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***On Perfect Weighted Coverings  
with Small Radius***

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# ON PERFECT WEIGHTED COVERINGS WITH SMALL RADIUS<sup>1</sup>

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**Abstract:** We extend the results of our previous paper [8] to the nonlinear case: The Lloyd polynomial of the covering has at least  $R$  distinct roots among  $1, \dots, n$ , where  $R$  is the covering radius. We investigate *PWC* with diameter 1, finding a partial characterization. We complete an investigation begun in [8] on linear *PMC* with distance 1 and diameter 2.

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# 1 Introduction

Much attention has been devoted to the problem of classifying perfect codes (See [13, 15]). Further generalizations of perfectness were introduced in [10, 2, 11, 14]. For all these codes the diameter of the covering spheres equals the covering radius of the code which by use of Delsarte's results leads to a very rigid set of possible parameters. This framework was broadened by introducing new types of perfect configurations [5, 6, 12, 16]. All these extensions fall under the concept of perfect weighted coverings (*PWC*) first considered in [8]. Although general, these definitions leave hope for a complete classification, at least for small diameter. The linear case with diameter at most 2 was considered in [8], where some motivation related to list decoding was given.

We are pleased to acknowledge that this problem arose in discussions with I. Honkala in Veldhoven in June, 1990.

## 2 Notations and known results

We denote by  $\mathbf{F}^n$  the vector space of binary  $n$ -tuples, by  $d(\cdot, \cdot)$  the Hamming distance, by  $C(n, K, d)R$  a code  $C$  with length  $n$ , size  $K$ , minimum distance  $d = d(C)$  and covering radius  $R$  [9], [7]. When  $C$  is linear, we write  $C[n, k, d]R$ , where  $k$  is the binary log of  $K$ . We denote the Hamming weight of  $x \in \mathbf{F}^n$  by  $|x|$ .

For  $x \in \mathbf{F}^n$ ,  $A(x) = (A_0(x), A_1(x) \dots A_n(x))$  will stand for the distance distribution of  $C$  with respect to  $x$ ; thus

$$A_i(x) := |\{c \in C : d(c, x) = i\}|.$$

For any  $(n + 1)$ -tuple  $M = (m_0, m_1, \dots, m_n)$  of weights, i.e., rational numbers, we define the  $M$ -density of  $C$  at  $x$  as

$$(2.1) \quad \theta(x) := \sum_{i=0}^n m_i A_i(x) = \langle M, A(x) \rangle.$$

We consider only *coverings*, i.e., codes  $C$  such that  $\theta(x) \geq 1$  for all  $x$ .

$$(2.2) \quad C \text{ is a } \textit{perfect } M\text{-covering if } \theta(x) = 1 \text{ for all } x.$$

We define the *diameter* of an  $M$ -covering as

$$\delta := \max\{i : m_i \neq 0\}.$$

To avoid trivial cases, we usually assume that  $m_i = 0$  for  $i \geq n/2$ , i.e.,  $\delta < n/2$ .

Here are the known special cases.

$$(2.3) \quad \text{Classical perfect codes: } m_i = 1 \text{ for } i = 0, 1, \dots, \delta.$$

$$(2.4) \quad \text{Perfect multiple coverings (PMC): } m_i = 1/j \text{ for } i = 0, 1, \dots, \delta \\ \text{where } j \text{ is a positive integer. See [16] and [5].}$$

(2.5) Perfect  $L$ -codes:  $m_i = 1$  for  $i \in L \subseteq \{0, 1, \dots, \lfloor n/2 \rfloor\}$ . See [12] and [6].

(2.6) Strongly uniformly packed codes:  
 $m_i = 1$  for  $i = 0, 1, \dots, e - 1$   
 $m_e = m_{e+1} = 1/r$  for some integer  $r$ . See [14].

(2.7) Uniformly packed codes [2, 11]. For these codes  $\delta(M) = R(C)$ , and the  $m_i$ 's are uniquely determined.

