family $\{x_i^n\}$ which satisfy the following conditions:

1. $$\bigoplus_{\hat{T}_n} x_i^n K\mathcal{Q} \cap \bigoplus_{\hat{T}_{n-1}} x_i^{n-1} I = \left(\bigoplus_{\hat{T}_{n+1}} x_i^{n+1} K\mathcal{Q}\right) \bigoplus \left(\bigoplus_{\hat{U}_{n+1}} x_j^{n+1'} K\mathcal{Q}\right)$$

2. $x_i^n \notin \bigoplus_{\hat{T}_n} x_j^{n-1} I$.

3. Each x_i^n is uniform.

4. For every n, the set $\{x_i^n\}$ is tip reduced.

Notice $\{\text{tip}(x_i^n)\}_{1 \leq i \leq l^n}$ is chosen uniquely. We may now apply the methods found in [16] to conclude

$$\ldots \bigoplus_{x_j^n \in \hat{T}_n} x_j^n K\mathcal{Q} \to \ldots \to \bigoplus_{x_j^1 \in \hat{T}_0} \to \bar{A} \to 0$$

is a projective (not necessarily minimal) resolution of $\bar{A}$ over A and we call this resolution the AGS resolution of $\bar{A}$ over A.

4.4 The Monomial Algebra Case

Let $\mathcal{Q}$ be a finite quiver and let I be an admissible monomial ideal of $K\mathcal{Q}$. Let $A = K\mathcal{Q}/I$. We know I is generated by $\{x_1^2, \ldots, x_{l^2}^2\}$ where $\{x_i^2\}$ is constructed as in the previous section.

In this section we find that there are necessary and sufficient conditions for $E(A)$ to be generated in degrees 0,1, and 2. To do so, we start with the following.

Proposition 4.17. *x_i^n is a monomial for all i, n.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial because $\{x_i^1\}$ is the set of arrows, each x_i^1 is a path. Assume now that for all $j \leq n$, x_i^j is a monomial. We show that x_s^{n+1} is monomial for all s. To do so, for every i, consider $x_i^n = x_k^{n-1} h_{k,i}^{n-1,n}$ for some path h and some index k. Suppose $q \in O(h_{k,i}^{n-1,n})$and p is a path such that $h_{k,i}^{n-1,n} p = q$. Consider the difference $d = x_i^n p - x_k^{n-1} q$. We assume x_i^n and x_k^{n-1} are monomials by induction and p and q are paths. This implies $d = 0$. Consequently, there is an index s such that $x_s^{n+1} = x_i^n p$ is a monomial and the proof is complete. □

It is a known fact, see [12], that for monomial algebras, the AGS resolution is always minimal. For the convenience of the reader, we provide a proof here. We do so by connecting $\{x_i^m\}$ to the m-chains described in 3.2. It suffices to prove that for every m, $\{x_i^m\} = \Gamma_m$.

Proposition 4.18. *For a monomial algebra $A = KQ/I$ where I is an admissible ideal, $\{x_i^m\} = \Gamma_m$ for every $m \geq 0$.*

Proof. We use induction on m. For the base case, notice $\{x_i^1\} = \Gamma_1$ as both are the set of arrows, so each x_i^1 is a path. Now assume for all $n < m$, $\{x_i^n\} = \Gamma_n$. Now consider x_i^m for some i. By the construction in the previous proof, we know $x_i^m = x_k^{m-1} p$ for some path p and some index k. More specifically, to obtain p, we write $x_k^{m-1} = x_j^{m-2} h$ for some path h where $hp \in O(h)$. This implies that some x_s^2 left divides hp. In other words, $x_i^m = x_j^{m-2} hp$ where, by the inductive hypothesis, $x_j^{m-2} \in \Gamma_{m-2}$ and $x_k^{m-1} = x_j^{m-2} h \in \Gamma_{m-1}$. Moreover, again by the induction hypothesis, $x_s^2 \in \Gamma_2$. It follows that hp contains a subpath in Γ_2. Thus $x_i^m \in \Gamma_m$.

Conversely, suppose $qrs \in \Gamma_m$, where $q \in \Gamma_{m-2}$, $qr \in \Gamma_{m-1}$, and rs contains a subpath in Γ_2. By the inductive hypothesis, we assume $q = x_j^{m-2}, qr = x_k^{m-1}$, and x_t^2 is a subpath of rs, for some j, k, t. Note that x_t^2 right divides rs, otherwise qrs would have a proper subpath which is also an $(m-1)$-prechain, contradicting our choice of qrs. In other words, $rs \in O(r)$. Thus $qrs = x_i^m$ for some i. □

Let $v_k^m = t(x_k^m)$. Now that we know $\{x_i^m\} = \Gamma_m$ for each m, we see immediately that

$$...\bigoplus_k v_k^m A \to \bigoplus v_k^{m-1} A \to ... \to \bigoplus v_k^1 A \to \bigoplus v_k^0 A \to \bar{A} \to 0$$

is a minimal projective resolution of $\bar{A}$ over A. As in Chapter 2, we may define basis elements of $E(A)$ to be of the form

$$(x_i^m)^* = (0, ..., 0, \pi_i^m, 0, .., 0) \in \mathrm{Hom}_A(\bigoplus v_k^m A, \bar{A})$$

which is the matrix with nonzero m^{th} entry π_i^m.

Now we introduce a set which will be of key importance in the next theorem.

Definition 4.19. Let $A = K\mathcal{Q}/I$ be a monomial algebra and let $S \subset K\mathcal{Q}$ be the set of paths p in $\mathcal{Q}$ such that

1. $x_k^2 p = t x_r^2$ for some path p where $l(t) \geq 1$ and $x_r^2 = wp$ for some path w such that $l(w) \geq 1$.

2. If p and p' are such that $x_k^2 p = r x_t^2$ and $x_k^2 p' = r' x_{t'}^2$ and p and p' left divides $x_t^2, x_{t'}^2$ respectively, then $p = p'$.

It is important to note that if $A = K\mathcal{Q}/I$ is a monomial algebra, then the reduced Gröbner basis of I is the unique minimal monomial generating set of I.

Theorem 4.20. *Let $A = K\mathcal{Q}/I$ be a monomial algebra where $\mathcal{G} = \langle x_i^2 \rangle_{1 \le i \le l^2}$ is the unique minimal monomial generating set of I. Then $E(A)$ is generated in degrees 0,1, and 2 if and only if for all paths $p \in S$, we have that if $q \in O(p)$, then either $q \in \mathcal{G}$ or $l(q) = l(p) + 1$.*

Before we prove the theorem, we provide some illustrating examples.

Example 4.21. Let $\mathcal{Q}$ be given by the following quiver

and let $A = K\mathcal{Q}/I$ where $I = \langle a^3, b^2, aba \rangle$ and

$$\mathcal{G} = \{a^3, b^2, aba\}$$

so we can set

$$x_1^2 = a^3$$
$$x_2^2 = b^2$$
$$x_3^2 = aba$$

It is easy to see $S = \{ba, a, b\}$ and consider the path $p = ba \in S$. Note that as paths,

$$a^2(aba) = (a^3)ba$$

and we compute $O(ba) = \{baba, ba^3\}$ Because $baba \notin \mathcal{G}$ and $l(baba) = 4 = l(ba) + 2$, by the above theorem, $E(A)$ cannot be generated in degrees 1 and 2.

Example 4.22. Let $\mathcal{Q}$ be

and let $A = K\mathcal{Q}/I$ where $I = \langle aba, bab, a^2, b^2\rangle$ and

$$\mathcal{G} = \{aba, bab, a^2, b^2\}$$

So we can set

$$x_1^2 = aba$$
$$x_2^2 = bab$$
$$x_3^2 = a^2$$
$$x_4^2 = b^2$$

Clearly, $S = \{a, b, ba, ab\}$.

1. $p = ba$. Indeed we see

$$a(aba) = a^2(ba)$$

and $O(ba) = \{bab, ba^2\}$. Notice $bab \in \mathcal{G}$ and $l(ba^2) = l(ba) + 1$.

2. $p = a$ Indeed we see

$$b(aba) = (bab)a$$

and $O(a) = \{aba, a^2\}$. Notice both aba and a^2 are elements in $\mathcal{G}$.

3. $p = b$. Indeed we see

$$b(b^2) = (b^2)b$$

and $O(b) = \{bab, b^2\}$. Notice both bab and b^2 are elements in $\mathcal{G}$.

4. $p = ab$. Indeed we see

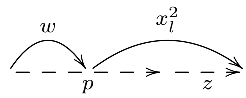
$$b(bab) = (b^2)ab$$

and $O(ab) = \{aba, ab^2\}$. Notice $aba \in \mathcal{G}$ and $l(ab^2) = l(ab) + 1$.

