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Abstract

We analyse the general Lévy insurance risk process for Lévy measures in the convolution equivalence class $\mathcal{S}^{(\alpha)}$, $\alpha > 0$, via a new kind of path decomposition. This yields a very general functional limit theorem as the initial reserve level $u \to \infty$, and a host of new results for functionals of interest in insurance risk. Particular emphasis is placed on the time to ruin, which is shown to have a proper limiting distribution, as $u \to \infty$, conditional on ruin occurring, under our assumptions. Existing asymptotic results under the $\mathcal{S}^{(\alpha)}$ assumption are synthesised and extended, and proofs are much simplified, by comparison with previous methods specific to the convolution equivalence analyses. Additionally, limiting expressions for penalty functions of the type introduced into actuarial mathematics by Gerber and Shiu, are derived as straightforward applications of our main results.

Keywords: Lévy insurance risk process, convolution equivalence, time to ruin, overshoot, expected discounted penalty function
AMS 2010 Subject Classifications: 60G51; 60F17; 91B30; 62P05.

1 Introduction

Let $X = \{X_t : t \geq 0\}$, $X_0 = 0$, be a Lévy process defined on $(\Omega, \mathcal{F}, P)$, with triplet $(\gamma, \sigma^2, \Pi_X)$, Π_X being the Lévy measure of X. Thus the characteristic function of X is given by the Lévy-Khintchine representation, $Ee^{i\theta X_t} = e^{t\Psi_X(\theta)}$, where

$$\Psi_X(\theta) = \mathrm{i}\theta\gamma - \sigma^2\theta^2/2 + \int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}})\Pi_X(\mathrm{d}x), \text{ for } \theta \in \mathbb{R}.$$

We will be concerned with the case where $X_t \to -\infty$ a.s. We have in mind an insurance risk model with premiums and other income producing a downward drift in X,

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while claims are represented by positive jumps. Thus the process X, called the *claim surplus process*, represents the excess in claims over premium. We think of an insurance company starting with an initial positive reserve u, and ruin occurring if this level is exceeded by X. We will refer to this as the *General Lévy Insurance Risk Model*. It is a generalisation of the classical *Cramér-Lundberg model*, which arises when the claim surplus process is taken to be

$$X_t = \sum_1^{N_t} U_i - rt \tag{1.1}$$

where N_t is a Poisson process, $U_i > 0$ form an independent i.i.d. sequence and $r > 0$. Here r represents the rate of premium inflow and U_i the size of the ith claim. The general model allows for income other than through premium inflow and a more realistic claims structure; see Section 2.7.1 of Kyprianou [19]. The assumption $X_t \to -\infty$ a.s. is a reflection of premiums being set to avoid almost certain ruin for finite u.

The primary focus of this paper is on when and how ruin occurs for large reserve levels, that is as $u \to \infty$. Introduce

$$\overline{X}_t = \sup_{0\le s\le t} X_s, \qquad G_t = \sup\{0 \le s \le t : X_s = \overline{X}_s\} \tag{1.2}$$

and

$$\tau(u) = \inf\{t \ge 0 : X_t > u\}. \tag{1.3}$$

(In cases where possible confusion might arise, we will indicate the dependence on the process under consideration by a superscript, as in G_t^X.) These variables play a central rôle in fluctuation theory for Lévy processes, and give rise to the main variables of interest in insurance risk:

- Ruin Time : $\tau(u)$,
- Shortfall at Ruin (Overshoot): $X_{\tau(u)} - u$,
- Surplus Immediately Prior to Ruin (Undershoot): $u - X_{\tau(u)-}$,
- Minimum Surplus Prior to Ruin: $u - \overline{X}_{\tau(u)-}$,
- Time of Minimum Surplus Prior to Ruin: $G_{\tau(u)-}$,
- Time Remaining to Ruin from the Time of Minimum Surplus: $\tau(u) - G_{\tau(u)-}$.

Our main interest is in the behaviour of the process when ruin occurs, that is when $\tau(u) < \infty$. Crucial questions, for example, are "how long does it take for ruin to occur?" and "what do the paths look like leading up to ruin?" We pay particular attention to these issues. We will exclude the trivial case that X is the negative of a subordinator, so $P(\tau(u) < \infty) > 0$ for finite u (cf. (2.16) below). On the other hand, the assumption $X_t \to -\infty$ a.s. implies $P(\tau(u) < \infty) \to 0$ as the initial level $u \to \infty$. Consequently it is convenient to introduce, by elementary means, a new probability measure $P^{(u)}$ given by

$$P^{(u)}(\,\cdot\,) = P(\,\cdot\,|\tau(u) < \infty),$$

and to state our results as limit theorems conditional on $\tau(u) < \infty$, that is under $P^{(u)}$.

Some further background is useful to place our results in context. The original work on the Cramér-Lundberg model was done under the *Cramér-Lundberg condition*

$$Ee^{\alpha X_1} = 1 \text{ for some } \alpha > 0, \tag{1.4}$$

which among other things implies $X_t \to -\infty$ a.s. Embrechts, Klüppelberg and Mikosch [14] call (1.4) the *small claims condition*. The major results in this area include a large deviation estimate for the probability of ruin:

$$e^{\alpha u} P(\tau(u) < \infty) \to C, \tag{1.5}$$

where C is a constant which can be identified, and $C > 0$ if

$$EX_1 e^{\alpha X_1} < \infty. \tag{1.6}$$

In addition, the asymptotic behavior under $P^{(u)}$ of several of the variables listed above is known; see, e.g., [1] or [14]. The ruin estimate (1.5) was extended to general Lévy insurance risk processes satisfying (1.4) by Bertoin and Doney [5].

A second regime under which the Cramér-Lundberg model has been studied is the *subexponential* or *large claims case*; see Asmussen and Klüppelberg [3]. In this scenario, the claim size distribution is subexponential, and, roughly speaking, ruin occurs solely due to the realisation of one extremely large claim.

The small and large claims models each have their various strengths and weaknesses. A third, intermediate, regime was introduced recently in the general model by Klüppelberg, Kyprianou and Maller [18]. To motivate this model, observe that in the small claims case (1.4) holds, while in the large claims (subexponential) case

$$Ee^{\alpha X_1} = \infty \text{ for all } \alpha > 0. \tag{1.7}$$

Thus to obtain a new model we must either consider processes whose distributions satisfy (1.7) and which are not subexponential, or processes which satisfy $Ee^{\alpha X_1} < \infty$ for some $\alpha > 0$ but for which (1.4) fails. It is the latter alternative that we will focus on. Since $X_t \to -\infty$ a.s., it is easy to see that such processes must satisfy that, for some $\alpha > 0$,

$$Ee^{\alpha X_1} < 1 \quad \text{and} \quad Ee^{(\alpha+\varepsilon)X_1} = \infty \text{ for all } \varepsilon > 0. \tag{1.8}$$

For example, those with distribution tails of the form

$$P(X_1 > x) \sim \frac{e^{-\alpha x}}{x^p} \text{ for } p > 1 \tag{1.9}$$

satisfy (1.8). A natural class of distributions which include those of the form (1.9) is the class of convolution equivalent distributions of index α, which we now briefly describe. As in [18], we will restrict ourselves to the nonlattice case, with the understanding that the alternative can be handled by obvious modifications. A distribution F on $[0, \infty)$ with tail $\overline{F} = 1 - F$ belongs to the *class* $\mathcal{S}^{(\alpha)}$, $\alpha > 0$, if $\overline{F}(u) > 0$ for all $u > 0$,

$$\lim_{u\to\infty} \frac{\overline{F}(u+x)}{\overline{F}(u)} = e^{-\alpha x}, \text{ for } x \in (-\infty, \infty), \tag{1.10}$$

and

$$\lim_{u\to\infty} \frac{\overline{F^{2*}}(u)}{\overline{F}(u)} \text{ exists and is finite,} \tag{1.11}$$

where $F^{2*} = F * F$. Distributions in $\mathcal{S}^{(\alpha)}$ are called *convolution equivalent* with index α. When $F \in \mathcal{S}^{(\alpha)}$, the limit in (1.11) must be of the form $2\delta_\alpha^F$, where $\delta_\alpha^F := \int_{[0,\infty)} e^{\alpha x} F(\mathrm{d}x)$ is finite. Much is known about the properties of such distributions. In particular, the class is closed under tail equivalence, that is, if $F \in \mathcal{S}^{(\alpha)}$ and G is a distribution function for which

$$\lim_{u\to\infty} \frac{\overline{G}(u)}{\overline{F}(u)} = c \text{ for some } c \in (0,\infty),$$

then $G \in \mathcal{S}^{(\alpha)}$.

Although the exponential distribution with parameter α is not in $\mathcal{S}^{(\alpha)}$, distributions in $\mathcal{S}^{(\alpha)}$ are "near to exponential"; for example, distributions with tails comparable to $x^{-p}e^{-\alpha x}$, where $p > 1$, are in $\mathcal{S}^{(\alpha)}$. The inverse Gaussian distributions, with appropriate choices of parameters, form an important class of distributions which are convolution equivalent. These in turn are a special case of the tempered stable distributions, which have been the subject of considerable recent activity. For further examples and more on convolution equivalence see [9], [12], [17], [20] and [21].

We can take the tail of any Lévy measure, assumed nonzero on some interval (x_0,∞), $x_0 > 0$, to be the tail of a distribution function on $[0,\infty)$, after renormalisation. With this convention, we say then that the measure (or its tail) is in $\mathcal{S}^{(\alpha)}$ if this is true for the distribution with the corresponding (renormalised) tail. The convolution equivalent model introduced in [18] is then one in which

$$\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)} \text{ and } Ee^{\alpha X_1} < 1, \text{ for some } \alpha > 0, \tag{1.12}$$

where Π_X^+ is the restriction of Π_X to $(0,\infty)$, and $\overline{\Pi}^+(x) = \Pi_X((x,\infty))$, $x > 0$. The condition $Ee^{\alpha X_1} < 1$ implies $e^{\alpha X_t}$ is a nonnegative supermartingale, from which it follows immediately that $X_t \to -\infty$ a.s. (This is also true when $Ee^{\alpha X_1} = 1$.)

By way of comparison with the small claims model, consider a one parameter family of Cramér-Lundberg models (1.1), in which the claim size distribution $U \in \mathcal{S}^{(\alpha)}$. Let

$$X_t^{(r)} = \sum_1^{N_t} U_i - rt, \ r \geq 0,$$

and set

$$r_L = \frac{\ln(Ee^{\alpha X_1^{(0)}})}{\alpha}.$$

Then $Ee^{\alpha X_1^{(r)}} = 1$ if $r = r_L$ and $Ee^{\alpha X_1^{(r)}} < 1$ if $r > r_L$. Thus the convolution equivalent models correspond to larger premium rates (faster drift of X to $-\infty$, lower probability of ruin), than under the small claims condition (1.4). In general, for any convolution equivalent model, there is an associated model in which (1.4) holds, obtained by adding an appropriate positive drift, which corresponds to decreasing the premium rate. However this change in premium rate leads to quite different behavior in the two models.

Conditional on ruin occurring, the qualitative behavior of the claims surplus process is very different in the convolution equivalent model as opposed to either the small or large claims models. In these latter two cases, the time to ruin, $\tau(u)$, is of order u as $u \to \infty$. In the small claims case, under mild assumptions, there is a constant $b > 0$ such that

$$\frac{\tau(u)}{u} \to b^{-1} \text{ in } P^{(u)} \text{ probability}$$

and

$$\sup_{t \in [0,1]} \left| \frac{X(t\tau(u))}{\tau(u)} - bt \right| \to 0 \text{ in } P^{(u)} \text{ probability },$$

indicating that ruin occurs owing to the build up of small claims which tend to cause X to behave as though it had positive drift; see [1] or [14]. In the subexponential case, the ruin time is again of order u (in distribution). However in this case the process evolves quite normally, that is, like a sample path for which ruin does not occur, until a very large claim suddenly causes ruin. This claim is so large that the shortfall $X_{\tau(u)} - u \xrightarrow{P^{(u)}} \infty$; see [3] or [14].

An obvious shortcoming of the small claims model is that it does not allow for disasters, that is large jumps, which are observed in real insurance data. On the other hand the subexponential model is very extreme and uninformative in the sense that paths leading to ruin look quite normal until suddenly a large claim occurs, which results in ruin with an arbitrarily large shortfall.

By contrast, the convolution equivalent model allows for disasters to occur, but they are not so ruinous as to be disproportionate in size relative to the reserve level. We will show that, in this model, asymptotically, ruin occurs in finite time (in distribution), and for ruin to occur, the claims surplus process must take a large jump from a neighbourhood of the origin to a neighbourhood of u. This jump may result in ruin, but if not, the process $X - u$ subsequently behaves like X conditioned to hit $(0, \infty)$. In either case, the shortfall at ruin converges in distribution to a finite random variable as $u \to \infty$. These results will follow from a path decomposition and asymptotic analysis of the distribution of X, conditional on ruin, in a way described below. The idea of studying ruin through a description of the entire path leading up to ruin, seems to have first appeared in Asmussen [1], where the small claims case for random walk is investigated. For work in the subexponential case, see Asmussen and Klüppelberg [3] .

2 Skorohod Space and Notation

Fix $\Delta \notin \mathbb{R}$ and let $E = \mathbb{R} \cup \{\Delta\}$. Define a metric d on E by

$$d(x,y) = \begin{cases} |x-y| \wedge 1, & x, y \in \mathbb{R} \\ 1, & x \in \mathbb{R}, y = \Delta \\ 0 & x = y = \Delta. \end{cases}$$

Thus Δ is an isolated point, which will act as a cemetery state, and for $x, y \in \mathbb{R}$, $|x-y| \to 0$ if and only if $d(x, y) \to 0$. Let D be the Skorohod space of functions on $[0, \infty)$, taking values in the metric space E, and which are right continuous with left limits. It is often convenient to assume that X is given as the coordinate process on D. We will interchangeably write X or w depending on which seems clearer in the context. The usual right continuous completion of the filtration generated by the coordinate maps will be denoted by $\{\mathcal{F}_t\}_{t\ge0}$. P_z denotes the probability measure induced on $\mathcal{F} = \vee_{t\ge0}\mathcal{F}_t$ by the Lévy process starting at $z \in \mathbb{R}$. We sometimes write just P for P_0. The shift operators $\theta_t : D \to D$, $t \ge 0$, are defined by $(\theta_t(w))_s = w(t+s)$.

For a given function $w = (w_t)_{t\ge0} \in D$, and $r \ge 0$, let $w_{[0,r)} = (w_{[0,r)}(t))_{t\ge0} \in D$ denote the killed path

$$w_{[0,r)}(t) = \begin{cases} w_t, & 0 \le t < r \\ \Delta & t \ge r. \end{cases}$$

For any $\rho : D \to [0, \infty]$ we then have the corresponding element $w_{[0,\rho)} \in D$ defined by $w_{[0,\rho)} = w_{[0,\rho(w))}$. For $x \in E$, let $c^x \in D$ be the constant path $c^x_t = x$ for all $t \ge 0$. If $w, w' \in D$, $w - w'$ denotes the path in D given by

$$(w - w')_t = \begin{cases} w_t - w'_t, & \text{if } t < \tau_\Delta(w) \wedge \tau_\Delta(w') \\ \Delta & \text{otherwise.} \end{cases}$$

Let

$$\tau_z = \tau_z(w) = \inf\{t > 0 : w_t > z\}, \qquad \tau_\Delta = \tau_\Delta(w) = \inf\{t > 0 : w_t = \Delta\}$$

For notational convenience we will interchangeably write w_t and $w(t)$, τ_z and $\tau(z)$ etc. Observe that for any $t \ge 0$ and $w \in D$

$$\tau_\Delta(w_{[0,t)}) = t \text{ if } \tau_\Delta(w) \ge t. \tag{2.1}$$

We adopt the following notation from [16] which is very standard in the area (cf. [4], [10] and [19]). Let $(L_s)_{s\ge0}$ denote the local time of X at its maximum, and $(L^{-1}_s, H_s)_{s\ge0}$ the weakly ascending bivariate ladder process. When $X_t \to -\infty$ a.s., L_∞ has an exponential distribution with some parameter $q > 0$, and the defective process (L^{-1}, H) may be obtained from a nondefective process $(\mathcal{L}^{-1}, \mathcal{H})$ by independent exponential killing at rate $q > 0$. Thus

$$\big((L^{-1}_s, H_s) : s < L_\infty\big) \stackrel{\mathrm{D}}{=} \big((\mathcal{L}^{-1}_s, \mathcal{H}_s) : s < e(q)\big) \tag{2.2}$$

where $e(q)$ is independent of $(\mathcal{L}^{-1}, \mathcal{H})$ and has exponential distribution with parameter q.

