Some Results on the Weight Structure of Cyclic Codes of Composite Length

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SOME RESULTS ON THE WEIGHT STRUCTURE OF CYCLIC CODES OF COMPOSITE LENGTH

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May 1975
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This work was supported by the National Science
Foundation Grant GK-37449,

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Abstract

In this work we investigate the weight structure of cyclic codes of composite length $n = n_1 n_2$, where $n_1$ and $n_2$ are relatively prime. The actual minimum distances of some classes of binary cyclic codes of composite length are derived. For other classes new lower bounds on the minimum distance are obtained. These new lower bounds improve on the BCH bound for a considerable number of binary cyclic codes.
I. Introduction

The problem of constructing cyclic product codes has been considered by Burton and Weldon [1] and by Abramson [2]. The factoring of cyclic codes was considered by Assmus and Mattson [3,4], and Goethals [5]. Goethals found new lower bounds on the minimum weight of a subclass of cyclic codes of composite length $n = n_1 n_2$ with $GCD(n_1, n_2) = 1$. Kasami [6] extended Goethals result. In both papers [5,6] a factorization is applied to the polynomials obtained from the Mattson-Solomon formulation [7].

By using a factorization applied directly to code vectors the actual minimum distances of some classes of binary cyclic codes of composite length are derived. For other classes new lower bounds on the minimum distance are obtained. The minimum distance and the lower bounds are given in terms of the minimum distance of cyclic codes of length $n_1$ and $n_2$. In many cases, the new lower bounds improve on the BCH bound, $d_0$ [8].

Some preliminaries are introduced in Section II. In Section III the minimum distances and the lower bounds are derived. In Section IV tables with numerical examples are presented. Concluding remarks are contained in Section V.
II. Preliminaries

Let $V_n$ be a cyclic code over $GF(q)$ of length $n = n_1 n_2$, $GCD(n_1, n_2) = 1$, and minimum distance $d$ generated by $g(x)$. Since $n_1$ and $n_2$ are relatively prime, there exist integers $a$ and $b$ such that

$$an_1 + bn_2 = 1.$$ 

Let $\beta$ be an element of order $n$ in an extension field $GF(q^m)$ of $GF(q)$ and let

$$\alpha = \beta^{bn_2}, \quad \gamma = \beta^{an_1}.$$

Then, $\alpha$ and $\gamma$ are primitive $n_1$th and $n_2$th roots of unity respectively, and

$$\alpha \gamma = \beta.$$

Let $\rho(\theta, \phi), \ 0 \leq \rho(\theta, \phi) < n$, be the unique solution of the following congruences given by the Chinese remainder theorem:

$$\rho(\theta, \phi) \equiv \begin{cases} \theta \pmod{n_1}, & 0 \leq \theta < n_1 \\ \phi \pmod{n_2}, & 0 \leq \phi < n_2 \end{cases}.$$

It follows that $\beta^{\rho(\theta, \phi)} = \alpha^\theta \gamma^\phi$.

Let

$$v(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{\rho(i,j)} x^\rho(i,j), \quad a_{\rho(i,j)} \in GF(q)$$

be a code vector of $V_n$. Associated with the polynomial $v(x)$, polynomials $V(y,z), V_j(y)$ and $\tilde{V}_i(z)$ are defined as follows:
\[ V(y, z) = \sum_{j=0}^{n_2-1} V_j(y) z^j \]
\[ = \sum_{i=0}^{n_1-1} \tilde{V}_i(z) y^i, \]

where
\[ V_j(y) = \sum_{i=0}^{n_1-1} a_{\rho(i, j)} y^i \]
and \[ \tilde{V}_i(z) = \sum_{j=0}^{n_2-1} a_{\rho(i, j)} z^j. \]

Similar to Kasami's derivation [6], it can be shown that
\[ V(\beta^\rho(\theta, \phi)) = V(\alpha^\theta, \gamma^\phi). \]

In the next section we will derive the minimum distances and the lower bounds on the minimum distance of some classes of cyclic codes of composite length \( n = n_1 n_2, \gcd(n_1, n_2) = 1. \)
III. Theorems

At first we will present a new lower bound on $d$ which is a generalization of Elias' bound [9] for cyclic product codes. In order to prove this bound we require two technical lemmas. The proofs of these lemmas are similar to the proofs of [10, Lemma 1] and [10, Lemma 2], respectively.

Lemma 1: If $V_j(a^\theta) = \sum_{i=0}^{n_1-1} a_{\rho(i,j)} a_i^\theta = 0$ for $\theta = 0,1,...,n_1-1$, then $a_{\rho(i,j)} = 0$ for $i = 0,1,...,n_1-1$.

Lemma 2: If $V_j(\beta^\rho(\theta,\phi)) = \sum_{j=0}^{n_2-1} V_j(a^\theta) v_j^\phi = 0$ for $\phi = 0,1,...,n_2-1$, then $V_j(a^\theta) = 0$ for $j = 0,1,...,n_2-1$.

Let $V_n$ be a cyclic code over $GF(q)$ of length $n = n_1n_2$ and minimum distance $d$ generated by $g(x)$, where $GCD(n_1,n_2) = 1$. For each $\theta$, $0 \leq \theta < n_1$, we define $J_\theta = \{\phi | g(\beta^\rho(\theta,\phi)) = 0\}$ and define $m_\theta$ to be the least nonzero integer such that $\theta q^{m_\theta} \equiv \theta (mod n_1)$. Thus if $\phi \in J_\theta$, then $\phi_1 \in J_\theta$, where $\phi_1 \equiv q^{m_\theta} \phi (mod n_2)$. Define $S_1 = \{\theta | J_{\theta} = \{0,1,...,n_2-1\}\}$. For each $\theta \notin S_1$ and such that $J_\theta$ is nonempty we define $V_{n_2}^{(\theta)}$ to be the cyclic code over $GF(q^{m_\theta})$ of length $n_2$ and minimum distance $d_2^{(\theta)}$ generated by

$$g_2^{(\theta)}(x) = \prod_{\phi \in J_\theta} (x-\gamma^\phi).$$

For each $\theta \notin S_1$ and such that $J_\theta$ is empty we define $V_{n_2}^{(\theta)}$ to be the cyclic code over $GF(q^{m_\theta})$ of length $n_2$ and minimum distance $d_2^{(\theta)} = 1$.
generated by 

\[ g_2^{(\theta)}(x) = 1. \]

Further, for each \( \theta \not\in S_1 \) define \( S_\theta = S_1 \cup S_1^{(\theta)} \), where

\[ S_1^{(\theta)} = \{ \hat{\theta} | 0 \leq \hat{\theta} < n_1; \hat{\theta} \not\in S_1 \text{ and } d_2^{(\hat{\theta})} > d_2^{(\theta)} \}. \]

Now, for each \( \theta \) such that \( S_\theta \) is nonempty define \( V_n^{(\theta)} \) to be the cyclic code over GF(q) of length \( n_1 \) and minimum distance \( d_1^{(\theta)} \) generated by

\[ g_1^{(\theta)}(x) = \text{LCM}\left\{ \prod_{i \in S_\theta} m_i(x) \right\} \]

where \( m_i(x) \) is the minimum polynomial of \( \alpha^i \) over GF(q). Further, for each \( \theta \) such that \( S_\theta \) is empty define \( V_n^{(\theta)} \) to be the cyclic code over GF(q) of length \( n_1 \) and minimum distance \( d_1^{(\theta)} = 1 \) generated by

\[ g_1^{(\theta)}(x) = 1. \]

We are now in the position to prove the following theorem.

**Theorem 1**: \( d > \min(d_1^{(\theta)}, d_2^{(\theta)}) | \theta \not\in S_1 \)

**Proof**: Let \( v(x) \) be a nonzero code polynomial of weight \( w \) in \( V_n \).

Then

\[ v(\beta^\rho(\theta, \phi)) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta) \gamma^j. \]

First, we note, by Lemma 2, that for each \( \theta \in S_1 \) we have

\[ V_j(\alpha^\theta) = 0 \] for \( j = 0, 1, \ldots, n_2-1 \). By Lemma 1, if \( V_j(\alpha^\theta) = 0 \) for \( j = 0, 1, \ldots, n_2-1 \) and for \( \theta = 0, 1, \ldots, n_1-1 \), then \( v(x) \equiv 0 \), contradicting the assumption that \( v(x) \) is a nonzero code polynomial of \( V_n \). Hence \( S_1 \neq \{0, 1, \ldots, n_1-1\} \) and there must exist at least one \( \theta \not\in S_1, 0 \leq \theta < n_1 \) such that \( V_j(\alpha^\theta) \neq 0 \) for some \( j, 0 \leq j < n_2 \). In
general for each $\theta$ such that $J_\theta$ is nonempty and $\theta \not\in S_1$ we have

$$v(\beta^\rho(\theta,\phi)) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta) y^j = 0$$

for $\phi \in J_\theta$. Now since $(V_j(\alpha^\theta))^q_{\Omega^\theta} = v_j(\alpha^\theta)$, $v_2(z) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta) z^j$
is a code polynomial of $V_{n_2}^{(\theta)}$. For cases where $v_j(\alpha^\theta) \neq 0$, for some

