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Gauge-Invariant Correlators In Quantum Gravity

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Abstract

Recent successes in Euclidean Dynamical Triangulations (EDT) motivate the further comparison of lattice observables to predictions of general relativity (GR) treated as an effective quantum theory. A particularly promising observable is the two-point function of the scalar curvature, which can be straightforwardly computed on the lattice and which in principle can also be computed from the Einstein-Hilbert path integral. Any such comparison should be between manifestly gauge-invariant observables, and will require that the GR predictions be analytically continued in a gauge-invariant manner to the Euclidean signature of the lattice. In this thesis I present my work toward this goal, namely: the construction of a set of relational observables, including the scalar invariantized scalar curvature; the calculation of the graviton propagator in a basis suitable for continuation; and the calculation of three manifestly gauge-invariant results in Lorentzian signature, as support of the coherence of the so-far developed machinery. I conclude by outlining the difficulties that remain in the evaluation of the scalar curvature two-point function at one loop, including the stubborn gauge dependence of the result and the difficulty in actually performing the analytic continuation to Euclidean signature.

Gauge-Invariant Correlators In Quantum Gravity

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Contents

1 Introduction

The formulation of a coherent and testable theory of quantum gravity continues to be elusive. The well-known nonrenormalizability of general relativity [1, 2] requires that any consistent theory of gravity differ at high energies from the perturbative quantization. For decades string theory has been seen as the most promising candidate for such a theory. However the stability of string theory requires that it be supersymmetric [3], and experimental searches have found no evidence of superpartners [4–13]. While these results do not exclude the possibility of high-mass superpartners the complete lack of experimental evidence for supersymmetry casts doubt on string theory and all other supersymmetric theories as viable UV completions of gravity.

That general relativity is not renormalizable in the perturbative sense means that there does not exist a finite set of counterterms which can be added to the action to absorb every divergence in the theory: at higher and higher order in the loop expansion, i.e. in the expansion in the gravitational coupling κ, new divergences inevitably arise, which must be absorbed by new counterterms. Since each new counterterm must be fixed by a new experiment such a theory has predictive power only at low energies, where we can restrict ourselves to finite order in the loop expansion, add a finite number of operators with associated couplings to the action, and absorb the finite number of singularities in the truncated loop expansion by renormalizing these finite new couplings [14–16].

This effective field theory approach allows us to perform perfectly well-defined quantum gravitational calculations over a limited range of energies, but does not solve the problem of formulating a theory of gravity which is valid at all scales. However this approach of building the full theory out of the classical theory by performing a series of perturbative calculations and adding counterterms to the action to absorb the divergences is not the only way for a theory to be renormalizable. As was first observed by Wilson and Kogut [17] a theory can also be UV-completed if its couplings flow to a fixed point in theory space at high energies. As Weinberg pointed out in [18], if such a fixed point exists for gravity then it can still be *nonperturbatively* renormalizable.

To see this let me be more clear about my terms. (My exposition here echoes the much fuller treatments in the literature, e.g. [19, 20].) Suppose we have some set of operators $\mathcal{O}_i$, finite or infinite in number. Then given this set of operators and a corresponding set of dimensionless coupling values g_i a theory is defined by the resulting action,

$$S = \sum_i \mu^{n_i} g_i \mathcal{O}_i, \tag{1.1}$$

including an energy scale μ^{n_i} for each term to account for the possibly varying dimensions of the operators. The set of all possible values of the couplings defines a manifold on which the couplings act as coordinates, and this *theory space* will be finite or infinite dimensional depending on the number of operators we're including. The renormalized couplings will generically depend on the scale μ at which they're measured, $g_i = g_i(\mu)$, meaning that the action S, and hence the theory, will vary depending on the scale at which renormalization is performed. Letting μ vary then defines a trajectory through theory space, and a given theory is *asymptotically safe* if as $\mu \to \infty$ its trajectory flows to a fixed point in theory space. The *surface of criticality* of a particular fixed point is the submanifold of theory space for which that fixed point is an attractor, i.e. the set of all asymptotically safe theories which tend toward that particular fixed point in the UV. A direction in theory space is called *relevant* if it is tangent to the surface of criticality.

Suppose now that a theory *is* asymptotically safe. Then it must lie on some surface of criticality, and it follows that the number of free couplings in the theory is given by precisely the dimension of that surface of criticality. For examples on either extremes, if the surface of criticality is a line, i.e. if there is one relevant direction, then there is only one free coupling in the theory, in terms of which all other couplings are determined; while if the surface of criticality is infinite-dimensional then there are infinite free couplings, i.e. infinite relevant directions, and we're no better off than we were with our perturbatively nonrenormalizable general relativity. It therefore follows that if a theory is asymptotically safe, and if the fixed point to which it tends in the UV has a finite-dimensional surface of criticality, then only a finite number of measurements are needed entirely determine the theory. It is in this sense that an asymptotically safe theory can be nonperturbatively renormalizable: even though an attempt to measure all the couplings by proceeding through higher- and higher-loop Feynman diagrams may well fail, there can still be in fact only a finite number of independent couplings, and the theory is therefore predictive. The wrinkle is that if the theory is not perturbatively renormalizable then these relationships between the couplings are not apparent on the level of Feynman diagrams.

Asymptotic safety therefore offers, at least in principle, a potential paradigm for a consistent quantum theory of gravity, as was noted in e.g. [21]. However to test whether gravity actually *is* asymptotically safe an investigative approach other than perturbation theory is needed. *Euclidean dynamical triangulations* (EDT) have recently proven fruitful in this regard [22–30].

EDT begins in a similar spirit to the more standard lattice field theories, in that the first step is to discretize spacetime as a collection of simplices. However EDT is a discretization of gravity, i.e. of a theory of dynamical spacetime itself, and it is therefore the geometry of spacetime, which manifests as the relationships between the simplices which compose the discretized spacetime, which EDT evolves. This is done as follows. The starting point is the Euclidean gravitational path integral:

$$Z = \int \mathcal{D}\boldsymbol{g}\mathrm{e}^{-S[\boldsymbol{g}]}, \quad S[\boldsymbol{g}] = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{g}(R-2\Lambda). \tag{1.2}$$

This is discretized as [31, 32]

$$Z = \sum_T \frac{1}{C_T}\left\{\prod_{i=1}^{N_2} \mathcal{O}(t_i)^\beta\right\}\mathrm{e}^{-S_{\mathrm{ER}}}, \tag{1.3}$$

in which the sum is over all triangulations; C_T is a symmetry factor which accounts for the fact that the vertices in a given triangulation T can be labelled in many different ways; N_2 is the number of triangles, i.e. two-simplices, in the triangulation; and $\mathcal{O}(t_i)$ is the number of four-simplices to which the triangle t_i belongs. The Einstein-Hilbert action is replaced by the Einstein-Regge action S_{ER} [33], and the lattice is updated via the Pachner moves [34–36], with each update accepted or rejected via the Metropolis algorithm.

In [22, 23] it was found that there exists a region of EDT parameter space containing extended lattices with spectral and Hausdorff dimensions approaching 4 at large distances. In [24] the phase diagram of the model was investigated; in [25] four-dimensional semiclassical geometries were obtained in the large distance limit, and it was argued that the number of relevant couplings in the continuum limit is one, making the theory maximally predictive. In [26, 27] scalar fields and Kähler-Dirac fermions were successfully implemented on the lattice, with the latter used in place of Dirac spinors because they can be defined without reference to any metric. In [28] the emergence of classical de Sitter space from the lattice was studied, and by making contact with the Hawking-Moss instanton [37] a value was extracted for Newton's constant G in lattice units. In [29] it was shown that in the continuum limit the interaction of two scalar particles on the lattice does behave as one would expect for Newtonian gravity, allowing for an independent determination of G in lattice units. Importantly the two values of G found in [28] and [29] agree within uncertainty, providing strong evidence that the lattice is in fact simulating a recognizably gravitational interaction. Finally in [30] an algorithmic improvement was presented which allowed for a more detailed investigation of the de Sitter solution of [28], and on these finer lattices the agreement with classical de Sitter space was stronger still.

EDT therefore shows promise as a nonperturbative and asymptotically safe formulation of gravity. To continue buttressing this case it is important to continue to test the correspondence between the continuum limit of the lattice and the known low-energy theory of general relativity, and to do so we must identify observables which can be calculated both on the lattice and in the low-energy theory.

A seemingly obvious candidate for such an observable is the scalar curvature R, whose discretized form, the Regge curvature [38], can be and has been [24] calculated on the lattice. In particular work is ongoing on the lattice to obtain the two-point function of the Regge curvature in the long-distance limit, and it therefore behooves us to obtain an analogous prediction of the two-point function of the scalar curvature in the low-energy effective theory of general relativity, again evaluated at long distances. Actually producing a result for this correlation function which can be compared to the lattice is a formidable task, and one which has not yet been accomplished. In this thesis I present the progress that has been made in this direction.

The fundamental obstacle, and the one from which all subsequent difficulties arise, is that the lattice to which I want to compare my result is Euclidean, while the known low-energy effective theory of general relativity is Lorentzian. The question of analytically continuing general relativity to Euclidean space has long been known to be subtle (see e.g. [39]), with the trouble arising from the sign of the kinetic term of the conformal mode of the graviton. However any such continuation certainly ought to be done in a gauge-invariant manner, meaning that we would like to be able to explicitly verify the gauge invariance of the continuation. To do so we should therefore begin with an explicitly gauge-invariant Lorentzian result, perform the continuation, and verify that the result remains gauge-invariant.

The definition of gauge-invariant observables in general relativity is itself a subtle one; I review and surmount this difficulty sec. 2, following and extending the recent development of *relational observables* [40–48]. The essential idea of relational observables is to define a shared "master" coordinate system X to which all other coordinate systems x refer their observations. These master coordinates are defined to be harmonic with respect to the full metric, $\nabla^2\mathsf{X} = 0$, and are in this sense therefore the "straightest possible" coordinates. Two-point functions of relational observables are then functions of the "master coordinate distance" between the points at which the relational observables are measured. Meanwhile correlators on the lattice are measured at fixed geodesic distance, which is the distance between two points measured along the "straightest possible" curve between those points. It may therefore be plausibly conjectured that the master coordinate distance on which relational observables depend is equivalent to the geodesic distance measured on the lattice, and therefore that (modulo a successful Euclidean continuation) the correlators of relational observables may be appropriately compared to the corresponding lattice results.[1]

In sec. 3 I obtain the propagators of the graviton and its Faddeev-Popov ghost, choosing a useful tensor basis which allows one to keep track of the contributions of the problematic conformal mode in loop calculations; since the difficulty of the continuation arises from the "wrong" sign of the conformal mode kinetic term, and since the sign of a field's kinetic term determines the direction in which its propagator's poles are deformed in the complex plane, any effective continuation scheme will surely need to do so. In sec. 4 I give the relevant Feynman rules, taking care to derive the new external vertices which arise from the use of relational observables, and in sec. 5 I apply this machinery to obtain three manifestly gauge-invariant correlators: the two-point function of the volume factor $\sqrt{-g}$ at tree level, the gravitational corrections to the mass of a minimally-coupled scalar at one loop, and the two-point function of a massless scalar at one loop. Finally in sec. 6 I discuss the work that remains to obtain a gauge-invariant Euclidean continuation of the scalar curvature two-point function.

Conventions

Throughout this thesis I will denote the d-dimensional spacetime manifold by M and a generic coordinate system on M as[2] $\mathsf{x} : M \to \mathbb{R}^d$. Note that the sans serif symbol x represents the *map* which takes a point $p \in M$ to its coordinates. The italic symbol x will rather denote an actual *value* of the coordinates:

$$\mathsf{x} : p \in M \mapsto \mathsf{x}(p) = x \in \mathbb{R}^d. \tag{1.4}$$

[1] A previous proposal [49] attempted to directly compute gauge-invariant correlators at fixed geodesic distance. However we have favored the relational approach due to its relatively straightforward implementation and comprehensible results when compared with [49].

[2] I will typically ignore the fact that coordinate systems are generally defined only on subsets of M, since (for the purposes of our discussion) we gain nothing but a little extra notation by keeping explicit track of the coordinate domains.

This may seem unnecessarily pedantic at the moment, but we will find this distinction useful in our discussion of relational observables in sec. 2. Abiding with the usual conventions I denote the coordinate frame by ∂_μ and the coordinate coframe by dx^μ. When I need a second coordinate system I'll denote it with tildes, so that e.g. the coordinate system $\tilde{\mathrm{x}} : M \to \mathbb{R}^d$ has coordinate frame $\tilde{\partial}_\mu$. I denote a generic diffeomorphism of M by $F : M \to M$ and assume that some such diffeomorphism relates x and $\tilde{\mathrm{x}}$ as $\tilde{\mathrm{x}} = \mathrm{x} \circ F^{-1}$. I also typically abbreviate "diffeomorphism" as "diff".

The ring of smooth real-valued functions on M I denote in the standard way as $C^\infty(M)$. The space of rank $\binom{k}{\ell}$ tensors[3] at a point $p \in M$ I denote by $\left(T^k_\ell\right)_p M$, with the usual special cases for the tangent and cotangent spaces $T_p M$ and $T^*_p M$. The rank $\binom{k}{\ell}$ tensor bundles I denote similarly as $T^k_\ell M$, and the space of smooth sections thereof in the usual way as $\Gamma^k_\ell M$, with the conventional notation for the spaces of vector and one-form fields $\mathcal{X}(M) \equiv \Gamma^1 M$ and $\mathcal{X}^*(M) \equiv \Gamma_1 M$.

I write the metric on M and its components as $\boldsymbol{g} = g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu$. The flat metric is denoted by $\boldsymbol{\eta} = \eta_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu$, and the metric perturbation of $\boldsymbol{g}$ about flat space is denoted by

$$\boldsymbol{g} = \boldsymbol{\eta} + \kappa \boldsymbol{h}, \quad \boldsymbol{h} = h_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu\,, \tag{1.5}$$

where κ is the gravitational coupling, given in terms of Newton's constant by $\kappa = \sqrt{32\pi G}$ in four dimensions. I use the space-negative convention, so that e.g. $\boldsymbol{\eta} = \mathrm{dt}^2 - \mathrm{d}\mathbf{x}^2$ in Cartesian coordinates. Objects related to the background metric are denoted by a bar, e.g. $\bar{\nabla}$ for the background gradient operator, while objects related to the full metric are not, e.g. ∇ for the full gradient operator. Indices are raised and lowered with the background metric, so that e.g. $h^\mu{}_\nu = \eta^{\mu\rho} h_{\rho\nu}$, with the sole exception being the inverse of the full metric, which I write as $\boldsymbol{g}^{-1} = g^{\mu\nu}\partial_\mu\partial_\nu$. Thus to first order in κ the inverse metric has components $g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}$.