We denote the bivariate Lévy measure of $(\mathcal{L}^{-1}, \mathcal{H})$ by $\Pi_{L^{-1},H}(\cdot,\cdot)$. The Laplace exponent $\kappa(a, b)$ of (L^{-1}, H), defined by

$$e^{-\kappa(a,b)} = E(e^{-aL^{-1}_1 - bH_1}; L_\infty > 1) = e^{-q} E e^{-a\mathcal{L}^{-1}_1 - b\mathcal{H}_1} \tag{2.3}$$

for values of $a, b \in \mathbb{R}$ for which the expectation is finite, may be written

$$\kappa(a, b) \;=\; q + \mathrm{d}_{L^{-1}} a + \mathrm{d}_H b + \int_{t\ge0}\int_{x\ge0} \left(1 - e^{-at-bx}\right) \Pi_{L^{-1},H}(\mathrm{d}t, \mathrm{d}x), \tag{2.4}$$

where $\mathrm{d}_{L^{-1}} \geq 0$ and $\mathrm{d}_H \geq 0$ are drift constants. The bivariate renewal function of (L^{-1}, H) is

$$V(t,x) = \int_0^\infty e^{-qs} P(\mathcal{L}_s^{-1} \leq t, \mathcal{H}_s \leq x)\mathrm{d}s. \tag{2.5}$$

Its Laplace transform is given by

$$\int_{t\geq 0}\int_{x\geq 0} e^{-at-bx} V(\mathrm{d}t, \mathrm{d}x) = \int_{s\geq 0} e^{-qs} E(e^{-a\mathcal{L}_s^{-1} - b\mathcal{H}_s})\mathrm{d}s = \frac{1}{\kappa(a,b)} \tag{2.6}$$

provided $\kappa(a,b) > 0$. We will also frequently consider the renewal function of H, defined on $\mathbb{R}$ by

$$V(x) = \int_0^\infty e^{-qs} P(\mathcal{H}_s \leq x)\mathrm{d}s = \lim_{t\to\infty} V(t,x). \tag{2.7}$$

Observe that $V(x) = 0$ for $x < 0$, while $V(0) > 0$ iff $\mathcal{H}$ is compound Poisson. Also

$$V(\infty) := \lim_{x\to\infty} V(x) = q^{-1}. \tag{2.8}$$

Let $\widehat{X}_t = -X_t$, $t \geq 0$ denote the dual process, and $(\widehat{L}^{-1}, \widehat{H})$ the corresponding *strictly* ascending bivariate ladder processes of $\widehat{X}$. This is the same as the weakly ascending process if $\widehat{X}$ is not compound Poisson. All quantities relating to $\widehat{X}$ will be denoted in the obvious way, for example $\Pi_{\widehat{L}^{-1},\widehat{H}}$, $\widehat{\kappa}$ and $\widehat{V}$. The reason for this choice of $(\widehat{L}^{-1}, \widehat{H})$ is that we may then, for any Lévy process, choose the normalisation of the local times L and $\widehat{L}$ so that the Wiener-Hopf factorisation takes the form

$$\kappa(a, -ib)\widehat{\kappa}(a, ib) = a - \Psi_X(b), \;\; a \geq 0, b \in \mathbb{R}. \tag{2.9}$$

Throughout the paper our principal assumption will be (1.12). In that case, by Proposition 5.1 of [18],

$$\kappa(a, -\alpha) > 0 \;\; \text{for } a \geq 0. \tag{2.10}$$

Furthermore, by analytic extension, it follows from (2.9) that

$$\kappa(a, -z)\widehat{\kappa}(a, z) = a - \Psi_X(-\mathrm{i}z) \text{ for } a \geq 0, 0 \leq \Re z \leq \alpha. \tag{2.11}$$

Set

$$\beta_1 = -\ln Ee^{\alpha X_1} = -\Psi_X(-\mathrm{i}\alpha) = \kappa(0,-\alpha)\widehat{\kappa}(0,\alpha), \quad \beta_2 = \frac{\kappa(0,-\alpha)}{q}, \quad \beta = \beta_1\beta_2. \tag{2.12}$$

Note that $\beta_1, \beta_2 \in (0, \infty)$ under (1.12). These constants appear in several formulas throughout the paper. For future reference we also note that

$$\beta_1 \int_0^\infty Ee^{\alpha X_t} \,\mathrm{d}t = \beta_1 \int_0^\infty (Ee^{\alpha X_1})^t \,\mathrm{d}t = 1, \tag{2.13}$$

and, letting $\overline{V}(z) = V(\infty) - V(z)$, $z \in \mathbb{R}$, we have by (2.6) and (2.8)

$$\begin{aligned} \beta_2 \int_z \alpha e^{-\alpha z} q\overline{V}(-z)\ \mathrm{d}z &= \beta_2 \left(1 + \int_{z\geq 0} \alpha e^{\alpha z} q\overline{V}(z)\ \mathrm{d}z\right) \\ &= \beta_2 q \int_{z\geq 0} e^{\alpha z} V(\mathrm{d}z) \\ &= \frac{\beta_2 q}{\kappa(0,-\alpha)} = 1. \end{aligned} \tag{2.14}$$

The following important asymptotic estimate can be found in [18]. Assuming (1.12),

$$\lim_{u\to\infty} \frac{\overline{\Pi}_X^+(u)}{q\overline{V}(u)} = \beta. \tag{2.15}$$

This provides information about the probability of eventual ruin through the *Pollacek-Khintchine formula*:

$$P(\tau(u) < \infty) = q\overline{V}(u). \tag{2.16}$$

A further useful estimate from [18], holding under (1.12), is

$$\lim_{u\to\infty} \frac{\overline{\Pi}_X^+(u)}{\overline{\Pi}_H(u)} = \widehat{\kappa}(0,\alpha) \in (0,\infty), \tag{2.17}$$

where Π_H is the Lévy measure of H, and $\overline{\Pi}_H$ is its tail. In particular, this implies

$$\overline{\Pi}_H \in \mathcal{S}^{(\alpha)}, \tag{2.18}$$

since $\mathcal{S}^{(\alpha)}$ is closed under tail equivalence; see Theorem 2.7 of [12].

3 Main Results

We next introduce the basic components of the limiting process, namely, processes W and Z, and a random variable ρ. These three random elements are independent. The distribution of W is given by

$$P(W \in \mathrm{d}w) = \beta_2 \int_{z\in\mathbb{R}} \alpha e^{-\alpha z} q\overline{V}(-z)\mathrm{d}z\, P_z(X \in \mathrm{d}w|\tau(0) < \infty),\ w \in D. \tag{3.1}$$

(Recall that $\overline{V}(y)) = q^{-1}$ for $y < 0$.) Thus W has the law of X conditioned on $\tau(0) < \infty$ and started with initial distribution

$$P(W_0 \in \mathrm{d}z) = \beta_2 \alpha e^{-\alpha z} q\overline{V}(-z)\ \mathrm{d}z, z \in \mathbb{R}. \tag{3.2}$$

Observe that (3.2) is indeed a probability distribution, by (2.14). Let Z be the Esscher transform of X, defined by

$$\begin{aligned} P(\{Z_t : 0 \leq t \leq s\} \in B_s, Z_s \in \mathrm{d}x) &= \frac{e^{\alpha x} P(\{X_t : 0 \leq t \leq s\} \in B_s, X_s \in \mathrm{d}x)}{E e^{\alpha X_s}} \\ &= e^{\beta_1 s} e^{\alpha x} P(\{X_t : 0 \leq t \leq s\} \in B_s, X_s \in \mathrm{d}x) \end{aligned} \tag{3.3}$$

where B_s is a Borel in $\mathbb{R}^{[0,s]}$ and $x \in \mathbb{R}$. Finally let

$$\rho \text{ be exponentially distributed with parameter } \beta_1. \tag{3.4}$$

Let $H : D \otimes D \to \mathbb{R}$ be measurable with respect to the product σ-algebra and set

$$G(w, z) = E_z[H(w, X); \tau(0) < \infty], \ \ w \in D, z \in \mathbb{R}. \tag{3.5}$$

We denote by $\mathcal{H}$ the class of such functions H which satisfy

$$H(w, w')e^{\theta w_{\tau(\Delta)-} I(w_{\tau(\Delta)-} \le 0)} \text{ is bounded for some } \theta \in [0, \alpha); \tag{3.6}$$

$$G(w, \cdot) \text{ is continuous a.e. on } (-\infty, \infty) \text{ for every } w \in D. \tag{3.7}$$

For example, if H is bounded and continuous in the product Skorohod topology on $D \otimes D$, these conditions hold with $\theta = 0$. More general conditions on H, which ensure that (3.7) holds, will be discussed below. Taking $\theta > 0$ in (3.6) allows for certain unbounded functions H, which will be used in Section 8.

Here is our main theorem:

Theorem 3.1 (Path Decomposition) *Assume* (1.12). *Then for any* $H \in \mathcal{H}$

$$\lim_{x\to\infty} \lim_{u\to\infty} E^{(u)} H(X_{[0,\tau(u-x))}, X \circ \theta_{\tau(u-x)} - c^u) = EH(Z_{[0,\rho)}, W). \tag{3.8}$$

The reason for introducing x and taking the limit, is to capture the difference in behaviour of the conditioned process before and after entering a neighbourhood of u. The heuristic meaning of the result is that the conditioned process, for large u, can be approximated as follows;

- run the process Z for times $0 \le t < \rho$;
- then, run the process $u + W$ from time ρ on, that is, at time $\rho + t$ the value of the process is $u + W_t$.

Thus the process behaves like Z up until an independent exponential time ρ, at which time it makes a large jump from a neighbourhood of 0 to a neighborhood of u. Its position immediately prior to the jump is $Z_{\rho-}$ and its position after the jump is $u + W_0$. If $W_0 > 0$ the process $X - u$ behaves like X started at W_0. If $W_0 \le 0$, the process $X - u$ behaves like X started at W_0 and conditioned on $\tau(0) < \infty$. This behavior is significantly different from the Cramér and subexponential cases discussed earlier.

It is apparent that many asymptotic results will flow from Theorem 3.1. We develop some of these in Sections 5-8. The literature to date has focused on deficit at ruin (overshoot) and surplus prior to ruin (undershoot). We use Theorem 3.1 to derive these and related results in Section 7. Of perhaps greater importance in insurance risk theory, though, is the probability of ruin occurring in finite time. So far this has been neglected in studies of this type (except, see the paper of Braverman [8] discussed below). We use Theorem 3.1 to give a completely explicit representation of the asymptotic distribution of the ruin time, in:

Theorem 3.2 (Asymptotic Distribution of Ruin Time) *Assume* (1.12)*. Then for* $t \geq 0$

$$\lim_{u\to\infty} P^{(u)}(\tau(u) \leq t) = P(\rho + \tau^W(0) \leq t) \\ = \beta_2 E(e^{\alpha \overline{X}_{t-\rho}}; \rho \leq t), \tag{3.9}$$

where ρ is independent of X and W, and has exponential distribution with parameter β_1.

We can compare this result with those of Braverman [8]. He assumes, as we do, that $\overline{\Pi}_X^+(x) \in \mathcal{S}^{(\alpha)}$ for an $\alpha > 0$, and his Theorem 2.1 can be used to deduce that

$$\lim_{u\to\infty} \frac{P(\tau(u) \leq t)}{\overline{\Pi}_X^+(u)}$$

exists for each $t > 0$, and hence, via (2.15), that $\lim_{u\to\infty} P^{(u)}(\tau(u) \leq t)$ also exists. However, the expressions thus obtained for these limits are highly inexplicit, and it is not at all clear from them whether or not the limiting distribution is proper (total mass 1). Theorem 3.2 gives a much simpler expression for the limiting distribution and establishes that it is indeed proper, being the convolution of two proper probability distributions.

4 Proof of Path Decomposition

Let $\mathcal{B}$ denote the Borel sets on $\mathbb{R}$, $\mathcal{B}([0,\infty))$ the Borel sets on $[0,\infty)$ and set $\mathcal{D} = D \otimes [0,\infty) \otimes (-\infty,\infty)$. For $K \in (-\infty,\infty]$ and $x \in [0,\infty]$, define measures μ_K and ν_x on $\mathcal{F} \otimes \mathcal{B}([0,\infty)) \otimes \mathcal{B}$ by

$$\mu_K(\mathrm{d}w, \mathrm{d}t, \mathrm{d}\phi) = \beta_1 I(\phi < K) e^{\alpha\phi} P(X_{[0,t)} \in \mathrm{d}w; X_{t-} \in \mathrm{d}\phi)\ \mathrm{d}t \tag{4.1}$$

and

$$\nu_x(\mathrm{d}w', \mathrm{d}r, \mathrm{d}z) = \beta_2 I(z > -x)\alpha e^{-\alpha z}\mathrm{d}z\ P_z(X \in \mathrm{d}w'; \tau(0) \in \mathrm{d}r). \tag{4.2}$$

We will write μ and ν for μ_∞ and ν_∞ respectively. These are finite measures and indeed μ and ν are probability measures on $\mathcal{D}$ since by (2.13)

$$\mu(\mathcal{D}) = \beta_1 \int_0^\infty E e^{\alpha X_t}\ \mathrm{d}t = 1, \tag{4.3}$$

and by (2.14) and (2.16)

$$\nu(\mathcal{D}) = \beta_2 \left(1 + \int_{z\leq 0} \alpha e^{-\alpha z}\ P_z(\tau(0) < \infty)\ \mathrm{d}z\right) \\ = \beta_2 \left(1 + \int_{z\geq 0} \alpha e^{\alpha z}\ P(\tau(z) < \infty)\ \mathrm{d}z\right) \\ = \beta_2 \left(1 + \int_{z\geq 0} \alpha e^{\alpha z} q\overline{V}(z)\ \mathrm{d}z\right) = 1. \tag{4.4}$$

In a slight abuse of notation we will denote the marginal measures in the obvious way. Thus for example

$$\begin{aligned} \mu_K(\mathrm{d}w, \mathrm{d}\phi) &= \beta_1 \int_0^\infty I(\phi < K) e^{\alpha\phi} P(X_{[0,t)} \in \mathrm{d}w; X_{t-} \in \mathrm{d}\phi)\ \mathrm{d}t \\ \nu_x(\mathrm{d}w') &= \beta_2 \int_{z>-x} \alpha e^{-\alpha z} \mathrm{d}z\ P_z(X \in \mathrm{d}w'; \tau(0) < \infty). \end{aligned} \tag{4.5}$$

From (4.3) and (4.4), $\mu(\mathrm{d}w)$ and $\nu(\mathrm{d}w')$ define probability measures on D. From (2.16) and (3.1), it is clear that $\nu(\mathrm{d}w') = P(W \in \mathrm{d}w')$. The following result identifies μ as the distribution of $Z_{[0,\rho)}$, where Z and ρ are given by (3.3) and (3.4) respectively.

