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Abstract

This paper considers the problem of hypotheses testing in a simple panel data regression model with random individual effects and serially correlated disturbances. Following Baltagi, Kao and Liu (2008), we allow for the possibility of non-stationarity in the regressor and/or the disturbance term. While Baltagi et al. (2008) focus on the asymptotic properties and distributions of the standard panel data estimators, this paper focuses on test of hypotheses in this setting. One important finding is that unlike the time series case, one does not necessarily need to rely on the "super-efficient" type AR estimator by Perron and Yabu (2009) to make inference in panel data. In fact, we show that the simple t-ratio always converges to the standard normal distribution regardless of whether the disturbances and/or the regressor are stationary.

Key Words: Panel Data, OLS, Fixed-Effects, First-Difference, GLS, t-ratio.

JEL Classification: C12; C33.

TEST OF HYPOTHESES IN PANEL DATA MODELS WHEN THE REGRESSOR AND DISTURBANCES ARE POSSIBLY NONSTATIONARY

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Abstract

This paper considers the problem of hypotheses testing in a simple panel data regression model with random individual effects and serially correlated disturbances. Following Baltagi, Kao and Liu (2008), we allow for the possibility of non-stationarity in the regressor and/or the disturbance term. While Baltagi *et al.* (2008) focus on the asymptotic properties and distributions of the standard panel data estimators, this paper focuses on test of hypotheses in this setting. One important finding, is that unlike the time series case, one does not necessarily need to rely on the "super-efficient" type AR estimator by Perron and Yabu (2009) to make inference in panel data. In fact, we show that the simple t-ratio always converges to the standard normal distribution regardless of whether the disturbances and/or the regressor are stationary.

1 Introduction

In the time series literature, estimation and test of hypotheses of the deterministic time trend model with serially correlated disturbances have been studied by Canjels and Watson (1997), Vogelsang (1998) and Perron and Yabu (2009) to mention a few. For the panel data model, Baltagi and Krämer (1997) and Kao and Emerson (2004, 2005) study the corresponding time trend model with unobservable individual effects and autoregressive remainder disturbances. Baltagi, Kao and Liu (2008) extend this analysis to the case of a panel data regression model with possible non-stationarity in the regressor and/or the disturbance term. They derive the asymptotic distributions of the standard panel data estimators including ordinary least squares (OLS), fixed effects (FE), first-difference (FD), and generalized least squares (GLS) estimators

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when both the time-series length (T) and the number of cross-sections (n) are large. They show that these estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressor and the remainder disturbances. Some of their important findings include the following: (i) When the disturbance term is $I(0)$ and the regressor is $I(1)$, the FE estimator is asymptotically equivalent to the GLS estimator and OLS is less efficient than GLS; (ii) When the disturbance term and the regressor are $I(1)$, GLS is more efficient than the FE estimator since GLS is $\sqrt{nT}$ consistent, while FE is $\sqrt{n}$ consistent. As a result, they recommend the GLS estimator as the preferred estimator, and they show using Monte Carlo experiments that the loss in efficiency of the OLS, FE, and FD estimators relative to true GLS can be substantial. This paper is a follow up paper which is concerned with *test of hypotheses* using these standard panel data estimators. One important finding, is that unlike the time series setting, one does not necessarily need to rely on the "*super-efficient*" type AR estimator by Perron and Yabu (2009) to make inference in panel data. In fact, we show that the simple t-ratio based on the FGLS estimator of Baltagi and Li (1991), will always converge to the standard normal distribution regardless of whether the disturbances and/or the regressor are stationary or not. We also show using Monte Carlo experiments that inference based on the OLS, FE, and FD estimators could be misleading relative to that based on feasible GLS. The outline of the paper is as follows: Section 2 considers a simple panel data regression model with unobserved individual effects and AR(1) remainder disturbances and derives the asymptotic distributions of the t statistics of the standard FE and FD estimators, respectively. This is done for four cases, corresponding to whether the remainder disturbances and/or the regressor are stationary or not. In Section 3, we derive the corresponding asymptotic distributions of the t statistic for the FGLS estimator under these four cases. Section 4 reports the finite sample properties of the proposed tests using Monte Carlo experiments. Section 5 concludes. All proofs are given in the supplemental appendix which are available in Baltagi, Kao and Na (2010).

Unless otherwise specified, for all the asymptotic results in this paper, we let n and T go to infinity simultaneously (i.e., $(n,T) \to \infty$), see Phillips and Moon (1999). We require $\frac{n}{T} \to 0$ in some cases. We write the integral $\int_0^1 W(s)ds$ as $\int W$ and $\bar{W}$ as $W - \int W$ when there is no ambiguity over limits. We use $\xrightarrow{p}$ to denote convergence in probability, $\xrightarrow{d}$ to denote convergence in distribution, $\otimes$ to denote Kronecker product, and $[x]$ to denote the largest integer $\leq x$.

2 The Model and Assumptions

Consider the following panel data regression model:

$$y_{it} = \gamma + \beta x_{it} + u_{it}, \quad i = 1, \dots, n, \quad t = 1, \dots, T \tag{1}$$

where $u_{it} = \mu_i + \nu_{it}$, and γ and β are scalars.[1] We assume that the individual effect μ_i is random with $\mu_i \sim iid(0, \sigma_\mu^2)$ and $\{\nu_{it}\}$ is an $AR(1)$

$$\nu_{it} = \rho\nu_{it-1} + e_{it}, \quad |\rho| \leq 1 \tag{2}$$

where e_{it} is a white noise process with variance σ_e^2. The μ_i is independent of the ν_{it} for all i and t.[2] Let $\{x_{it}\}$ be also an AR(1) such that

$$x_{it} = \lambda x_{it-1} + \varepsilon_{it}, \quad |\lambda| \leq 1 \tag{3}$$

where ε_{it} is a white noise process with variance σ_ε^2. In this paper we assume that

$$E(\mu_i \mid x_{it}) = 0. \tag{4}$$

The initialization of this system is $y_{i1} = x_{i1} = O_p(1)$ for all i. Baltagi *et al.* (2008) derive the asymptotic distributions of the standard panel data estimators including OLS, FE, FD, and GLS estimators of β when both T and n are large. They find that, when ν_{it} is $I(0)$ (i.e., $\rho < 1$), the FE[3] and the GLS estimators are both $\sqrt{n}T$ consistent and (asymptotically) equivalent. However, this asymptotic equivalence breaks down when ν_{it} is $I(1)$ (i.e., $\rho = 1$). In this case, the GLS and the FD[4] estimators are both $\sqrt{nT}$ consistent and more efficient than the FE estimator (which is $\sqrt{n}$ consistent).

Define the innovation vector $\mathbf{w}_{it} = (e_{it}, \varepsilon_{it})'$. We assume that $\mathbf{w}_{it}$ is a linear process that satisfies the following assumptions:

Assumption 1 *For each i, we assume:*

1. $\mathbf{w}_{it} = \Pi(L)\boldsymbol{\eta}_{it} = \sum_{j=0}^{\infty} \Pi_j \boldsymbol{\eta}_{it-j}$, $\sum_{j=0}^{\infty} j^a \|\Pi_j\| < \infty$, $|\Pi(1)| \neq 0$ for some $a > 1$.
2. For a given i, $\boldsymbol{\eta}_{it}$ is i.i.d. with zero mean and variance-covariance matrix Ξ, and finite fourth order cumulants.[5]

Assumption 2 *We assume $\boldsymbol{\eta}_{it}$ and $\boldsymbol{\eta}_{jt}$ are independent for $i \neq j$. That is, we assume cross-sectional independence.*

[1] For simplicity, we consider the case of one regressor, but our results can be extended to the multiple regressors case. In fact, we assume that for the multiple regressors case, $X'X$ is of full rank to avoid the complexity from possible cointegration.

[2] This model was studied by Baltagi and Li (1991) under stationarity of the regressors and the disturbances.

[3] The fixed effects estimator of β is given by,

$$\hat{\beta}_{FE} = \frac{\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - \bar{x}_{i.})(y_{it} - \bar{y}_{i.})}{\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - \bar{x}_{i.})^2}$$

where $\bar{x}_{i.} = \frac{1}{T}\sum_{t=1}^{T} x_{it}$ and $\bar{y}_{i.} = \frac{1}{T}\sum_{t=1}^{T} y_{it}$, see Hsiao (2003).

[4] The FD estimator is the OLS estimator of a first-differenced regression, see Hsiao (2003). That is,

$$\Delta y_{it} = \beta\Delta x_{it} + \Delta\nu_{it}.$$

[5] This paper does not allow the heteroskedasticity across the cross-sectional units. In a recent paper, Bresson, Hsiao and Pirotte (2011) propose a random coefficient approach to model the heteroskedasticity.

Assumption 3 *We also assume* $E(e_{it}\varepsilon_{i(t+k)}) = 0$ *for all* i *and* k *and* ε_{it} *and* e_{it} *are independent.*

Assumption 1 implies that the partial sum process $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\mathbf{w}_{it}$ satisfies the following properties:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\mathbf{w}_{it} \xrightarrow{d} \mathbf{B}_i(r) = \mathbf{BM}_i(\Omega) \text{ as } T \to \infty \text{ for all } i \tag{5}$$

where

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\epsilon i} \end{bmatrix}.$$

The long-run variance covariance matrix of $\{\mathbf{w}_{it}\}$ with Assumption 3 is given by

$$\begin{aligned} \Omega &= \sum_{j=-\infty}^{\infty} E(\mathbf{w}_{ij}\mathbf{w}_{i0}^{'}) \\ &= \Pi(1)\Xi\Pi(1)^{'} \\ &= \begin{bmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix}. \end{aligned}$$

Then $\mathbf{B}_i$ can be rewritten as

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\epsilon i} \end{bmatrix} = \begin{bmatrix} \sigma_e & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix} \begin{bmatrix} V_i \\ W_i \end{bmatrix} \tag{6}$$

where $\begin{bmatrix} V_i \\ W_i \end{bmatrix}$ is a standard Brownian motion.

2.1 The Fixed Effects and the First Difference Estimators

In this paper, we focus on testing the common slope β,

$$H_0 : \beta = \beta_0.$$

We start by investigating the asymptotic distributions of the t-statistics for H_0 based on the FE and FD estimators. Let us denote these by t_{FE} and t_{FD}, respectively. We derive these asymptotic distributions under four scenarios where the disturbances and the regressor are allowed to be $I(0)$ or $I(1)$.

If v_{it} is known to be $I(0)$,[6] the corresponding t-test for H_0 using the FE estimator $\widehat{\beta}_{FE}$, is given by:

$$t_{FE} = \frac{\widehat{\beta}_{FE} - \beta_0}{s_{FE}} \tag{7}$$

where $s_{FE} = \sqrt{\frac{\widehat{\sigma}_\nu^2}{\sum_{i=1}^n\sum_{t=1}^T(x_{it}-\bar{x}_{i.})^2}}$ with $\widehat{\sigma}_\nu^2 = \frac{\widehat{\sigma}_e^2}{1-\widehat{\rho}^2}$ and $\widehat{\sigma}_e^2 = \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\left(\widehat{\nu}_{it} - \widehat{\rho\nu}_{it-1}\right)^2$. Here $\widehat{\nu}_{it} = (y_{it} - \overline{y}_{i.}) - \widehat{\beta}_{FE}(x_{it} - \overline{x}_{i.})$ denotes the FE residuals from equation (1), and $\widehat{\rho} = \frac{\sum_{i=1}^n\sum_{t=2}^T\hat{\nu}_{it}\hat{\nu}_{it-1}}{\sum_{i=1}^n\sum_{t=2}^T\hat{\nu}_{it-1}^2}$ is the estimator of ρ suggested by Baltagi and Li (1991). Next, we derive the limiting distribution of $\widehat{\rho}$ when $|\rho| < 1$ as well as when $\rho = 1$.

[6]Note that the FE and the GLS estimators are asymptotically equivalent for this case, see Baltagi *et al.* (2008).

Lemma 1 *Assume* $(n,T) \to \infty$.

1. *If* $|\rho| < 1$ *with* $\frac{n}{T} \to 0$,
$$\sqrt{nT}\left(\widehat{\rho} - \rho\right) \xrightarrow{d} N\left(0, 1-\rho^2\right)$$
and
$$\widehat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2.$$

2. *If* $\rho = 1$,
$$T\left(\widehat{\rho} - 1\right) \xrightarrow{p} -3$$
and
$$\widehat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2.$$

Theorem 1 derives the corresponding asymptotic distribution of the t statistic based on the FE estimator (t_{FE}) under various scenarios involving the stationarity or non-stationarity of the regressor and the disturbances.

Theorem 1 *Assume* $(n,T) \to \infty$ *with* $\frac{n}{T} \to 0$

1. *If* $|\rho| < 1$, $|\lambda| < 1$,
$$t_{FE} \xrightarrow{d} N\left(0, \frac{1+\rho\lambda}{1-\rho\lambda}\right).$$

2. *If* $\rho = 1$, $|\lambda| < 1$,
$$t_{FE} \xrightarrow{d} N\left(0, \frac{1-\lambda}{1+\lambda}\right).$$

3. *If* $|\rho| < 1$, $\lambda = 1$,
$$t_{FE} \xrightarrow{d} N\left(0, \frac{1+\rho}{1-\rho}\right).$$

4. *If* $\rho = 1$, $\lambda = 1$,
$$\frac{t_{FE}}{\sqrt{T}} \xrightarrow{d} N\left(0, \frac{2}{5}\right).$$

The results of Theorem 1 show that, under the null, t_{FE} has a normal distribution if the disturbance term is $I(0)$ regardless of the stationarity or non-stationarity of the regressor. $\frac{t_{FE}}{\sqrt{T}} = O_p(1)$ in part 4 above has been pointed out by Kao (1999).

Next, we turn to the case of the FD estimator, $\widehat{\beta}_{FD}$.[7] The corresponding t-test for H_0 using the FD estimator $\widehat{\beta}_{FD}$, is given by:

$$t_{FD} = \frac{\widehat{\beta}_{FD} - \beta_0}{s_{FD}} \tag{8}$$

[7] Note that if v_{it} is known to be $I(1)$, the FD and the GLS estimators are asymptotically equivalent, see Baltagi *et al.* (2008).

where

$$s_{FD} = \sqrt{\frac{\widehat{\sigma}^2_{\triangle\nu}}{\sum\limits_{i=1}^{n}\sum\limits_{t=2}^{T}(x_{it} - x_{it-1})^2}}$$

with $\widehat{\sigma}^2_{\triangle\nu} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\Delta y_{it} - \widehat{\beta}_{FD} \triangle x_{it}\right)^2$.

Theorem 2 *Assume* $(n, T) \to \infty$ *and* $\frac{n}{T} \to 0$.

1. *If* $|\rho| < 1,\ |\lambda| < 1$,

$$t_{FD} \xrightarrow{d} N\left(0, \frac{(1+\rho)(1+\lambda)[(2-\rho-\lambda)^2 + \frac{(1-\rho)^3}{1+\rho} + \frac{(1-\lambda)^3}{1+\lambda}]}{4(1-\rho\lambda)^2}\right).$$

2. *If* $\rho = 1,\ |\lambda| < 1$,

$$t_{FD} \xrightarrow{d} N(0,1).$$

3. *If* $|\rho| < 1,\ \lambda = 1$,

$$t_{FD} \xrightarrow{d} N(0,1).$$

4. *If* $\rho = 1,\ \lambda = 1$,

$$t_{FD} \xrightarrow{d} N(0,1).$$

The results of Theorem 2 show that, under the null, t_{FD} has a normal distribution regardless of the stationarity or non-stationarity of the regressor and/or the disturbance term.

3 The Feasible GLS Estimator

We rewrite equation (1) in vector form

$$\mathbf{y} = \gamma\boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{u} = \gamma\boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{Z}_{\boldsymbol{\mu}}\boldsymbol{\mu} + \boldsymbol{\nu} \tag{9}$$

where $\mathbf{y}$ is $nT \times 1$, $\mathbf{x}$ is a vector of x_{it} of dimension $nT \times 1$, $\boldsymbol{\iota}_{nT}$ is a vector of ones of dimension nT, $\mathbf{u}$ is $nT \times 1$, $\boldsymbol{\mu}$ is a vector of μ_i, $\boldsymbol{\nu}$ is a vector of ν_{it} and $\mathbf{Z}_\mu = I_n \otimes \boldsymbol{\iota}_T$.

By the partitioned inverse rule, it can be shown, see Baltagi *et al.* (2008), that

$$\begin{aligned}\widehat{\beta}_{GLS} &= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}\right]^{-1} \\ &\quad\times\left[\mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{y}\right]\end{aligned} \tag{10}$$

Substituting (9), one gets:

$$\begin{aligned}\widehat{\beta}_{GLS} - \beta &= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}\right]^{-1} \\ &\quad\times\left[\mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}\right] = G_1^{-1}G_2\end{aligned} \tag{11}$$

where G_1 and G_2 are defined accordingly, see also the Appendix. The variance-covariance matrix is given by:

$$\Phi = E\left(\mathbf{u}\mathbf{u}'\right) = \sigma_\mu^2 \left(I_n \otimes \boldsymbol{\iota}_T \boldsymbol{\iota}_T'\right) + \sigma_e^2 \left(I_n \otimes \mathbf{A}\right) \tag{12}$$

where $\boldsymbol{\iota}_T$ is a vector of ones of dimension T. $\mathbf{A}$ is the variance-covariance matrix of ν_{it}, which for the $AR(1)$ is given by:

$$\mathbf{A} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix} \tag{13}$$

when $|\rho| < 1$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$$

when $\rho = 1$. Thus, it can be shown, see Baltagi *et al.* (2008), that

$$\Phi^{-1} = I_n \otimes \left[\frac{1}{\sigma_e^2}\left(\mathbf{A}^{-1} - \frac{\sigma_\mu^2}{\sigma_e^2 + \theta\sigma_\mu^2}\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\right)\right] \tag{14}$$

where $\theta = \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T$. When $|\rho| < 1$, this estimation is equivalent to the Prais-Winsten transformation method suggested by Baltagi and Li (1991) for the panel data model. One can easily verify that $\mathbf{A}^{-1} = \mathbf{C}'\mathbf{C}$, where

$$\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix} \tag{15}$$

is the well known Prais-Winsten transformation for the $AR(1)$ model. Baltagi and Li (1991) suggest premultiplying the panel model (9) by $(I_n \otimes \mathbf{C})$ to get rid of serial correlation in the remainder term, and then performing a Fuller and Battese (1973) transformation in the second step to take care of the random effects.

In order to obtain the FGLS estimator, $\widehat{\beta}_{FGLS}$, we use an estimate of ρ suggested by Baltagi and Li (1991) based on FE residuals given below equation (7). The asymptotic distribution of $\widehat{\rho}$ was derived in

Lemma 1. Define $\hat{\alpha} = \sqrt{(1+\widehat{\rho})/(1-\widehat{\rho})}$ and $\boldsymbol{\hat{\iota}}_T^{\alpha\prime} = \left(\hat{\alpha}, \boldsymbol{\iota}_{T-1}'\right)$, where $\boldsymbol{\iota}_{T-1}$ is a vector of ones of dimension $T-1$. Using a trick by Wansbeek and Kapteyn (1983), define $\widehat{\overline{J}}_T^{\alpha} = \boldsymbol{\hat{\iota}}_T^{\alpha}\boldsymbol{\hat{\iota}}_T^{\alpha\prime}/\hat{d}^2$, where $\hat{d}^2 = \boldsymbol{\hat{\iota}}_T^{\alpha\prime}\boldsymbol{\hat{\iota}}_T^{\alpha} = \frac{2\widehat{\rho}}{1-\widehat{\rho}} + T$. Then, $\hat{E}_T^{\alpha} = I_T - \widehat{\overline{J}}_T^{\alpha}$. Also let $\sigma_\alpha^2 = \theta\sigma_\mu^2 + \sigma_e^2$ where $\theta = (1-\rho)^2 d^2$. Estimates for σ_e^2 and σ_μ^2 can be obtained from

$$\hat{\sigma}_e^2 = \frac{1}{n(T-1)}\mathbf{\hat{u}}^{*\prime}\left(I_n \otimes \hat{E}_T^{\alpha}\right)\mathbf{\hat{u}}^*$$

and

$$\hat{\sigma}_\alpha^2 = \frac{1}{n}\mathbf{\hat{u}}^{*\prime}\left(I_n \otimes \widehat{\overline{J}}_T^{\alpha}\right)\mathbf{\hat{u}}^*$$

where $\mathbf{\hat{u}}^*$ are the Prais-Winsten transformed residuals (see Baltagi and Li (1991) for more details). Hence, $\hat{\sigma}_\mu^2$ can be estimated as

$$\hat{\sigma}_\mu^2 = \frac{1}{\hat{\theta}}\left(\hat{\sigma}_\alpha^2 - \hat{\sigma}_e^2\right).$$

Substituting $\hat{\sigma}_e^2$, $\hat{\sigma}_\mu^2$, and $\widehat{\rho}$ into equation (14), one obtains $\widehat{\beta}_{FGLS}$. The corresponding t-test for H_0 using the FGLS estimator $\widehat{\beta}_{FGLS}$, is given by:

$$t_{FGLS} = \frac{\hat{\beta}_{FGLS} - \beta}{\sqrt{var(\hat{\beta}_{FGLS})}} = \frac{\widehat{G}_1^{-1}\widehat{G}_2}{\sqrt{\widehat{G}_1^{-1}}} = \widehat{G}_1^{-1/2}\widehat{G}_2 \tag{16}$$

where $\widehat{G}_1$ and $\widehat{G}_2$ are given as equation (11) with the replacement of Φ by $\hat{\Phi}$.

3.1 Case 1: Without Individual Effects

We begin with a simple case where $\mu_i = 0$. That is, the individual effects are not included in the true model, but there is first order serial correlation. This is not realistic in panel data economic models, but we study it as a base case. The variance-covariance matrix given in (12) reduces to

$$\Phi = E(\mathbf{uu}') = \sigma_e^2(I_n \otimes \mathbf{A}) \tag{17}$$

with

$$\Phi^{-1} = \frac{1}{\sigma_e^2}(I_n \otimes \mathbf{A}^{-1}).$$

In this case, the FGLS estimator, $\widehat{\beta}_{FGLS}$, will be based on $\widetilde{\rho}$ and $\widetilde{\sigma}_e^2$ given by,

$$\widetilde{\rho} = \frac{\sum_{i=1}^n \sum_{t=2}^T \widehat{u}_{it}\widehat{u}_{it-1}}{\sum_{i=1}^n \sum_{t=2}^T \widehat{u}_{it-1}^2} \tag{18}$$

and

$$\widetilde{\sigma}_e^2 = \frac{1}{n(T-1)}\mathbf{\widehat{u}}^{*\prime}\mathbf{\widehat{u}}^*$$

where $\widehat{u}_{it}$ denotes the OLS residual.[8]

[8] Note that we use the OLS residuals instead of the FE residuals in this case. That is, $\widehat{u}_{it} = y_{it} - \widehat{\gamma}_{OLS} - \hat{\beta}_{OLS}x_{it} = (y_{it} - \overline{\overline{y}}) - \hat{\beta}_{OLS}(x_{it} - \overline{\overline{x}})$ with $\overline{\overline{y}} = \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T y_{it}$ and $\overline{\overline{x}} = \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T x_{it}$.

Lemma 2 *Assume $(n,T) \to \infty$ and $\frac{n}{T} \to 0$.*

1. *If $|\rho| < 1$, $|\lambda| < 1$,*
$$\sqrt{nT}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{(1+\rho\lambda)(1-\lambda^2)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right).$$

2. *If $\rho = 1$, $|\lambda| < 1$,*
$$\sqrt{n}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{(1+\lambda)^2\sigma_e^2}{2\sigma_\varepsilon^2}\right).$$

3. *If $|\rho| < 1$, $\lambda = 1$,*
$$\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{2\sigma_e^2}{(1-\rho)^2\sigma_\varepsilon^2}\right).$$

4. *If $\rho = 1$, $\lambda = 1$,*
$$\sqrt{n}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right).$$

As shown above, we have the same rate of converging speed as that assuming individual effects except for case (3). That is, in the panel cointegration case, we have the convergence rate $\sqrt{n}T$ which is the same as that of the GLS estimator and the FE estimator. However, note that once we add the individual effects, the OLS estimator has the slower convergence rate $\sqrt{nT}$ rather than $\sqrt{n}T$ because μ_i dominates v_i.[9]

Lemma 3 *Assume $(n,T) \to \infty$.*

1. *If $|\rho| < 1$ with $\frac{n}{T} \to 0$,*
$$\sqrt{nT}\left(\widetilde{\rho} - \rho\right) \xrightarrow{d} N\left(0, 1-\rho^2\right)$$
and
$$\widetilde{\sigma}_e^2 \xrightarrow{p} \sigma_e^2.$$

2. *If $\rho = 1$,*
$$T\left(\widetilde{\rho} - 1\right) \xrightarrow{p} 0$$
and
$$\widetilde{\sigma}_e^2 \xrightarrow{p} \sigma_e^2.$$

As can be seen in Lemma 3, we find that the limiting distribution of $\widetilde{\rho}$ is the same as that of $\widehat{\rho}$ using the FE residuals, when $|\rho| < 1$ with $\frac{n}{T} \to 0$. However, this limiting distribution is different when $\rho = 1$. Compare, $T\left(\widetilde{\rho} - 1\right) \xrightarrow{p} 0$ without individual effects with $T(\widehat{\rho} - 1) \xrightarrow{p} -3$ with individual effects. We also find that, in both cases, the consistency of $\widetilde{\sigma}_e^2$ can be achieved. Based on the above results, one can derive the asymptotic distribution of the t-ratio for each case.

[9] The limiting distribution of the OLS estimator with individual effects is given by
$$\sqrt{nT}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{4\sigma_\mu^2}{3\sigma_\varepsilon^2}\right)$$
e.g., see Baltagi *et al.* (2008) for details.

Theorem 3 *Assume* $(n,T) \to \infty$ *and* $\frac{n}{T} \to 0$. *Without individual effects,* $\widetilde{\rho}$ *always leads to* $t_{FGLS} \xrightarrow{d} N(0,1)$ *regardless of the stationarity or non-stationarity of the regressor and/or the disturbance term.*

Theorem 3 shows that t_{FGLS} always converges to the standard normal case whether the disturbance term is $I(0)$ or $I(1)$ and whether the regressor is $I(0)$ or $I(1)$. That is, without individual effects, the t-ratio based on the FGLS, can be used for inference using the standard normal distribution. Hence, in this case, one does not have to consider the "*super-efficient*" type estimator by Perron and Yabu (2009) which is designed to bridge the gap between $I(0)$ and $I(1)$.[10]

3.2 Case 2: With Individual Effects

This section derives the asymptotic distribution of t_{FGLS} given in (16) and discussed in Section 3.

Theorem 4 *Assume* $(n,T) \to \infty$ *and* $\frac{n}{T} \to 0$.

1. *If* $|\rho| < 1$, $|\lambda| < 1$

   (a)
   $$\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2,$$

   (b)
   $$\hat{\sigma}_\mu^2 \xrightarrow{p} \sigma_\mu^2,$$

   (c)
   $$t_{FGLS} \xrightarrow{d} N(0,1).$$

2. *If* $\rho = 1$, $|\lambda| < 1$

   (a)
   $$\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2,$$

   (b)
   $$(1-\hat{\rho})\,\hat{\sigma}_\mu^2 \xrightarrow{p} \frac{16}{25}\sigma_e^2,$$

   (c)
   $$t_{FGLS} \xrightarrow{d} N(0,1).$$

[10] One can define the "super-efficient" estimator $\widehat{\rho}_s$ as

$$\widehat{\rho}_s = \begin{cases} \widehat{\rho} & \text{if } |\widehat{\rho}-1| > \frac{\varepsilon}{T^\delta} \\ 1 & \text{if } |\widehat{\rho}-1| \leq \frac{\varepsilon}{T^\delta} \end{cases}$$

for some $\delta \in (0,1)$ and $\varepsilon > 0$. Hence, when $\widehat{\rho}$ is in a $T^{-\delta}$ neighborhood of 1, it is assigned a value of 1. For details, see Perron and Yabu (2009).

3. *If* $|\rho| < 1,\ \lambda = 1$

   (a)
$$\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2,$$

   (b)
$$\hat{\sigma}_\mu^2 \xrightarrow{p} \sigma_\mu^2,$$

   (c)
$$t_{FGLS} \xrightarrow{d} N(0,1).$$

4. *If* $\rho = 1,\ \lambda = 1$

   (a)
$$\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2,$$

   (b)
$$(1-\hat{\rho})\,\hat{\sigma}_\mu^2 \xrightarrow{p} \frac{16}{25}\sigma_e^2,$$

   (c)
$$t_{FGLS} \xrightarrow{d} N(0,1).$$

Theorem 4 implies that the t-ratio based on $\widehat{\rho}$ by Baltagi and Li (1991) asymptotically leads to the standard normal distribution regardless of the stationarity or non-stationarity of the regressor and/or the disturbance term. This is an interesting finding because despite the fact that we do not have a consistent estimate of σ_μ^2 when $\rho = 1$, we can still obtain t_{FGLS} converging to $N(0,1)$. Accordingly, we have a similar result to that of Theorem 3 except that one cannot expect consistent estimates for all the variance components when $\rho = 1$.

4 Monte Carlo Results

This section runs Monte Carlo experiments in order to study the finite sample properties of the t-statistics for $H_0 : \beta = \beta_0$; based on OLS, FE, FD, GLS, FGLS using Cochrane-Orcutt (GLS-CO), and FGLS using Prais-Winsten (GLS-PW) estimators. We denote these t-statistics by tOLS, tFE, tFD, tGLS, tGLSCO, and tGLSPW, respectively. The model is generated by

$$y_{it} = x_{it}\beta + \mu_i + \nu_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T \tag{19}$$

with $\beta = 10$. ν_{it} and x_{it} follow an AR(1) process as in (2) and (3) respectively with ρ and λ varying over the range $(0, 0.2, 0.4, 0.6, 0.8, 1)$. We set the variance from signal, see (3), at $\sigma_\varepsilon^2 = 5$. We also control the total variance from noise across experiments, see (2), to be $\sigma_\mu^2 + \sigma_e^2 = 10$. Hence, we have a fixed signal to noise ratio $\frac{\sigma_\varepsilon^2}{\sigma_\mu^2+\sigma_e^2} = \frac{1}{2}$ across experiments.[11] Next, we vary $\xi = \frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma_e^2}$ over the range $(0, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8)$, respectively. The sample sizes n and T are varied over the range $(20, 40, 60, 120, 240)$. In our experiments, ρ is estimated as the sample correlation coefficient between $\widehat{\nu}_{it}$ and $\widehat{\nu}_{it-1}$, i.e.,

$$\widehat{\rho} = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it} - \overline{\widehat{\nu}}\right)\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)}{\sqrt{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)^2}\sqrt{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it-1} - \overline{\widehat{\nu}}\right)^2}}$$

where $\overline{\widehat{\nu}}$ is the sample average of $\widehat{\nu}_{it}$. We choose the correlation coefficient estimator because it ensures that $\widehat{\rho}$ is always between 0 and 1.

For each experiment, we perform $10,000$ replications and compute the t-statistics using OLS, FE, FD, GLS-CO, GLS-PW, and true GLS. With this design we have 900 experiments. GAUSS 7.0.6 is used to perform the simulations. Random numbers for e_{it}, μ_i, and ε_{it} are generated by the GAUSS procedure RNDNS. We generate $n(T + 1000)$ random numbers and then split them into n series so that each series has the same mean and variance. The first $1,000$ observations are discarded for each series.

Tables 1 to 4 report the empirical size of these various t-statistics, when the true size is 5%, for $(\rho, \lambda) = (0.4, 0.4), (1, 0.4), (0.4, 1), (1, 1)$, respectively. Note that $(\rho, \lambda) = (0.4, 1)$ is the *panel cointegration* case and $(\rho, \lambda) = (1, 1)$ is the *spurious regression* case. Some of our findings are the following: (i) As expected, tOLS and tFE perform badly and their performance deteriorate as ρ or λ increase. For Table 1, the size of tOLS varies between 10 and 18%, while the size of tFE varies between 9 and 11%. This gets worse for the non-stationary disturbances case in Table 2, where the size of tOLS and tFE varies between 18 and 20%. For the non-stationary regressor case in Table 3, the size of tOLS varies between 24 and 80%, while the size of tFE varies between 17 and 20%. The spurious regression case in Table 4 gives the worst performance for tOLS with size varying between 59 and 83%. The size for tFE is also bad varying between 51 and 78%. (ii) In all cases, except case 1, tFD performs well with empirical size close to 5%. For case 1, tFD is slightly over-sized at 7 to 9%. (iii) tGLS gives the best performance, with empirical size not statistically different from 5%, for all cases considered. (iv) Both tGLSPW and tGLSCO perform well across experiments. In fact, for small sample sizes such as $(n, T) = (20, 20)$, they are undersized in case 2, and oversized in cases 3 and 4. However, as n and/or T increase, the empirical size of tGLSPW and tGLSCO improves considerably. For example, in case 4, tGLSPW and tGLSCO are oversized at about 10 to 12% for $(n, T) = (20, 20)$, but their empirical size improves to around 6% for $(n, T) = (120, 120)$.

We also note that the size of tOLS gets worse as the percentage of heterogeneity across individuals (ξ) increases. However, this heterogeneity measure does not affect the performance of tFE and tFD, since both estimators wipe out the individual effects. Theorems 3 and 4 also imply that the t-ratio using FGLS should

[11] Note that Baltagi and Li (1997) fix $\sigma_\mu^2 + \sigma_\nu^2$ across experiments. Here, one cannot obtain σ_ν^2 in the nonstationary case. Instead we fix $\sigma_\mu^2 + \sigma_e^2$ and our results are not sensitive to the choice of this sum. In fact, we tried 5, 10, and 20, and the results are similar.

converge to $N(0,1)$ whether or not the individual effects are included in the model. In fact, Figures 1 to 5 show the overlap of the $N(0,1)$ distribution and the distribution of tGLSPW for various sample sizes (fixing $\xi = 0.4$).