[3] There seems to be no settled convention in the literature as to whether e.g. the k in "rank $\binom{k}{\ell}$" means "takes k vector arguments" or "carries k superscripted indices", so to be clear: in this thesis a rank $\binom{k}{\ell}$ tensor has k superscripted and ℓ subscripted indices.

Finally I will note that I write $\boldsymbol{g}$ and $\boldsymbol{h}$ in boldface in order to align myself with convention and write $g = \det \boldsymbol{g}$ and $h = \operatorname{tr} \boldsymbol{h}$. All other tensors I'll just write as e.g. $C = C^\mu{}_\nu \partial_\mu \otimes \mathrm{dx}^\nu$.

Many expressions will feature multiple integrals over momenta and/or position variables. To simplify the notation in these cases I adopt the shorthand $\int \mathrm{d}^d x = \int_x$ and $\int \mathrm{d}^d p\,/\,(2\pi)^d = \int_p$.

Acknowledgement of tools

Throughout this thesis I make almost ubiquitous use of Mathematica [50], in particular xAct and its extensions [51–58] for abstract tensor manipulations and FeynCalc [59–61] for one-loop tensor reductions. I also make use of the LaTeX package tikz-feynman [62] to produce the inline Feynman diagrams on display.

2 Diffeomorphism-invariant observables

2.1 There are no diffeomorphism-invariant local observables

It is well known that in a diffeomorphism-invariant field theory there exist no invariant local observables. In this section I demonstrate this claim, from both the passive and active perspectives of diffeomorphisms.[4]

2.1.1 The passive perspective

Let's begin with the passive perspective. Let M be our spacetime manifold, $\mathsf{x} : M \to \mathbb{R}^d$ a coordinate system on M, and $F : M \to M$ a diffeomorphism. In the passive perspective we think of x and F as providing us with a new coordinate system $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1} : M \to \mathbb{R}^d$. The transformation between these coordinates is given explicitly by the transition map between them, defined by $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1} : \mathsf{x}(M) \to \tilde{\mathsf{x}}(M)$, which eats a value $x \in \mathsf{x}(M) \subseteq \mathbb{R}^d$, takes us to the point $p = \mathsf{x}^{-1}(x) \in M$ whose coordinates are x in the "old" coordinate system, and spits out the coordinates $\mathsf{T}(x) = \tilde{\mathsf{x}}(p) \in \mathbb{R}^d$ of that point in the "new" coordinate system. For an equivalent

[4]See Appendix A for details on these two perspectives.

interpretation of T observe that

$$\mathsf{T} = \mathsf{x} \circ F^{-1} \circ \mathsf{x}^{-1} = \tilde{\mathsf{x}} \circ F^{-1} \circ \tilde{\mathsf{x}}^{-1}, \tag{2.1}$$

meaning that T is the coordinate representation of the (inverse of the) inducing diffeomorphism F^{-1} in *either* coordinate system.

From this viewpoint the tensor fields of interest, say some arbitrary $C \in \Gamma^k_\ell M$, do not themselves change – rather we shift our focus from the components $C^{\mu\ldots\nu}{}_{\rho\ldots\sigma}$ of C in the old coordinates to its components $\tilde{C}^{\mu\ldots\nu}{}_{\rho\ldots\sigma}$ in the new coordinates, e.g. for a rank $\binom{1}{1}$ tensor

$$C = C^\mu{}_\nu \partial_\mu \otimes \mathrm{d}\mathsf{x}^\nu = \tilde{C}^\mu{}_\nu \tilde{\partial}_\mu \otimes \mathrm{d}\tilde{\mathsf{x}}^\nu . \tag{2.2}$$

From eq. (A.84) we can relate the components in the old and new coordinates via the inducing diff as

$$\tilde{C}^\mu{}_\nu(p) = \left(F^* C\right)^\mu{}_\nu \circ F^{-1}(p) = \left(\partial_\alpha \left(F^{-1}\right)^\mu\right)_p \left(\partial_\nu F^\beta\right)_{F^{-1}(p)} C^\alpha{}_\beta(p). \tag{2.3}$$

evaluating at some point $p \in M$ for clarify and writing $F^\mu = \mathsf{x}^\mu \circ F$ and $\left(F^{-1}\right)^\mu = \mathsf{x}^\mu \circ F^{-1}$. It may be further shown (see eq. (A.85)) that the matrices in the above can be given explicitly in terms of the transition map as

$$\left(\partial_\mu F^\nu\right)_{F^{-1}(p)} = \frac{\partial\left(\mathsf{T}^{-1}\right)^\nu}{\partial x^\mu}\left(\tilde{\mathsf{x}}(p)\right), \quad \left(\partial_\mu \left(F^{-1}\right)^\nu\right)_p = \frac{\partial \mathsf{T}^\nu}{\partial x^\mu}\left(\mathsf{x}(p)\right), \tag{2.4}$$

i.e. we evaluate the derivatives of the transition map at the old coordinates of the point p and the derivatives of its inverse at the new coordinates of that same point. Thus if we express the transformation rule (2.3) as a function of the coordinates (which as we will discuss below is the more physically meaningful expression, despite obscuring the basic geometric significance) we find

$$\tilde{C}^\mu{}_\nu \circ \tilde{\mathsf{x}}^{-1}(x) = \frac{\partial \mathsf{T}^\mu}{\partial x^\alpha}\left(\mathsf{T}^{-1}(x)\right) \frac{\partial\left(\mathsf{T}^{-1}\right)^\beta}{\partial x^\nu}(x) \left(C^\alpha{}_\beta \circ \mathsf{x}^{-1}\left(\mathsf{T}^{-1}(x)\right)\right). \tag{2.5}$$

Let's unpack the above. The short explanation for what we've done is that we've taken eq. (2.3) and substituted in $\tilde{\mathsf{x}}^{-1}(x)$ for p. The immediate result of doing so is that we obtain the "natural" new-coordinate representation of C on the left hand side: its components in the new coordinate frame and coframe, evaluated as a function of the new coordinates.[5] We wish to relate this to the natural old-coordinate representation of C on the right hand side, so since $\tilde{\mathsf{x}}^{-1} = \mathsf{x}^{-1} \circ \mathsf{T}^{-1}$ we obtain the old-coordinate representation, evaluated at the coordinate value $\mathsf{T}^{-1}(x)$. That we evaluate these components at *different* coordinate values, even though they're the components at the *same* spacetime point, is just the consequence of the fact that in the passive perspective we assign different coordinates to the same point before and after the transformation. Finally by similar logic we obtain the transformation matrices evaluated at those same distinct coordinate values. These coordinate values differ for a different reason: unlike the components, which are evaluated at the same point but in different coordinate systems, the matrices in eq. (2.3) are evaluated at the distinct points p and $F(p)$ in the same coordinate system, meaning that when we substitute $\tilde{\mathsf{x}}^{-1}(x)$ for p in eq. (2.4) we obtain the given differing arguments.

Now let's turn to the main topic of this section. To see why there are no diff-invariant local observables from this passive perspective let's first be explicit about what we mean by that phrase. In particular let's begin by discussing the term *observable*, by which I refer to any quantity which one might reasonably hope to measure in an experiment. In the context of this thesis, which is entirely concerned with theories of tensor fields on spacetime, we will use the word to refer most broadly to any component of a tensor field, or any polynomial and/or integral thereof.[6]

To be more specific about what an observable is as a mathematical object let's think about

[5] N.B. since x is just some arbitrary element of $\mathbb{R}^d$ I don't need to decorate it with a tilde to indicate that it's a value of the new coordinates - the fact that it's mapped into spacetime by $\tilde{\mathsf{x}}^{-1}$ is enough to indicate that fact.

[6] Note that I very deliberately don't exclude the components of tensor fields of nontrivial rank from this definition. For example one might reasonably hope to measure the total energy in a region, which is the integral of the $(0,0)$ component of the energy-momentum tensor over that region. I'll also note that this is the definition of a *classical* observable - when we move to the quantum theory it will be the correlation functions of these objects which are the actual physically observable quantities, although we will still often refer to the classical observables as just the "observables".

how one would actually perform a measurement in a field theory, say of a scalar field $\phi : M \to \mathbb{R}$. The observer would wish to record not just the measured value of ϕ but also the time and place at which it had that value. To record these latter the observer would set up their coordinate system (or make reference to one preestablished) and record the coordinates at which the measurement occurs. Importantly, the observer does *not* record the actual spacetime point $p \in M$ at which the measurement occurs - they instead record its coordinates $x = \mathsf{x}(p) \in M$. To emphasize this point consider the fact that if two observers who do not know each other make a measurement apiece, and then they meet to discuss their measurements, they would have no way of knowing whether they took their measurements at the same place and time unless they first established the rules by which to compare the coordinate values they recorded, i.e. unless they first determined the transition map between their coordinates.

All of the above is to say: it is *not* the actual map $\phi : M \to \mathbb{R}$ which is observable, but rather its representation $\phi \circ \mathsf{x}^{-1} : \mathsf{x}(M) \to \mathbb{R}$. More generally, when I refer to a component of a tensor field (or a polynomial and/or integral thereof) as an observable, I am referring specifically to its coordinate representation. Distinguishing between fields and their coordinate representations may seem unnecessarily pedantic now (and when we turn to the active perspective it certainly is), but it is critical to understanding precisely what is meant by a diff-invariant observable, to which we will come in a moment.

Before discussing diffeomorphism invariance we will briefly discuss the entirely straightforward notion of a *local* observable, which is one which requires a measurement at only a single point in space. In other words the value of a field at a point, e.g. $\phi \circ \mathsf{x}^{-1}(x)$, is a local observable, while e.g. the integral of the field over spacetime or some spacelike hypersurface is nonlocal. (As a spoiler: this is the property we will sacrifice in order to construct diff-invariant observables in sec. 2.3.)

Now, the basic definition of *diffeomorphism invariance* is what you likely expect: an observable is diff-invariant if it does not change under diffeomorphisms. From the passive perspective this means that the value of the observable in one coordinate system x agrees with its value in any other $\tilde{\mathrm{x}}$.

It is immediately apparent that, for example, the value at a point in spacetime of a component of a tensor field of nontrivial rank cannot be diff-invariant, since the components of such a tensor mix amongst themselves under the transformation (2.3). However one might think that any scalar quantity ought to be diff-invariant, since performing a coordinate transformation doesn't change e.g. the value $\phi(p)$ of a scalar field ϕ at a point $p \in M$. But this is precisely the point at which the distinction we made above - that the actual observable quantity is the *coordinate representation* of the field, instead of the field itself - becomes critical: in order for a local observable to be invariant, it must give the same value when measured at the same *coordinate values*, no matter the coordinate system used. In other words it doesn't matter that $\phi(p)$ is the same in all coordinate systems. What matters is that $\phi \circ \mathrm{x}^{-1}(x)$ and $\phi \circ \tilde{\mathrm{x}}^{-1}(x)$, i.e. the values of the distinct coordinate representations of ϕ at the same coordinate value x, are in general not equal, since they are the values of ϕ at the distinct points $p = \mathrm{x}^{-1}(x)$ and $\tilde{p} = \tilde{\mathrm{x}}^{-1}(x)$.

In other words, to summarize: from the passive perspective, even the value of a scalar field at a fixed coordinate value x changes under a diffeomorphism, since the diffeomorphism changes the spacetime point to which the coordinate value x is assigned. And since the components of tensors of nontrivial rank mix amongst themselves under diffeomorphisms, on top of the spacetime point at which the measurement is made changing, there are thus no local observables which are invariant under diffeomorphisms.

2.1.2 The active perspective

In this section we'll argue the same point, in much briefer fashion, from the active perspective. Again let M be our spacetime manifold, $\mathrm{x} : M \to \mathbb{R}^d$ a coordinate system on M, and $F : M \to M$ a diffeomorphism. In the active perspective we leave x alone and think of F as actually pulling the fields of interest around on M:

$$C \in \Gamma^k_\ell M \mapsto \tilde{C} \equiv F^* C. \tag{2.6}$$

Thus, while in the passive perspective the components $\tilde{C}^{\mu\ldots\nu}{}_{\rho\ldots\sigma}$ are of the original tensor C and evaluated in the new coordinates $\tilde{\mathsf{x}}$, in the active perspective they are the components of a new tensor $\tilde{C}$ and evaluated in the old coordinates x.

From eq. (2.3) we can obtain the active transformation rule as a function of the spacetime point p, simply by sending $p \mapsto F(p)$ in the second and third expressions:

$$\tilde{C}^{\mu}{}_{\nu}(p) = \left(F_* C\right)^{\mu}{}_{\nu}(p) = \left(\partial_\alpha \left(F^{-1}\right)^{\mu}\right)_{F(p)} \left(\partial_\nu F^{\beta}\right)_p C^{\alpha}{}_{\beta} \circ F(p) \tag{2.7}$$

again giving the case of a rank $\binom{1}{1}$ tensor for notational brevity. Eq. (2.7) seems to give an explicitly *different* expression for $\tilde{C}^{\mu}{}_{\nu}$ than eq. (2.3), since the right hand sides are the *same* functions evaluated at *different* spacetime points, and this might call into question the common claim that the active and passive perspectives are equivalent.

To see that this claim is not in fact in any jeopardy let's obtain the analogue of eq. (2.5), i.e. the more physically significant relationship of the coordinate representations. In the passive perspective the natural new-coordinate representation was $\tilde{C}^{\mu}{}_{\nu} \circ \tilde{\mathsf{x}}^{-1}$, so we substituted $\tilde{\mathsf{x}}^{-1}(x)$ for p in eq. (2.3) to obtain eq. (2.5). In the active perspective we only have one coordinate system x, so we ought to substitute $\mathsf{x}^{-1}(x)$ for p in eq. (2.7), and doing so yields

$$\tilde{C}^{\mu}{}_{\nu} \circ \mathsf{x}^{-1}(x) = \frac{\partial \mathsf{T}^{\mu}}{\partial x^{\alpha}}\left(\mathsf{T}^{-1}(x)\right) \frac{\partial \left(\mathsf{T}^{-1}\right)^{\beta}}{\partial x^{\nu}}(x) \left(C^{\alpha}{}_{\beta} \circ \mathsf{x}^{-1}\left(\mathsf{T}^{-1}(x)\right)\right). \tag{2.8}$$

which is precisely the same relationship as eq. (2.5).[7] Since the coordinate representations are the physical quantities this confirms the claim that the active and passive perspectives are physically equivalent.

