An N-Dimensional Version of the Beurling-Ahlfors Extension

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AN $N$-DIMENSIONAL VERSION OF THE BEURLING-AHLFORS EXTENSION

LEONID V. KOVALEV AND JANI ONNINEN

Abstract. We extend monotone quasiconformal mappings from dimension $n$ to $n+1$ while preserving both monotonicity and quasiconformality. The extension is given explicitly by an integral operator. In the case $n=1$ it yields a refinement of the Beurling-Ahlfors extension.

1. Introduction

Extension Problem. Given a mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{A}$, find $F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathcal{A}$ such that the restriction of $F$ to $\mathbb{R}^n$ agrees with $f$.

Let us introduce coordinate notation $x = (x^1, \ldots, x^n)$ and $f = (f^1, \ldots, f^n)$. By setting $F^i = f^i$ for $i = 1, \ldots, n$ and $F^{n+1} = x^{n+1}$ one immediately obtains a solution to the extension problem for many classes $\mathcal{A}$ such as continuous ($\mathcal{A} = C^0$), smooth ($\mathcal{A} = C^k$), homeomorphic, diﬀeomorphic, and (bi-)Lipschitz mappings.

When $\mathcal{A} = QC$, the class of quasiconformal mappings, the extension problem is much more diﬃcult. It was solved

• for $n = 1$ by Beurling and Ahlfors [4] in 1956,
• for $n = 2$ by Ahlfors [1] in 1964,
• for $n \leq 3$ by Carleson [8] in 1974, and
• for all $n \geq 1$ by Tukia and Väisälä [16] in 1982.

The Tukia-Väisälä extension uses, among other things, Sullivan’s theory [15] of deformations of Lipschitz embeddings. Our goal is to give an explicit extension for a subclass of $QC$. Quasiconformal mappings can be defined as orientation-preserving quasisymmetric mappings [11, 17].

Definition 1.1. A homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$ is quasisymmetric if there is a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that

\[
\frac{|f(x) - f(z)|}{|f(y) - f(z)|} \leq \eta \left( \frac{|x - z|}{|y - z|} \right).
\]

for $x, y, z \in \mathbb{R}^n$, $z \neq y$. 

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One can say that quasisymmetry is a three-point condition. But there are two subclasses of QC that are defined by two-point conditions, namely bi-Lipschitz class BL and the class of nonconstant delta-monotone mappings [2, Chapter 3]. Recall that a mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is monotone if
\begin{equation}
(f(x) - f(y), x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n.
\end{equation}

We called $f$ delta-monotone if there exists $\delta > 0$ such that
\begin{equation}
(f(x) - f(y), x - y) \geq \delta |f(x) - f(y)||x - y| \quad \text{for all } x, y \in \mathbb{R}^n.
\end{equation}

The class of nonconstant delta-monotone mappings is denoted by $\mathcal{DM}$.

Main Result. Let $n \geq 2$. For any mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{DM}$ there exists $F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathcal{DM}$ such that the restriction of $F$ to $\mathbb{R}^n$ agrees with $f$.

Our proof is by an explicit construction that can be viewed as an $n$-dimensional version of the Beurling-Ahlfors extension. Suppose $f \in \mathcal{DM}$.

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0, \infty)$ and
\begin{equation}
\phi(x) = (2\pi)^{-\frac{n}{2}} e^{-|x|^2/2}, \quad x \in \mathbb{R}^n.
\end{equation}

We define $F: \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ by
\begin{align}
F^i(x, t) &= \int_{\mathbb{R}^n} f^i(x + ty) \phi(y) dy \quad i = 1, \ldots, n \\
F^{n+1}(x, t) &= \int_{\mathbb{R}^n} (f(x + ty), y) \phi(y) dy
\end{align}

where $x \in \mathbb{R}^n$, $t \geq 0$ (see [3] for the convergence of these integrals). Observe that $F(x, 0) = (f(x), 0)$. Furthermore, $F^{n+1}(x, t) \geq 0$ because
\[
\int_{\mathbb{R}^n} (f(x + ty), y) \phi(y) dy = \int_{\mathbb{R}^n} (f(x + ty) - f(x), y) \phi(y) dy \geq 0
\]
due to the monotonicity of $f$. Finally, we extend $F$ to $\mathbb{R}^{n+1}$ by reflection
\[
F^i(x, t) = F^i(x, -t) \quad i = 1, \ldots, n \quad \text{and} \quad F^{n+1}(x, t) = -F^{n+1}(x, -t).
\]