In conclusion, we note that tGLS gives the best performance, but it is infeasible. We recommend tFGLS for testing $H_0 : \beta = \beta_0$ when the researcher has no perfect foresight on stationarity of the regressor and/or the error term. tFD is a viable alternative to tFGLS if either the regressor or the error is nonstationary. tOLS and tFE are not recommended in these cases.

4.1 Robustness to Heterogeneous AR Parameters and Heteroskedasticity[12]

In this section we check the robustness of our results to (i) heterogeneity in the AR parameters in both the regressor and the error term and also to (ii) heteroskedasticity in the error terms. To accomplish this we run two sets of Monte Carlo experiments. The first set of experiments allow the AR parameters to vary across individuals. More specifically, λ_i (for the regressor) and ρ_i (for the error term) are allowed to be uniformly distributed, i.e., $IIDU(0,1)$. The estimation and test procedure are the same as before while the Data Generating Process is different. Table 5 reports the empirical size of these new experiments. Interestingly, the t-statistics using FGLS turn out to be robust across these experiments. In fact, tGLSPW and tGLSCO have empirical size that varies between $4-5\%$. tOLS and tFE perform badly again. In fact, tOLS has empirical size that varies between 19% and 67%, while tFE has empirical size that varies between 16% and 34%. tFD is slightly oversized with empirical size that varies between 6% and 7%.

As for the presence of heteroskedasticity in the error terms, we generate the error terms using the following design:

$$\begin{aligned} e_{it} &= \sigma_i \zeta_{it}^1, \text{ and} \\ \varepsilon_{it} &= \sigma_i \zeta_{it}^2 \end{aligned}$$

where ζ_{it}^1 and ζ_{it}^2 are generated from $N(0,1)$, respectively. To incorporate heteroskedasticity, σ_i are generated as follows:

$$\sigma_i \begin{cases} = 1 \text{ for } i = 1, \ldots, \frac{4n}{5} \\ = c \text{ for } i = \frac{4n}{5} + 1, \ldots, n \end{cases}$$

where $c = \sqrt{2}$ or 10. The simulation results are reported in Table 6. For Case 1, i.e., $(\rho, \lambda) = (0.4, 0.4)$, we find the following: (i) Panel A reports the results under relatively low degree of heteroskedasticity ($c = \sqrt{2}$). tFGLS are slightly oversized. In fact, the size for tGLSPW and tGLSCO varies between 6 and 7% for various sample sizes. tOLS and tFE are bad with size varying between 12 to 18% and 11 to 12%, respectively. tFD is also oversized at 9-10%. (ii) Panel B presents the results under a higher degree of heteroskedasticity ($c = 10$). In this case all the t-statistics are way oversized. The size for tGLSPW and tGLSCO varies

[12] We would like to thank the referee for this suggestion.

between 36 and 40%.[13] Hence, we conclude that tFGLS is robust to heterogeneous AR parameters, but not to heteroskedasticity in the error terms.

5 Conclusion

This paper derived the limiting distribution of the t-statistic for $H_0 : \beta = \beta_0$; using different panel data estimators including FE, FD, and FGLS. This is done in the context of a linear panel data regression model with possible nonstationarity in the regressor and/or the error term. We showed that one can use t statistics based on the FGLS estimator regardless of the nonstationarity of the regressor and/or the disturbance term. This is unlike the time-series case, where one has to consider a "super-efficient" type AR estimator of Perron and Yabu (2009) to achieve the normal limiting distribution of the t-ratio. One caveat is that this may not be robust to heteroskedasticity of the error terms, but it is robust to heterogenous AR parameters across individuals.

[13] The case of heteroskedastic error terms remains to be studied in the future. For possible ideas on how to handle this problem, see, Baltagi and Kao (2000).

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Table 1: The Empirical Size (%) of Case 1 ($\rho = 0.4$, $\lambda = 0.4$) with True Size 5%

(n, T)	ξ	tGLS	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	5.11	5.63	5.64	10.77		
	0.1	5.02	5.51	5.54	11.07		
	0.2	5.02	5.44	5.54	11.97	10.65	8.60
	0.4	4.96	5.34	5.40	13.73		
	0.6	4.92	5.23	5.23	15.63		
	0.8	4.90	5.13	5.07	17.28		
(40, 40)	0.05	5.12	5.15	5.34	10.20		
	0.1	5.06	5.20	5.35	10.74		
	0.2	4.99	5.14	5.36	11.67	10.23	7.84
	0.4	4.94	5.16	5.27	13.32		
	0.6	5.03	5.18	5.30	15.21		
	0.8	5.03	5.14	5.25	17.16		
(40, 120)	0.05	5.04	5.05	4.98	10.12		
	0.1	5.03	5.05	4.99	10.53		
	0.2	5.07	5.04	5.04	11.18	9.89	7.37
	0.4	5.06	5.05	5.00	13.32		
	0.6	5.04	5.02	5.00	15.47		
	0.8	5.04	5.02	4.96	17.78		
(120, 40)	0.05	4.90	4.96	5.06	10.63		
	0.1	4.84	4.98	5.10	11.04		
	0.2	4.87	5.03	5.03	11.54	10.33	8.01
	0.4	4.94	5.04	5.08	13.62		
	0.6	4.92	5.16	5.08	15.63		
	0.8	4.92	5.15	5.00	18.02		
(120, 120)	0.05	4.92	4.96	5.00	10.06		
	0.1	4.94	4.95	4.98	10.53		
	0.2	4.90	4.91	4.97	11.48	9.68	7.72
	0.4	4.94	4.93	5.00	13.59		
	0.6	4.94	4.97	4.98	15.35		
	0.8	4.96	4.95	4.96	17.39		

Table 2: *The Empirical Size (%) of Case 2 ($\rho = 1.0$, $\lambda = 0.4$) with True Size 5%*

(n, T)	ξ	tGLS	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	5.23	3.28	3.46	18.91		
	0.1	5.21	3.28	3.45	18.90		
	0.2	5.24	3.27	3.43	18.84	19.99	5.78
	0.4	5.11	3.27	3.39	19.01		
	0.6	5.15	3.24	3.37	18.88		
	0.8	5.16	3.19	3.25	18.89		
(40, 40)	0.05	5.25	4.17	4.14	18.37		
	0.1	5.27	4.17	4.14	18.38		
	0.2	5.17	4.17	4.14	18.42	19.78	5.48
	0.4	5.08	4.17	4.14	18.51		
	0.6	5.13	4.15	4.13	18.39		
	0.8	5.08	4.09	4.08	18.73		
(40, 120)	0.05	5.08	4.43	4.21	19.41		
	0.1	5.04	4.42	4.21	19.46		
	0.2	5.03	4.42	4.22	19.52	20.29	5.11
	0.4	5.02	4.41	4.23	19.44		
	0.6	4.94	4.36	4.22	19.53		
	0.8	5.01	4.35	4.22	19.89		
(120, 40)	0.05	5.32	4.36	4.31	19.16		
	0.1	5.33	4.36	4.31	19.05		
	0.2	5.29	4.35	4.31	18.90	19.44	5.55
	0.4	5.29	4.34	4.29	18.93		
	0.6	5.29	4.33	4.30	19.21		
	0.8	5.24	4.34	4.29	19.26		
(120, 120)	0.05	5.18	4.72	4.77	19.30		
	0.1	5.20	4.72	4.77	19.27		
	0.2	5.21	4.71	4.77	19.22	20.11	5.24
	0.4	5.23	4.71	4.78	19.29		
	0.6	5.17	4.71	4.79	19.03		
	0.8	5.16	4.71	4.78	19.20		

Table 3: The Empirical Size (%) of Case 3 ($\rho = 0.4$, $\lambda = 1.0$) with True Size 5%

(n, T)	ξ	tGLS	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	5.21	7.79	7.57	24.06		
	0.1	5.09	7.67	7.42	28.40		
	0.2	5.16	7.25	7.44	34.85	17.55	5.83
	0.4	4.90	7.12	7.23	44.86		
	0.6	4.88	7.02	7.05	51.77		
	0.8	4.89	6.67	6.69	57.25		
(40, 40)	0.05	4.92	6.26	6.16	28.53		
	0.1	5.01	6.16	6.03	35.16		
	0.2	4.95	6.14	6.12	44.19	18.47	5.48
	0.4	4.85	6.08	5.94	55.58		
	0.6	4.75	5.89	5.98	62.41		
	0.8	4.77	5.77	5.79	66.73		
(40, 120)	0.05	4.78	5.57	5.61	41.13		
	0.1	4.87	5.55	5.75	51.45		
	0.2	4.99	5.62	5.66	62.38	19.76	4.96
	0.4	5.10	5.76	5.57	72.81		
	0.6	5.30	5.77	5.69	77.72		
	0.8	5.24	5.65	5.63	80.63		
(120, 40)	0.05	4.71	5.56	5.60	28.08		
	0.1	4.78	5.57	5.59	35.28		
	0.2	4.90	5.80	5.82	44.61	19.01	5.72
	0.4	4.90	6.08	5.96	56.25		
	0.6	4.88	5.81	5.91	62.60		
	0.8	4.83	5.73	5.94	67.72		
(120, 120)	0.05	5.06	5.57	5.54	40.54		
	0.1	5.13	5.57	5.73	51.57		
	0.2	5.01	5.51	5.84	62.51	19.62	5.17
	0.4	5.17	5.60	5.78	72.29		
	0.6	5.14	5.65	5.74	77.34		
	0.8	5.23	5.62	5.77	80.61		

Table 4: The Empirical Size (%) of Case 4 ($\rho = 1.0$, $\lambda = 1.0$) with True Size 5%

(n, T)	ξ	tGLS	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	5.15	12.22	11.91	59.95		
	0.1	5.17	12.08	11.93	59.77		
	0.2	5.15	12.11	11.76	59.75	51.43	5.78
	0.4	5.01	11.93	11.59	59.90		
	0.6	5.03	11.55	11.24	60.30		
	0.8	4.97	11.04	10.52	60.95		
(40, 40)	0.05	5.02	8.48	8.50	70.70		
	0.1	5.04	8.44	8.43	70.74		
	0.2	4.99	8.43	8.40	70.84	62.56	5.41
	0.4	4.96	8.41	8.33	70.76		
	0.6	4.93	8.24	8.20	70.55		
	0.8	4.98	7.95	8.04	71.72		
(40, 120)	0.05	4.98	5.99	5.90	82.67		
	0.1	4.94	6.00	5.90	82.74		
	0.2	4.94	5.99	5.96	82.73	77.77	4.96
	0.4	4.92	5.98	5.97	82.67		
	0.6	4.94	6.00	5.92	82.81		
	0.8	4.96	5.92	5.86	82.73		
(120, 40)	0.05	4.84	8.06	8.01	70.69		
	0.1	4.80	8.03	7.97	70.75		
	0.2	4.81	8.06	7.95	70.53	63.09	5.26
	0.4	4.85	7.99	7.92	70.65		
	0.6	4.90	7.90	7.82	70.48		
	0.8	4.81	7.69	7.49	70.46		
(120, 120)	0.05	5.06	6.22	6.19	82.25		
	0.1	5.10	6.21	6.20	82.32		
	0.2	5.13	6.16	6.19	82.49	78.46	5.18
	0.4	5.12	6.14	6.18	82.17		
	0.6	5.15	6.15	6.15	82.26		
	0.8	5.11	6.10	6.08	82.55		

Table 5: The Empirical Size (%) with Heterogeneous AR Parameters $\left(\lambda_i, \rho_i \overset{i.i.d.}{\sim} U(0,1)\right)$

(n, T)	ξ	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	5.22	5.30	19.51		
	0.1	5.22	5.33	20.36		
	0.2	5.11	5.29	22.36	16.17	7.70
	0.4	5.13	5.23	26.61		
	0.6	5.00	5.12	30.83		
	0.8	4.73	4.80	35.05		
(40, 40)	0.05	4.73	4.75	21.71		
	0.1	4.72	4.80	23.57		
	0.2	4.77	4.86	27.00	20.00	7.21
	0.4	4.91	4.89	33.78		
	0.6	4.86	4.96	40.15		
	0.8	4.79	4.91	46.78		
(40, 120)	0.05	4.34	4.20	22.97		
	0.1	4.31	4.23	25.83		
	0.2	4.28	4.25	31.38	21.12	6.37
	0.4	4.29	4.25	40.31		
	0.6	4.32	4.27	47.84		
	0.8	4.35	4.29	55.02		
(120, 40)	0.05	5.26	5.23	36.51		
	0.1	5.28	5.28	37.75		
	0.2	5.40	5.33	40.42	26.06	7.53
	0.4	5.38	5.38	44.92		
	0.6	5.47	5.35	49.40		
	0.8	5.37	5.29	54.34		
(120, 120)	0.05	5.51	5.62	42.24		
	0.1	5.51	5.59	43.76		
	0.2	5.50	5.53	47.65	34.91	6.79
	0.4	5.41	5.53	54.71		
	0.6	5.38	5.53	61.22		
	0.8	5.36	5.49	67.46		

Table 6: The Empirical Size (%) of Case 1 ($\rho = 0.4$, $\lambda = 0.4$) under Heteroskedasticity of the Error Terms

		Panel A $(c = \sqrt{2})$					Panel B $(c = 10)$				
(n, T)	ξ	tGLSPW	tGLSCO	tOLS	tFE	tFD	tGLSPW	tGLSCO	tOLS	tFE	tFD
(20, 20)	0.05	7.22	7.37	12.24			40.13	40.23	43.94		
	0.1	7.18	7.28	12.59			40.01	40.06	43.91		
	0.2	7.21	7.32	13.00	12.48	10.27	39.83	39.89	43.76	44.71	42.78
	0.4	7.10	7.25	14.15			39.67	39.74	43.74		
	0.6	7.12	7.25	15.67			39.52	39.67	43.07		
	0.8	7.01	7.15	17.67			39.35	39.43	41.87		
(40, 40)	0.05	6.66	6.80	12.25			37.45	37.51	43.98		
	0.1	6.66	6.74	12.61			37.27	37.32	43.92		
	0.2	6.60	6.71	13.24	11.92	10.35	37.16	37.21	43.81	44.44	41.92
	0.4	6.59	6.70	14.64			36.97	36.98	43.47		
	0.6	6.57	6.70	16.33			36.71	36.74	42.95		
	0.8	6.58	6.71	17.95			36.76	36.83	41.76		
(40, 120)	0.05	6.28	6.38	11.71			36.53	36.62	43.36		
	0.1	6.31	6.43	12.01			36.61	36.62	43.35		
	0.2	6.33	6.48	12.71	11.33	9.23	36.51	36.54	43.33	43.67	41.37
	0.4	6.38	6.53	14.14			36.55	36.56	43.24		
	0.6	6.42	6.58	15.99			36.51	36.52	42.88		
	0.8	6.42	6.53	17.63			36.55	36.65	41.67		
(120, 40)	0.05	6.72	6.72	12.31			36.44	36.44	43.67		
	0.1	6.69	6.63	12.77			36.45	36.48	43.49		
	0.2	6.62	6.56	13.49	12.45	9.51	36.35	36.37	43.40	43.98	41.40
	0.4	6.57	6.67	14.89			36.18	36.21	43.27		
	0.6	6.62	6.69	16.28			36.24	36.22	43.06		
	0.8	6.66	6.68	18.42			36.34	36.37	41.96		

Figure 1: $(n, T) = (20, 20)$

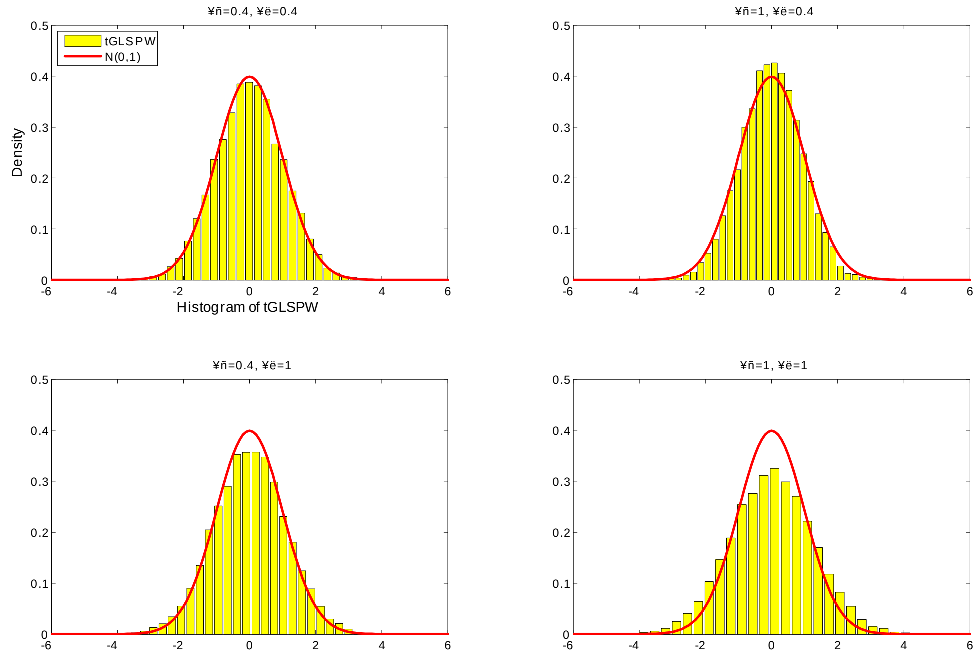


Figure 2: $(n, T) = (40, 40)$

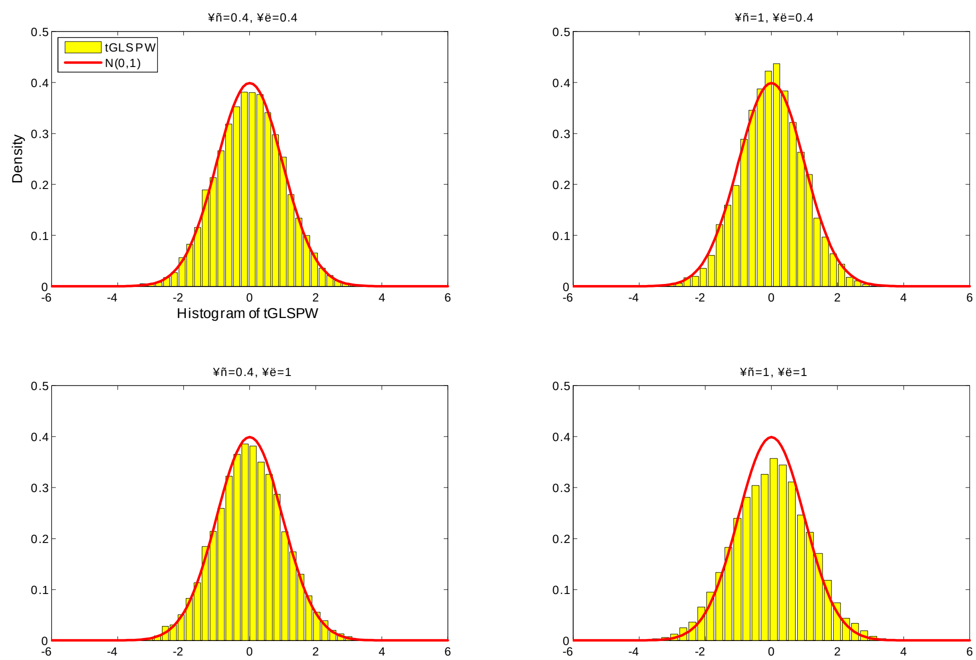


Figure 3: $(n, T) = (40, 120)$

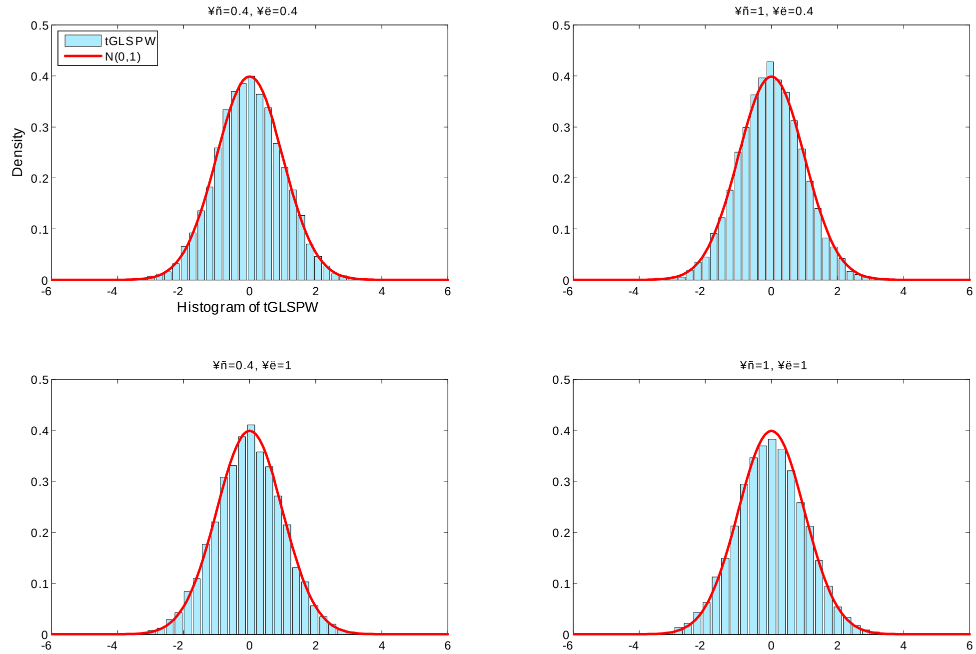


Figure 4: $(n, T) = (120, 40)$

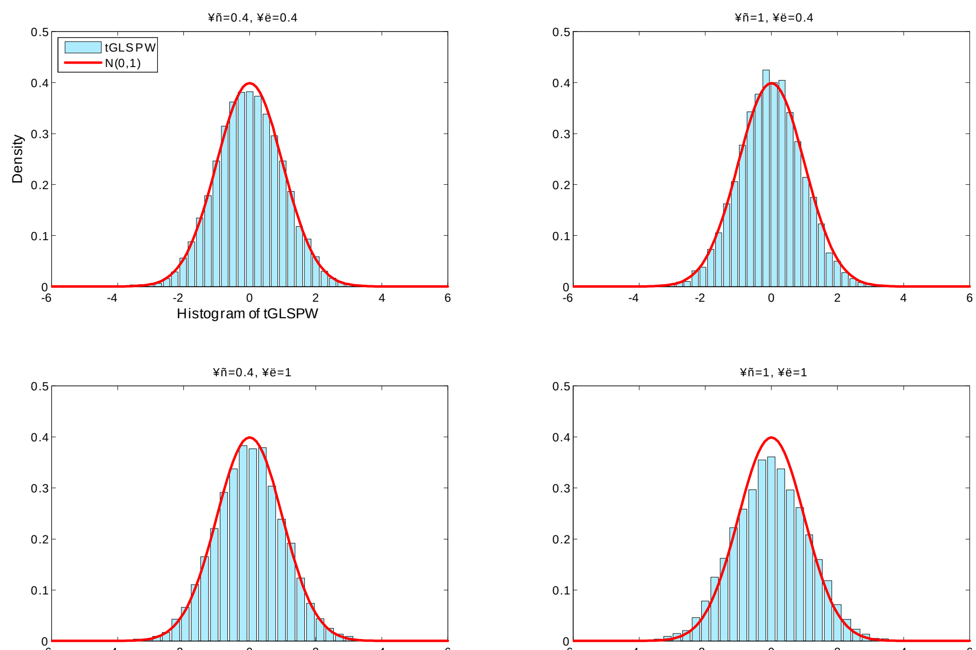
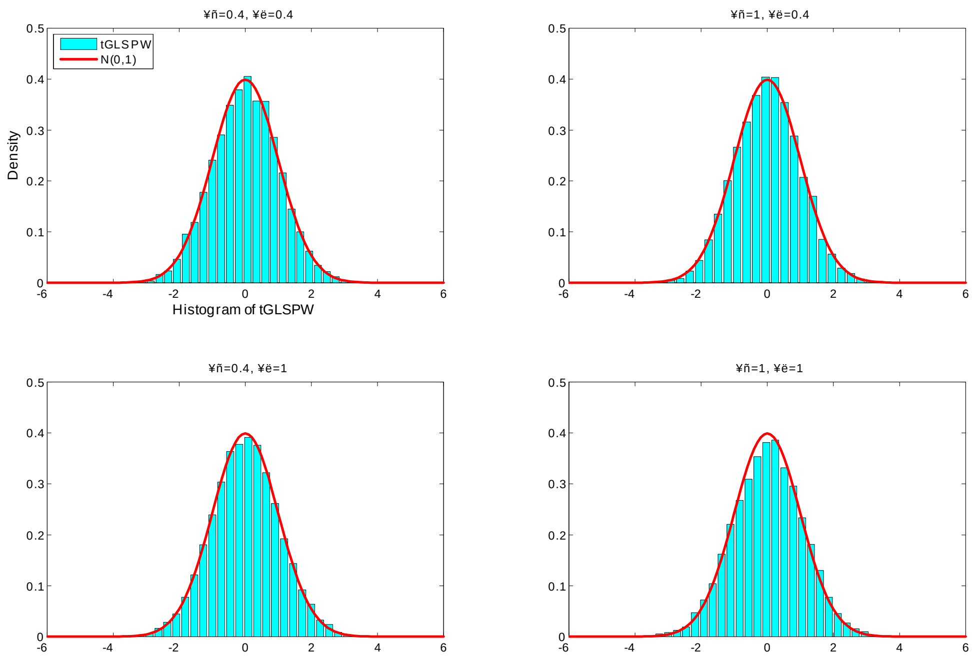


Figure 5: $(n, T) = (120, 120)$



SUPPLEMENTARY APPENDIX TO TEST OF HYPOTHESES IN PANEL DATA MODELS WHEN THE REGRESSOR AND DISTURBANCES ARE POSSIBLY NONSTATIONARY

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Abstract

This Appendix provides all the proofs for the lemmas and theorems in "Test of Hypotheses in Panel Data Models When the Regressor and Disturbances Are Possibly Nonstationary" by Baltagi, Kao and Na (2010).

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Appendix (for web posting and reference)

A Proof of Lemma 1

Proof. We investigate $|\rho| < 1$ and $\rho = 1$ cases, consecutively.

1. $|\rho| < 1$ case

   (a) $|\rho| < 1,\ |\lambda| < 1$ case
   Consider the limiting distribution of $\hat{\rho}$. Note that

$$\widehat{\rho} - \rho = \frac{\sum_{i=1}^{n} \sum_{t=2}^{T} \left(\widehat{\nu}_{it} - \rho\widehat{\nu}_{it-1}\right) \widehat{\nu}_{it-1}}{\sum_{i=1}^{n} \sum_{t=2}^{T} \widehat{\nu}_{it-1}^{2}}$$

and $\widehat{\nu}_{it} = (y_{it} - \bar{y}_{i.}) - \hat{\beta}_{FE}\,(x_{it} - \bar{x}_{i.}) = (\nu_{it} - \overline{\nu}_{i.}) - \left(\hat{\beta}_{FE} - \beta\right)(x_{it} - \bar{x}_{i.})$.
For the denominator,

$$\begin{aligned}
\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^{2} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1} - \overline{\nu}_{i.}) - (\hat{\beta}_{FE} - \beta)(x_{it-1} - \bar{x}_{i.})\right]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\nu}_{i.})^2 + \frac{1}{nT}\left\{\sqrt{nT}(\hat{\beta}_{FE} - \beta)\right\}^2\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1} - \bar{x}_{i.})^2\right\} \\
&\quad - \frac{2}{nT}(\hat{\beta}_{FE} - \beta)\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\nu}_{i.})(x_{it-1} - \bar{x}_{i.}) \\
&= I + II + III.
\end{aligned}$$

Consider II first. It is easy to see $II = O_p(\frac{1}{nT}) = o_p(1)$ because if $(n,T) \to \infty$ and $\frac{n}{T} \to 0$, then

$$\sqrt{nT}\left(\hat{\beta}_{FE} - \beta\right) = O_p\left(1\right)$$

by, e.g., Baltagi *et al.* (2008). Similarly, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1} - \bar{x}_{i.})^2 = O_p(1).$$

Next we show that $III = O_p(\frac{1}{nT})$. It can be shown that

$$\begin{aligned}
&\frac{2}{nT}(\hat{\beta}_{FE} - \beta)\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\nu}_{i.})(x_{it-1} - \bar{x}_{i.}) \\
= &\frac{2}{nT}(\hat{\beta}_{FE} - \beta)\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\nu_{it-1}x_{it-1} - \nu_{it-1}\bar{x}_{i.} - \overline{\nu}_{i.}x_{it-1} + \overline{\nu}_{i.}\bar{x}_{i.}\right] \\
= &O_p(\frac{1}{nT}) = o_p\left(1\right)
\end{aligned}$$

using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}x_{it-1} = O_p\left(1\right)$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_{it}\right)=O_p\left(1\right)$$

since ν_{it-1} and x_{it-1} are uncorrelated.
Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^{2}=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+O_p(\frac{1}{nT})\overset{p}{\to}\frac{\sigma_e^2}{1-\rho^2}$$

as $(n,T)\to\infty$.
For the numerator, it can be shown that $\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}=e_{it}-(\hat{\beta}_{FE}-\beta)\left\{(\lambda-\rho)x_{it-1}+\varepsilon_{it}\right\}+o_p(1)$ and accordingly we have

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}\\
=\;&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{\begin{array}{c}\left[e_{it}-(\hat{\beta}_{FE}-\beta)\left\{(\lambda-\rho)x_{it-1}+\varepsilon_{it}\right\}+o_p(1)\right]\\ \left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\overline{x}_{i.})\right]\end{array}\right\}\\
=\;&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\overline{\nu}_{i.}\right)e_{it}-\sqrt{nT}(\hat{\beta}_{FE}-\beta)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{(\lambda-\rho)x_{it-1}+\varepsilon_{it}\right\}\left(\nu_{it-1}-\overline{\nu}_{i.}\right)\\
&-\sqrt{nT}(\hat{\beta}_{FE}-\beta)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(x_{it-1}-\overline{x}_{i.})\\
&+\left\{\sqrt{nT}(\hat{\beta}_{FE}-\beta)\right\}^2\frac{1}{(nT)^{3/2}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{(\lambda-\rho)x_{it-1}+\varepsilon_{it}\right\}\left(x_{it-1}-\overline{x}_{i.}\right)+o_p(1)\\
=\;&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\left(\nu_{it-1}-\overline{\nu}_{i.}\right)e_{it}\right]+O_p(\frac{1}{\sqrt{nT}})
\end{aligned}$$

because $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}\nu_{it-1}=O_p(1)$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}\nu_{it-1}=O_p(1)$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}^2=O_p(1)$, and $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}e_{it}=O_p(1)$ as $(n,T)\to\infty$.
Therefore, we have

$$\begin{aligned}
&\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\left(\nu_{it-1}-\overline{\nu}_{i.}\right)e_{it}\right]+o_p\left(1\right)\\
=\;&\tfrac{1}{\sqrt{nT}}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}e_{it}-\tfrac{1}{\sqrt{nT}}\sum_{i=1}^{n}\left[\left(\tfrac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it}\right)\left(\tfrac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\right]+o_p\left(1\right)\\
\overset{d}{\to}\;&\sigma_e^2N\left(0,\tfrac{1}{1-\rho^2}\right)
\end{aligned}$$

if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$. Note that

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\right]\\
=\;&\sqrt{\frac{n}{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\right]\\
=\;&O_p\left(\sqrt{\frac{n}{T}}\right)O_p\left(1\right)\\
=\;&o_p\left(1\right)
\end{aligned}$$

with a condition $\frac{n}{T} \to 0$ since

$$\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\right] = O_p(1).$$

Finally, we conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\widehat{\rho}-\rho\right) = \frac{\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^{2}} \xrightarrow{d} \frac{\sigma_e^2 N\left(0,\frac{1}{1-\rho^2}\right)}{\frac{\sigma_e^2}{1-\rho^2}} = N\left(0,1-\rho^2\right).$$

Next we consider $\widehat{\sigma}_e^2$. Note that

$$\begin{aligned}
\widehat{\sigma}_e^2 &= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\rho}\widehat{\nu}_{it-1}\right)^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\begin{array}{c}\rho\nu_{it-1}+e_{it}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(\lambda x_{it-1}+\varepsilon_{it}-\bar{x}_{i.})\\ -\hat{\rho}\left(\nu_{it-1}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right)\end{array}\right]^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\begin{array}{c}e_{it}-(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}+(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\\ +(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}-(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right\}\end{array}\right]^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2+\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}\right]^2 \\
&\quad+\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\right]^2 \\
&\quad+\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\right]^2 \\
&\quad+\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right\}\right]^2 \\
&\quad-\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\} \\
&\quad+\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\} \\
&\quad+\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\} \\
&\quad-\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right\}+o_p(1) \\
&= I+II+III+IV+V+VI+VII+VIII+VIIII+o_p(1).
\end{aligned}$$

One can show that all the rest of terms except I is $o_p(1)$. For example, consider II.

$$\begin{aligned}
&\left\{\sqrt{nT}(\hat{\rho}-\rho)\right\}^2\frac{1}{(nT)^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}^2-2(\hat{\beta}_{FE}-\beta)\nu_{it-1}x_{it-1}+(\hat{\beta}_{FE}-\beta)^2x_{it-1}^2) \\
=\;& O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{n^2T^2}\right)+O_p\left(\frac{1}{n^2T^2}\right)=O_p\left(\frac{1}{nT}\right)=o_p(1)
\end{aligned}$$

using

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}^2 = O_p(1),$$

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}x_{it-1} = O_p(1)$$

and

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}^2 = O_p(1)$$

as $(n,T)\to\infty$.