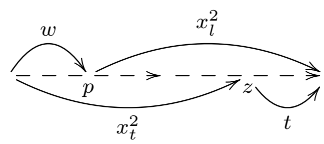
By the above theorem, we know $E(A)$ must be generated in degrees 0, 1 and 2. Moreover, we can even determine the generators of $E(A)$. Because we know $\{x_i^m\} = \Gamma_m$ and $\Gamma_m = \{f_i^j\}$ from Chapter 2, the reslts of Chapter 2 apply. Specifically, we can apply 2.10 to see both $(a^2)^* = a^*a^*$ and $(b^2)^* = b^*b^*$ are elements in $(\mathrm{Ext}_A^1(\bar{A}, \bar{A}))^2$. The same corollary tells us that $a^*b^*a^*$ and $b^*a^*b^*$ are not elements in $(\mathrm{Ext}_A^1(\bar{A}, \bar{A}))^2$, thus must be minimal generators of $E(A)$. Thus the set of generators for $E(A)$ are as follows:

$$v^*, a^*, b^*, (aba)^*, (bab)^*$$

Example 4.23. Truncated Monomial Algebras: Let $A = K\mathcal{Q}/I$ be a monomial algebra where I is the ideal generated by all the paths of length m. Let $|\mathcal{Q}_0| = n$ and $|\mathcal{Q}_1| = c$. Let p be *any* path in $\mathcal{Q}$ such that $p \notin I$. Suppose $O(p)$ is nonempty, and let $q \in O(p)$. Then for some index l and some paths w, z,



and we know $l(x_l^2) = m$, which implies $l(pz) \geq m$. If $l(pz) > m$, then $pz = x_t^2 t'$ where $l(x_t^2) = m$



which implies that $x_t^2 \in O(p)$ and x_t^2 left divides q. This contradicts our choice that $q \in O(p)$. Thus, $l(pz) = m$, which implies that $l(w) = 0$. Consequently, $q = x_l^2 \in \mathcal{G}$.

We have shown that for all paths $p \notin I$, if $q \in O(p)$, $q \in \mathcal{G}$. Because $S \cap I = \emptyset$, the same holds for all $p \in S$. Therefore $E(A)$ is generated in degrees 0,1, and 2.

If $m = 2$, by 2.10, we see that for all x_i^2, $(x_i^2)^* = (x_k^1)^*(x_s^1)^*$ for some indices k, s. Thus $E(A)$ is generated in degree 0 and 1. If $m > 2$, then, also by 2.10, we see that for all x_i^2, $(x_i^2)^* \notin \text{Ext}_A^1(\bar{A}, \bar{A})^2$, thus must be a minimal generator. Then the minimal generators of $E(A)$ are the following:

$$\{(x_1^0)^*, ..., (x_n^0)^*, (x_1^1)^*, ..., (x_c^1)^*, (x_1^2)^*, ..., (x_{l^2}^2)^*\}$$

This example shows it is not always necessary to compute S.

Example 4.24. Let

$$A = K[x_1, ..., x_n]/\langle x_j x_i, x_i^{m_i} \mid i \neq j \rangle$$

where

$$\mathcal{G} = \{x_j x_i, x_i^{m_i} \mid i \neq j\}$$

Then

$$S = \{x_i \mid 1 \leq i \leq n\}$$

because $x_i^{m_i} \in \mathcal{G}$. Notice that for all i, $q \in O(x_i)$ implies $q \in \{x_i^{m_i}, x_i x_k \mid k \neq i\} \subset \mathcal{G}$. Thus $E(A)$ is generated in degrees 0,1,2. By 2.8, we know that the minimal generators of degree 2 are of the form $(x_i^{m_i})^*$ where $m_i \geq 3$.

Example 4.25. Here is another local monomial algebra.

$$A = K[x_1, ..., x_n]/\langle x_j x_i, x_i^{m_i}, x_n^{m_n} \mid 1 \leq i < j \leq n\rangle$$

where

$$\mathcal{G} = \{x_j x_i, x_i^{m_i}, x_n^{m_n} \mid 1 \leq i < j \leq n\}$$

Then

$$S = \{x_i \mid 1 \leq i \leq n\}$$

because $x_i^{m_i} \in \mathcal{G}$. Notice that for all i, $q \in O(x_i)$ implies $q \in \{x_i^{m_i}, x_i x_k \mid 1 \leq k < i\} \subset \mathcal{G}$. Thus $E(A)$ is generated in degrees 0,1,2. By 2.8, we know that the minimal generators of degree 2 are of the form $(x_i^{m_i})^*$ where $m_i \geq 3$.

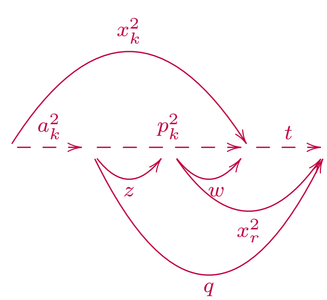
Remark 4.26. Recall 3.12, which states that given a monomial algebra, for every j and s, we have $(x_i^j)^*(x_r^s)^* = (x_k^{j+s})^*$ if and only if $x_i^j x_r^s = x_k^{j+s}$.

Lemma 4.27. *For every k, let $x_k^2 = aq$ where $a \in \mathcal{Q}_1$ and q is a path in $\mathcal{Q}$. Then $p \in S$ implies $qp \in O(p_k^2)$ for some k.*

Proof. Suppose $p \in S$ and $qp \notin O(q)$. Because $p \in S$, we know there exist indices k, t and a path r such that $x_k^2 p = r x_t^2$. Also, since $qp \notin O(q)$, we know there exists a path p' such that for some path z where $l(z) \geq 1$, we have $p'z = p$ and $qp' \in O(q)$. In other words, $qp = z x_s^2$ for some index s. This implies $x_k^2 p' = a z x_s^2$. However, by 4.19, this implies $p = p'$. □

We now prove the theorem 4.20.

Proof. Let $\mathcal{G} = \{x_i^2\}_{i=1}^{l2}$ be the reduced Gröbner basis for I. For each k, we may write $x_k^2 = a_k^2 p_k^2$ where $a_k^2 \in \mathcal{Q}_1$ and p_k^2 is a path in $\mathcal{Q}$. Recall how we construct the set $\{x_i^3\}$. For every i, $x_i^3 = a_k^2 q$ where $q \in O(p_k^2)$. From the algorithm we see that any x_i^3 is the dotted path in the following diagram.



where $q = z x_r^2 = p_k^2 t$ and $x_r^2 = wt$ and $l(a_k^1) = 1, l(p_k^2) \geq 1$, and $l(t) \geq 1$ (We know $l(t) \geq 1$ because $p_k^2 \notin I$ and $p_k^2 t \in I$).

Now suppose $p \in S$. We claim that $p = t$ for some x_i^3 in the diagram above. If $t \in S$, then $x_k^2 p = m x_r^2$ for some path m and some indices k, r. We can write $x_k^2 = a_k^2 p_k^2$. Because $p \in S$, we know $p_k^2 p \in O(p_k^2)$ by 4.27. Thus the claim holds and such an x_i^3 exists.

($\Rightarrow$) Assume $E(A)$ is generated in degree 0,1, and 2. Let $t \in S$ and x_i^3 chosen so that $x_i^3 = x_k^2 t$. By the above remark, one of the following three cases must occur:

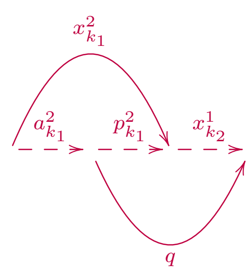
1. Case I: $(x_i^3)^* = (x_{k_1}^1)^*(x_{k_2}^1)^*(x_{k_3}^1)^*$ which implies $x_i^3 = x_{k_1}^1 x_{k_2}^1 x_{k_3}^1$.

2. Case II: $(x_i^3)^* = (x_{k_1}^2)^*(x_{k_2}^1)^*$ which implies $x_i^3 = x_{k_1}^2 x_{k_2}^1$.

3. Case III: $(x_i^3)^* = (x_{k_1}^1)^*(x_{k_2}^2)^*$ which implies $x_i^3 = x_{k_1}^1 x_{k_2}^2$.

Case I: Assume $x_i^3 = x_{k_1}^1 x_{k_2}^1 x_{k_3}^1$. Then $a_k^1 = x_{k_1}^1$ which implies that $q = x_{k_2}^1 x_{k_3}^1$. Because $q \in O(p_k^2)$, we know $q = zx_r^2$ for some path z and some r. Because $l(x_r^2) \geq 2$, we must have $q = x_r^2$.

Case II: Assume $x_i^3 = x_{k_1}^2 x_{k_2}^1$. Then consider the following diagram



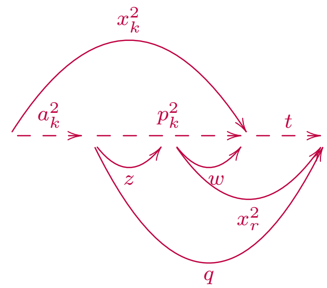
and we see $l(q) = l(p_{k_1}^2) + 1$.

Case III: Assume $x_i^3 = x_{k_1}^1 x_{k_2}^2$. Similar to Case 1, we must have $a_k^1 = x_{k_1}^1$ and $q = x_{k_2}^2 \in \mathcal{G}$.

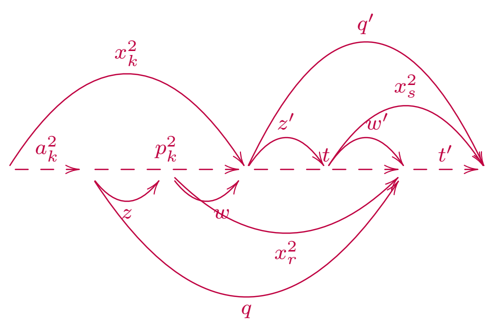
With these three cases, we have shown that for all $q \in O(p_k^2)$, either $l(q) = l(p_k^2)+1$ or $q \in \mathcal{G}$.