The following necessary and sufficient condition was already in [8] in the linear case. For a perfect  $M$ -covering  $C$  one gets from the definition:

$$\sum_{i=0}^n m_i A_i(x) = 1 \text{ for all } x.$$

Summing over all  $x$  in  $F^n$  and permuting sums, we get

$$\sum_{i=0}^n m_i \sum_{x \in F^n} A_i(x) = 2^n.$$

For  $i = 0$ , the second sum is  $|C| = K$ , for  $i = 1$  it is  $Kn$ , and so on. For the converse we use the condition  $\theta(x) \geq 1$ . Hence we get the following analog of the Hamming condition.

**Proposition 2.1** *A covering  $C$  is a perfect  $M$ -covering if and only if*

$$(2.8) \quad K \sum_{i=0}^n m_i \binom{n}{i} = 2^n.$$

□

### 3 A Lloyd theorem

In this section we prove

**Theorem 3.1** *Let  $C$  be a perfect weighted covering with  $M = (m_0, m_1, \dots, m_\delta)$ . Then the Lloyd polynomial of this covering,*

$$L(x) := \sum_{0 \leq i \leq \delta} m_i P_{n,i}(x)$$

*has at least  $R$  distinct integral roots among  $1, 2, \dots, n$ .*

**Proof.** (Adapted from [1], Chapter II, Section 1, which records A. M. Gleason's proof of the classical Lloyd theorem.) The first part of the proof is identical to that of [8, Thm. 4.1].

We use the group algebra  $\mathcal{A}$  of all formal polynomials

$$\sum_{a \in \mathbf{F}^n} \gamma_a X^a$$

with  $\gamma_a \in \mathbf{Q}$ , the field of rational numbers.

Define

$$(3.1) \quad S := \sum_{0 \leq i \leq \delta} m_i \sum_{|a|=i} X^a.$$

We let the symbol  $C$  for our code also stand for the corresponding element in  $\mathcal{A}$ , namely,

$$(3.2) \quad C := \sum_{c \in C} X^c.$$

Then we find that

$$(3.3) \quad SC = \sum_{c \in C} X^c \cdot S = \mathbf{F}^n := \sum_{a \in \mathbf{F}^n} X^a.$$

Characters on  $\mathbf{F}^n$  are group homomorphisms of  $(\mathbf{F}^n, +)$  into  $\{1, -1\}$ , the group of order 2 in  $\mathbf{Q}^\times$ . All characters have the form  $\chi_u$  for  $u \in \mathbf{F}^n$ , where  $\chi_u$  is defined as

$$\chi_u(v) = (-1)^{u \cdot v} \text{ for } u, v \in \mathbf{F}^n.$$

We use linearity to extend  $\chi_u$  to a linear functional defined on  $\mathcal{A}$ :

For all  $Y \in \mathcal{A}$  if  $Y = \sum_{a \in \mathbf{F}^n} \gamma_a X^a$ , then  $\chi_u(Y) := \sum \gamma_a \chi_u(a)$ .

It follows that

$$\chi_u(YZ) = \chi_u(Y)\chi_u(Z) \text{ for all } Y, Z \in \mathcal{A}.$$

It is known [1, 9] that for any  $u \in \mathbf{F}^n$ , if  $|u| = w$ , then

$$(3.4) \quad \chi_u \left( \sum_{|a|=i} X^a \right) = P_{n,i}(w).$$

It follows that

$$(3.5) \quad \chi_u(S) = L(w).$$

From (3.3), furthermore, we see that

$$\chi_u(SC) = \chi_u(S)\chi_u(C) = 0$$

for all  $u \neq 0$ .