Similarly, it can be easily shown that

$$\widehat{\sigma}_e^2=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2+o_p(1)\xrightarrow{p}\sigma_e^2$$

as $(n,T)\to\infty$.

(b) $|\rho|<1,\ \lambda=1$ case

From

$$\widehat{\rho}-\rho=\frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}}{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2},$$

we have

$$\begin{aligned}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right]^2\\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+\frac{1}{nT}\left\{\sqrt{n}T(\hat{\beta}_{FE}-\beta)\right\}^2\left\{\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\bar{x}_{i.})^2\right\}\\ &\quad -\frac{2}{n^{3/2}}\sqrt{n}T(\hat{\beta}_{FE}-\beta)\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.})\\ &= I+II+III.\end{aligned}$$

Using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008), it can be shown that $II=O_p(\frac{1}{nT})$ if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, which yields

$$\sqrt{n}T\left(\hat{\beta}_{FE}-\beta\right)=O_p\left(1\right).$$

Also note that as $(n,T)\to\infty$, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it-1}-\bar{x}_{i.})^2=O_p(1),$$

see equation (C.3) in Kao (1999).

Next we show that $III=O_p\left(\frac{1}{nT}\right)$. This follows because

$$\begin{aligned}&\frac{2}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.})\\ =&\ \frac{2}{nT}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})\frac{(x_{it-1}-\bar{x}_{i.})}{\sqrt{T}}\\ =&\ O_p\left(\frac{1}{nT}\right)\end{aligned}$$

and

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})\frac{(x_{it-1}-\bar{x}_{i.})}{\sqrt{T}}=O_p\left(1\right)$$

where v_{it} and x_{it} are not correlated. Hence,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+O_p\left(\frac{1}{nT}\right)\xrightarrow{p}\frac{\sigma_e^2}{1-\rho^2}$$

as $(n,T)\to\infty$.
For the numerator, $\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}=e_{it}-(\hat{\beta}_{FE}-\beta)\left\{(1-\rho)x_{it-1}+\varepsilon_{it}\right\}+o_p(1)$ and by using a similar argument, we get

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}\\
&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[e_{it}-(\hat{\beta}_{FE}-\beta)\left\{(1-\rho)x_{it-1}+\varepsilon_{it}\right\}+o_p(1)\right]\left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right]\\
&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\left(\nu_{it-1}-\overline{\nu}_{i.}\right)e_{it}\right]+o_p\left(1\right)\\
&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}e_{it}-\frac{1}{\sqrt{T}}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\nu_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\right]+o_p\left(1\right)\\
&\xrightarrow{d}\sigma_e^2N\left(0,\frac{1}{1-\rho^2}\right)
\end{aligned}$$

if $(n,T)\to\infty$ and $\frac{n}{T}\to0$.
We conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to0$, then

$$\sqrt{nT}\left(\widehat{\rho}-\rho\right)=\frac{\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\rho\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2}\xrightarrow{d}\frac{\sigma_e^2N\left(0,\frac{1}{1-\rho^2}\right)}{\frac{\sigma_e^2}{1-\rho^2}}=N\left(0,1-\rho^2\right),$$

which is the same result as in Lemma 1.(1).
Next consider $\widehat{\sigma}_e^2$.

$$\begin{aligned}
\widehat{\sigma}_e^2 &= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\rho}\widehat{\nu}_{it-1}\right)^2\\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}[\rho\nu_{it-1}+e_{it}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}+\varepsilon_{it}-\bar{x}_{i.})\\
&\quad -\hat{\rho}(\nu_{it-1}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.}))]^2\\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}[e_{it}-(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}+(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\\
&\quad +(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}-(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(1-\rho)x_{it-1}\right\}]^2\\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2+\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}\right]^2\\
&\quad +\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\right]^2\\
&\quad +\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\right]^2\\
&\quad +\tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(1-\rho)x_{it-1}\right\}\right]^2\\
&\quad -\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\rho}-\rho)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}\\
&\quad +\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\rho}-\rho)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\\
&\quad +\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\rho-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\\
&\quad -\tfrac{2}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(1-\rho)x_{it-1}\right\}+o_p(1)\\
&= I+II+III+IV+V+VI+VII+VIII+VIIII+o_p(1)
\end{aligned}$$

and it can be shown that

$$\widehat{\sigma}_e^2=I+o_p(1).$$

Let us consider II, for example.

$$\frac{\left\{\sqrt{nT}(\hat{\rho}-\rho)\right\}^2}{(nT)^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}^2-2(\hat{\beta}_{FE}-\beta)\nu_{it-1}x_{it-1}+(\hat{\beta}_{FE}-\beta)^2x_{it-1}^2)=O_p\left(\frac{1}{nT}\right)$$

because

$$\frac{1}{(nT)^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it}^2 = \frac{1}{nT}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it}^2 = O_p\left(\frac{1}{nT}\right)$$

using $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it}^2 \xrightarrow{p} \frac{\sigma_e^2}{1-\rho^2} = O_p(1)$.
Also

$$\frac{2}{n^{3/2}T^2}\sqrt{n}T(\hat{\beta}_{FE}-\beta)\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\nu_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}\right) = O_p\left(\frac{1}{n^{3/2}T^2}\right)$$

using $\frac{1}{n}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\nu_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}} = O_p(1)$
and

$$\frac{1}{(nT)^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\hat{\beta}_{FE}-\beta)^2 x_{it-1}^2 = \frac{(\sqrt{n}T(\hat{\beta}_{FE}-\beta))^2}{n^2T^2}\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}^2\right) = O_p\left(\frac{1}{n^2T^2}\right)$$

as $(n,T)\to\infty$ with the joint limit argument.
By a similar process, we have

$$\widehat{\sigma}_e^2 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2 + o_p(1) \xrightarrow{p} \sigma_e^2.$$

as $(n,T)\to\infty$.

2. $\rho = 1$ case

   (a) $\rho = 1$, $|\lambda| < 1$ case
   Since we have $\rho = 1$,

$$\widehat{\rho} - 1 = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}}{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2}.$$

   For the denominator,

$$\begin{aligned}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2 &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1}-\overline{\nu}_{i.}) - (\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right]^2 \\ &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2 + \frac{1}{nT}\left\{\sqrt{n}(\hat{\beta}_{FE}-\beta)\right\}^2\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\bar{x}_{i.})^2\right\} \\ &\quad -\sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{n^{3/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.}) \\ &= I + II + III.\end{aligned}$$

   Consider II first. Using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008), it can be shown that $II = O_p\left(\frac{1}{nT}\right)$ if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, which yields

$$\sqrt{n}\left(\hat{\beta}_{FE}-\beta\right) = O_p\left(1\right).$$

   Also as $(n,T)\to\infty$, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it-1}-\bar{x}_{i.})^2 = O_p(1).$$

Consider III. It can be shown that

$$\begin{aligned} III &= \sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{n^{3/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.}) \\ &= \frac{1}{nT}\sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\nu_{it-1}-\overline{\nu}_{i.}}{\sqrt{T}}\frac{x_{it-1}-\bar{x}_{i.}}{\sqrt{T}} \\ &= O_p(\frac{1}{nT}) \end{aligned}$$

as $(n,T)\to\infty$ using

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\nu_{it-1}-\overline{\nu}_{i.}}{\sqrt{T}}\frac{x_{it-1}-\bar{x}_{i.}}{\sqrt{T}}=O_p(1)$$

and

$$\sqrt{n}(\hat{\beta}_{FE}-\beta)=O_p(1).$$

Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+O_p\left(\frac{1}{nT}\right)\xrightarrow{p}\frac{\sigma_e^2}{6}$$

by equation (C.3) in Kao (1999).
For the numerator, $\widehat{\nu}_{it}-\widehat{\nu}_{it-1}=e_{it}-(\hat{\beta}_{FE}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}$. One can show that

$$\begin{aligned} &\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\widehat{\nu}_{it}-\widehat{\nu}_{it-1})\widehat{\nu}_{it-1} \\ &=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[e_{it}-(\hat{\beta}_{FE}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\right]\left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\overline{x}_{i.})\right] \\ &=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\begin{array}{c} e_{it}(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}(\nu_{it-1}-\overline{\nu}_{i.}) \\ -(\hat{\beta}_{FE}-\beta)e_{it}(x_{it-1}-\overline{x}_{i.})+(\hat{\beta}_{FE}-\beta)^2\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}(x_{it-1}-\overline{x}_{i.}) \end{array}\right] \\ &=I+II+III+IV. \end{aligned}$$

Consider II. It can be easily shown that

$$\begin{aligned} II &= \frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}(\nu_{it-1}-\overline{\nu}_{i.}) \\ &= \frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\lambda-1)x_{it-1}\nu_{it-1}-(\lambda-1)x_{it-1}\overline{\nu}_{i.}+\varepsilon_{it}\nu_{it-1}-\varepsilon_{it}\overline{\nu}_{i.}] \\ &= O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n}\right) \\ &= O_p\left(\frac{1}{n}\right) \end{aligned}$$

using

$$\begin{aligned} \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}\nu_{it-1} &= O_p(1), \\ \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}\nu_{it-1} &= O_p(1), \end{aligned}$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_{it-1}\right)\left(\frac{1}{T^{3/2}}\sum_{t=2}^{T}\nu_{it-1}\right)=O_p\left(1\right).$$

Consider III. It is also easy to see that

$$\begin{aligned} III &= -\frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}(x_{it-1}-\overline{x}_{i.}) \\ &= O_p\left(\frac{1}{n\sqrt{T}}\right)+O_p\left(\frac{1}{nT^{3/2}}\right)=O_p\left(\frac{1}{n\sqrt{T}}\right). \end{aligned}$$

Consider IV.

$$\begin{aligned} IV &= \frac{\left(\sqrt{n}(\hat{\beta}_{FE}-\beta)\right)^2}{n^2T}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{(\lambda-1)x_{it-1}+\varepsilon_{it}\right\}(x_{it-1}-\overline{x}_{i.}) \\ &= \frac{\left(\sqrt{n}(\hat{\beta}_{FE}-\beta)\right)^2}{n^2T}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\lambda-1)x_{it-1}^2-(\lambda-1)x_{it-1}\overline{x}_{i.}+\varepsilon_{it}x_{it-1}-\varepsilon_{it}\overline{x}_{i.}\right] \\ &= O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{n^{3/2}\sqrt{T}}\right)+O_p\left(\frac{1}{nT}\right) \\ &= O_p\left(\frac{1}{n}\right) \end{aligned}$$

using

$$\begin{aligned} \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}^2 &= O_p\left(1\right), \\ \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_{it}\right) &= O_p\left(1\right), \end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_{it}\right)=O_p\left(1\right).$$

We conclude that

$$\begin{aligned} &\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1} \\ &= I+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{n\sqrt{T}}\right)+O_p\left(\frac{1}{n}\right) \\ &= I+O_p\left(\frac{1}{n}\right) \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\left(\nu_{it}-\overline{\nu}_{i.}\right)e_{it}\right]+o_p\left(1\right)\xrightarrow{p}-\frac{\sigma_e^2}{2} \end{aligned}$$

using equation (C.5) in Kao (1999). Hence,

$$T\left(\widehat{\rho}-1\right)=\frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta\widehat{\nu}_{it}\widehat{\nu}_{it-1}}{\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2}\xrightarrow{p}-\frac{\frac{\sigma_e^2}{2}}{\frac{\sigma_e^2}{6}}=-3.$$

Next we consider $\widehat{\sigma}_e^2$.

$$\begin{aligned}
\widehat{\sigma}_e^2 &= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left(\widehat{\nu}_{it}-\widehat{\rho}\widehat{\nu}_{it-1}\right)^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left[\begin{array}{c}\nu_{it-1}+e_{it}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(\lambda x_{it-1}+\varepsilon_{it}-\bar{x}_{i.}) \\ -\hat{\rho}\left\{\nu_{it-1}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right\}\end{array}\right]^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left[\begin{array}{c}e_{it}-(\hat{\rho}-1)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\} \\ +(\hat{\rho}-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}-(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-1)x_{it-1}\right\}\end{array}\right]^2 \\
&= \tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T e_{it}^2+\tfrac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\left[(\hat{\rho}-1)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\}\right]^2 \\
&\quad +\tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left[(\hat{\rho}-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}\right]^2 \\
&\quad +\tfrac{1}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left[(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-1)x_{it-1}\right\}\right]^2 \\
&\quad -\tfrac{2}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T e_{it}(\hat{\rho}-1)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\} \\
&\quad +\tfrac{2}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T e_{it}(\hat{\rho}-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\} \\
&\quad -\tfrac{2}{nT}\textstyle\sum_{i=1}^n\sum_{t=2}^T e_{it}(\hat{\beta}_{FE}-\beta)\left\{\varepsilon_{it}+(\lambda-1)x_{it-1}\right\}+o_p(1) \\
&= I+II+III+IV+V+VI+VII+VIII+VIIII+o_p(1).
\end{aligned}$$

From above, it can be shown that

$$\widehat{\sigma}_e^2=\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T e_{it}^2+o_p(1)\overset{p}{\to}\sigma_e^2$$

as $(n,T)\to\infty$. We illustrate II only as an example. It can be shown that

$$\{T(\hat{\rho}-1)\}^2\,\frac{1}{nT^3}\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}^2-2(\hat{\beta}_{FE}-\beta)\nu_{it-1}x_{it-1}+(\hat{\beta}_{FE}-\beta)^2x_{it-1}^2)=O_p(\frac{1}{\sqrt{n}T})$$

as $(n,T)\to\infty$ because

$$\frac{1}{nT^3}\sum_{i=1}^n\sum_{t=2}^T\nu_{it-1}^2=O_p(\frac{1}{T})$$

with $\frac{1}{nT^2}\sum_{i=1}^n\sum_{t=2}^T\nu_{it-1}^2=O_p(1)$ and

$$\left(\sqrt{n}(\hat{\beta}_{FE}-\beta)\right)^2\frac{1}{nT^2}\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T x_{it-1}^2=O_p(\frac{1}{nT^2})$$

with $\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T x_{it-1}^2=O_p(1)$.
Also note that

$$-2\sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{1}{nT^2}\frac{1}{\sqrt{n}}\sum_{i=1}^n\sum_{t=2}^T\frac{\nu_{it-1}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_p(\frac{1}{nT^2})$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^n\sum_{t=2}^T\frac{\nu_{it-1}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_p(1)$.

(b) $\rho=1$, $\lambda=1$ case
Recall that

$$\widehat{\rho}-1=\frac{\sum_{i=1}^n\sum_{t=2}^T\left(\widehat{\nu}_{it}-\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}}{\sum_{i=1}^n\sum_{t=2}^T\widehat{\nu}_{it-1}^2}.$$

We have

$$\begin{aligned}
\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2 &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right]^2 \\
&= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+\frac{1}{n}\left\{\sqrt{n}(\hat{\beta}_{FE}-\beta)\right\}^2\left\{\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\bar{x}_{i.})^2\right\} \\
&\quad -\sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{n^{3/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.}) \\
&= I+II+III.
\end{aligned}$$

Consider II first. Using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008), it can be shown that $II=O_p(\frac{1}{n})$ if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, which yields

$$\sqrt{n}\left(\hat{\beta}_{FE}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\right)=O_p\left(1\right).$$

Also as $(n,T)\to\infty$, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it-1}-\bar{x}_{i.})^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{6}=O_p(1)$$

see equation (C.3) in Kao (1999).

Next consider III.

$$\begin{aligned}
III &= \sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{n^{3/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})(x_{it-1}-\bar{x}_{i.}) \\
&= \sqrt{n}(\hat{\beta}_{FE}-\beta)\frac{2}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\frac{\nu_{it-1}-\overline{\nu}_{i.}}{\sqrt{T}})(\frac{x_{it-1}-\bar{x}_{i.}}{\sqrt{T}})\frac{1}{T} \\
&= O_p(\frac{1}{n})
\end{aligned}$$

as $(n,T)\to\infty$ since

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\frac{\nu_{it-1}-\overline{\nu}_{i.}}{\sqrt{T}})(\frac{x_{it-1}-\bar{x}_{i.}}{\sqrt{T}})\frac{1}{T}=O_p\left(1\right)$$

where ν_{it} and x_{it} are not correlated.

Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^2=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})^2+o_p(1)\xrightarrow{p}\frac{\sigma_e^2}{6}$$

by equation (C.3) in Kao (1999).

For the numerator, $\widehat{\nu}_{it}-\widehat{\nu}_{it-1}=e_{it}-(\hat{\beta}_{FE}-\beta)\varepsilon_{it}$, and it can be shown that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}\\
=\;&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[e_{it}-(\hat{\beta}_{FE}-\beta)\varepsilon_{it}\right]\left[(\nu_{it-1}-\overline{\nu}_{i.})-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})\right]\\
=\;&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\begin{array}{c}(\nu_{it-1}-\overline{\nu}_{i.})e_{it}-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})e_{it}\\ -(\hat{\beta}_{FE}-\beta)(\nu_{it-1}-\overline{\nu}_{i.})\varepsilon_{it}+(\hat{\beta}_{FE}-\beta)^{2}\varepsilon_{it}(x_{it-1}-\bar{x}_{i.})\end{array}\right]\\
=\;&I+II+III+IV.
\end{aligned}$$

Consider II. It can be shown that

$$\begin{aligned}
II &= -\frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\bar{x}_{i.})e_{it}\\
&= -\frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}e_{it}+\frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\left(\frac{1}{T^{3/2}}\sum_{t=2}^{T}x_{it}\right)\\
&= O_{p}\left(\frac{1}{n}\right)+O_{p}\left(\frac{1}{n}\right)=O_{p}\left(\frac{1}{n}\right)
\end{aligned}$$

using

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}e_{it}=O_{p}\left(1\right)$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}e_{it}\right)\left(\frac{1}{T^{3/2}}\sum_{t=2}^{T}x_{it}\right)=O_{p}\left(1\right).$$

Consider III and IV. In a similar vein as II, one can see that

$$III=-\frac{\sqrt{n}(\hat{\beta}_{FE}-\beta)}{n^{3/2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\nu_{it-1}\varepsilon_{it}-\overline{\nu}_{i.}\varepsilon_{it}\right]=O_{p}\left(\frac{1}{n}\right)$$

and

$$IV=\frac{\left(\sqrt{n}(\hat{\beta}_{FE}-\beta)\right)^{2}}{n^{2}T}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\varepsilon_{it}x_{it-1}-\varepsilon_{it}\bar{x}_{i.}\right]=O_{p}\left(\frac{1}{n}\right).$$

We conclude that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\nu}_{it-1}\right)\widehat{\nu}_{it-1}\\
&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\nu}_{i.})e_{it}+O_{p}\left(\frac{1}{n}\right)\xrightarrow{p}-\frac{\sigma_{e}^{2}}{2}
\end{aligned}$$

using equation (C.5) in Kao (1999). Combining these results, we get

$$T\left(\widehat{\rho}-1\right)=\frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta\widehat{\nu}_{it}\widehat{\nu}_{it-1}}{\frac{1}{nT^{2}}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{\nu}_{it-1}^{2}}\xrightarrow{p}-\frac{\frac{\sigma_{e}^{2}}{2}}{\frac{\sigma_{e}^{2}}{6}}=-3$$

which is the required result.

Next we consider $\widehat{\sigma}_e^2$. Note that

$$\begin{aligned}
\widehat{\sigma}_e^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{\nu}_{it}-\widehat{\rho}\widehat{\nu}_{it-1}\right)^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{\begin{array}{c} \nu_{it-1}+e_{it}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}+\varepsilon_{it}-\bar{x}_{i.}) \\ -\hat{\rho}(\nu_{it-1}-\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)(x_{it-1}-\bar{x}_{i.})) \end{array}\right\}^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{\begin{array}{c} e_{it}-(\hat{\rho}-1)\left\{\nu_{it-1}-(\hat{\beta}_{FE}-\beta)x_{it-1}\right\} \\ +(\hat{\rho}-1)\left\{\overline{\nu}_{i.}-(\hat{\beta}_{FE}-\beta)\bar{x}_{i.}\right\}-(\hat{\beta}_{FE}-\beta)\varepsilon_{it} \end{array}\right\}^2.
\end{aligned}$$

After some tedious algebra, it can be easily shown that

$$\widehat{\sigma}_e^2=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2+o_p(1)\xrightarrow{p}\sigma_e^2.$$

■

B Proof of Theorem 1

Proof. Now we are ready to prove Theorem 1.

1. $|\rho|<1,\ |\lambda|<1$ case

   Recall

$$S_{FE}=\sqrt{\frac{\widehat{\sigma}_\nu^2}{\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\bar{x}_{i.}\right)^2}}.$$

   Now note that

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\bar{x}_{i.})^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{1-\lambda^2}$$

   and

$$\widehat{\sigma}_v^2=\frac{\hat{\sigma}_e^2}{1-\widehat{\rho}^2}\xrightarrow{p}\frac{\sigma_e^2}{(1-\rho^2)}$$

   since $1-\hat{\rho}^2=(1-\rho^2)+(1-\rho)(\hat{\rho}-\rho)-(\hat{\rho}-\rho)^2=(1-\rho^2)+o_p(1)$ using $\hat{\rho}-\rho=O_p\left(\frac{1}{\sqrt{nT}}\right)$. Therefore, if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$t_{FE}=\frac{\sqrt{nT}(\hat{\beta}_{FE}-\beta)}{\sqrt{\widehat{\sigma}_\nu^2/\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\bar{x}_{i.})^2\right)}}\xrightarrow{d}\frac{N\left(0,\frac{(1+\rho\lambda)(1-\lambda^2)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right)}{\sqrt{\left(\frac{\sigma_e^2}{1-\rho^2}\right)/\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)}}=N(0,\frac{1+\rho\lambda}{1-\rho\lambda}).$$

2. $\rho=1,\ |\lambda|<1$ case

   Consider $\widehat{\sigma}_\nu^2$.

$$\frac{\widehat{\sigma}_\nu^2}{T}=\frac{\hat{\sigma}_e^2}{T(1-\widehat{\rho}^2)}=\frac{\hat{\sigma}_e^2}{T(1-\hat{\rho})(1+\hat{\rho})}\xrightarrow{p}\frac{\sigma_e^2}{6}$$

   using

$$T\left(\widehat{\rho}-1\right)\xrightarrow{p}-3$$

and

$$\widehat{\rho} \xrightarrow{p} 1.$$

Now

$$\begin{aligned}
t_{FE} &= \frac{\sqrt{n}(\hat{\beta}_{FE} - \beta)}{\sqrt{\frac{\frac{\widehat{\sigma}_\nu^2}{T}}{\frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_{i.})^2}}} \xrightarrow{d} \frac{N\left(0, \frac{(1-\lambda)^2 \sigma_e^2}{6\sigma_\varepsilon^2}\right)}{\sqrt{\frac{\frac{\sigma_e^2}{6}}{\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)}}} \\
&= N\left(0, \frac{(1-\lambda)^2 \sigma_e^2}{6\sigma_\varepsilon^2}\right) \times \sqrt{\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}\right)\frac{6}{\sigma_e^2}} \\
&= N\left(0, \frac{(1-\lambda)^2}{1-\lambda^2}\right) \\
&= N\left(0, \frac{(1-\lambda)^2}{(1-\lambda)(1+\lambda)}\right) \\
&= N\left(0, \frac{1-\lambda}{1+\lambda}\right)
\end{aligned}$$

using

$$\sqrt{n}(\hat{\beta}_{FE} - \beta) \xrightarrow{d} N\left(0, \frac{(1-\lambda)^2 \sigma_e^2}{6\sigma_\varepsilon^2}\right).$$

3. $|\rho| < 1,\ \lambda = 1$ case.
   Because

$$\frac{1}{nT^2}\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_{i.})^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{6}$$

   and

$$\widehat{\sigma}_\nu^2 = \frac{\hat{\sigma}_e^2}{1-\widehat{\rho}^2} \xrightarrow{p} \frac{\sigma_e^2}{(1-\rho^2)}$$

   as shown in Theorem 1.(1), one can easily verify that if $(n,T) \to \infty$ and $\frac{n}{T} \to 0$, then

$$t_{FE} = \frac{\sqrt{n}T(\hat{\beta}_{FE} - \beta)}{\sqrt{\widehat{\sigma}_\nu^2 / \left(\frac{1}{nT^2}\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_{i.})^2\right)}} \xrightarrow{d} \frac{N\left(0, \frac{6\sigma_e^2}{(1-\rho)^2 \sigma_\varepsilon^2}\right)}{\sqrt{\left(\frac{\sigma_e^2}{1-\rho^2}\right) / \left(\frac{\sigma_\varepsilon^2}{6}\right)}} = N(0, \frac{1+\rho}{1-\rho}).$$

4. $\rho = 1,\ \lambda = 1$ case.
   Now

$$\begin{aligned}
\frac{t_{FE}}{\sqrt{T}} &= \frac{\sqrt{n}(\hat{\beta}_{FE} - \beta)}{\sqrt{\frac{\frac{\widehat{\sigma}_\nu^2}{T}}{\frac{1}{nT^2}\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_{i.})^2}}} \xrightarrow{d} \frac{N\left(0, \frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\right)}{\sqrt{\frac{\sigma_e^2}{\sigma_\varepsilon^2}}} \\
&= N\left(0, \frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\right) \times \sqrt{\frac{\sigma_\varepsilon^2}{\sigma_e^2}} \\
&= N\left(0, \frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\frac{\sigma_\varepsilon^2}{\sigma_e^2}\right) \\
&= N\left(0, \frac{2}{5}\right)
\end{aligned}$$

using

$$\sqrt{n}(\hat{\beta}_{FE}-\beta)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{5\sigma_\varepsilon^2}\right)$$

and

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\bar{x}_{i.})^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{6}.$$

■

C Proof of Theorem 2

Proof. Now we prove Theorem 2.

1. $|\rho|<1,\ |\lambda|<1$ case

   Recall

$$\Delta y_{it}=\beta\Delta x_{it}+\Delta\nu_{it}$$

with $\Delta y_{it}-\hat{\beta}_{FD}\Delta x_{it}=\Delta\nu_{it}-(\hat{\beta}_{FD}-\beta)\triangle x_{it}$.

Let us look at $\hat{\sigma}^2_{\triangle\nu}$ first.

$$\begin{aligned}
\hat{\sigma}^2_{\triangle\nu} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{\Delta\nu_{it}-(\hat{\beta}_{FD}-\beta)\triangle x_{it}\right\}^2\\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\{(\rho-1)\nu_{it-1}+e_{it}\}-(\hat{\beta}_{FD}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\right]^2\\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\rho-1)\nu_{it-1}+e_{it}]^2+\frac{1}{n^2T^2}[\sqrt{nT}(\hat{\beta}_{FD}-\beta)]^2\sum_{i=1}^{n}\sum_{t=2}^{T}[(\lambda-1)x_{it-1}+\varepsilon_{it}]^2\\
&\quad -\frac{2}{nT}\left[\sqrt{nT}(\hat{\beta}_{FD}-\beta)\right]\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\rho-1)\nu_{it-1}+e_{it}][(\lambda-1)x_{it-1}+\varepsilon_{it}]\\
&= I+II+III.
\end{aligned}$$

Consider I. It can be shown that

$$\begin{aligned}
I &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\rho-1)\nu_{it-1}+e_{it}]^2\\
&= \frac{(\rho-1)^2}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}^2+\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2+\frac{2(\rho-1)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}e_{it}\\
&= (\rho-1)^2\frac{\sigma_e^2}{1-\rho^2}+\sigma_e^2+O_p(\frac{1}{\sqrt{nT}})=\frac{2\sigma_e^2}{1+\rho}
\end{aligned}$$

as $(n,T)\to\infty$ using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}e_{it}=O_p(1).$$

For II and III, one can verify that $II=O_p\left(\frac{1}{nT}\right)$ and $III=O_p\left(\frac{1}{nT}\right)$ using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}(\hat{\beta}_{FD}-\beta)\xrightarrow{d}N(0,\frac{(1+\lambda)^2\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]\sigma_e^2}{4(1-\rho\lambda)^2\sigma_\varepsilon^2})=O_p(1).$$

This uses a similar argument as in Phillips and Moon (1999), also Corollary 5.1 in Baltagi *et al.* (2008). Hence, we have

$$\begin{aligned}
\hat{\sigma}^2_{\triangle\nu} &= (\rho-1)^2\frac{\sigma_e^2}{1-\rho^2}+\sigma_e^2+O_p(\frac{1}{\sqrt{nT}})+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{nT}\right)\\
&= (\rho-1)^2\frac{\sigma_e^2}{1-\rho^2}+\sigma_e^2+O_p(\frac{1}{\sqrt{nT}})\\
&\xrightarrow{p}\frac{2\sigma_e^2}{1+\rho}.
\end{aligned}$$

Now recall that

$$t_{FD}=\frac{\widehat{\beta}_{FD}-\beta_0}{s_{FD}}$$

with $s_{FD}=\sqrt{\frac{\hat{\sigma}^2_{\triangle\nu}}{\sum_{i=1}^n\sum_{t=1}^T(x_{it}-x_{it-1})^2}}$. Here one can easily see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T(x_{it}-x_{it-1})^2\\
=\ &\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T x_{it}^2+\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T x_{it-1}^2-\frac{2}{nT}\sum_{i=1}^n\sum_{t=2}^T x_{it}x_{it-1}\xrightarrow{p}\frac{2\sigma_\varepsilon^2}{1+\lambda}
\end{aligned}$$

as $(n,T)\to\infty$. We conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\begin{aligned}
t_{FD} &= \frac{\sqrt{nT}(\hat{\beta}_{FD}-\beta)}{\sqrt{\hat{\sigma}^2_{\triangle\nu}/\left(\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T(x_{it}-x_{it-1})^2\right)}}\\
&\xrightarrow{d}\frac{N(0,\frac{(1+\lambda)^2\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]\sigma_e^2}{4(1-\rho\lambda)^2\sigma_\varepsilon^2})}{\sqrt{\left(\frac{2\sigma_e^2}{1+\rho}\right)/\left(\frac{2\sigma_\varepsilon^2}{1+\lambda}\right)}}\\
&= N\left(0,\frac{(1+\lambda)(1+\rho)\left[(2-\rho-\lambda)^2+\frac{(1-\rho)^3}{1+\rho}+\frac{(1-\lambda)^3}{1+\lambda}\right]}{4(1-\rho\lambda)^2}\right).
\end{aligned}$$

2. $\rho=1$, $|\lambda|<1$ case

From $\Delta y_{it}-\hat{\beta}_{FD}\Delta x_{it}=\Delta\nu_{it}-(\hat{\beta}_{FD}-\beta)\triangle x_{it}$, one can show that

$$\begin{aligned}
\hat{\sigma}^2_{\triangle v} &= \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\left\{\Delta\nu_{it}-(\hat{\beta}_{FD}-\beta)\triangle x_{it}\right\}^2\\
&= \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\left[e_{it}-(\hat{\beta}_{FD}-\beta)\left\{(\lambda-1)x_{it-1}+\varepsilon_{it}\right\}\right]^2\\
&= \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T e_{it}^2+\frac{1}{n^2T^2}[\sqrt{nT}(\hat{\beta}_{FD}-\beta)]^2\sum_{i=1}^n\sum_{t=2}^T[(\lambda-1)x_{it-1}+\varepsilon_{it}]^2\\
&\quad -\frac{2}{\sqrt{nT}}\left[\sqrt{nT}(\hat{\beta}_{FD}-\beta)\right]\frac{1}{n\sqrt{T}}\sum_{i=1}^n\sum_{t=2}^T e_{it}\left[(\lambda-1)x_{it-1}+\varepsilon_{it}\right]\\
&= I+II+III.
\end{aligned}$$

Obviously, $I = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}^2 \xrightarrow{p} \sigma_e^2$.