Now we repeat our argument as we consider the construction of $\{x_i^4\}$. To construct x_j^4

for any j, we first start with an x_i^3,



Now we find $O(t)$. Suppose $q' \in O(t)$ Then $q' = tt'$ for some path t' such that $l(t') \geq 1$ such that $tt' = z'w'x_s^2$ for some s. Then x_j^4 can be visualized by the following diagram:



Because we assume $E(A)$ is generated in degrees 0,1, and 2, we may assume one of the following holds:

1. Case I: $(x_i^4)^* = (x_{k_1}^1)^*(x_{k_2}^1)^*(x_{k_3}^1)^*(x_{k_4}^1)^*$, which implies $x_i^4 = x_{k_1}^1 x_{k_2}^1 x_{k_3}^1 x_{k_4}^1$

2. Case II: $(x_i^4)^* = (x_{k_1}^1)^*(x_{k^2}^1)^*(x_{k_3}^2)^*$, which implies $x_i^4 = x_{k_1}^1 x_{k^2}^1 x_{k_3}^2$

3. Case III: $(x_i^4)^* = (x_{k_1}^1)^*(x_{k^2}^2)^*(x_{k_3}^1)^*$, which implies $x_i^4 = x_{k_1}^1 x_{k^2}^2 x_{k_3}^1$

4. Case IV: $(x_i^4)^* = (x_{k_1}^2)^*(x_{k_2}^1)^*(x_{k_3}^1)^*$, which implies $x_i^4 = x_{k_1}^2 x_{k_2}^1 x_{k_3}^1$

5. Case V: $(x_i^4)^* = (x_{k_1}^2)^*(x_{k_2}^2)^*$, which implies $x_i^4 = x_{k_1}^2 x_{k_2}^2$

Case I: Assume $x_i^4 = x_{k_1}^1 x_{k_2}^1 x_{k_3}^1 x_{k_4}^1$. Then $l(x_i^4) = 4$. Because $l(a_k^2) = 1, l(p_k^1) \geq 1, l(t) \geq 1$, and $l(t') \geq 1$, we must have a_k^2, p_k^2, t, and t' are all arrows and $a_k^2 = x_{k_1}^1, p_k^2 = x_{k_2}^1, t = x_{k_3}^1$ and $t' = x_{k_4}^1$. Consequently, $q' = x_s^2$.

Case II: Assume $x_i^4 = x_{k_1}^1 x_{k^2}^1 x_{k_3}^2$. Then, as before, we must have $x_{k_1}^1 = a_k^2$, and $x_s^2 = x_{k_3}^2$. Then $x_{k_2}^1 = p_k^2 z'$. Because $l(p_k^2) \geq 1$, we must have $p_k^2 = x_{k_2}^1$. Consequently, $q' = x_s^2$.

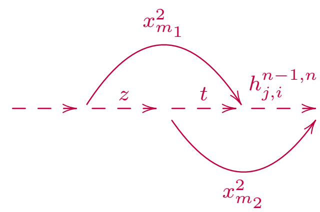
Case III: Assume $x_i^4 = x_{k_1}^1 x_{k^2}^2 x_{k_3}^1$. Then $a_k^2 = x_{k_1}^1$ as before. We must have $x_{k_2}^2 m_1 = z x_r^2 m_2$ for some paths m_1, m_2. However, if either $l(m_1), l(m_2)$, or $l(z) \geq 1$, then $\mathcal{G}$ is not a reduced Gröbner basis. Thus, we have $x_{k_2}^2 = x_r^2$. Consequently, $x_{k_3}^1 = t'$, which implies that $l(q') = l(q) + 1$.

Case IV: Assume $x_i^4 = x_{k_1}^2 x_{k_2}^1 x_{k_3}^1$. Then $x_{k_1}^2 = x_k^2$ and because $l(t) \geq 1$ and $l(t') \geq 1$, we must have $t = x_{k_2}^1$ and $t' = x_{k_3}^1$, and $x_s^2 = q'$. Thus $q' \in \mathcal{G}$.

Case V: Assume $x_i^4 = x_{k_1}^2 x_{k_2}^2$. Then $x_{k_1}^2 = x_k^2$ and $x_{k_2}^2 = x_s^2$. We then have $l(z') = 0$, which implies $q' = x_s^2$. Consequently, $q' \in \mathcal{G}$.

In all cases, we have $q' \in \mathcal{G}$ or $l(q') = l(q) + 1$. Because for all paths p such that $a x_r^2 = x_k^2 p$ there exists some x_i^3 with $p = t$ the claim holds.

To prove the converse, we proceed by induction on n to show that if $(x_i^n)^*$ is generated in degrees $0, 1, 2$, then for all k such that $x_k^{n+1} = x_i^n h_{i,k}^{n-1,n}$, $(x_k^{n+1})^*$ is also generated in degrees 0,1, and 2. The case where $n = 2$ is trivial because every $(x_i^2)^*$ is generated in degree 2. We start by visualizing $x_i^n = x_j^{n-1} h_{j,i}^{n-1,n}$.



for some indices m_1, m_2 and some paths z, t. Because $zx_{m_2}^2 = x_{m_1}^2 h_{j,i}^{n-1,n}$. we know for all $q \in O(h_{j,i}^{n-1,n})$ either $q \in \mathcal{G}$ or $l(q) = l(h_{j,i}^{n-1,n})$.

Case I: Assume $q \in \mathcal{G}$. Then $q = x_s^2$ for some s and $x_k^{n+1} = x_j^{n-1} x_s^2$. By the inductive hypothosis, we assume $(x_j^{n-1})^*$ is generated in degree 0,1, and 2. It follows that $(x_k^{n+1})^*$ is as well.

Case II: Assume $l(q) = l(h_{j,i}^{n-1,n}) + 1$. Then $x_k^{n+1} = x_i^n x_m^1$ for some index m. By the inductive hypothesis, we assume $(x_i^n)^*$ is generated in degree 0,1, and 2, so it follows that $(x_k^{n+1})^*$ is also generated in degrees 0,1, and 2. □

4.5 The Case Where the AGS Resolution is Minimal

In this section, suppose A is a finitely generated graded K-algebra and $A = K\mathcal{Q}/I$. We consider the important case where the AGS resolution is a minimal resolution of $\bar{A}$. In other words, we consider the case where for each j, $\{x_i^j\} = \{f_i^j\}$ from Chapter 2 and we wish to prove the following: Assuming the AGS resolution is minimal, then if $E(A_{\text{MON}})$ is finitely generated,then $E(A)$ is finitely generated.

First let us consider A_{MON}. Suppose $\mathcal{P}_{\text{MON}}$ is the AGS resolution of $\bar{A}_{\text{MON}}$ over A_{MON}.

$$...P^2_{\text{MON}} \to P^1_{\text{MON}} \to P^0_{\text{MON}} \to \bar{A} \to 0$$

Let $\{g^n_k\} = \Gamma_n$ be the n-chains, as given as in [16] and $v^n_k = t(g^n_k)$. Then $P^n_{\text{MON}} = \bigoplus v^n_k A_{\text{MON}}$. By [12], we know $\mathcal{P}_{\text{MON}}$ is minimal.

Consider now the AGS resolution of $\bar{A}$ over A. If $\{f^n_i\}$ is given by the AGS algorithm of $\bar{A}$ over A we know, by construction, that $\text{tip}(f^n_i) = g^n_i$ and thus $t(f^n_i) = v^n_i$. We may write the AGS resolution $\mathcal{P}$ of $\bar{A}$ as

$$...P^2 \to P^1 \to P^0 \to \bar{A} \to 0$$

where $P^n = \bigoplus v^n_k A$. Note that $\mathcal{P}$ need not be minimal in general. However, from here on, we assume that $\mathcal{P}$ is a minimal projective resolution.

Since we assume the AGS resolution is a minimal projective resolution of $\bar{A}$ over A, letting $\{(f^n_i)^*\}$ be the corresponding basis of $\text{Ext}^n_A(\bar{A}, \bar{A})$, we know we may write $(f^{n-z}_m)^*(f^z_i)^*$ as a linear combination of elements of of the form $(f^n_k)^*$. It turns out the coefficients have a particular form. Namely, if

$$f^n_k = f^{n-z}_1 h^{n-z,n}_{1,k} + ... + f^{n-z}_m h^{n-z,n}_{m,k} + ... + f^{n-z}_{l^{n-z}} h^{n-z,n}_{l^{n-z},k}$$

then we will show that

$$h^{n-z,n}_{m,k} = \sum_{\hat{T}_z} f^z_s b_s + \sum_{\hat{U}_z} f^{z'}_t c_t$$

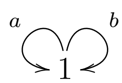
where $b_s, c_s \in K\mathcal{Q}$. Recall that $\hat{T}_z = \{f^z_k \mid 1 \leq k \leq l^z\}$.

