Extension of Plurisubharmonic Functions with Growth Control

Dan Coman
Syracuse University

Vincent Guedj
Université Aix-Marseille 1

Ahmed Zeriahi
Université Paul Sabatier,

Follow this and additional works at: https://surface.syr.edu/mat

Part of the Mathematics Commons

Recommended Citation
Coman, Dan; Guedj, Vincent; and Zeriahi, Ahmed, "Extension of Plurisubharmonic Functions with Growth Control" (2010). Mathematics - Faculty Scholarship. 22.
https://surface.syr.edu/mat/22

This Article is brought to you for free and open access by the Mathematics at SURFACE. It has been accepted for inclusion in Mathematics - Faculty Scholarship by an authorized administrator of SURFACE. For more information, please contact surface@syr.edu.
EXTENSION OF PLURISUBHARMONIC FUNCTIONS WITH GROWTH CONTROL

DAN COMAN, VINCENT GUEDJ AND AHMED ZERIAHI

Abstract. Suppose that $X$ is an analytic subvariety of a Stein manifold $M$ and that $\varphi$ is a plurisubharmonic (psh) function on $X$ which is dominated by a continuous psh exhaustion function $u$ of $M$. Given any number $c > 1$, we show that $\varphi$ admits a psh extension to $M$ which is dominated by $cu$ on $M$.

We use this result to prove that any $\omega$-psh function on a subvariety of the complex projective space is the restriction of a global $\omega$-psh function, where $\omega$ is the Fubini-Study Kähler form.

Introduction

Let $X \subset \mathbb{C}^n$ be a (closed) analytic subvariety. In the case when $X$ is smooth it is well known that a plurisubharmonic (psh) function on $X$ extends to a psh function on $\mathbb{C}^n$ [Sa] (see also [BL, Theorem 3.2]). Using different methods, Coltoiu generalized this result to the case when $X$ is singular [Co, Proposition 2].

In this article we follow Coltoiu’s approach and show that it is possible to obtain extensions with global growth control:

Theorem A. Let $X$ be an analytic subvariety of a Stein manifold $M$ and let $\varphi$ be a psh function on $X$. Assume that $u$ is a continuous psh exhaustion function on $M$ so that $\varphi(z) < u(z)$ for all $z \in X$. Then for every $c > 1$ there exists a psh function $\psi = \psi_e$ on $M$ so that $\psi|_X = \varphi$ and $\psi(z) < c \max\{u(z), 0\}$ for all $z \in M$.

We recall that a function $\varphi : X \to (-\infty, +\infty)$ is called psh if $\varphi \not\equiv -\infty$ on $X$ and if every point $z \in X$ has a neighborhood $U$ in $\mathbb{C}^n$ so that $\varphi = u|_U$ for some psh function $u$ on $U$. We refer to [FN] and [D2, section 1] for a detailed discussion of this notion. We note here that if $\varphi$ is not identically $-\infty$ on an irreducible component $Y$ of $X$ then $\varphi$ is locally integrable on $Y$ with respect to the area measure of $Y$. Let us stress that the more general notion of weakly psh function is not appropriate for the extension problem (see section 3).

We then look at a similar problem on a compact Kähler manifold $V$. Here psh functions have to be replaced by quasip plurisubharmonic (qpsh) ones. Given a Kähler form $\omega$, we let

$$PSH(V, \omega) = \{\varphi \in L^1(V, [-\infty, +\infty)) : \varphi \text{ upper semicontinuous, } dd^c\varphi \geq -\omega\}$$

denote the set of $\omega$-plurisubharmonic ($\omega$-psh) functions. If $X \subset V$ is an analytic subvariety, we define similarly the class $PSH(X, \omega|_X)$ of $\omega$-psh functions on $X$ (see section 2 for precise definitions).

2000 Mathematics Subject Classification. Primary 32U05; Secondary: 32C25, 32Q15, 32Q28.
First author is supported by the NSF Grant DMS-0900934.
By restriction, \( \omega \)-psh functions on \( V \) yield \( \omega|_X \)-psh functions on \( X \). Assuming that \( \omega \) is a Hodge form, i.e. a Kähler form with integer cohomology class, our second result is that every \( \omega|_X \)-psh function on \( X \) arises in this way.

**Theorem B.** Let \( X \) be a subvariety of a projective manifold \( V \) equipped with a Hodge form \( \omega \). Then any \( \omega|_X \)-psh function on \( X \) is the restriction of an \( \omega \)-psh function on \( V \).

Note that in the assumptions of Theorem B there exists a positive holomorphic line bundle \( L \) on \( V \) whose first Chern class \( c_1(L) \) is represented by \( \omega \). In this case the \( \omega \)-psh functions are in one-to-one correspondence with the set of (singular) positive metrics of \( L \) (see [GZ]). Thus an alternate formulation of Theorem B is the following:

**Theorem B’.** Let \( X \) be a subvariety of a projective manifold \( V \) and \( L \) be an ample line bundle on \( V \). Then any (singular) positive metric of \( L|_X \) is the restriction of a (singular) positive metric of \( L \) on \( V \).

Recall that it is possible to regularize qpsh functions on \( \mathbb{P}^n \), since it is a homogeneous manifold. Hence Theorem B has the following immediate corollary:

**Corollary C.** Let \( X \) be a subvariety of a projective manifold \( V \) equipped with a Hodge form \( \omega \). If \( \varphi \in \text{PSH}(X, \omega|_X) \) then there exists a sequence of smooth functions \( \varphi_j \in \text{PSH}(V, \omega) \) which decrease pointwise on \( V \) so that \( \lim \varphi_j = \varphi \) on \( X \).

When \( X \) is smooth this regularization result is well known to hold even when the cohomology class of \( \omega \) is not integral (see [D3], [BK]).

Corollary C allows to show that the singular Kähler-Einstein currents constructed in [EGZ1] have continuous potentials, a result that has been obtained recently in [EGZ2] by completely different methods (see also [DZ] for partial results in this direction).

We prove Theorem A in section 1. The compact setting is considered in section 2, where Theorem B is derived from Theorem A. In section 3 we discuss the special situation when \( X \) is an algebraic subvariety of \( \mathbb{C}^n \). As an application of Theorem B, we give a characterization of those psh functions in the Lelong class \( \mathcal{L}(X) \) which admit an extension in the Lelong class \( \mathcal{L}(\mathbb{C}^n) \) (see section 3 for the necessary definitions). In particular, we give simple examples of algebraic curves \( X \subset \mathbb{C}^2 \) and of functions \( \eta \in \mathcal{L}(X) \) which do not have extensions in \( \mathcal{L}(\mathbb{C}^2) \).

1. **Proof of Theorem A**

The following proposition will allow us to reduce the proof of Theorem A to the case \( M = \mathbb{C}^n \). We include its short proof for the convenience of the reader.