Proposition 4.1 *Let $\tilde{Z}$ have law given by $P(\tilde{Z} \in \mathrm{d}w) = \mu(\mathrm{d}w)$, and set $\tau_{\tilde{Z}} = \tau_\Delta(\tilde{Z}) = \inf\{t > 0 : \tilde{Z}_t = \Delta\}$. Then with ρ and Z as above,*

$$\{\tilde{Z}_t : t < \tau_{\tilde{Z}}\} \overset{d}{=} \{Z_t : t < \rho\}. \tag{4.6}$$

Proof of Proposition 4.1. For any $B_s \in \mathcal{B}([0, s])$

$$\begin{aligned} &P(\{\tilde{Z}_t : 0 \le t \le s\} \in B_s, \tilde{Z}_s \in \mathrm{d}x, s < \tau_{\tilde{Z}}) \\ &\quad= \beta_1 \int_{r>s} \int_\phi e^{\alpha\phi} P(\{X_t : 0 \le t \le s\} \in B_s, X_s \in \mathrm{d}x, X_{r-} \in \mathrm{d}\phi)\ \mathrm{d}r \\ &\quad= \beta_1 P(\{X_t : 0 \le t \le s\} \in B_s, X_s \in \mathrm{d}x) \int_{r>0} \int_\phi e^{\alpha\phi} P(X_{r-} \in \mathrm{d}\phi - x)\ \mathrm{d}r \\ &\quad= e^{\alpha x} P(\{X_t : 0 \le t \le s\} \in B_s, X_s \in \mathrm{d}x) \quad \text{(by (2.13))} \\ &\quad= P(\{Z_t : 0 \le t \le s\} \in B_s, Z_s \in \mathrm{d}x) e^{-\beta_1 s} \quad \text{(by (3.3))} \\ &\quad= P(\{Z_t : 0 \le t \le s\} \in B_s, Z_s \in \mathrm{d}x, s < \rho). \end{aligned}$$

Integrating out x completes the proof. □

Lemma 4.1 *Fix $x \in [0, \infty)$, $u > x$, $A \subset (-\infty, u - x]$ and $B \subset (u - x, \infty)$. Then for any $H \in \mathcal{H}$ which is nonnegative,*

$$\begin{aligned} &E[H(X_{[0,\tau(u-x))}, X \circ \theta_{\tau(u-x)} - c^u) : X_{\tau(u-x)-} \in A, X_{\tau(u-x)} \in B, \tau(u) < \infty] = \\ &\int_0^\infty \mathrm{d}t \int_{w\in D} \int_{\phi\in A} \int_{z\in B-u} G(w, z) \Pi_X^+(u - \phi + \mathrm{d}z) P(X_{[0,t)} \in \mathrm{d}w, X_{t-} \in \mathrm{d}\phi, \tau(u - x) \ge t), \end{aligned} \tag{4.7}$$

where G is defined by (3.5).

Proof of Lemma 4.1. By the strong Markov property

$$\begin{aligned} &E[H(X_{[0,\tau(u-x))}, X \circ \theta_{\tau(u-x)} - c^u) : X_{\tau(u-x)-} \in A, X_{\tau(u-x)} \in B, \tau(u) < \infty] = \\ &\quad E[G(X_{[0,\tau(u-x))}, X_{\tau(u-x)} - u) : X_{\tau(u-x)-} \in A, X_{\tau(u-x)} \in B, \tau(u - x) < \infty]. \end{aligned}$$

Since $AB = \emptyset$, $\Delta X_{\tau(u-x)} > 0$ on $\{X_{\tau(u-x)-} \in A, X_{\tau(u-x)} \in B\}$. Thus by the compensation formula (see [4], p.7)

$$
\begin{aligned}
&E[G(X_{[0,\tau(u-x))}, X_{\tau(u-x)} - u) : X_{\tau(u-x)-} \in A, X_{\tau(u-x)} \in B, \tau(u-x) < \infty] = \\
&E\sum_t G(X_{[0,t)}, X_{t-} + \Delta X_t - u) I(X_{t-} \in A, \tau(u-x) \geq t) I(X_{t-} + \Delta X_t \in B) = \\
&E\int_0^\infty \mathrm{d}t \int_\xi G(X_{[0,t)}, X_{t-} + \xi - u) I(X_{t-} \in A, \tau(u-x) \geq t) I(X_{t-} + \xi \in B) \Pi_X^+(\mathrm{d}\xi) = \\
&\int_0^\infty \mathrm{d}t \int_{w\in D}\int_{\phi\in A}\int_{\xi+\phi\in B} G(w, \phi + \xi - u)\Pi_X^+(\mathrm{d}\xi) P(X_{[0,t)} \in \mathrm{d}w, X_{t-} \in \mathrm{d}\phi, \tau(u-x) \geq t),
\end{aligned}
$$

and this is (4.7). □

In conjunction with Lemma 4.1 it is useful to note that, for $u > x \geq 0$,

$$
P(X_{\tau(u-x)-} < u - x, X_{\tau(u-x)} = u - x, \tau(u-x) < \infty) = 0; \tag{4.8}
$$

see for example Lemma 5.1 of [16].

We need two further observations before we come to the proof of Theorem 3.1. From (3.6) and (3.7), it follows immediately that for any K

$$
G(w, z) e^{\theta w_{\tau(\Delta)-} I(w_{\tau(\Delta)-} < K)} \tag{4.9}
$$

is bounded as a function of (w, z) and continuous a.e. in z for every $w \in D$.

Referring to (1.10), an important global bound on convolution equivalent distributions is obtained by applying Theorem 1.5.6(ii) of [7], to the function

$$
l(r) = (r \vee e)^\alpha \overline{F}(\ln(r \vee e))
$$

which is slowly varying as $r \to \infty$. This yields the following version of Potter's bounds for regularly varying functions. Assume (1.10); then for every $\varepsilon > 0$ there exists an $A = A_\varepsilon$ such that

$$
\frac{\overline{F}(u+x)}{\overline{F}(u)} \leq A[e^{-(\alpha-\varepsilon)x} \vee e^{-(\alpha+\varepsilon)x}] \text{ for all } u \geq 1, x \geq 1 - u. \tag{4.10}
$$

Clearly this estimate also applies to $\overline{\Pi}_X^+$ since we may take $\overline{F}(x) = \overline{\Pi}_X^+(x)/\overline{\Pi}_X^+(1)$ for $x \geq 1$. Similarly for $\overline{\Pi}_H$, since recall $\overline{\Pi}_H \in \mathcal{S}^{(\alpha)}$ from (2.18).

The key step in the proof of Theorem 3.1 is the following result:

Theorem 4.1 *Assume* (1.12), *and fix* $x \in [0, \infty)$ *and* $K \in (-\infty, \infty)$. *Then for any* $H \in \mathcal{H}$

$$
\begin{aligned}
&\lim_{u\to\infty} E^{(u)}[H(X_{[0,\tau(u-x))}, X \circ \theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-} < K] \\
&\qquad = \int_{w\in D}\int_{w'\in D} H(w, w')\mu_K(\mathrm{d}w)\nu_x(\mathrm{d}w').
\end{aligned} \tag{4.11}
$$

Proof of Theorem 4.1. We first show that the expression for the limit is finite. By Proposition 4.1, and independence of $Z_{[0,\rho)}$ and W,

$$P(Z_{[0,\rho)} \in \mathrm{d}w, W \in \mathrm{d}w') = \mu(\mathrm{d}w) \otimes \nu(\mathrm{d}w'). \tag{4.12}$$

Hence using (4.5)

$$\begin{aligned}\int_{w\in D}\int_{w'\in D} |H(w,w')|\mu_K(\mathrm{d}w)\nu_x(\mathrm{d}w') &\leq \int_{w\in D}\int_{w'\in D} |H(w,w')|\mu(\mathrm{d}w)\nu(\mathrm{d}w') \\ &= E|H(Z_{[0,\rho)}, W)| < \infty,\end{aligned} \tag{4.13}$$

where to verify that the final expectation is finite, it suffices by (3.6) to show that $Ee^{-\theta Z_{\rho-} - I(Z_{\rho-}\leq 0)} < \infty$. But by (3.3)

$$Ee^{-\theta Z_{\rho-}} = \int_0^\infty Ee^{-\theta Z_s} P(\rho \in \mathrm{d}s) = \beta_1 \int_0^\infty Ee^{(\alpha-\theta)X_s}\mathrm{d}s < \infty$$

if $0 \leq \theta < \alpha$.

We now prove convergence. Take u large enough that $K < u-x$, and set $A = (-\infty, K)$ and $B = (u-x, \infty)$ in (4.7). Then, recalling (4.8), we have

$$\begin{aligned}&E[H(X_{[0,\tau(u-x))}, X\circ\theta_{\tau(u-x)} - c^u) : X_{\tau(u-x)-} < K, \tau(u) < \infty] = \\ &\int_0^\infty \mathrm{d}t \int_{w\in D}\int_{\phi<K}\int_{z>-x} G(w,z)\Pi_X^+(u-\phi+\mathrm{d}z)P(X_{[0,t)} \in \mathrm{d}w, X_{t-}\in \mathrm{d}\phi, \tau(u-x)\geq t).\end{aligned}$$

Using that K and x are fixed, and that as $u \to \infty$, $I(\tau(u-x) \geq t) \to 1$ and

$$\frac{\Pi_X^+(u-\phi+\mathrm{d}z)}{\overline{\Pi}_X^+(u)} \to e^{\alpha\phi}\alpha e^{-\alpha z}\ \mathrm{d}z \tag{4.14}$$

in the sense of weak convergence on $(-x, \infty)$, we will show

$$\begin{aligned}\int_0^\infty \mathrm{d}t \int_{w\in D}\int_{\phi<K}\int_{z>-x} G(w,z)\frac{\Pi_X^+(u-\phi+\mathrm{d}z)}{\overline{\Pi}_X^+(u)}P(X_{[0,t)} \in \mathrm{d}w, X_{t-}\in \mathrm{d}\phi, \tau(u-x)\geq t) \\ \to \beta^{-1}\int_{w'\in D}\int_{w\in D} H(w,w')\mu_K(\mathrm{d}w)\nu_x(\mathrm{d}w').\end{aligned} \tag{4.15}$$

By (2.15) and (2.16), this will complete the proof.

Let

$$\Lambda_u(w,\phi) = \int_{z>-x} G(w,z)\frac{\Pi_X^+(u-\phi+\mathrm{d}z)}{\overline{\Pi}_X^+(u)}.$$

For fixed w, $G(w,\cdot)$ is bounded and continuous a.e. by (4.9). Thus by (4.14), for fixed (w,ϕ)

$$\Lambda_u(w,\phi) \to \int_{z>-x} G(w,z)\alpha e^{\alpha(\phi-z)}\ \mathrm{d}z =: \Lambda(w,\phi).$$

Next write

$$\frac{\Pi_X^+(u-\phi+\mathrm{d}z)}{\overline{\Pi}_X^+(u)} = \frac{\Pi_X^+(u-\phi+\mathrm{d}z)}{\overline{\Pi}_X^+(u-\phi-x)}\frac{\overline{\Pi}_X^+(u-\phi-x)}{\overline{\Pi}_X^+(u)}.$$

The first term is a probability measure on $(-x,\infty)$. For the second term, fix $\varepsilon > 0$ so that $\theta + 2\varepsilon < \alpha$. By (4.10), there exists an A so that

$$\frac{\overline{\Pi}_X^+(u-\phi-x)}{\overline{\Pi}_X^+(u)} \le A[e^{(\alpha-\varepsilon)(\phi+x)} \vee e^{(\alpha+\varepsilon)(\phi+x)}] \tag{4.16}$$

if $u \ge 1$ and $\phi + x \le u - 1$. Now for $\phi < K$,

$$e^{(\alpha-\varepsilon)\phi} \vee e^{(\alpha+\varepsilon)\phi} \le e^{(\alpha-\varepsilon)\phi} I(\phi < 0) + e^{(\alpha+\varepsilon)K} e^{(\alpha-\varepsilon)\phi} I(0 \le \phi < K) \le e^{(\alpha+\varepsilon)|K|} e^{(\alpha-\varepsilon)\phi}.$$

Thus if $u_0 =: (K + x + 1) \vee 1$, then for some constant C depending on H, K and x,

$$\sup_{u \ge u_0} |\Lambda_u(w,\phi)| \le C e^{(\alpha-\varepsilon)\phi} e^{-\theta w_{\tau(\Delta)-} I(w_{\tau(\Delta)-} < K)}, \quad \text{all } w \in D,\ \phi < K. \tag{4.17}$$

In particular, since $\alpha - \varepsilon - \theta > \varepsilon$, for every $t \ge 0$

$$\begin{aligned} \sup_{u \ge u_0} |\Lambda_u(X_{[0,t)}, X_{t-})| I(X_{t-} < K) &\le C e^{(\alpha-\varepsilon-\theta)X_{t-}} I(X_{t-} < K) \\ &\le C_1 e^{\varepsilon X_{t-}} I(X_{t-} < K), \end{aligned} \tag{4.18}$$

where $C_1 = Ce^{(\alpha-\varepsilon-\theta)|K|}$. Next observe that

$$\begin{aligned} \Phi_u(t) &=: \int_{w \in D} \int_{\phi < K} \Lambda_u(w,\phi) P(X_{[0,t)} \in \mathrm{d}w, X_{t-} \in \mathrm{d}\phi; \tau(u-x) \ge t) \\ &= E[\Lambda_u(X_{[0,t)}, X_{t-}); X_{t-} < K, \tau(u-x) \ge t] \\ &\to E[\Lambda(X_{[0,t)}, X_{t-}); X_{t-} < K] =: \Phi(t) \end{aligned}$$

as $u \to \infty$, by bounded convergence using (4.18). Further, again by (4.18), for any $t \ge 0$

$$\sup_{u \ge u_0} |\Phi_u(t)| \le C_1 E[e^{\varepsilon X_{t-}}; X_{t-} < K] \le C_1 \left(E e^{\alpha X_t}\right)^{\varepsilon/\alpha} = C_1 \left(E e^{\alpha X_1}\right)^{\varepsilon t/\alpha},$$

where recall $Ee^{\alpha X_1} < 1$ by (1.12). Thus dominated convergence gives

$$\int_0^\infty \Phi_u(t)\ \mathrm{d}t \to \int_0^\infty \Phi(t)\ \mathrm{d}t. \tag{4.19}$$

This is equivalent to (4.15) since the limit, which is expressed in (4.19) as an iterated integral, may be rewritten as

$$\begin{aligned}\int_0^\infty \Phi(t)\ \mathrm{d}t &= \int_t \mathrm{d}t \int_{w\in D}\int_{\phi<K}\int_{z>-x} G(w,z)\alpha e^{\alpha(\phi-z)}\ \mathrm{d}z P(X_{[0,t)}\in \mathrm{d}w, X_{t-}\in \mathrm{d}\phi)\\ &= \int_t \mathrm{d}t \int_{w\in D}\int_{\phi<K}\int_{z>-x} E_z[H(w,X);\tau(0)<\infty]\alpha e^{\alpha(\phi-z)}\ \mathrm{d}z P(X_{[0,t)}\in \mathrm{d}w, X_{t-}\in \mathrm{d}\phi)\\ &= \beta_2^{-1}\int_t \mathrm{d}t \int_{w\in D}\int_{\phi<K} e^{\alpha\phi} P(X_{[0,t)}\in \mathrm{d}w, X_{t-}\in \mathrm{d}\phi)\int_{w'\in D} H(w,w')\nu_x(\mathrm{d}w')\\ &= (\beta_1\beta_2)^{-1}\int_{w\in D}\int_{w'\in D} H(w,w')\mu_K(\mathrm{d}w)\nu_x(\mathrm{d}w').\end{aligned} \tag{4.20}$$

This calculation is justified by absolute convergence of the final integral, proved earlier in (4.13). □

Proof of Theorem 3.1. Assume (1.12). Then using (4.5) and dominated convergence, which is justified by (4.13), we have

$$\lim_{K,x\to\infty}\int_{w\in D}\int_{w'\in D} H(w,w')\mu_K(\mathrm{d}w)\nu_x(\mathrm{d}w') = \int_{w\in D}\int_{w'\in D} H(w,w')\mu(\mathrm{d}w)\nu(\mathrm{d}w').$$