Definition 4.28. For some $m, i, z \in \mathbb{Z}$, let $Z^{n-z,n}_{m,i}$ be the set containing all elements f^n_k which satisfy the following property:

- Writing $h^{n-z,n}_{m,k} = \sum_{s=1}^{l^z} f^z_s b_s + \sum_{\hat{U}_z} f^{z'}_t c_t$, then b_i has a term in K^*

If there is no index k such that $h^{n-z,n}_{m,k} = \sum_{s=1}^{l^z} f^z_s b_s + \sum_{\hat{U}_z} f^{z'}_t c_t$ and b_i has a term in K^*, then $Z^{n-z,n}_{m,i}$ is empty. To get a better handle on such a technical set, here is an example:

Example 4.29.



and let $A = K\mathcal{Q}/I$ where $I = \langle aba, bab, a^2, b^2 \rangle$

$f^0_1 = 1$	$f^1_1 = a$	$f^2_1 = a^2$	$f^3_1 = a^3$
	$f^1_2 = b$	$f^2_2 = b^2$	$f^3_2 = b^3$
		$f^2_3 = aba$	$f^3_3 = a^2ba$
		$f^2_4 = bab$	$f^3_4 = b^2ab$
		$f^{2'}_1 = ab^2$	$f^3_5 = aba^2$
		$f^{2'}_2 = ba^2$	$f^3_6 = bab^2$
			$f^3_7 = abab$
			$f^3_8 = baba$

Say we wish to construct $Z^{1,3}_{1,4}$. By definition, this is the set consisting of f^3_k which satisfy the following property: $h^{1,3}_{1,k} = \sum_{s=1}^{l^1} f^1_s b_s + \sum_{\hat{U}_z} f^{z'}_t c_t$, then b_4 has a term in K^*. Note that

$$f^3_1 = a^3 = f^1_1 f^2_1 \Rightarrow h^{1,3}_{1,1} = f^2_1$$

$$f^3_2 = b^3 = f^1_2 f^2_2 \Rightarrow h^{1,3}_{2,2} = f^2_2$$

$$f^3_3 = a^2ba = f^1_1 f^2_3 \Rightarrow h^{1,3}_{1,3} = f^2_3$$

$$f^3_4 = b^2ab = f^1_2 f^2_4 \Rightarrow h^{1,3}_{2,4} = f^2_4$$

$$f^3_5 = aba^2 = f^1_1 f^{2'}_2 \Rightarrow h^{1,3}_{1,5} = f^{2'}_2$$

$$f^3_6 = bab^2 = f^1_2 f^{2'}_1 \Rightarrow h^{1,3}_{2,6} = f^{2'}_1$$

$$f^3_7 = abab = f^1_1 f^2_4 \Rightarrow h^{1,3}_{1,7} = f^2_4$$

$$f^3_8 = baba = f^1_2 f^2_3 \Rightarrow h^{1,3}_{2,8} = f^2_3$$

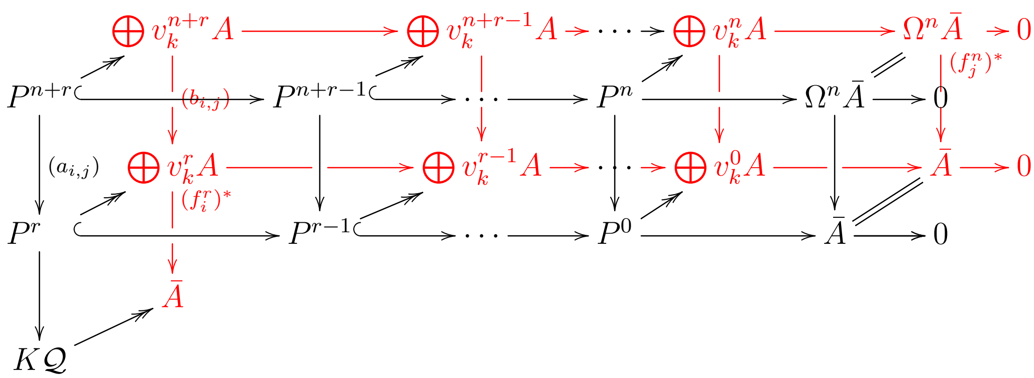
and the only f^3_k such that $h^{1,3}_{1,k} = f^2_4$ is f^3_7. Thus $Z^{1,3}_{1,4} = \{f^3_7\}$.

Lemma 4.30. *Let $\{f^n_i\}$ be given by the AGS algorithm and let $\{(f^n_i)^*\}$ the corresponding basis of $Ext^n_A(\bar{A}, \bar{A})$. Then for every m, i,*

$$(f^{n-z}_m)^*(f^z_i)^* = \sum_{f^n_k \in Z^{n-z,n}_{m,i}} \lambda_k (f^n_k)^*$$

where $\lambda_k \in K^*$.

Proof. First we look at the following diagram, in which each square commutes, found in section 3 of [15]. However, here we make our diagram specific to the module $\bar{A}$ and we use results found in section 4 of [14]. Let $v_k^n = t(f_k^n)$ and $v_k^{n'} = t(f_k^{n'})$.



where

$$P^n = \bigoplus_{k=1}^{l^n} v_k^n K\mathcal{Q} + \bigoplus_{\hat{U}_n} v_k^{n'} K\mathcal{Q}$$

To paraphrase [15], we do not want to complicate notation and assume the map $(f_i^r)^*$ denotes the map $P^r \to \bar{A}$ as well as its image modulo I, $\bigoplus v_k^r A \to \bar{A}$. However, we briefly describe the maps in detail. Note the map $P^r \to \bigoplus_{k=1}^{l^r} v_k^r A$ is the matrix $(Q, 0)$ where $Q : \bigoplus_{k=1}^{l^r} v_k^r K\mathcal{Q} \to \bigoplus_{k=1}^{l^r} v_k^r A$ is the canonical projection map. We know $\bigoplus_{\hat{U}_r} v_k^{r'} K\mathcal{Q}$ must map to 0 from corollary 14 of [14]. Thus we may define the map $P^r \to K\mathcal{Q}$ as the matrix $(N, 0)$ where $N : \bigoplus_{k=1}^{l^r} v_k^r K\mathcal{Q} \to K\mathcal{Q}$ is the map $(1, 1, ..., 1)$ and 0 is the zero map.

The red rows are minimal AGS resolutions of $\Omega^n \bar{A}$ and $\bar{A}$ over A, respectively. The red face of the parallelepiped is the front face modulo the ideal I. Suppose the lifting map $(a_{i,j}) : P^{n+r} \to P^r$ is such that for some i, j, $a_{i,j}$ is multiplication by an element in $x \in K\mathcal{Q}$ where x has a nonzero constant term λ. An easy computation shows this is true if and only if modulo I, the lifting map $(b_{i,j}) : v_k^{n+r} A \to v_k^r A$ satisfies that $b_{i,j}$ is also multiplication by an element $\bar{x} \in A$ with $\bar{\lambda}$ as a nonzero constant term.

Now assume $t(f_m^{n-z}) = v_1^0$ and $s(f_i^z) = v_1^0$ and consider the following commutative diagram

$$
\begin{array}{ccccc}
\bigoplus_{k=1}^{l^n} v_k^n K\mathcal{Q} + \bigoplus_{\hat{U}_n} v_k^{n'} K\mathcal{Q} & \xrightarrow{M_1} & \bigoplus_{k=1}^{l^{n-z}} v_k^{n-z} K\mathcal{Q} + \bigoplus_{\hat{U}_{n-z}} v_k^{n-z'} K\mathcal{Q} & & \\
\downarrow (a_{i,j}) & & \downarrow L^0 & \searrow (f_m^{n-z})^* & \\
\bigoplus_{k=1}^{l^z} v_k^z K\mathcal{Q} + \bigoplus_{\hat{U}_z} v_k^{z'} K\mathcal{Q} & \xrightarrow{M_2} & \bigoplus_{k=1}^{l^0} v_k^0 K\mathcal{Q} & \longrightarrow & \bar{A} \longrightarrow 0 \\
\downarrow (f_i^z)^* & & & & \\
\bar{A} & & & &
\end{array}
$$

where

$$
L^0 = \begin{pmatrix} 0 & 0 & ... & 1 & ... & 0 \\ 0 & ... & ... & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & ... & ... & ... & 0 \end{pmatrix}
$$

where the $(1, m)$ entry is 1,

$$
M_1 = \begin{pmatrix} (h^{n-z,n}) & D_1 \\ 0 & D_2 \end{pmatrix}
$$

$$
M_2 = \begin{pmatrix} (h^{0,z}) & C_1 \\ 0 & C_2 \end{pmatrix}
$$

and by $(f_i^z)^*$ we are denoting the composition

$$
P^r \to \bigoplus_{k=1}^{l^r} v_k^r A \to \bar{A}
$$

Because the diagram commutes, the $(1, k)$ entry of $L^0 M_1$ and $M_2(a_{i,j})$ are the same, so for $1 \le k \le l^n$,

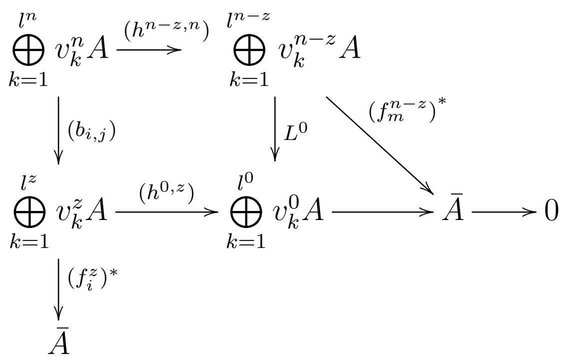
$$
h_{m,k}^{n-z,n} = \sum_j h_{1,j}^{0,z} a_{j,k} + \sum_s C_{1,s} a_{s,k} + \sum_t C_{1,t} a_{t,k}
$$

where $\sum_s C_{1,s}a_{s,k} + \sum_t C_{1,t}a_{t,k} \in \bigoplus_{\hat{U}_z} f_t^{z'} K\mathcal{Q}$. Also, notice the set $\{f_j^z\} = \{h_{i,j}^{0,z}\}$. We now compute

$$(f_i^z)^*(a_{i,j}) = (\pi_i^z a_{i,1}, \pi_i^z a_{i,2}, ..., \pi_i^z a_{i,l^n}, 0, ..., 0) = \sum_{\{k \le l^n \ | \ a_{i,k} \in K^*\}} a_{i,k}(f_k^n)^*$$

Notice that $\pi_i^z a_{i,k} \neq 0$ if and only if $a_{i,k}$ has a term in K^* because π_i^z vanishes on the radical. However, for $k \le l^z$, $a_{i,k}$ has a nonzero constant term if and only if $f_k^n \in Z_{m,i}^{n-z,n}$.