**Proposition 1.1.** Let \( V \) be a complex submanifold of \( \mathbb{C}^N \) and \( u \) be a continuous psh exhaustion function on \( V \). Then there exists a continuous psh exhaustion function \( \tilde{u} \) on \( \mathbb{C}^N \) so that \( \tilde{u}|_V = u \).

**Proof.** The argument is very similar to the one of Sadullaev ([Sa], [BL, Theorem 3.2]). By [Si], there exists an open neighborhood \( W \) of \( V \) in \( \mathbb{C}^N \) and a holomorphic retraction \( r : W \to V \). We can find an open neighborhood \( U \) of \( V \) so that \( U \subset W \) and \( \| r(z) - z \| < 2 \) for every \( z \in U \). Indeed, if \( B(p,r) \) denotes the open ball in \( \mathbb{C}^N \) centered at \( p \) and of radius \( r \), then \( U_p = r^{-1}(B(p,1)) \cap B(p,1) \) is an open
neighborhood of \( p \in V \), and we let \( U = \bigcup_{p \in V} U_p \). Since \( u \) is a continuous psh exhaustion function on \( V \), it follows that the function \( u(r(z)) \) is continuous psh on \( U \) and \( \lim_{z \to U, \|z\| \to +\infty} u(r(z)) = +\infty \).

It is well known that there exist entire functions \( f_0, \ldots, f_N \), so that \( V = \{ z \in \mathbb{C}^N : f_k(z) = 0, 0 \leq k \leq N \} \) (see [Ch, p.63]). The function \( \rho = \log(\sum |f_k|^2) \) is psh on \( \mathbb{C}^N \) and \( V = \{ \rho = -\infty \} \).

Let \( D \) be an open set so that \( V \subset D \subset \overline{D} \subset U \). Since \( \rho \) is continuous on \( \mathbb{C}^N \setminus V \), we can find a convex increasing function \( \chi \) on \([0, +\infty)\) which verifies for every \( R \geq 0 \) the following two properties:

\[
\begin{align*}
(i) \quad & \chi(R) > R - \rho(z) \quad \text{for all } \|z\| = R, \\
(ii) \quad & \chi(R) > u(r(z)) - \rho(z) \quad \text{for all } \|z\| = R.
\end{align*}
\]

Then \( \tilde{u}(z) = \begin{cases} 
\max\{u(r(z)), \chi(\|z\|) + \rho(z)\}, & \text{if } z \in D, \\
\chi(\|z\|) + \rho(z), & \text{if } z \in \mathbb{C}^N \setminus D,
\end{cases} \)
is a continuous psh exhaustion function on \( \mathbb{C}^N \) and \( \tilde{u} = u \) on \( V \).

Employing the methods of Coltoiu [Co] we now construct psh extensions with growth control over bounded sets in \( \mathbb{C}^n \).

**Proposition 1.2.** Let \( \chi \) be a psh function on a subvariety \( X \subset \mathbb{C}^n \) and let \( v \) be a continuous psh function on \( \mathbb{C}^n \) with \( \chi < v \) on \( X \). If \( R > 0 \), there exists a psh function \( \tilde{\chi} = \tilde{\chi}_R \) on \( \mathbb{C}^n \) so that \( \tilde{\chi}|_X = \chi \) and \( \tilde{\chi}(z) < v(z) \) for all \( z \in \mathbb{C}^n \) with \( \|z\| \leq R \).

**Proof.** We use a similar argument to the one in the proof of Proposition 2 in [Co]. Consider the subvariety \( A = (X \times \mathbb{C}) \cup (\mathbb{C}^n \times \{0\}) \subset \mathbb{C}^{n+1} \), and let
\[
D = \{(z, w) \in X \times \mathbb{C} : \log|w| + \chi(z) < 0\} \cup (\mathbb{C}^n \times \{0\}) \subset A.
\]

Since \( D \cap (X \times \mathbb{C}) \) is Runge in \( X \times \mathbb{C} \), it follows that \( D \) is Runge in \( A \). Let
\[
K = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) = \max\{\log^+(\|z\|/R), \log|w| + v(z)\} \leq 0\}.
\]

Since \( v \) is continuous, \( \rho \) is a continuous psh exhaustion function on \( \mathbb{C}^{n+1} \), so \( K \) is a polynomially convex compact set. As \( \chi < v \) on \( X \), we have \( K \cap A \subset D \). By [Co, Theorem 3] there exists a Runge domain \( \tilde{D} \subset \mathbb{C}^{n+1} \), with \( \tilde{D} \cap A = D \) and \( K \subset \tilde{D} \). Let \( \delta(z, w) \) denote the distance from \((z, w) \in \tilde{D}\) to \( \partial \tilde{D} \) in the \( w \)-direction. Since \( \tilde{D} \) is pseudoconvex, \(-\log \delta \) is psh on \( \tilde{D} \) (see e.g. [FS, Proposition 9.2]). Hence \( \tilde{\chi}(z) = -\log \delta(z, 0) \) is psh on \( \mathbb{C}^n \), as \( \mathbb{C}^n \times \{0\} \subset \tilde{D} \). Since \( \tilde{D} \cap A = D \), it follows that \( \tilde{\chi}|_X = \chi \). Moreover, \( K \subset \tilde{D} \) implies that \( \tilde{\chi}(z) < v(z) \) for all \( z \in \mathbb{C}^n \) with \( \|z\| \leq R \).

The proof of Theorem A proceeds like this. Given a partition
\[
\mathbb{C}^n = \bigcup\{m_{j-1} < u \leq m_j\},
\]
where \( m_j \not\nearrow +\infty \), we apply Proposition 1.2 inductively to construct an extension dominated in each “annulus” \( \{m_{j-1} < u \leq m_j\} \) by \( \gamma_j u \), where \( \gamma_j > 1 \) is an increasing sequence defined in terms of the \( m_j \)'s. Theorem A will follow by showing that it is possible to choose \( \{m_j\} \) rapidly increasing so that \( \lim \gamma_j \) is arbitrarily close to 1.

We fix next an increasing sequence \( \{m_j\}_{j \geq -1} \) so that
\[
m_{-1} = m_0 = 0 < m_1 < m_2 < \ldots, \{u < m_1\} \neq \emptyset, m_j \not\nearrow +\infty.
\]
Define inductively a sequence \( \{\gamma_j\}_{j \geq 0} \), as follows:

\[
\gamma_0 = 1, \quad \gamma_j(m_j - m_{j-1}) = \gamma_{j-1}(m_j - m_{j-2}) + 1 \quad \text{for} \quad j \geq 1.
\]

Clearly, \( \gamma_j > \gamma_{j-1} > 1 \) for all \( j > 1 \).