Thus by (4.11) and (4.12),

$$\lim_{K,x\to\infty}\lim_{u\to\infty} E^{(u)}[H(X_{[0,\tau(u-x))}, X\circ\theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-}<K] = EH(Z_{[0,\rho)}, W). \tag{4.21}$$

Taking $H\equiv 1$ in (4.21) gives

$$\lim_{K,x\to\infty}\lim_{u\to\infty} P^{(u)}(X_{\tau(u-x)-}<K) = 1. \tag{4.22}$$

Since H is bounded on $\{w_{\tau(\Delta)-}\geq K\}$ by (3.6), it follows that

$$\lim_{K,x\to\infty}\lim_{u\to\infty} E^{(u)}[H(X_{[0,\tau(u-x))}, X\circ\theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-}\geq K] = 0. \tag{4.23}$$

Combining (4.21) and (4.23) then proves (3.8). □

Remark 4.1 The limiting operations in this section are simpler than those in [18], not requiring the splitting of integrals over subdomains and associated delicate estimations. Further, many of the calculations do not require the full force of the $\mathcal{S}^{(\alpha)}$ condition. In particular the proof of (4.15) only uses (1.10) prior to equation (4.19). At this point the additional condition $Ee^{\alpha X_1}<1$ is needed to ensure that dominated convergence applies. Thus the proof actually shows that under (1.10), if $H\in\mathcal{H}$ is such that (4.19) holds, then for any $x\geq 0$ and $K\in(-\infty,\infty)$

$$\lim_{u\to\infty}\frac{E[H(X_{[0,\tau(u-x))}, X\circ\theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-}<K, \tau(u)<\infty]}{\overline{\Pi}_X^{+}(u)} = \int_0^\infty \Phi(t)\mathrm{d}t.$$

This is the case if, for example, Φ_u are dominated by an integrable function on $[0,\infty)$. If in addition

$$\int_{w\in D}\int_{w'\in D}|H(w,w')|\tilde{\mu}_K(\mathrm{d}w)\tilde{\nu}_x(\mathrm{d}w')<\infty,$$

where

$$\tilde{\mu}_K(\mathrm{d}w,\mathrm{d}t,\mathrm{d}\phi)=I(\phi<K)e^{\alpha\phi}P(X_{[0,t)}\in\mathrm{d}w;X_{t-}\in\mathrm{d}\phi)\ \mathrm{d}t \tag{4.24}$$

and

$$\tilde{\nu}_x(\mathrm{d}w',\mathrm{d}r,\mathrm{d}z)=I(z>-x)\alpha e^{-\alpha z}\mathrm{d}z\ P_z(X\in\mathrm{d}w';\tau(0)\in\mathrm{d}r), \tag{4.25}$$

then the limit may be rewritten as

$$\int_0^\infty\Phi(t)\mathrm{d}t=\int_{w\in D}\int_{w'\in D}H(w,w')\tilde{\mu}_K(\mathrm{d}w)\tilde{\nu}_x(\mathrm{d}w')$$

as demonstrated in (4.20). Comparing (4.24) and (4.25) with (4.1) and (4.2), note that the constants β_1 and β_2 must be excluded since they no longer need be finite and non-zero.

We briefly address conditions on H, beyond measurability, which ensure that (3.7) holds. It is natural that such conditions should relate to some type of continuity of H. We will assume that for each $w\in D$, $H(w,\cdot)$ is continuous from below on $\{\tau_0(w')<\infty\}$ a.s. P_z for every z, that is

$$\lim_{\varepsilon\downarrow 0}H(w,w'-c^{\varepsilon})=H(w,w')\ \text{ a.s. } P_z \text{ on } \{\tau_0(w')<\infty\},\ \text{ for all } w\in D, z\in\mathbb{R}. \tag{4.26}$$

This condition clearly holds if, for every $\omega\in D$, $H(w,\cdot)$ is continuous in the uniform topology on D, and so in particular if $H(w,\cdot)$ is continuous in any of the usual Skorohod topologies. Several examples of functionals of interest that satisfy (4.26) are given in Lemma 5.1. For a detailed discussion of the various topologies on Skorohod space, see [26].

Proposition 4.2 *If H is measurable and satisfies* (3.6) *and* (4.26)*, then* (3.7) *holds.*

Proof of Proposition 4.2. For $y<z$, we have

$$G(w,y)=E_y[H(w,X);\tau_0(X)<\infty]=E_z[H(w,X-c^{z-y});\tau_{z-y}(X)<\infty].$$

Next, by right continuity, $\tau_\varepsilon(w')\downarrow\tau_0(w')$ as $\varepsilon\downarrow 0$ for any $w'\in D$ with $\tau_0(w')<\infty$. Thus by (4.26), as $y\uparrow z$

$$H(w,X-c^{z-y})I(\tau_{z-y}(X)<\infty)\to H(w,X)I(\tau_0(X)<\infty)\ \text{ a.s } P_z. \tag{4.27}$$

Hence by bounded convergence, for each $w\in D$

$$G(w,y)\to G(w,z)$$

as $y\uparrow z$. Thus $G(w,\cdot)$ is left continuous and consequently continuous except at countably many points. □

Remark 4.2 Condition (4.26) can be weakened by requiring it to hold except for a discrete set of z. This would result in $G(w,\cdot)$ being left continuous except on a discrete set which again implies continuity except at countably many points.

One technical point should be mentioned. In order that the expression in (3.7) make sense, $H(w,\cdot)$ must be measurable. If $H(w,\cdot)$ is continuous in the uniform topology this need not be the case since there are open sets in the uniform topology which are not in the σ-algebra generated by the coordinate maps $\{w'_t : t \geq 0\}$. This is why we impose the blanket condition that H be measurable with respect to the product σ-algebra on $D \otimes D$.

For later reference we note that $H(w,w') = e^{-\theta w_{\tau(\Delta)-}}$ trivially satisfies (4.26), and if $\theta \in [0,\alpha)$, then H also satisfies (3.6). Thus by Proposition 4.2, $H \in \mathcal{H}$. Hence by taking $x = K = 0$ in Theorem 4.1, it follows that

$$\limsup_{u\to\infty} E^{(u)} e^{-\theta X_{\tau(u)-}} < \infty \quad \text{for every } \theta \in [0,\alpha). \tag{4.28}$$

We will later show that the limit exists and evaluate it; see Proposition 8.2.

5 General Marginal Convergence Results

In this section we provide a recipe for constructing conditional limit theorems for the fluctuation variables, by specialising Theorem 3.1. This gives, in Theorem 5.1, joint convergence of the main variables of interest in insurance risk. Again we need some preliminary results.

By convention we set $w'_{0-} = w'_0$ and $G_{0-}(w') = 0$. Also we define $\overline{w'}_t = \sup_{0\leq s\leq t} w'_s$.

Lemma 5.1 *Each of the following functions is continuous from below on $\{\tau_0(w') < \infty\}$ a.s. P_z, for all z:*

$$w'_0, \tau_0(w'), G_{\tau(0)-}(w'), \overline{w'}_{\tau(0)-}, w'_{\tau(0)-}, w'_{\tau(0)}$$

Proof of Lemma 5.1. Clearly w'_0 is continuous from below without any extra conditions. Now assume that $\tau_0(w') < \infty$. Let $\varepsilon > 0$ be sufficiently small that $\tau_\varepsilon(w') < \infty$. Then

$$\tau_0(w' - c^\varepsilon) = \tau_\varepsilon(w'). \tag{5.1}$$

Thus by right continuity

$$\tau_0(w' - c^\varepsilon) \downarrow \tau_0(w') \text{ as } \varepsilon \downarrow 0, \tag{5.2}$$

which proves $\tau_0(w')$ is continuous from below on $\{\tau_0(w') < \infty\}$. Next, from (5.1) we have

$$(w' - c^\varepsilon)_{\tau_0(w'-c^\varepsilon)\wedge\cdot} = w'_{\tau_\varepsilon(w')\wedge\cdot} - \varepsilon. \tag{5.3}$$

If $w'_{\tau(0)} > 0$, then $\tau_\varepsilon(w') = \tau_0(w')$ for all $\varepsilon \in (0, w'_{\tau(0)})$, so the result for the remaining functionals follows immediately from (5.3). Thus we assume $w'_{\tau(0)} = 0$, in which case $\tau_\varepsilon(w') > \tau_0(w')$ for all $\varepsilon > 0$. Now $P_z(w'_{\tau(0)} = 0) = 0$ if $z > 0$ so we only need consider

$z \leq 0$. If $z < 0$, then by Lemma 5.1 of [16], $w'_{\tau_0(w')-} = w'_{\tau_0(w')}$, and consequently also $G_{\tau_0(w')-}(w') = \tau_0(w')$, a.s. P_z. This continues to hold for $z = 0$, since applying the strong Markov property at time $\tau_0(w')$, shows $\tau_0(w') = 0$ a.s. P_0 when $w'_{\tau(0)} = 0$. Thus by right continuity, we have P_z a.s.

$$(w' - c^\varepsilon)_{\tau_0(w'-c^\varepsilon)-} = w'_{\tau_\varepsilon(w')-} - \varepsilon \to w'_{\tau_0(w')} = w'_{\tau_0(w')-}$$

and

$$G_{\tau_0(w'-c^\varepsilon)-}(w' - c^\varepsilon) = G_{\tau_\varepsilon(w')-}(w') \downarrow G_{\tau_0(w')}(w') = \tau_0(w') = G_{\tau_0(w')-}(w').$$

The proofs for the remaining functionals are similar. □

Remark 5.1 The above result is false if we replace continuous from below with continuous from above. For example if X is a Poisson process, then for any $\varepsilon > 0$

$$P_0(\tau_0(w' + c^\varepsilon) = 0) = 1, \quad P_0(\tau_0(w') = 0) = 0.$$

It will be convenient to write

$$Y = X \circ \theta_{\tau(u-x)} - c^u \quad \text{if } \tau(u-x) < \infty. \tag{5.4}$$

Thus $Y_t = X_{t+\tau(u-x)} - u$, $t \geq 0$, and in particular $Y_0 = X_{\tau(u-x)} - u$, when $\tau(u-x) < \infty$. Of course $Y = Y(u, x)$, but to simplify the notation, we suppress the dependence on u and x. From Theorem 3.1 we have that Y converges to W under $P^{(u)}$, as $x, u \to \infty$, in the sense specified there. Likewise, $X_{[0,\tau(u-x))}$ converges to $Z_{[0,\rho)}$ in the sense of Theorem 3.1, and in fact we have joint convergence. This provides us with a means for constructing limit theorems for the fluctuation variables. The first step is in the next proposition. Recall the definition of $\overline{X}$ in (1.2), and define $\overline{W}$ and $\overline{Z}$ analogously. Note that in (5.5) we replace the variables on the lefthand side with those on the righthand side in the limit, as just described. Since Z is a.s. continuous at ρ, one may further replace the subscripts $\rho-$ by ρ in (5.5), but we leave them in their present form to help emphasize the remark in the previous sentence.

Proposition 5.1 *Assume* (1.12) *and suppose* $f : \mathbb{R}^{10} \to \mathbb{R}$ *is bounded, measurable and jointly continuous in the last six arguments. Let* $0 \leq \theta < \alpha$ *and set*

$$H(w, w') = f(G_{\tau(\Delta)-}(w), \tau_\Delta(w), \overline{w}_{\tau(\Delta)-}, w_{\tau(\Delta)-}, w'_0, G_{\tau(0)-}(w'), \tau_0(w'), \overline{w'}_{\tau(0)-}, w'_{\tau(0)-}, w'_{\tau(0)})$$
$$\times e^{-\theta w_{\tau(\Delta)-} - I(w_{\tau(\Delta)-} \leq 0)} I(\tau_\Delta(w) < \infty, \tau_0(w') < \infty).$$

Then H satisfies (4.26) *and hence*

$$\lim_{x\to\infty} \lim_{u\to\infty} E^{(u)} f(G_{\tau(u-x)-}, \tau(u-x), \overline{X}_{\tau(u-x)-}, X_{\tau(u-x)-}, Y_0, G^Y_{\tau(0)-}, \tau^Y_0, \overline{Y}_{\tau(0)-}, Y_{\tau(0)-}, Y_{\tau(0)})$$
$$\times e^{-\theta X_{\tau(u-x)-} - I(X_{\tau(u-x)-} \leq 0)}$$
$$= E f(G^Z_{\rho-}, \rho, \overline{Z}_{\rho-}, Z_{\rho-}, W_0, G^W_{\tau(0)-}, \tau^W_0, \overline{W}_{\tau(0)-}, W_{\tau(0)-}, W_{\tau(0)}) e^{-\theta Z_{\rho-} - I(Z_{\rho-} \leq 0)}. \tag{5.5}$$

Proof of Proposition 5.1. H satisfies (4.26) by Lemma 5.1. Thus by Proposition 4.2 we may apply Theorem 3.1. Upon noting that $\tau_\Delta(X_{[0,\tau(u-x))}) = \tau(u-x)$, the result then follows immediately. □

In what is essentially a special case of the description of the limiting process given after Theorem 3.1, we can read off from Proposition 5.1 that the joint limiting distribution of the time of, the position prior to, and the position relative to u after, the large jump is $(\rho, Z_{\rho-}, W_0)$. To be precise: under (1.12) we have

$$\begin{aligned}\lim_{x\to\infty}\lim_{u\to\infty} P^{(u)}(\tau(u-x)\in \mathrm{d}t, X_{\tau(u-x)-}\in \mathrm{d}\phi, X_{\tau(u-x)}-u\in \mathrm{d}z)& \\ = P(\rho\in \mathrm{d}t, Z_{\rho-}\in \mathrm{d}\phi, W_0\in \mathrm{d}z)& \\ = \beta_1 e^{\alpha\phi}P(X_{t-}\in \mathrm{d}\phi)\ \mathrm{d}t\ \beta_2\alpha e^{-\alpha z} q\overline{V}(-z)\mathrm{d}z,&\end{aligned} \tag{5.6}$$

where the last equality follows from (3.2), (3.3) and (3.4). The exact meaning of this convergence is given by (3.8), which by (5.5), is stronger that the usual weak convergence.

Observe that on $(0,\infty)$, $P(W_0\in \mathrm{d}z) = \beta_2\alpha e^{-\alpha z}\mathrm{d}z$ is the limiting distribution of the overshoot $X_{\tau(u)}-u$ when the overshoot is due to the large jump. The limiting probability that the large jump results in an overshoot of u is $P(W_0>0)=\beta_2$. A further discussion of the overshoot is given in Section 7. Note also that (5.6) describes the joint limiting distribution of the ruin time, the claim surplus immediately prior to ruin and the shortfall at ruin, when ruin is due to a large claim. This makes precise the "intuitively obvious" asymptotic independence observed after Theorem 11 in [11], and extends it to also include the ruin time.

The next step in our recipe is to transfer from the variables on the left hand side of (5.5) to the fluctuation variables. The key point is to observe that if $\tau(u)<\infty$ and $x<u$, then

$$\begin{aligned} G_{\tau(u)-} &= G_{\tau(u-x)-}I(Y_0>0) + (\tau(u-x)+G^Y_{\tau(0)-})I(Y_0\le 0),\\ \tau(u)-G_{\tau(u)-} &= (\tau(u-x)-G_{\tau(u-x)-})I(Y_0>0) + (\tau^Y_0 - G^Y_{\tau(0)-})I(Y_0\le 0),\\ X_{\tau(u)}-u &= Y_0 I(Y_0>0) + Y_{\tau(0)}I(Y_0\le 0) = Y_{\tau(0)},\\ \overline{X}_{\tau(u)-} - X_{\tau(u)-} &= (\overline{X}_{\tau(u-x)-} - X_{\tau(u-x)-})I(Y_0>0) + (\overline{Y}_{\tau(0)-} - Y_{\tau(0)-})I(Y_0\le 0),\\ \overline{X}_{\tau(u)-} &= \overline{X}_{\tau(u-x)-}I(Y_0>0) + (u+\overline{Y}_{\tau(0)-})I(Y_0\le 0),\\ u-\overline{X}_{\tau(u)-} &= (u-\overline{X}_{\tau(u-x)-})I(Y_0>0) - \overline{Y}_{\tau(0)-}I(Y_0\le 0),\ \text{and}\\ X_{\tau(u)-} &= X_{\tau(u-x)-}I(Y_0>0) + (u+Y_{\tau(0)-})I(Y_0\le 0).\end{aligned} \tag{5.7}$$

Since some limiting variables have mass at infinity, we will consider weak convergence on $\mathbb{R}\cup\{\infty\}$. To be precise we will consider functions $f:\mathbb{R}^4\otimes(\mathbb{R}\cup\{\infty\})\to\mathbb{R}$ which are jointly continuous in the sense that $f(\mathbf{x}_n,y_n)\to f(\mathbf{x},y)$ as $(\mathbf{x}_n,y_n)\to(\mathbf{x},y)$ for $\mathbf{x}_n,\mathbf{x}\in\mathbb{R}^4$ and $y_n,y\in(\mathbb{R}\cup\{\infty\})$.