$j, 0 \leq j < n_2$, we actually must have $V_j(\alpha^\theta) \neq 0$ for $j = j_1, j_2, \ldots, j_\mu$

where $\mu \geq d_2^{(\theta)}$. So it is possible that $V_j(y) \neq 0$ for $j = j_1, j_2, \ldots, j_\mu$

for $u = d_2^{(\theta)}$ and $V_j(y) = 0$ for $j = j_{\mu+1}, j_{\mu+2}, \ldots, j_{n_2}$. But in this
case $V_j(\alpha^\theta) = 0$ for $j = 0, 1, \ldots, n_2 - 1$ and for all $\theta \in S_\theta$. Thus,$V_j(y)$ is a code polynomial of $V_{n_2}^{(\theta)}$ for $j = 0, 1, \ldots, n_2 - 1$. Hence the
weight of $V_j(y)$, for $j = j_1, j_2, \ldots, j_\mu$ is at least $d_1^{(\theta)}$. It follows
that $w \geq d_1^{(\theta)} d_2^{(\theta)}$. The case where $V_j(y) \neq 0$ for $j = j_1\theta^{(\theta_1)}_1, \ldots, j_\mu$

for $u > d_2^{(\theta)}$ is considered when we analyze the case $V_j(\alpha^1) \neq 0$ for
some $j, 0 \leq j < n_2$, and $\theta_1$ is such that $d_2^{(\theta_1)} > d_2^{(\theta)}$. By a similar
argument, when $\theta$ is such that $J_\theta$ is empty and $\theta \not\in S_1$, we obtain
$w \geq d_1^{(\theta)}$ since $d_2^{(\theta)} = 1$. Thus we conclude that

$$d \geq \min(d_1^{(\theta)} d_2^{(\theta)} | \theta \not\in S_1)$$

Q.E.D.

We remark that [10, Theorem 2] is a weak version of this theorem.

We now give an example of the application of Theorem 1.

Example 1: Consider the $(55,35)$ binary BCH code generated by

$g(x) = m_1(x)$. For this code $n_1 = 5, n_2 = 11, J_0 = \text{empty set},$

$J_1 = J_4 = \{1,3,4,5,9\}, J_2 = J_3 = \{2,6,7,8,10\}, S_1 = \text{empty set},$ and
\( d_0 = 4. \) Thus \( V_{n_2}^{(0)} \) is the \((11, 11)\) binary cyclic code with 
\[ d_1^{(0)} = 1, \quad V_{n_2}^{(1)} = V_{n_2}^{(4)} \] is a \((11, 6)\) quadratic residue code over \( \mathbb{GF}(4) \) with 
\[ d_2^{(1)} = d_2^{(4)} = 5 \] and \( V_{n_2}^{(2)} = V_{n_2}^{(3)} \) is also a \((11, 6)\) cyclic code 
over \( \mathbb{GF}(4) \) with \( d_2^{(2)} = d_2^{(3)} = 5 \) since it is equivalent to \( V_{n_2}^{(1)} \).

Thus we obtain the following table:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( d_1^{(\theta)} )</th>
<th>( d_2^{(\theta)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence, by Theorem 1, \( d \geq 5 \). We remark that for this example the 
generalized BCH bound [11] also gives \( d \geq 5 \) and that in this case 
both bounds achieve the actual minimum distance [12]. If we apply 
[10, Theorem 2] to this code we obtain only \( d \geq 1 \).

We are now interested in the investigation of the minimum 
weight of odd-weight code vectors and the minimum weight of even-
weight code vectors of binary cyclic codes of composite length 
\( n = n_1 n_2, \ GCD(n_1, n_2) = 1. \) Thus from now on we assume \( q = 2. \)

In order to continue our development we need to introduce some 
definitions. Let \( d_{\text{odd}}, d_{\text{even}} \) be the minimum weight of odd-weight 
and the minimum weight of even-weight code vectors of \( V_n \), 
respectively. Further, for \( i = 1 \) and \( 2 \), let \( d_i \) be the minimum 
distance of \( V_{n_i} \) and let \( d_{i, \text{odd}}, d_{i, \text{even}} \) be the minimum weight of
odd-weight and minimum weight of even-weight code vectors of $V_{n_i}$, respectively. Where $V_{n_i}$ is a binary cyclic code of length $n_i$ generated by $g_i(x)$.

The next theorem gives the exact value on $d_{\text{odd}}$ and $d_{\text{even}}$ when $V_n$ is a binary cyclic product code of $V_{n_1}$ and $V_{n_2}$.

**Theorem 2:** Let $V_n$ be the binary cyclic product code of $V_{n_1}$ and $V_{n_2}$ generated by $g(x)$ such that $g(1) \neq 0$. Then,

$$d_{\text{odd}} = d_{1\text{odd}} d_{2\text{odd}}$$

and

$$d_{\text{even}} = \min(d_{1\text{even}}, d_2, d_1 d_{2\text{even}})$$

**Proof:** Let $v(x)$ be a nonzero code vector of $V_n$. Thus, we have that

$$v(\beta^\rho(\theta, \phi)) = V(\alpha^\theta, \gamma^\phi) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta)\gamma^j \phi .$$

Let

$$v_2(z) = \sum_{j=0}^{n_2-1} v_j(1)z^j ,$$

then

$$v_2(\gamma^\phi) = v(\beta^\rho(0, \phi)) .$$

According to [13, Theorem 3] we have that

$$v(\beta^\rho(0, \phi)) = 0 \text{ for } \phi \in S_2$$

where $S_2 = \{\phi | g(\beta^\rho(\theta, \phi)) = 0 \text{ for } \theta = 0, 1, \ldots, n_1-1\}$. Thus, by [13] $v_2(z)$ is a code polynomial of $V_{n_2}$. Furthermore, by [13, Theorem 3]

$$v(\beta^\rho(\theta, \phi)) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta)\gamma^j \phi = 0$$
for \( \theta \in S_1 \), where \( S_1 = \{ \theta \mid g(\beta^\theta(0, \phi)) = 0 \text{ for } \phi = 0, 1, \ldots, n_2-1 \} \).

Thus, by Lemma 2, \( V_j(\alpha^\theta) = 0 \) for \( \theta \in S_1 \) and \( j = 0, 1, \ldots, n_2-1 \).

Hence by [13], \( V_j(y) \) is a code polynomial of \( V_{n_1} \) for \( j = 0, 1, \ldots, n_2-1 \).

First let us assume that \( v(x) \) has odd weight. Hence, \( V_j(1) \neq 0 \) for at least one \( j, 0 \leq j < n_2 \). So, \( v_2(z) \) has weight at least \( d_{2\text{odd}} \) and since \( V_j(y) \) is a code polynomial of \( V_{n_1} \) we can conclude that

\[
d_{\text{odd}} \geq d_{1\text{odd}} d_{2\text{odd}}.
\]

Now we assume that \( v(x) \) has even weight. Two cases must be analyzed, \( V_j(1) \neq 0 \) for some \( j, 0 \leq j < n_2 \) and \( V_j(1) = 0 \) for \( j = 0, 1, \ldots, n_2-1 \). If \( V_j(1) \neq 0 \) for some \( j, 0 \leq j < n_2 \), then \( v_2(z) \neq 0 \) and \( v_2(z) \) must have even weight. Thus,

\[
d_{\text{even}} \geq d_{1\text{odd}} d_{2\text{even}}.
\]

Now, if \( V_j(1) = 0 \) for \( j = 0, 1, \ldots, n_2-1 \), then

\[
v(\beta^\theta(0, \phi)) = \sum_{j=0}^{n_2-1} V_j(1) y^j \phi = 0
\]

for \( \phi = 0, 1, \ldots, n_2-1 \). Thus, \( v(x) \) is divisible by \( x^{n_2} + 1 \). According to [13] \( v(x) \) is a code polynomial of the binary cyclic product code of \( V_{n_2} \) and \( V(\mathbb{E}) \), where \( V(\mathbb{E}) \) is the binary cyclic code of length \( n_1 \) generated by \( (x+1)g_1(x) \), \( g_1(x) \) is the generator of \( V_{n_1} \). Hence, by the Elias bound [9] for cyclic product codes we obtain

\[
d_{\text{even}} \geq d_{1\text{even}} d_{2}.
\]

In conclusion

\[
d_{\text{even}} \geq \min(d_{1\text{odd}} d_{2\text{even}}, d_{1\text{even}} d_{2}) = \min(d_{1} d_{2\text{even}}, d_{1\text{even}} d_{2}).
\]

Now we will show that if \( w(v_1(x)) = w_1 \) is the Hamming weight of \( v_1(x) \), a nonzero code polynomial of \( V_{n_1} \) and \( w(v_2(x)) = w_2 \) is the Hamming weight of \( v_2(x) \), a nonzero code polynomial of \( V_{n_2} \), then there exists
a code polynomial \( v(x) \) of \( V_n \) such that \( w(v(x)) = w_1 w_2 \). Let

\[
v_1(x) = 1 + \sum_{i=1}^{w_1-1} x^i, \quad 0 < k_i < n_1,
\]

\[
v_2(x) = 1 + \sum_{j=1}^{w_2-1} x^j, \quad 0 < \ell_j < n_2,
\]

\( M_1 = \{0, k_1, k_2, \ldots, k_{w_1-1}\} \) and \( M_2 = \{0, \ell_1, \ell_2, \ldots, \ell_{w_2-1}\} \).