N.B. the symbol T appears throughout eq. (2.8), even though in the active perspective we have only one coordinate system and therefore we should have no transition map at all. The symbol is instead showing up here in its other guise, as you might recall from the previous sec-

[7] There is an apparent difference: in eq. (2.5) we have $\tilde{C}^{\mu}{}_{\nu} \circ \tilde{\mathsf{x}}^{-1}$ on the left hand side, while in eq. (2.5) we have $\tilde{C}^{\mu}{}_{\nu} \circ \mathsf{x}^{-1}$. However this is not a difference in fact, since in either case the object at hand is the relevant natural coordinate representation of $\tilde{C}^{\mu}{}_{\nu}$.

tion, as the coordinate representation of the diffeomorphism: $\mathsf{T} = \mathsf{x} \circ F^{-1} \circ \mathsf{x}^{-1}$. In this context it is, again, *not* playing the role of the transition map, i.e. eating a coordinate value x, taking us to the spacetime point $p = \mathsf{x}^{-1}(x)$ which is assigned those coordinates under the old system, and telling us the coordinates $\tilde{\mathsf{x}}(p) = \mathsf{T}(x)$ of that point under the new system. Rather it eats a coordinate value x, takes us to the spacetime point p with those coordinates, moves us from p to $F^{-1}(p)$, and then tells us the coordinates $\mathsf{x}\big(F^{-1}(p)\big) = \mathsf{T}(x)$ of that new point, all working in the same solitary coordinate system x. Of course as a map $\mathbb{R}^d \to \mathbb{R}^d$, and as a representation of F, this is precisely the same T we've been using all this time - the only difference is that, not having any new coordinate system to transition to, we must think of the intermediate step as moving us around in M, instead of changing the coordinates we're using on M.

Let us now return to demonstrating that in this picture there are no diff-invariant local observables. We can directly import our definition of the term from the previous section, with the simplification that, since in the active perspective we're concerned with only one coordinate system, we need not worry about distinguishing between the field as a function of spacetime and the field as a function of the coordinates. In other words in this picture a local observable is diff-invariant if and only if its value at an arbitrary point $p \in M$ is unchanged under diffeomorphisms.

In fact in this perspective it is nearly self-evident that there exist no diff-invariant local observables. As before we can immediately discount the components of any tensor field of nontrivial rank, leaving scalar fields as our only hope. But even scalar fields change under diffs in the active picture: the pullback of a scalar field $\phi : M \to \mathbb{R}$ by F is given by $F^*\phi = \phi \circ F$, meaning that $F^*\phi \neq \phi$ for arbitrary p outside of the specific cases in which ϕ is a constant or F is the identity map. And therefore, as previously claimed, in the active perspective there are also no diff-invariant local observables.

2.2 THE COORDINATE SCALARS

In sec. 2.1 we discussed at length the impossibility of diff-invariant local observables. In this and the next section I will describe a recently-developed program [40–48], called the *relational approach*, by which to construct, given any local observable, a corresponding diff-invariant but nonlocal observable.

To understand the relational approach let's start by recapping the narrative of the previous section, from the passive perspective. Suppose that two observers wish to measure some observable, which for simplicity we'll take to be a scalar field $\phi : M \to \mathbb{R}$. Then each would set up their coordinate systems and make their measurements and record where and when they did so. Thus each would measure some value $\phi \circ \mathrm{x}^{-1}(x)$ and $\phi \circ \tilde{\mathrm{x}}^{-1}(\tilde{x})$, where x and $\tilde{x}$ are the coordinate *values* at which each observer made their measurement and x and $\tilde{\mathrm{x}}$ are the coordinate *systems* each has set up. If these observers wished they could then sit down later and talk about it, and they could (in principle) figure out whether their different sets of numbers x and $\tilde{x}$ corresponded to the same point in space. However this doesn't change the fact that in order for the observable to be diff-invariant it must appear the same to them in their own frames, without them sitting down and figuring out how to translate from one of their systems to the other, and if $\mathrm{x} \neq \tilde{\mathrm{x}}$ and $x = \tilde{x}$ (i.e. if the observers are distinct and make their measurements at the same coordinate values) then their measurements cannot in general be the same in both frames.

The above also points to a resolution to the problem: if it were somehow possible to "signpost" each point in spacetime, so that observers in different coordinate systems could still agree on the point at which to make their measurements, then it would certainly be possible to make diff-invariant observations - each observer would simply mark down the signpost at which the measurement was made, instead of the coordinates in their own system.[8] To put this more quantitatively, we would hope to construct a "master" coordinate system, and pro-

[8] This is all framed from the passive perspective. From the active perspective we would want our signposts to get pushed around by diffeomorphisms in the same manner as the fields of the theory.

vide to every observer the means to obtain these master coordinates given only information in their frame. In this section I will review the construction of such a coordinate system [47, 48]. In the next I review the resulting general construction of nonlocal diffeomorphism-invariant observables and obtain the explicit perturbative expansions of the invariantized volume factor $\sqrt{-g}$ and scalar curvature R, which to the best of my knowledge have not previously been obtained in the literature.

2.2.1 The coordinate scalars as a function of the background coordinates

Our first order of business is to construct the master coordinates. We denote these by $\mathsf{X} : M \to \mathbb{R}^d$, with the italic symbol X referring to an arbitrary value, i.e. $\mathsf{X}(p) = X$ for $p \in M$. Since X are not "coordinates" in the usual sense, with instead each component X^μ transforming under diffeomorphisms in precisely the same way as an arbitrary scalar field, we will refer to these as the *coordinate scalars*.

The construction of the coordinate scalars depends on the setting in which they are constructed. In our case we are interested in perturbations about flat space, which implies the following.

- We assume some pre-existing coordinate system $\mathsf{x} = (\mathsf{t}, \mathbf{x}) : M \to \mathbb{R}^d$, an arbitrary value of which is denoted $\mathsf{x}(p) = x = (t, \boldsymbol{x}) \in \mathbb{R}^d$. We'll call these the *background coordinates*.

- We assume the existence of a metric $\boldsymbol{g}$ on M, whose perturbation about flat space is denoted in the usual way,

$$\boldsymbol{g} = \boldsymbol{\eta} + \kappa \boldsymbol{h}, \tag{2.9}$$

  with $\boldsymbol{\eta} = \eta_{\mu\nu}\, \mathrm{d}\mathsf{x}^\mu\, \mathrm{d}\mathsf{x}^\nu = \mathrm{dt}^2 - \mathrm{d}\mathbf{x}^2$ the flat metric. As discussed more fully in sec. A.6 the background metric is itself not a well-defined geometric object on M in either the active or passive pictures: in the active picture the background metric is unaffected by diffeomorphisms while all other fields (including the full metric) are pulled around, while in the passive picture the background metric is defined to have the same components in any coordinate system.

Following [47] we will begin from the observation that the background coordinates are harmonic with respect to the background metric, $\bar{\nabla}^2 \mathsf{x}^\mu = 0$. Since we are perturbing about flat space we then define the coordinate scalars (a) to be harmonic with respect to the perturbed (full) metric, and (b) to reduce to the background coordinates when the metric perturbation vanishes. In other words we define the coordinate scalars to satisfy

$$\nabla^2 \mathsf{X}^\mu = 0 \tag{2.10}$$

(recalling that ∇ denotes the gradient operator with respect to the *full* metric $\boldsymbol{g}$) and construct them perturbatively as

$$\mathsf{X} = \sum_{a=0}^{\infty} \kappa^a \mathsf{X}_a, \quad \mathsf{X}_0(p) = \mathsf{x}(p). \tag{2.11}$$

Note that the coordinate scalars depend nontrivially, by construction, on the background coordinates from which we build them: if we first proceed through the next paragraphs and then perform a passive coordinate transformation $\mathsf{x} \mapsto \tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}$ the explicitly constructed coordinate scalars will obey $\mathsf{X}_0 = \mathsf{x}$ and *not* $\mathsf{X}_0 = \tilde{\mathsf{x}}$. Equivalently if we perform an active transformation then the coordinate scalars will themselves change as any other scalar field, $\mathsf{X}^\mu \mapsto \tilde{\mathsf{X}}^\mu = F^* \mathsf{X}^\mu$, meaning that $\tilde{\mathsf{X}}_0(p) = \mathsf{X}_0 \circ F(p) \neq \mathsf{x}(p)$. (N.B. even though the components of X carry an index which looks superficially like a vector index they are all individually scalars, not the components of a vector.)

We can reexpress the equation (2.10) in a more perturbatively useful form by recalling [63] that we can write the full Laplacian ∇^2 in terms of the full metric $\boldsymbol{g}$, its determinant g, and the coordinate frame ∂_μ of the x coordinates as

$$\nabla^2 \mathsf{X}^\mu = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta \mathsf{X}^\mu \right). \tag{2.12}$$

We may then expand this expression in κ and solve for the X_a's order by order, which proceeds as follows. Expanding the various pieces gives

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha{}^\nu - \kappa^3 h^{\mu\alpha} h^{\nu\beta} h_{\alpha\beta} + \kappa^4 h^{\mu\alpha} h^{\nu\beta} h_\alpha{}^\gamma h_{\beta\gamma} + \mathcal{O}(\kappa^5) \tag{2.13}$$

for the inverse metric and

$$\begin{aligned}\sqrt{-g} = 1 + \tfrac{1}{2}\kappa h + \kappa^2\Big(\tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\Big) + \kappa^3\Big(\tfrac{1}{6}h^{\mu\nu}h_\mu{}^\alpha h_{\alpha\nu} - \tfrac{1}{8}hh_{\mu\nu}h^{\mu\nu} + \tfrac{1}{48}h^3\Big)\\ + \kappa^4\Big(-\tfrac{1}{8}h^{\mu\nu}h_\mu{}^\alpha h_\nu{}^\beta h_{\alpha\beta} + \tfrac{1}{12}hh^{\mu\nu}h_\mu{}^\alpha h_{\alpha\nu} + \tfrac{1}{32}(h_{\mu\nu}h^{\mu\nu})^2 - \tfrac{1}{32}h^2 h_{\mu\nu}h^{\mu\nu} + \tfrac{1}{384}h^4\Big) + \mathcal{O}(\kappa^5)\end{aligned} \tag{2.14}$$

for the volume factor. I review the derivations of these expansions in sec. 4.1.1 and 4.1.2. I also make use of the MATHEMATICA [50] package XACT [51–58] to confirm my work, as well as for the lengthier perturbative expansions to come. The $\mathcal{O}(\kappa^0)$ term in $\nabla^2 \mathsf{X}^\mu$ is then

$$\nabla^2\mathsf{X}^\mu = \partial_\alpha\big(\eta^{\alpha\beta}\partial_\beta\mathsf{X}_0^\mu\big) + \mathcal{O}(\kappa) = \bar{\nabla}^2\mathsf{X}_0^\mu + \mathcal{O}(\kappa). \tag{2.15}$$

Since we impose $\nabla^2\mathsf{X}^\mu = 0$ this term must vanish, although we in fact already knew this, since we also impose that $\mathsf{X}_0 = \mathsf{x}$.

Things get less trivial at $\mathcal{O}(\kappa)$. When we expand the right hand side of eq. (2.12) at this order we find one term featuring X_1^μ and with every other factor evaluated at $\mathcal{O}(\kappa^0)$, and the rest of the terms feature $\mathsf{X}_0^\mu = \mathsf{x}^\mu$ and various factors of $\boldsymbol{h}$. The X_1^μ term reduces to $\bar{\nabla}^2\mathsf{X}_1^\mu$, and (since $\partial_\mu\mathsf{X}_0^\nu = \partial_\mu\mathsf{x}^\nu = \delta_\mu^\nu$) the rest form a linear polynomial in $\boldsymbol{h}$, which we may denote J_1^μ. Setting $\nabla^2\mathsf{X}^\mu = 0$ at this order then implies that X_1^μ satisfies an equation of the form

$$\bar{\nabla}^2\mathsf{X}_1^\mu = J_1^\mu, \tag{2.16}$$

and using the expansions (2.13) and (2.14) and turning the crank yields

$$J_1^\mu = \partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h. \tag{2.17}$$

The objects in eqs. (2.16) and (2.17) are functions of spacetime, e.g. $\mathsf{X}_1^\mu : M \to \mathbb{R}$. To obtain an explicit expression for X_1 we need to rewrite these as functions of the background coordinates $\mathsf{x} : M \to \mathbb{R}^d$. Let's therefore denote the coordinate representations of these objects with hats, e.g. $\hat{\mathsf{X}} = \mathsf{X} \circ \mathsf{x}^{-1}$, so that $\hat{\mathsf{X}}(x)$ tells us the value of the coordinate scalars X at the spacetime point $p \in M$ whose background coordinates are $x \in \mathbb{R}^d$. The right hand side of eq. (2.16) expands to the standard coordinate representation of the flat-space Laplacian,

$$\left(\bar{\nabla}^2 \mathsf{X}_1^\mu\right) \circ \mathsf{x}^{-1}(x) = \eta^{\alpha\beta} \frac{\partial^2 \hat{\mathsf{X}}_1^\mu}{\partial x^\alpha \partial x^\beta}(x) = \left(\Box \hat{\mathsf{X}}_1^\mu\right)(x), \tag{2.18}$$

denoting by $\Box$ the explicit coordinate representation $\eta^{\alpha\beta}\, \partial^2/\partial x^\alpha \partial x^\beta$ of the flat-space Laplacian $\bar{\nabla}^2$. Eq. (2.16) then becomes

$$\Box \hat{\mathsf{X}}_1^\mu = \hat{J}_1^\mu, \tag{2.19}$$

meaning that, given a Green function $G(x, x')$ of $\Box$, we obtain the explicit solution

$$\hat{\mathsf{X}}_1^\mu(x) = \int \mathrm{d}^d x'\, G(x, x') \hat{J}_1^\mu(x'). \tag{2.20}$$

Similar logic applies at $\mathcal{O}(\kappa^2)$. There is one term featuring X_2^μ and with every other factor evaluated at $\mathcal{O}(\kappa^0)$, which term reduces to $\bar{\nabla}^2 \mathsf{X}_2^\mu$. There is then a collection of terms featuring X_1^μ, in each of which one of the other factors is evaluated at $\mathcal{O}(\kappa)$ and the rest are at $\mathcal{O}(\kappa^0)$. This collection takes the form of an $\boldsymbol{h}$-linear scalar differential operator acting on X_1^μ, which we may denote $K_1 \mathsf{X}_1^\mu$. Finally there is another collection of terms featuring X_0^μ, and this collection reduces to a quadratic polynomial in $\boldsymbol{h}$, which we will denote J_2^μ. Setting $\nabla^2 \mathsf{X}^\mu = 0$ at this order therefore implies that X_2^μ satisfies an equation of the form

$$\bar{\nabla}^2 \mathsf{X}_2^\mu = J_2^\mu + K_1 \mathsf{X}_1^\mu, \tag{2.21}$$

and turning the crank yields

$$K_1 = h^{\alpha\beta}\partial_\alpha\partial_\beta + J_1^\alpha\partial_\alpha, \quad J_2^\mu = \frac{1}{2}\Big(h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + h^{\alpha\mu}\partial_\alpha h\Big) - \partial_\alpha\Big(h^{\alpha\beta}h_\beta{}^\mu\Big). \tag{2.22}$$

By the same logic as for X_1 we can then obtain an explicit solution for X_2^μ as

$$\hat{\mathsf{X}}_2^\mu(x) = \int \mathrm{d}^d x'\, G(x,x')\Big(\hat{J}_2^\mu(x') + \hat{K}_1\hat{\mathsf{X}}_1^\mu(x')\Big), \tag{2.23}$$

again with $G(x,x')$ a Green function of $\Box$ and with hats denoting the coordinate representations in x.