Theorem 1.2. Let $n \geq 2$. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is $\delta$-monotone, then $F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is $\delta_1$-monotone where $\delta_1$ depends only on $\delta$ and $n$. In addition, $F: \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ is bi-Lipschitz in the hyperbolic metric.

Here $\mathbb{H}^{n+1} = \mathbb{R}^n \times (0, \infty)$ and the hyperbolic metric on $\mathbb{H}^{n+1}$ is $|dx|/x^{n+1}$. Theorem 1.2 can be also formulated for $n = 1$, in which case it becomes a refinement of the Beurling-Ahlfors extension theorem.
Proposition 1.3. If \( f: \mathbb{R} \to \mathbb{R} \) is increasing and quasisymmetric, then \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) is \( \delta_1 \)-monotone where \( \delta_1 \) depends only on \( \eta \) in Definition 1.1. Furthermore, \( F: \mathbb{H}^2 \to \mathbb{H}^2 \) is bi-Lipschitz in the hyperbolic metric.

Fefferman, Kenig and Pipher [9, Lemma 4.4] proved that \( F \) in Proposition 1.3 is quasiconformal. Proposition 1.3 was originally proved in [12] using their result. In this paper we give a direct proof.

Theorem 1.2 has an application to mappings with a convex potential [7], i.e., those of the form \( f = \nabla u \) with \( u \) convex. The basic properties and examples of quasiconformal mappings with a convex potential are given in [13].

Corollary 1.4. Suppose that \( f: \mathbb{R}^n \to \mathbb{R}^n, n \geq 2, \) is a \( K \)-quasiconformal mapping with a convex potential. Then \( f \) can be extended to a \( K_1 \)-quasiconformal mapping \( F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) with a convex potential, where \( K_1 \) depends only on \( K \) and \( n \).

2. Preliminaries

Let \( e_1, \ldots, e_{n+1} \) be the standard basis of \( \mathbb{R}^{n+1} \). All vectors are treated as column vectors. The transpose of a vector \( v \) is denoted by \( v^T \). We use the operator norm \( \| \cdot \| \) for matrices. A Borel measure \( \mu \) on \( \mathbb{R}^n \) is doubling if there exists \( D \mu \), called the doubling constant of \( \mu \), such that

\[
\mu(2B) \leq D \mu \mu(B)
\]

for all balls \( B = B(x, r) \). Here \( 2B = B(x, 2r) \).

The geometric definition of class \( QC \) given in the introduction is equivalent to the following analytic definition [11, 17].

Definition 2.1. A homeomorphism \( f: \mathbb{R}^n \to \mathbb{R}^n \) (\( n \geq 2 \)) is quasiconformal if \( f \in W^{1,n}_{loc} (\mathbb{R}^n, \mathbb{R}^n) \) and there exists a constant \( K \) such that the differential matrix \( Df(x) \) satisfies the distortion inequality

\[
\|Df(x)\|^n \leq K \det Df(x) \quad \text{a.e. in } \mathbb{R}^n.
\]

Delta-monotone mappings also have an analytic definition.

Lemma 2.2. Let \( \Omega \) be a convex domain in \( \mathbb{R}^n, n \geq 2 \). Suppose \( f \in W^{1,1}_{loc} (\Omega, \mathbb{R}^n) \) is continuous. The following are equivalent:

(i) \( f \) is \( \delta \)-monotone in \( \Omega \) for some \( \delta > 0 \); that is, \( (1.3) \) holds for all \( x, y \in \Omega \);

(ii) there exists \( \delta > 0 \) such that for a.e. \( x \in \Omega \) the matrix \( Df(x) \) satisfies

\[
v^T Df(x) v \geq \delta |Df(x) v| \quad \text{for every vector } v \in \mathbb{R}^n;
\]