For II, using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008), it can be shown that $II = O_p\left(\frac{1}{nT}\right)$ because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, which yields

$$\sqrt{nT}(\hat{\beta}_{FD}-\beta) \xrightarrow{d} N(0, \frac{(1+\lambda)\sigma_e^2}{2\sigma_\varepsilon^2}) = O_p(1).$$

Also note that

$$\frac{(\lambda-1)^2}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T} x_{it-1}^2 = O_p(1),$$
$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}^2 = O_p(1),$$

and

$$\frac{2(\lambda-1)}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T} x_{it-1}\varepsilon_{it} = O_p(1).$$

For III, it is easy to see that

$$III = \frac{2}{nT}\left[\sqrt{nT}(\hat{\beta}_{FD}-\beta)\right]\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}\left[(\lambda-1)x_{it-1}+\varepsilon_{it}\right] = O_p(\frac{1}{nT})$$

using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}x_{it-1} = O_p(1)$$

and

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}\varepsilon_{it} = O_p(1).$$

Then we have,

$$\begin{aligned}\hat{\sigma}_{\triangle v}^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}^2 + O_p\left(\frac{1}{nT}\right) + O_p(\frac{1}{nT}) \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T} e_{it}^2 + O_p(\frac{1}{\sqrt{nT}}) \xrightarrow{p} \sigma_e^2.\end{aligned}$$

Also recall that

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it}-x_{it-1})^2 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it}^2 + \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}^2 - \frac{2}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it}x_{it-1} \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda}$$

as $(n,T)\to\infty$. Hence, we conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$t_{FD} = \frac{\sqrt{nT}(\hat{\beta}_{FD}-\beta)}{\sqrt{\hat{\sigma}_{\triangle\nu}^2/\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-x_{it-1})^2\right)}} \xrightarrow{d} \frac{N(0, \frac{(1+\lambda)\sigma_e^2}{2\sigma_\varepsilon^2})}{\sqrt{\sigma_e^2/\left(\frac{2\sigma_\varepsilon^2}{1+\lambda}\right)}} = N(0,1).$$

3. $|\rho| < 1,\ \lambda = 1$ case

By a similar argument as above, it can be easily shown that

$$\begin{aligned}
\hat{\sigma}^2_{\triangle v} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{\Delta\nu_{it} - (\hat{\beta}_{FD} - \beta)\triangle x_{it}\right\}^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\{(\rho-1)\nu_{it-1} + e_{it}\} - (\hat{\beta}_{FD} - \beta)\varepsilon_{it}\right]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\rho-1)\nu_{it-1} + e_{it}]^2 + \frac{1}{n^2T^2}[\sqrt{nT}(\hat{\beta}_{FD} - \beta)]^2\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}^2 \\
&\quad -\frac{2}{nT}\left[\sqrt{nT}(\hat{\beta}_{FD} - \beta)\right]\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}[(\rho-1)\nu_{it-1} + e_{it}]\,\varepsilon_{it} \\
&= \frac{2\sigma_e^2}{1+\rho} + O_p\left(\frac{1}{nT}\right) + O_p\left(\frac{1}{nT}\right) = \frac{2\sigma_e^2}{1+\rho} + O_p\left(\frac{1}{nT}\right).
\end{aligned}$$

This is because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}(\hat{\beta}_{FD} - \beta) \xrightarrow{d} N\left(0, \frac{2\sigma_e^2}{(1+\rho)\sigma_\varepsilon^2}\right) = O_p(1)$$

using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008). Also recall that

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it} - x_{it-1})^2 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}^2 \xrightarrow{p} \sigma_\varepsilon^2.$$

Hence, if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, we have

$$t_{FD} = \frac{\sqrt{nT}(\hat{\beta}_{FD} - \beta)}{\sqrt{\hat{\sigma}^2_{\triangle\nu}/\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - x_{it-1})^2\right)}} \xrightarrow{d} \frac{N(0, \frac{2\sigma_e^2}{(1+\rho)\sigma_\varepsilon^2})}{\sqrt{\left(\frac{2\sigma_e^2}{1+\rho}\right)/\sigma_\varepsilon^2}} = N(0,1).$$

4. $\rho = 1,\ \lambda = 1$ case

Again, it can be easily shown that

$$\begin{aligned}
\hat{\sigma}^2_{\triangle\nu} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[\Delta\nu_{it} - (\hat{\beta}_{FD} - \beta)\triangle x_{it}\right]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}[e_{it} - (\hat{\beta}_{FD} - \beta)\varepsilon_{it}]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2 + \frac{1}{n^2T^2}[\sqrt{nT}(\hat{\beta}_{FD} - \beta)]\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}^2 - \frac{2}{nT}[\sqrt{nT}(\hat{\beta}_{FD} - \beta)]\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}\varepsilon_{it} \\
&= \sigma_e^2 + O_p\left(\frac{1}{nT}\right) + O_p\left(\frac{1}{nT}\right) \\
&= \sigma_e^2 + O_p\left(\frac{1}{nT}\right) \xrightarrow{p} \sigma_e^2
\end{aligned}$$

because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}(\hat{\beta}_{FD}-\beta)\xrightarrow{d} N(0,\frac{\sigma_e^2}{\sigma_\varepsilon^2})=O_p(1)$$

using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008). Hence, we conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$t_{FD}=\frac{\sqrt{nT}(\hat{\beta}_{FD}-\beta)}{\sqrt{\hat{\sigma}^2_{\triangle\nu}/\left(\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-x_{it-1}\right)^2\right)}}\xrightarrow{d}\frac{N(0,\sigma_e^2/\sigma_\varepsilon^2)}{\sqrt{\sigma_e^2/\sigma_\varepsilon^2}}=N(0,1).$$

■

D Proof of Lemma 2

The OLS estimator of β is given by

$$\widehat{\beta}_{OLS}=\frac{\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)\left(y_{it}-\overline{\overline{y}}\right)}{\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)^2}$$

where $\overline{\overline{x}}=\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T x_{it}$ and $\overline{\overline{y}}=\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T y_{it}$. Rewriting the equation, we have

$$\widehat{\beta}_{OLS}-\beta=\frac{\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)\left(v_{it}-\overline{\overline{v}}\right)}{\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)^2}.$$

Proof. We consider the denominator first and then move to the numerator to prove Lemma 2.[1]

1. The denominator

   (a) when $|\lambda|<1$,

   $$\begin{aligned}\tfrac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)^2 &= \tfrac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T x_{it}^2-\left(\tfrac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T x_{it}\right)^2\\ &\xrightarrow{p} \tfrac{\sigma_\varepsilon^2}{1-\lambda^2}\end{aligned}$$

   as $(n,T)\to\infty$ using $\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T x_{it}^2=\frac{\sigma_\varepsilon^2}{1-\lambda^2}$ and $\frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=1}^T x_{it}=O_p(1)$.

   (b) When $\lambda=1$,

   $$\begin{aligned}\tfrac{1}{nT^2}\sum_{i=1}^n\sum_{t=1}^T\left(x_{it}-\overline{\overline{x}}\right)^2 &= \tfrac{1}{n}\sum_{i=1}^n\sum_{t=1}^T\left(\tfrac{x_{it}}{\sqrt{T}}\right)^2\tfrac{1}{T}-\left[\tfrac{1}{n}\sum_{i=1}^n\sum_{t=1}^T\tfrac{x_{it}}{\sqrt{T}}\tfrac{1}{T}\right]^2\\ &\xrightarrow{p} \tfrac{\sigma_\varepsilon^2}{2}\end{aligned}$$

   as $(n,T)\to\infty$.

2. The numerator

[1] Note that μ_i is not included in error term here.

(a) If $|\rho| < 1$, $|\lambda| < 1$, it can be shown that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\nu_{it}-\overline{\overline{\nu}}\right)=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\nu_{it}+O_p(\frac{1}{\sqrt{nT}})$$

and

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\nu_{it}\xrightarrow{d}N(0,\frac{\sigma_\varepsilon^2}{(1-\lambda^2)}\frac{\sigma_e^2}{(1-\rho^2)}\frac{(1+\rho\lambda)}{(1-\rho\lambda)})$$

as $(n,T)\to\infty$.

(b) If $\rho=1$, $|\lambda|<1$, it can be shown that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\nu_{it}-\overline{\overline{\nu}}\right)=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\nu_{it}+O_p(\frac{1}{\sqrt{n}})$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\frac{\nu_{it}}{\sqrt{T}}\xrightarrow{d}N(0,\frac{\sigma_\varepsilon^2\sigma_e^2}{2(1-\lambda)^2})$$

as $(n,T)\to\infty$.

(c) If $|\rho|<1$, $\lambda=1$, again we have

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\nu_{it}-\overline{\overline{\nu}}\right)=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\nu_{it}+O_p(\frac{1}{\sqrt{n}})$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\frac{\nu_{it}}{\sqrt{T}}\xrightarrow{d}N(0,\frac{\sigma_e^2\sigma_\varepsilon^2}{2(1-\rho)^2})$$

as $(n,T)\to\infty$.

(d) If $\rho=1$, $\lambda=1$,

$$\frac{1}{\sqrt{nT^2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\nu_{it}-\overline{\overline{\nu}}\right)=\frac{1}{\sqrt{nT^2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\nu_{it}+O_p(\frac{1}{\sqrt{n}}).$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\frac{\nu_{it}}{\sqrt{T}}\frac{1}{T}\xrightarrow{d}N(0,\frac{\sigma_e^2\sigma_\varepsilon^2}{6})$$

as $(n,T)\to\infty$.

■

Using the results above, the proof of Lemma 2 is straightforward

E Proof of Lemma 3

In this section, we consider the limiting distribution of ρ using OLS residuals and we check the consistency of σ_e^2 under nonstationarity of both the error term and the regressor.

Proof. Assume $(n,T)\to\infty$.

1. $|\rho|<1$ case

(a) $|\rho| < 1$, $|\lambda| < 1$ case
Recall

$$\widetilde{\rho} - \rho = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}(\widehat{u}_{it} - \rho\widehat{u}_{it-1})\widehat{u}_{it-1}}{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^{2}}$$

where $\widehat{u}_{it} = (y_{it} - \bar{\bar{y}}) - \hat{\beta}_{OLS}(x_{it} - \bar{\bar{x}}) = \left(\nu_{it} - \overline{\overline{\nu}}\right) - \left(\hat{\beta}_{OLS} - \beta\right)(x_{it} - \bar{\bar{x}})$.
For the denominator,

$$\begin{aligned}
\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^{2} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1} - \overline{\overline{\nu}}) - (\hat{\beta}_{OLS} - \beta)(x_{it-1} - \bar{\bar{x}})\right]^{2} \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\overline{\nu}})^{2} + \frac{1}{nT}\left(\sqrt{nT}(\hat{\beta}_{OLS} - \beta)\right)^{2}\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1} - \bar{\bar{x}})^{2}\right\} \\
&\quad - \frac{2}{\sqrt{nT}}\left(\sqrt{nT}(\hat{\beta}_{OLS} - \beta)\right)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\overline{\nu}})(x_{it-1} - \bar{\bar{x}}) \\
&= I + II + III.
\end{aligned}$$

Consider II first. It is easy to see that $II = O_p\left(\frac{1}{nT}\right)$ because if $(n,T) \to \infty$ and $\frac{n}{T} \to 0$, then

$$\sqrt{nT}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, \frac{(1+\rho\lambda)(1-\lambda^{2})\sigma_{e}^{2}}{(1-\rho\lambda)(1-\rho^{2})\sigma_{\varepsilon}^{2}}\right).$$

Also, by Lemma 2.(1), we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1} - \bar{\bar{x}})^{2} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it}^{2} - \left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it}\right]^{2} \xrightarrow{p} \frac{\sigma_{\varepsilon}^{2}}{1-\lambda^{2}}.$$

$III = O_p\left(\frac{1}{nT}\right)$ because

$$\frac{1}{nT}\left(\sqrt{nT}(\hat{\beta}_{OLS} - \beta)\right)\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\overline{\nu}})(x_{it-1} - \bar{\bar{x}}) = O_p\left(\frac{1}{nT}\right)$$

as $(n,T) \to \infty$ using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1} - \overline{\overline{\nu}})(x_{it-1} - \bar{\bar{x}}) = O_p(1)$$

since ν_{it} and x_{it} are not correlated.
Hence,

$$\begin{aligned}
\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^{2} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(v_{it-1} - \bar{\bar{v}})^{2} + O_p\left(\frac{1}{nT}\right) + O_p\left(\frac{1}{nT}\right) \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(v_{it-1} - \bar{\bar{v}})^{2} + O_p\left(\frac{1}{nT}\right) \xrightarrow{p} \frac{\sigma_{e}^{2}}{1-\rho^{2}}.
\end{aligned}$$

Let us look at the numerator. Because $\widehat{u}_{it} - \rho\widehat{u}_{it-1} = e_{it} - (\hat{\beta}_{OLS} - \beta)\{(\lambda - \rho)x_{it-1} + \varepsilon_{it}\} + o_p(1)$, we have

$$
\begin{aligned}
& \tfrac{1}{\sqrt{nT}} \textstyle\sum_{i=1}^{n} \sum_{t=2}^{T} \left(\widehat{u}_{it} - \rho \widehat{u}_{it-1} \right) \widehat{u}_{it-1} \\
= & \tfrac{1}{\sqrt{nT}} \textstyle\sum_{i=1}^{n} \sum_{t=2}^{T} \left[e_{it} - (\hat{\beta}_{OLS} - \beta) \left\{ (\lambda - \rho) x_{it-1} + \varepsilon_{it} \right\} \right] \left[\left(\nu_{it-1} - \overline{\overline{\nu}} \right) - \left(\hat{\beta}_{OLS} - \beta \right) \left(x_{it-1} - \bar{\bar{x}} \right) \right] \\
= & \tfrac{1}{\sqrt{nT}} \textstyle\sum_{i=1}^{n} \sum_{t=2}^{T} \left(\nu_{it-1} - \overline{\overline{\nu}} \right) e_{it} - \sqrt{nT} (\hat{\beta}_{OLS} - \beta) \tfrac{1}{n} \sum_{i=1}^{n} \tfrac{1}{T} \sum_{t=2}^{T} \left\{ (\lambda - \rho) x_{it-1} + \varepsilon_{it} \right\} \left(\nu_{it-1} - \overline{\overline{\nu}} \right) \\
& - \sqrt{nT} \left(\hat{\beta}_{OLS} - \beta \right) \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \tfrac{1}{T} \sum_{t=2}^{T} \left(x_{it-1} - \bar{\bar{x}} \right) e_{it} \\
& + \left(\sqrt{nT} \left(\hat{\beta}_{OLS} - \beta \right) \right)^2 \tfrac{1}{n^{3/2}} \textstyle\sum_{i=1}^{n} \tfrac{1}{T^{3/2}} \sum_{t=2}^{T} \left\{ (\lambda - \rho) x_{it-1} + \varepsilon_{it} \right\} \left(x_{it-1} - \bar{\bar{x}} \right) \\
= & I + II + III + IV.
\end{aligned}
$$

Consider I. One can see that

$$
\begin{aligned}
I &= \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \left[\left(\nu_{it-1} - \overline{\overline{\nu}} \right) e_{it} \right] \\
&= \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1} e_{it} - \frac{1}{\sqrt{nT}} \left[\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it} \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} e_{it} \right) \right] \\
&= \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1} e_{it} + O_p \left(\frac{1}{\sqrt{nT}} \right) \xrightarrow{d} \sigma_e^2 N \left(0, \frac{1}{1 - \rho^2} \right).
\end{aligned}
$$

Consider II.

$$
\begin{aligned}
II &= -\sqrt{nT} (\hat{\beta}_{OLS} - \beta) \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=2}^{T} \left\{ (\lambda - \rho) x_{it-1} + \varepsilon_{it} \right\} \left(\nu_{it-1} - \overline{\overline{\nu}} \right) \\
&= -\sqrt{nT} (\hat{\beta}_{OLS} - \beta) \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=2}^{T} \left[(\lambda - \rho) x_{it-1} \nu_{it-1} - (\lambda - \rho) x_{it-1} \overline{\overline{\nu}} + \varepsilon_{it} \nu_{it-1} - \varepsilon_{it} \overline{\overline{\nu}} \right] \\
&= O_p \left(\frac{1}{\sqrt{nT}} \right) + O_p \left(\frac{1}{nT} \right) + O_p \left(\frac{1}{\sqrt{nT}} \right) + O_p \left(\frac{1}{nT} \right) = O_p \left(\frac{1}{\sqrt{nT}} \right)
\end{aligned}
$$

as $(n, T) \to \infty$ using

$$
\begin{gathered}
\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} x_{it-1} \nu_{it-1} = O_p(1), \\
\left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} x_{it-1} \right) \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1} \right) = O_p(1), \\
\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \varepsilon_{it} \nu_{it-1} = O_p(1),
\end{gathered}
$$

and

$$
\left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \varepsilon_{it} \right) \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1} \right) = O_p(1).
$$

Consider III. Using a similar argument, it can be shown that

$$
III = -\sqrt{nT} \left(\hat{\beta}_{OLS} - \beta \right) \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=2}^{T} \left(x_{it-1} - \bar{\bar{x}} \right) e_{it} = O_p \left(\frac{1}{\sqrt{nT}} \right).
$$

Consider IV.

$$\begin{aligned}
& \left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2 \frac{1}{n^{3/2}T^{3/2}} \sum_{i=1}^{n}\sum_{t=2}^{T} \left\{(\lambda-\rho)x_{it-1}+\varepsilon_{it}\right\}\left(x_{it-1}-\bar{\bar{x}}\right) \\
= & \left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2 \frac{1}{\sqrt{nT}}\frac{1}{nT} \sum_{i=1}^{n}\sum_{t=2}^{T} \left[(\lambda-\rho)x_{it-1}^2-(\lambda-\rho)x_{it-1}\bar{\bar{x}}+\varepsilon_{it}x_{it-1}-\varepsilon_{it}\bar{\bar{x}}\right] \\
= & O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{n^{3/2}T^{3/2}}\right)+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{n^{3/2}T^{3/2}}\right)=O_p\left(\frac{1}{\sqrt{nT}}\right)
\end{aligned}$$

as $(n,T)\to\infty$.
Hence, we have

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\rho\widehat{u}_{it-1}\right)\widehat{u}_{it-1}+O_p\left(\frac{1}{\sqrt{nT}}\right)\xrightarrow{d}\sigma_e^2 N\left(0,\frac{1}{1-\rho^2}\right)$$

as $(n,T)\to\infty$.
Therefore, we conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\widetilde{\rho}-\rho\right)=\frac{\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\rho\widehat{u}_{it-1}\right)\widehat{u}_{it-1}}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2}\xrightarrow{d}\frac{\sigma_e^2 N\left(0,\frac{1}{1-\rho^2}\right)}{\frac{\sigma_e^2}{1-\rho^2}}=N\left(0,1-\rho^2\right).$$

Next we show $\widetilde{\sigma}_e^2$ is a consistent estimator. Note that

$$\widehat{\mathbf{u}}=\mathbf{y}-\widehat{\gamma}_{OLS}\boldsymbol{\iota}_{nT}-\mathbf{x}\hat{\beta}_{OLS}=E_{nT}\left[\boldsymbol{\nu}+\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right]$$

where $E_{nT}=I_{nT}-\bar{J}_{nT}$ and $\bar{J}_{nT}=\boldsymbol{\iota}_{nT}\boldsymbol{\iota}_{nT}'/nT$. Hence,

$$\begin{aligned}
\widetilde{\sigma}_e^2 &= \frac{1}{n(T-1)}\widehat{\mathbf{u}}^{*\prime}\widehat{\mathbf{u}}^{*}\approx\frac{1}{nT}\widehat{\mathbf{u}}^{*\prime}\widehat{\mathbf{u}}^{*} \\
&= \frac{1}{nT}\widehat{\mathbf{u}}'\left(I_n\otimes\hat{C}'\right)\left(I_n\otimes\hat{C}\right)\widehat{\mathbf{u}} \\
&= \frac{1}{nT}\left[\boldsymbol{\nu}+\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right]'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\left[\boldsymbol{\nu}+\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \\
&= \frac{1}{nT}\boldsymbol{\nu}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu}+\frac{1}{nT}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x} \\
&\quad +\frac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu} \\
&= I+II+III
\end{aligned}$$

using $\widehat{\mathbf{u}}^*=\left(I_n\otimes\hat{C}\right)\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}}=E_{nT}\left[\boldsymbol{\nu}+\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right]$.
To rearrange the terms, note that

$$\begin{aligned}
E_{nT} &= I_{nT}-\bar{J}_{nT} \\
&= E_n\otimes I_T+\bar{J}_n\otimes I_T-\bar{J}_n\otimes\bar{J}_T \\
&= E_n\otimes I_T+\bar{J}_n\otimes E_T
\end{aligned}$$

and accordingly

$$\begin{aligned}
E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT} &= \left(E_n\otimes I_T+\bar{J}_n\otimes E_T\right)\left(I_n\otimes\hat{C}'\hat{C}\right)\left(E_n\otimes I_T+\bar{J}_n\otimes E_T\right) \\
&= E_n\otimes\hat{C}'\hat{C}+\bar{J}_n\otimes E_T\hat{C}'\hat{C}E_T.
\end{aligned}$$

Consider I.

$$\begin{aligned}
I &= \frac{1}{nT}\boldsymbol{\nu}' E_{nT}\left(I_n \otimes \hat{C}'\hat{C}\right) E_{nT}\boldsymbol{\nu} \\
&= \frac{1}{nT}\boldsymbol{\nu}'\left(E_n \otimes \hat{C}'\hat{C}\right)\boldsymbol{\nu} + \frac{1}{nT}\boldsymbol{\nu}'\left(\bar{J}_n \otimes E_T\hat{C}'\hat{C}E_T\right)\boldsymbol{\nu} \\
&= \frac{1}{nT}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{C}\right)\boldsymbol{\nu} - \frac{1}{nT}\boldsymbol{\nu}'\left(\bar{J}_n \otimes \hat{C}'\hat{C}\right)\boldsymbol{\nu} + \frac{1}{nT}\boldsymbol{\nu}'\left(\bar{J}_n \otimes E_T\hat{C}'\hat{C}E_T\right)\boldsymbol{\nu} \\
&\approx \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it} - \widetilde{\rho}\nu_{it-1}\right)^2 - \left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it} - \widetilde{\rho}\nu_{it-1}\right)\right\}^2 \\
&\quad + \left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left((\nu_{it} - \overline{\nu}_{i.}) - \widetilde{\rho}\left(\nu_{it-1} - \overline{\nu}_{i.}\right)\right)\right\}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2 - 2\left(\widetilde{\rho} - \rho\right)\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1} + \left(\widetilde{\rho} - \rho\right)^2\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^2\right] \\
&\quad - \left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it} - \left(\widetilde{\rho} - \rho\right)\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]\right\}^2 \\
&\quad + \left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it} - \frac{1}{T}\sum_{t=1}^{T}\left(\widetilde{\rho} - \rho\right)\nu_{it-1} - \frac{1}{T}\sum_{t=1}^{T}(1 - \widetilde{\rho})\overline{\nu}_{i.}\right]\right\}^2.
\end{aligned}$$

Now it is easy to see that

$$I = \sigma_e^2 + o_p(1)$$

using

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}^2 \xrightarrow{p} \sigma_e^2,$$

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}^2 \xrightarrow{p} \frac{\sigma_e^2}{(1 - \rho^2)},$$

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it} = O_p(1),$$

and

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(1 - \widetilde{\rho})\overline{\nu}_{i.} = O_p(1)$$

as $(n, T) \to \infty$ with $\sqrt{nT}\left(\widetilde{\rho} - \rho\right) = O_p(1)$.

Consider II. In a similar vein as I, it is easy to see that

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}' E_{nT}\left(I_n \otimes \hat{C}'\hat{C}\right) E_{nT}\mathbf{x} \\
&= \frac{1}{nT}\mathbf{x}'\left(E_n \otimes \hat{C}'\hat{C}\right)\mathbf{x} + \frac{1}{nT}\mathbf{x}'\left(\bar{J}_n \otimes E_T\hat{C}'\hat{C}E_T\right)\mathbf{x} \\
&= \frac{1}{nT}\mathbf{x}'\left(I_n \otimes \hat{C}'\hat{C}\right)\mathbf{x} - \frac{1}{nT}\mathbf{x}'\left(\bar{J}_n \otimes \hat{C}'\hat{C}\right)\mathbf{x} + \frac{1}{nT}\mathbf{x}'\left(\bar{J}_n \otimes E_T\hat{C}'\hat{C}E_T\right)\mathbf{x}.
\end{aligned}$$

Expanding this equation, one can show that

$$
\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^{2}-\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)\right\}^{2} \\
& +\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\left(x_{it}-\overline{x}_{i.}\right)-\widetilde{\rho}\left(x_{it-1}-\overline{x}_{i.}\right)\right)\right\}^{2} \\
= & \frac{1}{nT}\sum_{i=1}^{n}\left\{\begin{array}{c}\sum_{t=1}^{T}\varepsilon_{it}^{2}+(\widetilde{\rho}-\rho)^{2}\sum_{t=1}^{T}x_{it-1}^{2}+(\lambda-\rho)^{2}\sum_{t=1}^{T}x_{it-1}^{2}-2\left(\widetilde{\rho}-\rho\right)\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} \\ +2(\lambda-\rho)\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}-2\left(\widetilde{\rho}-\rho\right)\left(\lambda-\rho\right)\sum_{t=1}^{T}x_{it-1}^{2}\end{array}\right\} \\
& -\left[\frac{1}{nT}\sum_{i=1}^{n}\left\{\sum_{t=1}^{T}\varepsilon_{it}-\left(\widetilde{\rho}-\rho\right)\sum_{t=1}^{T}x_{it-1}+(\lambda-\rho)\sum_{t=1}^{T}x_{it-1}\right\}\right]^{2} \\
& +\left[\frac{1}{nT}\sum_{i=1}^{n}\left\{\sum_{t=1}^{T}\varepsilon_{it}-\sum_{t=1}^{T}\left(\widetilde{\rho}-\rho\right)x_{it-1}+(\lambda-\rho)\sum_{t=1}^{T}x_{it-1}-\sum_{t=1}^{T}(1-\widetilde{\rho})\overline{x}_{i.}\right\}\right]^{2} \\
= & \sigma_{\varepsilon}^{2}+\frac{\sigma_{\varepsilon}^{2}(\lambda-\rho)^{2}}{\left(1-\lambda^{2}\right)}+o_{p}(1)
\end{aligned}
$$

as $(n,T)\to\infty$. This is because in the first term

$$
\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}\xrightarrow{p}\sigma_{\varepsilon}^{2}, \\
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}=O_{p}\left(1\right), \\
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}\xrightarrow{p}\sigma_{\varepsilon}^{2}/\left(1-\lambda^{2}\right),
\end{aligned}
$$

and $\sqrt{nT}\left(\widetilde{\rho}-\rho\right)=O_{p}\left(1\right)$. Also note that $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}=O_{p}\left(1\right)$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}=O_{p}\left(1\right)$, and $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(1-\widetilde{\rho})\overline{x}_{i.}=O_{p}\left(1\right)$ as $(n,T)\to\infty$. Now we get

$$
II\approx\frac{1}{nT}\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^{2}\left[\frac{1}{nT}\mathbf{x}'E_{nT}\left(I_{n}\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x}\right]=O_{p}\left(\frac{1}{nT}\right)=o_{p}\left(1\right)
$$

because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$
\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{(1+\rho\lambda)(1-\lambda^{2})\sigma_{e}^{2}}{(1-\rho\lambda)(1-\rho^{2})\sigma_{\varepsilon}^{2}}\right)
$$

and accordingly

$$
\frac{1}{nT}\mathbf{x}'E_{nT}\left(I_{n}\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x}=O_{p}\left(1\right).
$$

Consider III.

$$
\begin{aligned}
III & = \frac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{x}'E_{nT}\left(I_{n}\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu} \\
& \leq 2\sqrt{\left[\frac{1}{nT}\boldsymbol{\nu}'E_{nT}\left(I_{n}\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu}\right]\left[\frac{1}{nT}\left(\hat{\beta}_{OLS}-\beta\right)^{2}\mathbf{x}'E_{nT}\left(I_{n}\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x}\right]} \\
& = 2\sqrt{I\times II}=2\sqrt{\sigma_{e}^{2}\times 0}=0
\end{aligned}
$$

as $(n,T)\to\infty$ using the Cauchy-Schwarz inequality.
Summarizing, we proved that

$$\widetilde{\sigma}_e^2 \xrightarrow{p} \sigma_e^2.$$

(b) $|\rho|<1$, $\lambda=1$ case

This is the *panel cointegration* case. Consider $\widetilde{\rho}-\rho=\frac{\sum_{i=1}^n\sum_{t=2}^T(\widehat{u}_{it}-\rho\widehat{u}_{it-1})\widehat{u}_{it-1}}{\sum_{i=1}^n\sum_{t=2}^T\widehat{u}_{it-1}^2}$.

For the denominator,

$$\begin{aligned}
\tfrac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\widehat{u}_{it-1}^2 &= \tfrac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\left[(\nu_{it-1}-\overline{\overline{\nu}})-(\hat{\beta}_{OLS}-\beta)(x_{it-1}-\overline{\overline{x}})\right]^2\\
&= \tfrac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})^2\\
&\quad +\tfrac{1}{nT}\left\{\sqrt{n}T(\hat{\beta}_{OLS}-\beta)\right\}^2\left\{\tfrac{1}{nT^2}\sum_{i=1}^n\sum_{t=2}^T(x_{it-1}-\overline{\overline{x}})^2\right\}\\
&\quad -2\sqrt{n}T(\hat{\beta}_{OLS}-\beta)\tfrac{1}{n^{3/2}T^2}\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})(x_{it-1}-\overline{\overline{x}})\\
&= I+II+III.
\end{aligned}$$

Consider II first. With the joint limit, one can verify $II=O_p\left(\frac{1}{nT}\right)$ using that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}T\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{(1-\rho)^2\sigma_\varepsilon^2}\right)$$

by Lemma 2.(3). Also note that as $(n,T)\to\infty$, we have

$$\frac{1}{nT^2}\sum_{i=1}^n\sum_{t=2}^T(x_{it-1}-\overline{\overline{x}})^2=\left[\frac{1}{n}\sum_{i=1}^n\frac{1}{T^2}\sum_{t=2}^T x_{it-1}^2\right]-\left[\frac{1}{n}\sum_{i=1}^n\frac{1}{T^{3/2}}\sum_{t=2}^T x_{it}\right]^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{2}=O_p(1).$$

Next, $III=O_p\left(\frac{1}{nT}\right)$ using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=2}^T\left(\nu_{it-1}-\overline{\overline{\nu}}\right)\left(\frac{x_{it-1}-\overline{\overline{x}}}{\sqrt{T}}\right)=O_p(1)$$

since ν_{it} and x_{it} are not correlated.
Hence, we have

$$\begin{aligned}
\frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T\widehat{\nu}_{it-1}^2 &= \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})^2+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{nT}\right)\\
&= \frac{1}{nT}\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})^2+O_p\left(\frac{1}{nT}\right)\\
&\xrightarrow{p}\frac{\sigma_e^2}{1-\rho^2}
\end{aligned}$$

as $(n,T)\to\infty$.
For the numerator, $\widehat{u}_{it}-\rho\widehat{u}_{it-1}=e_{it}-(\hat{\beta}_{OLS}-\beta)\{(1-\rho)x_{it-1}+\varepsilon_{it}\}+o_p(1)$ and

$$\begin{aligned}
&\tfrac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=2}^T\left(\widehat{u}_{it}-\rho\widehat{u}_{it-1}\right)\widehat{u}_{it-1}\\
=&\tfrac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=2}^T\left[e_{it}-(\hat{\beta}_{OLS}-\beta)\{(1-\rho)x_{it-1}+\varepsilon_{it}\}+o_p(1)\right]\left[\left(\nu_{it-1}-\overline{\overline{\nu}}\right)-\left(\hat{\beta}_{OLS}-\beta\right)(x_{it-1}-\overline{\overline{x}})\right]\\
=&\tfrac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=2}^T\left(\nu_{it-1}-\overline{\overline{\nu}}\right)e_{it}-\tfrac{\sqrt{n}T\left(\hat{\beta}_{OLS}-\beta\right)}{nT^{3/2}}\sum_{i=1}^n\sum_{t=2}^T\{(1-\rho)x_{it-1}+\varepsilon_{it}\}\left(\nu_{it-1}-\overline{\overline{\nu}}\right)\\
&-\tfrac{\sqrt{n}T\left(\hat{\beta}_{OLS}-\beta\right)}{nT^{3/2}}\sum_{i=1}^n\sum_{t=2}^T\left(x_{it-1}-\overline{\overline{x}}\right)e_{it}\\
&+\tfrac{\left\{\sqrt{n}T\left(\hat{\beta}_{OLS}-\beta\right)\right\}^2}{n^{3/2}T^{5/2}}\sum_{i=1}^n\sum_{t=2}^T\{(1-\rho)x_{it-1}+\varepsilon_{it}\}\left(x_{it-1}-\overline{\overline{x}}\right)\\
=&I+II+III+IV.
\end{aligned}$$

Consider II first. Again, it can be shown that $II = O_p\left(\frac{1}{\sqrt{nT}}\right)$ because

$$\begin{aligned}
& \frac{1}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=2}^{T} \{(1-\rho)x_{it-1} + \varepsilon_{it}\} \left(\nu_{it-1} - \overline{\overline{\nu}}\right) \\
= \; & \frac{(1-\rho)}{\sqrt{nT}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=2}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{\nu_{it-1}}{\sqrt{T}} - \frac{(1-\rho)}{n\sqrt{T}} \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1}\right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=2}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{1}{T}\right) \\
& + \frac{1}{\sqrt{nT}} \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \varepsilon_{it}\nu_{it-1} - \frac{1}{nT^{3/2}} \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \varepsilon_{it}\right) \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} \nu_{it-1}\right) \\
= \; & O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{n\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{nT^{3/2}}\right) \\
= \; & O_p\left(\frac{1}{\sqrt{nT}}\right)
\end{aligned}$$

as $(n, T) \to \infty$.
Consider III.

$$\begin{aligned}
III &= -\frac{\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right)}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=2}^{T} \left(x_{it-1} - \overline{\overline{x}}\right) e_{it} \\
&= -\frac{\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right)}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=2}^{T} x_{it-1}e_{it} + \frac{\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right)}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=2}^{T} e_{it}\overline{\overline{x}} \\
&= -\frac{\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right)}{\sqrt{nT}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=2}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{e_{it}}{\sqrt{T}} \\
&\quad + \frac{\sqrt{n}T\left(\hat{\beta}_{OLS} - \beta\right)}{n\sqrt{T}} \left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=2}^{T} e_{it}\right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{x_{it}}{\sqrt{T}} \frac{1}{T}\right) \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{n\sqrt{T}}\right) = O_p\left(\frac{1}{\sqrt{nT}}\right).
\end{aligned}$$

Consider IV.

$$\begin{aligned}
IV &= \frac{\left\{\sqrt{n}T(\hat{\beta}_{OLS} - \beta)\right\}^2}{n^{3/2}T^{5/2}} \sum_{i=1}^{n} \sum_{t=2}^{T} \{(1-\rho)x_{it-1} + \varepsilon_{it}\} \left(x_{it-1} - \overline{\overline{x}}\right) \\
&= \frac{\left\{\sqrt{n}T(\hat{\beta}_{OLS} - \beta)\right\}^2}{n^{3/2}} \sum_{i=1}^{n} \frac{1}{T^{5/2}} \sum_{t=2}^{T} \left[(1-\rho)x_{it-1}^2 - (1-\rho)x_{it-1}\overline{\overline{x}} + \varepsilon_{it}x_{it-1} - \varepsilon_{it}\overline{\overline{x}}\right] \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right)
\end{aligned}$$

using

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^2} \sum_{t=2}^{T} x_{it-1}^2 &= O_p(1), \\
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=2}^{T} x_{it-1} &= O_p(1),
\end{aligned}$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}x_{it-1}=O_p(1).$$