Now we construct the following polynomial

\[
\hat{v}(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_p(i,j) x^i y^j
\]

such that \( a_p(i,j) = 1 \) if \( i \in M_1 \) and \( j \in M_2 \), otherwise \( a_p(i,j) = 0 \). Hence \( w(\hat{v}(x)) = w_1 w_2 \). Associated with the polynomial \( \hat{v}(x) \) we have

\[
\hat{v}(y,z) = \sum_{j=0}^{n_2-1} \hat{v}_j(y) z^j,
\]

where

\[
\hat{v}_j(y) = \sum_{i=0}^{n_1-1} a_p(i,j) y^i.
\]

In this case we have

\[
\hat{v}_0(y) = \hat{v}_{k_1}(y) = \ldots = \hat{v}_{k_{w_1-1}}(y) = 1 + \sum_{i=1}^{w_1-1} y^i
\]

and \( \hat{v}_j(y) \equiv 0 \) for \( j \not\in M_2 \) and \( 0 < j < n_2 \). Hence

\[
\hat{v}(y,z) = (1 + \sum_{i=1}^{w_1-1} y^i)(1 + \sum_{j=1}^{w_2-1} z^j)
\]

and

\[
\hat{v}(\alpha^\theta, \gamma^\phi) = \hat{v}(\alpha^\theta, \gamma^\phi) = (1 + \sum_{i=1}^{w_1-1} \alpha^{\theta i})(1 + \sum_{j=1}^{w_2-1} \gamma^{\phi j}).
\]
Thus, \( \hat{v}(\beta^\rho(\theta, \phi)) = 0 \) for \( \theta \in S_1 \) and \( \phi = 0,1, \ldots, n_2-1 \); and for \( \phi \in S_2 \) and \( \theta = 0,1, \ldots, n_1-1 \). According to [13], \( \hat{v}(x) \) is a code polynomial of \( V_n \). Hence \( d_{\text{odd}} \leq d_{\text{1odd}} d_{\text{2odd}} \) and \( d_{\text{even}} \leq \min \{ d_{\text{leven}} d_2, d_1 d_{\text{2even}} \} \).

Q.E.D.

Example 2: As an example of application of Theorem 2 let us consider the \((105,44)\) binary cyclic product code of the \((7,4)\) and the \((15,11)\) binary cyclic codes. In this example \( d_{\text{1odd}} = 3 \), \( d_{\text{leven}} = 4 \), \( d_{\text{2odd}} = 3 \) and \( d_{\text{2even}} = 4 \). Thus, by Theorem 2
\[
 d_{\text{odd}} = 9 \text{ and } d_{\text{even}} = 12. \quad \text{The BCH bound gives } d_{\text{odd}} \geq 7 \text{ and } d_{\text{even}} \geq 10.
\]

In order to avoid proving special cases of the following theorems, we define the following three quantities to be infinity: the minimum distance of the \((n,0)\) code, the minimum weight of even-weight code vectors of the binary cyclic \((n,1)\) code and the minimum weight of odd-weight code vectors of the binary cyclic codes which have 1 as roots of their generator polynomial.

The binary cyclic product code of \( V_{n_1} \) and \( V_{n_2} \), where \( V_{n_1} \) is the \((n_1,n_1)\) binary cyclic code will be called the one-dimensional product code of \( V_n \). A lower bound on the minimum distance of a subcode of a one-dimensional product code can now be derived.

Let \( V_n \) be a subcode of the one-dimensional product code of \( V_{n_2} \), generated by \( g(x) \) such that \( g(1) \neq 0 \). Let \( J_0 = \{ \theta | g(\beta^\rho(\theta,0)) = 0 \} \), \( J_0 = \{ \phi | g(\beta^\rho(0,\phi)) = 0 \} \) and \( S_2 = \{ \phi | g(\beta^\rho(\theta,\phi)) = 0 \text{ for } \theta = 0,1, \ldots, n_1-1 \} \). According to [13], \( V_{n_2} \) is generated by \( g_2(x) = \prod_{\phi \in S_2} (x+\gamma^\phi) \). Now
we define $V_{n_1}$ to be the binary cyclic code of length $n_1$ generated by $g_1(x) = \prod_{\theta \in J_0} (x + \alpha^\theta)$ and $V^{(0)}_{n_2}$ to be the binary cyclic code generated by $g^{(0)}_2 = \prod_{\phi \in J_0} (x + \gamma^\phi)$. Finally we let $d^{(0)}_{2\text{odd}}$, $d^{(0)}_{2\text{even}}$ to be the minimum weight of odd-weight and of even-weight code vectors of $V^{(0)}_{n_2}$, respectively. Now we are in the position to prove the following theorem:

**Theorem 3:** $d^{(0)}_{\text{odd}} \geq \max(d^{(0)}_{1\text{odd}}, d^{(0)}_{2\text{odd}})$ and $d^{(0)}_{\text{even}} \geq \min(d^{(0)}_{2\text{even}}, 2d^{(0)}_{2\text{even}}, d^{(0)}_{\text{even}})$.

**Proof:** Let $v(x)$ be a nonzero code polynomial of $V_n$. Thus

$$v(\beta^\theta(\theta, \phi)) = \sum_{i=0}^{n_1-1} \bar{V}_i(y^{\phi}) \alpha^{i\theta} = 0$$

for $\phi \in S_2$ and $\theta = 0, 1, \ldots, n_1-1$. So, by Lemma 2 $\bar{V}_i(y^{\phi}) = 0$ for $\phi \in S_2$. Hence, $\bar{V}_i(z)$ is a code polynomial of $V_{n_2}^{(0)}$. Let

$$v_1(y) = \sum_{i=0}^{n_1-1} \bar{V}_i(1)y^{i}.$$ 

We note that $v_1(y)$ is a code polynomial of $V_{n_1}$. If $v(x)$ has odd weight, then, similar to the proof of Theorem 2, we obtain

$$d^{\text{odd}} \geq d^{\text{1odd}} d^{\text{2odd}}.$$ 

By [14, Theorem 3] $d^{\text{odd}} \geq d^{(0)}_{2\text{odd}}$. Hence

$$d^{\text{odd}} \geq \max(d^{\text{1odd}}, d^{\text{2odd}}, d^{(0)}_{2\text{odd}}).$$

If $v(x)$ has even weight, then we

*If $J_0$ is empty, then $g_1(x) = 1$. \*
consider two cases. $\tilde{V}_i(1) \neq 0$ for some $i$, $0 \leq i < n_1$ and $\tilde{V}_i(1) = 0$ for $i = 0, 1, \ldots, n_1 - 1$. If $\tilde{V}_i(1) \neq 0$ for some $i$, $0 \leq i < n_1$, then as in the proof of Theorem 2, we obtain $d_{\text{even}} \geq d_{\text{even}} \oplus d_{\text{2odd}}$.

Similarly, if $\tilde{V}_i(1) = 0$ for $i = 0, 1, \ldots, n_1 - 1$ then $x^{n_1} + 1$ divides $v(x)$. Thus, by [13] $v(x)$ is a code polynomial of the binary cyclic one-dimensional product code of $V_{n_2}^{(E)}$, where $V_{n_2}^{(E)}$ is the binary cyclic code of length $n_2$ generated by $(x+1)g_2(x)$, and $g_2(x)$ is the generator of $V_{n_2}$. For this $v(x)$ we can also write

$$v(\beta^\phi(0, \phi)) = v(\gamma^\phi) = \sum_{j=0}^{n_2-1} V_j(1) \gamma^j \phi = 0$$

for $\phi \in J_0 \cup \{0\}$. Let us define

$$v_2(z) = \sum_{j=0}^{n_2-1} V_j(1) z^j.$$

Thus, $v_2(z)$ is code polynomial of even weight of $V_{n_2}^{(0)}$. Now if $V_j(1) \neq 0$ for some $j$, $0 \leq j < n_2$, then $d_{\text{even}} \geq d_{\text{even}}^{(0)}$. If $V_j(1) = 0$ for $j = 0, 1, \ldots, n_2 - 1$, then $x^{n_2 - 1}$ divides $v(x)$ and, by [13], $v(x)$ is a code polynomial of the binary cyclic product code of $V_{n_1}^{(E)}$, the binary cyclic code of length $n_1$ generated by $g_1(x) = (x+1)$, and $V_{n_2}^{(E)}$, the binary cyclic code of length $n_2$ generated by $(x+1)g_2(x)$. Thus $d_{\text{even}} \geq 2d_{\text{even}}$. Hence

$$d_{\text{even}} \geq \min(d_{\text{even}}^{(0)}, 2d_{\text{2even}}^2, d_{\text{even}} d_{\text{2odd}}).$$

Q.E.D.

**Example 3:** As an application of Theorem 3 let us consider the $(21, 7)$ binary cyclic code generated by $g(x) = m_1(x)m_3(x)m_7(x)m_9(x)$. 
For this case $n_1 = 3$, $n_2 = 7$, $J_0 = \{1,2\}$, $J_0 = \{1,2,3,4,5,6\}$ and $S_2 = \{1,2,4\}$. Thus, $V_{n_2}$ is a $(7,4)$ binary cyclic code, $V_{n_2}^{(0)}$ is the $(7,1)$ binary cyclic code. Since $d_{1\text{odd}} = 3$, $d_{1\text{even}} = \infty$, $d_{2\text{odd}} = 3$, $d_{2\text{even}} = 4$, $d_{2\text{odd}}^{(0)} = 7$ and $d_{2\text{even}}^{(0)} = \infty$, by Theorem 3 $d_{\text{odd}} \geq 9$ and $d_{\text{even}} \geq 8$. The BCH bound gives $d_{\text{odd}} \geq 5$ and $d_{\text{even}} \geq 6$.