The expressions (2.20) and (2.23) make manifest the tradeoff we're making in this construction. As we'll see below the X's do allow us to define diffeomorphism forms of tensor components of arbitrary rank. However these "invariantized" tensor components will be written in terms of the X's, which contain explicit integrations over all of spacetime, and hence the gauge-invariant observables so defined will be nonlocal. This is to be expected given the discussion of sec. 2.1.

In fact we can make systematic the above construction to all orders as follows. Let's define

$$D^\mu \equiv \frac{1}{\sqrt{-g}}\partial_\alpha\big(\sqrt{-g}\,g^{\alpha\mu}\big) \equiv -\sum_{n=0}^{\infty}\kappa^n J_n^\mu. \tag{2.24}$$

Since $g^{\alpha\mu} = g^{\alpha\beta}\partial_\beta x^\mu$ we can interpret D^μ as the Laplacian of the background coordinate component x^μ. As we will see below the J_n^μ's defined here include precisely the J_1^μ and J_2^μ we've already met. Let's also define, given any κ-independent function $f \in \mathrm{C}^\infty(M)$,

$$\nabla^2 f \equiv -\sum_{n=0}^{\infty}\kappa^2 K_n f. \tag{2.25}$$

Again we will see that the differential operators K_n include the same K_1 as before.

We can relate the K's and J's, so defined, by expanding the Laplacian of our κ-independent

f:

$$\nabla^2 f = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\mu}\partial_\mu f\right) = D^\mu\partial_\mu f + g^{\alpha\mu}\partial_\alpha\partial_\mu f. \tag{2.26}$$

Defining the perturbative expansion of the full inverse metric by

$$g^{\mu\nu} = \sum_{n=0}^{\infty}\kappa^n \tilde{g}_n^{\mu\nu} \tag{2.27}$$

we then have

$$\nabla^2 f = \sum_{n=0}^{\infty}\kappa^n\Big(-J_n^\mu\partial_\mu + \tilde{g}_n^{\alpha\mu}\partial_\alpha\partial_\mu\Big)f, \tag{2.28}$$

from which we can conclude that

$$K_n = J_n^\mu\partial_\mu - \tilde{g}_n^{\alpha\mu}\partial_\alpha\partial_\mu. \tag{2.29}$$

So far we have only considered the action of the Laplacian on the κ-independent function f. However what we are actually interested in is the action of the Laplacian on the coordinate scalars, which are not κ-independent, but rather infinite series in κ. This action can still be represented in terms of the K's, and therefore in terms of the J's, by expanding both the Laplacian and the coordinate scalar itself:

$$\nabla^2\mathrm{X}^\mu = -\sum_{r=0}^{\infty}\kappa^r K_r\mathrm{X}^\mu = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\kappa^{r+s}K_r\mathrm{X}_s^\mu = -\sum_{n=0}^{\infty}\kappa^n\sum_{r=0}^{n}K_r\mathrm{X}_{n-r}^\mu, \tag{2.30}$$

or to $\mathcal{O}(\kappa^2)$.

$$\nabla^2\mathrm{X}^\mu = -K_0\mathrm{X}_0^\mu - \kappa\Big(K_0\mathrm{X}_1^\mu + K_1\mathrm{X}_0^\mu\Big) - \kappa^2\Big(K_0\mathrm{X}_2^\mu + K_1\mathrm{X}_1^\mu + K_2\mathrm{X}_0^\mu\Big) + \mathcal{O}(\kappa^3). \tag{2.31}$$

To find the explicit forms of the K's we need the expansions of $g^{\mu\nu}$ and $\sqrt{-g}$, which are given by eqs. (2.13) and (2.14). At zeroth order we therefore have

$$D^\mu = \partial_\alpha\eta^{\alpha\mu} + \mathcal{O}(\kappa) = 0 + \mathcal{O}(\kappa) \implies J_0^\mu = 0, \tag{2.32}$$

and hence

$$K_0 = -\tilde{g}_0^{\mu}\nu\partial_\mu\partial_\nu = -\bar{\nabla}^2, \tag{2.33}$$

as it must. Thus at $\mathcal{O}(\kappa^0)$ and using our assumption $\mathsf{X}_0^\mu = x^\mu$ we find that the condition $\nabla^2\mathsf{X}^\mu = 0$ reduces to the harmonic gauge condition on the background coordinates, $\bar{\nabla}^2 x^\mu = 0$, as expected, and at arbitrary $\mathcal{O}(\kappa^n)$ the same condition yields a differential equation of the form

$$\bar{\nabla}^2\mathsf{X}_n^\mu = \sum_{r=1}^{n} K_r \mathsf{X}_{n-r}^\mu. \tag{2.34}$$

At first order we have

$$D^\mu = \partial_\alpha\Big(\big(1+\tfrac{1}{2}\kappa h\big)\big(\eta^{\alpha\mu}-\kappa h^{\alpha\mu}\big)\Big)+\mathcal{O}(\kappa^2) = \kappa\Big(\tfrac{1}{2}\partial^\mu h - \partial_\alpha h^{\alpha\mu}\Big)+\mathcal{O}(\kappa^2) \implies J_1^\mu = \partial_\alpha h^{\alpha\mu} - \tfrac{1}{2}\partial^\mu h, \tag{2.35}$$

in terms of which

$$K_1 = J_1^\mu\partial_\mu + h^{\mu\nu}\partial_\mu\partial_\nu, \tag{2.36}$$

both of which agree with the prior results (2.17) and (2.22). From eq. (2.34) we then find the equation for X_1^μ,

$$\bar{\nabla}^2\mathsf{X}_1^\mu = K_1\mathsf{X}_0^\mu = J_1^\alpha\partial_\alpha x^\mu + h^{\alpha\beta}\partial_\alpha\partial_\beta x^\mu = J_1^\mu, \tag{2.37}$$

in agreement with (2.16). Proceeding similarly at second order we have

$$\begin{aligned} D^\mu &= \big(1-\tfrac{1}{2}\kappa h\big)\partial_\alpha\Big\{\Big(1+\tfrac{1}{2}\kappa h+\kappa^2\big(\tfrac{1}{8}h^2-\tfrac{1}{4}h_{\rho\sigma}h^{\rho\sigma}\big)\Big)\Big(\eta^{\alpha\mu}-\kappa h^{\alpha\mu}+\kappa^2 h^{\mu\nu}h_\nu{}^\alpha\Big)\Big\} \\ &= \mathcal{O}(\kappa)+\kappa^2\Big\{-\tfrac{1}{2}h_{\alpha\beta}\partial^\mu h^{\alpha\beta}-\tfrac{1}{2}h^{\alpha\mu}\partial_\alpha h+\partial_\alpha\big(h^{\mu\nu}h_\nu{}^\alpha\big)\Big\}+\mathcal{O}(\kappa^3), \end{aligned} \tag{2.38}$$

from which we can read off

$$J_2^\mu = \tfrac{1}{2}\big(h_{\alpha\beta}\partial^\mu h^{\alpha\beta}+h^{\alpha\mu}\partial_\alpha h\big)-\partial_\alpha\big(h^{\alpha\beta}h_\beta{}^\mu\big), \tag{2.39}$$

in agreement with eq. (2.22). In terms of J_2 we then have

$$K_2 = J_2^\mu \partial_\mu - h^{\mu\alpha} h_\alpha{}^\nu \partial_\mu \partial_\nu, \tag{2.40}$$

and the general condition (2.34) at this order gives us the differential equation (2.21) for X_2,

$$\bar{\nabla}^2 \mathsf{X}_2^\mu = K_1 \mathsf{X}_1^\mu + K_2 \mathsf{X}_0^\mu = K_1 \mathsf{X}_1^\mu + J_2^\mu, \tag{2.41}$$

again using the fact that $\mathsf{X}_0^\mu = x^\mu$.

2.2.2 Toy: inverting a perturbatively-constructed function

In sec. 2.2.1 we obtained a perturbative expression for the coordinate scalars X as a function of the background coordinates x,

$$\hat{\mathsf{X}}(x) = x + \kappa \hat{\mathsf{X}}_1(x) + \kappa^2 \hat{\mathsf{X}}_2(x) + \mathcal{O}(\kappa^3), \tag{2.42}$$

where $\hat{\mathsf{X}} \equiv \mathsf{X} \circ \mathsf{x}^{-1}$ is the background-coordinate representation of $\mathsf{X} : M \to \mathbb{R}^d$ and the $\hat{\mathsf{X}}_a$'s are given in eqs. (2.20) and (2.23). However our goal is to express the tensor fields of a theory, which we know as functions of the background coordinates, in terms of the coordinate scalars, and thus our goal is to invert the relationship $\hat{\mathsf{X}}(x)$ to obtain the background coordinates as a function of the coordinate scalars. In fact we may obtain this inverse in terms of the same $\hat{\mathsf{X}}_a$'s as above. To make this procedure clear I will in this section demonstrate the analogous logic as applied to a simple function $\mathbb{R} \to \mathbb{R}$.

Suppose therefore that we have some $f : \mathbb{R} \to \mathbb{R}$, analogous to $\hat{\mathsf{X}}(x)$, which is known to us as a Taylor expansion in some parameter κ and which at $\mathcal{O}(\kappa^0)$ is the identity map:

$$f(x) = \sum_a \kappa^a f_a(x) = x + \kappa f_1(x) + \kappa^2 f_2(x) + \mathcal{O}(\kappa^3). \tag{2.43}$$

Our goal is to obtain an expression for the inverse of f, which I will denote $g = f^{-1} : \mathbb{R} \to \mathbb{R}$, in terms of the f_a's. We begin from the fact that $f \circ g$ is the identity map by definition and then use our defining expansion of f, evaluated at $g(y)$ (writing an arbitrary element of the domain of f as x and an arbitrary element of its range as y):

$$y = (f \circ g)(y) = g(y) + \kappa f_1\big(g(y)\big) + \kappa^2 f_2\big(g(y)\big) + \mathcal{O}(\kappa^3), \tag{2.44}$$

or

$$g(y) = y - \kappa f_1\big(g(y)\big) - \kappa^2 f_2\big(g(y)\big) + \mathcal{O}(\kappa^3). \tag{2.45}$$

We can systematically eliminate the explicit dependence on the unknown g on the right hand side as follows. The above tells us that at $\mathcal{O}(\kappa^0)$ the function g is just the identity map:

$$g(y) = y + \mathcal{O}(\kappa). \tag{2.46}$$

The full function g contains $\mathcal{O}(\kappa^n)$ terms for, in principle, arbitrarily large $n \geq 0$, so the superficially $\mathcal{O}(\kappa)$ term in eq. (2.45), $-\kappa f_1\big(g(y)\big)$, in fact contributes at all orders $n \geq 1$. However by using eq. (2.46) in the argument we may explicitly isolate the $\mathcal{O}(\kappa)$ contribution:

$$f_1\big(g(y)\big) = f_1(y) + \mathcal{O}(\kappa), \tag{2.47}$$

which yields an explicit expression for $g(y)$ up to $\mathcal{O}(\kappa)$ in terms of the (assumed known) f_a's:

$$g(y) = y - \kappa f_1(y) + \mathcal{O}(\kappa^2). \tag{2.48}$$

Now that we know $g(y)$ to $\mathcal{O}(\kappa)$ we may isolate the $\mathcal{O}(\kappa^2)$ contribution to $g(y)$ in a similar manner from the $\mathcal{O}(\kappa)$ contribution to $f_1\big(g(y)\big)$ and the $\mathcal{O}(\kappa^0)$ contribution to $f_2\big(g(y)\big)$. For the former we find

$$f_1\big(g(y)\big) = f_1\big(y - \kappa f_1(y)\big) + \mathcal{O}(\kappa^2) = f_1(y) - \kappa f_1(y) f_1'(y) + \mathcal{O}(\kappa^2), \tag{2.49}$$

and for the latter

$$f_2\big(g(y)\big) = f_2(y) + \mathcal{O}(\kappa), \tag{2.50}$$

which yields to $\mathcal{O}(\kappa^2)$

$$g(y) = y - \kappa f_1(y) + \kappa^2\Big(f_1(y)f_1'(y) - f_2(y)\Big) + \mathcal{O}(\kappa^3). \tag{2.51}$$

This procedure may in principle be continued to arbitrary order in κ to obtain an expression for $g(y)$ in terms of the expansion functions f_a, although for our purposes $\mathcal{O}(\kappa^2)$ is sufficient. Once the calculation has performed to some $\mathcal{O}(\kappa^n)$ an explicit calculation will verify that $g = f^{-1}$ to that same order. For example combining eqs. (2.43) and (2.51) yields the expected results

$$(f \circ g)(y) = y + \mathcal{O}(\kappa^3), \quad (g \circ f)(x) = x + \mathcal{O}(\kappa^3). \tag{2.52}$$

N.B. in eq. (2.51) the functions $f_a(y)$ are the exact same as the functions $f_a(x)$ which appear in eq. (2.43) – if $f_2(x) = x^2$ in the latter, then $f_2(y) = y^2$ in the former. This may not seem like a point worth making at the moment, but it'll be important in the next section.

2.2.3 The background scalars as a function of the background coordinates

Now let's return to the task at hand. We have an expression for the coordinate scalars $\mathsf{X} : M \to \mathbb{R}^d$ as a function of the background coordinates $\mathsf{x} : M \to \mathbb{R}^d$,

$$\hat{\mathsf{X}}(x) = x + \kappa \hat{\mathsf{X}}_1(x) + \kappa^2 \hat{\mathsf{X}}_2(x) + \mathcal{O}(\kappa^3), \tag{2.53}$$

where $\hat{\mathsf{X}} = \mathsf{X} \circ \mathsf{x}^{-1}$; the italic x is an arbitrary value of the background coordinates; and the functions $\hat{\mathsf{X}}_a(x)$ are given in eqs. (2.20) and (2.23). Our goal is to obtain an expression $\hat{\mathsf{x}} = \mathsf{x} \circ \mathsf{X}^{-1}$ for the background coordinates as a function of the coordinate scalars, i.e. to invert $\hat{\mathsf{X}}(x)$ for $\hat{\mathsf{x}}(X)$, where the italic X is an arbitrary value of the coordinate scalars. And this is hardly any more complicated than the toy calculation of the previous section! Since we are working entirely in terms of the coordinate representations $\hat{\mathsf{X}} = \mathsf{X} \circ \mathsf{x}^{-1} : x \in \mathbb{R}^d \mapsto X \in \mathbb{R}^d$ and $\hat{\mathsf{x}} = \mathsf{x} \circ \mathsf{X}^{-1} : X \in \mathbb{R}^d \mapsto x \in \mathbb{R}^d$ the problem is simply the d-dimensional generalization of the previous, and is in itself oblivious to the geometrical origins of these functions – the fact that $\hat{\mathsf{X}}$ and $\hat{\mathsf{x}}$ both take a pitstop in spacetime on their way between values of the background coordinates and the coordinate scalars is completely irrelevant to the process of inverting $\hat{\mathsf{X}}$.