Now we consider the diagram



and compute

$$(f_i^r)^*(b_{i,j}) = (\pi_i^z b_{i,1}, \pi_i^z b_{i,2}, ..., \pi_i^z b_{i,l^n}) = \sum_{\{k \ | \ b_{i,k} \in K^*\}} b_{i,k}(f_k^n)^*$$

$\pi_i^z b_{i,k} \neq 0$ if and only if $b_{i,k}$ has a term in K^* because π_i^z vanishes on the radical. Recall that $b_{i,k}$ contains a term in K^* if and only if $a_{i,k}$ does, which implies

$$(f_m^n)^*(f_i^z)^* = \sum_{Z_{m,i}^{n-z,n}} a_{i,k}(f_k^n)^*$$

where $Z^{n-z,n}_{m,i}$ is defined in 4.28. □

Corollary 4.31. If $f^n_k \notin Z^{n-z,z}_{m,i}$ for any m, i, z, then as a basis element of $E(A_{\text{MON}})$, $(g^n_k)^*$ is a minimal generator.

Proof. For any z, we may write $\text{tip}(f^n_k) = \text{tip}(f^{n-z}_{i_z})\,\text{tip}(h^{n-z,n}_{i_z,k})$ for some index i_z. If $\text{tip}(h^{n-z,n}_{i_z,k}) = f^z_s$ for any index s, then $f^n_k \in Z^{n-z,n}_{i_z,s}$, a contradiction. Now consider $g^n_k = g^{n-z}_{i_z} h$ for h a path in $\mathcal{Q}$. Because $h = \text{tip}(h^{n-z,n}_{i_z,k})$, we know $h \neq g^z_s$ for any index s. Thus $(g^n_k)^* \neq (g^{n-z}_{i_z})^*(g^z_s)^*$ for any z, i_z, and s and $(g^n_k)^*$ must be a minimal generator of $E(A_{\text{MON}})$. □

Assume now that $E(A_{\text{MON}})$ is finitely generated in degrees $0, ..., m$. Then, for every $n > m$, we have $g^n_i = g^{n-z}_k g^z_t$ where $z \in \{1, 2, ...m\}$. This is because we are using the characterization of generators of the Ext-algebra of a mnomial algebra as found in [16]. This allows us to partition the set $\{g^n_i\}$ into m sets:

$$S_1 = \{(g^n_i)^* \mid g^n_i = g^{n-1}_k g^1_t, k, t \in \mathbb{N}\}$$

$$S_2 = \{(g^n_i)^* \mid g^n_i = g^{n-2}_k g^2_t, k, t \in \mathbb{N}\} - S_1$$

$$\vdots$$

$$S_z = \{(g^n_i)^* \mid g^n_i = g^{n-z}_k g^z_t, k, t \in \mathbb{N}\} - \bigcup_{w=1}^{z-1} S_w$$

for $1 \leq z \leq m$. It is important to note that S_z could be empty.

Because $\text{tip}(f^n_i) = g^n_i$ and each f^n_i is homogeneous, for every $n > m$ we partition the set

$\{(f_i^n)^*\}$ into m sets as well:

$$T_z = \{(f_i^n)^* \mid \operatorname{tip}(f_i^n) \in S_z\}$$

for $1 \leq z \leq m$. It is important to note that T_z could be empty.

We may now define, for $n > m$ a map of vector spaces

$$\Phi_n : \operatorname{Ext}_A^n(\bar{A}, \bar{A}) \longrightarrow \operatorname{Ext}_A^n(\bar{A}, \bar{A})$$

$$\Phi_n((f_s^n)^*) = (f_k^{n-z})^* (f_t^z)^*$$

if $(f_s^n)^* \in T_z$ then extended by linearity.

From here, we show that $\{\Phi_n((f_s^n)^*)\}$ is linearly independent. To do so, we need the following remark and proposition.

Remark 4.32. Suppose $g_s^n = g_k^{n-z} g_t^z$ where $z \in \{1, 2, ..., m\}$. Then $f_s^n \in Z_{k,t}^{n-z,n}$ because $\operatorname{tip}(h_{k,s}^{n-z,n}) = \operatorname{tip}(f_t^z)$.

Proposition 4.33. *Suppose* $f_i^n = f_1^{n-z} h_{1,i}^{n-z,n} + ... + f_k^{n-z} h_{k,i}^{n-z,n} + ... + f_{l^{n-z}}^{n-z} h_{l^{n-z},i}^{n-z,n}$ *where* $z \in \{1, 2, ..., m\}$. *Then* $\operatorname{tip}(f_j^{n-z}) \operatorname{tip}(h_{j,i}^{n-1,n}) \leq \operatorname{tip}(f_i^n)$ *for all* j.

Proof. Without loss of generality, assume $i = 1$. Note for all $j \neq r$,

$$\operatorname{tip}(f_r^{n-z}) \operatorname{tip}(h_{r,1}^{n-z,n}) \neq \operatorname{tip}(f_j^{n-z}) \operatorname{tip}(h_{j,1}^{n-z,n})$$

This is because, if they were equal,

$$\operatorname{tip}(f_r^{n-z})\operatorname{tip}(h_{r,1}^{n-z,n}) = \operatorname{tip}(f_j^{n-z})\operatorname{tip}(h_{j,1}^{n-z,n})$$

implies either $\operatorname{tip}(f_r^{n-z})$ left divides $\operatorname{tip}(f_j^{n-z})$, $\operatorname{tip}(f_j^{n-z})$ left divides $\operatorname{tip}(f_r^{n-z})$, or $\operatorname{tip}(f_j^{n-z}) = \operatorname{tip}(f_r^{n-z})$. However, this cannot happen by the construction of $\{f^{n-z}\}$. Thus $\operatorname{tip}(f_1^n) = \operatorname{tip}(f_j^{n-z})\operatorname{tip}(h_{j,1}^{n-z,n})$ for a unique index j such that $\operatorname{tip}(f_j^{n-z})\operatorname{tip}(h_{j,1}^{n-z,n}) > \operatorname{tip}(f_r^{n-z})\operatorname{tip}(h_{r,1}^{n-z,n})$ for all $r \neq j$. □

Proposition 4.34. *For every n, the image of $\{f_i^n\}$ under Φ_n is linearly independent.*

Proof. Suppose for $\beta_s \in K$,

$$\begin{aligned} 0 &= \sum_{s=1}^{l^n} \beta_s \Phi_n((f_s^n)^*) \\ &= \sum_{z=1}^{m} \left(\sum_{(f_s^n)^* \in T_z} \beta_s (f_k^{n-z})^* (f_t^z)^* \right) \\ &= \sum_{z=1}^{m} \left(\sum_{(f_s^n)^* \in T_z} \beta_s \left(\sum_{f_u^n \in Z_{k,t}^{n-2,n}} \lambda_u (f_u^n)^* \right) \right) \end{aligned} \tag{4.2}$$

By reindexing if necessary, we may assume that $\operatorname{tip}(f_s^n) < \operatorname{tip}(f_t^n)$ if and only if $s < t$. This is because $\{f_i^n\}$ is tip-reduced. Consequently, $\operatorname{tip}(f_1^n) < \operatorname{tip}(f_k^n)$ for all $k > 1$. From here, we would like to show that a nonzero scalar multiple of $(f_1^n)^*$ appears as a nonzero term of $\Phi((f_s^n)^*)$ if and only if $s = 1$. This will imply that $\beta_1 = 0$ by (4.2).

Suppose $\operatorname{tip}(f_1^n) = \operatorname{tip}(f_i^{n-x})\operatorname{tip}(f_j^x)$ for some pair or indices i, j and $x \in \{1, 2, ..., m\}$. By 4.32, we know $f_1^n \in Z_{i,j}^{n-x,n}$.

Now suppose $(k, t) \neq (i, j)$ and there is an index s such that

$$\text{tip}(f_s^n) = \text{tip}(f_k^{n-y})\,\text{tip}(f_t^y) \tag{4.3}$$

for $y \in \{1, 2, ..., m\}$. Then

$$\Phi((f_s^n)^*) = \sum_{Z_{k,t}^{n-y,n}} \lambda_w (f_w^n)^*$$

We claim that $f_1^n \notin Z_{k,t}^{n-y,n}$. To prove our claim, suppose $f_1^n \in Z_{k,t}^{n-y,n}$ and write

$$f_1^n = f_1^{n-y} h_{1,1}^{n-y,n} + ... + f_k^{n-y} h_{k,1}^{n-y,n} + ... + f_{l^{n-y}}^{n-y} h_{l^{n-y},1}^{n-y,n}$$

where, by 4.33, $\text{tip}(f_j^{n-y})\,\text{tip}(h_{j,1}^{n-y,n}) \leq \text{tip}(f_1^n)$. Because $f_1^n \in Z_{k,t}^{n-y,n}$, we know $\alpha_1 \,\text{tip}(f_t^z)$ is a term of $(h_{k,1}^{n-y,n})$ for some constant $\alpha_1 \in K^*$. Thus $f_k^{n-y} h_{k,1}^{n-y,n}$ has $\alpha_1 \,\text{tip}(f_k^{n-y})\,\text{tip}(f_t^y)$ as a nonzero term. Thus $\text{tip}(f_k^{n-y})\,\text{tip}(h_{k,1}^{n-y,n}) \geq \text{tip}(f_k^{n-y})\,\text{tip}(f_t^y)$. However, $\text{tip}(f_k^{n-y})\,\text{tip}(f_t^y) = \text{tip}(f_s^n)$ by (4.3). Because $\text{tip}(f_s^n) > \text{tip}(f_1^s)$, we conclude $\text{tip}(f_k^{n-y})\,\text{tip}(h_{k,1}^{n-y,n}) > \text{tip}(f_1^n)$, which contradicts the 4.33. Thus $f_1^n \notin Z_{k,t}^{n-y,n}$.