Theorem 5.1 *Assume* (1.12). *Let* $f : \mathbb{R}^4 \otimes (\mathbb{R} \cup \{\infty\}) \to \mathbb{R}$ *be bounded and jointly continuous. Then for* $0 \le \theta < \alpha$

$$\begin{aligned}
&\lim_{u\to\infty} E^{(u)} f(G_{\tau(u)-}, \tau(u) - G_{\tau(u)-}, X_{\tau(u)} - u, \overline{X}_{\tau(u)-} - X_{\tau(u)-}, \overline{X}_{\tau(u)-})e^{-\theta X_{\tau(u)-} I(X_{\tau(u)-}\le 0)} \\
&\quad = E[f(G^Z_{\rho-}, \rho - G^Z_{\rho-}, W_0, \overline{Z}_{\rho-} - Z_{\rho-}, \overline{Z}_{\rho-})e^{-\theta Z_{\rho-} I(Z_{\rho-}\le 0)}; W_0 > 0] \\
&\qquad + E[f(\rho + G^W_{\tau(0)-}, \tau^W_0 - G^W_{\tau(0)-}, W_{\tau(0)}, \overline{W}_{\tau(0)-} - W_{\tau(0)-}, \infty); W_0 \le 0]
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\lim_{u\to\infty} E^{(u)} f(G_{\tau(u)-}, \tau(u) - G_{\tau(u)-}, X_{\tau(u)} - u, \overline{X}_{\tau(u)-} - X_{\tau(u)-}, u - \overline{X}_{\tau(u)-})e^{-\theta X_{\tau(u)-} I(X_{\tau(u)-}\le 0)} \\
&\quad = E[f(G^Z_{\rho-}, \rho - G^Z_{\rho-}, W_0, \overline{Z}_{\rho-} - Z_{\rho-}, \infty)e^{-\theta Z_{\rho-} I(Z_{\rho-}\le 0)}; W_0 > 0] \\
&\qquad + E[f(\rho + G^W_{\tau(0)-}, \tau^W_0 - G^W_{\tau(0)-}, W_{\tau(0)}, \overline{W}_{\tau(0)-} - W_{\tau(0)-}, -\overline{W}_{\tau(0)-}); W_0 \le 0].
\end{aligned} \tag{5.9}$$

Thus, under $P^{(u)}$,

$$\begin{aligned}
&(G_{\tau(u)-}, \tau(u) - G_{\tau(u)-}, X_{\tau(u)} - u, \overline{X}_{\tau(u)-} - X_{\tau(u)-}, \overline{X}_{\tau(u)-}) \\
&\quad \to (G^Z_{\rho-}, \rho - G^Z_{\rho-}, W_0, \overline{Z}_{\rho-} - Z_{\rho-}, \overline{Z}_{\rho-}) I(W_0 > 0) \\
&\qquad + (\rho + G^W_{\tau(0)-}, \tau^W_0 - G^W_{\tau(0)-}, W_{\tau(0)}, \overline{W}_{\tau(0)-} - W_{\tau(0)-}, \delta_\infty) I(W_0 \le 0)
\end{aligned} \tag{5.10}$$

in the sense of weak convergence on $\mathbb{R}^4 \otimes (\mathbb{R} \cup \{\infty\})$, *and similarly for* (5.9).

Proof of Theorem 5.1. We only prove (5.8), as the proof of (5.9) is similar. Write the expectation on the left side of (5.8) as the sum of two expectations, one over $Y_0 > 0$ and the other over $Y_0 \le 0$. Convergence of the expectation over $Y_0 > 0$ to the first term on the right side of (5.8), as $u \to \infty$ then $x \to \infty$, follows easily from (5.7) and Proposition 5.1, since $P(W_0 = 0) = 0$ and f is bounded and jointly continuous. For the expectation over $Y_0 \le 0$, first observe that if $Y_0 \le 0$, then $X_{\tau(u)-} = u + Y_{\tau(0)-}$, and so on $\{Y_0 \le 0\}$

$$e^{-\theta X_{\tau(u)-} I(X_{\tau(u)-}\le 0)} = I(u + Y_{\tau(0)-} > 0) + e^{-\theta X_{\tau(u)-}} I(u + Y_{\tau(0)-} \le 0).$$

Convergence of the expectation over $\{u + Y_{\tau(0)-} > 0, Y_0 \le 0\}$ to the second term on the right side of (5.8), as $u \to \infty$ then $x \to \infty$, again follows from (5.7) and Proposition 5.1 since

$$\lim_{x\to\infty} \lim_{u\to\infty} P^{(u)}(u + Y_{\tau(0)-} \le 0, Y_0 \le 0) = 0. \tag{5.11}$$

Since f is bounded, it thus remains to show

$$\lim_{x\to\infty} \lim_{u\to\infty} E^{(u)}(e^{-\theta X_{\tau(u)-}}; u + Y_{\tau(0)-} \le 0, Y_0 \le 0) = 0.$$

For this it suffices by (5.11) and Hölder's inequality, to show that for some $\theta' > \theta$,

$$\limsup_{u\to\infty} E^{(u)} e^{-\theta' X_{\tau(u)-}} < \infty,$$

which in turn holds for any $\theta' \in (\theta, \alpha)$ by (4.28). $\square$

Theorem 5.1 provides a general convergence result for the variables of primary interest in insurance risk, in the convolution equivalent case. It contains and extends many previous results in the literature as will be explained in Sections 6–8. The two components that make up the limiting distributions in Theorem 5.1 arise as a consequence of the process either overshooting or undershooting the boundary at the time of the large jump. We now give alternate characterisations of these distributions in terms of quantities arising in fluctuation theory.

Recall the definitions of κ and V in (2.4) and (2.5), and of $\widehat{\kappa}$ and $\widehat{V}$ in the paragraph following (2.8). To avoid introducing further notation, there is clearly no harm in assuming that the random elements (W, Z, ρ) are independent of X. In particular ρ has an exponential distribution with parameter β_1 and is independent of X. Then by the Wiener-Hopf Factorisation Theorem, $(G_\rho, \overline{X}_\rho)$ and $(\rho - G_\rho, \overline{X}_\rho - X_\rho)$ are independent with Laplace transforms given by

$$Ee^{-aG_\rho - b\overline{X}_\rho} = \frac{\kappa(\beta_1, 0)}{\kappa(\beta_1 + a, b)}, \quad Ee^{-a(\rho - G_\rho) - b(\overline{X}_\rho - X_\rho)} = \frac{\widehat{\kappa}(\beta_1, 0)}{\widehat{\kappa}(\beta_1 + a, b)} \tag{5.12}$$

for $a, b > 0$; see Section 6.4 of [19].

Before stating the next result, we wish to make clear the meaning of the notation $|V(\mathrm{d}t - r, z - \mathrm{d}y)|$ below. It is the measure defined on Borel sets in $\mathbb{R}^2$ by

$$\int\int_{(t,y)} 1_A(t, y)|V(\mathrm{d}t - r, z - \mathrm{d}y)| = \int\int_{(t,y)} 1_A(t + r, z - y)V(\mathrm{d}t, \mathrm{d}y).$$

Some authors omit the absolute values signs. We include them to emphasize that the function $V(t - r, z - y)$ is increasing in t and decreasing in y, hence the Stieltjes measure associated with it, which assigns mass

$$V(t_1 - r, z - y_1) - V(t_1 - r, z - y_0) - V(t_0 - r, z - y_1) + V(t_0 - r, z - y_0)$$

to rectangles $(t_0, t_1] \times [y_0, y_1)$, is negative.

Theorem 5.2 *For $\gamma > 0, t \geq 0, s \geq 0, \theta \geq 0, \phi \geq 0$*

$$\begin{aligned} &P(G^Z_{\rho-} \in \mathrm{d}t, \rho - G^Z_{\rho-} \in \mathrm{d}s, W_0 \in \mathrm{d}\gamma, \overline{Z}_{\rho-} - Z_{\rho-} \in \mathrm{d}\phi, \overline{Z}_{\rho-} \in \mathrm{d}\theta; W_0 > 0) \\ &\quad = \beta\alpha e^{-\alpha(\gamma+\phi-\theta)}V(\mathrm{d}t, \mathrm{d}\theta)\widehat{V}(\mathrm{d}s, \mathrm{d}\phi)\mathrm{d}\gamma, \end{aligned} \tag{5.13}$$

where β is given by (2.12).

For $\gamma \geq 0, t \geq 0, s \geq 0, v \geq 0, y \geq 0$

$$\begin{aligned} P(\rho + G^W_{\tau(0)-} \in \mathrm{d}t, \tau^W_0 - G^W_{\tau(0)-} \in \mathrm{d}s, W_{\tau(0)} \in \mathrm{d}\gamma, \overline{W}_{\tau(0)-} - W_{\tau(0)-} \in \mathrm{d}v,& \\ - \overline{W}_{\tau(0)-} \in \mathrm{d}y; W_0 \leq 0)& \\ = \beta I(\gamma > 0)\int_{r\leq t} e^{-\beta_1 r}\mathrm{d}r \int_{z\geq y} \alpha e^{\alpha z}\mathrm{d}z|V(\mathrm{d}t - r, z - \mathrm{d}y)|\widehat{V}(\mathrm{d}s, \mathrm{d}v)\Pi_X(\mathrm{d}\gamma + v + y)& \\ + \beta \mathrm{d}_H \int_{r\leq t} e^{-\beta_1 r}\mathrm{d}r \int_{z\geq 0} \alpha e^{\alpha z}V(\mathrm{d}t - r, \mathrm{d}z)\delta_0(\mathrm{d}s, \mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y)& \end{aligned} \tag{5.14}$$

where δ_0 denotes a point mass at the origin.

Proof of Theorem 5.2. The form of the limit in (5.14) follows from an extension of Doney and Kyprianou's [11] quintuple law to include creeping, as given in Griffin and Maller [16]. For $\gamma \geq 0, t \geq 0, r \geq 0, v \geq 0, y \geq 0$, we have by, (3.1), and Theorems 3.1 (ii) and 3.2 of [16],

$$\begin{aligned}
&P(G^W_{\tau(0)-} \in \mathrm{d}r, \tau^W_0 - G^W_{\tau(0)-} \in \mathrm{d}s, W_{\tau(0)} \in \mathrm{d}\gamma, \overline{W}_{\tau(0)-} - W_{\tau(0)-} \in \mathrm{d}v, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \overline{W}_{\tau(0)-} \in \mathrm{d}y; W_0 \leq 0) \\
&\quad = \beta_2 \int_{z \geq 0} \alpha e^{\alpha z} \mathrm{d}z P(G_{\tau(z)-} \in \mathrm{d}r, \tau(z) - G_{\tau(z)-} \in \mathrm{d}s, X_{\tau(z)} - z \in \mathrm{d}\gamma, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \overline{X}_{\tau(z)-} - X_{\tau(z)-} \in \mathrm{d}v, z - \overline{X}_{\tau(z)-} \in \mathrm{d}y) \\
&\quad = \beta_2 I(\gamma > 0) \int_{z \geq 0} \alpha e^{\alpha z} \mathrm{d}z I(y \leq z) |V(\mathrm{d}r, z - \mathrm{d}y)| \widehat{V}(\mathrm{d}s, \mathrm{d}v) \Pi_X(\mathrm{d}\gamma + v + y) \qquad (5.15) \\
&\qquad\qquad + \beta_2 \mathrm{d}_H \int_{z \geq 0} \alpha e^{\alpha z} \mathrm{d}z \frac{\partial_-}{\partial_- z} V(\mathrm{d}t, z) \delta_0(\mathrm{d}s, \mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y) \\
&\quad = \beta_2 I(\gamma > 0) \int_{z \geq y} \alpha e^{\alpha z} \mathrm{d}z |V(\mathrm{d}r, z - \mathrm{d}y)| \widehat{V}(\mathrm{d}s, \mathrm{d}v) \Pi_X(\mathrm{d}\gamma + v + y) \\
&\qquad\qquad + \beta_2 \mathrm{d}_H \int_{z \geq 0} \alpha e^{\alpha z} V(\mathrm{d}t, \mathrm{d}z) \delta_0(\mathrm{d}s, \mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y).
\end{aligned}$$

Convolving with the exponential distribution of ρ gives (5.14).

For (5.13), using (3.2), (3.3), and independence of W, Z and ρ, we have

$$\begin{aligned}
&P(G^Z_{\rho-} \in \mathrm{d}t, \rho - G^Z_{\rho-} \in \mathrm{d}s, W_0 \in \mathrm{d}\gamma, \overline{Z}_{\rho-} - Z_{\rho-} \in \mathrm{d}\phi, \overline{Z}_{\rho-} \in \mathrm{d}\theta; W_0 > 0) \\
&= \beta_2 \alpha e^{-\alpha\gamma} \mathrm{d}\gamma P(G^Z_{\rho-} \in \mathrm{d}t, \rho - G^Z_{\rho-} \in \mathrm{d}s, \overline{Z}_{\rho-} - Z_{\rho-} \in \mathrm{d}\phi, \overline{Z}_{\rho-} \in \mathrm{d}\theta) \\
&= \beta_2 \alpha e^{-\alpha\gamma} \mathrm{d}\gamma P(G^Z_{(t+s)-} \in \mathrm{d}t, Z_{(t+s)-} \in \theta - \mathrm{d}\phi, \overline{Z}_{(t+s)-} \in \mathrm{d}\theta)\ \beta_1 e^{-\beta_1(t+s)} \mathrm{d}s \\
&= \beta_2 \alpha e^{-\alpha(\gamma+\phi-\theta)} \mathrm{d}\gamma\ e^{\beta_1(t+s)} P(G_{(t+s)-} \in \mathrm{d}t, X_{(t+s)-} \in \theta - \mathrm{d}\phi, \overline{X}_{(t+s)-} \in \mathrm{d}\theta) \beta_1 e^{-\beta_1(t+s)} \mathrm{d}s \\
&= \beta_2 \alpha e^{-\alpha(\gamma+\phi-\theta)} \mathrm{d}\gamma\ e^{\beta_1(t+s)} P(G_{\rho-} \in \mathrm{d}t, \overline{X}_{\rho-} \in \mathrm{d}\theta, \rho - G_{\rho-} \in \mathrm{d}s, \overline{X}_{\rho-} - X_{\rho-} \in \mathrm{d}\phi) \\
&= \beta_2 \alpha e^{-\alpha(\gamma+\phi-\theta)} \mathrm{d}\gamma\ e^{\beta_1 t} P(G_{\rho-} \in \mathrm{d}t, \overline{X}_{\rho-} \in \mathrm{d}\theta)\ e^{\beta_1 s} P(\rho - G_{\rho-} \in \mathrm{d}s, \overline{X}_{\rho-} - X_{\rho-} \in \mathrm{d}\phi),
\end{aligned}$$

by independence of the Wiener-Hopf factors. Further

$$e^{\beta_1 t} P(G_{\rho-} \in \mathrm{d}t, \overline{X}_{\rho-} \in \mathrm{d}\theta) = \kappa(\beta_1, 0) V(\mathrm{d}t, \mathrm{d}\theta)$$

and

$$e^{\beta_1 s} P(\rho - G_{\rho-} \in \mathrm{d}s, \overline{X}_{\rho-} - X_{\rho-} \in \mathrm{d}\phi) = \widehat{\kappa}(\beta_1, 0) \widehat{V}(\mathrm{d}s, \mathrm{d}\phi),$$

as can be seen by taking the Laplace transforms and using (2.6) and (5.12). (5.13) then follows since $\kappa(\beta_1, 0)\widehat{\kappa}(\beta_1, 0) = \beta_1$ by (2.9). □