We can therefore follow the exact same steps as in sec. 2.2.2, which I will here outline in brief. Starting from eq. (2.53) we use the fact that $X = \big(\hat{\mathsf{X}} \circ \hat{\mathsf{x}}\big)(X)$ to obtain

$$\hat{\mathsf{x}}^\mu(X) = X^\mu - \kappa \hat{\mathsf{X}}_1^\mu\big(\hat{\mathsf{x}}(X)\big) - \kappa^2 \hat{\mathsf{X}}_2^\mu\big(\hat{\mathsf{x}}(X)\big) + \mathcal{O}(\kappa^3), \tag{2.54}$$

analogous to eq. (2.45). From this we have

$$\hat{\mathsf{x}}^\mu(X) = X^\mu + \mathcal{O}(\kappa), \tag{2.55}$$

analogous to eq. (2.46), using which in the $\mathcal{O}(\kappa)$ term yields

$$\hat{\mathsf{x}}^\mu(X) = X^\mu - \kappa \hat{\mathsf{X}}_1^\mu(X) + \mathcal{O}(\kappa^2), \tag{2.56}$$

analogous to eq. (2.48), and using which in turn in the $\mathcal{O}(\kappa)$ and $\mathcal{O}(\kappa^2)$ terms yields

$$\hat{\mathsf{x}}^\mu(X) = X^\mu - \kappa \hat{\mathsf{X}}_1^\mu(X) + \kappa^2 \left(\hat{\mathsf{X}}_1^\alpha(X) \frac{\partial \hat{\mathsf{X}}_1^\mu}{\partial x^\alpha}(X) - \hat{\mathsf{X}}_2^\mu(X) \right) + \mathcal{O}(\kappa^3), \tag{2.57}$$

analogous to eq. (2.51).

It is here that the point raised at the end of the previous section becomes important. Recall that in the toy model I emphasized that the $f_a(y)$'s which appear in the expansion of $g(y)$ have the same functional dependence on y as the $f_a(x)$'s have on x in the expansion of $f(x)$. In the exact same way, the $\hat{\mathsf{X}}_a^\mu(X)$'s which appear in the above expansion of $\hat{\mathsf{x}}^\mu(X)$ have the same functional dependence on the coordinate scalar value X as the $\hat{\mathsf{X}}_a^\mu(x)$'s have on the background coordinate value x in the expansion of $\hat{\mathsf{X}}^{\mu}(x)$. In other words there is *nothing* implicit in eq. (2.57): $\hat{\mathsf{X}}_1^{\mu}(X)$ (for instance) means the function $\hat{\mathsf{X}}_1^{\mu} : \mathbb{R}^d \to \mathbb{R}$ evaluated at $X \in \mathbb{R}^d$, and nothing more. In particular, even though X represents an arbitrary value of the *coordinate scalars,* we're feeding it directly into $\hat{\mathsf{X}}_1^{\mu} = \mathsf{X}_1^{\mu} \circ \mathsf{x}^{-1}$ in the slot where we would expect to put a value of the *background coordinates.* While this may not feel right, it is in fact critical to the usefulness of this whole construction – we have explicit expressions for $\hat{\mathsf{X}}_1^{\mu}$ and $\hat{\mathsf{X}}_2^{\mu}$ in eqs. (2.20) and (2.23) as functions of the background coordinates, and eq. (2.57) tells us how to use these exact same results, with the desired value of the coordinate scalars playing the role of the background coordinates, to obtain (a second-order approximation of) the value of the background coordinates which corresponds to that value of the coordinate scalars.

2.2.4 Derivatives of and with respect to the background coordinates and the coordinate scalars

So we have constructed two coordinate systems on spacetime: the background coordinates $\mathsf{x} : M \to \mathbb{R}^d$, $p \mapsto \mathsf{x}(p) = x$, and the coordinate scalars $\mathsf{X} : M \to \mathbb{R}^d$, $p \mapsto \mathsf{X}(p) = X$. In this section I will carefully discuss the basis frames each of these coordinate systems.

It is important to keep in mind for this discussion that I am engaging in a slight abuse of notation here: namely, in this thesis the lowercase italic symbol x refers both to a generic value of the background coordinates and to the canonical coordinates on $\mathbb{R}^d$ themselves. This is directly relevant in the construction of the basis frames as follows. As discussed at length in the appendix (see sec. A.1.3), the basis frame of any coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ is given by the pushforward by x^{-1} of the canonical coordinate frame on $\mathbb{R}^d$:

$$\partial_{\mu} = \left(\mathsf{x}^{-1}\right)_* \frac{\partial}{\partial x^{\mu}} \implies \left(\partial_{\mu} f\right)_p = \frac{\partial\left(f \circ \mathsf{x}^{-1}\right)}{\partial x^{\mu}}\left(\mathsf{x}(p)\right). \tag{2.58}$$

Changing the coordinate system whose frame you're interested in does *not* change the basis frame on $\mathbb{R}^d$ which you push forward – it only changes the map x^{-1} by which you push it forward. Thus the basis frame of the coordinate scalars is the pushforward of the *same* coordinate frame $\partial/\partial x^\mu \in \mathfrak{X}(\mathbb{R}^d)$, just by X^{-1} this time:[9]

$$D_\mu = \left(\mathsf{X}^{-1}\right)_* \frac{\partial}{\partial x^\mu} \implies \left(D_\mu f\right)_p = \frac{\partial\left(f \circ \mathsf{X}^{-1}\right)}{\partial x^\mu}\left(\mathsf{X}(p)\right). \tag{2.59}$$

I make this point to emphasize that the denominator in eq. (2.59) should *not* be a capital X^μ – we are differentiating the coordinate scalar representation $f \circ \mathsf{X}^{-1} : \mathbb{R}^d \to \mathbb{R}$ with respect to the *same* coordinates on $\mathbb{R}^d$ as those with respect to which we differentiate the background coordinate representation $f \circ \mathsf{x}^{-1} : \mathbb{R}^d \to \mathbb{R}$ in eq. (2.58). The only differences are the coordinate representations $f \circ \mathsf{x}^{-1}$ and $f \circ \mathsf{X}^{-1}$ themselves, and the coordinate values $\mathsf{x}(p)$ and $\mathsf{X}(p)$ at which we evaluate the derivatives.

This is directly relevant to explicit calculations in that, if we did write $\partial/\partial X^\mu$ instead of $\partial/\partial x^\mu$, that would then mistakenly suggest that we need an extra factor of $\partial\hat{\mathsf{X}}^\mu/\partial x^\nu$ to relate D_μ and ∂_μ, and including this extra factor would lead to outright errors in our calculations. (This is especially important when we construct the relational Christoffel symbols – including an extra $\partial\hat{\mathsf{X}}^\mu/\partial x^\nu$ next to the partial derivatives in that construction would then lead to an incorrect invariantized Ricci scalar.)

2.3 Relational observables

We now come to the crux of this section: the construction, given any tensor field $C \in \Gamma^k_\ell M$, of a set of corresponding diffeomorphism-invariant observables.

[9] I use D_μ for the basis from of the coordinate scalars in keeping with the general theming of “lowercase for background, uppercase for scalars”. N.B. D_μ does *not* in this thesis refer to the gauge covariant derivative of some Yang-Mills theory.

2.3.1 Defining relational observables

The *relational observable* $\mathscr{C}^\mu{}_\nu$ corresponding to any component $C^\mu{}_\nu$ of C is defined [47, 48] to be that component in the coordinate system defined by the coordinate scalars:

$$C \equiv \mathscr{C}^\mu{}_\nu D_\mu \otimes \mathrm{d}\mathsf{X}^\nu . \tag{2.60}$$

If the tensor field has a name then the corresponding set of relational observables is its *invariantized* form (e.g. the invariantized metric in sec. 2.3.4).

In terms of the components $C^\mu{}_\nu$ of C in the background coordinates the invariantized form is found by transformating from x to X as one would transform between any coordinate systems, namely

$$\mathscr{C}^\mu{}_\nu = \left(\partial_\alpha \mathsf{X}^\mu\right)\left(D_\nu \mathsf{x}^\beta\right) C^\alpha{}_\beta. \tag{2.61}$$

N.B. the above is just the standard rule (A.62) for the passive transformation of the components of a tensor field, with the coordinate scalars X^μ and the corresponding frame D_μ playing the role of the "new" coordinates and frame $\tilde{\mathsf{x}}$ and $\tilde{\partial}_\mu$.

Evaluating eq. (2.61) at a point $p \in M$ yields

$$\mathscr{C}^\mu{}_\nu(p) = \left(\partial_\alpha \mathsf{X}^\nu\right)_p \left(D_\nu \mathsf{x}^\beta\right)_p C^\alpha{}_\beta(p). \tag{2.62}$$

Let's rewrite the above more explicitly in terms of functions of the coordinates, starting with the transformation matrices. This is just the calculation of sec. A.2.1, with X in place of $\tilde{\mathsf{x}}$ and $\hat{\mathsf{X}} = \mathsf{X} \circ \mathsf{x}^{-1}$ in place of the arbitrary transition map $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1}$. For the first matrix we find

$$\left(\partial_\mu \mathsf{X}^\nu\right)_p = \left(\mathsf{x}^{-1}\right)_* \left(\left.\frac{\partial}{\partial x^\mu}\right|_{\mathsf{x}(p)}\right) \mathsf{X}^\nu = \frac{\partial\left(\mathsf{X}^\nu \circ \mathsf{x}^{-1}\right)}{\partial x^\mu}\left(\mathsf{x}(p)\right) = \frac{\partial \hat{\mathsf{X}}^\nu}{\partial x^\mu}\left(\mathsf{x}(p)\right), \tag{2.63}$$

and similarly for the second

$$\left(D_\mu \mathsf{x}^\nu\right)_p = \left(\mathsf{X}^{-1}\right)_* \left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{X}(p)}\right)\mathsf{x}^\nu = \frac{\partial \hat{\mathsf{x}}^\nu}{\partial x^\mu}\left(\mathsf{X}(p)\right). \tag{2.64}$$

In sec. 2.3.2 we'll expand these transformation matrices in terms of the $\hat{\mathsf{X}}^\mu_a$'s.

In the literature these matrices are often written more concisely as $\partial X^\nu/\partial x^\mu$ and $\partial x^\nu/\partial X^\mu$. However I will emphasize once again that I am not making a mistake by leaving the denominator in the latter lowercase – in both matrices we differentiate the transition map and its inverse using the *same* basis frame on $\mathbb{R}^d$, but we evaluate the matrices at the different coordinate values $\mathsf{x}(p)$ and $\mathsf{X}(p)$. It is this latter difference which is more concisely indicated by the differing denominators in the literature. I make the distinction here to make it clear that there is no extra factor of $\partial X^\mu/\partial x^\nu$ needed to relate the derivatives in the two matrices.

Let's return to the question of writing eq. (2.62) in terms of functions of the coordinates. Since the invariantized $\mathscr{C}^\mu{}_\nu$ is a component of the tensor C in the X coordinate system its natural coordinate representation is as a function of the background coordinates:

$$\hat{\mathscr{C}}^\mu{}_\nu \equiv \mathscr{C}^\mu{}_\nu \circ \mathsf{X}^{-1}. \tag{2.65}$$

We should therefore compose both sides of eq. (2.62) with X^{-1}:

$$\hat{\mathscr{C}}^\mu{}_\nu(X) = \left(\partial_\alpha \mathsf{X}^\mu\right)_{\mathsf{X}^{-1}(X)}\left(D_\nu \mathsf{x}^\beta\right)_{\mathsf{X}^{-1}(X)} C^\alpha{}_\beta \circ \mathsf{X}^{-1}(X). \tag{2.66}$$

From eqs. (2.63) and (2.64) we can simplify the derivative matrices. For the first we find

$$\left(\partial_\mu \mathsf{X}^\nu\right)_{\mathsf{X}^{-1}(X)} = \frac{\partial \hat{\mathsf{X}}^\nu}{\partial x^\mu}\left(\hat{\mathsf{x}}(X)\right), \tag{2.67}$$

i.e. the μ^{th} derivative of $\hat{\mathsf{X}}^\nu$, evaluated at the background coordinate value of the point with coordinate scalar value X. For the second we find

$$\left(D_\mu \mathsf{x}^\nu\right)_{\mathsf{X}^{-1}(X)} = \frac{\partial \hat{\mathsf{x}}^\nu}{\partial x^\mu}(X). \tag{2.68}$$

i.e. the μ^{th} derivative of $\hat{\mathsf{x}}^\nu$, evaluated directly at the coordinate scalar value X. (To reiterate ad infinitum, there is no mistake in the denominator being lowercase.) Finally we can rewrite $C^\alpha{}_\beta \circ \mathsf{X}^{-1}(X)$ in terms of the natural coordinate representation[10] $\hat{C}^\mu{}_\nu \equiv C^\mu{}_\nu \circ \mathsf{x}^{-1}$ as

$$C^\alpha{}_\beta \circ \mathsf{X}^{-1}(X) = \hat{C}^\alpha{}_\beta\big(\hat{\mathsf{x}}(X)\big). \tag{2.69}$$

Thus in terms of the natural coordinate representations eq. (2.62) becomes

$$\hat{\mathscr{C}}^\mu{}_\nu(X) = \frac{\partial \hat{\mathsf{X}}^\mu}{\partial x^\alpha}\big(\hat{\mathsf{x}}(X)\big)\frac{\partial \hat{\mathsf{x}}^\beta}{\partial x^\nu}\big(X\big)\hat{C}^\alpha{}_\beta\big(\hat{\mathsf{x}}(X)\big). \tag{2.70}$$

2.3.2 Perturbative expansion of the transformation matrices

To obtain an explicit expression for a relational observable we need the derivative matrices which transform tensor components from the background coordinates x to the coordinate scalars X.