Now we return to the original equation (4.2),

$$\begin{aligned} 0 &= \sum_{s=1}^{l^n} \beta_s \Phi_n((f_s^n)^*) \\ &= \sum_{z=1}^{m} \left(\sum_{(f_s^n)^* \in T_z} \beta_s (f_k^{n-z})^* (f_t^z)^* \right) \\ &= \sum_{z=1}^{m} \left(\sum_{(f_s^n)^* \in T_z} \beta_s \left(\sum_{f_u^n \in Z_{k,t}^{n-2,n}} \lambda_u (f_u^n)^* \right) \right) \end{aligned}$$

and now know that

1. $f_1^n \in Z_{i,j}^{n-x,n}$

2. $f_1^n \notin Z_{k,t}^{n-z,n}$ for all $z \neq x$

3. $f_1^n \notin Z_{k,t}^{n-x,n}$ for all $(k,t) \neq (i,j)$.

Consequently, we have $(f_1^n)^*$ appearing in a single term of (4.2) with the coefficient β_1. As the AGS resolution is minimal, we know $\{(f_i^n)^*\}$ is a basis of $\mathrm{Ext}_A^n(\bar{A}, \bar{A})$, which means the set is linearly independent. Because we have shown $(f_1^n)^*$ appears as a nonzero term in $\Phi_n((f_1^n)^*)$ but not as a nonzero term in $(f_k^n)^*$ for any $k \neq 1$, we can conclude that $\beta_1 = 0$. We may now repeat the argument with $(f_2^n)^*$ in place of $(f_1^n)^*$ to conclude $\beta_2 = 0$. $\{(f_i^n)^*\}$ is a finite set, so we will eventually find that all $\beta_k = 0$. □

Therefore, for each n, we must have $\{\Phi_n((f_i^n)^*)\}$ is a basis of $\mathrm{Ext}_A^n(\bar{A}, \bar{A})$. Consequently, we may now state the following theorem:

Theorem 4.35. *Let A be a finite-dimensional length graded K-algebra. Suppose the AGS resolution is minimal. Then if $E(A_{MON})$ is finitely generated in degrees $0, 1, ..., m$ for some m, then $E(A)$ is also generated in degrees $0, 1, ..., m$.*

Note the theorem need not be true if the AGS resolution is not minimal. Consider the following:

Example 4.36. Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is given by the following quiver.

$$1 \underset{a}{\overset{b}{\leftrightarrows}} 2 \underset{c}{\overset{d}{\leftrightarrows}} 3$$

and $I = \langle ac, db, ba - cd \rangle$. A reduced Gröbner basis for I is given by

$$\mathcal{G} = \{ac, db, cd - ba, aba, bab\}$$

Using m-chains, it is not difficult to show that $E(A_{\text{MON}})$ is generated in degres 0,1,2. However, in [6], it is shown that $E(A)$ is generated in degrees 0,1,3.

It is believed that the converse to the theorem also holds. That is, it is conjectured the following statement is true: Let A be a finite-dimensional length graded K-algebra. Suppose the AGS resolution is minimal. Then if $E(A)$ is finitely generated in degrees $0, 1, ..., m$ for some m, then $E(A_{\text{MON}})$ is also generated in degrees $0, 1, ..., m$. However, it is unproven.

We now ask ourselves for which graded algebras A does $\bar{A}$ have a minimal AGS resolution. The following proposition is known by Green and Marcos, and for the convenience of the reader we prove it here.

Proposition 4.37. *If $A = K\mathcal{Q}/I$ is a length graded algebra and $\mathcal{G}$ is a Gröbner basis for I and a minimal generating set for I, then the AGS resolution of $\bar{A}$ will be minimal.*

Proof. Suppose $\mathcal{G} = \{f_i^1, ...f_{l^2}^2\}$ is a minimal generating set for I and a Gröbner basis for I. Now construct $\{f_i^n\}$ with the AGS algorithm. We proceed by induction on n to show that $im(h_{i,k}^{n-1,n}) \in P^{n-1}\underline{r}$. For $n = 2$, it is obvious. So suppose $\{f_i^{n-1}\}$ is such that $im(h_{i,k}^{n-2,n-1}) \subset P^{n-2}\mathfrak{r}$. In other words, suppose $\bigoplus f_i^{n-1}A = P^{n-1}$ in a minimal projective resolution of $\bar{A}$. Now consider $\{f_i^n\}$. We claim for every i, $f_i^n \in \bigoplus f_i^{n-1}J \cap \bigoplus f_i^{n-2}I$. So, suppose not. Then there exists some i such that

$$f_i^n = \sum_{k=1}^{t} f_k^{n-1} r_k = \sum_{j=1}^{u} f_j^{n-2} s_j$$

where $r_k \in K\mathcal{Q}$ and $s_j \in I$ are all homogeneous elements. Moreover, we assume there exists at least one k such that $r_k \notin J$. Without loss of generality, suppose $r_1, ..., r_v \notin J$ and $r_{v+1}, ..., r_t \in J$. Then

$$\sum_{k=1}^{v} f_k^{n-1} r_k = \sum_{j=1}^{u} f_j^{n-2} s_j - \sum_{k=v+1}^{t} f_k^{n-1} r_k$$

which implies

$$\sum_{k=1}^{v} f_k^{n-1} r_k \in \bigoplus f_i^{n-2} I + \bigoplus f_k^{n'} J$$

which, by [15, theorem 2.4], implies $\{f_i^{n-1}\} \neq P^{n-1}$ in a minimal projective resolution of $\bar{A}$. That is a contradiction and the claim holds. Because $f_i^n \in \bigoplus f_i^{n-1} J \cap \bigoplus f_i^{n-2} I$, we can see that $im(h_{i,k}^{n-1,n}) \subset P^{n-1}\mathfrak{r}$. □

Example 4.38. Let

$$A = K[x_1, ..., x_n]/\langle\{x_j x_i - \lambda x_i x_j, x_i^{m_i}, x_n^{m_n} \mid 1 \leq i < j \leq n\}\rangle$$

for some $\lambda \in K^*$. Using theorem 2.3 of [8], we see that

$$\mathcal{G} = \{x_j x_i - \lambda x_i x_j, x_i^{m_i}, x_n^{m_n} \mid 1 \leq i < j \leq n\lambda \in K^*\}$$

is a reduced Gröbner basis of I. Thus $\mathcal{G}$ is a reduced Gröbner basis of the ideal I, consequently a minimal generating set for $\langle\{x_j x_i - \lambda x_i x_j, x_i^{m_i}, x_n^{m_n} \mid 1 \leq i < j \leq n\}\rangle$, which implies that the AGS resolution of $\bar{A}$ is minimal. By 4.25, we know that $E(A_{\text{MON}})$ is generated in degrees 0,1,2. By the above theorem, we see that $E(A)$ must also be generated in degrees 0,1,2.

We end this section by posing a more general question: Suppose $A = K\mathcal{Q}/I$ where I is admissible. If $E(A_{\text{MON}})$ is finitely generated, is $E(A)$ finitely generated? This question is still unsolved.

4.6 2-d-Determined Algebras

We assume throughout this section that A is a 2-d-determined algebra. These algebras were introduced by Green and Marcos in the paper "d-Koszul algebras, 2-d-determined algebras and 2-d-Koszul algebras." These algebras are a generalization of Koszul algebras and d-Koszul algebras. We restrict ourselves to the case where the AGS resolution of $\bar{A}$ over A is minimal.

Definition 4.39. Let $A = K\mathcal{Q}/I$ where I is a homogeneous ideal generated in degrees 2 and d for some $d \geq 3$. We define the function $\delta : \mathbb{N} \to \mathbb{N}$ by

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1 & \text{if } n \text{ is odd} \end{cases}$$

and let

$$\ldots \to P^2 \to P^1 \to P^0 \to \bar{A} \to 0$$

be a minimal graded projective resolution of $\bar{A}$. We say that A is *2-d-determined* if for each $n \geq 0$, P^n is generated by elements of degree not greater than $\delta(n)$.

Here is a simple lemma which describes the length of f_i^3 for each i.

Lemma 4.40. $l(f_i^3) \in \{3, d+1\}$

Proof. From the AGS algorithm (from [14]), we know that $\text{tip}(f_i^3) = \text{tip}(f_s^2)p$ for a unique index s and some path p in $\mathcal{Q}$. If $f_s^2 = a_1...a_m$ where $a_i \in \mathcal{Q}_1$ then we can also write $\text{tip}(f_i^3) = a_1(a_2...a_m p)$ where $a_2...a_m p \in O(a_2...a_m)$. Thus $a_2...a_m p = q\,\text{tip}(f_r^2)$ for an index r and, if $l(q) > 0$, $q = a_2...a_j$ for $2 \leq j < m$.

1. Case I: $l(f_s^2) = 2$ and $l(f_r^2) = 2$. Then $j < 2$ implies that $l(q) = 0$. Then $f_r^2 = a_2 p$, forcing $l(p) = 1$. Thus $l(f_i^3) = 3$.

2. Case II: $l(f_s^2) = 2$ and $l(f_r^2) = d$. Then $j < 2$ implies that $l(q) = 0$. Then $f_r^2 = a_2 p$. Thus $l(f_i^3) = d+1$.