Now we will investigate the weight structure of a class of binary cyclic codes which will be called the class of binary cyclic quasi-product codes. These codes are defined in the following manner:

Consider the binary cyclic product code of $V_{n_1}$, with $d_1 \geq 2$ and $g_1(l) \neq 0$, and $V_{n_2}$, with $d_2 \geq 2$ and $g_2(l) \neq 0$, generated by $g(x)$, such that $d_1 d_2 > 4$. Let $\overline{g}(x) = \text{GCD}(g(x), (x^{n_1+1})(x^{n_2+1}))$. The binary cyclic code of length $n$ generated by $g'(x) = (g(x)/\overline{g}(x))$ is defined to be the binary cyclic quasi-product code of $V_{n_1}$ and $V_{n_2}$. That is, if $S_1 = \{\theta | g(\beta^\theta(\theta, \phi)) = 0 \text{ for } \phi = 0,1,\ldots,n_2-1\}$ and $S_2 = \{\phi | g(\beta^\phi(\theta, \phi)) = 0 \text{ for } \theta = 0,1,\ldots,n_1-1\}$, then $g'(\beta^\theta(\theta, \phi)) = 0$ for $\theta \in S_1$ and $\phi = 1,2,\ldots,n_2-1$, and $g'(\beta^\phi(\theta, \phi)) = 0$ for $\phi \in S_2$ and $\theta = 1,2,\ldots,n_1-1$.

We are now in the position to prove the following theorem.

**Theorem 4:** Let $V_n$ be the binary cyclic quasi-product code of $V_{n_1}$ and $V_{n_2}$. Then

$$d_{\text{odd}} = \min(n_1, n_2, d_{1\text{odd}}, d_{2\text{odd}})$$

and

$$d_{\text{even}} = \min(2n_1, 2n_2, d_{1\text{even}}, d_1 d_{2\text{even}}, n_2 (d_{1\text{odd}} - 2) d_{2\text{odd}}, n_1 (d_{2\text{odd}} - 2) x d_{1\text{odd}}).$$
Proof: Let
\[ v(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{\rho(i,j)} x^\rho(i,j) \]
be a nonzero code polynomial of \( V_n \). Hence
\[ v(\beta^\rho(\theta, \phi)) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta) \gamma^j, \]
where \( v_j(y) = \sum_{i=0}^{n_1-1} a_{\rho(i,j)} y^i \).

Since \( v(\beta^\rho(\theta, 0)) \), \( \theta \in S_1 \), can be zero or nonzero and since
\( v(\beta^\rho(0, \phi)) \), \( \phi \in S_2 \), can be zero or nonzero, we must inspect several cases.

Case 1. In this case we consider the possibility of having
\[ v(\beta^\rho(\theta, 0)) = 0 \quad \text{for} \quad \theta \in S_1 \]
and \( v(\beta^\rho(0, \phi)) = 0 \quad \text{for} \quad \phi \in S_2 \). Hence,
\[ v(\beta^\rho(\theta, \phi)) = 0 \quad \text{for} \quad \theta \in S_1 \]
and \( \phi = 0, 1, \ldots, n_2 - 1 \) and \( v(\beta^\rho(\theta, \phi)) = 0 \)
for \( \phi \in S_2 \) and \( \theta = 0, 1, \ldots, n_1 - 1 \). This implies that \( v(x) \) is a code
polynomial of the binary cyclic product code of \( V_{n_1} \) and \( V_{n_2} \) [13].

Thus, by Theorem 2, \( d_{\text{odd}} = d_{1, \text{odd}} d_{2, \text{odd}} \) and \( d_{\text{even}} = \min(d_{1, \text{even}} d_2, d_1 d_{2, \text{even}}) \).

Case 2. In this case we consider the possibility of having
\[ v(\beta^\rho(\theta, 0)) \neq 0 \quad \text{for} \quad \theta \in S_1 \]
and \( v(\beta^\rho(0, \phi)) = 0 \quad \text{for} \quad \phi \in S_2 \). Hence,
\[ v(\beta^\rho(\theta, \phi)) = 0 \quad \text{for} \quad \phi \in S_2 \]
and \( \theta = 0, 1, \ldots, n_1 - 1 \). Let us define
\[ v_2^{(\theta)}(z) = \sum_{j=0}^{n_2-1} v_j(\alpha^\theta) z^j. \]

Thus, \( v_2^{(\theta)}(\gamma^\phi) = 0 \quad \text{for} \quad \theta \in S_1 \)
and \( \phi = 1, 2, \ldots, n_2 - 1 \). This implies
that for \( \theta \in S_1 \), \( v_2^{(\theta)}(z) \) is divisible by \( z^{n_2-1} + z^{n_2-2} + \ldots + z + 1 \).
Hence, for \( \theta \in S_1 \), \( v_2^{(\theta)}(z) = v_3(\alpha^\theta) (z^{n_2-1} + \sum_{j=0}^{n_2-1} z^j). \) Since
\[ v_2^{(\theta)}(l) = v(\beta^\theta(\theta,0)) \neq 0 \] for \( \theta \in S_1 \), we can conclude that
\[ V_0(\alpha^\theta) = V_1(\alpha^\theta) = \ldots = V_{n_2-1}(\alpha^\theta) \neq 0 \] (1)
for \( \theta \in S_1 \). So, the Hamming weight of \( V_j(y) \), \( w(V_j(y)) \), is at least one for \( j = 0,1,\ldots,n_2-1 \). Hence \( w(v(x)) \geq n_2 \), which implies that \( d_{\text{odd}} \geq n_2 \) and \( d_{\text{even}} \geq n_2+1 \). Now let us obtain a better bound for \( d_{\text{even}} \), for this we assume that \( v(x) \) has even weight. Thus \( w(v(x)) \geq n_2+1 \). Furthermore let us assume that \( v(x) \) is a code polynomial such that there exists at least one \( j, 0 \leq j < n_2 \), satisfying \( w(V_j(y)) = 1 \). Because of the cyclic property we can, without loss of generality, assume that \( V_0(y) = 1 \). Thus, by Equation 1
\[ V_0(\alpha^\theta) = V_1(\alpha^\theta) = \ldots = V_{n_2-1}(\alpha^\theta) = 1 \]
for \( \theta \in S_1 \). Now based on the code polynomial \( v(x) \) we construct the following polynomial:
\[ v'(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{i,j} x^i y^j \]
where \( a_{i,j} = a_{i,j} \) for \( i = 1,2,\ldots,n_1-1 \) and \( j = 0,1,\ldots,n_2-1 \) \( a_{0,j} = a_{0,j} + 1 \) for \( j = 0,1,\ldots,n_2-1 \). Associated with the polynomial \( v'(x) \), polynomials \( V'(y,z) \), \( V'_j(y) \) and \( \bar{V}_i(z) \) are defined as follows:
\[ V'(y,z) = \sum_{j=0}^{n_2-1} V'_j(y) z^j = \sum_{i=0}^{n_1-1} \bar{V}_i(z) y^i \]
where \( V'_j(y) = \sum_{i=0}^{n_1-1} a_{i,j} y^i \) and \( \bar{V}_i(z) = \sum_{j=0}^{n_2-1} a_{i,j} z^j \).
Thus
\[ v'(y,z) = \sum_{j=0}^{n_2-1} (V'_j(y)+1) z^j \]
and \( v'(\beta^\rho(\theta, \phi)) = v'(\alpha^\theta, \gamma^\phi) \). This implies that

\[
v'(\alpha^\theta, \gamma^\phi) = \sum_{j=0}^{n_2-1} (V_j(\alpha^\theta)+1) \gamma^j\phi.
\]

Hence, \( v'(\beta^\rho(\theta, \phi)) = 0 \) for \( \theta \in S_1 \) and \( \phi = 0, 1, \ldots, n_2-1 \). In addition

\[
v'(\alpha^\theta, \gamma^\phi) = \sum_{j=0}^{n_2-1} V_j(\alpha^\theta) \gamma^j\phi + \sum_{j=0}^{n_2-1} \gamma^j\phi = \sum_{i=0}^{n_1-1} \tilde{V}_i(\gamma^\phi) \alpha^i\theta + \frac{\gamma^{n_2\phi}+1}{\gamma^\phi+1}.
\]

Thus for \( \phi \neq 0 \) we have

\[
v'(\beta^\rho(\theta, \phi)) = \sum_{i=0}^{n_1-1} \tilde{V}_i(\gamma^\phi) \alpha^i\theta.
\]

Now, since \( v(\beta^\rho(\theta, \phi)) = 0 \) for \( \phi \in S_2 \) and \( \theta = 0, 1, \ldots, n_1-1 \), by Lemma 2, \( \tilde{V}_i(\gamma^\phi) = 0 \) for \( \phi \in S_2 \) and \( i = 0, 1, \ldots, n_1-1 \). Hence, since \( 0 \notin S_2 \) because \( g_2(1) \neq 0 \), we can conclude that \( v'(\beta^\rho(\theta, \phi)) = 0 \) for \( \phi \in S_2 \) and \( \theta = 0, 1, \ldots, n_1-1 \). By [13] \( v'(x) \) is a code polynomial of the binary cyclic product code of \( V_{n_1} \) and \( V_{n_2} \). \( v'(x) \) has odd weight because \( v(x) \) has even weight. Now we investigate the \( w(v'(x)) \).