Let's begin with the "forward" derivative $\partial_\mu \mathsf{X}^\nu$, whose coordinate representation we know from eq. (2.67). To explicitly write it in terms of the $\hat{\mathsf{X}}^\mu_a$'s we start by differentiating the expansion of $\hat{\mathsf{X}}(x)$ in κ,

$$\frac{\partial \hat{\mathsf{X}}^\nu}{\partial x^\mu}(x) = \delta^\nu_\mu + \kappa \frac{\partial \hat{\mathsf{X}}^\nu_1}{\partial x^\mu}(x) + \kappa^2 \frac{\partial \hat{\mathsf{X}}^\nu_2}{\partial x^\mu}(x) + \mathcal{O}(\kappa^3). \tag{2.71}$$

Now evaluate the above at $x = \hat{\mathsf{x}}(X)$, using the expansion (2.57). In fact since the $\mathcal{O}(\kappa^0)$ term in

[10] For the interested reader I will note that it may be straightforwardly verified that this definition of $\hat{C}^\mu{}_\nu$ is equivalent to defining $\hat{C} = \left(\mathsf{x}^{-1}\right)^* C \in \Gamma^k_\ell \mathbb{R}^d$ and taking the components of the result in the canonical basis frame and coframe on $\mathbb{R}^d$. (An analogous statement holds for $\hat{\mathscr{C}}$ and X.)

eq. (2.71) is independent of x we only need $\hat{\mathsf{x}}(X)$ to $\mathcal{O}(\kappa)$,

$$\hat{\mathsf{x}}(X) = X - \kappa\hat{\mathsf{X}}_1(X) + \mathcal{O}(\kappa), \tag{2.72}$$

from which we find

$$\frac{\partial\hat{\mathsf{X}}^\nu}{\partial x^\mu}\big(\hat{\mathsf{x}}(X)\big) = \delta^\nu_\mu + \kappa\frac{\partial\hat{\mathsf{X}}^\nu_1}{\partial x^\mu}(X) + \kappa^2\left(\frac{\partial\hat{\mathsf{X}}^\nu_2}{\partial x^\mu}(X) - \hat{\mathsf{X}}^\alpha_1(X)\frac{\partial^2\hat{\mathsf{X}}^\nu_1}{\partial x^\alpha\partial x^\mu}(X)\right) + \mathcal{O}(\kappa^3). \tag{2.73}$$

N.B. while on the left hand side of eq. (2.73) the coordinate scalar value X is converted to a background coordinate value by $\hat{\mathsf{x}}$, there is no such $\hat{\mathsf{x}}$ implicit on the right hand side. For example $\hat{\mathsf{X}}^\nu_1 : \mathbb{R}^d \to \mathbb{R}$ is a function of the background coordinates $x \in \mathbb{R}^d$, which we differentiate with respect to the μ^{th} canonical coordinate x^μ on $\mathbb{R}^d$ to obtain $\partial\hat{\mathsf{X}}^\nu_1/\partial x^\mu : \mathbb{R}^d \to \mathbb{R}$, and we then plug the coordinate scalar value $X \in \mathbb{R}^d$ directly into this function.

For the "backward" derivative $D_\mu\mathsf{x}^\nu$ we similarly use the coordinate representation (2.68) and the expansion (2.57) of $\hat{\mathsf{x}}(X)$ in terms of the $\hat{\mathsf{X}}^\mu_a$'s, from which we obtain

$$\frac{\partial\hat{\mathsf{x}}^\nu}{\partial x^\mu}(X) = \delta^\nu_\mu - \kappa\frac{\partial\hat{\mathsf{X}}^\nu_1}{\partial x^\mu}(X) + \kappa^2\left(\hat{\mathsf{X}}^\alpha_1(X)\frac{\partial^2\hat{\mathsf{X}}^\nu_1}{\partial x^\alpha\partial x^\mu}(X) + \frac{\partial\hat{\mathsf{X}}^\alpha_1}{\partial x^\mu}(X)\frac{\partial\hat{\mathsf{X}}^\nu_1}{\partial x^\alpha}(X) - \frac{\partial\hat{\mathsf{X}}^\nu_2}{\partial x^\mu}(X)\right) + \mathcal{O}(\kappa^3). \tag{2.74}$$

Note that it may be straightforwardly checked that the above results satisfy the condition

$$\frac{\partial x^\nu}{\partial x^\mu}(X) = \frac{\partial\big(\hat{\mathsf{X}}\circ\hat{\mathsf{x}}\big)^\nu}{\partial x^\mu}(X) = \frac{\partial\hat{\mathsf{X}}^\nu}{\partial x^\alpha}\big(\hat{\mathsf{x}}(X)\big)\frac{\partial\hat{\mathsf{x}}^\alpha}{\partial x^\mu}(X) \tag{2.75}$$

to $\mathcal{O}(\kappa^2)$, as they must.

2.3.3 Invariantized scalars

The simplest example of a relational observable is the invariantized form $\Phi = \phi\circ\mathsf{X}^{-1} : \mathbb{R}^d \to \mathbb{R}$ of a real scalar field $\phi : M \to \mathbb{R}$. Our goal is to obtain an explicit expression for Φ entirely in terms of quantities which are known in the background coordinate system, namely:

- the coordinate representation of the scalar field, $\hat{\phi} = \phi \circ \mathsf{x}^{-1} : \mathbb{R}^d \to \mathbb{R}$.

- the coordinate representation of the perturbative expansion of the coordinate scalars, i.e. the $\hat{\mathsf{X}}^\mu_a$'s.

Before proceeding I will note that for the scalar field we have only three distinct quantities – the original scalar $\phi : M \to \mathbb{R}$, the background coordinate representation $\hat{\phi} = \phi \circ \mathsf{x}^{-1}$, and the invariantized scalar $\Phi = \phi \circ \mathsf{X}^{-1}$, which I am here conflating with its own coordinate representation. This is in contrast with a tensor field of nontrivial rank, for which there are four distinct quantities – the original tensor components $C^\mu{}_\nu : M \to \mathbb{R}$, the background coordinate representation $\hat{C}^\mu{}_\nu = C^\mu{}_\nu \circ \mathsf{x}^{-1}$ of those components, the invariantized components $\mathscr{C}^\mu{}_\nu : M \to \mathbb{R}$, and the coordinate representation $\hat{\mathscr{C}}^\mu{}_\nu = \mathscr{C}^\mu{}_\nu \circ \mathsf{X}^{-1}$ of those invariantized components. For the scalar we may conflate the latter two simply because a scalar field does not have different components in different coordinate systems, so the only new quantity introduced by the relational program is the coordinate representation $\Phi = \phi \circ \mathsf{X}^{-1}$ of the scalar field with respect to the coordinate scalars.

Anyway, to business. We use the fact that $\mathsf{X}^{-1} = \mathsf{x}^{-1} \circ \hat{\mathsf{x}}$ to write

$$\Phi = \hat{\phi} \circ \hat{\mathsf{x}}, \tag{2.76}$$

and expand $\Phi(X)$ using the expansion (2.57) of $\hat{\mathsf{x}}(X)$ in terms of the $\hat{\mathsf{X}}^\mu_a$'s:

$$\begin{aligned} \Phi(X) &= \hat{\phi}\left(X - \kappa \hat{\mathsf{X}}_1(X) + \kappa^2 \left[\hat{\mathsf{X}}^\alpha_1(X) \frac{\partial \hat{\mathsf{X}}_1}{\partial x^\alpha}(X) - \hat{\mathsf{X}}_2(X)\right]\right) + \mathcal{O}(\kappa^3) \\ &= \hat{\phi} - \kappa \hat{\mathsf{X}}^\alpha_1 \frac{\partial \hat{\phi}}{\partial x^\alpha} + \kappa^2 \left(\frac{1}{2} \hat{\mathsf{X}}^\alpha_1 \hat{\mathsf{X}}^\beta_1 \frac{\partial^2 \hat{\phi}}{\partial x^\alpha \partial x^\beta} + \hat{\mathsf{X}}^\alpha_1 \frac{\partial \hat{\mathsf{X}}^\beta_1}{\partial x^\alpha} \frac{\partial \hat{\phi}}{\partial x^\beta} - \hat{\mathsf{X}}^\alpha_2 \frac{\partial \hat{\phi}}{\partial x^\alpha}\right) + \mathcal{O}(\kappa^3), \end{aligned} \tag{2.77}$$

where every quantity in the last line is evaluated at the coordinate scalar value X.

N.B. the above applies to *any* scalar field, including one which is built out of a tensor or tensors of nontrivial rank. In particular the invariantized Ricci scalar $\mathscr{R}(X)$ is obtained from the coordinate representation $\hat{R} = R \circ \mathrm{x}^{-1}$ of the Ricci scalar R in the exact same way:

$$\mathscr{R}(X) = \hat{R} - \kappa \hat{\mathrm{X}}_1^\alpha \frac{\partial \hat{R}}{\partial x^\alpha} + \kappa^2 \left(\frac{1}{2} \hat{\mathrm{X}}_1^\alpha \hat{\mathrm{X}}_1^\beta \frac{\partial^2 \hat{R}}{\partial x^\alpha \partial x^\beta} + \hat{\mathrm{X}}_1^\alpha \frac{\partial \hat{\mathrm{X}}_1^\beta}{\partial x^\alpha} \frac{\partial \hat{R}}{\partial x^\beta} - \hat{\mathrm{X}}_2^\alpha \frac{\partial \hat{R}}{\partial x^\alpha} \right) + \mathcal{O}(\kappa^3), \tag{2.78}$$

every quantity on the right hand side again being evaluated at the coordinate scalar value X. In secs. 2.3.4 and 2.3.6 I'll verify this result in the context of perturbation theory by properly constructing the invariantized metric and the resulting Christoffel symbols.

2.3.4 The invariantized metric

In this section I obtain the explicit expansions of the invariantized metric and its inverse, whose coordinate representations are given by

$$\hat{\mathscr{G}}_{\mu\nu}(X) = \frac{\partial \hat{\mathrm{x}}^\alpha}{\partial x^\mu}(X) \frac{\partial \hat{\mathrm{x}}^\beta}{\partial x^\nu}(X) \hat{g}_{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big), \quad \hat{\mathscr{G}}^{\mu\nu}(X) = \frac{\partial \hat{\mathrm{X}}^\mu}{\partial x^\alpha}\big(\hat{\mathrm{x}}(X)\big) \frac{\partial \hat{\mathrm{X}}^\nu}{\partial x^\beta}\big(\hat{\mathrm{x}}(X)\big) \hat{g}^{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big), \tag{2.79}$$

in which $\hat{g}_{\mu\nu} = g_{\mu\nu} \circ \mathrm{x}^{-1}$ is the background coordinate representation of the components of $\boldsymbol{g}$ and analogously for $\hat{g}^{\mu\nu}$ and $\boldsymbol{g}^{-1}$.

These calculations are a bit more complicated than the analogous calculation for the invariantized scalar field Φ. Recall that in the scalar case we needed only evaluate the background coordinate representation $\hat{\phi}$ at the background coordinate value $\hat{\mathrm{x}}(X)$ of the spacetime point whose coordinate scalar value is X and use our known expansion of $\hat{\mathrm{x}}(X)$ to obtain an expansion of Φ in terms of quantities which are known in the background coordinates. We still need do that when we invariantize the metric and its inverse – that's how we handle the $\hat{g}_{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big)$ and $\hat{g}^{\alpha\beta}\big(\hat{x}(X)\big)$ factors – but we then also need to multiply that result by expansions of the transformation matrices, which we found eqs. (2.73) and (2.74).

However on the bright side this process is simplified somewhat by the fact that we are interested in obtaining expressions for $\hat{\mathscr{G}}_{\mu\nu}$ and $\hat{\mathscr{G}}^{\mu\nu}$ not in terms of the full metric $g_{\mu\nu}$ but the metric perturbation $h_{\mu\nu}$, in terms of which the metric and its inverse are

$$\hat{g}_{\alpha\beta} = \eta_{\alpha\beta} + \kappa \hat{h}_{\alpha\beta}, \quad \hat{g}^{\alpha\beta} = \eta^{\alpha\beta} - \kappa \hat{h}^{\alpha\beta} + \kappa^2 \hat{h}^{\alpha\sigma} \hat{h}_\sigma{}^\beta + \mathcal{O}(\kappa^2). \tag{2.80}$$

Thus expanding $\hat{g}_{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big)$ and $\hat{g}^{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big)$ in κ consists of two steps: first, apply the expansion of the argument, which proceeds identically to the steps which led to the invariantized scalar field (2.77) and hence yields identical results but with $\hat{g}_{\alpha\beta}$ and $\hat{g}^{\alpha\beta}$ in place of $\hat{\phi}$; and second, apply the expansions (2.80). This latter step simplifies things a great deal, since (a) all partial derivatives of $\eta_{\alpha\beta}$ vanish and (b) we need only keep the terms up to $\mathcal{O}(\kappa)$ in eq. (2.80) when calculating the $\mathcal{O}(\kappa)$ terms in eq. (2.77), and even better we need only keep the $\mathcal{O}(\kappa^0)$ terms in the former – whose derivatives, again, vanish – when calculating the $\mathcal{O}(\kappa^2)$ terms in the latter, meaning that all the terms in brackets in eq. (2.77) actually vanish. We're left with the reasonable results

$$\begin{aligned} \hat{g}_{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big) &= \eta_{\alpha\beta} + \kappa \hat{h}_{\alpha\beta} - \kappa^2 \hat{\mathrm{X}}_1^\sigma \frac{\partial \hat{h}_{\alpha\beta}}{\partial x^\sigma} + \mathcal{O}(\kappa^3), \\ \hat{g}^{\alpha\beta}\big(\hat{\mathrm{x}}(X)\big) &= \eta^{\alpha\beta} - \kappa \hat{h}^{\alpha\beta} + \kappa^2 \left(\hat{h}^{\alpha\sigma} \hat{h}_\sigma{}^\beta + \hat{\mathrm{X}}_1^\sigma \frac{\partial \hat{h}^{\alpha\beta}}{\partial x^\sigma} \right) + \mathcal{O}(\kappa^3), \end{aligned} \tag{2.81}$$

again with all quantities on the right hand sides evaluated directly at X. As a check it's straightforward to verify that the above satisfy $\hat{g}_{\alpha\sigma}\hat{g}^{\sigma\beta} = \delta_\alpha^\beta + \mathcal{O}(\kappa^3)$.