Theorems 5.1 and 5.2 extend Theorems 10 and 11 in [11]. To see the connection between (5.14) and Theorem 10 of [11], set

$$m(\mathrm{d}t, \mathrm{d}y) = \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge y} e^{\alpha z}\mathrm{d}z|V(\mathrm{d}t - r, z - \mathrm{d}y)|,$$
$$n(\mathrm{d}t, \mathrm{d}y) = \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge y} e^{\alpha z}\mathrm{d}z V(\mathrm{d}t - r, \mathrm{d}z)\delta_0(\mathrm{d}y).$$

For any $a > 0, b > \alpha$,

$$\begin{aligned}\int_{t\ge 0}\int_{y\ge 0} e^{-at-by} m(\mathrm{d}t, \mathrm{d}y) &= \int_{t\ge 0}\int_{y\ge 0} e^{-at-by} \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge y} e^{\alpha z}\mathrm{d}z|V(\mathrm{d}t - r, z - \mathrm{d}y)| \\ &= \int_{r\ge 0} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge 0} e^{\alpha z}\mathrm{d}z \int_{t\ge r}\int_{0\le y\le z} e^{-at-by}|V(\mathrm{d}t - r, z - \mathrm{d}y)| \\ &= \int_{r\ge 0} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge 0} e^{\alpha z}\mathrm{d}z \int_{t\ge 0}\int_{0\le y\le z} e^{-a(t+r)-b(z-y)} V(\mathrm{d}t, \mathrm{d}y) \\ &= \int_{r\ge 0} e^{-(\beta_1+a)r}\mathrm{d}r \int_{y\ge 0}\int_{t\ge 0} e^{-at+by} V(\mathrm{d}t, \mathrm{d}y) \int_{z\ge y} e^{-(b-\alpha)z}\mathrm{d}z \\ &= \frac{1}{(\beta_1 + a)\kappa(a, -\alpha)(b - \alpha)}.\end{aligned} \tag{5.16}$$

Similarly

$$\int_{t\ge 0}\int_{y\ge 0} e^{-at-by} n(\mathrm{d}t, \mathrm{d}y) = \frac{1}{(\beta_1 + a)\kappa(a, -\alpha)}. \tag{5.17}$$

Setting $a = 0$ and inverting shows that

$$\begin{aligned}\int_{t\ge 0} m(\mathrm{d}t, \mathrm{d}y) &= \frac{e^{\alpha y}\mathrm{d}y}{\beta_1\kappa(0, -\alpha)} = \frac{e^{\alpha y}\mathrm{d}y}{\beta q}, \\ \int_{t\ge 0} n(\mathrm{d}t, \mathrm{d}y) &= \frac{\delta_0(\mathrm{d}y)}{\beta_1\kappa(0, -\alpha)} = \frac{\delta_0(\mathrm{d}y)}{\beta q}\end{aligned} \tag{5.18}$$

from (2.12). Thus after integrating out t, (5.14) reduces to

$$\begin{aligned}&P(\tau_0^W - G^W_{\tau(0)-} \in \mathrm{d}s, W_{\tau(0)} \in \mathrm{d}\gamma, \overline{W}_{\tau(0)-} - W_{\tau(0)-} \in \mathrm{d}v, -\overline{W}_{\tau(0)-} \in \mathrm{d}y; W_0 \le 0) \\ &\quad = I(\gamma > 0)q^{-1}\alpha e^{\alpha y}\mathrm{d}y\widehat{V}(\mathrm{d}s, \mathrm{d}v)\Pi_X(\mathrm{d}\gamma + v + y) + q^{-1}\alpha \mathrm{d}_H\delta_0(\mathrm{d}s, \mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y),\end{aligned} \tag{5.19}$$

for $\gamma \ge 0, s \ge 0, v \ge 0, y \ge 0$. Thus we may conclude that for $\gamma \ge 0, s \ge 0, v \ge y \ge 0$

$$\begin{aligned}&\lim_{u\to\infty} P^{(u)}(\tau(u) - G_{\tau(u)-} \in \mathrm{d}s, X_{\tau(u)} - u \in \mathrm{d}\gamma, u - X_{\tau(u)-} \in \mathrm{d}y, u - \overline{X}_{\tau(u)-} \in \mathrm{d}v) \\ &\quad = I(\gamma > 0)q^{-1}\alpha e^{\alpha y}\mathrm{d}y\widehat{V}(\mathrm{d}s, \mathrm{d}v - y)\Pi_X(\mathrm{d}\gamma + v) + q^{-1}\alpha \mathrm{d}_H\delta_0(\mathrm{d}s, \mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y)\end{aligned} \tag{5.20}$$

in the sense of vague convergence. For $\gamma > 0$, this is Doney and Kyprianou's expression in Theorem 10 of [11], for the vague limit when X does not creep over the boundary.

The connection between (5.13) and Theorem 11 of [11] is similar but easier to see. It is worth emphasizing that the convergence in Theorem 5.1 is stronger than the convergence in (5.20). In particular convergence of the marginals does not follow from the vague convergence of (5.20), indeed it need not be the case, but it does follow from the weak convergence in Theorem 5.1. For example, marginal convergence of the overshoot in (5.20) would imply

$$\begin{aligned}
&\lim_{u\to\infty} P^{(u)}(X_{\tau(u)} - u \in \mathrm{d}\gamma) \\
&\quad = I(\gamma > 0) q^{-1} \alpha e^{\alpha y} \mathrm{d}y \int_{y\ge 0}\int_{v\ge y}\int_{s\ge 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v - y)\Pi_X(\mathrm{d}\gamma + v) + q^{-1}\alpha \mathrm{d}_H \delta_0(\mathrm{d}\gamma) \\
&\quad = q^{-1}\alpha\big[\mathrm{d}_H\delta_0(\mathrm{d}\gamma) + \int_{y\ge 0} e^{\alpha y}\Pi_H(\mathrm{d}\gamma + y)\mathrm{d}y\big]
\end{aligned}$$

by Vigon's équation amicale inversée; see (6.7) below. However by Theorem 5.1, in which marginal convergence does hold, we find that

$$\begin{aligned}
\lim_{u\to\infty} P^{(u)}(X_{\tau(u)} - u \in \mathrm{d}\gamma) &= P(W_0 I(W_0 > 0) + W_{\tau(0)} I(W_0 \le 0) \in \mathrm{d}\gamma) \\
&= \beta_2 \alpha e^{-\alpha\gamma}\mathrm{d}\gamma + q^{-1}\alpha\big[\mathrm{d}_H\delta_0(\mathrm{d}\gamma) + \int_{y\ge 0} e^{\alpha y}\Pi_H(\mathrm{d}\gamma + y)\mathrm{d}y\big],
\end{aligned}$$

as discussed in Section 7. We will make frequent use of marginal convergence in Theorem 5.1 in the subsequent sections.

6 The Ruin Time

By taking f constant in the spatial variables in Theorem 5.1, we obtain marginal convergence in the time variables. We begin with

Proof of Theorem 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Then by (5.8) (or (5.9)) with $\theta = 0$

$$\lim_{u\to\infty} E^{(u)} f(\tau(u)) = E[f(\rho); W_0 > 0] + E[f(\rho + \tau^W(0)); W_0 \le 0] = Ef(\rho + \tau^W(0))$$

which proves the first equality in (3.9). Since $\rho + \tau^W(0)$ has a continuous distribution

$$\begin{aligned}
P(\rho + \tau^W(0) \le t) &= \int_{s\le t} \beta_1 e^{-\beta_1 s}\mathrm{d}s\ \beta_2 \int_z \alpha e^{-\alpha z}\mathrm{d}z\ P_z(\tau(0) < t - s) \\
&= \int_{s\le t} \beta_1 e^{-\beta_1 s}\mathrm{d}s\ \beta_2 [1 + \int_{z>0} \alpha e^{\alpha z}\mathrm{d}z\ P(\overline{X}_{t-s} > z)] \\
&= \beta_2 \int_{s\le t} \beta_1 e^{-\beta_1 s} E e^{\alpha \overline{X}_{t-s}}\mathrm{d}s \\
&= \beta_2 E(e^{\alpha\overline{X}_{t-\rho}}; \rho \le t)
\end{aligned} \tag{6.1}$$

which completes the proof. □

Our derivation of the limiting distribution of the ruin time is based on splitting the distribution at the time of the large jump. One of the points of distinction between the path decomposition approach to studying ruin and that of [11], is that in [11] the split is at $G_{\tau(u)-}$, the time of the last maximum prior to passage over the boundary. This is a very natural approach given the fluctuation theory as developed in [4] Ch. VI, for example. We now show how the path decomposition approach can be used to easily derive the joint limiting distribution of the fluctuation variables $\big(G_{\tau(u)-}, \tau(u) - G_{\tau(u)-}\big)$ under $P^{(u)}$, thus extending the results in [11].

Introduce the measures on $[0,\infty)$ given by

$$\delta_\alpha^V(\mathrm{d}t) = \int_{\theta\ge 0} e^{\alpha\theta} V(\mathrm{d}t,\mathrm{d}\theta), \qquad \delta_{-\alpha}^{\widehat{V}}(\mathrm{d}s) = \int_{\phi\ge 0} e^{-\alpha\phi}\widehat{V}(\mathrm{d}s,\mathrm{d}\phi) \tag{6.2}$$

$$K(\mathrm{d}s) = \int_{z\ge 0}(e^{\alpha z}-1)\Pi_{L^{-1},H}(\mathrm{d}s,\mathrm{d}z) = \int_{z\ge 0}\alpha e^{\alpha z}\mathrm{d}z\overline{\Pi}_{L^{-1},H}(\mathrm{d}s,z)$$

and their respective (improper) distribution functions $\delta_\alpha^V(t)$, $\delta_{-\alpha}^{\widehat{V}}(s)$ and $K(s)$, where

$$\overline{\Pi}_{L^{-1},H}(\mathrm{d}s,z) = \int_{y>z}\overline{\Pi}_{L^{-1},H}(\mathrm{d}s,\mathrm{d}y).$$

Theorem 6.1 *Assume* (1.12)*. Then for all* $s,t,\ge 0$*, we have*

$$\begin{aligned}\lim_{u\to\infty} P^{(u)}&\left(G_{\tau(u)-}\in\mathrm{d}t, \tau(u)-G_{\tau(u)-}\in\mathrm{d}s\right)\\ &= \beta\Big[\delta_\alpha^V(\mathrm{d}t)\delta_{-\alpha}^{\widehat{V}}(\mathrm{d}s) + \big(K(\mathrm{d}s)+\alpha\mathrm{d}_H\delta_0(\mathrm{d}s)\big)\int_{0\le r\le t} e^{-\beta_1 r}\delta_\alpha^V(\mathrm{d}t-r)\mathrm{d}r\Big]\end{aligned} \tag{6.3}$$

in the sense of weak convergence of probability measures on $[0,\infty)^2$.

Proof of Theorem 6.1. From Theorem 5.1 we have

$$\begin{aligned}\lim_{u\to\infty} P^{(u)}&\left(G_{\tau(u)-}\in\mathrm{d}t, \tau(u)-G_{\tau(u)-}\in\mathrm{d}s\right)\\ &= P(G_{\rho-}^Z\in\mathrm{d}t, \rho-G_{\rho-}^Z\in\mathrm{d}s; W_0>0)+\\ &\qquad P(\rho+G_{\tau(0)-}^W\in\mathrm{d}t, \tau_0^W-G_{\tau(0)-}^W\in\mathrm{d}s; W_0\le 0).\end{aligned} \tag{6.4}$$

Integrating out γ,θ and ϕ in (5.13) gives

$$\begin{aligned}P(G_{\rho-}^Z&\in\mathrm{d}t, \rho-G_{\rho-}^Z\in\mathrm{d}s; W_0>0)\\ &= \beta\int_{\gamma>0}\int_{\theta\ge 0}\int_{\phi\ge 0}\alpha e^{-\alpha(\gamma+\phi-\theta)}V(\mathrm{d}t,\mathrm{d}\theta)\widehat{V}(\mathrm{d}s,\mathrm{d}\phi)\mathrm{d}\gamma\\ &= \beta\delta_\alpha^V(\mathrm{d}t)\delta_{-\alpha}^{\widehat{V}}(\mathrm{d}s).\end{aligned} \tag{6.5}$$

Integrating out γ, y and v in the first term of (5.14) gives

$$\begin{aligned}
&P(\rho + G^W_{\tau(0)-} \in \mathrm{d}t, \tau^W_0 - G^W_{\tau(0)-} \in \mathrm{d}s, W_{\tau(0)} > 0; W_0 \le 0) \\
&\quad = \beta \int_{\gamma>0}\int_{y\ge 0}\int_{v\ge 0}\int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge y} \alpha e^{\alpha z}\mathrm{d}z |V(\mathrm{d}t - r, z - \mathrm{d}y)| \widehat{V}(\mathrm{d}s, \mathrm{d}v)\Pi_X(\mathrm{d}\gamma + v + y).
\end{aligned} \tag{6.6}$$

By Doney and Kyprianou's extension of Vigon's équation amicale inversée, it follows that

$$\overline{\Pi}_{L^{-1},H}(\mathrm{d}s, y) = \int_{v\ge 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v)\overline{\Pi}^+_X(v+y), \quad s \ge 0, y \ge 0. \tag{6.7}$$

Thus continuing the equalities in (6.6)

$$\begin{aligned}
&= \beta \int_{y\ge 0}\int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge y} \alpha e^{\alpha z}\mathrm{d}z |V(\mathrm{d}t - r, z - \mathrm{d}y)| \overline{\Pi}_{L^{-1},H}(\mathrm{d}s, y) \\
&= \beta \int_{z\ge 0} \alpha e^{\alpha z}\mathrm{d}z \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{y\le z} |V(\mathrm{d}t - r, z - \mathrm{d}y)| \overline{\Pi}_{L^{-1},H}(\mathrm{d}s, y) \\
&= \beta \int_{z\ge 0} \alpha e^{\alpha z}\mathrm{d}z \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{y\le z} V(\mathrm{d}t - r, \mathrm{d}y) \overline{\Pi}_{L^{-1},H}(\mathrm{d}s, z - y) \\
&= \beta \int_{y\ge 0}\int_{r\le t} e^{-\beta_1 r}\mathrm{d}r V(\mathrm{d}t - r, \mathrm{d}y) \int_{z\ge y} \alpha e^{\alpha z}\mathrm{d}z \overline{\Pi}_{L^{-1},H}(\mathrm{d}s, z - y) \\
&= \beta \int_{y\ge 0} e^{\alpha y} \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r V(\mathrm{d}t - r, \mathrm{d}y) \int_{z\ge 0} \alpha e^{\alpha z}\mathrm{d}z \overline{\Pi}_{L^{-1},H}(\mathrm{d}s, z) \\
&= \beta K(\mathrm{d}s) \int_{0\le r\le t} e^{-\beta_1 r}\delta^V_\alpha(\mathrm{d}t - r)\mathrm{d}r.
\end{aligned} \tag{6.8}$$

Integrating out γ, y and v in the second term of (5.14) gives

$$\begin{aligned}
&P(\rho + G^W_{\tau(0)-} \in \mathrm{d}t, \tau^W_0 - G^W_{\tau(0)-} \in \mathrm{d}s, W_{\tau(0)} = 0; W_0 \le 0) \\
&\quad = \beta \mathrm{d}_H \int_{r\le t} e^{-\beta_1 r}\mathrm{d}r \int_{z\ge 0} \alpha e^{\alpha z} V(\mathrm{d}t - r, \mathrm{d}z)\delta_0(\mathrm{d}s) \\
&\quad = \beta\alpha \mathrm{d}_H \delta_0(\mathrm{d}s) \int_0^t e^{-\beta_1 r}\delta^V_\alpha(\mathrm{d}t - r)\mathrm{d}r.
\end{aligned} \tag{6.9}$$

Adding the three terms in (6.5), (6.8) and (6.9) gives (6.3). □

7 Overshoots and Undershoots

By taking f constant in the time variables we obtain joint convergence of overshoots and undershoots.