**Proposition 1.3.** Let \( X, \varphi, u \) be as in Theorem A with \( M = \mathbb{C}^n \), and let \( \{m_j\} \), \( \{\gamma_j\} \) be as above. There exists a psh function \( \psi \) on \( \mathbb{C}^n \) so that \( \psi|_X = \varphi \) and for all \( z \in \mathbb{C}^n \) we have

\[
\psi(z) < \left\{ \begin{array}{ll}
\gamma_j u(z), & \text{if } m_{j-1} < u(z) \leq m_j, \ j \geq 2, \\
\gamma_1 \max\{u(z), 0\}, & \text{if } u(z) \leq m_1.
\end{array} \right.
\]

**Proof.** We introduce the sets

\[
D_j = \{z \in \mathbb{C}^n : u(z) < m_j\}, \quad K_j = \{z \in \mathbb{C}^n : u(z) \leq m_j\}.
\]

Since \( u \) is a continuous psh exhaustion function, \( K_j \) is a compact set. Let

\[
\rho_j = \gamma_j \max\{u - m_{j-1}, 0\} - j, \ j \geq 0.
\]

Then \( \rho_j \) is psh on \( \mathbb{C}^n \) and (1) implies that

\[
(2) \quad \rho_j(z) = \rho_{j-1}(z) \text{ if } u(z) = m_j, \ j \geq 1.
\]

We claim that

\[
(3) \quad \rho_j(z) \geq u(z) \text{ if } z \in \mathbb{C}^n \setminus D_j, \ j \geq 0.
\]

Indeed, since \( \gamma_j \geq 1 \) and using (1) we obtain

\[
\rho_j(z) - u(z) = (\gamma_j - 1)u(z) - \gamma_j m_{j-1} - j \geq (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j
\]

\[
= (\gamma_{j-1} - 1)m_j - \gamma_{j-1} m_{j-2} - j + 1
\]

\[
\geq (\gamma_{j-1} - 1)m_{j-1} - \gamma_{j-1} m_{j-2} - (j - 1).
\]

So \( x_j := (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j \geq x_0 = 0 \), and (3) is proved.

Let \( \varphi_j = \max\{\varphi, -j\} \). We construct by induction on \( j \geq 1 \) a sequence of continuous psh functions \( \psi_j \) on \( \mathbb{C}^n \) with the following properties:

\[
(4) \quad \psi_j(z) > \varphi_j(z) \text{ for } z \in X, \quad \int_{X \cap K_{j-1}} (\psi_j - \varphi_j) < 2^{-j}.
\]

\[
(5) \quad \psi_j(z) \geq \rho_j(z) \text{ for } z \in D_j, \quad \psi_j(z) = \rho_j(z) \text{ for } z \in \mathbb{C}^n \setminus D_j.
\]

\[
(6) \quad \psi_j(z) < \psi_{j-1}(z) \text{ for } z \in K_{j-1}, \text{ where } \psi_0 = \rho_0 = \max\{u, 0\}.
\]

Here the integral in (4) is with respect to the area measure on each irreducible component, i.e.

\[
\int_{X \cap K} f := \sum \int_{Y \cap K} f \beta^{\text{dim}Y},
\]

where the sum is over all irreducible components \( Y \) of \( X \) which intersect \( K \) and \( \beta \) is the standard Kähler form on \( \mathbb{C}^n \). (Note that this is a finite sum.)

Assume that the function \( \psi_{j-1} \) is constructed with the desired properties. We construct \( \psi_j \) by applying Proposition 1.2 with \( \chi = \varphi_j \) and \( v = \psi_{j-1} \). (If \( j = 1 \), \( \psi_1 \) is constructed in the same way by applying Proposition 1.2 with \( \chi = \varphi_1 \) and \( v = \psi_0 \).) By (4), \( \varphi_j \leq \varphi_{j-1} < \psi_{j-1} \) on \( X \) (and for \( j = 1 \), clearly \( \varphi_1 < \psi_0 \) on \( X \)). Therefore Proposition 1.2 yields a psh function \( \tilde{\varphi}_j \) on \( \mathbb{C}^n \) so that \( \tilde{\varphi}_j|_X = \varphi_j \) and \( \tilde{\varphi}_j < \psi_{j-1} \) on \( K_j \). Using the standard regularization of \( \tilde{\varphi}_j \) and the dominated
convergence theorem (as \( \varphi_j \geq -j \)) we obtain a continuous psh function \( \tilde{\psi}_j \) on \( \mathbb{C}^n \) which verifies

\[
\tilde{\psi}_j(z) > \varphi_j(z) \quad \text{for} \quad z \in X, \quad \int_{X \cap K_j} (\tilde{\psi}_j - \varphi_j) < 2^{-j}.
\]

Moreover, since \( \psi_{j-1} \) is continuous, we can ensure by the Hartogs lemma that we also have \( \tilde{\psi}_j(z) < \psi_{j-1}(z) \) for \( z \in K_j \).

We now define

\[
\psi_j(z) = \begin{cases} 
\max\{\tilde{\psi}_j(z), \rho_j(z)\}, & \text{if} \ z \in D_j, \\
\rho_j(z), & \text{if} \ z \in \mathbb{C}^n \setminus D_j.
\end{cases}
\]

By (5) and (2) we have \( \tilde{\psi}_j < \psi_{j-1} = \rho_{j-1} = \rho_j \) on \( \partial D_j \) (for \( j = 1 \), recall that \( \psi_0 = \rho_0 \) by definition). So \( \psi_j \) is a continuous psh function on \( \mathbb{C}^n \) which verifies (5).

On \( X \setminus D_j \) we have by (3) that \( \psi_j = \rho_j \geq u > \varphi_j \), while on \( X \cap D_j \), \( \psi_j \geq \tilde{\psi}_j > \varphi_j \).

Since \( \rho_j = -j \leq \varphi_j < \tilde{\psi}_j \) on \( X \cap K_{j-1} \), we see that \( \psi_j = \tilde{\psi}_j \) on \( X \cap K_{j-1} \) so

\[
\int_{X \cap K_{j-1}} (\psi_j - \varphi_j) \leq \int_{X \cap K_j} (\tilde{\psi}_j - \varphi_j) < 2^{-j}.
\]

Hence \( \psi_j \) verifies (4). Finally, we have by (5), \( \rho_j = -j < \rho_{j-1} \leq \psi_{j-1} \) on \( K_{j-1} \) (and for \( j = 1 \), \( \rho_1 = -1 < \psi_0 = 0 \) on \( K_0 \)). Since \( \psi_j < \psi_{j-1} \) on \( K_j \) we conclude that \( \psi_j < \psi_{j-1} \) on \( K_{j-1} \), so (6) is verified.

So we have constructed a sequence of continuous psh functions \( \psi_j \) on \( \mathbb{C}^n \) verifying properties (4)-(6). Since \( \bigcup_{j \geq 1} D_j = \mathbb{C}^n \), we have by (6) that the function

\[
\psi(z) = \lim_{j \to \infty} \psi_j(z)
\]

is well defined and psh on \( \mathbb{C}^n \). As \( \ldots < \psi_{j+2} < \psi_{j+1} < \psi_j \) on \( K_j \), it follows that \( \psi < \psi_j \) on \( K_j \).