However we're not done at eqs. (2.81) – it remains to plug these results into the definitions (2.79) of the invariantized metric and inverse metric and apply the expansions (2.73) and (2.74) of the derivative matrices. It is at this point that the arithmetic gets moderately heinous without becoming particularly interesting, so in the interest of clarity and brevity I will leave out the intermediate steps and organize the results by defining

$$\hat{\mathcal{G}}_{\mu\nu} \equiv \sum_a \kappa^a \hat{\mathcal{G}}^a_{\mu\nu}, \quad \hat{\mathcal{G}}^{\mu\nu} \equiv \sum_a \kappa^a \hat{\mathcal{G}}^{\mu\nu}_a, \tag{2.82}$$

in terms of which we find (expectedly) at zeroth order

$$\hat{\mathscr{G}}^0_{\mu\nu} = \eta_{\mu\nu}, \quad \hat{\mathscr{G}}_0^{\mu\nu} = \eta^{\mu\nu}; \tag{2.83}$$

at first order

$$\hat{\mathscr{G}}^1_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{\partial \hat{\mathsf{X}}_{1\nu}}{\partial x^\mu} - \frac{\partial \hat{\mathsf{X}}_{1\mu}}{\partial x^\nu}, \quad \hat{\mathscr{G}}_1^{\mu\nu} = -\left(\hat{h}^{\mu\nu} - \frac{\partial \hat{\mathsf{X}}_1^\nu}{\partial x_\mu} - \frac{\partial \hat{\mathsf{X}}_1^\mu}{\partial x_\nu}\right); \tag{2.84}$$

and at second order

$$\begin{aligned}
\hat{\mathscr{G}}^2_{\mu\nu} = &\left\{\frac{\partial \hat{\mathsf{X}}_1^\sigma}{\partial x^\mu}\frac{\partial \hat{\mathsf{X}}_{1\nu}}{\partial x^\sigma} + \frac{\partial \hat{\mathsf{X}}_1^\sigma}{\partial x^\nu}\frac{\partial \hat{\mathsf{X}}_{1\mu}}{\partial x^\sigma} + \hat{\mathsf{X}}_1^\sigma \frac{\partial^2 \hat{\mathsf{X}}_{1\nu}}{\partial x^\sigma \partial x^\mu} + \hat{\mathsf{X}}_1^\sigma \frac{\partial^2 \hat{\mathsf{X}}_{1\mu}}{\partial x^\sigma \partial x^\nu} + \frac{\partial \hat{\mathsf{X}}_1^\sigma}{\partial x^\mu}\frac{\partial \hat{\mathsf{X}}_{1\sigma}}{\partial x^\nu}\right\} \\
&- \left\{\frac{\partial \hat{\mathsf{X}}_{2\nu}}{\partial x^\mu} + \frac{\partial \hat{\mathsf{X}}_{2\mu}}{\partial x^\nu}\right\} - \left\{\hat{\mathsf{X}}_1^\sigma \frac{\partial \hat{h}_{\mu\nu}}{\partial x^\sigma} + \hat{h}_{\mu\sigma}\frac{\partial \hat{\mathsf{X}}_1^\sigma}{\partial x^\nu} + \hat{h}_{\sigma\nu}\frac{\partial \hat{\mathsf{X}}_1^\sigma}{\partial x^\mu}\right\}, \\
\hat{\mathscr{G}}_2^{\mu\nu} = &\left\{\frac{\partial \hat{\mathsf{X}}_1^\mu}{\partial x^\sigma}\frac{\partial \hat{\mathsf{X}}_1^\nu}{\partial x_\sigma} - \hat{\mathsf{X}}_1^\sigma \frac{\partial^2 \hat{\mathsf{X}}_1^\nu}{\partial x^\sigma \partial x_\mu} - \hat{\mathsf{X}}_1^\sigma \frac{\partial^2 \hat{\mathsf{X}}_1^\mu}{\partial x^\sigma \partial x_\nu}\right\} + \left\{\frac{\partial \hat{\mathsf{X}}_2^\nu}{\partial x_\mu} + \frac{\partial \hat{\mathsf{X}}_2^\mu}{\partial x_\nu}\right\} \\
&+ \left\{\hat{h}^{\mu\sigma}\hat{h}_\sigma{}^\nu + \hat{\mathsf{X}}_1^\sigma \frac{\partial \hat{h}^{\mu\nu}}{\partial x^\sigma} - \hat{h}^{\mu\sigma}\frac{\partial \hat{\mathsf{X}}_1^\nu}{\partial x^\sigma} - \hat{h}^{\sigma\nu}\frac{\partial \hat{\mathsf{X}}_1^\mu}{\partial x^\sigma}\right\},
\end{aligned} \tag{2.85}$$

using brackets purely to visually separate distinct classes of terms (those quadratic in $\hat{\mathsf{X}}_1$, those linear in $\hat{\mathsf{X}}_2$, and those containing at least one factor of $\hat{h}$).

I will first note that it may be (somewhat laboriously) verified by hand that the above do indeed satisfy $\hat{\mathscr{G}}_{\mu\alpha}\hat{\mathscr{G}}^{\alpha\nu} = \delta^\nu_\mu$, as they must. Additionally, if one wishes to stay a little organized, I will note that given the expansion coefficients $\hat{\mathscr{G}}^a_{\mu\nu}$ for the invariantized metric and the straightforward results that at zeroth order both $\hat{\mathscr{G}}_{\mu\nu}$ and $\hat{\mathscr{G}}^{\mu\nu}$ are flat it is straightforward to show that the expansion coefficients $\hat{\mathscr{G}}_a^{\mu\nu}$ for the invariantized inverse metric are given by

$$\hat{\mathscr{G}}_1^{\mu\nu} = -\hat{\mathscr{G}}^{1\mu\nu}, \quad \hat{\mathscr{G}}_2^{\mu\nu} = \hat{\mathscr{G}}^{1\mu\alpha}\hat{\mathscr{G}}^{1\nu}_\alpha - \hat{\mathscr{G}}^{2\mu\nu}, \tag{2.86}$$

and it may be shown that these relations are satisfied by the above.

2.3.5 The invariantized metric perturbation and volume factor

From the invariantized metric we can immediately define the invariantized metric perturbation [48] to be the metric perturbation in the coordinate system X:

$$\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + \kappa \mathcal{H}_{\mu\nu}, \quad \text{i.e.} \quad \mathcal{H}_{\mu\nu} = \frac{1}{\kappa}\left(\mathcal{G}_{\mu\nu} - \eta_{\mu\nu}\right). \tag{2.87}$$

In terms of the metric expansion coefficients defined above the coordinate representation of the invariantized metric perturbation is therefore

$$\begin{aligned}\hat{\mathcal{H}}_{\mu\nu} &= \hat{\mathcal{G}}^1_{\mu\nu} + \kappa\hat{\mathcal{G}}^2_{\mu\nu} + \mathcal{O}(\kappa^2) \\ &= \left\{\hat{h}_{\mu\nu} - \frac{\partial \hat{\mathsf{X}}_{1\nu}}{\partial x^\mu} - \frac{\partial \hat{\mathsf{X}}_{1\mu}}{\partial x^\nu}\right\} + \kappa\left\{\frac{\partial \hat{\mathsf{X}}^\sigma_1}{\partial x^\mu}\frac{\partial \hat{\mathsf{X}}_{1\nu}}{\partial x^\sigma} + \frac{\partial \hat{\mathsf{X}}^\sigma_1}{\partial x^\nu}\frac{\partial \hat{\mathsf{X}}_{1\mu}}{\partial x^\sigma} + \hat{\mathsf{X}}^\sigma_1 \frac{\partial^2 \hat{\mathsf{X}}_{1\nu}}{\partial x^\sigma \partial x^\mu} + \hat{\mathsf{X}}^\sigma_1 \frac{\partial^2 \hat{\mathsf{X}}_{1\mu}}{\partial x^\sigma \partial x^\nu} + \frac{\partial \hat{\mathsf{X}}^\sigma_1}{\partial x^\mu}\frac{\partial \hat{\mathsf{X}}_{1\sigma}}{\partial x^\nu}\right. \\ &\qquad \left. - \frac{\partial \hat{\mathsf{X}}_{2\nu}}{\partial x^\mu} - \frac{\partial \hat{\mathsf{X}}_{2\mu}}{\partial x^\nu} - \hat{\mathsf{X}}^\sigma_1 \frac{\partial \hat{h}_{\mu\nu}}{\partial x^\sigma} - \hat{h}_{\mu\sigma}\frac{\partial \hat{\mathsf{X}}^\sigma_1}{\partial x^\nu} - \hat{h}_{\sigma\nu}\frac{\partial \hat{\mathsf{X}}^\sigma_1}{\partial x^\mu}\right\} + \mathcal{O}(\kappa^2).\end{aligned} \tag{2.88}$$

In keeping with the convention that $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} \neq (g^{\mu\nu} - \eta^{\mu\nu})/\kappa$ we can also define $\mathcal{H}^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\mathcal{H}_{\alpha\beta}$, although we will not need this.

The invariantized metric perturbation was recently used in [48] to compute gauge-invariant corrections to the Newtonian potential. For our purposes its usefulness is in obtaining the invariantized volume factor $\sqrt{-\det\mathcal{G}}$, which we may do as follows. In any coordinate system the volume factor $\sqrt{-\det \boldsymbol{g}}$ can be expanded in the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$:

$$\sqrt{-\det \boldsymbol{g}} = 1 + \tfrac{1}{2}\kappa h + \kappa^2\left(\tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(\kappa^3). \tag{2.89}$$

(I review the details of this expansion in sec. 4.1.2.) Eq. (2.89) provides the expansion of the volume factor $\sqrt{-\det \boldsymbol{g}}$ in terms of the metric perturbation $h_{\mu\nu}$, evaluated in any coordinate system. It therefore follows that the invariantized volume factor is given by the exact same equation, evaluated in the X-coordinate system:

$$\sqrt{-\det \mathcal{G}} = 1 + \tfrac{1}{2}\kappa\mathcal{H} + \kappa^2\Big(\tfrac{1}{8}\mathcal{H}^2 - \tfrac{1}{4}\mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu}\Big) + \mathcal{O}(\kappa^3), \tag{2.90}$$

where $\mathcal{H} = \mathcal{H}^{\mu}{}_{\mu}$. Using eq. (2.88) we can write this in terms of the graviton and the X's as

$$\begin{aligned}\sqrt{-\det \mathcal{G}} = 1 + \kappa\Big(\tfrac{1}{2}h - \partial_\mu \mathrm{X}_1^\mu\Big) + \kappa^2\Big(&\tfrac{1}{2}\partial_\mu \mathrm{X}_1^\mu \partial_\nu \mathrm{X}_1^\nu + \mathrm{X}_1^\mu \partial_\mu\partial_\nu \mathrm{X}_1^\nu + \tfrac{1}{2}\partial_\mu \mathrm{X}_{1\nu}\partial^\nu \mathrm{X}_1^\mu - \partial_\mu \mathrm{X}_2^\mu \\ &- \tfrac{1}{2}\mathrm{X}_1^\mu \partial_\mu h - \tfrac{1}{2}h\partial_\mu \mathrm{X}_1^\mu + \tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\Big) + \mathcal{O}(\kappa^3).\end{aligned} \tag{2.91}$$

2.3.6 The invariantized Christoffel symbols and Ricci scalar

Finally let's return to my claim at the end of sec. 2.3.3 that the invariantized Ricci scalar may be obtained from the invariantized metric.

THE RICCI SCALAR IN PERTURBATION THEORY. Let's begin by obtaining an expansion of the standard (non-invariantized) Ricci scalar from the expansion of the metric about flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. The Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\big(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\big), \tag{2.92}$$

where ∂_μ is the frame of the coordinate system in which the metric components are $g_{\mu\nu}$, and the Riemann tensor is

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\sigma\mu} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\alpha}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\nu}, \tag{2.93}$$

from which the Ricci tensor and Ricci scalar are obtained as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \quad R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu}. \tag{2.94}$$

Since every term in the Christoffel symbol contains at least one partial derivative on the metric, and the partial derivatives of $\eta_{\mu\nu}$ vanish, it follows that the Christoffel symbols begin at $\mathcal{O}(\kappa)$. Following the definitions through it follows that the Ricci scalar also begins at this order, meaning that we may write

$$R = \kappa R_1 + \kappa^2 R_2 + \mathcal{O}(\kappa^3). \tag{2.95}$$

An explicit calculation yields for the expansion terms

$$\begin{aligned} R_1 &= \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h, \\ R_2 &= h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{4}\partial_\mu h \partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\rho h_\nu{}^\rho + \partial^\mu h \partial_\nu h_\mu{}^\nu - 2h^{\mu\nu}\partial_\nu\partial_\rho h_\mu{}^\rho \\ &\qquad + h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_\nu h_{\mu\rho}\partial^\rho h^{\mu\nu} + \frac{3}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}. \end{aligned} \tag{2.96}$$

THE INVARIANTIZED RICCI SCALAR, OBTAINED AS A SCALAR FIELD. Using the above expansion in eq. (2.78) yields an expression for the invariantized Ricci scalar in terms of the expansions of both the metric and the coordinate scalars:

$$\mathscr{R} = \kappa \hat{R}_1 + \kappa^2 \left(\hat{R}_2 - \hat{\mathsf{X}}_1^\alpha \frac{\partial \hat{R}_1}{\partial x^\alpha} \right) + \mathcal{O}(\kappa^3), \tag{2.97}$$

where $\mathscr{R} = R \circ \mathsf{X}^{-1}$ and $\hat{R}_a = R_a \circ \mathsf{x}^{-1}$.

N.B. this result, which is the correct one, does not in itself rely at all on the fact that R is defined in terms of any higher-rank tensor field – given any scalar field ϕ known as an expansion in κ and whose $\mathcal{O}(\kappa^0)$ contribution vanishes, the invariantized ϕ would have this exact same form. In what follows I will show that this form may also be obtained by correctly constructing the invariantized curvature tensors from the invariantized metric.

A TEMPTING BUT INCORRECT DERIVATION FROM THE INVARIANTIZED METRIC. Before proceeding to the correct derivation I will briefly demonstrate the problem with the formulation which is most tempting in the standard more concise notation. It is most common to conflate the basis frame ∂_μ on M corresponding to a coordinate system with the partial derivative

$\partial/\partial x^\mu$ with respect to those coordinates, and to conflate a tensor field component $g_{\mu\nu}$ with its coordinate representation $\hat{g}_{\mu\nu}$. In this notation one might then think to construct the Christoffel symbols in the X-coordinate system as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\mathscr{G}^{\rho\alpha}\left(\frac{\partial\mathscr{G}_{\alpha\nu}}{\partial X^\mu} + \frac{\partial\mathscr{G}_{\alpha\mu}}{\partial X^\nu} - \frac{\partial\mathscr{G}_{\mu\nu}}{\partial X^\alpha}\right), \tag{2.98}$$

the Riemann tensor as

$$\mathscr{R}^\mu{}_{\nu\rho\sigma} = \frac{\partial\Gamma^\mu_{\sigma\nu}}{\partial X^\rho} - \frac{\partial\Gamma^\mu_{\rho\nu}}{\partial X^\sigma} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\rho\nu}, \tag{2.99}$$

and the Ricci tensor and scalar as

$$\mathscr{R}_{\mu\nu} = \mathscr{R}^\rho{}_{\mu\rho\nu}, \quad \mathscr{R} = \mathscr{G}^{\mu\nu}\mathscr{R}_{\mu\nu}. \tag{2.100}$$

If one wished to then expand the Ricci scalar in κ one would then rightly use the known expansion of $\mathscr{G}_{\mu\nu}$ and $\mathscr{G}^{\mu\nu}$.