Hence, one can see that

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\rho\widehat{u}_{it-1}\right)\widehat{u}_{it-1}\\&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\overline{\overline{\nu}}\right)e_{it}+o_p(1)\end{aligned}$$

and we conclude that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\xrightarrow{d}\frac{\sigma_e^2 N\left(0,\frac{1}{1-\rho^2}\right)}{\frac{\sigma_e^2}{1-\rho^2}}=N\left(0,1-\rho^2\right).$$

We check the consistency of $\widetilde{\sigma}_e^2$ next. From Lemma 3.1.(a), we know

$$I\to\sigma_e^2.$$

Moreover, it can be easily shown that $II=o_p(1)$ because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}T\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{(1-\rho)^2\sigma_\varepsilon^2}\right)$$

and

$$\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\xrightarrow{d}N\left(0,1-\rho^2\right).$$

Also note that $III=o_p(1)$ with Cauchy-Schwarz inequality.
We proved

$$\widetilde{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

2. $\rho=1$ case

   (a) $\rho=1$, $|\lambda|<1$ case
   Here we have

$$\widetilde{\rho}-1=\frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1}}{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2}.$$

   For the denominator,

$$\begin{aligned}\tfrac{1}{nT^2}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2 &= \tfrac{1}{nT^2}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[(\nu_{it-1}-\overline{\overline{\nu}})-(\hat{\beta}_{OLS}-\beta)(x_{it-1}-\overline{\overline{x}})\right]^2\\ &= \tfrac{1}{nT^2}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\overline{\nu}})^2\\ &\quad+\left\{\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right\}^2\tfrac{1}{n^2T^2}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\overline{\overline{x}})^2\\ &\quad-\sqrt{n}(\hat{\beta}_{OLS}-\beta)\tfrac{2}{n^{3/2}T^2}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\overline{\nu}})(x_{it-1}-\overline{\overline{x}})\\ &= I+II+III.\end{aligned}$$

   Consider II first. It is easy to see $II=O_p\left(\frac{1}{nT}\right)$ because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{(1+\lambda)^2\sigma_e^2}{2\sigma_\varepsilon^2}\right)=O_p(1)$$

by Lemma 2.(2) and because as $(n,T)\to\infty$, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\bar{\bar{x}})^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2}.$$

Next consider III. One can show that

$$\begin{aligned} III &= -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{2}{n^{3/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it}-\bar{\bar{\nu}})(x_{it}-\bar{\bar{x}}) \\ &= -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{nT}\frac{2}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it}-\bar{\bar{x}})(\frac{\nu_{it}-\bar{\bar{\nu}}}{\sqrt{T}}) \\ &= O_p\left(\frac{1}{nT}\right) \end{aligned}$$

since ν_{it} and x_{it} are not correlated and accordingly $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it}-\bar{\bar{x}})(\frac{\nu_{it}-\bar{\bar{\nu}}}{\sqrt{T}})=O_p(1)$. Hence, we have

$$\begin{aligned} \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2 &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\bar{\bar{\nu}})^2+O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{nT}\right) \\ &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\bar{\bar{\nu}})^2+O_p\left(\frac{1}{nT}\right)\xrightarrow{p}\frac{\sigma_e^2}{6} \end{aligned}$$

as $(n,T)\to\infty$ by, e.g., equation (C.3) in Kao (1999).
For the numerator, $\widehat{u}_{it}-\widehat{u}_{it-1}=e_{it}-(\hat{\beta}_{OLS}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}$ and it can be shown that

$$\begin{aligned} & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1} \\ = & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[e_{it}-(\hat{\beta}_{OLS}-\beta)\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\right]\left[\left(\nu_{it-1}-\bar{\bar{\nu}}\right)-\left(\hat{\beta}_{OLS}-\beta\right)\left(x_{it-1}-\bar{\bar{x}}\right)\right] \\ = & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\bar{\bar{\nu}}\right)e_{it}-\sqrt{n}(\hat{\beta}_{OLS}-\beta)\tfrac{1}{n^{3/2}}\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\left(\nu_{it-1}-\bar{\bar{\nu}}\right) \\ & -\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\tfrac{1}{n^{3/2}}\textstyle\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\left(x_{it-1}-\bar{\bar{x}}\right)e_{it} \\ & +\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\tfrac{1}{n^2}\textstyle\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\left(x_{it-1}-\bar{\bar{x}}\right) \\ = & I+II+III+IV. \end{aligned}$$

Consider II.

$$\begin{aligned}
II &= -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\left(\nu_{it-1}-\overline{\overline{\nu}}\right) \\
&= -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{T}\sum_{t=2}^{T}(\lambda-1)x_{it-1}\nu_{it-1}-\frac{1}{T}\sum_{t=2}^{T}(\lambda-1)x_{it-1}\overline{\overline{\nu}} \\ +\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\nu_{it-1}-\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\overline{\overline{\nu}} \end{array}\right\} \\
&= -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{(\lambda-1)}{n}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\right) \\
&\quad +\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{(\lambda-1)}{n^{3/2}}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\nu_{it}}{\sqrt{T}}\frac{1}{T}\right) \\
&\quad -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\right) \\
&\quad +\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n^{3/2}}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\nu_{it}}{\sqrt{T}}\frac{1}{T}\right) \\
&= O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}}\right)=O_p\left(\frac{1}{n}\right)
\end{aligned}$$

using

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)=O_p\left(1\right)$$

as $(n,T)\to\infty$ with the joint CLT.
Consider III.

$$\begin{aligned}
III &= -\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\left(x_{it-1}e_{it}-\overline{\overline{x}}e_{it}\right) \\
&= -\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\frac{1}{n\sqrt{T}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}x_{it-1}e_{it} \\
&\quad +\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\frac{1}{n^{3/2}T}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}\right)\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right) \\
&= O_p\left(\frac{1}{n\sqrt{T}}\right)+O_p\left(\frac{1}{n^{3/2}T}\right)=O_p\left(\frac{1}{n\sqrt{T}}\right).
\end{aligned}$$

Consider IV. After some algebra, it can be shown that

$$\begin{aligned}
IV &= \left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\{(\lambda-1)x_{it-1}+\varepsilon_{it}\}\left(x_{it-1}-\overline{\overline{x}}\right) \\
&= \left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\left[(\lambda-1)x_{it-1}^2-(\lambda-1)x_{it-1}\overline{\overline{x}}+\varepsilon_{it}x_{it-1}-\varepsilon_{it}\overline{\overline{x}}\right] \\
&= O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^2T}\right)+O_p\left(\frac{1}{n^{3/2}\sqrt{T}}\right)+O_p\left(\frac{1}{n^2T}\right)=O_p\left(\frac{1}{n}\right).
\end{aligned}$$

Lastly, consider I.

$$\begin{aligned}
I &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\overline{\overline{\nu}}\right)e_{it} \\
&= \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\nu_{it-1}}{\sqrt{T}}\frac{e_{it}}{\sqrt{T}}-\frac{1}{n}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}\right)\left(\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}\right) \\
&= O_p\left(\frac{1}{\sqrt{n}}\right)+O_p\left(\frac{1}{n}\right)=O_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Therefore,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1}=o_p\left(1\right).$$

We finally have

$$T\left(\widetilde{\rho}-1\right)=\frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta\widehat{u}_{it}\widehat{u}_{it-1}}{\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2}\xrightarrow{p}\frac{0}{\frac{\sigma_e^2}{6}}=0.$$

Next we show $\widetilde{\sigma}_e^2$ is a consistent estimator. Again we have

$$\begin{aligned}
\widetilde{\sigma}_e^2 &= \frac{1}{nT}\boldsymbol{\nu}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu}+\frac{1}{nT}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x} \\
&\quad +\frac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu} \\
&= I+II+III.
\end{aligned}$$

Consider I.

$$\begin{aligned}
I &= \frac{1}{nT}\boldsymbol{\nu}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\boldsymbol{\nu} \\
&\approx \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it}-\widetilde{\rho}\nu_{it-1}\right)^2-\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it}-\widetilde{\rho}\nu_{it-1}\right)\right\}^2 \\
&\quad +\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(\left(\nu_{it}-\overline{\nu}_{i.}\right)-\widetilde{\rho}\left(\nu_{it-1}-\overline{\nu}_{i.}\right)\right)\right\}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2-2T\left(\widetilde{\rho}-1\right)\frac{1}{T^2}\sum_{t=1}^{T}e_{it}\nu_{it-1}+\left\{T\left(\widetilde{\rho}-1\right)\right\}^2\frac{1}{T^3}\sum_{t=1}^{T}\nu_{it-1}^2\right] \\
&\quad -\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-T\left(\widetilde{\rho}-1\right)\frac{1}{T^2}\sum_{t=1}^{T}\nu_{it-1}\right]\right\}^2 \\
&\quad +\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-\frac{T\left(\widetilde{\rho}-1\right)}{T^2}\sum_{t=1}^{T}\nu_{it-1}+\frac{T\left(\widetilde{\rho}-1\right)}{T^2}\sum_{t=1}^{T}\overline{\nu}_{i.}\right]\right\}^2.
\end{aligned}$$

Now it is easy to see that

$$I\to\sigma_e^2$$

as $(n,T)\to\infty$ using $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^2\xrightarrow{p}\sigma_e^2$, $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\nu_{it-1}=O_p(1)$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}^2=O_p(1)$, and $T\left(\widetilde{\rho}-1\right)=o_p\left(1\right)$ with the joint limit.

Consider II. After some tedious algebra, it can be shown that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^2-\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)\right]^2 \\
&+\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\left(x_{it}-\overline{x}_{i.}\right)-\widetilde{\rho}\left(x_{it-1}-\overline{x}_{i.}\right)\right)\right]^2 \\
= \;& \sigma_\varepsilon^2+\frac{\sigma_\varepsilon^2(\lambda-1)^2}{\left(1-\lambda^2\right)}+o_p(1)
\end{aligned}$$

as $(n,T)\to\infty$. Now one can see that

$$II\approx\frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n}\left[\frac{1}{nT}\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x}\right]=O_p\left(\frac{1}{n}\right)=o_p(1)$$

using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{(1+\lambda)^2\sigma_e^2}{2\sigma_\varepsilon^2}\right).$$

Because $III=o_p(1)$ by the Cauchy-Schwarz inequality, we get

$$\widetilde{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

(b) $\rho=1$, $\lambda=1$ case

Note that

$$\widetilde{\rho}-1=\frac{\sum_{i=1}^n\sum_{t=2}^T\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1}}{\sum_{i=1}^n\sum_{t=2}^T\widehat{u}_{it-1}^2}.$$

For the denominator,

$$\begin{aligned}
\tfrac{1}{nT^2}\textstyle\sum_{i=1}^n\sum_{t=2}^T\widehat{u}_{it-1}^2 &= \tfrac{1}{nT^2}\textstyle\sum_{i=1}^n\sum_{t=2}^T\left[(\nu_{it-1}-\overline{\overline{\nu}})-(\hat{\beta}_{OLS}-\beta)(x_{it-1}-\overline{\overline{x}})\right]^2 \\
&= \tfrac{1}{nT^2}\textstyle\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})^2 \\
&\quad+\left(\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right)^2\tfrac{1}{n^2T^2}\textstyle\sum_{i=1}^n\sum_{t=2}^T(x_{it-1}-\overline{\overline{x}})^2 \\
&\quad-\left(\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right)\tfrac{2}{n^{3/2}T^2}\textstyle\sum_{i=1}^n\sum_{t=2}^T(\nu_{it-1}-\overline{\overline{\nu}})(x_{it-1}-\overline{\overline{x}}) \\
&= I+II+III.
\end{aligned}$$

Consider II first. With the joint limit, one can see that

$$\begin{aligned}
&\left(\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right)^2\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(x_{it-1}-\overline{\overline{x}})^2 \\
=\;&\left(\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right)^2\frac{1}{n}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\frac{x_{it-1}}{\sqrt{T}}\right)^2\frac{1}{T}-\left(\sqrt{n}(\hat{\beta}_{OLS}-\beta)\right)^2\frac{1}{n^{3/2}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{1}{T}\right)^2 \\
=\;&O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}}\right)=O_p\left(\frac{1}{n}\right)
\end{aligned}$$

as $(n,T)\to\infty$ using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right)=O_p\left(1\right)$$

by Lemma 2.(4).
Next $III = O_p(\frac{1}{n})$ because

$$\begin{aligned}
& \sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{2}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\overline{\nu}})(x_{it-1}-\overline{\overline{x}}) \\
= & \sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{2}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\frac{\nu_{it-1}-\overline{\overline{\nu}}}{\sqrt{T}})(\frac{x_{it-1}-\overline{\overline{x}}}{\sqrt{T}})\frac{1}{T} \\
= & O_p(\frac{1}{n})
\end{aligned}$$

using

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}(\frac{\nu_{it-1}-\overline{\overline{\nu}}}{\sqrt{T}})(\frac{x_{it-1}-\overline{\overline{x}}}{\sqrt{T}})\frac{1}{T} = O_p(1)$$

where ν_{it} and x_{it} are not correlated.
Accordingly, we conclude that

$$\begin{aligned}
\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2 &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\overline{\nu}})^2 + O_p\left(\frac{1}{n}\right) + O_p(\frac{1}{n}) \\
&= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}(\nu_{it-1}-\overline{\overline{\nu}})^2 + O_p(\frac{1}{n}) \xrightarrow{p} \frac{\sigma_e^2}{6}
\end{aligned}$$

by equation (C.3) in Kao (1999).
For the numerator, $\widehat{u}_{it}-\widehat{u}_{it-1} = e_{it}-(\hat{\beta}_{OLS}-\beta)\varepsilon_{it}+o_p(1)$. Hence

$$\begin{aligned}
& \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1} \\
= & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left[e_{it}-(\hat{\beta}_{OLS}-\beta)\varepsilon_{it}\right]\left[\left(\nu_{it-1}-\overline{\overline{\nu}}\right)-\left(\hat{\beta}_{OLS}-\beta\right)\left(x_{it-1}-\overline{\overline{x}}\right)\right] \\
= & \tfrac{1}{nT}\textstyle\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\overline{\overline{\nu}}\right)e_{it}-\sqrt{n}(\hat{\beta}_{OLS}-\beta)\tfrac{1}{n^{3/2}}\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\left(\nu_{it-1}-\overline{\overline{\nu}}\right) \\
& -\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\tfrac{1}{n^{3/2}}\textstyle\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\left(x_{it-1}-\overline{\overline{x}}\right)e_{it} \\
& +\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\tfrac{1}{n^2}\textstyle\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\left(x_{it-1}-\overline{\overline{x}}\right) \\
= & I+II+III+IV.
\end{aligned}$$

Consider I. One can verify that

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\nu_{it-1}-\overline{\overline{\nu}}\right)e_{it} = O_p\left(\frac{1}{\sqrt{n}}\right)$$

using $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=2}^{T}\nu_{it-1}e_{it} = O_p(1)$ and $\frac{1}{T}\sum_{i=1}^{n}\sum_{t=2}^{T}e_{it}\overline{\overline{\nu}} = O_p(1)$ as $(n,T)\to\infty$.
Consider II.

$$\begin{aligned}
II &= \sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\left(\nu_{it-1}-\overline{\overline{\nu}}\right) \\
&= \sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=2}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\right) \\
&\quad -\sqrt{n}(\hat{\beta}_{OLS}-\beta)\frac{1}{n^{3/2}}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=2}^{T}\varepsilon_{it}\right)\left(\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right) \\
&= O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}}\right) = O_p\left(\frac{1}{n}\right)
\end{aligned}$$

as $(n,T)\to\infty$.
Consider III and IV. In a similar vein as II, it is easy to see that

$$III=-\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\left(x_{it-1}-\bar{\bar{x}}\right)e_{it}=O_p\left(\frac{1}{n}\right)$$

and

$$IV=\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}\left(x_{it-1}-\bar{\bar{x}}\right)=O_p\left(\frac{1}{n^{3/2}}\right).$$

Therefore,

$$\begin{aligned}&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\widehat{u}_{it}-\widehat{u}_{it-1}\right)\widehat{u}_{it-1}\\ =\;& O_p(\frac{1}{\sqrt{n}})+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}}\right)\\ =\;& O_p(\frac{1}{\sqrt{n}})=o_p\left(1\right).\end{aligned}$$

Summarizing, we have

$$T\left(\widetilde{\rho}-1\right)=\frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta\widehat{u}_{it}\widehat{u}_{it-1}}{\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^2}\xrightarrow{p}\frac{0}{\frac{\sigma_e^2}{6}}=0.$$

Next we show $\widetilde{\sigma}_e^2$ is a consistent estimator. It is clear that $I\to\sigma_e^2$ as $(n,T)\to\infty$ as shown already.
Consider II. In a similar process as above, one can show that

$$\begin{aligned}&\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widehat{\rho}x_{it-1}\right)^2-\left\{\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\widehat{\rho}x_{it-1}\right)\right\}^2\\ &+\left\{\tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=1}^{T}\left(\left(x_{it}-\overline{x}_{i.}\right)-\widehat{\rho}\left(x_{it-1}-\overline{x}_{i.}\right)\right)\right\}^2\\ =\;&\tfrac{1}{n}\sum_{i=1}^{n}[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2+\tfrac{\{T(\widehat{\rho}-1)\}^2}{T^3}\sum_{t=1}^{T}x_{it-1}^2-\tfrac{2T(\widehat{\rho}-1)}{T^2}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}]\\ &-\left[\tfrac{1}{n}\sum_{i=1}^{n}\left\{\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}-\tfrac{T(\widehat{\rho}-1)}{T^2}\sum_{t=1}^{T}x_{it-1}\right\}\right]^2\\ &+\left\{\tfrac{1}{n}\sum_{i=1}^{n}\left[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}-\tfrac{T(\widehat{\rho}-1)}{T^2}\sum_{t=1}^{T}x_{it-1}+\tfrac{T(\widehat{\rho}-1)}{T^2}\sum_{t=1}^{T}\overline{x}_{i.}\right]\right\}^2\\ =\;&\sigma_\varepsilon^2+o_p(1)\end{aligned}$$

as $(n,T)\to\infty$. Hence we have

$$II\approx\frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n}\left[\frac{1}{nT}\mathbf{x}'E_{nT}\left(I_n\otimes\hat{C}'\hat{C}\right)E_{nT}\mathbf{x}\right]=O_p(\frac{1}{n})=o_p\left(1\right)$$

since if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right).$$

Also because $III=o_p(1)$ by the Cauchy-Schwarz inequality, we get

$$\widetilde{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

■

F Proof of Theorem 3

Preparation: Note that from equation (9), we have

$$\mathbf{y} = \gamma \boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{u} = \gamma \boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{Z}_{\mu}\boldsymbol{\mu} + \boldsymbol{\nu}$$

where $\mathbf{y}$ is $nT \times 1$, $\mathbf{x}$ is a vector of x_{it} of dimension $nT \times 1$, $\boldsymbol{\iota}_{nT}$ is a vector of ones of dimension nT, $\mathbf{u}$ is $nT \times 1$, $\boldsymbol{\mu}$ is a vector of μ_i, $\boldsymbol{\nu}$ is a vector of ν_{it} and $\mathbf{Z}_{\mu} = I_n \otimes \boldsymbol{\iota}_T$. Also recall from equation (13) that

$$\Phi^{-1} = I_n \otimes \left[\frac{1}{\sigma_e^2} \left(\mathbf{A}^{-1} - \frac{\sigma_\mu^2}{\sigma_e^2 + \theta\sigma_\mu^2} \mathbf{A}^{-1} \boldsymbol{\iota}_T \boldsymbol{\iota}_T' \mathbf{A}^{-1} \right) \right].$$

Here we define $\mathbf{z} = [\boldsymbol{\iota}_{nT}, \mathbf{x}]$, then

$$\begin{aligned}
\begin{pmatrix} \widehat{\gamma}_{GLS} \\ \widehat{\beta}_{GLS} \end{pmatrix} &= \left(\mathbf{z}'\Phi^{-1}\mathbf{z}\right)^{-1} \left(\mathbf{z}'\Phi^{-1}\mathbf{y}\right) \\
&= \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT} & \mathbf{x} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1}\mathbf{y} \right) \\
&= \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \\ \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \mathbf{x}'\Phi^{-1}\mathbf{x} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{12}\mathbf{x}'\Phi^{-1}\mathbf{y} \\ F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}\mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
F_{11} &= \left[\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} - \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \left(\mathbf{x}'\Phi^{-1}\mathbf{x}\right)^{-1} \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \right]^{-1}, \\
F_{12} &= -\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \right]^{-1}, \\
F_{21} &= -\left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \right]^{-1} \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1},
\end{aligned}$$

and

$$F_{22} = \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \right]^{-1}.$$

Hence, we have

$$\begin{aligned}
\widehat{\beta}_{GLS} &= F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}'\mathbf{x}'\Phi^{-1}\mathbf{y} \\
&= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \right]^{-1} \\
&\quad \times \left[\mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \right]
\end{aligned}$$

and

$$\widehat{\beta}_{GLS} - \beta = G_1^{-1} G_2$$

where

$$G_1 = \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}$$

and

$$G_2 = \mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}$$

respectively.

Proof. Following Baltagi *et al.* (2008), we first define matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ which replace ρ in the matrix A and C in equation (12) and (14) with $\widetilde{\rho}$ given by,

$$\widetilde{\rho} = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it}\widehat{u}_{it-1}}{\sum_{i=1}^{n}\sum_{t=2}^{T}\widehat{u}_{it-1}^{2}}$$

where $\widehat{u}_{it}$ denotes the it-th OLS residual. Using the definition of Φ^{-1} in equation (13) and $\widetilde{\sigma}_e^2$ given by,

$$\widetilde{\sigma}_e^2 = \frac{1}{n(T-1)}\widehat{\mathbf{u}}^{*\prime}\widehat{\mathbf{u}}^{*}$$

where $\widehat{\mathbf{u}}^{*} = (I_n \otimes \hat{C})\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}}$ denotes the $nT \times 1$ vector of the OLS residuals, one obtains:

$$\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\hat{\mathbf{A}}_i^{-1}\mathbf{x}_i\right),$$

$$\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}\right),$$

$$\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}\right),$$

$$\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\nu} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}\right),$$

and

$$\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\nu} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}\right)$$

where

$$\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\mathbf{x}_i = \mathbf{x}_i'\hat{\mathbf{C}}'\hat{\mathbf{C}}\mathbf{x}_i \approx \sum_{t=1}^{T}\left(x_{it} - \widetilde{\rho}x_{it-1}\right)^2,$$

$$\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i = \mathbf{x}_i'\hat{\mathbf{C}}'\hat{\mathbf{C}}\boldsymbol{\nu}_i \approx \sum_{t=1}^{T}\left(x_{it} - \widetilde{\rho}x_{it-1}\right)\left(\nu_{it} - \widetilde{\rho}\nu_{it-1}\right),$$

$$\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T = \mathbf{x}_i'\hat{\mathbf{C}}'\hat{\mathbf{C}}\boldsymbol{\iota}_T \approx (1-\widetilde{\rho})\sum_{t=1}^{T}\left(x_{it} - \widetilde{\rho}x_{it-1}\right),$$

$$\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i = \boldsymbol{\iota}_T'\widehat{\mathbf{C}}'\widehat{\mathbf{C}}\boldsymbol{\nu}_i \approx (1-\widetilde{\rho})\sum_{t=1}^{T}\left(\nu_{it} - \widetilde{\rho}\nu_{it-1}\right),$$

and

$$\theta = \boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T = \boldsymbol{\iota}_T'\widehat{\mathbf{C}}'\widehat{\mathbf{C}}\boldsymbol{\iota}_T = (1-\widetilde{\rho}^2) + (T-1)(1-\widetilde{\rho})^2 \approx \sum_{t=1}^{T}(1-\widetilde{\rho})^2 = T\left(1-\widetilde{\rho}\right)^2.$$

In this section, we assume that $(n,T) \to \infty$ and $\frac{n}{T} \to 0$ unless otherwise specified.

1. $|\rho| < 1$, $|\lambda| < 1$ case

(a) Define

$$\frac{1}{nT}\widehat{G}_1 = \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{nT}.$$

First we consider

$$\begin{aligned} & \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} \\ = & \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i. \end{aligned}$$

Expanding this equation, we will show that

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i &\approx \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\lambda x_{it-1}-\widetilde{\rho}x_{it-1}\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{T}\sum_{t=2}^{T}\varepsilon_{it}^2+\left(\widetilde{\rho}-\rho\right)^2\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2+(\lambda-\rho)^2\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2 \\ -\left(\widetilde{\rho}-\rho\right)\frac{2}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}+(\lambda-\rho)\frac{2}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} \\ -\left(\widetilde{\rho}-\rho\right)(\lambda-\rho)\frac{2}{T}\sum_{t=1}^{T}x_{it-1}^2 \end{array}\right\} \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2+\frac{(\lambda-\rho)^2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2+I+II+III+IV \\ &= \frac{\left(1-2\rho\lambda+\rho^2\right)}{\left(1-\lambda^2\right)}\sigma_\varepsilon^2+o_p(1) \end{aligned}$$

as $(n,T)\to\infty$.

Consider I.

$$I = \left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)^2\frac{1}{nT}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 = O_p\left(\frac{1}{nT}\right)$$

using

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 = O_p(1).$$

Consider II.

$$II = -2\left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)\frac{1}{nT}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = O_p\left(\frac{1}{nT}\right)$$

using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = O_p(1)$$

and

$$\sqrt{nT}\left(\widetilde{\rho}-\rho\right) = O_p(1).$$

Consider III.

$$2(\lambda-\rho)\frac{1}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = O_p\left(\frac{1}{\sqrt{nT}}\right).$$

Consider IV.

$$-2\left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)(\lambda-\rho)\frac{1}{\sqrt{nT}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}=O_p(\frac{1}{\sqrt{nT}}).$$

Hence, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^{2} \\
=\;& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}+\frac{(\lambda-\rho)^2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}+O_p(\frac{1}{nT})+O_p(\frac{1}{nT})+O_p(\frac{1}{\sqrt{nT}})+O_p(\frac{1}{\sqrt{nT}}) \\
=\;& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}+\frac{(\lambda-\rho)^2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}+O_p(\frac{1}{\sqrt{nT}}).
\end{aligned}$$

Because

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}\xrightarrow{p}\sigma_{\varepsilon}^{2}$$

and

$$\frac{(\lambda-\rho)^2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}\xrightarrow{p}\frac{(\lambda-\rho)^2\sigma_{\varepsilon}^{2}}{(1-\lambda^2)},$$

one concludes that

$$\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}=\frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i\xrightarrow{p}\frac{(1-2\rho\lambda+\rho^2)\sigma_{\varepsilon}^{2}}{\left(1-\lambda^2\right)\sigma_e^2}.$$

Next consider

$$\begin{aligned}
\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\right) \\
&\approx (1-\widetilde{\rho})\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}-(\widetilde{\rho}-\rho)x_{it-1}\right] \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{nT}\right)=O_p\left(\frac{1}{\sqrt{nT}}\right)
\end{aligned}$$

as $(n,T)\to\infty$.

Also note

$$\frac{1}{nT}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\approx\frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}T\left(1-\widetilde{\rho}\right)^2\xrightarrow{p}\frac{\left(1-\rho\right)^2}{\sigma_e^2}=O_p\left(1\right).$$

Hence,

$$\begin{aligned}
\frac{1}{nT}\widehat{G}_1 &= \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}-\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{nT} \\
&\xrightarrow{p}\frac{(1-2\rho\lambda+\rho^2)\sigma_{\varepsilon}^{2}}{\left(1-\lambda^2\right)\sigma_e^2}
\end{aligned}$$

as $(n,T)\to\infty$.

(b) Now we investigate $\widehat{G}_2$.

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}}$$

Consider first

$$\begin{aligned} & \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} \\ = \; & \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}. \end{aligned}$$

Here

$$\begin{aligned} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)\left(\nu_{it}-\widetilde{\rho}\nu_{it-1}\right) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}-(\widetilde{\rho}-\rho)x_{it-1}\right]\left[e_{it}-(\widetilde{\rho}-\rho)\nu_{it-1}\right] \\ &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right]e_{it} \\ &\quad -\frac{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right]\nu_{it-1} \\ &\quad -\frac{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}e_{it}+\frac{(\sqrt{nT}\left(\widetilde{\rho}-\rho\right))^2}{n^{3/2}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}\nu_{it-1} \\ &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right]e_{it}+I+II+III. \end{aligned}$$

Consider I. Note that we have

$$\begin{aligned} I &= -\frac{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1}-(\lambda-\rho)\frac{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}\nu_{it-1} \\ &= O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right)=O_p\left(\frac{1}{\sqrt{nT}}\right) \end{aligned}$$

using $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1}=O_p\left(1\right)$ and $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}\nu_{it-1}=O_p\left(1\right)$.
By a similar process, it can be shown that

$$II=-\frac{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}e_{it}=O_p\left(\frac{1}{\sqrt{nT}}\right)$$

and

$$III=\frac{(\sqrt{nT}\left(\widetilde{\rho}-\rho\right))^2}{nT}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}\nu_{it-1}=O_p\left(\frac{1}{nT}\right).$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}\right]e_{it}+O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{nT}\right) \\ &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)\varepsilon_{it-1}+\lambda(\lambda-\rho)\varepsilon_{it-2}+\cdots\right]e_{it}+O_p\left(\frac{1}{\sqrt{nT}}\right). \end{aligned}$$

Because

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-\rho)\varepsilon_{it-1}+\lambda(\lambda-\rho)\varepsilon_{it-2}+\cdots\right]e_{it}\xrightarrow{d}N(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_e^2\sigma_\varepsilon^2}{1-\lambda^2}),$$

we get

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu}=\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\xrightarrow{d}N(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{\left(1-\lambda^2\right)\sigma_e^2})$$

Next consider

$$\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}=\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}.$$

From

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{(1-\widetilde{\rho})}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\nu_{it}-\widetilde{\rho}\nu_{it-1})\\ &= (1-\widetilde{\rho})\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(e_{it}-(\widetilde{\rho}-\rho)\nu_{it-1})\\ &= (1-\widetilde{\rho})\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}-(1-\widetilde{\rho})\frac{\sqrt{nT}(\widetilde{\rho}-\rho)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1},\end{aligned}$$

it is easy to see that

$$(1-\widetilde{\rho})\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\xrightarrow{d}(1-\rho)N(0,\sigma_e^2)$$

and

$$(1-\widetilde{\rho})\frac{\sqrt{nT}(\widetilde{\rho}-\rho)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}=O_p\left(\frac{1}{\sqrt{nT}}\right).$$

Accordingly, we have

$$\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}\xrightarrow{d}(1-\rho)N(0,\frac{1}{\sigma_e^2}).$$

Also recall that $\frac{1}{nT}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}=O_p(\frac{1}{\sqrt{nT}})$; $\frac{1}{nT}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{(1-\rho)^2}{\sigma_e^2}$ as shown above. Hence, we have

$$\begin{aligned}\frac{1}{\sqrt{nT}}\widehat{G}_2 &= \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu}-\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}}\\ &\xrightarrow{d}N(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{(1-\lambda^2)\sigma_e^2})\end{aligned}$$

as $(n,T)\to\infty$.

(c) We conclude that

$$t_{FGLS}=\left[\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{\left(1-\lambda^2\right)\sigma_e^2}\right]^{-1/2}N\left(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{(1-\lambda^2)\sigma_e^2}\right)=N(0,1).$$

2. $\rho=1$, $|\lambda|<1$ case

(a) Let

$$\frac{1}{nT}\widehat{G}_1 = \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} - T(1-\widetilde{\rho})\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT(1-\widetilde{\rho})}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n(1-\widetilde{\rho})}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{nT(1-\widetilde{\rho})}.$$

Using a similar argument as above, we first consider

$$\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} = \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i.$$

Expanding this equation, we get

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i &\approx \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\varepsilon_{it}+\lambda x_{it-1}-\widetilde{\rho}x_{it-1}\right)^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 + \frac{1}{T^2}\left\{T(\widetilde{\rho}-1)\right\}^2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 \\
&\quad +(\lambda-1)^2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 - \frac{2}{T}T(\widetilde{\rho}-1)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} \\
&\quad +(\lambda-1)\frac{2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} - \frac{2}{T}T(\widetilde{\rho}-1)(\lambda-1)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2 + (\lambda-1)^2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 + I + II + III + IV.
\end{aligned}$$

Consider I. With the joint limit, we have

$$I = (\widetilde{\rho}-1)^2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 = o_p\left(\frac{1}{T^2}\right)$$

using

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2 = O_p(1)$$

and

$$(\widetilde{\rho}-1) = o_p\left(\frac{1}{T}\right).$$

Consider II. In a similar vein as I,

$$II = -\frac{2}{\sqrt{nT}}(\widetilde{\rho}-1)\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = o_p\left(\frac{1}{\sqrt{n}T^{3/2}}\right)$$

using $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = O_p(1)$.
Consider III.

$$III = (\lambda-1)\frac{1}{\sqrt{nT}}\frac{2}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} = O_p\left(\frac{1}{\sqrt{nT}}\right)$$

as $(n,T)\to\infty$.

Consider IV.

$$IV = -2\left(\widetilde{\rho}-1\right)\left(\lambda-1\right)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2} = o_p\left(\frac{1}{T}\right).$$

Finally, because we know

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2} \xrightarrow{p} \sigma_{\varepsilon}^{2}$$

and

$$(\lambda-1)^2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2} \xrightarrow{p} \frac{(\lambda-1)^2\sigma_{\varepsilon}^{2}}{1-\lambda^2},$$

we get

$$\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} = \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i \xrightarrow{p} \frac{2\sigma_{\varepsilon}^{2}}{(1+\lambda)\sigma_e^2}$$

as $(n,T)\to\infty$.