Theorem 7.1 *Assume* (1.12). *Let* $f : \mathbb{R}^2 \otimes (\mathbb{R} \cup \{\infty\}) \to \mathbb{R}$ *be bounded and jointly continuous. Then for* $0 \le \theta < \alpha$

$$\begin{aligned}
&\lim_{u\to\infty} E^{(u)} f(X_{\tau(u)} - u, \overline{X}_{\tau(u)-} - X_{\tau(u)-}, \overline{X}_{\tau(u)-}) e^{-\theta X_{\tau(u)-} I(X_{\tau(u)-} \le 0)} \\
&\quad = E[f(W_0, \overline{Z}_{\rho-} - Z_{\rho-}, \overline{Z}_{\rho-}) e^{-\theta Z_{\rho-} I(Z_{\rho-} \le 0)}; W_0 > 0] \\
&\qquad + E[f(W_{\tau(0)}, \overline{W}_{\tau(0)-} - W_{\tau(0)-}, \infty); W_0 \le 0]
\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}
&\lim_{u\to\infty} E^{(u)} f(X_{\tau(u)} - u, \overline{X}_{\tau(u)-} - X_{\tau(u)-}, u - \overline{X}_{\tau(u)-}) e^{-\theta X_{\tau(u)-} I(X_{\tau(u)-} \le 0)} \\
&\quad = E[f(W_0, \overline{Z}_{\rho-} - Z_{\rho-}, \infty) e^{-\theta Z_{\rho-} I(Z_{\rho-} \le 0)}; W_0 > 0] \\
&\qquad + E[f(W_{\tau(0)}, \overline{W}_{\tau(0)-} - W_{\tau(0)-}, -\overline{W}_{\tau(0)-}); W_0 \le 0].
\end{aligned} \tag{7.2}$$

For $\gamma > 0,\ \theta \ge 0, \phi \ge 0$

$$P(W_0 \in \mathrm{d}\gamma, \overline{Z}_{\rho-} - Z_{\rho-} \in \mathrm{d}\phi, \overline{Z}_{\rho-} \in \mathrm{d}\theta; W_0 > 0) = \beta\alpha e^{-\alpha(\gamma+\phi-\theta)} V(\mathrm{d}\theta)\widehat{V}(\mathrm{d}\phi)\mathrm{d}\gamma. \tag{7.3}$$

For $\gamma \ge 0, v \ge 0, y \ge 0$

$$\begin{aligned}
&P(W_{\tau(0)} \in \mathrm{d}\gamma, \overline{W}_{\tau(0)-} - W_{\tau(0)-} \in \mathrm{d}v, -\overline{W}_{\tau(0)-} \in \mathrm{d}y; W_0 \le 0) \\
&\quad = I(\gamma > 0) q^{-1}\alpha e^{\alpha y} \mathrm{d}y \widehat{V}(\mathrm{d}v)\Pi_X(\mathrm{d}\gamma + v + y) + q^{-1}\alpha \mathrm{d}_H \delta_0(\mathrm{d}\gamma, \mathrm{d}v, \mathrm{d}y).
\end{aligned} \tag{7.4}$$

Proof of Theorem 7.1. This follows immediately from (5.8), (5.9), (5.13) and (5.19). □

Theorem 7.1 contains all results we know of in the literature on convergence of individual overshoots and undershoots, under a convolution equivalent assumption. The only marginal limiting distribution in Theorem 7.1 which is proper is that of the overshoot, and this is given by $W_0 I(W_0 > 0) + W_{\tau(0)} I(W_0 \le 0)$. An easy calculation from (7.3) and (7.4), using (6.7), gives

Overshoot: Assume (1.12). Then for $\gamma \ge 0$

$$P^{(u)}(X_{\tau(u)} - u \in \mathrm{d}\gamma) \to \beta_2 \alpha e^{-\alpha\gamma}\mathrm{d}\gamma + q^{-1}\alpha\big[\mathrm{d}_H\delta_0(\mathrm{d}\gamma) + \int_{y\ge 0} e^{\alpha y}\Pi_H(\mathrm{d}\gamma + y)\mathrm{d}y\big]. \tag{7.5}$$

Observe that the limiting distribution has mass $\alpha \mathrm{d}_H q^{-1}$ at the origin, and for $x > 0$

$$P^{(u)}(X_{\tau(u)} - u > x) \to \beta_2 e^{-\alpha x} + q^{-1} e^{-\alpha x} \int_{y > x} (e^{\alpha y} - e^{\alpha x})\Pi_H(\mathrm{d}y). \tag{7.6}$$

This is the form of the limiting distribution given in [18] and [11]. In [11], it is indicated that the limiting distribution on $(0, \infty)$ arises as a consequence of either an arbitrarily large jump from a finite position after a finite time, or a finite jump from a finite distance relative to the boundary after an arbitrarily large time. This is not quite correct. From the path decomposition, the latter component of the limiting distribution arises as a consequence of a large jump from a finite position to within a finite distance of the boundary after a finite time, followed by a finite jump a finite time later. The atom at 0 in the limiting distribution is a consequence of creeping across the boundary when the large jump undershoots the boundary.

The other marginal limits in Theorem 7.1 are improper, thus in each instance below, convergence is in the vague sense with remaining mass escaping to $+\infty$. We leave the calculations to the reader.

Undershoots: Assume (1.12). Then for $x \geq 0$

$$P^{(u)}(u - X_{\tau(u)-} \in \mathrm{d}x) \to q^{-1}\alpha \mathrm{d}_H \delta_0(\mathrm{d}x) + q^{-1}\alpha e^{\alpha x}\overline{\Pi}_X(x)\mathrm{d}x \int_{0 \leq v \leq x} e^{-\alpha v}\widehat{V}(\mathrm{d}v), \quad (7.7)$$

while for $y \geq 0$

$$P^{(u)}(u - \overline{X}_{\tau(u)-} \in \mathrm{d}y) \to q^{-1}\alpha \mathrm{d}_H \delta_0(\mathrm{d}y) + q^{-1}\alpha e^{\alpha y}\overline{\Pi}_H(y)\mathrm{d}y. \quad (7.8)$$

Remark 7.1 An alternative formulation of (7.7) appears in Theorem 3.2 of [22]. (7.8) corrects an oversight in Theorem 3.3 of [22]. The first term in (7.8), representing possible mass at 0 if creeping over the boundary occurs, was omitted.

Positions Prior to Overshoot: Assume (1.12). Then for $\zeta \in (-\infty, \infty)$

$$P^{(u)}(X_{\tau(u)-} \in \mathrm{d}\zeta) \to \beta e^{\alpha\zeta} V_X(\mathrm{d}\zeta), \quad (7.9)$$

where V_X is the potential measure of X, while for $\theta \geq 0$

$$P^{(u)}(\overline{X}_{\tau(u)-} \in \mathrm{d}\theta) \to q\beta_2^2 e^{\alpha\theta} V(\mathrm{d}\theta).$$

8 Laplace Transforms and Penalty Functions

Expected discounted penalty functions (EDPF's) were introduced into risk theory by Gerber and Shiu [15]. As an example consider

$$E^{(u)} e^{-\nu G_{\tau(u)-} - \zeta(\tau_u - G_{\tau(u)-}) - \eta(X_{\tau_u} - u) - \lambda(u - \overline{X}_{\tau_u -})} \quad (8.1)$$

where $\nu \geq 0$, $\zeta \geq 0$, $\eta > -\alpha$, $\lambda \geq 0$. In this case penalization is more severe when the shortfall at ruin is greater (if $\eta < 0$), but this is moderated by a later occurrence of ruin or by a larger minimum surplus prior to ruin. Among other things, EDPF's provide a natural approach to studying solvency requirements, and more generally to valuing cashflows related to first passage over a barrier; see for example the discussion in Biffis and Morales [6]. In this section we use our previous results to calculate the limit, as $u \to \infty$, of (8.1) and other related EDPF's and Laplace transforms.

If $\eta \geq 0$, then the limit in (8.1) can be found by using Theorem 5.1. To include the case $-\alpha < \eta < 0$ it will suffice, by uniform integrability, to show that

$$\limsup_{u\to\infty} E^{(u)} e^{-\eta(X_{\tau(u)}-u)} < \infty, \quad \eta > -\alpha. \tag{8.2}$$

A stronger version of (8.2) is in Park and Maller [22]. Since our weaker version is easy to prove, we give a direct proof that does not involve delicate estimation of convolution equivalent integrals as in [22]. Combined with convergence of the overshoot, this weaker result is in fact equivalent to Park and Maller's *a priori* stronger result on convergence of the mgf of the overshoot.

Lemma 8.1 *Let F and G be distribution functions with $F(0-) = G(0-) = 0$, $F \in \mathcal{S}^{(\alpha)}$ and*

$$\limsup_{u\to\infty} \frac{\overline{G}(u)}{\overline{F}(u)} < \infty. \tag{8.3}$$

Then

$$\limsup_{u\to\infty} \int \frac{\overline{F}(u-y)}{\overline{F}(u)} G(\mathrm{d}y) < \infty \tag{8.4}$$

Proof of Lemma 8.1. (8.3) implies $\sup_u \overline{G}(u)/\overline{F}(u) \leq C$ for some $C < \infty$, so the lemma follows easily from (1.11) since

$$\begin{aligned}\int \overline{F}(u-y)G(\mathrm{d}y) &= \int \overline{G}(u-y)F(\mathrm{d}y)\\ &\leq C\int \overline{F}(u-y)F(\mathrm{d}y)\\ &= C\overline{F^{*2}}(u).\end{aligned}$$

□

In the following lemma, C denotes an unimportant constant which may change in value from one usage to the next.

Lemma 8.2 *For any $\eta > -\alpha$*

$$\limsup_{u\to\infty} E^{(u)} e^{-\eta(X_{\tau(u)}-u)} < \infty. \tag{8.5}$$

Proof of Lemma 8.2. Let $T(u) = \inf\{t : H_t > u\}$. Then $\tau(u) = L^{-1}_{T(u)}$ and $X_{\tau(u)} = H_{T(u)}$. Hence applying the killed version of Proposition III.2 of [4] (see Theorem 5.6 of [19]), for $x \geq 0$

$$\begin{aligned}P^{(u)}(X_{\tau(u)} - u > x) &= \frac{P(H_{T(u)} - u > x, T(u) < \infty)}{P(\tau(u) < \infty)}\\ &= \frac{\overline{\overline{\Pi}}_H(u)}{P(\tau(u) < \infty)} \int_{0\leq y\leq u} \frac{\overline{\overline{\Pi}}_H(u-y+x)}{\overline{\overline{\Pi}}_H(u)} V(\mathrm{d}y).\end{aligned} \tag{8.6}$$

Fix $\varepsilon > 0$ so that $\alpha - \varepsilon + \eta > 0$. Applying (4.10),

$$\int_{0\le y\le u-1} \frac{\overline{\Pi}_H(u-y+x)}{\overline{\Pi}_H(u)} V(\mathrm{d}y) \le Ae^{-(\alpha-\varepsilon)x} \int_{0\le y\le u-1} \frac{\overline{\Pi}_H(u-y)}{\overline{\Pi}_H(u)} V(\mathrm{d}y) \le ACe^{-(\alpha-\varepsilon)x} \tag{8.7}$$

if $u \ge 2$, since by (2.15), (2.17) and (2.18), Lemma 8.1 may be applied to the distributions $F(\mathrm{d}y) = I(y>1)\Pi_H(\mathrm{d}y)/\overline{\Pi}_H(1)$ and $G(\mathrm{d}y) = V(\mathrm{d}y)/V(\infty)$. On the other hand

$$\int_{u-1\le y\le u} \frac{\overline{\Pi}_H(u-y+x)}{\overline{\Pi}_H(u)} V(\mathrm{d}y) \le \overline{\Pi}_H(x) \frac{\overline{V}(u-1)}{\overline{\Pi}_H(u)} \le Ce^{-\alpha x}, \tag{8.8}$$

as $\Pi_H \in \mathcal{S}^{(\alpha)}$. Since the ratio in front of the integral in (8.6) is bounded by (2.15)–(2.17), the result follows from (8.7) and (8.8). □

As preparation for calculating the limit of (8.1) we need,

Proposition 8.1 *Let $\nu \ge 0$, $\zeta \ge 0$, $\eta > -\alpha$, $\lambda \ge 0$. Then*

$$E[e^{-\nu G^Z_{\rho-} - \zeta(\rho - G^Z_{\rho-}) - \eta W_0 - \lambda \overline{Z}_{\rho-}}; W_0 > 0] = \frac{\beta\alpha\, \kappa(\zeta, -\alpha)}{(\alpha+\eta)(\zeta+\beta_1)\kappa(\nu, \lambda-\alpha)} \tag{8.9}$$

If in addition $\lambda \ne \alpha + \eta$, then

$$E[e^{-\nu(\rho + G^W_{\tau(0)-}) - \zeta(\tau^W_0 - G^W_{\tau(0)-}) - \eta W_{\tau(0)} + \lambda \overline{W}_{\tau(0)-}}; W_0 \le 0] = \frac{\beta\alpha\, [\kappa(\zeta, \lambda-\alpha) - \kappa(\zeta, \eta)]}{(\beta_1+\nu)(\lambda-\alpha-\eta)\kappa(\nu, -\alpha)} \tag{8.10}$$

Proof of Proposition 8.1. Fix $\nu \ge 0$, $\zeta \ge 0$, $\eta > -\alpha$, $\lambda \ge 0$. Then by (5.13)

$$\begin{aligned}
&E[e^{-\nu G^Z_{\rho-} - \zeta(\rho - G^Z_{\rho-}) - \eta W_0 - \lambda \overline{Z}_{\rho-}}; W_0 > 0] \\
&\quad = \beta\alpha \int_{t\ge0}\int_{s\ge0}\int_{\gamma>0}\int_{\theta\ge0}\int_{\phi\ge0} e^{-\nu t - \zeta s - \eta\gamma - \lambda\theta} e^{-\alpha(\gamma+\phi-\theta)} V(\mathrm{d}t, \mathrm{d}\theta)\widehat{V}(\mathrm{d}s, \mathrm{d}\phi)\mathrm{d}\gamma \\
&\quad = \beta\alpha \int_{\gamma>0} e^{-(\alpha+\eta)\gamma}\mathrm{d}\gamma \int_{t\ge0}\int_{\theta\ge0} e^{-\nu t - (\lambda-\alpha)\theta} V(\mathrm{d}t, \mathrm{d}\theta) \int_{s\ge0}\int_{\phi\ge0} e^{-\zeta s - \alpha\phi}\widehat{V}(\mathrm{d}s, \mathrm{d}\phi) \\
&\quad = \frac{\beta\alpha}{(\alpha+\eta)\kappa(\nu, \lambda-\alpha)\widehat{\kappa}(\zeta, \alpha)},
\end{aligned}$$

since $\kappa(\nu, \lambda-\alpha) > 0$ by (2.10) and $\widehat{\kappa}(\zeta, \alpha) > 0$ trivially. Thus (8.9) follows from (2.11) and (2.12).