Suppose now that \( z \in K_j \setminus D_{j-1} \), for some \( j \geq 2 \), so \( m_{j-1} \leq u(z) \leq m_j \). By the above construction and property (5), we have

\[
\tilde{\psi}_j(z) < \psi_{j-1}(z) = \rho_{j-1}(z) \implies \psi(z) < \psi_j(z) \leq \max\{\rho_{j-1}(z), \rho_j(z)\} \leq \gamma_j u(z).
\]

Similarly, for \( z \in K_1 \) we have

\[
\psi(z) < \psi_1(z) \leq \max\{\rho_0(z), \rho_1(z)\} \leq \gamma_1 \max\{u(z), 0\}.
\]

Hence \( \psi \) satisfies the desired global upper estimates on \( \mathbb{C}^n \).

Property (4) implies that \( \psi(z) \geq \varphi(z) \) for every \( z \in X \). Let \( K \) be a compact in \( \mathbb{C}^n \) and \( Y \) be an irreducible component of \( X \) so that \( \varphi|_Y \neq -\infty \). By (4) we have that for all \( j \) sufficiently large

\[
0 \leq \int_{Y \cap K} (\psi_j - \varphi) = \int_{Y \cap K} (\tilde{\psi}_j - \varphi) + \int_{Y \cap K} (\varphi - \varphi_j) \leq 2^{-j} + \int_{Y \cap K} (\varphi_j - \varphi).
\]

Hence by dominated convergence, \( \int_{Y \cap K} (\varphi - \varphi_j) = 0 \), which shows that \( \psi = \varphi \) on \( Y \).

Assume now that \( Y \) is an irreducible component of \( X \) so that \( \varphi|_Y \equiv -\infty \). Then using (4) and the monotone convergence theorem we conclude that

\[
\int_{Y \cap K} \psi = \lim_{j \to \infty} \int_{Y \cap K} \psi_j = \lim_{j \to \infty} \left( \int_{Y \cap K} (\psi_j - \varphi_j) + \int_{Y \cap K} \varphi_j \right) = -\infty,
\]

so \( \psi|_Y \equiv -\infty \). Therefore \( \psi = \varphi \) on \( X \), and the proof is finished. \( \square \)
Proof of Theorem A. We consider first the case \( M = \mathbb{C}^n \). Fix \( c > 1 \). We define inductively a sequence \( \{m_j\} \) with the following properties: \( m_{j-1} = m_0 = 0 < m_1 \), \( \{u < m_1\} \neq \emptyset \), and for \( j \geq 1 \), \( m_j > m_{j-1} \) is chosen large enough so that
\[
a_j = \frac{m_{j-1} - m_{j-2} + 1}{m_j - m_{j-1}} \leq \frac{\log c}{2^j}.
\]
Since \( \gamma_j \geq \gamma_0 = 1 \) we have by (1),
\[
\gamma_j(m_j - m_{j-1}) \leq \gamma_{j-1}(m_j - m_{j-2} + 1) \Rightarrow \gamma_j \leq \gamma_{j-1}(1 + a_j).
\]
Thus
\[
\gamma_j < \gamma = \prod_{j=1}^{\infty} (1 + a_j), \quad \log \gamma \leq \sum_{j=1}^{\infty} a_j \leq \log c.
\]

Let \( \psi = \psi_c \) be the psh extension of \( \varphi \) provided by Proposition 1.3 for this sequence \( \{m_j\} \). Then for every \( z \in \mathbb{C}^n \) we have
\[
\psi(z) < \gamma \max\{u(z), 0\} \leq c \max\{u(z), 0\}.
\]

Assume now that \( M \) is a Stein manifold of dimension \( n \). Then \( M \) can be properly embedded in \( \mathbb{C}^{2n+1} \), hence we may assume that \( M \) is a complex submanifold of \( \mathbb{C}^{2n+1} \) (see e.g. [Ho, Theorem 5.3.9]). Proposition 1.1 implies the existence of a continuous psh exhaustion function \( \tilde{u} \) on \( \mathbb{C}^{2n+1} \) so that \( \tilde{u} = u \) on \( M \). By what we already proved, given \( c > 0 \) there exists a psh function \( \tilde{\psi} \) on \( \mathbb{C}^{2n+1} \) which extends \( \varphi \) and such that \( \tilde{\psi} < c \max\{\tilde{u}, 0\} \) on \( \mathbb{C}^{2n+1} \). We let \( \psi = \tilde{\psi} |_M \). \( \square \)

We end this section by noting that some hypothesis on the growth of \( u \) is necessary in Theorem A. Indeed, suppose that \( X \) is a submanifold of \( \mathbb{C}^n \) for which there exists a non-constant negative psh function \( \varphi \) on \( X \). Then any psh extension of \( \varphi \) to \( \mathbb{C}^n \) cannot be bounded above. However, by Theorem A, given any \( \varepsilon > 0 \) there exists a psh function \( \psi = \psi_\varepsilon \) so that \( \psi |_X = \varphi \) and \( \psi(z) < \varepsilon \log^+ \|z\| \) on \( \mathbb{C}^n \).

2. Extension of Qpsh functions

Let \( V \) be a compact Kähler manifold equipped with a Kähler form \( \omega \). We let \( PSH(V, \omega) \) denote the set of \( \omega \)-psh functions on \( V \). These are upper semicontinuous functions \( \varphi \in L^1(V, [-\infty, +\infty)) \) such that \( \omega + dd^c \varphi \geq 0 \), where \( d = \partial + \bar{\partial} \) and \( d^c = \frac{1}{2\pi i}(\partial - \bar{\partial}) \). We refer the reader to [GZ] for basic properties of \( \omega \)-psh functions.

Let \( X \) be an analytic subvariety of \( V \). Recall that an upper semicontinuous function \( \varphi : X \to [-\infty, +\infty) \) is called \( \omega |_X \)-psh if \( \varphi \not\equiv -\infty \) on \( X \) and if there exist an open cover \( \{U_i\}_{i \in I} \) of \( X \) and psh functions \( \varphi_i, \rho_i \) defined on \( U_i \), where \( \rho_i \) is smooth and \( dd^c \rho_i = \omega \), so that \( \rho_i + \varphi = \varphi_i \) holds on \( X \cap U_i \), for every \( i \in I \). Moreover, \( \varphi \) is called strictly \( \omega |_X \)-psh if it is \( (1-\varepsilon)\omega |_X \)-psh for some small \( \varepsilon > 0 \). The current \( \omega |_X + dd^c \varphi \) is then called a Kähler current on \( X \) (see [EGZ1, section 5.2]). We denote by \( PSH(X, \omega |_X) \), resp. \( PSH^+(X, \omega |_X) \), the class of \( \omega |_X \)-psh, resp. strictly \( \omega |_X \)-psh functions on \( X \).

Every \( \omega \)-psh function \( \varphi \) on \( V \) yields, by restriction, an \( \omega |_X \)-psh function \( \varphi |_X \) on \( X \), as soon as \( \varphi |_X \not\equiv -\infty \). The question we address here is whether this restriction operator is surjective. In other words, is there equality
\[
PSH(X, \omega |_X) \cong PSH(V, \omega |_X).
\]
2.1. **The smooth case.** We start with the elementary observation that smooth strictly $\omega$-psh functions can easily be extended.