Similar to the proof of Theorem 2 we can conclude that \( V_j(\gamma) \) is a code polynomial of \( V_{n_1} \) for \( j = 0, 1, \ldots, n_2-1 \) and that

\[
v_2'(z) = \sum_{j=0}^{n_2-1} V_j(\gamma) z^j \quad \text{is a code polynomial of } V_{n_2}.
\]

Since \( v'(x) \) has odd weight, there exists at least one \( j, 0 \leq j < n_2 \), such that \( V_j(\gamma) \neq 0 \). This implies that we must have \( V_j(1) \neq 0 \) for \( \ell = 1, 2, \ldots, r \), with \( d_{2\text{odd}} \leq r \leq n_2 \) and \( r \) odd. Let us assume that \( V_{j_2}(1) = 0 \), with \( V_{j_2}(y) \neq 0 \), for \( \ell = r+1, r+2, \ldots, r+s \), with
r \leq r + s \leq n_2. Since V'_j(y) is a code polynomial of V_{n_1},

\[ w(v'(x)) \geq r \, d_{\text{lodd}} + s \, d_{\text{leven}}. \]

Since \( v(x) = v'(x) + 1 + x_{n_1+1} x_{n_1+2} \ldots x_{n_1+(n_2-1)n_1} \), then, for a given \( w(v'(x)) \), the minimum weight of \( v(x) \) is going to be achieved when \( w(v_0'(z)) \) is maximum. \( w(v_0'(z)) \) is maximum when for each \( V_j(y) \neq 0 \) we have \( a'_j(0,j) = 1 \).

Now the number of \( j \) such that \( V_j(y) \neq 0 \) is \( r+s \). Thus

\[ w(v(x)) \geq r(d_{\text{lodd}} - 1) + s(d_{\text{leven}} - 1) + n_2 - (r+s) =
\]

\[ = r(d_{\text{lodd}} - 2) + s(d_{\text{leven}} - 2) + n_2. \]

Since \( d_{\text{lodd}} \geq 3, d_{\text{leven}} \geq 2, r \geq d_{\text{2odd}} \) and \( s \geq 0 \), the minimum is achieved for \( r = d_{\text{2odd}} \) and \( s = 0 \).

Thus for this case we have shown that \( d_{\text{odd}} \geq n_2 \) and \( d_{\text{even}} \geq \min(n_2 + (d_{\text{lodd}} - 2) d_{\text{2odd}}, 2n_2) \). Now we will show the existence of \( v(x) \) with Hamming weights \( n_2, 2n_2 \) and \( n_2 + d_{\text{2odd}}(d_{\text{lodd}} - 2) \).

At first consider \( \hat{v}(x) = 1 + x_{n_1+1} + x_{n_1+2} \ldots x_{n_1+(n_2-1)n_1} \). Let us show that \( \hat{v}(x) \) is a code polynomial of \( V_n \) of weight \( n_2 \). Now

\[ \hat{v}(\beta(\theta, \phi)) = \frac{\phi_{n_1+1}}{\phi_{n_1+1}}. \]

Since \( \text{GCD}(n_1, n_2) = 1 \) and \( 0 \leq \phi < n_2 \),

\[ \phi_{n_1+1} = 0 \] if and only if \( \phi = 0 \). Thus, \( \hat{v}(\beta(\theta, \phi)) = 0 \) for \( \theta = 0, 1, \ldots, n_1-1 \) and \( \phi = 1, 2, \ldots, n_2-1 \) which implies that \( \hat{v}(x) \) is a code polynomial of \( V_n \). Now we consider the following polynomial of weight \( 2n_2 \): \( \hat{v}(x) = (1 + x_{n_1+1} + \ldots + x_{n_1+(n_2-1)n_1})(1+x) \). By a similar procedure we can show that \( \hat{v}(x) \) is a code polynomial of \( V_n \). To prove the existence of a code polynomial of weight \( n_2 + (d_{\text{lodd}} - 2) d_{\text{2odd}} \) we will show that if \( w(v_1(x)) = w_1 \), where \( v_1(x) \) is a nonzero code polynomial of \( V_{n_1} \), and \( w(v_2(x)) = w_2 \), where \( v_2(x) \) is a nonzero code
polynomial of $V_{n_2}$, then there is a code polynomial of $V_n$ with weight $n_2 + (w_1 - 2)w_2$. Let

$$v_1(x) = 1 + \sum_{i=1}^{w_1-1} x^i, \quad 0 < k_i < n_1$$

$$v_2(x) = 1 + \sum_{j=1}^{w_2-1} x^j, \quad 0 < k_j < n_2,$$

$M_1 = \{0, k_1, k_2, \ldots, k_{w_1-1}\}$ and $M_2 = \{0, \ell_1, \ell_2, \ldots, \ell_{w_2-1}\}$. Now we construct the following polynomial.

$$\hat{v}(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{\phi}(i,j)x^i + \sum_{k=0}^{\gamma^{n_1-1}} x^k$$

such that $a_{\phi}(i,j) = 1$ if $i \in M_1$ and $j \in M_2$ otherwise $a_{\phi}(i,j) = 0$. Hence $w(\hat{v}(x)) = n_2 + (w_1 - 2)w_2$. Now

$$\hat{v}(\beta^\rho(\theta, \phi)) = \tilde{v}(\beta^\rho(\theta, \phi)) + \frac{\gamma^{\phi n_1+1}}{\phi n_1! + 1}, \quad \text{where} \quad \tilde{v}(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{\phi}(i,j)x^i.$$  

As shown in the proof of Theorem 2 we can conclude that

$$\tilde{v}(\beta^\rho(\theta, \phi)) = 0 \text{ for } \theta \in S_1 \text{ and } \phi = 0, 1, \ldots, n_2 - 1; \quad \text{and for } \phi \in S_2 \text{ and } \theta = 0, 1, \ldots, n_1 - 1.$$  

We also know that $\frac{\gamma^{\phi n_1+1}}{\phi n_1! + 1} = 0$ for $\phi = 1, 2, \ldots, n_2 - 1$. Thus, since $0 \notin S_2$, we can conclude that $\hat{v}(\beta^\rho(\theta, \phi)) = 0$ for $\theta \in S_1$ and $\phi = 1, 2, \ldots, n_2 - 1$; and $\hat{v}(\beta^\rho(\theta, \phi)) = 0$ for $\phi \in S_2$ and $\theta = 0, 1, \ldots, n_1 - 1$. Thus $\hat{v}(x)$ is a code polynomial of $V_n$. We have shown for this case that $d_{\text{odd}} = n_2$ and $d_{\text{even}} = \min(2n_2, n_2 + (d_{1\text{odd}} - 2)d_{2\text{odd}})$. 
Case 3. In this case we consider the possibility of having $v(\beta^0(\theta,0)) = 0$ for $\theta \in S_{11}$ and $v(\beta^0(\theta,0)) \neq 0$ for $\theta \in S_{12}$, where $S_{11}$ and $S_{12}$ form a partition of $S_1$. By the same argument used in the analysis of the previous two cases we can conclude that $V_j(\alpha^\theta) = 0$ for $\theta \in S_{11}$ and $j = 0, 1, \ldots, n_2 - 1$ and $V_0(\alpha^\theta) = V_1(\alpha^\theta) = \ldots = V_{n_2 - 1}(\alpha^\theta) \neq 0$ for $\theta \in S_{12}$. Hence $w(v(x)) \geq 2n_2$.

Case 4. In this case we consider the possibility of having $v(\beta^0(\theta,0)) = 0$ for $\theta \in S_1$ and $v(\beta^0(0,\phi)) \neq 0$ for $\phi \in S_2$. As proved in Case 2 we can show that $d_{\text{odd}} = n_1$ and $d_{\text{even}} = \min(2n_1, n_1 + (d_{2\text{odd}} - 2)d_{1\text{odd}})$.

Case 5. In this case we consider the possibility of having $v(\beta^0(0,\phi)) = 0$ for $\phi \in S_{21}$ and $v(\beta^0(0,\phi)) \neq 0$ for $\phi \in S_{22}$, where $S_{21}$ and $S_{22}$ form a partition of $S_2$. As proved in Case 3 we can show that $w(v(x)) \geq 2n_1$.

Case 6. At last we consider the possibility of having $v(\beta^0(\theta,0)) \neq 0$ for $\theta \in S_1$ and $v(\beta^0(0,\phi)) \neq 0$ for $\phi \in S_2$. As proved in Case 2 we can show that $V_0(\alpha^\theta) = V_1(\alpha^\theta) = \ldots = V_{n_2 - 1}(\alpha^\theta) \neq 0$ \hspace{1cm} (2)

for $\theta \in S_1$, and $\bar{V}_0(\gamma^\phi) = \bar{V}_1(\gamma^\phi) = \ldots = \bar{V}_{n_2 - 1}(\gamma^\phi) \neq 0$ \hspace{1cm} (3)

for $\phi \in S_2$. Hence, $w(v(x)) \geq \max(n_1, n_2)$, which implies that $d_{\text{odd}} \geq \max(n_1, n_2)$ and $d_{\text{even}} \geq \max(n_1, n_2) + 1$. Now let us obtain a better bound for $d_{\text{even}}$, for this we assume that $v(x)$ is a code polynomial of even weight such that there exists at least one $j$,
0 \leq j < n_2$, satisfying $w(V_j(y)) = 1$; and also there exists at least one $i$, $0 \leq i < n_1$, satisfying $w(V_i(z)) = 1$. Because of the cyclic property we can, without loss of generality assume that $V_0(y) = 1$.