3. Case III: $l(f_s^2) = d$. Then $m = d$. Because $l(p) \geq 1$, we know $l(f_i^3) \geq d+1$. However, because A is 2-d-determined, we know $l(f_i^3) \leq d+1$. Thus $l(f_i^3) = d+1$.

□

Recall we assumed the AGS resolution of $\bar{A}$ over A is minimal. So, we know that for every n,

$$\{l(f_i^n) \mid 1 \leq i \leq l^n\} = \{l(p) \mid p \in \Gamma_n\}$$

where Γ_n is the set of n-chains of A_{MON}. This is because $\Gamma_n = \{\text{tip}(f_i^n)\}$. Thus, if the AGS resolution of $\bar{A}$ over A is minimal and A is 2-d-determined, the minimal projective resolution of $\bar{A}_{\text{MON}}$ is of the form

$$\cdots \to P^2 \to P^1 \to P^0 \to \bar{A}_{\text{MON}} \to 0$$

where P^2 is generated in degrees 2 and d and P^3 are generated in degrees 3 and $d+1$. We use two results from [12].

Proposition 4.41. [12, Corollary 15] *Let $A = K\mathcal{Q}/I$ be a monomial algebra where I is generated by elements of length exactly 2 or d. Let*

$$\cdots \to P^2 \to P^1 \to P^0 \to \bar{A}_{MON} \to 0$$

be a minimal graded projective resolution of $\bar{A}$. The following are equivalent:

1. *A is 2-d-determined.*

2. *P^3 can be generated in degrees bounded above by $d+1$.*

which shows us that if A is a 2-d-determined algebra and the AGS resolution in minimal, then A_{MON} is a 2-d-determined algebra as well. Moreover, we may apply the following to A_{MON}:

Theorem 4.42. [12, Theorem 16] *If A_{MON} is 2-d-determined, then $E(A_{MON})$ is generated in degrees 0,1,2*

For the convenience of the reader, we have included the proof of the preceding lemma using the terminology of the AGS algorithm. We do so with a series of lemmas:

Lemma 4.43. *Let A be a 2-d-determined algebra and suppose $\mathrm{tip}(f_m^2) = a_1 ... a_d$ where $a_i \in \mathcal{Q}_1$. If $q = w\,\mathrm{tip}(f_k^2) \in O(a_i....a_d)$ for some index $2 \le i \le d$ where $l(f_k^2) = d$, then $l(w) = 0$.*

Proof. We may write $wf_k^2 = a_i....a_j...a_d...a_{d+j-1}$ where $\mathrm{tip}(f_k^2) = a_j...a_{d+j-1}$. If $a_2....a_{d+j-1} \in O(a_2...a_d)$, then $a_1....a_{d+j-1} = \mathrm{tip}(f_s^3)$ for some index s. Then $l(f_s^3) = d+j-1$. Because

$l(f_m^2) = d = l(f_k^2)$, we know $l(f_s^3) \geq d+1$. Because A is 2-d- determined, $l(f_s^3) = d+1$. Solving for j, we see $j = 2$. Thus $l(w) = 0$.

So suppose $a_2...a_{d+j-1} \notin O(a_2...a_d)$. Then there exists a proper subpath $p \in O(a_2...a_d)$ and an index s such that $\text{tip}(f_s^3) = a_1 p$. Because $l(f_s^3) > l(a_1a_2...a_d)$ and $l(f_s^3) \leq d+1$, we must have $l(f_s^3) = d+1$, forcing $p = a_2...a_{d+1}$. If $a_da_{d+1} = \text{tip}(f_r^2)$ for some r, then a_da_{d+1} is a subpath of $a_j...a_{d+j-1}$, which cannot happen because $\{f_i^2\}$ form a Gröbner basis of I.

So suppose $a_2...a_{d+1} = p\,\text{tip}(f_{r_2}^2) \in O(a_2...a_d)$ for some index r_2, and $l(f_{r_2}^2) \neq 2$. Then $l(f_{r_2}^2) = d$ which implies $a_2...a_{d+1} = \text{tip}(f_{r_2}^2)$.

We now repeat the argument with the path $a_2...a_{d+1}$ in place of $a_1...a_d$. Then there exists an r_3 such that one of two possibilities emerge: Either $i = j = 3$ and $a_3....a_{d+2} = \text{tip}(f_{r_3}^2)$, which implies $l(w) = 0$, or $4 \leq i \leq j$ and $a_3...a_{d+2} = \text{tip}(f_{r_3}^2)$. In the latter case, we now repeat the argument with $a_3....a_{d+3}$ in place of $a_2...a_{d+1}$. We continue with these iterations until we reach an index $t-1$ such that $i \leq t$. Then there exists an r_t such that one of the two possibilities occur: Either $i = j = t$ and $a_t....a_{d+t-1} = \text{tip}(f_{r_t}^2)$, which implies $l(w) = 0$, or $t+1 \leq i \leq j$ and $a_t...a_{d+t-1} = \text{tip}(f_{r_t}^2)$. Because we assume $i \leq t$, we must assume $i = j = t$ and $a_t....a_{d+t-1} = \text{tip}(f_{r_t}^2)$, which implies $l(w) = 0$. □

Lemma 4.44. *For all n and k, either $\text{tip}(f_k^n) = \text{tip}(f_m^{n-1})\,\text{tip}(f_i^1)$ for some unique $m \in \{1,...,l^{n-1}\}$ and $i \in \{1,...,l^1\}$, or $\text{tip}(f_k^n) = \text{tip}(f_m^{n-2})\,\text{tip}(f_i^2)$ for some unique $m \in \{1,...,l^{n-2}\}$ and $i \in \{1,...,l^2\}$.*

Proof. We may assume that $k = 1$. By the construction of $\{f_i^n\}$, we may write

$$\text{tip}(f_1^n) = \text{tip}(f_{k_1}^{n-1})p$$

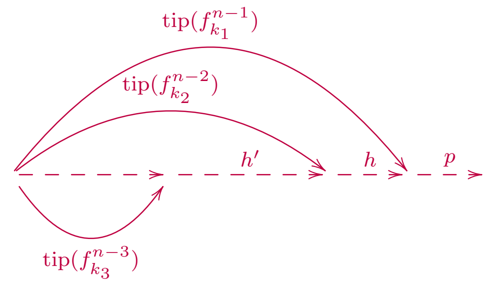
where p is a path such that $l(p) \geq 1$ and k_1 is some index. Similarly, we write

$$\operatorname{tip}(f^{n-1}_{k_1}) = \operatorname{tip}(f^{n-2}_{k_2})h$$

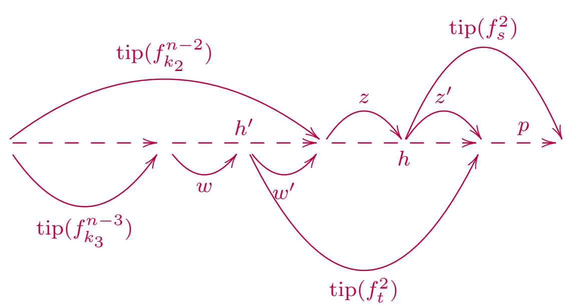
for the path $h = \operatorname{tip}(h^{n-1,n}_{k_2,k_1})$ and some index k_2. Notice $l(h) \geq 1$. We may also write

$$\operatorname{tip}(f^{n-2}_{k_2}) = \operatorname{tip}(f^{n-3}_{k_3})h'$$

where $h' = \operatorname{tip}(h^{n-2,n-1}_{k_3,k_2})$ and $l(h') \geq 1$ and k_3 is some index. We may visualize the path $\operatorname{tip}(f^n_1)$ with the following diagram:



Also by construction, we have $h'h \in O(h')$, which implies $h'h = w \operatorname{tip}(f^2_t)$ for some path w and some index t where $h' = ww'$ for some path w' such that $l(w') \geq 1$. Similarly, we know $hp \in O(h)$ implies $hp = z \operatorname{tip}(f^2_s)$ for some index s and $h = zz'$ for some path z' such that $l(z') \geq 1$. Visually, we can represent that in the following picture,



Because $l(f_t^2) \in \{2, d\}$, we consider the following two cases:

1. Case I: $l(f_t^2) = 2$.

   Because $w'h = \text{tip}(f_t^2)$ and $l(w') \geq 1$ and $l(h) \geq l(z') \geq 1$, we must have $l(w') = l(h) = l(z') = 1$, which implies $l(z) = 0$. Then $\text{tip}(f_m^n) = \text{tip}(f_{k_2}^{n-2})\,\text{tip}(f_s^2)$, proving the claim.

2. Case II: $l(f_t^2) = d$.

   If $l(f_s^2) = 2$, then, because $l(p) \geq 1$ and $l(z') \geq 1$, we must have $l(p) = 1, l(z') = 1$,. However, $l(p) = 1$ implies that $p = f_x^1$ for some index x. Thus $\text{tip}(f_m^n) = \text{tip}(f_{k_1}^{n-1})\,\text{tip}(f_x^1)$, proving our claim.

   If $l(f_s^2) = d$, then by the previous lemma, we have $l(z) = 0$. Consequently, $\text{tip}(f_m^n) = \text{tip}(f_{k_2}^{n-2})\,\text{tip}(f_s^2)$, proving the claim.