**Proposition 2.1.** Let $V$ be a compact Kähler manifold equipped with a Kähler form $\omega$, and let $X$ be a complex submanifold of $V$. Then

$$PSH^+(X,\omega|_X) \cap C^\infty(X,\mathbb{R}) = (PSH^+(V,\omega) \cap C^\infty(V,\mathbb{R}))|_X.$$  

We include a proof for the convenience of the reader, although this is probably part of the “folklore” (see e.g. [Sch] for the case where $\omega$ is a Hodge form).

**Proof.** Let $\varphi \in C^\infty(X,\mathbb{R})$ be such that $(1 - \varepsilon)\omega|_X + dd^c\varphi \geq 0$ on $X$, for some $\varepsilon > 0$. We first choose $\tilde{\varphi}$ to be any smooth extension of $\varphi$ to $V$. Consider

$$\psi := \tilde{\varphi} + A\chi \text{dist}(\cdot, X)^2,$$

where $\chi$ is a test function supported in a small neighborhood of $X$ and such that $\chi \equiv 1$ near $X$. Here dist is any Riemannian distance on $V$, for instance the distance associated to the Kähler metric $\omega$. Then $\psi$ is yet another smooth extension of $\varphi$ to $V$, which now satisfies $(1 - \varepsilon/2)\omega + dd^c\psi \geq 0$ near $X$, if $A$ is chosen large enough.

The function $\log(\text{dist}(\cdot, X)^2)$ is well defined andqpsh in a neighborhood of $X$. Let $\chi$ be a test function supported in this neighborhood so that $\chi \equiv 1$ near $X$. The function $u = \chi \log(\text{dist}(\cdot, X)^2)$ is $N\omega$-psh on $V$ for a large integer $N$. Moreover, $\exp(u)$ is smooth and $X = \{u = -\infty\}$. Replacing $\omega$ by $N\omega$, $\varphi$ by $N\varphi$, and $\psi$ by $N\psi$, we may assume that $N = 1$. Set now

$$\psi_C := \frac{1}{2} \log \left[ e^{2\psi} + e^{u+C} \right].$$

This again is a smooth extension of $\varphi$, and a straightforward computation yields

$$dd^c\psi_C \geq \frac{2e^{2\psi}(dd^c\psi + e^{u+C}dd^c u)}{2(e^{2\psi} + e^{u+C})}.$$  

Hence

$$\left(1 - \frac{\varepsilon}{2}\right)\omega + dd^c\psi_C \geq \frac{2e^{2\psi}\left[(1 - \varepsilon)\omega + dd^c\psi\right] + (1 - \varepsilon)e^{u+C}\omega}{2(e^{2\psi} + e^{u+C})} \geq 0,$$

if $C$ is chosen large enough. 

This proof breaks down when $\varphi$ is singular and hence a different approach is needed. We consider in the next section the particular case when $\omega$ is a Hodge form.

2.2. **Proof of Theorem B.** We assume here that $\omega$ is a Hodge form, i.e. that the cohomology class $\{\omega\}$ belongs to $H^2(V,\mathbb{Z})$ (more precisely to the image of $H^2(V,\mathbb{Z})$ in $H^2(V,\mathbb{R})$ under the mapping induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$). We prove the following more precise version of Theorem B.

**Theorem 2.2.** Let $X$ be a subvariety of a projective manifold $V$ equipped with a Hodge form $\omega$. If $\varphi \in PSH(X,\omega|_X)$ then given any constant $a > 0$ there exists $\psi \in PSH(V,\omega)$ so that $\psi|_X = \varphi$ and $\max_V \psi < \max_X \varphi + a$. 


In the assumptions of Theorem 2.2 there exists a positive holomorphic line bundle \( L \) on \( V \) whose first Chern class \( c_1(L) \) is represented by \( \omega \). By Kodaira’s embedding theorem \( L \) is ample, hence for large \( k \) there exists an embedding \( \pi : V \hookrightarrow \mathbb{P}^n \) such that \( L^k = \pi^* \mathcal{O}(1) \).

Replacing \( \omega \) by \( k\omega \), \( \varphi \) by \( k\varphi \), we can assume that \( L = \mathcal{O}(1) \), \( V \) is an algebraic submanifold of the complex projective space \( \mathbb{P}^n \), and \( \omega = \omega_{FS} \mid_V \) is the Fubini-Study Kähler form. Hence \( X \) is an algebraic subvariety of \( \mathbb{P}^n \), and Theorem 2.2 follows if we show that \( \omega_{FS} \)-psh functions on \( X \) extend to \( \omega_{FS} \)-psh functions on \( \mathbb{P}^n \).

Therefore we assume in the sequel that \( X \subset V = \mathbb{P}^n \) and \( \omega \) is the Fubini-Study Kähler form on \( \mathbb{P}^n \). Let \([z_0 : \ldots : z_n]\) denote the homogeneous coordinates. Without loss of generality, we may assume that they are chosen so that no coordinate hyperplane \( \{z_j = 0\} \) contains any irreducible component of \( X \).

Let

\[
\theta(z) = \log \frac{\max\{|z_0|, \ldots, |z_n|\}}{\sqrt{|z_0|^2 + \cdots + |z_n|^2}}, \quad z = [z_0 : \ldots : z_n] \in \mathbb{P}^n.
\]

This is an \( \omega \)-psh function and for all \( z \in \mathbb{P}^n \),

\[-m \leq \theta(z) \leq 0, \quad \text{where} \quad m = \log \sqrt{n+1}.
\]

We start by noting that Theorem A yields special subextensions of \( \omega \)-psh functions on \( X \).

**Lemma 2.3.** Let \( \varepsilon \geq 0 \) and \( u \) be a continuous \((1 + \varepsilon)\omega\)-psh function on \( \mathbb{P}^n \) so that \( u(z) \leq 0 \) for all \( z \in \mathbb{P}^n \). If \( c > 1 \) and \( \varphi \) is an \( \omega \)-psh function on \( X \) so that \( \varphi < u \), then there exists a \( c\omega \)-psh function \( \psi \) on \( \mathbb{P}^n \) so that

\[
\frac{1}{c} \psi(z) \leq \frac{1}{1 + \varepsilon} u(z), \quad \forall z \in \mathbb{P}^n,
\]

and

\[
\psi(z) = \varphi(z) + (c - 1)\theta(z) + (c - 1) \min_{\zeta \in \mathbb{P}^n} u(\zeta), \quad \forall z \in X.
\]

**Proof.** Let

\[ M = -\min_{\zeta \in \mathbb{P}^n} u(\zeta) \geq 0. \]

We work first in an affine chart \( \{z_j = 1\} \equiv \mathbb{C}^n \). Let \( X_j = X \cap \{z_j = 1\} \) and let \( \rho_j \geq 0 \) be the potential of \( \omega \) in this chart with \( \rho_j(0) = 0 \). Then \( \varphi + \rho_j \) is psh on \( X_j \) and since \( u \leq 0 \),

\[
\varphi + \rho_j + M < u + \rho_j + M \leq \frac{1}{1 + \varepsilon} u + \rho_j + M \quad \text{on} \quad X_j.
\]

Note that \((1 + \varepsilon)^{-1}u + \rho_j + M \geq 0\) is a continuous psh exhaustion function on \( \mathbb{C}^n \).