Thus, by Equation 2
\[
V_0(\alpha^\theta) = V_1(\alpha^\theta) = \ldots = V_{n_2-1}(\alpha^\theta) = 1
\] (4)
for $\theta \in S_1$ and by Equation 3
\[
\tilde{V}_0(\gamma^\phi) = \tilde{V}_1(\gamma^\phi) = \ldots = \tilde{V}_{n_1-1}(\gamma^\phi) = \gamma^{j_2}, 0 \leq j_2 < n_2,
\] (5)
for $\phi \in S_2$.

Now, based on the code polynomial $v(x)$ we construct the following polynomial
\[
v'(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a'_\rho(i,j)x^\rho(i,j)
\]
where $a'_\rho(0,j) = a_\rho(0,j) + 1$ for $j = 0,1,\ldots,j_2-1, j_2+1,\ldots,n_2-1$;
$a'_\rho(i,j_2) = a_\rho(i,j_2) + 1$ for $i = 1,2,\ldots,n_1-1$; $a'_\rho(0,j_2) = a_\rho(0,j_2)$ and $a'_\rho(i,j) = a_\rho(i,j)$ for $i = 1,2,\ldots,n_1-1$ and $j = 0,1,\ldots,j_2-1, j_2+1,\ldots,n_2-1$. Associated with the polynomial $v'(x)$, polynomials $v'(y,z)$, $V'_j(y)$ and $\tilde{V}'_i(z)$ are defined as follows:
\[
v'(y,z) = \sum_{j=0}^{n_2-1} v'_j(y)z^j = \sum_{i=0}^{n_1-1} \tilde{V}'_i(z)y^i
\]

Thus,
\[
v'(\beta^\rho(\theta,\phi)) = v'(\alpha^\theta, \gamma^\phi)
\]
and
\[ V'(y, z) = \sum_{j=0}^{n-1} (V_j(y) + 1)z^j + (V_{j_2}(y) + y + y^2 + \ldots + y^{n_1-1})z^{j_2} \]

For \( j \neq j_2 \),
\[ n_1-1 \]
\[ = \sum_{i=1}^{n_2-1} (V_1(z) + z^{j_2})y^i + (V_0(z) + 1 + z + \ldots + z^{j_2-1} + z^{j_2+1} + \ldots + z^{n_2-1}) \]

Hence,
\[ V' (\beta^0 (\theta, \phi)) = \sum_{j=0}^{n_2-1} (V_j (\alpha^0) + 1) y^j \phi (V_{j_2} (\alpha^0) + \alpha^0 + \alpha^2 \theta + \ldots + \alpha^{(n_1-1)\theta}) \phi^{j_2} \]

\( j \neq j_2 \)

But for \( \theta \neq 0 \), \( \alpha^0 + \alpha^2 \theta + \ldots + \alpha^{(n_1-1)\theta} = 1 \), which implies
\[ V' (\beta^0 (\theta, \phi)) = \sum_{j=0}^{n_2-1} (V_j (\alpha^0) + 1) y^j \phi \]

for \( \theta \neq 0 \). Since \( 0 \notin S_1 \) because \( g_1 (1) \neq 0 \), we conclude, by Equation 4, that \( V' (\beta^0 (\theta, \phi)) = 0 \) for \( \theta \in S_1 \) and \( \phi = 0, 1, \ldots, n_2-1 \). We also know that
\[ V' (\beta^0 (\theta, \phi)) = \sum_{i=1}^{n_1-1} (\bar{V}_1 (\gamma^i) + \gamma^{j_2}) \alpha^i \theta + \gamma^{(j_2-1)\phi} (j_2+1) + \gamma^{(j_2+1)\phi} \]

Thus, by a similar procedure we can show that \( V' (\beta^0 (\theta, \phi)) = 0 \) for \( \phi \in S_2 \) and \( \theta = 0, 1, \ldots, n_1-1 \). So, by \([13]\) \( \nu'(x) \) is a code polynomial of the binary product code of \( V_{n_1} \) and \( V_{n_2} \). \( \nu'(x) \) has even weight because \( \nu(x) \) has even weight and \( \nu(x) = \nu'(x) + \sum_{j=0}^{n_2-1} x^\rho (0, j) + \sum_{i=1}^{n_1-1} x^\rho (i, j_2) \) . If \( \nu'(x) \equiv 0 \), then \( w(\nu(x)) = n_1 + n_2 - 2 \).
If \( v'(x) \neq 0 \), then we must consider two cases: \( a'_p(0,j_2) = 0 \) and \( a'_p(0,j_2) = 1. \) At first let us assume \( a'_p(0,j_2) = 0. \) Now if

\[
\omega(\nu_0(z)) = \omega(\nu_{j_2}^i(y)) = 0, \text{ then } \omega(v(x)) \geq n_1 + n_2 - 2 + d_1 d_2 > n_1 + n_2 - 2.
\]

If \( \omega(\nu_0(z)) = 0 \) and \( \omega(\nu_{j_2}^i(y)) = w_1, \) then there exists \( i_k, \)

\[0 < i_k < n_1, \text{ } k = 1, 2, \ldots, w_1, \text{ such that } \omega(\nu_{i_k}^i(z)) \geq d_2. \text{ Thus,} \]

\[
\omega(v(x)) \geq n_2 - 1 + w_1(d_2 - 1) + n_1 - 1 - w_1 \geq n_1 + n_2 - 2 + d_1(d_2 - 2) \geq n_1 + n_2 - 2.
\]

Similarly, if \( \omega(\nu_0(z)) = w_2 \) and \( \omega(\nu_{j_2}^i(y)) = 0, \) then

\[
\omega(v(x)) \geq n_1 + n_2 - 2 + d_2(d_1 - 2) \geq n_1 + n_2 - 2. \text{ If } \omega(\nu_0(z)) = w_2 \text{ and} \]

\[
\omega(v_{j_2}^i(y)) = w_1, \text{ then there exists } i_k, \text{ } 0 < i_k < n_1, \text{ } k = 1, 2, \ldots, w_1, \text{ such that } \omega(\nu_{i_k}^i(z)) \geq d_2 \text{ and also there exists j_\ell,} \]

\[0 < j_\ell < n_2, \text{ } \ell = 1, 3, 4, \ldots, w_2 + 1, \text{ such that } \omega(\nu_{j_\ell}^i(y)) \geq d_1. \]

Thus, \( \omega(v'(x)) \geq w_1 d_2 + w_2 + (w_2 - (d_2 - 1))(d_1 - 1) = w_1 d_2 + w_2 d_1 - (d_2 - 1)(d_1 - 1). \)

Hence \( \omega(v(x)) \geq n_1 + n_2 - 2 + w_1(d_2 - 2) + w_2(d_1 - 2) - (d_2 - 1)(d_1 - 1) \geq n_1 + n_2 - 2 + d_1(d_2 - 2) - (d_2 - 1)(d_1 - 1) = n_1 + n_2 - 2 + (d_2 - 1)(d_1 - 1) - 2 \geq n_1 + n_2 - 2, (d_1 d_2 > 4). \) At last we assume \( a'_p(0,j_2) = 1. \) Thus, we have only to inspect the case \( \omega(\nu_0(z)) = w_2 \) and \( \omega(\nu_{j_2}^i(y)) = w_1. \) So,

\[
\omega(v'(x)) \geq w_1 d_2 + (w_2 - d_2)d_1 = w_1 d_2 + w_2 d_1 - d_2 d_1. \text{ Hence, } \omega(v(x)) \geq n_1 + n_2 - 2 + w_1(d_2 - 2) + w_2(d_1 - 2) - d_2 d_1 + 4 \geq n_1 + n_2 - 2 + d_1(d_2 - 2) + d_2(d_1 - 2) - d_2 d_1 + 4 = n_1 + n_2 - 2 + (d_1 - 2)(d_2 - 2) \geq n_1 + n_2 - 2. \text{ Since } n_1 + n_2 - 2 \geq \min(2n_1, 2n_2), \text{ we can conclude that for Case 6 } d_{\text{even}} \geq \min(2n_1, 2n_2). \]

This completes the proof of Theorem 4.

Q.E.D.

Example 4: As an application of Theorem 4 let us consider the
(119,47) binary quasi-product code generated by
\[ g(x) = m_1(x)m_{11}(x)m_{13}(x). \]
In this case \( n_1 = 7, n_2 = 17, S_1 = \{1,2,4\} \)
and \( S_2 = \{1,2,4,8,9,13,15,16\} \). Hence, \( V_{n_1} \) is the (7,4) binary
code with \( d_{\text{odd}} = 3 \) and \( d_{\text{even}} = 4 \); and \( V_{n_2} \) is the (17,9) binary
code with \( d_{\text{odd}} = 5 \) and \( d_{\text{even}} = 6 \). Thus by Theorem 4,
\[ d_{\text{odd}} = 7 \] and \( d_{\text{even}} = 14 \). The BCH bound gives \( d_{\text{odd}} \geq 7 \) and
\( d_{\text{even}} \geq 10 \).