Next, it can be shown that

$$\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT\left(1-\widetilde{\rho}\right)} = \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{1}{T\left(1-\widetilde{\rho}\right)}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\right) = o_p\left(1\right)$$

as $(n,T)\to\infty$ because

$$\begin{aligned}
\frac{1}{T\left(1-\widetilde{\rho}\right)}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T &\approx \frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}-\left(\widetilde{\rho}-1\right)x_{it-1}\right]
\end{aligned}$$

and accordingly

$$\begin{aligned}
\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT\left(1-\widehat{\rho}\right)} &= \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{1}{T\left(1-\widetilde{\rho}\right)}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\right) \\
&\approx \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}-\left(\widetilde{\rho}-1\right)x_{it-1}\right] \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right)+o_p\left(\frac{1}{\sqrt{n}T^{3/2}}\right).
\end{aligned}$$

We also know that

$$\frac{1}{n\left(1-\widetilde{\rho}\right)}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left[\frac{2\widetilde{\rho}(1-\widetilde{\rho})}{(1-\widetilde{\rho})}+\frac{T\left(1-\widetilde{\rho}\right)^2}{(1-\widetilde{\rho})}\right] \xrightarrow{p} \frac{2}{\sigma_e^2}.$$ [2]

Hence, we get

$$\frac{1}{nT}\widehat{G}_1 \xrightarrow{p} \frac{2\sigma_{\varepsilon}^{2}}{(1+\lambda)\sigma_e^2}$$

as $(n,T)\to\infty$.

[2] Note that we are using the entire form including the first observation, not the abbreviated form. That is,

$$\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left[2\widetilde{\rho}(1-\widetilde{\rho})+T\left(1-\widetilde{\rho}\right)^2\right].$$

(b) Now we investigate $\widehat{G}_2$. Recall that

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} - T(1-\widetilde{\rho})\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT(1-\widetilde{\rho})}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n(1-\widetilde{\rho})}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}(1-\widetilde{\rho})}.$$

Firstly, consider

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} = \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}.$$

Here

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\widetilde{\rho}x_{it-1})(\nu_{it}-\widetilde{\rho}\nu_{it-1}) \\
&= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}-(\widetilde{\rho}-1)x_{it-1}\right]\left[e_{it}-(\widetilde{\rho}-1)\nu_{it-1}\right] \\
&= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\begin{array}{c}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]e_{it}-(\widetilde{\rho}-1)\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]\nu_{it-1} \\ -(\widetilde{\rho}-1)x_{it-1}e_{it}+(\widetilde{\rho}-1)^2x_{it-1}\nu_{it-1}\end{array}\right\} \\
&= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]e_{it}+I+II+III.
\end{aligned}$$

Consider I. It can be shown that

$$\begin{aligned}
I &= (\widetilde{\rho}-1)\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]\nu_{it-1} \\
&= \frac{T(\widetilde{\rho}-1)}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}+\frac{T(\widetilde{\rho}-1)}{\sqrt{T}}\frac{(\lambda-1)}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}} \\
&= o_p\left(\frac{1}{\sqrt{T}}\right)+o_p\left(\frac{1}{\sqrt{T}}\right)=o_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}} = O_p(1)$ and $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}} = O_p(1)$ with $(\widetilde{\rho}-1) = o_p\left(\frac{1}{T}\right)$.

Consider II and III. One can easily verify that

$$II = (T(\widetilde{\rho}-1))\frac{1}{T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}e_{it} = o_p\left(\frac{1}{T}\right)$$

and

$$III = (T(\widetilde{\rho}-1))^2\frac{1}{T^{3/2}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}} = o_p\left(\frac{1}{T^{3/2}}\right).$$

We conclude that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]e_{it} \\
&\quad +o_p\left(\frac{1}{\sqrt{T}}\right)+o_p\left(\frac{1}{T}\right)+o_p\left(\frac{1}{T^{3/2}}\right) \\
&= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\varepsilon_{it}+(\lambda-1)x_{it-1}\right]e_{it}+o_p\left(\frac{1}{\sqrt{T}}\right) \\
&\quad \xrightarrow{d} N\left(0,\frac{2\sigma_e^2\sigma_\varepsilon^2}{1+\lambda}\right).
\end{aligned}$$

Accordingly, we have

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu}=\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\xrightarrow{d}\frac{1}{\sigma_e^2}N(0,\frac{2\sigma_e^2\sigma_\varepsilon^2}{1+\lambda}).$$

Next consider

$$\frac{1}{\sqrt{nT}\,(1-\widetilde{\rho})}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}=\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\,(1-\widetilde{\rho})}.$$

One can see that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\,(1-\widetilde{\rho})} &\approx \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(e_{it}+(1-\widetilde{\rho})\nu_{it-1})\\ &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}+\frac{T(1-\widetilde{\rho})}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}\\ &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}+o_p\left(1\right)\xrightarrow{d}\sigma_e N(0,1)\end{aligned}$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\nu_{it-1}}{\sqrt{T}}\frac{1}{T}=O_p\left(1\right)$.
Therefore,

$$\frac{1}{\sqrt{nT}\,(1-\widetilde{\rho})}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}\xrightarrow{d}N(0,\frac{1}{\sigma_e^2}).$$

Also note that

$$\frac{1}{nT\,(1-\widetilde{\rho})}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0;\ \frac{1}{n\,(1-\widetilde{\rho})}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}=O_p\left(1\right)\xrightarrow{p}\frac{2}{\sigma_e^2},$$

as has been shown already.
Therefore,

$$\frac{1}{\sqrt{nT}}\widehat{G}_2\xrightarrow{d}\frac{1}{\sigma_e^2}N\left(0,\frac{2\sigma_e^2\sigma_\varepsilon^2}{1+\lambda}\right)=N\left(0,\frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2}\right)$$

as $(n,T)\to\infty$.

(c) Summarizing, we have

$$t_{FGLS}=\left[\frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2}\right]^{-1/2}N\left(0,\frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2}\right)=N(0,1).$$

3. $|\rho|<1$, $\lambda=1$ case

(a) This is the *panel cointegration* case. Note that

$$\frac{1}{nT^2}\widehat{G}_1=\frac{1}{nT^2}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}-\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT^{3/2}}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{nT^{3/2}}.$$

First we consider

$$\frac{1}{nT^2}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}=\frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T^2}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i.$$

Expanding this equation, we get

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\mathbf{x}_{i} \\
\approx \;& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^{2}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\sum_{t=1}^{T}\left[\varepsilon_{it}+(1-\rho)x_{it-1}-(\widetilde{\rho}-\rho)x_{it-1}\right]^{2} \\
= \;& \frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{c}\frac{1}{T^{2}}\sum_{t=1}^{T}\varepsilon_{it}^{2}+\left\{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right\}^{2}\frac{1}{nT^{3}}\sum_{t=1}^{T}x_{it-1}^{2} \\ +(1-\rho)^{2}\frac{1}{T^{2}}\sum_{t=1}^{T}x_{it-1}^{2}-\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\frac{2}{\sqrt{n}T^{5/2}}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} \\ +(1-\rho)\frac{2}{T^{2}}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}-\sqrt{nT}\left(\widetilde{\rho}-\rho\right)(1-\rho)\frac{2}{\sqrt{n}T^{5/2}}\sum_{t=1}^{T}x_{it-1}^{2}\end{array}\right\} \\
= \;& (1-\rho)^{2}\frac{1}{nT^{2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}+I+II+III+IV+V.
\end{aligned}$$

Consider I.

$$I=\frac{1}{T}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}=O_{p}\left(\frac{1}{T}\right)$$

using $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}=O_{p}(1)$.
Consider II.

$$II=\left\{\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right\}^{2}\frac{1}{nT}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\sum_{t=1}^{T}x_{it-1}^{2}=O_{p}\left(\frac{1}{nT}\right)$$

using $\sqrt{nT}\left(\widetilde{\rho}-\rho\right)=O_{p}(1)$ and $\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\frac{x_{it-1}}{\sqrt{T}}\right)^{2}\frac{1}{T}=O_{p}(1)$.
Consider III.

$$III=-2\left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)\frac{1}{nT^{3/2}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_{p}\left(\frac{1}{nT^{3/2}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_{p}(1)$.
Consider IV and V. It is easy to see that

$$IV=(1-\rho)\frac{2}{\sqrt{nT}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_{p}\left(\frac{1}{\sqrt{nT}}\right)$$

and

$$V=-(1-\rho)\left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)\frac{1}{\sqrt{nT}}\frac{1}{n}\sum_{i=1}^{n}\frac{2}{T^{2}}\sum_{t=1}^{T}x_{it-1}^{2}=O_{p}\left(\frac{1}{\sqrt{nT}}\right).$$

Therefore,

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\mathbf{x}_{i} \\
\approx \;& (1-\rho)^{2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\sum_{t=1}^{T}x_{it-1}^{2}+O_{p}\left(\frac{1}{T}\right)+O_{p}\left(\frac{1}{nT}\right)+O_{p}\left(\frac{1}{nT^{3/2}}\right)+O_{p}\left(\frac{1}{\sqrt{nT}}\right)+O_{p}\left(\frac{1}{\sqrt{nT}}\right) \\
= \;& (1-\rho)^{2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{2}}\sum_{t=1}^{T}x_{it-1}^{2}+\max\left\{O_{p}\left(\frac{1}{T}\right),O_{p}\left(\frac{1}{\sqrt{nT}}\right)\right\}.
\end{aligned}$$

Finally, because we know

$$(1-\rho)^2 \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2(1-\rho)^2}{2},$$

we have

$$\frac{1}{nT^2}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} = \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T^2}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i \xrightarrow{p} \frac{\sigma_\varepsilon^2(1-\rho)^2}{2\sigma_e^2}$$

as $(n,T)\to\infty$.

Next note that

$$\begin{aligned}
\frac{1}{nT^{3/2}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{1}{T^{3/2}}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\right) \\
&\approx \frac{(1-\widetilde{\rho})}{\widetilde{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_{it}-\widetilde{\rho}x_{it-1}) \\
&= \frac{(1-\widetilde{\rho})}{\widetilde{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}[\varepsilon_{it}-(1-\widetilde{\rho})x_{it-1}] \\
&= I + II.
\end{aligned}$$

For I, one can see that

$$\frac{(1-\widetilde{\rho})}{\widehat{\sigma}_e^2}\frac{1}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it} = O_p\left(\frac{1}{\sqrt{nT}}\right).$$

For II,

$$\frac{(1-\widetilde{\rho})^2}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1} = O_p\left(\frac{1}{\sqrt{n}}\right).$$

Therefore, we have

$$\frac{1}{nT^{3/2}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = O_p\left(\frac{1}{\sqrt{n}}\right) = o_p(1).$$

Also note that

$$\frac{1}{nT}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} \approx \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}T(1-\widetilde{\rho})^2 \xrightarrow{p} \frac{(1-\rho)^2}{\sigma_e^2}.$$

Hence, we have

$$\begin{aligned}
\frac{1}{nT^2}\widehat{G}_1 &= \frac{1}{nT^2}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT^{3/2}}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{nT^{3/2}} \\
&\xrightarrow{p} \frac{(1-\rho)^2\sigma_\varepsilon^2}{2\sigma_e^2}
\end{aligned}$$

as $(n,T)\to\infty$.

(b) Now we investigate $\widehat{G}_2$. Note that

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT^{3/2}}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}}.$$

We first consider

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu}=\frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{T}.$$

Here

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{T}\\
\approx\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)\left(\nu_{it}-\widetilde{\rho}\nu_{it-1}\right)\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\varepsilon_{it}+(1-\rho)x_{it-1}-(\widetilde{\rho}-\rho)x_{it-1}\right]\left[e_{it}-(\widetilde{\rho}-\rho)\nu_{it-1}\right]\right\}\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\begin{array}{c}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}e_{it}+\frac{1}{T}\sum_{t=1}^{T}(1-\rho)x_{it-1}e_{it}-\frac{\sqrt{nT}(\widetilde{\rho}-\rho)}{\sqrt{n}}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1}\\-(1-\rho)\frac{\sqrt{nT}(\widetilde{\rho}-\rho)}{\sqrt{n}}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\nu_{it-1}-\frac{\sqrt{nT}(\widetilde{\rho}-\rho)}{\sqrt{n}}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}e_{it}\\+\frac{(\sqrt{nT}(\widetilde{\rho}-\rho))^2}{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}\nu_{it-1}\end{array}\right\}\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(1-\rho)x_{it-1}e_{it}+I+II+III+IV+V.
\end{aligned}$$

Consider I. One can see that

$$I=\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it}=O_p\left(\frac{1}{\sqrt{T}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it}=O_p\left(1\right)$.
Consider II.

$$II=\frac{1}{\sqrt{nT}}\left(\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\right)\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1}=O_p\left(\frac{1}{\sqrt{nT}}\right)$$

using $\sqrt{nT}\left(\widetilde{\rho}-\rho\right)=O_p\left(1\right)$ and $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1}=O_p\left(1\right)$.
Consider III.

$$III=-(1-\rho)\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\frac{1}{\sqrt{nT}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}=O_p\left(\frac{1}{\sqrt{nT}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}=O_p\left(1\right)$.
Consider IV and V. In a similar vein as above, it is easy to see that

$$IV=-\sqrt{nT}\left(\widetilde{\rho}-\rho\right)\frac{1}{\sqrt{nT}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{e_{it}}{\sqrt{T}}=O_p\left(\frac{1}{\sqrt{nT}}\right)$$

and

$$V=(\sqrt{nT}\left(\widetilde{\rho}-\rho\right))^2\frac{1}{nT}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}=O_p\left(\frac{1}{nT}\right).$$

Therefore,

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\mathbf{x}_i' \widehat{\mathbf{A}}^{-1} \boldsymbol{\nu}_i}{T} \\
\approx \quad & \frac{(1-\rho)}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{e_{it}}{\sqrt{T}} \\
& + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right) + O_p\left(\frac{1}{nT}\right) \\
= \quad & \frac{(1-\rho)}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{e_{it}}{\sqrt{T}} + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Because it can be shown that

$$\frac{(1-\rho)}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{x_{it-1}}{\sqrt{T}} \frac{e_{it}}{\sqrt{T}} \xrightarrow{d} (1-\rho) N(0, \frac{\sigma_\varepsilon^2 \sigma_e^2}{2}),$$

one concludes that

$$\frac{1}{\sqrt{nT}} \mathbf{x}' \widehat{\Phi}^{-1} \boldsymbol{\nu} = \frac{1}{\widetilde{\sigma}_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\mathbf{x}_i' \widehat{\mathbf{A}}^{-1} \boldsymbol{\nu}_i}{T} \xrightarrow{d} (1-\rho) N(0, \frac{\sigma_\varepsilon^2}{2\sigma_e^2})$$

as $(n,T) \to \infty$.

Next, recall that

$$\frac{1}{\sqrt{nT}} \boldsymbol{\iota}_{nT}' \widehat{\Phi}^{-1} \boldsymbol{\nu} \xrightarrow{d} (1-\rho) N(0, \frac{1}{\sigma_e^2})$$

as shown in 1.(2). Also, $\frac{1}{nT^{3/2}} \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \xrightarrow{p} 0$; $\frac{1}{nT} \boldsymbol{\iota}_{nT}' \Phi^{-1} \boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{(1-\rho)^2}{\sigma_e^2}$ as shown above. Hence, we get

$$\frac{1}{\sqrt{nT}} \widehat{G}_2 \xrightarrow{d} N(0, \frac{(1-\rho)^2 \sigma_\varepsilon^2}{2\sigma_e^2})$$

as $(n,T) \to \infty$.

(c) Finally, we conclude that

$$t_{FGLS} = \left[\frac{(1-\rho)^2 \sigma_\varepsilon^2}{2\sigma_e^2} \right]^{-1/2} N(0, \frac{(1-\rho)^2 \sigma_\varepsilon^2}{2\sigma_e^2}) = N(0,1).$$

4. $\rho = 1$, $\lambda = 1$ case

(a) Recall

$$\frac{1}{nT} \widehat{G}_1 = \frac{1}{nT} \mathbf{x}' \widehat{\Phi}^{-1} \mathbf{x} - T\,(1-\widetilde{\rho})\, \frac{\mathbf{x}' \widehat{\Phi}^{-1} \boldsymbol{\iota}_{nT}}{nT\,(1-\widetilde{\rho})} \left(\frac{\boldsymbol{\iota}_{nT}' \widehat{\Phi}^{-1} \boldsymbol{\iota}_{nT}}{n\,(1-\widetilde{\rho})} \right)^{-1} \frac{\boldsymbol{\iota}_{nT}' \widehat{\Phi}^{-1} \mathbf{x}}{nT\,(1-\widetilde{\rho})}.$$

In a similar process as above, we consider first

$$\frac{1}{nT} \mathbf{x}' \widehat{\Phi}^{-1} \mathbf{x} = \frac{1}{n} \frac{1}{\widetilde{\sigma}_e^2} \sum_{i=1}^{n} \frac{1}{T} \mathbf{x}_i' \widehat{\mathbf{A}}^{-1} \mathbf{x}_i.$$

Expanding this equation, we get

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i &\approx \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+x_{it-1}-\widetilde{\rho}x_{it-1}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2+\frac{1}{T^2}\left\{T\left(\widetilde{\rho}-1\right)\right\}^2\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2-\frac{2}{T}T\left(\widetilde{\rho}-1\right)\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}\right\} \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2+I+II.
\end{aligned}$$

Consider I. It is easy to see that

$$I=\left\{T\left(\widetilde{\rho}-1\right)\right\}^2\frac{1}{T}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2=o_p\left(\frac{1}{T}\right)$$

using

$$T\left(\widetilde{\rho}-1\right)=o_p\left(1\right)$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\frac{x_{it-1}}{\sqrt{T}}\right)^2\frac{1}{T}=O_p\left(1\right).$$

For II,

$$II=-2T\left(\widetilde{\rho}-1\right)\frac{1}{\sqrt{n}T}\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}=o_p\left(\frac{1}{\sqrt{n}T}\right)$$

because $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{x_{it-1}}{\sqrt{T}}=O_p\left(1\right)$. Hence,

$$\begin{aligned}
\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} &= \frac{1}{n}\frac{1}{\widetilde{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i \\
&\approx \frac{1}{nT}\frac{1}{\hat{\sigma}_e^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2+o_p\left(\frac{1}{T}\right)+o_p\left(\frac{1}{\sqrt{n}T}\right) \\
&= \frac{1}{nT}\frac{1}{\hat{\sigma}_e^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2+o_p\left(\frac{1}{T}\right) \\
&\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\sigma_e^2}
\end{aligned}$$

as $(n,T)\to\infty$.

Next note that

$$\begin{aligned}
\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT\left(1-\widehat{\rho}\right)} &= \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T\left(1-\widetilde{\rho}\right)}\right) \\
&\approx \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it} - T\left(\widetilde{\rho}-1\right)\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}\right\} \\
&= \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{nT}}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\right) - \frac{T\left(\widetilde{\rho}-1\right)}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{nT}}\left(\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}\right) \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right) + o_p\left(\frac{1}{\sqrt{nT}}\right) \xrightarrow{p} 0
\end{aligned}$$

because $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it} = O_p\left(1\right)$ and $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{1}{T} = O_p\left(1\right)$ as $(n,T)\to\infty$. We also know

$$\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n\left(1-\widetilde{\rho}\right)} \xrightarrow{p} \frac{2}{\sigma_e^2}$$

as shown in 2.(a). Hence,

$$\frac{1}{nT}\widehat{G}_1 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{\sigma_e^2}$$

as $(n,T)\to\infty$.

(b) Now we turn to $\widehat{G}_2$.

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} - T\left(1-\widetilde{\rho}\right)\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT\left(1-\widehat{\rho}\right)}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n\left(1-\widetilde{\rho}\right)}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}\left(1-\widetilde{\rho}\right)}.$$

First, we consider

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} = \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}.$$

Here

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} &\approx \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-\widetilde{\rho}x_{it-1}\right)\left(\nu_{it}-\widetilde{\rho}\nu_{it-1}\right)\right\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\varepsilon_{it}-(\widetilde{\rho}-1)x_{it-1}\right]\left[e_{it}-(\widetilde{\rho}-1)\nu_{it-1}\right]\right\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\begin{array}{c}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it} - \frac{T(\widetilde{\rho}-1)}{T^{3/2}}\sum_{t=1}^{T}\varepsilon_{it}\nu_{it-1} \\ -\frac{T(\widetilde{\rho}-1)}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}e_{it} + \frac{\{T(\widetilde{\rho}-1)\}^2}{T^{5/2}}\sum_{t=1}^{T}x_{it-1}\nu_{it-1}\end{array}\right\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it} + I + II + III.
\end{aligned}$$

Consider I.

$$I = -T\left(\widetilde{\rho}-1\right)\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\right) = o_p\left(\frac{1}{\sqrt{T}}\right)$$

using

$$T\left(\widetilde{\rho}-1\right) = o_p\left(1\right)$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\varepsilon_{it}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}=O_p\left(1\right).$$

Consider II. In a similar vein as I, one can also verify that

$$II=-T\left(\widetilde{\rho}-1\right)\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{e_{it}}{\sqrt{T}}=o_p\left(\frac{1}{\sqrt{T}}\right)$$

using

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{e_{it}}{\sqrt{T}}=O_p\left(1\right).$$

Consider III.

$$III=\left\{T\left(\widetilde{\rho}-1\right)\right\}^2\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\frac{1}{T}\right)=o_p\left(\frac{1}{\sqrt{T}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{\nu_{it-1}}{\sqrt{T}}\frac{1}{T}=O_p\left(1\right)$.

We conclude that

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\nu} &= \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \\
&\approx \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}e_{it}+o_p\left(\frac{1}{\sqrt{T}}\right)+o_p\left(\frac{1}{\sqrt{T}}\right)+o_p\left(\frac{1}{\sqrt{T}}\right) \\
&= \frac{1}{\widetilde{\sigma}_e^2}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}e_{it}+o_p\left(\frac{1}{\sqrt{T}}\right) \\
&\xrightarrow{d} N(0,\frac{\sigma_\varepsilon^2}{\sigma_e^2})
\end{aligned}$$

as $(n,T)\to\infty$.

Also recall that $\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{nT(1-\widetilde{\rho})}\xrightarrow{p}0$, $\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n(1-\widetilde{\rho})}\xrightarrow{p}\frac{2}{\sigma_e^2}$ from above and $\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\nu}}{\sqrt{nT}(1-\widetilde{\rho})}\xrightarrow{d}N(0,\frac{1}{\sigma_e^2})$ from 2.(b).

Hence, the second term of $\frac{1}{\sqrt{nT}}\widehat{G}_2$ is $o_p(1)$ and we conclude that

$$\frac{1}{\sqrt{nT}}\widehat{G}_2\xrightarrow{d}N(0,\frac{\sigma_\varepsilon^2}{\sigma_e^2})$$

as $(n,T)\to\infty$.

(c) Finally, we have

$$t_{FGLS}=\left[\frac{\sigma_\varepsilon^2}{\sigma_e^2}\right]^{-1/2}N(0,\frac{\sigma_\varepsilon^2}{\sigma_e^2})=N(0,1).$$

■

G Proof of Theorem 4

We study the following lemmas before proving Theorem 4.

Lemma 1 (B)

$$\hat{\sigma}_e^2 = I + II + III$$

where

$$\begin{aligned} I &= \tfrac{1}{n}\sum_{i=1}^n \left[\tfrac{1}{T}\sum_{t=1}^T e_{it}^2 - 2\left(\widehat{\rho}-\rho\right)\tfrac{1}{T}\sum_{t=1}^T e_{it}\nu_{it-1} + \left(\widehat{\rho}-\rho\right)^2 \tfrac{1}{T}\sum_{t=1}^T \nu_{it-1}^2\right] \\ &\quad -\tfrac{T}{\hat{d}^2}\tfrac{1}{n}\sum_{i=1}^n \left[\tfrac{1}{T}\sum_{t=1}^T e_{it} - \left(\widehat{\rho}-\rho\right)\tfrac{1}{T}\sum_{t=1}^T \nu_{it-1}\right]^2, \end{aligned}$$

$$\begin{aligned} II &= \tfrac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^n [\tfrac{1}{T}\sum_{t=1}^T \varepsilon_{it}^2 + \tfrac{(\widehat{\rho}-\rho)^2}{T}\sum_{t=1}^T x_{it-1}^2 + \tfrac{(\lambda-\rho)^2}{T}\sum_{t=1}^T x_{it-1}^2 \\ &\quad -\tfrac{2(\widehat{\rho}-\rho)}{T}\sum_{t=1}^T \varepsilon_{it}x_{it-1} + \tfrac{2(\lambda-\rho)}{T}\sum_{t=1}^T \varepsilon_{it}x_{it-1} - \tfrac{2(\widehat{\rho}-\rho)(\lambda-\rho)}{T}\sum_{t=1}^T x_{it-1}^2] \\ &\quad -\tfrac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\left(\tfrac{T}{\hat{d}^2}\right)\sum_{i=1}^n \left[\tfrac{1}{T}\sum_{t=1}^T \varepsilon_{it} - \left(\widehat{\rho}-\rho\right)\tfrac{1}{T}\sum_{t=1}^T x_{it-1} + (\lambda-\rho)\tfrac{1}{T}\sum_{t=1}^T x_{it-1}\right]^2, \end{aligned}$$

and $III \leq \sqrt{I \times II}$.

Proof. Note that

$$\hat{\mathbf{u}} = \mathbf{y} - \widehat{\gamma}_{OLS}\boldsymbol{\iota}_{nT} - \mathbf{x}\hat{\beta}_{OLS} = E_{nT}\left[\mathbf{u} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right].$$

Because

$$\begin{aligned} \left(I_n \otimes \hat{E}_T^\alpha\right)\hat{\mathbf{u}}^* &= \left(I_n \otimes \hat{E}_T^\alpha \hat{C}\right)\hat{\mathbf{u}} \\ &= \left(I_n \otimes \hat{E}_T^\alpha \hat{C}\right)\left(I_{nT} - \bar{J}_{nT}\right)\left[\mathbf{u} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \\ &= \left(I_n \otimes \hat{E}_T^\alpha \hat{C}\right)\left[\mathbf{u} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \\ &= \left(I_n \otimes \hat{E}_T^\alpha \hat{C}\right)\left[\left(I_n \otimes \boldsymbol{\iota}_T\right)\boldsymbol{\mu} + \boldsymbol{\nu} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \\ &= \left(I_n \otimes \hat{E}_T^\alpha \hat{C}\right)\left[\boldsymbol{\nu} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \end{aligned}$$

using

$$\hat{E}_T^\alpha \hat{C}\boldsymbol{\iota}_T = (1-\widehat{\rho})\hat{E}_T^\alpha \hat{\boldsymbol{\iota}}_T^\alpha = 0,$$

one can show that

$$\begin{aligned} \hat{\sigma}_e^2 &= \frac{1}{n\left(T-1\right)}\hat{\mathbf{u}}^{*\prime}\left(I_n \otimes \hat{E}_T^\alpha\right)\hat{\mathbf{u}}^* \\ &= \frac{1}{n\left(T-1\right)}\hat{\mathbf{u}}'\left(I_n \otimes \hat{C}'\hat{E}_T^\alpha \hat{C}\right)\hat{\mathbf{u}} \\ &= \frac{1}{n\left(T-1\right)}\left[\boldsymbol{\nu} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right]'\left(I_n \otimes \hat{C}\hat{E}_T^\alpha \hat{C}\right)\left[\boldsymbol{\nu} + \mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right] \\ &= \frac{1}{n\left(T-1\right)}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}\hat{E}_T^\alpha \hat{C}\right)\boldsymbol{\nu} + \frac{1}{n\left(T-1\right)}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'\left(I_n \otimes \hat{C}\hat{E}_T^\alpha \hat{C}\right)\mathbf{x} \\ &\quad + \frac{2}{n\left(T-1\right)}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{x}'\left(I_n \otimes \hat{C}\hat{E}_T^\alpha \hat{C}\right)\boldsymbol{\nu} \\ &= I + II + III. \end{aligned}$$

Consider I.

$$\begin{aligned} I &= \frac{1}{n\left(T-1\right)}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{E}_T^\alpha \hat{C}\right)\boldsymbol{\nu} \\ &= \frac{1}{n\left(T-1\right)}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{C}\right)\boldsymbol{\nu} - \frac{1}{n\left(T-1\right)\hat{d}^2}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{\boldsymbol{\iota}}_T^\alpha \hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\boldsymbol{\nu}. \end{aligned}$$

The first term in I is

$$\begin{aligned}
& \frac{1}{n(T-1)}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{C}\right)\boldsymbol{\nu} \\
\approx\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\widehat{\rho}\nu_{it-1}\right)^2\right] \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2 - 2\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1} + \left(\widehat{\rho}-\rho\right)^2\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^2\right].
\end{aligned}$$

The second term in I is

$$\begin{aligned}
\frac{1}{n(T-1)\hat{d}^2}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\hat{\boldsymbol{\iota}}_T^{\alpha}\hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\boldsymbol{\nu} &\approx \frac{T}{n\hat{d}^2}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\widehat{\rho}\nu_{it-1}\right)\right]^2 \\
&= \frac{T}{n\hat{d}^2}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it} - \left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
I \approx\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2 - 2\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1} + \left(\widehat{\rho}-\rho\right)^2\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^2\right] \\
& -\frac{T}{\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it} - \left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]^2.
\end{aligned}$$

Consider II. In a similar vein as I, we get

$$\begin{aligned}
II =\ & \frac{1}{n(T-1)}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'\left(I_n \otimes \hat{C}\hat{E}_T^{\alpha}\hat{C}\right)\mathbf{x} \\
\approx\ & \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widehat{\rho}x_{it-1}\right)^2\right] - \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2 T}{n\hat{d}^2}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\widehat{\rho}x_{it-1}\right)\right]^2 \\
=\ & \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left[\begin{array}{c} \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2 + \frac{(\widehat{\rho}-\rho)^2}{T}\sum_{t=1}^{T}x_{it-1}^2 + \frac{(\lambda-\rho)^2}{T}\sum_{t=1}^{T}x_{it-1}^2 \\ -\frac{2(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} + \frac{2(\lambda-\rho)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} - \frac{2(\widehat{\rho}-\rho)(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1}^2 \end{array}\right] \\
& -\frac{\left(\hat{\beta}_{OLS}-\beta\right)^2 T}{n\hat{d}^2}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it} - \left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}x_{it-1} + (\lambda-\rho)\frac{1}{T}\sum_{t=1}^{T}x_{it-1}\right]^2.
\end{aligned}$$

Consider III. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
III =\ & \frac{2}{n(T-1)}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{x}'\left(I_n \otimes \hat{C}\hat{E}_T^{\alpha}\hat{C}\right)\boldsymbol{\nu} \\
\leq\ & \sqrt{\left[\frac{1}{n(T-1)}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}\hat{E}_T^{\alpha}\hat{C}\right)\boldsymbol{\nu}\right]\left[\frac{1}{n(T-1)}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'\left(I_n \otimes \hat{C}\hat{E}_T^{\alpha}\hat{C}\right)\mathbf{x}\right]} \\
=\ & \sqrt{I \times II}.
\end{aligned}$$

■

Lemma 2 (B)

$$\frac{1}{T}\hat{\sigma}_{\alpha}^2 = I + II + III + IV + V + VI$$

where

$$I=(1-\hat{\rho})^2\,\frac{\hat{d}^2}{T}\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^2\right],$$

$$\begin{aligned} II &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]^2+\frac{(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2\\ &\quad -\frac{2\,(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right], \end{aligned}$$

$$\begin{aligned} III &= \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\hat{\rho}x_{it-1}\right)\right]^2\\ &\quad +\frac{\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]^2\\ &\quad -\frac{2\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\hat{\rho}x_{it-1}\right)\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right], \end{aligned}$$

and $IV\le\sqrt{I\times II}$, $V\le\sqrt{I\times III}$, $VI\le\sqrt{II\times III}$.