Theorem A yields a psh function \( \tilde{\psi} \) on \( \mathbb{C}^n \) so that

\[
\tilde{\psi} < \frac{c}{1 + \varepsilon} u + c\rho_j + cM \quad \text{on} \quad \mathbb{C}^n, \quad \tilde{\psi} = \varphi + \rho_j + M \quad \text{on} \quad X_j.
\]

The function \( \psi_j = \tilde{\psi} - c\rho_j - cM \) extends uniquely to a \( c\omega \)-psh function on \( \mathbb{P}^n \) which verifies

\[
\psi_j \leq \frac{c}{1 + \varepsilon} u \quad \text{on} \quad \mathbb{P}^n.
\]

Moreover on \( X \cap \{z_j = 1\} \) we have

\[
\psi_j = \varphi - (c - 1)\rho_j - (c - 1)M = \varphi + (c - 1)\theta_j - (c - 1)M,
\]
where
\[ \theta_j(z) = \log \frac{|z_j|}{\sqrt{|z_0|^2 + \ldots + |z_n|^2}}. \]

Hence \( \psi_j = -\infty \) on \( X \cap \{z_j = 0\} \).

We finally let \( \psi = \max\{\psi_0, \ldots, \psi_n\} \). This is a \( c\omega \)-psh function on \( \mathbb{P}^n \) which verifies the desired conclusions, since \( \theta = \max\{\theta_0, \ldots, \theta_n\} \). \( \square \)

**Proof of Theorem 2.2.** Fix \( a > 0 \). Replacing \( \varphi \) by \( \varphi - \max_X \varphi - a \) we may assume that \( \max_X \varphi = -a \). We will show that there exists a sequence of smooth \( \omega \)-psh functions \( \varphi_j \) on \( \mathbb{P}^n \) which decrease pointwise on \( \mathbb{P}^n \) to a negative \( \omega \)-psh function \( \psi \) so that \( \psi = \varphi \) on \( X \).

Let \( X' \) be the union of the irreducible components \( W \) of \( X \) so that \( \varphi|_W \not\equiv -\infty \).

We first construct by induction on \( j \geq 1 \) a sequence of numbers \( \varepsilon_j \searrow 0 \) and a sequence of negative smooth \( (1 + \varepsilon_j)\omega \)-psh functions \( \psi_j \) on \( \mathbb{P}^n \) so that for all \( j \geq 2 \)
\[ \frac{\psi_j}{1 + \varepsilon_j} < \frac{\psi_{j-1}}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi_j > \varphi \text{ on } X, \quad \int_{X'} (\psi_j - \varphi) < \frac{1}{j}, \quad \int_W \psi_j < -j, \]

for every irreducible component \( W \) of \( X \) where \( \varphi|_W \equiv -\infty \). Here the integrals are with respect to the area measure on each irreducible component \( X_j \) of \( X \), i.e.
\[ \int_X f := \sum_{X_j} \int_{X_j} f \omega_{\text{dim } X_j}. \]

Let \( \varepsilon_1 = 1, \psi_1 = 0 \), and assume that \( \varepsilon_{j-1}, \psi_{j-1} \), where \( j \geq 2 \), are constructed with the above properties. Since \( \varphi < \psi_{j-1}|_X \) and the latter is continuous on the compact set \( X \), we can find \( \delta > 0 \) so that \( \varphi < \psi_{j-1} - \delta \) on \( X \).

Let \( c > 1 \). By Lemma 2.3, there exists a \( c\omega \)-psh function \( \psi_c \) so that
\[ \frac{\psi_c}{c} \leq \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi_c = \varphi + (c-1)\theta - (c-1)M_{j-1} \text{ on } X, \]
where
\[ M_{j-1} = \delta - \min_{\zeta \in \mathbb{P}^n} \psi_{j-1}(\zeta) \geq 0. \]

We canregularize \( \psi_c \) on \( \mathbb{P}^n \); there exists a sequence of smooth \( c\omega \)-psh functions decreasing to \( \psi_c \) on \( \mathbb{P}^n \). Therefore we can find a smooth \( c\omega \)-psh function \( \psi'_c \) on \( \mathbb{P}^n \) so that
\[ \frac{\psi'_c}{c} < \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi'_c > \varphi + (c-1)\theta - (c-1)M_{j-1} \geq \varphi - (c-1)(m + M_{j-1}) \text{ on } X. \]

By dominated, resp. monotone convergence, we can in addition ensure that
\[ \int_{X'} (\psi'_c - \varphi) \leq \int_{X'} (\psi'_c - \varphi - (c-1)\theta + (c-1)M_{j-1}) < c - 1, \]
\[ \int_W \psi'_c < -j - (c-1)(m + M_{j-1})|W|, \]
for every irreducible component \( W \) of \( X \) where \( \varphi|_W \equiv -\infty \). Here \( |W| \) denotes the (projective) area of \( W \).
Now let $\psi''_c = \psi'_c + (c - 1)(m + M_{j-1})$. Then on $\mathbb{P}^n$ we have

$$
\frac{\psi''_c}{c} < \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} + \frac{(c - 1)(m + M_{j-1})}{c} < \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} - \frac{\delta}{4} + (c - 1)(m + M_{j-1}).
$$

Moreover, $\psi''_c > \varphi$ on $X$ and

$$
\int_{X'} (\psi''_c - \varphi) = \int_{X'} (\psi'_c - \varphi) + (c - 1)(m + M_{j-1})|X'| < (c - 1)(1 + m|X'| + M_{j-1}|X'|),
$$

$$
\int_W \psi''_c = \int_W \psi'_c + (c - 1)(m + M_{j-1})|W| < -j,
$$

for every irreducible component $W$ of $X$ where $\varphi|_W \equiv -\infty$.

We take $c = 1 + \varepsilon_j$ and $\psi_j = \psi''_c$, where $\varepsilon_j > 0$ is so that

$$
\varepsilon_j < \varepsilon_{j-1}/2, \quad \varepsilon_j(m + M_{j-1}) < \frac{\delta}{4}, \quad \varepsilon_j(1 + m|X'| + M_{j-1}|X'|) < \frac{1}{j}.
$$

Then $\varepsilon_j, \psi_j$ have the desired properties.