Let  $u_0, u_1, \dots, u_R$  be translate-leaders for  $C$  such that  $|u_i| = i$ . Define

$$C_i := X^{u_i} C.$$

Then

$$(3.6) \quad S C_i = \mathbf{F}^n.$$

Define the **symmetric subring**  $\overline{\mathcal{A}}$  of  $\mathcal{A}$  as the set of all elements  $Y$  of  $\mathcal{A}$  in which the coefficient of  $X^a$  depends only on the weight of  $a$ :

$$(3.7) \quad Y = \sum_{a \in \mathbf{F}^n} \gamma_a X^a \in \overline{\mathcal{A}} \text{ iff } \forall a, b \in \mathbf{F}^n, |a| = |b| \rightarrow \gamma_a = \gamma_b.$$

The mapping  $T : \mathcal{A} \rightarrow \overline{\mathcal{A}}$  defined by

$$T(Y) := \frac{1}{n!} \sum_{\varphi} \varphi(Y),$$

where  $\varphi$  runs over all  $n!$  permutations of the  $n$  coordinates of  $\mathbf{F}^n$ , maps  $\overline{\mathcal{A}}$  onto  $\overline{\mathcal{A}}$ . Furthermore, as the reader may easily verify,

$$(3.8) \quad \forall Y \in \overline{\mathcal{A}}, \forall Z \in \mathcal{A}, T(YZ) = YT(Z).$$

Define  $\overline{C}_i := T(C_i)$ . Applying (3.8) to (3.6), we see that

$$S\overline{C}_i = \mathbf{F}^n$$

since, of course,  $S \in \overline{\mathcal{A}}$ . Define also

$$(3.9) \quad K := \{Z; Z \in \overline{\mathcal{A}}, SZ = 0\}.$$

Thus  $K$  is the kernel of the linear mapping from  $\overline{\mathcal{A}}$  to  $\overline{\mathcal{A}}$  defined by  $Y \mapsto SY$  for all  $Y \in \overline{\mathcal{A}}$ .

It follows from (3.8) that for any character  $\chi_u$  such that  $\chi_u(S) \neq 0$ ,

$$\forall Z \in K, \chi_u(Z) = 0.$$

Since  $\overline{\mathcal{A}}$  has dimension  $n + 1$ , its space of linear functionals also has dimension  $n + 1$ . Since every linear functional on  $\overline{\mathcal{A}}$  can be extended to one on  $\mathcal{A}$ , the  $n + 1$  linear functionals on  $\overline{\mathcal{A}}$  obtained by restricting the  $\chi_u$  to  $\overline{\mathcal{A}}$ , as

$$\chi_u|_{\overline{\mathcal{A}}} =: \chi_w \text{ for } |u| = w$$

$$w = 0, 1, \dots, n,$$

are linearly independent.

Suppose that  $\rho$  is the exact number of values of  $w \in \{0, 1, \dots, n\}$  for which

$$\chi_w(S) \neq 0.$$

Since  $\chi_w(S)\chi_w(K) = 0$  for all  $w$ , it follows that  $\chi_w(K) = 0$  for  $\rho$  values of  $w$ . Since  $S\overline{C}_i = \mathbf{F}^n$  for  $i = 0, 1, \dots, R$ , we see that

$$S(\overline{C}_i - \overline{C}_0) = 0 \text{ for } i = 1, \dots, R.$$

The elements  $\overline{C}_i - \overline{C}_0$  are linearly independent because  $\overline{C}_i$  contains elements of weight  $i$  but of no smaller weight. We find that

$$R \leq \dim_{\mathbf{Q}} K \leq n + 1 - \rho,$$

since  $K$  is included in the intersection of the  $t$  kernels of the  $\chi_w$  mentioned above. But  $n + 1 - \rho$  is the number of  $\chi_w$ 's which vanish on  $S$ ; therefore  $\chi_w(S) = 0$  for at least  $R$  values of  $w$ .