(iii) there exists \( \gamma > 0 \) such that for a.e. \( x \in \Omega \) the matrix \( Df(x) \) satisfies

\[
v^T Df(x) v \geq \gamma |Df(x) v|^2 \quad \text{for every vector } v \in \mathbb{R}^n.
\]

The constants \( \delta \) and \( \gamma \) depend only on each other.
Proof. The equivalence of (i) and (ii), with the same constant $\delta$, was proved in [12, p. 397]. It is obvious that (iii) implies (ii) with $\delta = \gamma$. It remains to establish the converse implication (ii) $\Rightarrow$ (iii). To this end we need the following

**Claim:** if a real square matrix $A$ satisfies

$$v^T Av \geq \delta |Av||v|$$

for every $v \in \mathbb{R}^n$ then

$$(2.1) \quad |Av| \geq c\|A\||v| \quad c = c(\delta) > 0.$$ 

Although this claim is known, even with a sharp constant [3], we give a proof for the sake of completeness. It suffices to estimate $|Av|$ from below under the assumptions that $Av \neq 0$ and $\|A\| = 1 = |v|$. Let $u$ be a unit vector in $\mathbb{R}^n$ such that $|Au| = 1$. Replacing $u$ by $-u$ if necessary we may assume that $u^T Av + v^T Au \leq 0$. Let $\lambda = \sqrt{|Av|}$. On one hand we have

$$(2.2) \quad (\lambda u + v)^T A(\lambda u + v) \leq \lambda^2 u^T Au + v^T Av \leq \lambda^2 + \lambda^2 = 2\lambda^2.$$ 

On the other hand

$$(2.3) \quad (\lambda u + v)^T A(\lambda u + v) \geq \delta |\lambda Au + Av| |\lambda u + v| \geq \delta(\lambda - \lambda^2)(1 - \lambda).$$

Combining (2.2) and (2.3) we obtain $2\lambda \geq \delta(1 - \lambda)^2$, hence

$$\lambda \geq \delta^{-1} + 1 - \sqrt{(\delta^{-1} + 1)^2 - 1} > 0.$$ 

This proves the claim. $\square$

3. Delta-monotone mappings and doubling measures

The following result shows that $\mathcal{DM} \subset \mathcal{QC}$. In particular, $f \in \mathcal{DM}$ implies that $f$ is a continuous Sobolev mapping, and therefore (iii)–(iii) of Lemma 2.2 hold.

**Proposition 3.1.** [12, Theorem 6] Every nonconstant $\delta$-monotone mapping is $\eta$-quasisymmetric where $\eta$ depends only on $\delta$.

It is well-known that quasisymmetric mappings are closely related to doubling measures [11]. The following lemma is another instance of this relation.

**Lemma 3.2.** For any nonconstant $\delta$-monotone mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) the measure $\mu = ||Df(x)|| \, dx$ is doubling. The doubling constant $D_\mu$ depends only on $\delta$ and $n$.

**Proof.** Recall that $f$ is quasisymmetric. Lemma 3.2 in [14] implies the existence of a constant $C = C(\delta, n)$ such that

$$(3.1) \quad C^{-1} \frac{\text{diam } f(B)}{\text{diam } B} \leq \frac{1}{|B|} \int_B ||Df|| \, dx \leq C \frac{\text{diam } f(B)}{\text{diam } B}$$

for all balls $B \subset \mathbb{R}^n$. Since $\text{diam } f(2B) \leq C \text{ diam } f(B)$ with $C = C(\eta)$, the lemma follows. $\square$
Recall that \( \phi : \mathbb{R}^n \to (0, \infty) \) is the Gaussian kernel (1.4). Let \( B = B(0, 1) \) be the open unit ball in \( \mathbb{R}^n \).

**Lemma 3.3.** Let \( \mu \) be a doubling measure in \( \mathbb{R}^n \) and \( p \geq 0 \). Let \( \Omega \) be either \( \mathbb{R}^n \) or the half space \( \{ y : \langle y, \xi \rangle \geq 0 \} \) for some \( \xi \in \mathbb{R}^n \). Then

\[
C^{-1} \mu(B) \leq \int_\Omega |y|^p \phi(y) \, d\mu(y) \leq C \mu(B)
\]

where the constant \( C \) depends only on \( \mathcal{D}_\mu \), \( p \) and \( n \).