We conclude that $\varphi_j = (1 + \varepsilon_j)^{-1}\psi_j$ is a decreasing sequence of smooth negative $\omega$-psh function on $\mathbb{P}^n$, so that $\varphi_j > (1 + \varepsilon_j)^{-1}\varphi > \varphi$ on $X$. Hence $\psi = \lim_{j \to \infty} \varphi_j$ is a negative $\omega$-psh function on $\mathbb{P}^n$ and $\psi \geq \varphi$ on $X$. Note that

$$
\int_{X'} (\varphi_j - \varphi) = \frac{1}{1 + \varepsilon_j} \int_{X'} (\psi_j - \varphi) - \frac{\varepsilon_j}{1 + \varepsilon_j} \int_{X'} \varphi < \frac{1}{j} - \frac{\varepsilon_j}{1 + \varepsilon_j} \int_{X'} \varphi,
$$

$$
\int_W \varphi_j = \frac{1}{1 + \varepsilon_j} \int_W \psi_j < -\frac{j}{2},
$$

for every irreducible component $W$ of $X$ where $\varphi|_W \equiv -\infty$. It follows that $\psi = \varphi$ on $X$ and the proof of Theorem 2.2 is finished. \(\square\)

3. Algebraic subvarieties of $\mathbb{C}^n$

If $X$ is an analytic subvariety of $\mathbb{C}^n$ and $\gamma$ is a positive number, we denote by $\mathcal{L}_\gamma(X)$ the Lelong class of psh functions $\varphi$ on $X$ which verify $\varphi(z) \leq \gamma \log^+ \|z\| + C$ for all $z \in X$, where $C$ is a constant that depends on $\varphi$. We let $\mathcal{L}(X) = \mathcal{L}_1(X)$. By Theorem A, functions $\varphi \in \mathcal{L}(X)$ admit a psh extension in each class $\mathcal{L}_\gamma(\mathbb{C}^n)$, for every $\gamma > 1$. \(^1\)

We assume in the sequel that $X$ is an algebraic variety of $\mathbb{C}^n$ and address the question whether it is necessary to allow the arbitrarily small additional growth. More precisely, it is true that

$$
\mathcal{L}(X) \overset{?}{=} \mathcal{L}(\mathbb{C}^n)|_X,
$$

i.e. is every psh function with logarithmic growth on $X$ the restriction of a globally defined psh function with logarithmic growth? We will give a criterion for this to hold, but show that in general this is not the case.

\(^1\)If $X$ is algebraic this result is claimed in [BL, Proposition 3.3], but there is a gap in their proof.
3.1. Extension preserving the Lelong class. Consider the standard embedding 
\[ z \in \mathbb{C}^n \hookrightarrow [1 : z] \in \mathbb{P}^n, \]
where \([t : z]\) denote the homogeneous coordinates on \(\mathbb{P}^n\). Let \(\omega\) be the Fubini-Study Kähler form and let 
\[ \rho(t, z) = \log \sqrt{|t|^2 + \|z\|^2} \]
be its logarithmically homogeneous potential on \(\mathbb{C}^{n+1}\).

We denote by \(\overline{X}\) the closure of \(X\) in \(\mathbb{P}^n\), so \(\overline{X}\) is an algebraic subvariety of \(\mathbb{P}^n\).

It is well known that the class \(PSH(\mathbb{P}^n, \omega)\) is in one-to-one correspondence with the Lelong class \(\mathcal{L}(\mathbb{C}^n)\) (see [GZ]). Let us look at the connection between \(\omega\)-psh \(L\) functions on \(\overline{X}\) and the class \(\mathcal{L}(X)\).

The mapping 
\[ F_X : PSH(\overline{X}, \omega|_{\overline{X}}) \rightarrow \mathcal{L}(X), \quad (F_X \varphi)(z) = \rho(1, z) + \varphi([1 : z]), \]
is well defined and injective. However, it is in general not surjective, as shown by Examples 3.2 and 3.3 that follow.

Conversely, a function \(\eta \in \mathcal{L}(X)\) induces an upper semicontinuous function \(\tilde{\eta}\) on \(\overline{X}\) defined in the obvious way:

\[ \tilde{\eta}(\overline{t : z}) = \begin{cases} 
\eta(z) - \rho(1, z), & \text{if } t = 1, \; z \in X, \\
\limsup_{[1: \zeta] \to [0: z], \zeta \in X} (\eta(\zeta) - \rho(1, \zeta)), & \text{if } t = 0, \; [0 : z] \in \overline{X} \setminus X.
\end{cases} \]

The function \(\tilde{\eta}\) is in general only weakly \(\omega\)-psh on \(\overline{X}\), i.e., it is bounded above on \(\overline{X}\) and it is \(\omega|_{\overline{X}_r}\)-psh on the set \(\overline{X}_r\) of regular points of \(\overline{X}\). This notion is in direct analogy to that of weakly psh function on an analytic variety (see [D2, section 1]). We do not pursue it any further here.

Note that \(\eta \in F_X \left(PSH(\overline{X}, \omega|_{\overline{X}})\right)\) if and only if \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\). The following simple characterization is a consequence of Theorem B.

**Proposition 3.1.** Let \(\eta \in \mathcal{L}(X)\). The following are equivalent:

(i) There exists \(\psi \in \mathcal{L}(\mathbb{C}^n)\) so that \(\tilde{\psi} = \eta\) on \(X\).

(ii) \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\).

(iii) For every point \(a \in \overline{X} \setminus X\) the following holds: if \((X_j, a)\) are the irreducible components of the germ \((\overline{X}, a)\) then the value

\[ \limsup_{X_j \ni [1: \zeta] \to a} (\eta(\zeta) - \rho(1, \zeta)) \]

is independent of \(j\).

In particular, if the germs \((\overline{X}, a)\) are irreducible for all points \(a \in \overline{X} \setminus X\) then \(\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X\).

**Proof.** Assume that (i) holds. It follows that \(\tilde{\eta} = \varphi|_{\overline{X}}\), where

\[ \varphi([t : z]) := \begin{cases} 
\psi(z) - \rho(1, z), & \text{if } t = 1, \\
\limsup_{[1: \zeta] \to [0: z]} (\psi(\zeta) - \rho(1, \zeta)), & \text{if } t = 0,
\end{cases} \]
is an \(\omega\)-psh function on \(\mathbb{P}^n\). Hence \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\).
Conversely, if (ii) holds then by Theorem B there exists an ω-psh function ϕ on \( \mathbb{P}^n \) which extends \( \tilde{\eta} \). Hence \( \psi(z) = \rho(1, z) + \varphi([1 : z]) \) is an extension of \( \eta \) and \( \psi \in \mathcal{L}(\mathbb{C}^n) \).

The equivalence of (ii) and (iii) follows easily from [D2, Theorem 1.10]. \( \square \)

### 3.2. Explicit examples.

In view of section 3.1, it is easy to construct examples of algebraic curves \( X \subset \mathbb{C}^2 \) and functions in \( \mathcal{L}(X) \) which do not admit an extension in \( \mathcal{L}(\mathbb{C}^2) \). We write \( z = (x, y) \in \mathbb{C}^2 \).