Because we know that $\{\text{tip}(f_i^n)\}$ are the n-chains in a minimal projective resolution of $\bar{A}_{\text{MON}}$, the proof of the theorem is complete. □

We may now apply our main result of this section:

Theorem 4.45. *Suppose A is a 2-d-determined algebra such that the AGS resolution of $\bar{A}$ over A is minimal. Then $E(A)$ is generated in degrees 0,1,2.*

Proof. If A is a 2-d-determined algebra such that the AGS resolution of $\bar{A}$ over A is minimal, then $E(A_{\text{MON}})$ is generated in degrees 0,1,2. Apply theorem 4.35 to see that $E(A)$ is generated in degrees 0,1,2. □

In section 5 of [12], the following question is posed: If A is 2-d-determined, and $E(A)$ is finitely generated, is it generated in degrees 0,1,2? The answer is negative, a counterexample was found in [7]. However, their counterexample is an algebra A which is not a finitely generated K-algebra. Thus the question could be rephrased as the following: If A is a finitely generated, 2-d-determined algebra, and $E(A)$ is finitely generated, is it generated in degrees 0,1,2? However, the answer is still no, see below. If the AGS resolution fails to be minimal, then $E(A)$ need not be generated in degrees 0,1,2. Consider the following example

Example 4.46. Let $A = K\mathcal{Q}/I$ where $\mathcal{Q}$ is given by the following quiver.

$$1 \underset{a}{\overset{b}{\leftrightarrows}} 2 \underset{c}{\overset{d}{\leftrightarrows}} 3$$

and $I = \langle ac, db, ba - cd \rangle$. A reduced Gröbner basis for I is given by

$$\mathcal{G} = \{ac, db, cd - ba, aba, bab\}$$

Notice $A_{\text{MON}} = K\mathcal{Q}/I_{\text{MON}}$ where $I_{\text{MON}} = \langle ac, db, cd, aba, bab \rangle$. Moreover, an easy computation shows that the 3-chains of A_{MON} are the following: $acd, dbab, cdb, abab, abac, baba$, which implies that in a minimal projective resolution of A_{MON}

$$\cdots P^n \to P^{n-1} \to \cdots P^3 \to P^2 \to P^1 \to P^0 \to \bar{A} \to 0$$

that P^3 is generated in degrees ≤ 4.

If we apply Corollary 15 of [12] to A_{MON}, we see that A_{MON} is 2-d-determined with $d = 3$. Applying theorem 16 of [12], we see that A_{MON} is 2-d-Koszul. Now we apply Proposition 17 of [12] to conclude A is 2-d-determined. However, according to [6], we know that $E(A)$

is generated in degrees 0,1, and 3. In other words, we have found a finitely generated 2-d-determined algebra A such that $E(A)$ is not generated in degrees 0,1,2.

Bibliography

[1] David J. Anick and Edward L. Green, *On the homology of quotients of path algebras*, Comm. Algebra **15** (1987), no. 1-2, 309–341, DOI 10.1080/00927878708823422.

[2] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[3] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[4] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527, DOI 10.1090/S0894-0347-96-00192-0.

[5] Klaus Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. (2) **28** (1983), no. 3, 461–469, DOI 10.1112/jlms/s2-28.3.461.

[6] Peter Brown, *The Ext-algebra of a representation-finite biserial algebra*, J. Algebra **221** (1999), no. 2, 611–629, DOI 10.1006/jabr.1999.8001.

[7] Thomas Cassidy and Christopher Phan, *Quotients of Koszul algebras and 2-D-determined algebras*, Comm. Algebra **42** (2014), no. 9, 3742–3752, DOI 10.1080/00927872.2013.793695.

[8] E L. Green, *Noncommutative Gröbner bases, and projective resolutions*, Computational methods for representations of groups and algebras (Essen, 1997), 1999, pp. 29–60.

[9] Edward L. Green, *Multiplicative bases, Gröbner bases, and right Gröbner bases*, J. Symbolic Comput. **29** (2000), no. 4-5, 601–623, DOI 10.1006/jsco.1999.0324. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

[10] Robert and Green Gordon Edward L., *Graded Artin algebras*, J. Algebra **76** (1982), no. 1, 111–137, DOI 10.1016/0021-8693(82)90240-X.

[11] Edward L. Green, Kirkman Ellen, and James Kuzmanovich, *Finitistic dimensions of finite-dimensional monomial algebras*, J. Algebra **136** (1991), no. 1, 37–50, DOI 10.1016/0021-8693(91)90062-D.

[12] Edward L. Green and E. N Marcos, *d-Koszul algebras, 2-d-determined algebras and 2-d-Koszul algebras*, J. Pure Appl. Algebra **215** (2011), no. 4, 439–449, DOI 10.1016/j.jpaa.2010.04.028.

[13] Edward L. Green and Roberto Martínez Villa, *Koszul and Yoneda algebras*, Representation theory of algebras (Cocoyoc, 1994), 1996, pp. 247–297.

[14] Edward L. Green and Øyvind Solberg, *An algorithmic approach to resolutions*, J. Symbolic Comput. **42** (2007), no. 11-12, 1012–1033, DOI 10.1016/j.jsc.2007.05.002.

[15] E. L. Green, Ø Solberg, and D. Zacharia, *Minimal projective resolutions*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2915–2939 (electronic), DOI 10.1090/S0002-9947-01-02687-3.

[16] E. L. Green and D. Zacharia, *The cohomology ring of a monomial algebra*, Manuscripta Math. **85** (1994), no. 1, 11–23, DOI 10.1007/BF02568180.

[17] Clas Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Lecture Notes in Math., vol. 1183, Springer, Berlin, 1986.

[18] Roberto Martínez-Villa, *Introduction to Koszul algebras*, Rev. Un. Mat. Argentina **48** (2007), no. 2, 67–95 (2008).

[19] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005.

[20] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[21] S. Paul Smith, *Some finite-dimensional algebras related to elliptic curves*, Representation theory of algebras and related topics (Mexico City, 1994), 1996, pp. 315–348.

Melissa M. DiMarco Curriculum Vitae

4.7 Contact Information

Mathematics Department
215 Carnegie Building
Syracuse University Syracuse, NY 13244-1150

Phone: (315) 443-3312
Email: mmdimarc@syr.edu

4.8 Education

- Ph.D. in Mathematics, Syracuse University, August 2016
  - Title of dissertation: *Finite Generation of Ext-Algebras of Finite Dimensional Algebras and Associated Monomial Algebras.*
  - Advisor: Dan Zacharia
- M.S. in Mathematics, Syracuse University, May 2011
- B.A. in English & Mathematics, Binghamton University, May 2009

4.9 Teaching

Primary Instructor of Section (Syracuse University)

- MAT 221: Elementary Probability and Statistics I (7 sections)
- MAT 295: Calculus I (4 semesters)
- MAT 296: Calculus II (2 semesters)
- MAT 397: Calculus III: Multivariable Calculus (2 semesters)

Responsible for all classroom instruction, office hours, review sessions, writing syllabi, writing and grading exams, quizzes, and homework assignments.

Teaching Assistant (Syracuse University)

- MAT 285: Business Calculus
- MAT 295: Calculus I
- MAT 296: Calculus II
- MAT 397: Calculus III

Tutor

- Math clinic (Syracuse University, 2010–2011), tutored for courses on probability and statistics, calculus, linear algebra, differential equations.

- Privately tutored at levels from high school Algebra and Geometry (including Common Core) to proof-based undergraduate courses to graduate level algebra.

Undergraduate Course Assistant (Watson School of Engineering, Binghamton University)

- WTSN 103: Technical Communications I (Aug-Dec 2007)

- WTSN 104: Technical Communications II (Jan-May 2008)

Responsible for grading assignments and giving feedback on writing samples.

Writing Center Tutor (Binghamton University)

- Tutor (Aug 2006-May 2007)

Teach effective writing techniques, Work with ESL students.

4.10 Research Interests

Representation Theory, Finite Dimensional K-Algebras, Homological Algebra.

4.11 Awards

Kibbey Prize, Syracuse University Mathematics Department (April 2016)

4.12 Presentations

- *The Art Gallery Theorem*, Mathematics Job Candidate Presentation, St. Bonaventure University. (February 2016)

- *Maximal Green Sequences*, AMS Mathematical Research Communities Workshop, Snowbird, Utah. (June 2014)

- *The Finitistic Dimension Conjecture*, 39^{th} Annual New York Regional Graduate Mathematics Conference, Syracuse University. (April 2014)

- *How Can Two Rings be Equivalent?*, Algebra Seminar, Syracuse University. (February 2014)

- *Much Ado about Idempotents*, Graduate Algebra Seminar, Syracuse University. (November 2014)

- *Stable Representation Quivers*, 38^{th} Annual New York Regional Graduate Mathematics Conference, Syracuse University. (April 2013)

- *The Structure of Stable Representation Quivers*, Algebra Seminar, Syracuse University. (April 2013)

4.13 Service

- Graduate Representative, The Undergraduate Committee, Mathematics Department, Syracuse University (April 2013-April 2014)
- Member, Mathematics Graduate Organization
  - Secretary (April 2011–April 2014)
  - Co-organizer of 39^{th} Annual New York Regional Graduate Mathematics Conference, Syracuse University (April 2014)
  - Co-organizer of 38^{th} Annual New York Regional Graduate Mathematics Conference, Syracuse University (April 2013)

4.14 Professional Organization

American Mathematical Society
Association for Women in Mathematics

4.15 Invited Workshops

- BIRS Workshop 16w5023, Women in Noncommutative Algebra and Representation Theory (WINART) (March 2016)
- AMS Mathematical Research Communities: Cluster Algebras (June 2014)

4.16 References

- Dan Zacharia
  Professor of Mathematics
  Syracuse University
  zacharia@syr.edu
  315-443-1580
- Mark Kleiner
  Professor of Mathematics
  Syracuse University
  mkleiner@syr.edu
  315-443-1497
- William Wylie
  Assistant Professor of Mathematics
  Syracuse University
  wwylie@syr.edu
  315-443-1469