**Proof.** We begin by estimating the integral in (3.2) from above as follows

\[
\int_{\mathbb{R}^n} |y|^p \phi(y) \, d\mu(y) = \int_B |y|^p \phi(y) \, d\mu(y) + \sum_{k=0}^{\infty} \int_{2^k < |y| \leq 2^{k+1}} |y|^p \phi(y) \, d\mu(y),
\]

where

\[
\int_B |y|^p \phi(y) \, d\mu(y) \leq \phi(0) \mu(B) = (2\pi)^{-\frac{n}{2}} \mu(B)
\]

and

\[
\int_{2^k < |y| \leq 2^{k+1}} |y|^p \phi(y) \, d\mu(y) \leq 2^{p(k+1)} (2\pi)^{-\frac{n}{2}} e^{-2^{k+1}} \mu(B(0, 2^{k+1}))
\]

\[
\leq 2^{p(k+1)} (2\pi)^{-\frac{n}{2}} e^{-2^{k+1}} \mathcal{D}_\mu (k+1) \mu(B).
\]

Summing over \( k = 0, 1, 2 \ldots \) we obtain

\[
\int_{\mathbb{R}^n} \phi(y) \, d\mu(y) \leq C \mu(B)
\]

where \( C = C(\mathcal{D}_\mu, p, n) > 0 \).

We turn to the left side of (3.2). The inequality

\[
|y|^p \phi(y) \geq \frac{e^{-1/2}}{2^p (2\pi)^{n/2}} \quad \text{for} \quad \frac{1}{2} \leq |y| \leq 1
\]

implies

\[
\int_\Omega |y|^p \phi(y) \, d\mu(y) \geq \frac{e^{-1/2}}{2^p (2\pi)^{n/2}} \mu(\Omega \cap \{1/2 \leq |y| \leq 1\}).
\]

Since \( \mu(\Omega \cap \{1/2 \leq |y| \leq 1\}) \geq \mathcal{D}_\mu^{-3} \mu(B) \), the left side of (3.2) follows. \( \square \)

### 4. Proof of main results

**Proof of Theorem 1.2.** Since \( f \) is quasisymmetric by Proposition 3.1 it satisfies the growth condition \( |f(x)| \leq \alpha|x|^p + \beta \) for some constants \( \alpha, \beta, p \), see [11, Theorem 11.3]. Therefore, the integrals (1.5) and (1.6) converge and \( F \) is \( C^\infty \)-smooth in \( \mathbb{H}^{n+1} \). Let \( \gamma = \gamma(\delta) > 0 \) be as in part (iii) of Lemma 2.2.

Our first step is to prove that for \( (x, t) \in \mathbb{H}^{n+1} \) the matrix \( \mathcal{B} := DF(x, t) \) satisfies the condition

\[
w^T \mathcal{B} w \geq \gamma_1 \| \mathcal{B} \| |w|^2 \quad \text{for every vector} \quad w \in \mathbb{R}^{n+1}
\]

\[
w^T \mathcal{B} w \geq \gamma_1 \| \mathcal{B} \| |w|^2 \quad \text{for every vector} \quad w \in \mathbb{R}^{n+1}
\]
where \( \gamma_1 = \gamma_1(\delta, n) > 0 \). Fix \( x \in \mathbb{R}^n \) and \( t > 0 \). We compute the partial derivatives of \( F \) at \( (x, t) \in \mathbb{R}^{n+1} \) as follows.

\[
\frac{\partial F^i}{\partial x_j} = \int_{\mathbb{R}^n} f^i_j(x + ty)\phi(y)dy, \quad 1 \leq i, j \leq n;
\]
\[
\frac{\partial F^i}{\partial t} = \int_{\mathbb{R}^n} \sum_{j=1}^{n} f^i_j(x + ty)y^j\phi(y)dy, \quad 1 \leq i \leq n;
\]
\[
\frac{\partial F^{n+1}}{\partial x_j} = \int_{\mathbb{R}^n} \sum_{i=1}^{n} f^i_j(x + ty)y^i\phi(y)dy, \quad 1 \leq j \leq n;
\]
\[
\frac{\partial F^{n+1}}{\partial t} = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \sum_{j=1}^{n} f^i_j(x + ty)y^i y^j\phi(y)dy.
\]