Now assume $\lambda \neq \alpha + \eta$, then by (3.1)

$$
\begin{aligned}
&E[e^{-\nu(\rho+G^W_{\tau(0)-})-\zeta(\tau^W_0-G^W_{\tau(0)-})-\eta W_{\tau(0)}+\lambda\overline{W}_{\tau(0)-}}; W_0 \leq 0] \\
&\quad = \beta_2\alpha\int_{z\leq 0} e^{-\alpha z}\mathrm{d}z E_z[e^{-\nu(\rho+G_{\tau(0)-})-\zeta(\tau(0)-G_{\tau(0)-})-\eta X_{\tau(0)}+\lambda\overline{X}_{\tau(0)-}}; \tau(0)<\infty] \\
&\quad = \frac{\beta\alpha}{\beta_1+\nu}\int_{z>0} e^{\alpha z}\mathrm{d}z E[e^{-\nu G_{\tau(z)-}-\zeta(\tau(z)-G_{\tau(z)-})-\eta(X_{\tau(z)}-z)-\lambda(z-\overline{X}_{\tau(z)-})}; \tau(z)<\infty] \\
&\quad = \frac{\beta\alpha\,[\kappa(\zeta,\lambda-\alpha)-\kappa(\zeta,\eta)]}{(\beta_1+\nu)(\lambda-\alpha-\eta)\kappa(\nu,-\alpha)}
\end{aligned}
$$

by the extension of the Second Factorisation Identity in Theorem 3.5 of [16]. □

We are now ready to calculate the limit of (8.1) and a related penalty function.

Theorem 8.1 *Fix $\nu \geq 0$, $\zeta \geq 0$, $\eta > -\alpha$, $\lambda > 0$. Then*

$$
\lim_{u\to\infty} E^{(u)} e^{-\nu G_{\tau(u)-}-\zeta(\tau_u-G_{\tau(u)-})-\eta(X_{\tau(u)}-u)-\lambda\overline{X}_{\tau(u)-}} = \frac{\beta\alpha\;\kappa(\zeta,-\alpha)}{(\alpha+\eta)(\zeta+\beta_1)\kappa(\nu,\lambda-\alpha)}. \tag{8.11}
$$

If in addition $\lambda \neq \alpha+\eta$, then

$$
\lim_{u\to\infty} E^{(u)} e^{-\nu G_{\tau(u)-}-\zeta(\tau(u)-G_{\tau(u)-})-\eta(X_{\tau(u)}-u)-\lambda(u-\overline{X}_{\tau(u)-})} = \frac{\beta\alpha\,[\kappa(\zeta,\lambda-\alpha)-\kappa(\zeta,\eta)]}{(\beta_1+\nu)(\lambda-\alpha-\eta)\kappa(\nu,-\alpha)}. \tag{8.12}
$$

Proof of Theorem 8.1. Since (8.11) and (8.12) follow in a similar manner from (8.9) and (8.10) respectively, we only prove (8.11). Let

$$
g(t,s,\gamma,y) = e^{-\nu t-\zeta s-\eta\gamma-\lambda y}.
$$

By (5.10), (8.5) and uniform integrability

$$
\begin{aligned}
&\lim_{u\to\infty} E^{(u)} g(G_{\tau(u)-}, \tau(u)-G_{\tau(u)-}, X_{\tau(u)}-u, \overline{X}_{\tau(u)-}) \\
&\qquad = E[g(G^Z_{\rho-}, \rho-G^Z_{\rho-}, W_0, \overline{Z}_{\rho-}); W_0>0] \\
&\qquad\quad + E[g(\rho+G^W_{\tau(0)-}, \tau^W_0-G^W_{\tau(0)-}, W_{\tau(0)}, \infty); W_0\leq 0] \\
&\qquad = E[e^{-\nu G^Z_{\rho-}-\zeta(\rho-G^Z_{\rho-})-\eta W_0+\lambda\overline{Z}_{\rho-}}; W_0>0],
\end{aligned}
$$

since $\lambda > 0$. Thus (8.11) follows from (8.9). □

Setting $\eta = \nu = \zeta = 0$ in (8.11) gives

$$
\begin{aligned}
\lim_{u\to\infty} e^{-\lambda u} E^{(u)} e^{\lambda(u-\overline{X}_{\tau(u)-})} &= \lim_{u\to\infty} E^{(u)} e^{-\lambda\overline{X}_{\tau(u)-}} \\
&= \frac{\beta_2\;\kappa(0,-\alpha)}{\kappa(0,\lambda-\alpha)}.
\end{aligned} \tag{8.13}
$$

This gives a transparent explanation of the mgf result in Theorem 3.3 of [22], and extends it to all $\lambda > 0$. Note that letting $\lambda \downarrow 0$ in the final expression of (8.13), reflects that in the limit, $\overline{X}_{\tau(u)-}$ has mass $1-\beta_2$ at infinity under $P^{(u)}$. Similarly setting $\eta = \nu = \zeta = 0$ in (8.12) gives the growth in the mgf of $\overline{X}_{\tau(u)-}$ as measured from the origin; for every $\lambda > 0$

$$\begin{aligned} \lim_{u\to\infty} e^{-\lambda u} E^{(u)} e^{\lambda \overline{X}_{\tau(u)-}} &= \frac{\beta_2 \alpha\ [\kappa(0,\lambda-\alpha)-q]}{(\lambda-\alpha)\kappa(0,-\alpha)} \\ &= \frac{\alpha\ [\kappa(0,\lambda-\alpha)-q]}{(\lambda-\alpha)q}. \end{aligned} \tag{8.14}$$

In this case letting $\lambda \downarrow 0$ reflects that in the limit, $u-\overline{X}_{\tau(u)-}$ has mass β_2 at infinity under $P^{(u)}$.

Observe that (8.11) and (8.12) are both false when $\lambda = 0$, as can be seen from (8.13) and (8.14). In this case the limit is obtained by adding the corresponding expressions in (8.9) and (8.10).

Theorem 8.2 *Fix $\nu \ge 0$, $\zeta \ge 0$, $\eta > -\alpha$, then*

$$\begin{aligned} &\lim_{u\to\infty} E^{(u)} e^{-\nu G_{\tau(u)-} - \zeta(\tau(u)-G_{\tau(u)-}) - \eta(X_{\tau(u)}-u)} \\ &= \frac{\beta\alpha}{(\alpha+\eta)\kappa(\nu,-\alpha)} \left[\frac{\kappa(\zeta,-\alpha)}{\beta_1+\zeta} + \frac{\kappa(\zeta,\eta)-\kappa(\zeta,-\alpha)}{\beta_1+\nu} \right]. \end{aligned} \tag{8.15}$$

Proof of Theorem 8.2. By Theorem 5.1 and (8.5)

$$\begin{aligned} &\lim_{u\to\infty} E^{(u)} e^{-\nu G_{\tau(u)-} - \zeta(\tau(u)-G_{\tau(u)-}) - \eta(X_{\tau(u)}-u)} \\ &\qquad E[e^{-\nu G^Z_{\rho-} - \zeta(\rho - G^Z_{\rho-}) - \eta W_0}; W_0 > 0] + E[e^{-\nu(\rho + G^W_{\tau(0)-}) - \zeta(\tau^W_0 - G^W_{\tau(0)-}) - \eta W_{\tau(0)}}; W_0 \le 0]. \end{aligned}$$

The result now follows by setting $\lambda = 0$ in (8.9) and (8.10) and adding. □

As a special case of (8.15), with $\nu = \zeta$, we obtain the limit of the joint transform of the overshoot and ruin time;

$$\lim_{u\to\infty} E^{(u)} e^{-\zeta\tau(u) - \eta(X_{\tau(u)}-u)} = \frac{\beta\alpha\ \kappa(\zeta,\eta)}{(\alpha+\eta)(\beta_1+\zeta)\kappa(\zeta,-\alpha)}. \tag{8.16}$$

Setting $\zeta = 0$ in (8.16) evaluates the limit in (8.2). With $\eta = 0$, (8.16) reflects the description of the limiting distribution in Theorem 3.2.

We now briefly describe an application of the EDPF in (8.16) when $\eta > 0$. Fix $\zeta \ge 0$ and choose $\eta = \eta(\zeta)$ so that $e^{-\zeta t - \eta X_t}$ is a martingale. In actuarial terms, η is a solution to *Lundberg's fundamental equation* see, e.g., Gerber-Shiu [15], p.51. To see that such a η exists and is unique in our setup, first observe that by (2.9), this is equivalent to

$$\kappa(\zeta,\eta)\widehat{\kappa}(\zeta,-\eta) = 0. \tag{8.17}$$

Now for $\zeta \geq 0$,

$$e^{-\kappa(\zeta,\eta)} = e^{-q} E e^{-\zeta \mathcal{L}_1^{-1} - \eta \mathcal{H}_1} \leq e^{-q} E e^{-\eta \mathcal{H}_1} = \begin{cases} < 1, & \eta \geq -\alpha \\ = \infty, & \eta < -\alpha \end{cases}$$

by Proposition 5.1 of [18]. Thus in order that (8.17) hold it must be that $\widehat{\kappa}(\zeta, -\eta) = 0$. Since $\widehat{\kappa}(\zeta, 0) \geq 0$ and $\widehat{\kappa}(\zeta, -\eta) \downarrow -\infty$ as $\eta \uparrow \infty$, this equation has a unique solution $\eta \geq 0$. Then by (8.16), if $\zeta > 0$ and $\eta = \eta(\zeta)$

$$E^{(u)} \frac{e^{-\zeta \tau(u)}(1 - e^{-\eta(X_{\tau(u)} - u)})}{\zeta} \to \frac{\beta\alpha}{\zeta(\beta_1 + \zeta)\kappa(\zeta, -\alpha)} \left(\frac{\kappa(\zeta, 0)}{\alpha} - \frac{\kappa(\zeta, \eta)}{\alpha + \eta} \right)$$

In the spectrally positive case, Gerber and Shiu [15], interpret this in terms of the expected present value of a deferred continuous annuity at a rate of 1 per unit time, starting at the time of ruin and ending as soon as the shortfall returns to zero.

The standard form of the EDPF's introduced by Gerber and Shiu is

$$E^{(u)}[e^{-\zeta \tau(u)} g(X_{\tau(u)} - u, u - X_{\tau(u)-})] \tag{8.18}$$

for suitably chosen functions g. We have chosen to formulate the results in this section in terms of exponential penalty functions using the undershoot of the maximum $u - \overline{X}_{\tau(u)-}$ instead of $u - X_{\tau(u)-}$. It is clear that more general penalty functions could have been used, and the resulting limits could then be found using Theorem 5.1 and Theorem 5.2. For the Gerber-Shiu penalty function in (8.18), under the appropriate conditions on g so that Theorem 5.1 applies, we have

$$\begin{aligned} &\lim_{u \to \infty} E^{(u)}[e^{-\zeta \tau(u)} g(X_{\tau(u)} - u, u - X_{\tau(u)-})] \\ &\quad = E[e^{-\zeta \rho} g(W_0, \infty); W_0 > 0] + E[e^{-\zeta(\rho + \tau_0^W)} g(W_{\tau(0)}, -W_{\tau(0)-}); W_0 \leq 0]. \end{aligned} \tag{8.19}$$

A natural example would be

$$\lim_{u \to \infty} E^{(u)} e^{-\zeta \tau(u) - \eta(X_{\tau(u)} - u) - \lambda(u - X_{\tau(u)-})} = E e^{-\zeta \rho} E[e^{-\zeta \tau_0^W - \eta W_{\tau(0)} + \lambda W_{\tau(0)-}}; W_0 \leq 0] \tag{8.20}$$

for $\zeta \geq 0$, $\eta > -\alpha$ and $\lambda > 0$. The limit can then be calculated using Theorem 5.2, although the resulting expression obtained is not as simple as those obtained in Theorem 8.1. Quite different behaviour occurs if $\lambda < 0$ in (8.20);

Proposition 8.2 *Let $\nu \geq 0$, $\zeta \geq 0$, $\eta > -\alpha$, $0 < \theta < \alpha$ and assume that $\theta - \eta < \alpha$. Then*

$$\lim_{u \to \infty} E^{(u)} e^{-\nu G_{\tau(u)-} - \zeta(\tau(u) - G_{\tau(u)-}) - \eta(X_{\tau(u)} - u) - \theta X_{\tau(u)-}} = \frac{\beta\alpha}{(\alpha + \eta)\kappa(\nu, \theta - \alpha)\widehat{\kappa}(\zeta, \alpha - \theta)}. \tag{8.21}$$

Proof of Proposition 8.2. With η and θ as above, we first observe that

$$\limsup_{u\to\infty} E^{(u)} e^{-\eta(X_{\tau(u)}-u)-\theta X_{\tau(u)-}I(X_{\tau(u)-}\le 0)} < \infty. \tag{8.22}$$

This follows immediately from (4.28) if $\eta \ge 0$, consequently we may assume $-\alpha < \eta < 0$. By considering separately the cases $X_{\tau(u)} - u > |X_{\tau(u)-}|$ and $X_{\tau(u)} - u \le |X_{\tau(u)-}|$, one finds

$$e^{-\eta(X_{\tau(u)}-u)-\theta X_{\tau(u)-}I(X_{\tau(u)-}\le 0)} \le e^{(\theta-\eta)(X_{\tau(u)}-u)} + e^{-(\theta-\eta)X_{\tau(u)-}I(X_{\tau(u)-}\le 0)},$$

and so (8.22) again follows from (4.28) and (8.5), since $\theta-\eta < \alpha$. Hence $e^{-\eta(X_{\tau(u)}-u)-\theta X_{\tau(u)-}}$ is uniformly integrable if $\eta > -\alpha$, $0 < \theta < \alpha$ and $\theta - \eta < \alpha$. Thus by Theorems 5.1 and 5.2

$$\begin{aligned}
&\lim_{u\to\infty} E^{(u)} e^{-\nu G_{\tau(u)-}-\zeta(\tau(u)-G_{\tau(u)-})-\eta(X_{\tau(u)}-u)-\theta X_{\tau(u)-}} \\
&\quad = E[e^{-\nu G^Z_{\rho-}-\zeta(\rho-G^Z_{\rho-})-\eta W_0-\theta Z_{\rho-}}; W_0 > 0] \\
&\quad = \beta\alpha \int_{t\ge 0}\int_{s\ge 0}\int_{\gamma>0}\int_{\phi\ge 0}\int_{\xi\ge -\phi} e^{-\nu t-\zeta s-\eta\gamma-\theta\xi} e^{-\alpha(\gamma-\xi)} V(\mathrm{d}t, \phi+\mathrm{d}\xi)\widehat{V}(\mathrm{d}s, \mathrm{d}\phi)\mathrm{d}\gamma \\
&\quad = \frac{\beta\alpha}{\alpha+\eta}\int_{t\ge 0}\int_{\xi\ge 0} e^{-\nu t+(\alpha-\theta)\xi} V(\mathrm{d}t, \mathrm{d}\xi)\int_{s\ge 0}\int_{\phi\ge 0} e^{-\zeta s-(\alpha-\theta)\phi} V(\mathrm{d}t, \mathrm{d}\theta)\widehat{V}(\mathrm{d}s, \mathrm{d}\phi)
\end{aligned}$$

which gives (8.21). □

Setting $\nu = \zeta$, using (2.11), and rewriting (8.21) in terms of the undershoot gives

$$\lim_{u\to\infty} e^{-\theta u} E^{(u)} e^{-\zeta\tau(u)-\eta(X_{\tau(u)}-u)+\theta(u-X_{\tau(u)-})} = \frac{\beta\alpha}{(\alpha+\eta)(\zeta-\Psi(i(\theta-\alpha)))}. \tag{8.23}$$

The special case of (8.23) with $\zeta = \eta = 0$, is given in Theorem 3.2 of [22]. Results related to (8.23) for the case of a Cramér-Lundberg model with bounded claims density can be found in Corollary 3.2 of Tang and Wei [24]. When $\theta = 0$, (8.21) fails just as (8.11) fails when $\lambda = 0$. Observe though that letting $\theta \downarrow 0$ on the RHS of (8.21) and $\lambda \downarrow 0$ on the RHS of (8.11) results in the same limit, as one would expect.

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