Now we will investigate the weight structure of another class
of binary cyclic codes which will be called the class of binary
cyclic semi-quasi-product codes. These codes are defined in the
following manner: consider the binary cyclic product code of \( V_{n_1} \),
with \( g_1(1) \neq 0 \), and \( V_{n_2} \), with \( d_{\text{even}} \geq 2 \), generated by \( g(x) \). Let
\[ \overline{g}(x) = \text{GCD}(g(x),(x+1)^n). \]
The binary cyclic code of length \( n \)
generated by \( g'(x) = (g(x)/\overline{g}(x)) \) is defined to be the binary cyclic
semi-quasi-product code of \( V_{n_1} \) and \( V_{n_2} \). That is, if
\[ S_1 = \{ \theta | g(\beta^\theta(\theta,\phi)) = 0 \text{ for } \phi = 0,1,\ldots,n_2-1 \} \] and \( S_2 = \{ \phi | g(\beta^\phi(\theta,\phi)) = 0 \text{ for } \theta \in S_1 \}
\] and \( \phi = 0,1,\ldots,n_2-1 \) and \( g'(\beta^\phi(\theta,\phi)) = 0 \text{ for } \phi \in S_2 \) and \( \theta = 1,2,\ldots,n_1-1 \).

We are now in the position to prove the following theorem:

**Theorem 5:** Let \( V_n \) be the binary cyclic semi-quasi-product code of
\( \{ V_{n_1} \} \) and \( V_{n_2} \). Then
\[ d_{\text{odd}} = \min(n_1, d_{\text{odd}}, d_{\text{odd}}') \]
and
\[ d_{\text{even}} = \min(2n_1, d_{\text{even}}, d_{\text{even}}', n_1 + (d_{\text{odd}}'-2)d_{\text{odd}}) \]
Proof: Let
\[ v(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a^\rho(i,j)x^\rho(i,j) \]
be a nonzero code polynomial of \( V_n \). Since \( v(\beta^\rho(0,\phi)) \), \( \phi \in S_2 \), can be zero or nonzero, we must inspect 3 cases.

Case 1. In this case we consider the possibility of having \( v(\beta^\rho(0,\phi)) = 0 \) for \( \phi \in S_2 \). As proved in Case 1 of Theorem 4 we can conclude that
\[ d_{\text{odd}} = d_{1\text{odd}} d_{2\text{odd}} \text{ and } d_{\text{even}} = \min(d_{\text{1even}} d_2, d_1 d_{2\text{even}}). \]

Case 2. In this case we consider the possibility of having \( v(\beta^\rho(0,\phi)) = 0 \) for \( \phi \in S_{21} \) and \( v(\beta^\rho(0,\phi)) \neq 0 \) for \( \phi \in S_{22} \), where \( S_{21} \) and \( S_{22} \) are a partition of \( S_2 \). As proved in Case 3 of Theorem 4 we can conclude that \( w(v(x)) \geq 2n_1 \).

Case 3. In this case we consider the possibility of having \( v(\beta^\rho(0,\phi)) \neq 0 \) for \( \phi \in S_2 \). As proved in Case 2 of Theorem 4 we can conclude that \( d_{\text{odd}} = n_1 \) and \( d_{\text{even}} = \min(2n_1 n_1 + (d_{2\text{odd}} - 2) d_{1\text{odd}}) \).

Q.E.D.

Let \( V_n \) be the binary cyclic semi-quasi-product code of \( V_{n_1} \) and \( V_{n_2} \). If \( V_{n_1} \) is the \( (n_1,n_1) \) binary cyclic code we will call \( V_n \) the one-dimensional quasi-product code of \( V_{n_2} \). The minimum distance of this class of codes is specified by the following corollary.

Corollary 1: Let \( V_n \) be the one-dimensional quasi-product code of \( V_{n_2} \). Then
\[ d_{\text{odd}} = \min(n_1, d_{2\text{odd}}). \]
and
\[ d_{even} = \min(2n_1, d_{2even}, n_1 + d_{2odd} - 2). \]

**Example 5:** As an application of Theorem 5 let us consider the 
(119, 39) binary semi-quasi-product code generated by 
\[ g(x) = m_1(x)m_{11}(x)m_{13}(x)m_{21}(x). \]
In this case 
\[ n_1 = 17, n_2 = 7, S_1 = \{1, 2, 4, 8, 9, 13, 15, 16\} \text{ and } S_2 = \{1, 2, 4\}. \]
Hence, 
\[ V_{n_1} \] is the (17, 9) binary cyclic code, with \( d_{1odd} = 5 \) and 
\[ d_{1even} = 6; \] and \( V_{n_2} \) is the (7, 4) binary cyclic code, with 
\[ d_{2odd} = 3 \text{ and } d_{2even} = 4. \]

Thus by Theorem 5, \( d_{odd} = 15 \) and 
\( d_{even} = 18. \) The BCH bound gives \( d_{odd} \geq 13 \) and \( d_{even} \geq 14. \)

At last we will derive a lower bound on the minimum distance of a subcode of a one-dimensional quasi-product code of \( V_{n_2}. \) Let 
\( V_{n_1} \) be a subcode of the one-dimensional quasi-product code of \( V_{n_2}, \) 
generated by \( g(x) \) such that \( g(1) \neq 0. \) Let 
\[ \mathcal{J}_0 = \{ \theta | g(\rho(\theta, 0)) = 0 \}, \]
\[ J_0 = \{ \phi | g(\rho(0, \phi)) = 0 \}, \]
\[ S_2 = \{ \phi | g(\rho(\theta, \phi)) = 0 \text{ for } \theta = 1, 2, \ldots, n_1 - 1 \}, \]
\[ N_2 = S_2 \cap J_0 \] and 
\[ P_2 = S_2 \cup J_0. \]
Thus, by the definition of \( V_{n_1}, V_{n_2} \) is generated by 
\[ g_2(x) = \prod_{\phi \in S_2} (x + v_\phi). \]
We define \( V_{n_1} \) to be the binary cyclic code 
of length \( n_1 \) generated by \( g_1(x) = \prod_{\theta \in \mathcal{J}_0} (x + v_\theta)^*; \)
\( V_{n_2}^{(0)} \) to be the binary cyclic code of length \( n_2 \) generated by 
\[ g_2^{(0)}(x) = \prod_{\phi \in J_0} (x + v_\phi)^*. \]

*If \( J_0 \) is empty, then \( g_1(x) = 1. \)

**If \( J_0 \) is empty, then \( g_2^{(0)}(x) = 1. \)
\( V'_n \) to be the binary cyclic code of length \( n_2 \) generated by
\[
g'_2(x) = \prod_{\phi \in N_2} (x+y^\phi),
\]
* and \( V''_n \) to be the binary cyclic code of length \( n_2 \) generated by \( g''_2(x) = \prod_{\phi \in P_2} (x+y^\phi) \). \( d^{(0)}_{2\text{odd}} \) is defined as before.

At last we let \( d'^{\text{odd}}_2 \) (\( d'^{\text{even}}_2 \)), \( d''^{\text{even}}_2 \) (\( d''^{\text{even}}_2 \)) be the minimum weight of odd-weight, even-weight code vectors of \( V'_n \) (\( V''_n \)), respectively.

Now we are in the position to prove the following theorem.

**Theorem 6**: Let \( \overline{d}_{\text{odd}} = \max(d^{\text{odd}}_1, d^{\text{odd}}_2, d'^{\text{odd}}_2) \) and
\[
\overline{d}_{\text{even}} = \min(d''^{\text{even}}_2, 2d^{\text{even}}_2, d^{\text{even}}_1 d^{\text{odd}}_2). \]
If \( d^{\text{odd}}_2 > d^{\text{even}}_2 \) then \( d^{\text{odd}}_2 \geq \min(\overline{d}_{\text{odd}}, \max(n_1 d^{\text{even}}_2 + (d^{\text{even}}_2 - d^{\text{odd}}_2) d^{(0)}_{\text{odd}})) \)
and \( d^{\text{even}}_2 \geq \min(\overline{d}_{\text{even}}, n_1 d^{\text{even}}_2) \). If \( d^{\text{odd}}_2 < d^{\text{even}}_2 \), then
\[
d^{\text{odd}}_2 \geq \min(\overline{d}_{\text{odd}}, \max(n_1 d^{\text{odd}}_2, d^{(0)}_{\text{odd}})); d^{\text{even}}_2 \geq \min(\overline{d}_{\text{even}}, n_1 d^{\text{odd}}_2 + (d^{\text{even}}_2 - d^{\text{odd}}_2) d^{(0)}_{\text{odd}}) \text{ for } N_2 \text{ nonempty}
\]
and \( d^{\text{even}}_2 \geq \min(\overline{d}_{\text{even}}, 2n_1, n_1 + (d^{\text{odd}}_2 - 2) d^{(0)}_{\text{odd}}) \text{ for } N_2 \text{ empty.} \)

**Proof**: Let \( v(x) \) be a nonzero code polynomial of \( V_n \). Thus
\[
v(\beta^\phi(\theta, \phi)) = \sum_{i=0}^{n_1-1} V_i(\gamma^\phi) a^i = 0
\]
for \( \phi \in S_2 \) and \( \theta = 1, 2, \ldots, n_1-1 \). Since \( v(\beta^\phi(0, \phi)) \), \( \phi \in S_2 \), can be zero or nonzero we must inspect the following cases:

**Case 1.** In this case we consider the possibility of having \( v(\beta^\phi(0, \phi)) = 0 \) for \( \phi \in S_2 \). As proved in Theorem 3 we can conclude that \( d^{\text{odd}} \geq \max(d^{\text{odd}}_1, d^{\text{odd}}_2, d'^{\text{odd}}_2) \) and \( d^{\text{even}} \geq \min(d''^{\text{even}}_2, 2d^{\text{even}}_2, d^{\text{even}}_1 d^{\text{odd}}_2) \).