To simplify formulas we write \( A(y) = Df(x + ty) \) and let \( B(y) \) be the \((n + 1) \times (n + 1)\) matrix written in block form below.

\[
B(y) = \begin{pmatrix}
A(y) & A(y)y \\
y^T A(y) & y^T A(y)y
\end{pmatrix}.
\]

With this notation we have

\[
DF(x, t) = \int_{\mathbb{R}^n} B(y)\phi(y) dy.
\]

First we show that the norm of \( \mathcal{B} \) is dominated by the quantity

\[
\alpha := \int_{B(0,1)} \|A(y)\| dy.
\]

Indeed,

\[
\|\mathcal{B}\| \leq \int_{\mathbb{R}^n} \|B(y)\|\phi(y) dy \leq \int_{\mathbb{R}^n} \|A(y)\|(1 + |y|)^2\phi(y) dy.
\]

By Lemma 3.2 the measure \( \mu = \|A(y)\| dy \) is doubling. Applying Lemma 3.3 we obtain

\[
\|\mathcal{B}\| \leq C\alpha, \quad C = C(\delta, n).
\]

Next we estimate the quadratic form \( w \mapsto w^T \mathcal{B} w \) generated by \( \mathcal{B} \) from below. For this we fix a vector \( w \in \mathbb{R}^{n+1} \), written as \( w = v + se_{n+1} \) with \( v \in \mathbb{R}^n \) and \( s \in \mathbb{R} \). It is easy to see that

\[
w^T B(y)w = (v + sy)^T A(y)(v + sy).
\]
Let \( \Omega = \{ y \in \mathbb{R}^n : \langle v, sy \rangle \geq 0 \} \). Then
\[
w^T B w = \int_{\mathbb{R}^n} \{ (v + sy)^T A(y)(v + sy) \} \phi(y) \, dy
\]
\[
\geq \gamma \int_{\mathbb{R}^n} \|A(y)\| v + sy \|^2 \phi(y) \, dy
\]
\[
\geq \gamma \int_{\Omega} \|A(y)\| v + sy \|^2 \phi(y) \, dy
\]
\[
\geq \gamma |v|^2 \int_{\Omega} \|A(y)\| \phi(y) \, dy + \gamma s^2 \int_{\Omega} \|A(y)\| y \|^2 \phi(y) \, dy.
\]

Applying Lemma 3.3 with \( \mu = \|A(y)\| \, dy \) we obtain
\[
(4.5) \quad w^T B w \geq c \alpha \gamma (|v|^2 + s^2) = c \alpha \gamma |w|^2, \quad c = c(\delta, n).
\]

Combining (4.4) and (4.5) we obtain (4.1) with \( \gamma_1 = (c/C) \gamma \). By virtue of Lemma 2.2 \( F \) is \( \delta_1 \)-monotone in the upper half-space \( \mathbb{H}^{n+1} \) where \( \delta_1 = \delta_1(\delta, n) \). By symmetry, \( F \) is also \( \delta_1 \)-monotone in the lower half-space.

To prove that \( F \) is \( \delta_1 \)-monotone in the entire space \( \mathbb{R}^{n+1} \), we consider two points \( a, b \in \mathbb{R}^{n+1} \) such that the line segment \([a, b]\) crosses the hyperplane \( \mathbb{R}^n \) at some point \( c \). We have
\[
\langle F(a) - F(b), a - b \rangle = \langle f(a) - f(c), a - b \rangle + \langle F(c) - F(b), a - b \rangle
\]
\[
\geq \delta_1 |F(a) - F(c)| |a - b| + \delta_1 |F(c) - F(b)| |a - b|
\]
\[
\geq \delta_1 |F(a) - F(b)| |a - b|
\]

Therefore, \( F \in DM \).