Proof. It can be shown that

$$\begin{aligned} \left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\right)\hat{\mathbf{u}}^* &= \left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\hat{\mathbf{u}}\\ &= \left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\left[\left(I_n\otimes\boldsymbol{\iota}_T\right)\boldsymbol{\mu}+\boldsymbol{\nu}+\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right)\right]\\ &= (1-\hat{\rho})\left(E_n\otimes\hat{\boldsymbol{\iota}}_T^{\alpha}\right)\boldsymbol{\mu}+\left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)E_{nT}\boldsymbol{\nu}+\left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)E_{nT}\mathbf{x}\left(\hat{\beta}_{OLS}-\beta\right). \end{aligned}$$

using

$$\begin{aligned} &\left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n\otimes\boldsymbol{\iota}_T\right)\boldsymbol{\mu}\\ =\ &\left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_n\otimes\boldsymbol{\iota}_T-\bar{J}_n\otimes\boldsymbol{\iota}_T\right)\boldsymbol{\mu}\\ =\ &\left(E_n\otimes\widehat{\bar{J}}_T^{\alpha}\hat{C}\boldsymbol{\iota}_T\right)\boldsymbol{\mu}\\ =\ &(1-\hat{\rho})\left(E_n\otimes\hat{\boldsymbol{\iota}}_T^{\alpha}\right)\boldsymbol{\mu} \end{aligned}$$

where $\hat{C}\iota_T = (1-\hat{\rho})\,\hat{\boldsymbol{\iota}}_T^{\alpha}$ and $\widehat{\bar{J}}_T^{\alpha}\hat{\boldsymbol{\iota}}_T^{\alpha} = \hat{\boldsymbol{\iota}}_T^{\alpha}$. Therefore,

$$
\begin{aligned}
& \frac{1}{T}\hat{\sigma}_{\alpha}^{2} \\
= \; & \frac{1}{nT}\hat{\mathbf{u}}^{*\prime}\left(I_n \otimes \widehat{\bar{J}}_T^{\alpha}\right)\hat{\mathbf{u}}^{*} \\
= \; & \frac{1}{nT}(1-\hat{\rho})^2\hat{d}^2\boldsymbol{\mu}' E_n\boldsymbol{\mu} + \frac{1}{nT}\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\
& + \frac{1}{nT}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\
& + \frac{2}{nT}(1-\hat{\rho})\boldsymbol{\mu}'\left(E_n \otimes \hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\
& + \frac{2}{nT}(1-\hat{\rho})\left(\hat{\beta}_{OLS}-\beta\right)\boldsymbol{\mu}'\left(E_n \otimes \hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\
& + \frac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\mathbf{v}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\
= \; & I + II + III + IV + V + VI.
\end{aligned}
$$

Consider I.

$$
\begin{aligned}
I &= \frac{1}{nT}(1-\hat{\rho})^2\hat{d}^2\boldsymbol{\mu}' E_n\boldsymbol{\mu} \\
&= (1-\hat{\rho})^2\frac{\hat{d}^2}{T}\left(\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{n} - \frac{\boldsymbol{\mu}'\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{\mu}}{n^2}\right) \\
&= (1-\hat{\rho})^2\frac{\hat{d}^2}{T}\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^2\right].
\end{aligned}
$$

Consider II.

$$
\begin{aligned}
II &= \frac{1}{nT}\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\
&= \frac{1}{nT}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\boldsymbol{\nu} + \frac{1}{nT}\boldsymbol{\nu}'\bar{J}_{nT}\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\bar{J}_{nT}\boldsymbol{\nu} - \frac{2}{nT}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\bar{J}_{nT}\boldsymbol{\nu} \\
&= \frac{1}{nT}\boldsymbol{\nu}'\left(I_n \otimes \hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\boldsymbol{\nu} + \frac{(1-\hat{\rho})^2\hat{d}^2}{nT^2}\boldsymbol{\nu}'\bar{J}_{nT}\boldsymbol{\nu} - \frac{2(1-\hat{\rho})}{nT^2}\boldsymbol{\nu}'\left(\bar{J}_n \otimes \hat{C}'\hat{\boldsymbol{\iota}}_T^{\alpha}\boldsymbol{\iota}_T'\right)\boldsymbol{\nu}.
\end{aligned}
$$

This can be simplified as follows:

$$
\begin{aligned}
II \approx \; & \frac{1}{nT}\frac{1}{\hat{d}^2}\sum_{i=1}^{n}\left[\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]^2 + \frac{(1-\hat{\rho})^2\hat{d}^2}{nT^2}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2 \\
& - \frac{2(1-\hat{\rho})}{nT^2}\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right] \\
= \; & \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]^2 + \frac{(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2 \\
& - \frac{2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\nu_{it}-\hat{\rho}\nu_{it-1}\right)\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right].
\end{aligned}
$$

Consider IV.

$$
\begin{aligned}
IV &= \frac{2}{nT}(1-\hat{\rho})\boldsymbol{\mu}'\left(E_n \otimes \hat{\imath}_T^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\
&\leq \sqrt{\left[\frac{1}{nT}(1-\hat{\rho})^2\hat{d}^2\boldsymbol{\mu}'E_n\boldsymbol{\mu}\right]\left[\frac{1}{nT}\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n\otimes\hat{C}'\hat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu}\right]} \\
&= \sqrt{I\times II}
\end{aligned}
$$

by the Cauchy-Schwarz inequality. Consider III next. In a similar process as II, one can easily verify that

$$
\begin{aligned}
III \approx\; & \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\hat{\rho}x_{it-1}\right)\right]^2 \\
&+\frac{\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]^2 \\
&-\frac{2\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\hat{\rho}x_{it-1}\right)\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right].
\end{aligned}
$$

We can obtain V and VI as well by using the Cauchy-Schwarz inequality. ■

Lemma 3 (B)

$$
\begin{aligned}
\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x} &= \frac{1}{\hat{\sigma}_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\mathbf{x}_i-\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2}\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\mathbf{x}_i\right), \\
\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\hat{\sigma}_\alpha^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\right), \\
\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{n\hat{\theta}}{\hat{\sigma}_\alpha^2}, \\
\mathbf{x}'\hat{\Phi}^{-1}\mathbf{u} &= \frac{1}{\hat{\sigma}_e^2}\sum_{i=1}^{n}\left[\frac{\mu_i\hat{\sigma}_e^2}{\hat{\sigma}_\alpha^2}\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T+\left(\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i-\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2}\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i\right)\right], \\
\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\mathbf{u} &= \sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_\alpha^2}\left(\hat{\theta}\mu_i+\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i\right)\right].
\end{aligned}
$$

Proof. See Baltagi *et al.* (2008). ■

Now we are ready to prove Theorem 4.

Proof. Assume $(n,T)\to\infty$ and $\frac{n}{T}\to 0$.

1. When $|\rho|<1$, $|\lambda|<1$, if $\widehat{\rho}\xrightarrow{p}\rho$

   (a) First, let us show that $\hat{\sigma}_e^2$ is a consistent estimator.
   From Lemma 1 (B), we consider I first. It can be shown that

$$
\begin{aligned}
I &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2-2\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1}+\left(\widehat{\rho}-\rho\right)^2\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^2\right] \\
&\quad-\left(\frac{T}{\hat{d}^2}\right)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]^2 \\
&\xrightarrow{p}\sigma_e^2
\end{aligned}
$$

as $(n,T)\to\infty$. This is because in the first term,

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^{2}-2\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1}+\left(\widehat{\rho}-\rho\right)^{2}\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^{2}\right] \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^{2}-2\sqrt{nT}\left(\widehat{\rho}-\rho\right)\frac{1}{nT}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\nu_{it-1} \\
& +\left(\sqrt{nT}\left(\widehat{\rho}-\rho\right)\right)^{2}\frac{1}{nT}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}^{2} \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^{2}+O_{p}\left(\frac{1}{nT}\right)+O_{p}\left(\frac{1}{nT}\right) \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^{2}+O_{p}\left(\frac{1}{nT}\right)\xrightarrow{p}\sigma_{e}^{2}
\end{aligned}$$

using $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\nu_{it-1}=O_p(1)$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}^{2}=O_p(1)$, and $\sqrt{nT}\left(\widehat{\rho}-\rho\right)=O_p(1)$.
Let us look at the second term. One can verify that

$$\begin{aligned}
& \left(\frac{T}{\hat{d}^{2}}\right)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-\left(\widehat{\rho}-\rho\right)\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]^{2} \\
= & \left(\frac{T}{\hat{d}^{2}}\right)\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^{2}+\left(\frac{T}{\hat{d}^{2}}\right)\left(\sqrt{nT}\left(\widehat{\rho}-\rho\right)\right)^{2}\frac{1}{nT^{2}}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right)^{2} \\
& -2\left(\frac{T}{\hat{d}^{2}}\right)\frac{\sqrt{nT}\left(\widehat{\rho}-\rho\right)}{\sqrt{n}T^{3/2}}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right) \\
= & O_{p}(\frac{1}{T})+O_{p}(\frac{1}{nT^{2}})+O_{p}(\frac{1}{\sqrt{n}T^{3/2}})=O_{p}(\frac{1}{T})
\end{aligned}$$

using

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^{2} &= O_{p}(1), \\
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right)^{2} &= O_{p}(1),
\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right)=O_{p}(1).$$

Also note that $\frac{T}{\hat{d}^{2}}=\frac{T}{\frac{2\hat{\rho}}{1-\hat{\rho}}+T}\xrightarrow{p}1$ as $T\to\infty$.
Consider II. It can be also shown that

$$\begin{aligned}
II = & \frac{\left(\hat{\beta}_{OLS}-\beta\right)^{2}}{n}\sum_{i=1}^{n}[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{2}+\tfrac{(\widehat{\rho}-\rho)^{2}}{T}\sum_{t=1}^{T}x_{it-1}^{2}+\tfrac{(\lambda-\rho)^{2}}{T}\sum_{t=1}^{T}x_{it-1}^{2} \\
& -\tfrac{2(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}+\tfrac{2(\lambda-\rho)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}-\tfrac{2(\widehat{\rho}-\rho)(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1}^{2}] \\
& -\frac{\left(\hat{\beta}_{OLS}-\beta\right)^{2}T}{n\hat{d}^{2}}\sum_{i=1}^{n}\left[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}-\left(\widehat{\rho}-\rho\right)\tfrac{1}{T}\sum_{t=1}^{T}x_{it-1}+(\lambda-\rho)\tfrac{1}{T}\sum_{t=1}^{T}x_{it-1}\right]^{2} \\
= & O_{p}(\tfrac{1}{nT}).
\end{aligned}$$

This follows because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d} N\left(0,\frac{\sigma_\mu^2}{\sigma_\varepsilon^2}+\frac{(1+\rho\lambda)(1-\lambda^2)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right)$$

using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008).
Consider III. From Lemma 1 (B), we know that

$$III\leq\sqrt{I\times II}\xrightarrow{p} 0.$$

Hence,

$$\hat{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

(b) Next, we show that $\hat{\sigma}_\mu^2$ is a consistent estimator of σ_μ^2. From Lemma 2 (B), one can see that

$$I=(1-\hat{\rho})^2\frac{\hat{d}^2}{T}\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^2\right]\xrightarrow{p}(1-\rho)^2\,\sigma_\mu^2.$$

Consider II next. It can be shown that

$$\begin{aligned} II &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}(\nu_{it}-\hat{\rho}\nu_{it-1})\right]^2+\frac{(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2 \\ &\quad -\frac{2\,(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(\nu_{it}-\hat{\rho}\nu_{it-1})\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right] \\ &= O_p(\frac{1}{T})+O_p(\frac{1}{n^{3/2}T^{3/2}})+O_p(\frac{1}{n^2T})=O_p(\frac{1}{T}). \end{aligned}$$

For the first term, from (a), we know that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}(\nu_{it}-\hat{\rho}\nu_{it-1})\right]^2=O_p(\frac{1}{T}).$$

For the second term,

$$\begin{aligned} &\frac{(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2 \\ =\ &\frac{(1-\hat{\rho})^2}{n^{3/2}T^{3/2}}\frac{\hat{d}^2}{T}\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right]^2=O_p(\frac{1}{n^{3/2}T^{3/2}}) \end{aligned}$$

using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}=O_p(1).$$

Let us look at the last term. We have

$$\begin{aligned}
& \frac{2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(\nu_{it}-\hat{\rho}\nu_{it-1})\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right] \\
= & \frac{2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}-(\widehat{\rho}-\rho)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right] \\
= & \frac{2(1-\hat{\rho})}{n^2T}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right) \\
& -\frac{2(1-\hat{\rho})}{n^{5/2}T^{3/2}}\sqrt{nT}(\widehat{\rho}-\rho)\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}\right)\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it}\right) \\
= & O_p(\frac{1}{n^2T})+O_p(\frac{1}{n^{5/2}T^{3/2}})=O_p(\frac{1}{n^2T}).
\end{aligned}$$

Accordingly, we have

$$IV \leq \sqrt{I\times II}\xrightarrow{p}0.$$

Finally, consider III.

$$\begin{aligned}
III = & \frac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\hat{\rho}x_{it-1})\right]^2 \\
& +\frac{\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})^2}{nT}\frac{\hat{d}^2}{T}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]^2 \\
& -\frac{2\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})}{n}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\hat{\rho}x_{it-1})\right]\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right].
\end{aligned}$$

It can be easily shown that $III=o_p(1)$ as $(n,T)\to\infty$ in a similar way as above. This follows because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{\sigma_\mu^2}{\sigma_\varepsilon^2}+\frac{(1+\rho\lambda)(1-\lambda^2)\sigma_e^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2}\right).$$

Hence, by the fact that $V\leq\sqrt{I\times III}\xrightarrow{p}0$ and $VI\leq\sqrt{II\times III}\xrightarrow{p}0$, we get

$$\frac{1}{T}\hat{\sigma}_\alpha^2\xrightarrow{p}(1-\rho)^2\sigma_\mu^2.$$

One concludes that

$$\hat{\sigma}_\mu^2=\frac{1}{\hat{\theta}}\left(\hat{\sigma}_\alpha^2-\hat{\sigma}_e^2\right)=\frac{T}{\hat{\theta}}\left(\frac{\hat{\sigma}_\alpha^2}{T}-\frac{\hat{\sigma}_e^2}{T}\right)\xrightarrow{p}\frac{1}{(1-\rho)^2}\left[(1-\rho)^2\sigma_\mu^2-0\right]=\sigma_\mu^2$$

using

$$\begin{aligned}
\frac{1}{T}\hat{\theta} = & \frac{1}{T}(1-\hat{\rho})^2\hat{d}^2 \\
= & \frac{1}{T}(1-\hat{\rho})^2\left[\left(\frac{2\hat{\rho}}{1-\hat{\rho}}\right)+T\right] \\
= & (1-\hat{\rho})^2\left[\frac{1}{T}\left(\frac{2\hat{\rho}}{1-\hat{\rho}}\right)+1\right]\xrightarrow{p}(1-\rho)^2.
\end{aligned}$$

(c) Let us calculate the term $\widehat{G}_1$ in equation (14) first.

$$\frac{1}{nT}\widehat{G}_1 = \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{n}.$$

We investigate

$$\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} = \frac{1}{n\hat{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\mathbf{x}_i}{T} - \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\mathbf{x}_i}{T}\right)$$

from Lemma 3 (B). As shown in Theorem 3.1.(a), one can see that

$$\frac{1}{n\hat{\sigma}_e^2}\sum_{i=1}^{n}\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i \xrightarrow{p} \frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{(1-\lambda^2)\,\sigma_e^2}.$$

Next, it can be shown that

$$\begin{aligned}
&\frac{1}{n}\frac{1}{\widehat{\sigma}_e^2}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i}{T}\right)\\
=&\ \frac{(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it} + \frac{(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1} - \frac{(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right]^2\\
=&\ \frac{(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\right)^2 + \left(\frac{(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right)^2 + \left(\frac{(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right)^2\\ 2\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right) - 2\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right)\\ -2\left(\frac{(\lambda-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right)\left(\frac{(\widehat{\rho}-\rho)}{T}\sum_{t=1}^{T}x_{it-1}\right)\end{array}\right]\\
=&\ I + II + III + IV + V + VI.
\end{aligned}$$

Consider I and II. One can see that

$$I = \frac{(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)^2 = O_p\left(\frac{1}{T}\right)$$

and

$$II = \frac{(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\lambda-\rho)}{\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right)^2 = O_p\left(\frac{1}{T}\right).$$

Consider III.

$$III = \frac{(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\left(\sqrt{nT}(\widehat{\rho}-\rho)\right)^2\frac{1}{nT^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right)^2 = O_p\left(\frac{1}{nT^2}\right).$$

Consider IV.

$$IV = \frac{2(1-\widehat{\rho})^2}{\widehat{\sigma}_e^2}\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{(\lambda-\rho)}{\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right) = O_p\left(\frac{1}{T}\right)$$

using

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{(\lambda-\rho)}{\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right) = O_p(1).$$

Lastly, consider V and VI. It can be shown that

$$V = -\frac{2\left(1-\widehat{\rho}\right)^{2}}{\widehat{\sigma}_{e}^{2}} \frac{\sqrt{nT}(\widehat{\rho}-\rho)}{\sqrt{n}T^{3/2}} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_{it} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it-1} \right) = O_{p}\left(\frac{1}{\sqrt{n}T^{3/2}} \right)$$

and

$$VI = -\frac{2\left(1-\widehat{\rho}\right)^{2}}{\widehat{\sigma}_{e}^{2}} \frac{\sqrt{nT}(\widehat{\rho}-\rho)}{\sqrt{n}T^{3/2}} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{(\lambda-\rho)}{\sqrt{T}} \sum_{t=1}^{T} x_{it-1} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it-1} \right) = O_{p}\left(\frac{1}{\sqrt{n}T^{3/2}} \right).$$

Therefore, we conclude that

$$\begin{aligned} & \frac{1}{n} \frac{1}{\widehat{\sigma}_{e}^{2}} \sum_{i=1}^{n} \left(\frac{\mathbf{x}_{i}' \hat{\mathbf{A}}^{-1} \boldsymbol{\iota}_{T}}{T} \frac{\boldsymbol{\iota}_{T}' \hat{\mathbf{A}}^{-1} \mathbf{x}_{i}}{T} \right) \\ = \; & O_{p}\left(\frac{1}{T}\right) + O_{p}\left(\frac{1}{T}\right) + O_{p}\left(\frac{1}{nT^{2}}\right) + O_{p}\left(\frac{1}{T}\right) + O_{p}\left(\frac{1}{\sqrt{n}T^{3/2}}\right) + O_{p}\left(\frac{1}{\sqrt{n}T^{3/2}}\right) \\ = \; & O_{p}\left(\frac{1}{T}\right). \end{aligned}$$

Then we have

$$\frac{1}{nT} \mathbf{x}' \widehat{\Phi}^{-1} \mathbf{x} \xrightarrow{p} \frac{(1-2\rho\lambda+\rho^{2})\sigma_{\varepsilon}^{2}}{\left(1-\lambda^{2}\right)\sigma_{e}^{2}}.$$

One can also verify that

$$\frac{1}{n} \mathbf{x}' \hat{\Phi}^{-1} \boldsymbol{\iota}_{nT} = \frac{1}{\sqrt{nT}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{1}{\hat{\sigma}_{\alpha}^{2}/T} \frac{\mathbf{x}_{i}' \hat{\mathbf{A}}^{-1} \boldsymbol{\iota}_{T}}{\sqrt{T}} \right) = O_{p}\left(\frac{1}{\sqrt{nT}} \right)$$

and

$$\frac{1}{n} \boldsymbol{\iota}_{nT}' \hat{\Phi}^{-1} \boldsymbol{\iota}_{nT} = \frac{\hat{\theta}/T}{\hat{\sigma}_{\alpha}^{2}/T} \xrightarrow{p} \frac{\left(1-\rho\right)^{2}}{\left(1-\rho\right)^{2}\sigma_{\mu}^{2}} = \frac{1}{\sigma_{\mu}^{2}}.$$

Finally, we get

$$\frac{1}{nT} \widehat{G}_{1} \xrightarrow{p} \frac{(1-2\rho\lambda+\rho^{2})\sigma_{\varepsilon}^{2}}{\left(1-\lambda^{2}\right)\sigma_{e}^{2}}$$

as $(n,T) \to \infty$.

Now we turn to $\widehat{G}_{2}$. Note that

$$\frac{1}{\sqrt{nT}} \widehat{G}_{2} = \frac{1}{\sqrt{nT}} \mathbf{x}' \widehat{\Phi}^{-1} \mathbf{u} - \frac{1}{\sqrt{T}} \frac{\mathbf{x}' \widehat{\Phi}^{-1} \boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}' \widehat{\Phi}^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}_{nT}' \widehat{\Phi}^{-1} \mathbf{u}}{\sqrt{n}}.$$

Consider first

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \mathbf{x}' \hat{\Phi}^{-1} \mathbf{u} \\ = \; & \frac{1}{\hat{\sigma}_{e}^{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{\sqrt{T}} \frac{\hat{\sigma}_{e}^{2}}{\hat{\sigma}_{\alpha}^{2}/T} \mu_{i} \frac{\mathbf{x}_{i}' \hat{\mathbf{A}}^{-1} \boldsymbol{\iota}_{T}}{T} + \left(\frac{\mathbf{x}_{i}' \hat{\mathbf{A}}^{-1} \boldsymbol{\nu}_{i}}{\sqrt{T}} - \frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}/T} \frac{\mathbf{x}_{i}' \hat{\mathbf{A}}^{-1} \boldsymbol{\iota}_{T}}{T} \frac{\boldsymbol{\iota}_{T}' \hat{\mathbf{A}}^{-1} \boldsymbol{\nu}_{i}}{\sqrt{T}} \right) \right]. \end{aligned}$$

For the first term, one can show that

$$\begin{aligned}
&\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_\alpha^2/T}\mu_i\right)\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T}\right)\\
=&\ (1-\hat{\rho})\left(\frac{1}{\hat{\sigma}_\alpha^2/T}\right)\frac{1}{T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i\left[x_{it}-\hat{\rho}x_{it-1}\right]\\
=&\ (1-\hat{\rho})\left(\frac{1}{\hat{\sigma}_\alpha^2/T}\right)\frac{1}{T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i\left[\varepsilon_{it}+(\lambda-\rho)x_{it-1}-\sqrt{nT}(\hat{\rho}-\rho)\frac{x_{it-1}}{\sqrt{nT}}\right]\\
=&\ O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{\sqrt{n}T^{3/2}}\right)=O_p\left(\frac{1}{T}\right)
\end{aligned}$$

using

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i\varepsilon_{it}=O_p(1)$$

and

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i x_{it-1}=O_p(1).$$

Also recall from Theorem 3.1.(b) that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\xrightarrow{d}N\left(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_e^2\sigma_\varepsilon^2}{1-\lambda^2}\right)$$

as $(n,T)\to\infty$.

For the last term, it can be shown that

$$\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}=O_p\left(\frac{1}{\sqrt{T}}\right)$$

in a similar way as above.

Therefore, we conclude that

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{u} &= \frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)\\
&= \frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}+O_p\left(\frac{1}{\sqrt{T}}\right)\\
&\xrightarrow{d}N\left(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{\left(1-\lambda^2\right)\sigma_e^2}\right).
\end{aligned}$$

Next, recall that $\frac{1}{n}\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$ from above.

Finally, it can be shown that

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\mathbf{u}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_e^2+\hat{\theta}\hat{\sigma}_\mu^2}\left(\hat{\theta}\mu_i+\iota_T'\hat{\mathbf{A}}^{-1}\nu_i\right)\\
=&\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\left[\left(\frac{\hat{\theta}}{T}\right)\mu_i+\frac{1}{\sqrt{T}}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\\
=&\ I+II.
\end{aligned}$$

Consider I.

$$\begin{aligned} I &= \left(\frac{\hat{\theta}}{T}\right)\left(\frac{1}{\hat{\sigma}_\alpha^2/T}\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i \\ &\xrightarrow{d} \frac{1}{(1-\rho)^2\sigma_\mu^2}(1-\rho)^2 N\left(0,\sigma_\mu^2\right) \end{aligned}$$

using $\frac{\hat{\theta}}{T} \to (1-\rho)^2$ and $\hat{\sigma}_\alpha^2/T \to (1-\rho)^2\sigma_\mu^2$.
Consider II.

$$\begin{aligned} II &= \frac{1}{\hat{\sigma}_\alpha^2/T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i = \frac{(1-\widehat{\rho})}{\hat{\sigma}_\alpha^2/T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\nu_{it}-\widehat{\rho}\nu_{it-1}) \\ &= \frac{(1-\widehat{\rho})}{\hat{\sigma}_\alpha^2/T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[e_{it}-(\widehat{\rho}-\rho)\nu_{it-1}\right] \\ &= \frac{(1-\widehat{\rho})}{\hat{\sigma}_\alpha^2/T}\frac{1}{\sqrt{T}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it} - \frac{(1-\widehat{\rho})}{\hat{\sigma}_\alpha^2/T}\frac{\sqrt{nT}(\widehat{\rho}-\rho)}{\sqrt{nT}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1} \\ &= O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right) = O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\mathbf{u} &= \left(\frac{\hat{\theta}}{T}\right)\left(\frac{1}{\hat{\sigma}_\alpha^2/T}\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i + O_p\left(\frac{1}{\sqrt{T}}\right) \\ &\xrightarrow{d} \frac{1}{(1-\rho)^2\sigma_\mu^2}(1-\rho)^2 N\left(0,\sigma_\mu^2\right) = N\left(0,\frac{1}{\sigma_\mu^2}\right). \end{aligned}$$

Summarizing, we have

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 \xrightarrow{d} N(0,\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{(1-\lambda^2)\sigma_e^2})$$

as $(n,T)\to\infty$. Finally,

$$t_{FGLS} = \left(\frac{1}{nT}\widehat{G}_1\right)^{-1/2}\left(\frac{1}{\sqrt{nT}}\widehat{G}_2\right) \xrightarrow{d} N(0,1).$$

2. When $\rho = 1$, $|\lambda|<1$, if $T(\widehat{\rho}-1) \xrightarrow{p} \kappa$

   (a) First, let us show that $\hat{\sigma}_e^2$ is a consistent estimator of σ_e^2.
   From Lemma 1 (B), it can be shown, in a similar way as 1.(a) that

$$\begin{aligned} I &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2 - \frac{2\left[T(\widehat{\rho}-1)\right]}{T}\left(\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1}\right)+\frac{\left[T(\widehat{\rho}-1)\right]^2}{T}\left(\frac{1}{T^2}\sum_{t=1}^{T}\nu_{it-1}^2\right)\right] \\ &\quad -\left(\frac{T}{\hat{d}^2}\right)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-\frac{1}{\sqrt{T}}\left[T(\widehat{\rho}-1)\right]\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\right]^2 \\ &\xrightarrow{p} \sigma_e^2 \end{aligned}$$

as $(n,T)\to\infty$ using $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^{2}\xrightarrow{p}\sigma_{e}^{2}$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}\nu_{it-1}=O_{p}(1)$, $\frac{1}{nT^{2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}^{2}=O_{p}(1)$, and $\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\nu_{it-1}=O_{p}(1)$. Also note that $T(\widehat{\rho}-1)\xrightarrow{p}\kappa$ and

$$\frac{T}{\hat{d}^{2}}=\frac{T}{\frac{1+\widehat{\rho}}{1-\widehat{\rho}}+T-1}=\frac{T(1-\widehat{\rho})}{2\widehat{\rho}+T(1-\widehat{\rho})}\xrightarrow{p}\frac{-\kappa}{2-\kappa}.$$

Consider II. From Lemma 1 (B), we have

$$\begin{aligned}
&\tfrac{1}{n}\sum_{i=1}^{n}\Big[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{2}+\tfrac{(\widehat{\rho}-1)^{2}}{T}\sum_{t=1}^{T}x_{it-1}^{2}+\tfrac{(\lambda-1)^{2}}{T}\sum_{t=1}^{T}x_{it-1}^{2}\\
&-\tfrac{2(\widehat{\rho}-1)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}+\tfrac{2(\lambda-1)}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}-\tfrac{2(\widehat{\rho}-1)(\lambda-1)}{T}\sum_{t=1}^{T}x_{it-1}^{2}\Big]\\
&-\tfrac{T}{n\hat{d}^{2}}\sum_{i=1}^{n}\left[\tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}-(\widehat{\rho}-1)\tfrac{1}{T}\sum_{t=1}^{T}x_{it-1}+(\lambda-1)\tfrac{1}{T}\sum_{t=1}^{T}x_{it-1}\right]^{2}\\
=\;&2\sigma_{\varepsilon}^{2}/(1+\lambda)+o_{p}(1)
\end{aligned}$$

as $(n,T)\to\infty$. This is because $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^{2}\xrightarrow{p}\sigma_{\varepsilon}^{2}$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}=O_{p}(1)$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^{2}\xrightarrow{p}\sigma_{\varepsilon}^{2}/(1-\lambda^{2})$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}=O_{p}(1)$, and $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}=O_{p}(1)$.
Hence,

$$II\approx\frac{\left[\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right]^{2}}{n}\frac{1}{nT}\mathbf{x}'\left(I_{n}\otimes\hat{C}\hat{E}_{T}^{\alpha}\hat{C}\right)\mathbf{x}=O_{p}(\frac{1}{n})=o_{p}(1)$$

using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{(1-\lambda)^{2}\sigma_{e}^{2}}{2\sigma_{\varepsilon}^{2}}\right).$$

This follows from a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008). Since $III\leq\sqrt{I\times II}\xrightarrow{p}0$, we conclude that

$$\hat{\sigma}_{e}^{2}\xrightarrow{p}\sigma_{e}^{2}.$$

(b) Let us show that $\hat{\sigma}_{\mu}^{2}$ is not a consistent estimator of σ_{μ}^{2}.
Using Lemma 2 (B), we have

$$\begin{aligned}
\tfrac{1}{T(1-\hat{\rho})}\hat{\sigma}_{\alpha}^{2}&=\tfrac{1}{nT(1-\hat{\rho})}\hat{\mathbf{u}}^{*\prime}\left(I_{n}\otimes\widehat{\bar{J}}_{T}^{\alpha}\right)\hat{\mathbf{u}}^{*}\\
&=\tfrac{(1-\hat{\rho})\hat{d}^{2}}{nT}\boldsymbol{\mu}'E_{n}\boldsymbol{\mu}+\tfrac{1}{nT(1-\hat{\rho})}\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_{n}\otimes\hat{C}'\widehat{\bar{J}}_{T}^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu}\\
&\quad+\tfrac{1}{nT(1-\hat{\rho})}\left(\hat{\beta}_{OLS}-\beta\right)^{2}\mathbf{x}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_{n}\otimes\hat{C}'\widehat{\bar{J}}_{T}^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x}\\
&\quad+\tfrac{2}{nT}\boldsymbol{\mu}'\left(E_{n}\otimes\hat{\boldsymbol{\iota}}_{T}^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu}\\
&\quad+\tfrac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\boldsymbol{\mu}'\left(E_{n}\otimes\hat{\boldsymbol{\iota}}_{T}^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x}\\
&\quad+\tfrac{1}{nT(1-\hat{\rho})}\left(\hat{\beta}_{OLS}-\beta\right)\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_{n}\otimes\hat{C}'\widehat{\bar{J}}_{T}^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x}\\
&=I+II+III+IV+V+VI.
\end{aligned}$$

Consider I. It is easy to see that

$$I=\frac{(1-\hat{\rho})\hat{d}^{2}}{T}\left[\frac{1}{n}\sum_{i=1}^{n}\mu_{i}^{2}-\left(\frac{1}{n}\sum_{i=1}^{n}\mu_{i}\right)^{2}\right]\xrightarrow{p}0$$

using $(1-\hat{\rho})\hat{d}^{2}=2\widehat{\rho}+T(1-\widehat{\rho})\xrightarrow{p}2-\kappa$.

Consider II.

$$
\begin{aligned}
II &= \tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{(1-\hat{\rho})\hat{d}^2}\left[\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}+\left[T\left(1-\hat{\rho}\right)\right]\left(\tfrac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\right]^2 \\
&\quad +\tfrac{(1-\hat{\rho})\hat{d}^2}{nT}\left[\tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right]^2 \\
&\quad -\tfrac{2T(1-\hat{\rho})}{nT}\left[\tfrac{1}{n}\sum_{i=1}^{n}\left(\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}+T\left(1-\hat{\rho}\right)\tfrac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\right]\left[\tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right].
\end{aligned}
$$

Let us look at the first term. It can be shown that

$$
\begin{aligned}
&\frac{1}{\left(1-\hat{\rho}\right)\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}+\left[T\left(1-\hat{\rho}\right)\right]\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\right]^2 \\
&= \frac{1}{\left(1-\hat{\rho}\right)\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^2+\left[T\left(1-\hat{\rho}\right)\right]^2\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)^2 \\ +2T\left(1-\hat{\rho}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\end{array}\right] \\
&= \frac{1}{\left(1-\hat{\rho}\right)\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^2+\frac{\left(T\left(1-\hat{\rho}\right)\right)^2}{\left(1-\hat{\rho}\right)\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)^2 \\
&\quad +\frac{2T\left(1-\hat{\rho}\right)}{\left(1-\hat{\rho}\right)\hat{d}^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right) \\
&\quad \xrightarrow{p} \frac{1}{2-\kappa}\sigma_e^2\left(\frac{\kappa^2}{3}-\kappa+1\right)
\end{aligned}
$$

since

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^2 &\xrightarrow{p} \sigma_e^2, \\
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)^2 &\xrightarrow{p} \frac{\sigma_e^2}{3},
\end{aligned}
$$

and

$$
\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right)\xrightarrow{p}\frac{\sigma_e^2}{2}.
$$

Also note that the second and third terms of II are $o_p\left(1\right)$ as $(n,T)\to\infty$.
Therefore,

$$
II \xrightarrow{p} \frac{\kappa^2-3\kappa+3}{3\left(2-\kappa\right)}\sigma_e^2
$$

and from Lemma 2 (B),

$$
IV \leq \sqrt{I\times II}\xrightarrow{p} 0.
$$

Consider III next.

$$
\begin{aligned}
III &= \tfrac{\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\sum_{i=1}^{n}\left(\tfrac{1}{(1-\hat{\rho})\hat{d}^2}\right)\left[\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-\tfrac{T(\hat{\rho}-1)}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}+\tfrac{(\lambda-1)}{\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right]^2 \\
&\quad +\tfrac{\left(\hat{\beta}_{OLS}-\beta\right)^2(1-\hat{\rho})\hat{d}^2}{nT^2}\left[\tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{T}\sum_{t=1}^{T}x_{it}\right]^2 \\
&\quad -\tfrac{2\left(\hat{\beta}_{OLS}-\beta\right)^2}{n}\left[\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\hat{\rho}x_{i,t-1}\right)\right]\left[\tfrac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right].
\end{aligned}
$$

In a similar process as in II, one can verify that $III=O_p(\frac{1}{n})$ as $(n,T)\to\infty$ using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$
\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d} N\left(0,\frac{(1-\lambda)^2\sigma_e^2}{2\sigma_\varepsilon^2}\right)
$$

and accordingly that $V \leq \sqrt{I \times III} \xrightarrow{p} 0$, $VI \leq \sqrt{II \times III} \xrightarrow{p} 0$, respectively.
Summarizing, we have

$$\frac{\hat{\sigma}_\alpha^2}{T(1-\hat{\rho})} \xrightarrow{p} \frac{\kappa^2 - 3\kappa + 3}{3(2-\kappa)}\sigma_e^2.$$

Since $\hat{\sigma}_\mu^2 = \frac{1}{\hat{\theta}}\left(\hat{\sigma}_\alpha^2 - \hat{\sigma}_e^2\right)$ and $\hat{\theta} = (1-\hat{\rho})^2 \hat{d}^2$, we have

$$\begin{aligned}
(1-\hat{\rho})\,\hat{\sigma}_\mu^2 &= (1-\hat{\rho})\tfrac{1}{\hat{\theta}}\left(\hat{\sigma}_\alpha^2 - \hat{\sigma}_e^2\right) \\
&= \tfrac{1}{(1-\hat{\rho})\hat{d}^2}\left(\hat{\sigma}_\alpha^2 - \hat{\sigma}_e^2\right) \\
&= \left(\tfrac{T}{\hat{d}^2}\right)\left(\tfrac{\hat{\sigma}_\alpha^2}{T(1-\hat{\rho})}\right) - \tfrac{\hat{\sigma}_e^2}{(1-\hat{\rho})\hat{d}^2} \\
&\xrightarrow{p} \left(\tfrac{-\kappa}{2-\kappa}\right)\left(\tfrac{\kappa^2-3\kappa+3}{3(2-\kappa)}\sigma_e^2\right) - \left(\tfrac{1}{2-\kappa}\sigma_e^2\right) = \tfrac{-\kappa^3+3\kappa^2-6}{3(2-\kappa)^2}\sigma_e^2.
\end{aligned}$$

If we plug $k = -3$ into this equation, we get

$$(1-\hat{\rho})\,\hat{\sigma}_\mu^2 \xrightarrow{p} \frac{48}{75}\sigma_e^2 = \frac{16}{25}\sigma_e^2.$$

(c) We start from $\widehat{G}_1$ in equation (14). Let us define

$$\frac{1}{nT}\widehat{G}_1 = \frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{T}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{n}.$$

From Lemma 3 (B), we have

$$\frac{1}{nT}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x} = \frac{1}{n\hat{\sigma}_e^2}\sum_{i=1}^n\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\mathbf{x}_i}{T} - T(1-\widehat{\rho})^2\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T(1-\widehat{\rho})}\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\mathbf{x}_i}{T(1-\widehat{\rho})}\right).$$

Firstly, recall from Theorem 3.2.(a) that

$$\frac{1}{n}\frac{1}{\hat{\sigma}_e^2}\sum_{i=1}^n\frac{1}{T}\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\mathbf{x}_i \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2}.$$

Note also that

$$\frac{1}{n\hat{\sigma}_e^2}\sum_{i=1}^n T(1-\widehat{\rho})^2\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T(1-\widehat{\rho})}\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\mathbf{x}_i}{T(1-\widehat{\rho})} = O_p(\frac{1}{T})$$

using

$$\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T(1-\widehat{\rho})} = O_p(\frac{1}{\sqrt{T}})$$

as shown in 1.(c) and

$$T(1-\widehat{\rho})^2\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2} = \frac{(1-\widehat{\rho})\,\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2/T(1-\widehat{\rho})} \xrightarrow{p} \frac{\frac{-\kappa^3+3\kappa^2-6}{3(2-\kappa)^2}\sigma_e^2}{\frac{\kappa^2-3\kappa+3}{3(2-\kappa)}\sigma_e^2} = \frac{-k^3+3k^2-6}{(2-k)(k^2-3k+3)}.$$

Hence,

$$\frac{1}{nT}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x} \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2}$$

as $(n, T) \to \infty$.