Notice now that

$$\chi_w(S) = \sum_{0 \leq i \leq \delta} m_i P_{n,i}(w).$$

This finishes the proof. □

## 4 A construction

**Definition 4.1** Let  $C(n, K, d)R$  and  $C'(n', K', d')R'$  be two codes. Set

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ 1 & \text{otherwise.} \end{cases}$$

We extend  $\chi_C$  to a mapping  $\chi : \mathbf{F}^{nn'} \rightarrow \mathbf{F}^{n'}$  by setting

$$\chi(x) := (\chi_C(x_1), \chi_C(x_2), \dots, \chi_C(x_{n'}))$$

where the  $x_i$ 's are in  $\mathbf{F}^n$ , for  $1 \leq i \leq n'$ , and  $x = (x_1, x_2, \dots, x_{n'})$  is their concatenation. We are now ready to define  $C \otimes C'$  as follows:

$$C \otimes C' = \{z \in \mathbf{F}^{nn'} : \chi(z) \in C'\}.$$

**Proposition 4.1**  $C \otimes C'$  has length  $nn'$ , minimum distance  $\min\{d, d'\}$  and covering radius  $RR'$ .

The proof is immediate. □

**Proposition 4.2** Let  $x$  and  $x'$  be such that  $d(x, C) = R, d(x', C') = R'$ . Suppose that  $A_R(x)$  and  $A_{R'}(x')$  are independent of  $x$ . Then for  $C \otimes C'$  the coefficient  $A_{RR'}(z)$  is the same for any  $z$  such that  $d(z, C \otimes C') = RR'$  and one has

$$A_{RR'} = A_R A_{R'}.$$

□

## 5 PWC with diameter 1

Let us denote such a PWC by  $(n, m_0, m_1)$ . From (2.2),  $A_1(x) = 1/m_1$  for any  $x$  not in  $C$ . Hence  $m_1 = 1/p$ , where  $p$  is an integer. This means that every two noncodewords have the same number of codewords at distance 1.

For  $c \in C$ , we get:  $m_0 + A_1(c)/p = 1$ , hence

$$A_1(c) = p(1 - m_0)$$

is a constant independent of  $c$ . Since  $A_1(c)$  is an integer, so is  $m_0 p$ .

Now the Hamming analogue (2.9) gives

$$K(pm_0 + n) = p2^n,$$

which implies

$$(5.1) \quad n = p'2^i - m_0 p, \text{ with } p' \mid p.$$

The case  $m_0 = 1/p$  corresponds to the PMC mentioned in (2.4); it is solved in [16] and [8].

Let us give a few general constructions.



**Proposition 5.1** *If there exists a PWC  $C(n, m_0, m_1)$ , then for any  $l \geq 0$  there exists a PWC  $C'(n + l, m_0 - lm_1, m_1)$ .*

**Proof.** Let us define  $C'$  as the set of vectors  $(c, f)$  in  $\mathbf{F}^{n+l}$ , where  $c \in C$  and  $f \in \mathbf{F}^l$ . Let  $A$  be the distance distribution for  $C_1$  and  $A'$  that for  $C'$ . There are two possibilities for an arbitrary  $(x, f) \in \mathbf{F}^{n+l}$ :

- (a)  $x \in C$ . Then  $A'_1((x, f)) = A_1(x) + l$ . Evidently  $A'_0((x, f)) = 1$ .
- (b)  $x \notin C$ . Then  $A'_0((x, f)) = 0$  by construction and  $A'_1((x, f)) = A_1(x)$ . □

**Proposition 5.2** *If there exists a PWC  $C(n, m_0, m_1)$ , then there exists a PWC  $C'(ns, m_0, m_1/s)$ .*

**Proof.** Apply construction  $\otimes$  (Def. 4.1) with outer code  $C(n, m_0, m_1)$  and inner code the  $[s, s - 1]$  parity code. □

**Proposition 5.3** *If there exists a PWC  $C(n, m_0, m_1)$ , then there exists a PWC  $C'(n, m_0/i, m_1/i)$ , for  $i$  a positive integer.*

**Proof.** Take the union of  $i$  cyclic shifts of code  $C$ . □

Let us now turn to the special case when  $m_0 = 1$ .