It remains to show that \( F: \mathbb{H}^{n+1} \to \mathbb{H}^{n+1} \) is bi-Lipschitz in the hyperbolic metric. Since \( F \in QC \) and \( \mathbb{H}^{n+1} \) is a geodesic space, it suffices to prove that
\[
(4.6) \quad \|DF(x, t)\| \approx \frac{F^{n+1}(x, t)}{t}.
\]

Here \( X \approx Y \) means that \( X \) and \( Y \) are comparable, i.e., \( C^{-1}Y \leq X \leq CY \) where \( C = C(\delta, n) \). It follows from (4.4) and (4.5) that \( \|DF(x, t)\| \) is comparable to the integral average of \( \|Df\| \) over the ball \( B(x, t) \). By (3.1) this average is comparable to \( t^{-1} \text{diam} f(B(x, t)) \). The quasisymmetry of \( F \) implies (cf. [11, 11.18])
\[
\text{diam} f(B(x, t)) \approx \|F(x, t) - F(x, t/2)\| \approx F^{n+1}(x, t).
\]

This proves (4.6).

**Proof of Proposition 4.3** The proof of Theorem 1.2 also works in the case \( n = 1 \) with the following interpretation. Since quasisymmetric mappings on the line need not be absolutely continuous [4], the derivative \( f' \) must be understood in the sense of distributions. In fact, \( \mu := f' \) is a positive doubling measure with \( \mathcal{D}_\mu = \mathcal{D}_\mu(\eta) \) [11, 13.20]. Lemma 3.2 is not needed in this case. The rest of the proof carries over with \( \gamma = 1 \) and \( \gamma_1 = \gamma_1(\mathcal{D}_\mu) \). □
Proof of Corollary 1.4. According to [12, Lemma 18], a $K$-quasiconformal mapping with a convex potential is also $\delta$-monotone with $\delta = \delta(K, n)$. Let $F$ be the $\delta_1$-monotone extension of $f$ provided by Theorem 1.2. Since the differential matrix $Df$ is symmetric, the formulas (4.2) and (4.3) show that $DF$ is symmetric as well. In addition, $DF$ is positive semidefinite by Lemma 2.2. Thus, $F = \nabla U$ for some convex function $U : \mathbb{R}^{n+1} \to \mathbb{R}$. □

5. Concluding Remarks

Both classes $QC$ (quasiconformal) and $BL$ (bi-Lipschitz) are groups under composition. However, the class of delta-monotone mappings $DM$ is not closed under composition (consider the rotation of the complex plane given by $z \mapsto e^{i\theta}z$ where $|\theta| < \pi/2$). Let $QC_d \subset QC$ be the group generated by $BL$ and $DM$. In other words, $f$ belongs to $QC_d$ if it can be decomposed into bi-Lipschitz and delta-monotone mappings. This should be compared with the notion of polar factorization of mappings introduced by Brenier [6].

Theorem 1.2 together with the trivial extension of bi-Lipschitz mappings yield a solution to the extension problem for $QC_d$.

Corollary 5.1. Let $n \geq 2$. For any mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $QC_d$ there exists $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $QC_d$ such that the restriction of $F$ to $\mathbb{R}^n$ agrees with $f$.

It seems likely that $QC_d$ is a proper subset of $QC$. This motivates the following question:

Question 5.2. Which quasiconformal mappings are decomposable?

Both bi-Lipschitz and delta-monotone mappings take smooth curves into rectifiable curves [2, Theorem 3.11.7]. This is no longer true for their composition. More precisely, for any $1 < \alpha < 2$ one can construct a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \in QC_d$ and $f(\mathbb{R})$ has Hausdorff dimension at least $\alpha$. To this end, one first finds a bi-Lipschitz mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(\mathbb{R})$ contains a planar Cantor set $E$ of dimension $0 < \beta < 1$ (see Lemma 3.1 [5] and the comment after its proof). Second, there is a delta-monotone mapping $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that the Hausdorff dimension of $h(E)$ is equal to $\alpha$ (see the construction in [10, Theorem 5]). Finally, let $f = h \circ g$.

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