Next, one can shown in a similar way that

$$\begin{aligned}\frac{1}{n}\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T\left(1-\widehat{\rho}\right)}\right)\\ &= \frac{1}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\frac{1}{\sqrt{nT}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}\left(1-\widehat{\rho}\right)}\right)\\ &= O_p(\frac{1}{\sqrt{nT}}).\end{aligned}$$

Also

$$\frac{T}{n}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{\hat{\theta}/\left(1-\widehat{\rho}\right)}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)} \xrightarrow{p} \frac{2-k}{\frac{\kappa^2-3\kappa+3}{3(2-\kappa)}\sigma_e^2} = \frac{1}{\sigma_e^2}\frac{3\left(2-\kappa\right)^2}{\kappa^2-3\kappa+3}.$$

Therefore,

$$\frac{1}{nT}\widehat{G}_1 \xrightarrow{p} \frac{2\sigma_{\varepsilon}^2}{(1+\lambda)\sigma_e^2}.$$

Now we turn to $\widehat{G}_2$**.**

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u} - \frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{T}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\frac{\sqrt{T}}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{u}.$$

We have

$$\begin{aligned}\frac{1}{\sqrt{nT}}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{u} &= \frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\right)\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T\left(1-\widehat{\rho}\right)}\right)\mu_i\\ &\quad+\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\\ &\quad-T\left(1-\widehat{\rho}\right)^2\frac{\hat{\sigma}_{\mu}^2}{\hat{\sigma}_{\alpha}^2}\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T\left(1-\widehat{\rho}\right)}\right)\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\left(1-\widehat{\rho}\right)}\right)\\ &= I+II+III.\end{aligned}$$

For I, with the joint CLT we have

$$\begin{aligned}I &= \frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\right)\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T\left(1-\widehat{\rho}\right)}\right)\mu_i\\ &= \left(\frac{1}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\right)\frac{1}{T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i\left[x_{it}-\hat{\rho}x_{it-1}\right]\\ &= \left(\frac{1}{\hat{\sigma}_{\alpha}^{2}/T\left(1-\widehat{\rho}\right)}\right)\frac{1}{T}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\mu_i\left[\varepsilon_{it}+(\lambda-1)x_{it-1}-T(\hat{\rho}-1)\frac{x_{it-1}}{T}\right]\\ &= O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{\sqrt{n}T^2}\right)=O_p\left(\frac{1}{T}\right).\end{aligned}$$

For II, recall from Theorem 3 that

$$\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \xrightarrow{d} N(0,\frac{2\sigma_{\varepsilon}^2}{(1+\lambda)\,\sigma_e^2}).$$

For III, it is easy to see $III = O_p\left(\frac{1}{\sqrt{T}}\right)$ using

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}\,(1-\widehat{\rho})}\right)\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\,(1-\widehat{\rho})}\right) = O_p\,(1)\,.$$

Also, as shown already, $T\,(1-\widehat{\rho})^2\,\frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2} \xrightarrow{p} \frac{-k^3+3k^2-6}{(2-k)(k^2-3k+3)}$ and $\hat{\sigma}_\alpha^2/T\,(1-\widehat{\rho}) \xrightarrow{p} \frac{\kappa^2-3\kappa+3}{3(2-\kappa)}\sigma_e^2$.
Finally, we conclude that

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{u} \xrightarrow{d} N(0, \frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2})$$

as $(n,T)\to\infty$.
Next it can be shown that

$$\begin{aligned}\frac{\sqrt{T}}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\mathbf{u} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\right)\frac{\hat{\theta}/\,(1-\hat{\rho})}{\hat{\sigma}_\alpha^2/T\,(1-\hat{\rho})}\mu_i + \frac{1}{\hat{\sigma}_\alpha^2/T\,(1-\hat{\rho})}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\,(1-\widehat{\rho})}\right)\right] \\ &= O_p\,(1)\end{aligned}$$

using

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}\,(1-\widehat{\rho})} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it} + \frac{T\,(1-\widehat{\rho})}{T^{3/2}}\sum_{t=1}^{T}\nu_{it-1}\right] \\ &= O_p\,(1)\end{aligned}$$

and

$$\begin{aligned}\hat{\theta}/\,(1-\hat{\rho}) &\xrightarrow{p} 2-k, \\ \frac{\hat{\sigma}_\alpha^2}{T\,(1-\hat{\rho})} &\xrightarrow{p} \frac{\kappa^2-3\kappa+3}{3\,(2-\kappa)}\sigma_e^2,\end{aligned}$$

respectively.
Therefore,

$$\frac{1}{\sqrt{nT}}\widehat{G}_2 \xrightarrow{d} N(0, \frac{2\sigma_\varepsilon^2}{(1+\lambda)\sigma_e^2})$$

as $(n,T)\to\infty$ using $\frac{1}{n}\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{T}{n}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_e^2}\frac{3(2-\kappa)^2}{\kappa^2-3\kappa+3}$ from above.
We conclude that

$$t_{FGLS} = \left(\frac{1}{nT}\widehat{G}_1\right)^{-1/2}\left(\frac{1}{\sqrt{nT}}\widehat{G}_2\right) \xrightarrow{d} N\,(0,1)\,.$$

3. When $|\rho|<1$, $\lambda=1$, if $\widehat{\rho} \xrightarrow{p} \rho$

   (a) First, let us show that $\hat{\sigma}_e^2$ is a consistent estimator of σ_e^2. From Lemma 1 (B), we have

$$\begin{aligned}I &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2 - 2\,(\widehat{\rho}-\rho)\,\frac{1}{T}\sum_{t=1}^{T}e_{it}\nu_{it-1} + (\widehat{\rho}-\rho)^2\,\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}^2\right] \\ &\quad -\left(\frac{T}{\hat{d}^2}\right)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it} - (\widehat{\rho}-\rho)\,\frac{1}{T}\sum_{t=1}^{T}\nu_{it-1}\right]^2 \\ &\xrightarrow{p} \sigma_e^2\end{aligned}$$

as shown in 1.(a).

Consider II. It can be shown similarly that

$$\begin{aligned} II &= \frac{\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\sum_{i=1}^{n}\left[\begin{array}{c} \frac{1}{T^2}\sum_{t=1}^{T}\varepsilon_{it}^2+\frac{\left(\sqrt{nT}(\widehat{\rho}-\rho)\right)^2}{nT^3}\sum_{t=1}^{T}x_{it-1}^2 \\ +\frac{(1-\rho)^2}{T^2}\sum_{t=1}^{T}x_{it-1}^2-\frac{2\sqrt{nT}(\widehat{\rho}-\rho)}{\sqrt{n}T^{5/2}}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1} \\ +\frac{2(1-\rho)}{T^2}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}-\frac{2\sqrt{nT}(\widehat{\rho}-\rho)(1-\rho)}{\sqrt{n}T^{5/2}}\sum_{t=1}^{T}x_{it-1}^2 \end{array}\right] \\ &\quad -\frac{\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\frac{T}{\hat{d}^2}\sum_{i=1}^{n}\left[\begin{array}{c} \frac{1}{T^{3/2}}\sum_{t=1}^{T}\varepsilon_{it}-\sqrt{nT}\left(\widehat{\rho}-\rho\right)\frac{1}{\sqrt{n}T^2}\sum_{t=1}^{T}x_{it-1} \\ +(1-\rho)\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1} \end{array}\right]^2 \\ &= o_p(1). \end{aligned}$$

This follows because, if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, we get

$$\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{4\sigma_\mu^2}{3\sigma_\varepsilon^2}\right)$$

using a similar argument as in Phillips and Moon (1999) and Baltagi *et al.* (2008). Also note that

$$\sqrt{nT}\left(\widehat{\rho}-\rho\right)\xrightarrow{d}N\left(0,1-\rho^2\right)$$

from Lemma 1. This is because $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}^2=O_p(1)$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}^2=O_p(1)$, $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}=O_p(1)$, and $\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it-1}=O_p(1)$.

Also note that from Lemma 1 (B),

$$III\leq\sqrt{I\times II}\xrightarrow{p}0.$$

We conclude that

$$\hat{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

(b) Next, let us show that $\hat{\sigma}_\mu^2$ is a consistent estimator of σ_μ^2. From Lemma 2 (B), we know that $I\xrightarrow{p}(1-\rho)^2\sigma_\mu^2$, $II\xrightarrow{p}0$, and accordingly $IV\leq\sqrt{I\times II}\xrightarrow{p}0$ as shown 1.(b).

Let us look at III.

$$\begin{aligned} III &= \frac{\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\sum_{i=1}^{n}\left(\frac{T}{\hat{d}^2}\right)\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_{it}-\hat{\rho}x_{it-1})\right]^2 \\ &\quad +\frac{\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2(1-\hat{\rho})^2}{n^2}\frac{\hat{d}^2}{T}\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]^2 \\ &\quad -\frac{2\left(\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2(1-\hat{\rho})}{n^2}\left[\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\hat{\rho}x_{it-1})\right]\left[\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]. \end{aligned}$$

With a similar process to 2.(b), it can be shown that $III=o_p(1)$ because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{nT}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{4\sigma_\mu^2}{3\sigma_\varepsilon^2}\right).$$

Hence, with $V\leq\sqrt{I\times III}\xrightarrow{p}0$ and $VI\leq\sqrt{II\times III}\xrightarrow{p}0$, we finally have

$$\frac{1}{T}\hat{\sigma}_\alpha^2\xrightarrow{p}(1-\rho)^2\sigma_\mu^2$$

and accordingly

$$\hat{\sigma}_{\mu}^{2}=\frac{1}{\hat{\theta}}\left(\hat{\sigma}_{\alpha}^{2}-\hat{\sigma}_{e}^{2}\right)=\frac{T}{\hat{\theta}}\left(\frac{\hat{\sigma}_{\alpha}^{2}}{T}-\frac{\hat{\sigma}_{e}^{2}}{T}\right)\xrightarrow{p}\frac{1}{(1-\rho)^{2}}\left[(1-\rho)^{2}\sigma_{\mu}^{2}-0\right]=\sigma_{\mu}^{2}.$$

(c) Let us start from the term $\widehat{G}_1$ in equation (14). Recall

$$\frac{1}{nT^{2}}\widehat{G}_{1}=\frac{1}{nT^{2}}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}-\frac{1}{T}\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{n\sqrt{T}}.$$

From Lemma 3 (B), we have

$$\frac{1}{nT^{2}}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x}=\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T^{2}}-\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}/T}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T^{3/2}}\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T^{3/2}}\right).$$

From Theorem 3.3.(a), we have

$$\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\frac{1}{T^{2}}\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\mathbf{x}_{i}\xrightarrow{p}(1-\rho)^{2}\frac{\sigma_{\varepsilon}^{2}}{2\sigma_{e}^{2}}.$$

Next, it can be shown that

$$\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}/T}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T^{3/2}}\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T^{3/2}}\xrightarrow{p}(1-\rho)^{2}\frac{\sigma_{\varepsilon}^{2}}{3\sigma_{e}^{2}}$$

using the fact that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T^{3/2}}\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T^{3/2}}\\
=&\frac{1}{n}\sum_{i=1}^{n}\left[(1-\widehat{\rho})\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_{it}-\widehat{\rho}x_{it-1})\right]^{2}\\
=&\frac{1}{n}\sum_{i=1}^{n}\left[(1-\widehat{\rho})\frac{1}{T^{3/2}}\sum_{t=1}^{T}((1-\widehat{\rho})x_{it-1}+\varepsilon_{it})\right]^{2}\\
=&\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c}(1-\widehat{\rho})^{4}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)^{2}+(1-\widehat{\rho})^{2}\frac{1}{T^{2}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)^{2}\\(1-\widehat{\rho})^{3}\frac{1}{T}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\end{array}\right]\\
=&\frac{1}{n}\sum_{i=1}^{n}(1-\widehat{\rho})^{4}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)^{2}+O_{p}\left(\frac{1}{T^{2}}\right)+O_{p}\left(\frac{1}{T}\right)\\
=&(1-\widehat{\rho})^{4}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)^{2}+O_{p}\left(\frac{1}{T}\right)\xrightarrow{p}(1-\rho)^{4}\frac{\sigma_{\varepsilon}^{2}}{3}
\end{aligned}$$

and $\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}/T}\xrightarrow{p}\frac{1}{(1-\rho)^{2}}$. Hence,

$$\begin{aligned}
\frac{1}{nT^{2}}\mathbf{x}'\Phi^{-1}\mathbf{x}&=\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\left(\frac{1}{T^{2}}\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\mathbf{x}_{i}-\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}/T}\frac{\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T^{3/2}}\frac{\boldsymbol{\iota}_{T}'\widehat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T^{3/2}}\right)\\
&\xrightarrow{p}(1-\rho)^{2}\frac{\sigma_{\varepsilon}^{2}}{2\sigma_{e}^{2}}-(1-\rho)^{2}\frac{\sigma_{\varepsilon}^{2}}{3\sigma_{e}^{2}}=\frac{(1-\rho)^{2}\sigma_{\varepsilon}^{2}}{6\sigma_{e}^{2}}
\end{aligned}$$

as $(n,T)\to\infty$.

Next, one can verify that

$$\frac{1}{n\sqrt{T}}\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\right) = O_p(\frac{1}{\sqrt{n}})$$

as shown in Theorem 3.3.(a). Also recall that

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$$

as shown in 1.(c).

Hence, we have

$$\frac{1}{nT^2}\widehat{G}_1\xrightarrow{p}\frac{(1-\rho)^2\,\sigma_\varepsilon^2}{6\sigma_e^2}$$

as $(n,T)\to\infty$.

Now we investigate $\widehat{G}_2$. Let

$$\frac{1}{\sqrt{n}T}\widehat{G}_2=\frac{1}{\sqrt{n}T}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u}-\frac{1}{\sqrt{T}}\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{1}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{u}.$$

Consider that

$$\begin{aligned}&\frac{1}{\sqrt{n}T}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u}\\ =\;&\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\widehat{\sigma}_e^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\right)\mu_i+\left(\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\widehat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right].\end{aligned}$$

Firstly, in a similar vein as 1.(c) it can be shown that

$$\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left(\frac{\widehat{\sigma}_e^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\right)\mu_i=O_p\left(\frac{1}{\sqrt{T}}\right)$$

using

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\widehat{\sigma}_e^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\right)\mu_i=O_p(1).$$

Next, recall from Theorem 3.3.(b) that

$$\frac{1}{\hat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{T}\xrightarrow{d}N(0,\frac{(1-\rho)^2\sigma_\varepsilon^2}{2\sigma_e^2}).$$

Lastly, we consider

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \\
\approx\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\left[(1-\widehat{\rho})\sum_{t=1}^{T}(x_{it}-\widehat{\rho}x_{it-1})\right]\left[(1-\widehat{\rho})\sum_{t=1}^{T}(\nu_{it}-\widehat{\rho}\nu_{it-1})\right] \\
=\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{(1-\widehat{\rho})^2}{T^2}\left[\sum_{t=1}^{T}((1-\widehat{\rho})\,x_{it-1}+\varepsilon_{it})\right]\left[\sum_{t=1}^{T}(e_{it}-(\widehat{\rho}-\rho)\,\nu_{it-1})\right] \\
=\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{(1-\widehat{\rho})^2}{T^2}\left[\begin{array}{c}\sum_{t=1}^{T}(1-\widehat{\rho})\,x_{it-1}\sum_{t=1}^{T}e_{it}+\sum_{t=1}^{T}\varepsilon_{it}\sum_{t=1}^{T}e_{it} \\ -\sum_{t=1}^{T}(1-\widehat{\rho})\,x_{it-1}\sum_{t=1}^{T}(\widehat{\rho}-\rho)\,\nu_{it-1}-\sum_{t=1}^{T}(\widehat{\rho}-\rho)\,\varepsilon_{it}\sum_{t=1}^{T}\nu_{it-1}\end{array}\right] \\
=\ & I+II+III+IV.
\end{aligned}$$

Consider II first.

$$\begin{aligned}
II &= \frac{1}{T}\frac{(1-\widehat{\rho})^2}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right) \\
&= O_p\left(\frac{1}{T}\right)
\end{aligned}$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)=O_p(1)$ where ε_{it} and e_{it} are not correlated. Consider III and IV. It is easy to see that

$$\begin{aligned}
III &= -(1-\widehat{\rho})^3\frac{1}{\sqrt{nT}}\frac{\sqrt{nT}\,(\widehat{\rho}-\rho)}{\sqrt{n}}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right) \\
&= O_p\left(\frac{1}{\sqrt{nT}}\right)
\end{aligned}$$

using

$$\frac{\sqrt{nT}\,(\widehat{\rho}-\rho)}{\sqrt{n}}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right)$$

and

$$\begin{aligned}
IV &= -(1-\widehat{\rho})^2\frac{1}{\sqrt{n}T^{3/2}}\frac{\sqrt{nT}\,(\widehat{\rho}-\rho)}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it-1}\right) \\
&= O_p\left(\frac{1}{\sqrt{n}T^{3/2}}\right).
\end{aligned}$$

Lastly, consider I. it can be shown that

$$\begin{aligned}
I &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\widehat{\rho})^3\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right) \\
&\xrightarrow{d}(1-\rho)^3N(0,\frac{\sigma_\varepsilon^2\sigma_e^2}{3}).
\end{aligned}$$

Therefore,

$$\frac{1}{\widehat{\sigma}_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\widehat{\sigma}_\mu^2}{\hat{\sigma}_\alpha^2/T}\frac{\mathbf{x}_i'\widehat{\mathbf{A}}^{-1}\boldsymbol{\iota}_T}{T^{3/2}}\frac{\boldsymbol{\iota}_T'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\xrightarrow{d}(1-\rho)\,N(0,\frac{\sigma_\varepsilon^2}{3\sigma_e^2})$$

and accordingly it can be shown that

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u}\xrightarrow{d}N\left(0,\frac{(1-\rho)^2\sigma_\varepsilon^2}{6\sigma_e^2}\right)$$

by using a similar process as in Phillips and Moon (1999).
Next consider

$$\begin{aligned}
\frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{u} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\left[\left(\frac{\hat{\theta}}{T}\right)\mu_i+\frac{1}{\sqrt{T}}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\left(\frac{\hat{\theta}}{T}\right)\mu_i+\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right) \\
&= I+II.
\end{aligned}$$

Consider I. Recall from 1.(c) that

$$I=\frac{1}{\hat{\sigma}_\alpha^2/T}\left(\frac{\hat{\theta}}{T}\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\xrightarrow{d}\frac{1}{(1-\rho)^2\sigma_\mu^2}(1-\rho)^2N\left(0,\sigma_\mu^2\right).$$

Consider II.

$$II=\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\hat{\sigma}_\alpha^2/T}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)=O_p\left(\frac{1}{\sqrt{T}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\boldsymbol{\iota}_T'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)=O_p(1)$. We conclude that

$$\frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{u}\xrightarrow{d}N\left(0,\frac{1}{\sigma_\mu^2}\right).$$

Because we also know that $\frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved above, we have

$$\frac{1}{\sqrt{nT}}\widehat{G}_2\xrightarrow{d}N\left(0,\frac{(1-\rho)^2\sigma_\varepsilon^2}{6\sigma_e^2}\right).$$

Finally,

$$t_{FGLS}=\left(\frac{1}{nT^2}\widehat{G}_1\right)^{-1/2}\left(\frac{1}{\sqrt{nT}}\widehat{G}_2\right)\xrightarrow{d}N(0,1).$$

4. When $\rho=1$, $\lambda=1$ if $T(\widehat{\rho}-1)\xrightarrow{p}\kappa$

(a) First, let us show that $\hat{\sigma}_e^2$ is a consistent estimator of σ_e^2. From Lemma 1 (B), we have

$$\begin{aligned}
I &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}^2-\frac{2\left[T(\widehat{\rho}-1)\right]}{T}\left(\frac{1}{T}\sum_{t=1}^{T}e_{it}v_{it-1}\right)+\frac{\left[T(\widehat{\rho}-1)\right]^2}{T}\left(\frac{1}{T^2}\sum_{t=1}^{T}v_{it-1}^2\right)\right] \\
&\quad -\left(\frac{T}{\hat{d}^2}\right)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}-\frac{1}{\sqrt{T}}\left[T(\widehat{\rho}-1)\right]\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}v_{it-1}\right)\right]^2 \\
&\xrightarrow{p}\sigma_e^2
\end{aligned}$$

as $(n,T)\to\infty$, as shown already in 2.(a).

Consider II. Using a similar argument, one can easily show that

$$\begin{aligned} II &= \frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2+\frac{(T(\widehat{\rho}-1))^2}{T}\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}^2-\frac{2T(\widehat{\rho}-1)}{T}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}x_{it-1}\right] \\ &\quad -\frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\left(\frac{T}{\hat{d}^2}\right)\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-\frac{T(\widehat{\rho}-1)}{\sqrt{T}}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right]^2 \\ &= o_p(1) \end{aligned}$$

because if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right).$$

Consider III. From Lemma 1 (B), we know

$$III\le\sqrt{I\times II}\xrightarrow{p}0.$$

We conclude that

$$\hat{\sigma}_e^2\xrightarrow{p}\sigma_e^2.$$

(b) Next, we investigate $\hat{\sigma}_\mu^2$. From Lemma 2 (B), we have

$$\begin{aligned} \frac{1}{T(1-\hat{\rho})}\hat{\sigma}_\alpha^2 &= \frac{1}{nT(1-\hat{\rho})}\hat{\mathbf{u}}^{*\prime}\left(I_n\otimes\widehat{\bar{J}}_T^{\alpha}\right)\hat{\mathbf{u}}^* \\ &= \frac{(1-\hat{\rho})\hat{d}^2}{nT}\boldsymbol{\mu}'E_n\boldsymbol{\mu}+\frac{1}{nT(1-\hat{\rho})}\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n\otimes\hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\ &\quad+\frac{1}{nT(1-\hat{\rho})}\left(\hat{\beta}_{OLS}-\beta\right)^2\mathbf{x}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n\otimes\hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\ &\quad+\frac{2}{nT}\boldsymbol{\mu}'\left(E_n\otimes\hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\boldsymbol{\nu} \\ &\quad+\frac{2}{nT}\left(\hat{\beta}_{OLS}-\beta\right)\boldsymbol{\mu}'\left(E_n\otimes\hat{\boldsymbol{\iota}}_T^{\alpha\prime}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\ &\quad+\frac{1}{nT(1-\hat{\rho})}\left(\hat{\beta}_{OLS}-\beta\right)\boldsymbol{\nu}'\left(I_{nT}-\bar{J}_{nT}\right)\left(I_n\otimes\hat{C}'\widehat{\bar{J}}_T^{\alpha}\hat{C}\right)\left(I_{nT}-\bar{J}_{nT}\right)\mathbf{x} \\ &= I+II+III+IV+V+VI. \end{aligned}$$

Consider I.

$$I=\frac{(1-\hat{\rho})\,\hat{d}^2}{T}\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^2\right]\xrightarrow{p}0$$

as $(n,T)\to\infty$ with $(1-\hat{\rho})\,\hat{d}^2=2\widehat{\rho}+T\left(1-\widehat{\rho}\right)\xrightarrow{p}2-\kappa$.
Consider II. As shown in 2.(b), we have

$$II\xrightarrow{p}\frac{\kappa^2-3\kappa+3}{3\left(2-\kappa\right)}\sigma_e^2$$

and accordingly

$$IV\le\sqrt{I\times II}\xrightarrow{p}0.$$

Consider III next.

$$\begin{aligned} III &= \frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\sum_{i=1}^{n}\left(\frac{1}{(1-\hat{\rho})\hat{d}^2}\right)\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-\frac{T(\widehat{\rho}-1)}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right]^2 \\ &\quad+\frac{\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2(1-\hat{\rho})\hat{d}^2}{n^2}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\right]^2 \\ &\quad-\frac{2\left(\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\right)^2}{n^2}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-\frac{T(\widehat{\rho}-1)}{T^{3/2}}\sum_{t=1}^{T}x_{it-1}\right)\right]\left[\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}\right]. \end{aligned}$$

One can show that $III=o_p(1)$ as $(n,T)\to\infty$ using the fact that if $(n,T)\to\infty$ and $\frac{n}{T}\to 0$, then

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,\frac{2\sigma_e^2}{3\sigma_\varepsilon^2}\right)$$

and that $V \leq \sqrt{I \times III} \xrightarrow{p} 0$, $VI \leq \sqrt{II \times III} \xrightarrow{p} 0$, respectively.
Summarizing, we have the same result as 2.(b),

$$\frac{\hat{\sigma}_{\alpha}^{2}}{T(1-\hat{\rho})} \xrightarrow{p} \frac{\kappa^{2}-3\kappa+3}{3(2-\kappa)}\sigma_{e}^{2}$$

and

$$(1-\hat{\rho})\hat{\sigma}_{\mu}^{2} \xrightarrow{p} \frac{-\kappa^{3}+3\kappa^{2}-6}{3(2-\kappa)^{2}}\sigma_{e}^{2}.$$

With $k=-3$,

$$(1-\hat{\rho})\hat{\sigma}_{\mu}^{2} \xrightarrow{p} \frac{16}{25}\sigma_{e}^{2}.$$

(c) Let us first look at $\widehat{G}_1$ in equation (14). Define

$$\frac{1}{nT}\widehat{G}_{1}=\frac{1}{nT}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{x}-\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{T}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{x}}{n}.$$

From Lemma 3 (B), we have

$$\frac{1}{nT}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x}=\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T}-T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T(1-\widehat{\rho})}\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T(1-\widehat{\rho})}\right).$$

Note that

$$\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T} \xrightarrow{p} \frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}}$$

as shown in Theorem 3.4.(a).
Next, one can easily see that

$$\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T(1-\widehat{\rho})}\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T(1-\widehat{\rho})}=O_{p}\left(\frac{1}{T}\right)$$

using

$$\begin{aligned}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T(1-\widehat{\rho})} &\approx \frac{1}{\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(x_{it}-\hat{\rho}x_{it-1})\\ &= \frac{1}{\sqrt{T}}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}+T(1-\widehat{\rho})\sum_{t=1}^{T}\frac{x_{it-1}}{\sqrt{T}}\frac{1}{T}\right]=O_{p}(\frac{1}{\sqrt{T}})\end{aligned}$$

and $T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}} \xrightarrow{p} \frac{-k^{3}+3k^{2}-6}{(2-k)(k^{2}-3k+3)}$ as shown already.
Hence, we have

$$\frac{1}{nT}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{x}=\frac{1}{n\hat{\sigma}_{e}^{2}}\sum_{i=1}^{n}\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\mathbf{x}_{i}}{T}+O_{p}\left(\frac{1}{T}\right) \xrightarrow{p} \frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}}$$

as $(n,T)\to\infty$.
Note also that

$$\begin{aligned}\frac{1}{n}\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\sqrt{nT}}\frac{1}{\hat{\sigma}_{\alpha}^{2}/T(1-\widehat{\rho})}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{\sqrt{T}(1-\widehat{\rho})}\right)\\ &= O_{p}(\frac{1}{\sqrt{nT}}).\end{aligned}$$

Lastly, recall that

$$\frac{T}{n}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT}=\frac{\hat{\theta}/(1-\widehat{\rho})}{\hat{\sigma}_{\alpha}^{2}/T(1-\widehat{\rho})}\xrightarrow{p}\frac{2-k}{\frac{\kappa^{2}-3\kappa+3}{3(2-\kappa)}\sigma_{e}^{2}}=\frac{1}{\sigma_{e}^{2}}\frac{3(2-\kappa)^{2}}{\kappa^{2}-3\kappa+3}.$$

Therefore, we conclude that

$$\frac{1}{nT}\widehat{G}_{1}\xrightarrow{p}\frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}}.$$

Now we turn to $\widehat{G}_2$. Let

$$\frac{1}{\sqrt{nT}}\widehat{G}_{2}=\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u}-\frac{\mathbf{x}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{T}{n}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\frac{\sqrt{T}}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\widehat{\Phi}^{-1}\mathbf{u}.$$

Consider

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\hat{\Phi}^{-1}\mathbf{u}=\frac{1}{\hat{\sigma}_{e}^{2}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\begin{array}{c}\frac{1}{\sqrt{T}}\left(\frac{\hat{\sigma}_{e}^{2}}{\hat{\sigma}_{\alpha}^{2}/T(1-\widehat{\rho})}\right)\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T(1-\widehat{\rho})}\right)\mu_{i}\\+\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}}{\sqrt{T}}-T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{T(1-\widehat{\rho})}\right)\left(\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}}{\sqrt{T}(1-\widehat{\rho})}\right)\right)\end{array}\right].$$

Note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\left(\frac{1}{\hat{\sigma}_{\alpha}^{2}/T(1-\widehat{\rho})}\right)\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{\sqrt{T}(1-\widehat{\rho})}\right)\mu_{i}=O_{p}\left(\frac{1}{T}\right)$$

and recall from Theorem 3.4.(b) that

$$\frac{1}{\hat{\sigma}_{e}^{2}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{x}_{i}'\widehat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}}{\sqrt{T}}\xrightarrow{d}N(0,\frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}}).$$

Lastly, it can be shown that

$$\frac{1}{\hat{\sigma}_{e}^{2}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{T}}\sum_{i=1}^{n}T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{\sqrt{T}(1-\widehat{\rho})}\right)\left(\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}}{\sqrt{T}(1-\widehat{\rho})}\right)=O_{p}\left(\frac{1}{\sqrt{T}}\right)$$

using $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_{i}'\hat{\mathbf{A}}^{-1}\boldsymbol{\iota}_{T}}{\sqrt{T}(1-\widehat{\rho})}\right)\left(\frac{\boldsymbol{\iota}_{T}'\hat{\mathbf{A}}^{-1}\boldsymbol{\nu}_{i}}{\sqrt{T}(1-\widehat{\rho})}\right)=O_{p}(1)$ and $T(1-\widehat{\rho})^{2}\frac{\hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\alpha}^{2}}\xrightarrow{p}\frac{-k^{3}+3k^{2}-6}{(2-k)(k^{2}-3k+3)}$. Therefore,

$$\frac{1}{\sqrt{nT}}\mathbf{x}'\widehat{\Phi}^{-1}\mathbf{u}\xrightarrow{d}N(0,\frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}}).$$

Also, using the results above, $\frac{1}{n}\mathbf{x}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT}=O_{p}(\frac{1}{\sqrt{nT}})$ and $\frac{T}{n}\boldsymbol{\iota}_{nT}'\hat{\Phi}^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_{e}^{2}}\frac{3(2-\kappa)^{2}}{\kappa^{2}-3\kappa+3}$. Summarizing, we have

$$\frac{1}{\sqrt{nT}}\widehat{G}_{2}\xrightarrow{d}N(0,\frac{\sigma_{\varepsilon}^{2}}{\sigma_{e}^{2}})$$

and accordingly,

$$t_{FGLS}=\left(\frac{1}{nT}\widehat{G}_{1}\right)^{-1/2}\left(\frac{1}{\sqrt{nT}}\widehat{G}_{2}\right)\xrightarrow{d}N(0,1).$$

■