**Proposition 5.4** *A PWC with  $\delta = m_0 = 1$  exists for  $n = p(2^i - 1)$ ,  $m_1 = 1/p$ . It can be achieved by a linear code.* □

See [8] for a proof of this result. In contrast to the linear case, [8, Prop. 5.4], we cannot characterize PWC with  $\delta = m_0 = 1$  here. However, we have a partial characterization:

**Proposition 5.5** *A PWC  $(n, 1, 2^{-q})$  exists if and only if for some  $i$   $n = 2^q(2^i - 1)$ . Such a PWC can be achieved by a linear code.*

**Proof.** If  $m_1 = 2^{-q}$ , then  $p' = 2^{q'}$ ,  $q' \leq q$ , and (5.1) gives  $n = 2^q(2^{i+q'-q} - 1)$ . The converse stems from Proposition 5.4. □

We would like to point out that for some parameters satisfying (5.1) there is no corresponding code.

Consider the case  $m_0 = 1$ ,  $m_1 = 1/3$ . Proposition 5.4 gives the sequence of lengths  $n = 3 \cdot 2^i - 3$ . The other possibility is  $n = 2^i - 3$ . The first code in this sequence would be a PWC with  $n = 5$  and  $K = 12$ . Let us show its nonexistence.

**Proposition 5.6** *A  $(5, 1, 1/3)$  PWC does not exist.*

**Proof.** We may assume the code contains the zero vector. Furthermore, it does not contain vectors of weight 1, since the minimum distance is 2 for  $m_0 = 1$ . Every vector of weight 1 has to be covered by exactly two codewords of weight 2. There are exactly 5 codewords of weight 2, because if we consider the matrix of all such codewords, we see that each column has sum 2 (by the “coverage” condition just mentioned). Let  $x$  be any vector in  $\mathbf{F}^5$  of weight 3. Each “1” in  $x$  is covered by two codewords of weight 2. That makes six codewords of weight 2. By the pigeonhole principle, two are equal, say to  $c \in C$ . Then  $x$  is at distance 1 from  $c$ .

So the code does not contain vectors of weight 1 and 3, and we cannot cover vectors of weight 2. □

## 6 Linear PMC with diameter 2 ( $m_0 = m_1 = m_2 = 1/j$ )

The purpose of this section is to summarize and extend results from [8].

### 6.1 The case $s = 1$

**Proposition 6.1** [8] *The only PMC with  $s = 1, d = 2$  is the  $[2, 1, 2]$  code with  $j = 2$ .  $\square$*

We assume now that  $d$  is equal to 1. To set the stage, we repeat some material from [8]:

We find that the only possibility for the check matrix is the  $t$ -fold repetition of  $g(S_i)$  (generator matrix of a simplex code of length  $2^i - 1$ ) with  $l$  zero-columns appended, yielding  $n = t(2^i - 1) + l$ . It amounts to appending all possible tails of length  $l$  to codewords described in Proposition 5.2. It is easy to check that there are 2 kinds of covering equalities (namely, vectors coinciding with, or being at distance 1 from, codewords on the first  $t(2^i - 1)$  coordinates):

$$\begin{aligned} m_0 + lm_1 + \binom{t}{2} (2^i - 1)m_2 + \binom{l}{2} m_2 &= 1 \\ tm_1 + (2^{i-1} - 1)t^2m_2 + tlm_2 &= 1. \end{aligned}$$

This implies

$$(6.1) \quad t^2 - t(2^i + 1 + 2l) + (l^2 + l + 2) = 0$$

which has discriminant

$$(6.2) \quad D = (2^i + 1)^2 + 2^{i+2}l - 8.$$

We get a PMC iff  $D = x^2$  has integer solutions. For example, the values  $i = 3, l = 3, t = 14$  yield the PMC  $[101, 98]$  with  $j = 644$ . Of course, for  $i = t$  we get  $8l + 1 = x^2$  having all odd  $x$  as solutions.