*If \( N_2 \) is empty, then \( g'_2(x) = 1 \).*
Case 2. In this case we consider the possibility of having 
\( v(\beta^0(y, \phi)) \neq 0 \) for \( \phi \in S_2' \), where \( S_2' = S_2 - N_2 \). Let 
\[
 v_1(\phi)(y) = \sum_{i=0}^{n_1-1} \overline{v}_i(\gamma^\phi)y^i.
\]

Thus, \( v_1(\phi)(y) = 0 \) for \( \phi \in S_2 \) and \( \theta = 1, 2, \ldots, n_1-1 \). Hence 
\( \overline{v}_0(\gamma^\phi) = \overline{v}_1(\gamma^\phi) = \ldots = \overline{v}_{n_1-1}(\gamma^\phi) \neq 0 \) for \( \phi \in S_2' \) and by Lemma 2 
\( \overline{v}_i(\gamma^\phi) = 0 \) for \( \phi \in N_2 \), that is, \( \overline{v}_i(z) \) is a nonzero code polynomial 
of \( V_1 \), \( i = 0, 1, \ldots, n_1-1 \). We notice that if \( w(v(x)) \) is odd, then 
\( v(0)(y) \) has odd weight and if \( w(v(x)) \) is even, then \( w(v(0)(y)) \) is 
even. Thus, if \( \overline{v}_0(z) = \overline{v}_1(z) = \ldots = \overline{v}_{n_1-1}(z) \), then 
\[
 d_{\text{odd}} \geq n_1d_{\text{2odd}} \quad \text{and} \quad d_{\text{even}} \geq n_1d_{\text{2even}}. \]

If not all \( \overline{v}_i(z) \), 
i = 0, 1, \ldots, n_1-1, are equal, then 
\[
 d_{\text{odd}} \geq w_{\text{1odd}}d_{\text{2odd}} + (n_1-w_{\text{1odd}}) \times d_{\text{2even}} \quad \text{and} \quad d_{\text{even}} \geq w_{\text{1even}}d_{\text{2odd}} + (n_1-w_{\text{1even}}) d_{\text{2even}}, \]

where \( w_{\text{1odd}}, w_{\text{1even}} \) is the weight of an odd-weight, even-weight code word 
of \( V_1 \), respectively. Hence, 
\[
 d_{\text{odd}} \geq n_1d_{\text{2odd}} + (d_{\text{2odd}} - d_{\text{2even}}) \times w_{\text{1odd}}. \]

If \( d_{\text{2odd}} > d_{\text{2even}} \), then 
\[
 d_{\text{odd}} \geq n_1d_{\text{2even}} + (d_{\text{2odd}} - d_{\text{2even}}) \times d_{\text{1odd}}. \]

If \( d_{\text{2odd}} < d_{\text{2even}} \), then 
\[
 d_{\text{odd}} \geq n_1d_{\text{2odd}}. \]

For the even-weight code polynomials we obtain 
\[
 d_{\text{even}} \geq n_1d_{\text{2even}} + (d_{\text{2even}} - d_{\text{2odd}}) \times (n_1-d_{\text{1odd}}) = n_1d_{\text{2odd}} + (d_{\text{2even}} - d_{\text{2odd}}) d_{\text{1odd}}. \]

For this case we can conclude that if \( d_{\text{2odd}} > d_{\text{2even}} \), then 
\[
 d_{\text{odd}} \geq \max(n_1d_{\text{2even}} + (d_{\text{2odd}} - d_{\text{2even}}) d_{\text{1odd}}, d_{\text{2odd}}) \quad \text{and} \quad d_{\text{even}} \geq n_1d_{\text{2even}}. \]

Now if \( d_{\text{2odd}} < d_{\text{2even}} \), then 
\[
 d_{\text{odd}} \geq \max(n_1d_{\text{2odd}}, d_{\text{2odd}}) \quad \text{and} \quad d_{\text{even}} \geq n_1d_{\text{2odd}} + (d_{\text{2even}} - d_{\text{2odd}}) d_{\text{1odd}}. \]

These bounds are
valid for $N_2$ empty or $N_2$ nonempty. However when $N_2$ is empty we can obtain a better bound for $d_{\text{even}}$ as follows: assume we have some $i$, $0 \leq i < n_1$, such that $w(\tilde{V}_i(z)) = 1$, without loss of generality we assume $\tilde{V}_0(z) = 1$. Thus, $\tilde{V}_i(z) + 1$ is a code polynomial of $V_{n_2}$, $i = 0, 1, \ldots, n_1 - 1$. Since $w(v(x)) > n_1$, there exists at least one $i$, $0 < i < n_1$, such that $w(V_i(z) + 1) \neq 0$. Remembering that $V_i^{(0)}(y) \neq 0$ and has even weight we can conclude that

$$w(v(x)) \geq w_{\text{even}} + (n_1 - w_{\text{even}})(d_{2\text{odd}} - 1) =$$

$$= n_1(d_{2\text{odd}} - 1) - w_{\text{even}}(d_{2\text{odd}} - 2) \geq n_1(d_{2\text{odd}} - 1) - (n_1 - d_{\text{odd}})(d_{2\text{odd}} - 2) =$$

$$= n_1 + (d_{2\text{odd}} - 2)d_{\text{odd}}. \text{ Thus, } d_{\text{even}} \geq \min(2n_1, n_1 + (d_{2\text{odd}} - 2)d_{\text{odd}}).$$

**Case 3.** In this case we consider the possibility of having $v(\beta(0, \phi)) \neq 0$ for $\phi \in S'_{21}$ and $v(\beta(0, \phi)) = 0$ for $\phi \in S'_{22}$, where $S'_{21}$ and $S'_{22}$ are a partition of $S_2'$. In this case

$$\tilde{V}_0(y^\phi) = \tilde{V}_1(y^\phi) = \ldots = \tilde{V}_{n_1 - 1}(y^\phi) \neq 0 \text{ for } \phi \in S'_{21} \text{ and}$$

$$\tilde{V}_i(y^\phi) = 0 \text{ for } \phi \in N_2 \cup S'_{22}. \text{ Thus, } w(v(x)) \text{ is lower bounded by the bounds found in the analyses of the last case.} \quad \text{Q.E.D.}$$

**Example 6:** As an application of Theorem 6 we consider the (105, 46) binary cyclic code generated by

$$g(x) = m_1(x)m_3(x)m_7(x)m_9(x)m_{15}(x)m_{17}(x)m_{49}(x).$$

In this case

$$n_1 = 7, \; n_2 = 15, \; \tilde{J}_0 = \{1, 2, 4\}, \; J_0 = \{1, 2, 4, 7, 8, 11, 13, 14\},$$

$$S_2 = \{1, 2, 3, 4, 6, 8, 9, 12\}, \; N_2 = \{1, 2, 4, 8\},$$

$$P_2 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14\}. \text{ Thus}$$

$$d_{2\text{odd}} = 5, \; d_{2\text{even}} = 6, \; d_{2\text{odd}}^{(0)} = 3, \; d_{2\text{even}}^{(0)} = 6, \; d_{2\text{odd}}' = 3, \; d_{\text{even}}' = 4,$$

$$d_{2\text{odd}}'' = 5 \text{ and } d_{2\text{even}}''' = 10. \text{ By Theorem 6, } d_{\text{odd}} \geq 15 \text{ and } d_{\text{even}} \geq 10.$$
The BCH bound gives $d_{\text{odd}} \geq 7$ and $d_{\text{even}} \geq 8$.

In the next section we present numerical results obtained from the application of the theorems proved in this section.
IV. Numerical Results

In Table I we give numerical results obtained from the application of Theorem 1, Theorem 3 and Theorem 6. Numerical results obtained from the application of Theorem 2, Theorem 4, Theorem 5 and Corollary 1 are given in Table II. The symbols for the tables are the following:

- \(n\) = code length
- \(k\) = number of information digits
- \(\text{roots}\) = the powers of \(\beta\) that specify the generator polynomial
- \(d_{0\text{odd}}\) = BCH lower bound on the minimum distance of odd-weight code words
- \(d_{0\text{even}}\) = BCH lower bound on the minimum distance of even-weight code words
- \(d_{\text{odd}}\) = actual minimum weight of odd-weight code words
- \(d_{\text{even}}\) = actual minimum weight of even-weight code words
- \(T - a\) = by Theorem a
- \(C - a\) = by Corollary a
Table I

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<th>d_{0odd}</th>
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V. Conclusions

By exploiting the minimum distance relationship between codes of related lengths, the actual minimum distances of some classes of binary cyclic codes of composite length has been obtained. For other classes we were able to obtain new lower bounds on the minimum distance. These new lower bounds are useful in obtaining better estimates on the minimum distance of many new cyclic codes. The simplicity of the application of the theorems is apparent from the examples. In the examples of Table II the BCH bound gives a good estimate on the minimum distance.
References


The assistance of Mrs. Ruth Turnpaugh in the typing of the manuscript is gratefully acknowledged.