**Example 3.2.** Let \( X = \{y = 0\} \cup \{y = 1\} \subset \mathbb{C}^2 \) and \( \eta \in \mathcal{L}(X) \), where
\[
\eta(z) = \begin{cases} 
\rho(1, z), & \text{if } z = (x, 0), \\
\rho(1, z) + 1, & \text{if } z = (x, 1).
\end{cases}
\]

The function \( \tilde{\eta} \) is not ω-psh on \( \mathcal{X} = \{y = 0\} \cup \{y = t\} \), hence \( \eta \) does not have an extension in \( \mathcal{L}(\mathbb{C}^2) \). Indeed, the maximum principle is violated along \( \{y = 0\} \) near the point \( a = [0 : 1 : 0] \), since \( \tilde{\eta}([t : 1 : 0]) = 0 \) for \( t \neq 0 \), while \( \tilde{\eta}([t : 1 : 1]) = 1 \).

With a little more effort we can give an example as above where \( X \) is an irreducible curve. Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

**Example 3.3.** Let \( X \subset \mathbb{C}^2 \) be the irreducible cubic with equation \( xy = x^3 + 1 \). Then
\[
\mathcal{X} = \{[t : x : y] \in \mathbb{P}^2 : xyt = x^3 + t^3\}, \quad \mathcal{X} = X \cup \{a\}, \quad a = [0 : 0 : 1].
\]

The germ \((\mathcal{X}, a)\) has two irreducible components \( X_1, X_2 \), both are smooth at \( a \), \( X_1 \) being tangent to the line \( \{x = 0\} \), and \( X_2 \) to the line \( \{t = 0\} \).

Note that in fact \( X \subset \mathbb{C}^* \times \mathbb{C} \) is the graph of the rational function \( y = x^2 + x^{-1}, x \in \mathbb{C}^* \). If \((x, y) \in X \) and \( x \to 0 \) then \( (x, y) \to a \) along \( X_1 \), while as \( x \to \infty \) then \( (x, y) \to a \) along \( X_2 \). The function
\[
u(x, y) = \max \{-\log |x|, 2 \log |x| + 1\}
\]
is psh in \( \mathbb{C}^* \times \mathbb{C} \). It is easy to check that \( \eta := \nu|_X \in \mathcal{L}(X) \) and
\[
\limsup_{X_1 \ni [t : \zeta] \to a} (\eta(\zeta) - \rho(1, \zeta)) = 0, \quad \limsup_{X_2 \ni [t : \zeta] \to a} (\eta(\zeta) - \rho(1, \zeta)) = 1.
\]

Hence \( \eta \) does not admit an extension in \( \mathcal{L}(\mathbb{C}^2) \).

We conclude this section with an example of a cubic \( X \in \mathbb{C}^2 \) and a psh function on \( X \) of the form \( \eta = \log |P| \), where \( P \) is a polynomial, so that \( \eta \) admits a “transcendental” extension with exactly the same growth, but small additional growth is necessary if we look for an “algebraic” extension.

**Proposition 3.4.** Let \( X = \{x = y^3\} \) and \( \eta(x, y) = \log |1 + y| \), so \( \eta|_X \in \mathcal{L}_{\frac{1}{3}}(X) \).

Given \( k \geq 1 \), there is a polynomial \( Q_k(x, y) \) of degree \( k + 1 \) so that \( Q_k(y^k, y) = (y + 1)^{3k} \). In particular, \( \psi_k = \frac{1}{3k} \log |Q_k| \in \mathcal{L}_{(k+1)/3k}(\mathbb{C}^2) \) is an extension of \( \eta|_X \).

There exists no polynomial \( Q(x, y) \) of degree \( k \) so that \( Q(y^3, y) = (y + 1)^{3k} \). However, \( \eta|_X \) has an extension in \( \mathcal{L}_{\frac{1}{3}}(\mathbb{C}^2) \).

**Proof.** We construct \( Q_k \) by replacing \( y^3 \) by \( x \) in the polynomial
\[
(y + 1)^{3k} = \sum_{j=0}^{3k} \binom{3k}{j} y^j.
\]
Since \( j = 3[j/3] + r_j \), \( r_j \in \{ 0, 1, 2 \} \), it follows that

\[
Q_k(x, y) = \sum_{j=0}^{\lfloor 3k/3 \rfloor} \binom{3k}{j} x^{[j/3]} y^{r_j} = 3kx^{k-1}y^2 + \text{d.t.}.
\]

We now check that there is no polynomial \( Q(x, y) \) of degree \( k \) so that \( Q(y^3, y) = (y + 1)^{3k} \). Indeed, if \( Q(x, y) = \sum_{j+l \leq k} c_{jl}x^jy^l \) then

\[
Q(y^3, y) = c_{k0}y^{3k} + c_{k-1, 1}y^{3k-2} + \text{d.t.}
\]

does not contain the monomial \( y^{3k-1} \).

Note that \( \mathbb{X} = \{ xt^2 = y^3 \} = X \cup \{ a \} \), where \( a = [0 : 1 : 0] \), so the germ \( (\mathbb{X}, a) \) is irreducible. Proposition 3.1 implies that \( \eta \big|_X \) has an extension in \( \mathcal{L}_{1/3}(\mathbb{C}^2) \). \( \square \)

We conclude with some remarks regarding our last example. If \( X \) is an algebraic subvariety of \( \mathbb{C}^n \) and \( f \) is a holomorphic function on \( X \), \( f \) is said to have polynomial growth if there is an integer \( N(f) \) and a constant \( A \) so that

\[
|f(z)| \leq A(1 + ||z||)^{N(f)}, \quad \forall z \in X.
\]

Then it is well known that there exists a polynomial \( P \) of degree at most \( N(f) + \varepsilon(X) \) so that \( P \big|_X = f \), where \( \varepsilon(X) > 0 \) is a constant depending only on \( X \) (see e.g. [Bj] and references therein). However, if \( \mathbb{X} \subset \mathbb{P}^N \) is irreducible at each of its points at infinity then by Proposition 3.1 the psh function \( \eta = N(f)^{-1}\log|f| \in \mathcal{L}(X) \) has a psh extension in the Lelong class \( \mathcal{L}(\mathbb{C}^n) \).

On the other hand, Demailly [D1] has shown that in the case of the transcendental curve \( X = \{ e^x + e^y = 1 \} \) any holomorphic function \( f \) on \( X \), of polynomial growth, has a polynomial extension of the same degree to \( \mathbb{C}^n \). Hence it is natural to ask if for this curve one has that \( \mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n) \big|_X \).

References


D. Coman: dcoman@syr.edu, Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, USA

V. Guedj: guedj@cmi.univ-mrs.fr, Université Aix-Marseille 1, LATP, 13453 Marseille Cedex 13, FRANCE

A. Zeriahi: zeriahi@picard.ups-tlse.fr, Laboratoire Emile Picard, UMR 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 04, FRANCE