Now we can characterize the solutions of  $D = x^2$ . We need the following result:

**Proposition 6.2**  $(2^{i+1} - 7)$  is a square mod  $2^{i+2}$ .

**Proof.** Proof by induction on  $i$ . If  $x$  is a solution for some  $i$ , i.e., for  $\alpha \in \mathbf{N}$ ,  $x^2 = \alpha 2^{i+2} + 2^{i+1} - 7$ , then for any  $\beta \in \mathbf{N}$  to be chosen later on, and  $i \geq 3$ :

$$\begin{aligned} (x + 2^{i+1}\beta + 2^i)^2 &= x^2 + 2^{i+2}(x\beta + \alpha) + 2^{i+1}x + 2^{i+1} - 7 + 2^{2i}(1 + 4\beta^2 + 4\beta) \\ &\equiv 2^{i+2} \left( x\beta + \alpha + \frac{x-1}{2} \right) + 2^{i+2} - 7 \pmod{2^{i+3}}. \end{aligned}$$

Since  $x$  is odd, we can certainly find  $\beta$  to make  $x\beta + \alpha + \frac{x-1}{2}$  even. Then  $x + 2^{i+1}\beta + 2^i$  is a solution for  $i + 1$ . For  $i \leq 2$ , the proposition is easily checked.  $\square$

The first proof of this proposition was given by I. Shparlinski during the present Workshop.

Obviously, the congruence

$$x^2 \equiv 2^i - 7 \pmod{2^{i+2}}$$

has 4 roots. Denoting by  $a$  the one which lies in  $[0, 2^{i+1}]$ , they are

$$a, 2^{i+1} - a, 2^{i+1} + a, 2^{i+2} - a.$$

Now direct calculations lead to the solution of (6.2), giving the possible  $l$ . Then  $t$  is derived from (6.1).

**Theorem 6.1** *Linear PMC with  $m_0 = m_1 = m_2 = 1/j$ ,  $d = 1$  exist only for the following sets of parameters:*

$$l = (\gamma^2 2^{2i+2} \pm 2^{i+2} \gamma a + a^2 - 2^{i+1} + 7 - 2^{2i}) / 2^{i+2}$$

$$t = (2^i + 1 + 2l \pm \sqrt{(2^i + 1)^2 + 2^{i+2}l - 8}) / 2$$

$$n = t(2^i - 1) + l$$

$$k = n - i$$

$$j = (2^{i-1} - 1)t^2 + t(1 + l),$$

for  $\gamma \in \mathbf{Z}$ , provided  $l \in \mathbf{N}$ . □

## 6.2 The case $s = 2$

We have found the following *PMC* codes  $C$  in this case ( $d = s = \delta = 2$ ); see [8] for constructions.

$C$	$j$	$C^\perp$
[5, 1; 5]	$j = 1$	[5, 4; 2, 4]
[5, 2, 2]	$j = 2$	[5, 3; 2, 4]
[5, 3, 2]	$j = 4$	[5, 2; 2, 4]
[10, 7, 2]	$j = 7$	[10, 3; 4, 7]
[37, 32, 2]	$j = 22$	[37, 5; 16, 22]
[8282, 8269, 2]	$j = 4187$	[8282, 13; 4096, 4187]

The first is a classical perfect code. The notation  $[n, k; w_1, w_2, \dots]$  stands for an  $[n, k]$  code in which all nonzero weights are among  $w_1, w_2, \dots$ . In the above codes  $C^\perp$ , since  $s = 2$ , both weights are present. All the above codes  $C$  are *PMC* codes.

**Conjecture 6.1** *We conjecture the nonexistence of PMC with  $d = s = \delta = 2$  other than those in the table.*

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