The problem with this notation is that one would also think that, in order to reduce the expression to one involving only functions we know in the background coordinates – namely, partial derivatives of background coordinate functions with respect to the background coordinates – one must also convert the $\partial/\partial X^\mu$'s to $\partial/\partial x^\mu$'s via the chain rule:

$$\begin{aligned}\Gamma^\rho_{\mu\nu} &= \frac{1}{2}\mathscr{G}^{\rho\alpha}\left(\frac{\partial x^\beta}{\partial X^\mu}\frac{\partial\mathscr{G}_{\alpha\nu}}{\partial x^\beta} + \frac{\partial x^\beta}{\partial X^\nu}\frac{\partial\mathscr{G}_{\alpha\mu}}{\partial x^\beta} - \frac{\partial x^\beta}{\partial X^\alpha}\frac{\partial\mathscr{G}_{\mu\nu}}{\partial X^\beta}\right),\\ \mathscr{R}^\mu{}_{\nu\rho\sigma} &= \frac{\partial x^\alpha}{\partial X^\rho}\frac{\partial\Gamma^\mu_{\sigma\nu}}{\partial x^\alpha} - \frac{\partial x^\alpha}{\partial X^\sigma}\frac{\partial\Gamma^\mu_{\rho\nu}}{\partial x^\alpha} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\rho\nu}.\end{aligned} \tag{2.101}$$

That this construction is incorrect may be seen directly by following it through and observing that the result disagrees with the result (2.97) obtained from treating R like any other scalar field. This is by itself damning: all we are really doing in obtaining the invariantized scalar field is transforming from an arbitrary coordinate system x to the specified coordinate system

X, meaning that if the relationship between R and $\mathscr{R}$ differs from the relationship between a generic scalar field ϕ and its invariantized form Φ then the Ricci scalar does not transform like a scalar at all, in contradiction with, for example, a century and a half of well-established mathematics.

However the problem with the derivative prescription above may also be seen by considering of the actual meaning of a partial derivative with respect to coordinates on a manifold. This is most apparent by comparing to the more careful development below, but we may also understand it as follows. Suppose we have a function $f : M \to \mathbb{R}^d$ and some coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ with frame ∂_μ. If we wish to take the μ^{th} partial derivative of f with respect to this coordinate system then we "think of f as a function of the coordinates", i.e. construct its coordinate representation $f \circ \mathsf{x}^{-1}$, and then take the μ^{th} derivative of that function. If we also have another coordinate system $\tilde{\mathsf{x}}$ with frame $\tilde{\partial}_\mu$ and we want to take the μ^{th} partial derivative of f with respect to these other coordinates then we do the same thing: we construct the coordinate representation $f \circ \tilde{\mathsf{x}}^{-1}$ and take its μ^{th} derivative.

The key point here is that, once we have the coordinate representations $f_{\mathsf{x}} \equiv f \circ \mathsf{x}^{-1}$ and $f_{\tilde{\mathsf{x}}} \equiv f \circ \tilde{\mathsf{x}}^{-1}$, we do *the exact same thing* to each – we're differentiating these different coordinate representations with respect to *the same* coordinates on $\mathbb{R}^d$, and therefore we do not a priori need any extra chain-rule factor to relate the two derivatives. More explicitly, evaluating $\partial_\mu f$ and $\tilde{\partial}_\mu f$ at $p \in M$ such that $\mathsf{x}(p) = x$ and $\tilde{\mathsf{x}}(p) = \tilde{x}$, we have

$$\left(\partial_\mu f\right)_p = \frac{\partial f_{\mathsf{x}}}{\partial x^\mu}(x), \quad \left(\tilde{\partial}_\mu f\right)_p = \frac{\partial f_{\tilde{\mathsf{x}}}}{\partial x^\mu}(\tilde{x}). \tag{2.102}$$

Of course we can then relate the two derivatives by the chain rule if we wish by writing $f_{\tilde{\mathsf{x}}}(\tilde{x}) = f_{\mathsf{x}}\big(\hat{\mathsf{x}}(\tilde{x})\big)$ with $\hat{\mathsf{x}} = \mathsf{x} \circ \tilde{\mathsf{x}}^{-1}$, so that

$$\left(\tilde{\partial}_\mu f\right)_p = \frac{\partial f_{\tilde{\mathsf{x}}}}{\partial x^\mu}(\tilde{x}) = \frac{\partial \hat{\mathsf{x}}^\alpha}{\partial x^\mu}(\tilde{x}) \frac{\partial f_{\mathsf{x}}}{\partial x^\alpha}\big(\hat{\mathsf{x}}(\tilde{x})\big) = \frac{\partial \hat{\mathsf{x}}^\alpha}{\partial x^\mu}(\tilde{x}) \big(\partial_\mu f\big)_p, \tag{2.103}$$

but N.B. the expression containing the partial derivative matrix does not also contain the new coordinate representation of the function f.

In short, the problem with the intuitive construction (2.98) is that, implicitly, we are simultaneously including the partial derivative matrix *and* differentiating the *new* coordinate representation, when really we should be doing one or the other. Thus the correct invariantized Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\mathscr{G}^{\rho\alpha}\left(\frac{\partial \mathscr{G}_{\alpha\nu}}{\partial x^{\mu}} + \frac{\partial \mathscr{G}_{\alpha\mu}}{\partial x^{\nu}} - \frac{\partial \mathscr{G}_{\mu\nu}}{\partial x^{\alpha}}\right), \tag{2.104}$$

in terms of which the correct invariantized Riemann tensor is

$$\mathscr{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial \Gamma^{\mu}_{\sigma\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\mu}_{\rho\nu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\alpha}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\nu}. \tag{2.105}$$

To more rigorously justify the above results I will now obtain the above from the more careful construction in which spacetime- and coordinate-dependent objects are not conflated.

THE CORRECT DERIVATION FROM THE INVARIANTIZED METRIC. In a general coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ with coordinate frame $\partial_\mu = (\mathsf{x}^{-1})_*\left(\partial/\partial x^{\mu}\right)$ and in which the metric has components $\boldsymbol{g} = g_{\mu\nu}\,\mathrm{d}\mathsf{x}^{\mu}\,\mathrm{d}\mathsf{x}^{\nu}$ the Christoffel symbols are defined by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right). \tag{2.106}$$

We want to write down the Christoffel symbols in the coordinate system $\mathsf{X} : M \to \mathbb{R}^d$. Thus, not conflating anything and being careful to write $D_\mu = (\mathsf{X}^{-1})_*\left(\partial/\partial x^{\mu}\right)$ for the frame of this coordinate system and $\boldsymbol{g} = \mathscr{G}_{\mu\nu}\,\mathrm{d}\mathsf{X}^{\mu}\,\mathrm{d}\mathsf{X}^{\nu}$ for the metric components, the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\mathscr{G}^{\rho\alpha}\left(D_\mu \mathscr{G}_{\alpha\nu} + D_\nu \mathscr{G}_{\alpha\mu} - D_\alpha \mathscr{G}_{\mu\nu}\right). \tag{2.107}$$

N.B. in the above each metric component $\mathscr{G}_{\mu\nu}$ is a real-valued function of spacetime and hence *distinct* from its coordinate representation $\hat{\mathscr{G}}_{\mu\nu} = \mathscr{G}_{\mu\nu} \circ \mathsf{X}^{-1} = \left((\mathsf{X}^{-1})^*\boldsymbol{g}\right)_{\mu\nu}$, which is a real-valued function of the coordinate scalars.

To write the Christoffel symbols in this coordinate system as a function of the coordinates let's evaluate at a point p. For a representative derivative term we find

$$\left(D_\alpha \mathscr{G}_{\mu\nu}\right)_p = \frac{\partial\left(\mathscr{G}_{\mu\nu}\circ \mathsf{X}^{-1}\right)}{\partial x^\alpha}\left(\mathsf{X}(p)\right) = \frac{\partial \hat{\mathscr{G}}_{\mu\nu}}{\partial x^\alpha}\left(\mathsf{X}(p)\right), \tag{2.108}$$

meaning that the coordinate representation $\hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}\circ \mathsf{X}^{-1}$ of the Christoffel symbols is

$$\hat{\Gamma}^\rho_{\mu\nu} = \frac{1}{2}\hat{\mathscr{G}}^{\rho\alpha}\left(\frac{\partial \hat{\mathscr{G}}_{\alpha\nu}}{\partial x^\mu} + \frac{\partial \hat{\mathscr{G}}_{\alpha\mu}}{\partial x^\nu} - \frac{\partial \hat{\mathscr{G}}_{\mu\nu}}{\partial x^\alpha}\right), \tag{2.109}$$

in agreement with eq. (2.104). Similarly the invariantized Riemann tensor is

$$\mathscr{R}^\mu{}_{\nu\rho\sigma} = D_\rho \Gamma^\mu_{\sigma\nu} - D_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\rho\nu}, \tag{2.110}$$

which yields the coordinate representation

$$\hat{\mathscr{R}}^\mu{}_{\nu\rho\sigma} = \frac{\partial \hat{\Gamma}^\mu_{\sigma\nu}}{\partial x^\rho} - \frac{\partial \hat{\Gamma}^\mu_{\rho\nu}}{\partial x^\sigma} + \hat{\Gamma}^\mu_{\rho\alpha}\hat{\Gamma}^\alpha_{\sigma\nu} - \hat{\Gamma}^\mu_{\sigma\alpha}\hat{\Gamma}^\alpha_{\rho\nu}, \tag{2.111}$$

in agreement with eq. (2.105).

The invariantized Ricci scalar is, finally, given by

$$\mathscr{R}(X) = \hat{\mathscr{G}}^{\mu\nu}(X)\hat{\mathscr{R}}^\rho{}_{\mu\rho\nu}(X). \tag{2.112}$$

To turn this into an evaluable expression for $\mathscr{R}$ in terms of quantities known in the background coordinate system one would (i) use eqs. (2.109) and (2.111) to write the Riemann tensor in terms of the invariantized metric, yielding an expression for $\mathscr{R}$ entirely in terms of $\mathscr{G}_{\mu\nu}$ and $\mathscr{G}^{\mu\nu}$; and then (ii) use eqs. (2.82) through (2.85) to expand this result in terms of $\hat{h}_{\mu\nu}$ and the $\hat{\mathsf{X}}^\mu_a$'s, which are themselves given in terms of $\hat{h}_{\mu\nu}$ by eqs. (2.17) through (2.23). Doing so confirms that this construction agrees with the result (2.97) obtained by treating the Ricci scalar like any other scalar field.

2.4 SUMMARIZING AND CLEANING UP THE NOTATION

Throughout this section I have used a careful distinction between objects defined on spacetime and their coordinate representations to clarify certain subtle points in the construction of relational observables. For the rest of this thesis we will not need to be quite so explicit, so I will bring my notation more in line with convention as follows.

- I will gleefully conflate functions of spacetimes and their coordinate representations, meaning that I will drop all the hats and write things like $h_{\mu\nu}(x)$ and $\mathsf{X}(x)$.
- A partial derivative, e.g. ∂_μ, may denote either the coordinate frame (which acts on functions of spacetime) or the actual partial derivative (which acts on functions of the coordinates).

Additionally, in sec. 3 and beyond we will make frequent reference to the scalar modes of the metric tensor, one of which I denote Φ. Thus from here on out I will denote the invariantized scalar field by $\hat{\phi} = \phi \circ \mathsf{X}^{-1}$, since I no longer need to distinguish notationally between ϕ and its background coordinate representation $\phi \circ \mathsf{x}^{-1}$.

Finally, in the interest of clarity, I will summarize the main results of this section in this cleaned up notation. The coordinate scalars as a function of the background coordinates are

$$\mathsf{X}(x) = x + \kappa \mathsf{X}_1(x) + \kappa^2 \mathsf{X}_2(x) + \mathcal{O}(\kappa^3). \tag{2.113}$$

The expansion terms are

$$\mathsf{X}_1(x) = \int \mathrm{d}^d x' \, G(x, x') J(x'), \tag{2.114}$$

where $G(x, x')$ is a Green function of ∂^2 and

$$J_1^\mu = \partial_\alpha h^{\alpha\mu} - \tfrac{1}{2} \partial^\mu h; \tag{2.115}$$

and

$$\mathsf{X}_2(x) = \int \mathrm{d}^d x'\, G(x,x')\Big(J_2(x') + K_1 \mathsf{X}_1(x')\Big), \tag{2.116}$$

where

$$K_1 = h^{\alpha\beta}\partial_\alpha\partial_\beta + J_1^\alpha \partial_\alpha, \quad J_2^\mu = \tfrac{1}{2}\Big(h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + h^{\alpha\mu}\partial_\alpha h\Big) - \partial_\alpha\Big(h^{\alpha\beta}h_\beta{}^\mu\Big). \tag{2.117}$$

Given a scalar field ϕ its invariantized form is

$$\hat{\phi} = \phi \circ \mathsf{X}^{-1} = \phi - \kappa \mathsf{X}_1^\alpha \partial_\alpha \phi + \kappa^2\Big(\tfrac{1}{2}\mathsf{X}_1^\alpha \mathsf{X}_1^\beta \partial_\alpha\partial_\beta\phi + \mathsf{X}_1^\alpha \partial_\alpha \mathsf{X}_1^\beta \partial_\beta \phi - \mathsf{X}_2^\alpha \partial_\alpha \phi\Big) + \mathcal{O}(\kappa^3), \tag{2.118}$$

and considering in particular the scalar curvature yields

$$\mathscr{R} = R \circ \mathsf{X}^{-1} = \kappa R_1 + \kappa^2\Big(R_2 - \mathsf{X}_1^\alpha \partial_\alpha R_1\Big) + \mathcal{O}(\kappa^3), \tag{2.119}$$

where

$$\begin{aligned}
R_1 &= \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h, \\
R_2 &= h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{4}\partial_\mu h \partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\rho h_\nu{}^\rho + \partial^\mu h \partial_\nu h_\mu{}^\nu - 2h^{\mu\nu}\partial_\nu\partial_\rho h_\mu{}^\rho \\
&\qquad + h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_\nu h_{\mu\rho}\partial^\rho h^{\mu\nu} + \frac{3}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}.
\end{aligned} \tag{2.120}$$

3 Propagators

In the previous section I reviewed the relational program for constructing gauge-invariant nonlocal observables corresponding to the components of any tensor field, and explicitly constructed the invariantized volume factor, scalar field, and scalar curvature by perturbing about flat space. To calculate correlation functions of these quantities we need, among other things, the propagators of the scalar and gravitational fields. We're considering a minimally coupled scalar, so its propagator is the same as it always is:

$$D(p) = \frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon}. \tag{3.1}$$

However it takes a little more work to obtain the graviton propagator, and doing so also leads naturally to the introduction of the Faddeev-Popov ghost which removes the diffeomorphism redundancy in the gravitational functional measure, whose propagator is also necessary. In this section I obtain these propagators.

As mentioned previously, the eventual goal of this program is to obtain a manifestly gauge-invariant for $\langle \mathscr{R}(x)\mathscr{R}(y)\rangle$ which can be continued in a well-defined manner to Euclidean space. The well-known trouble posed by the scalar mode of gravity in this rotation suggests that, in any eventual Euclidean continuation procedure, the contributions to any amplitude of that scalar will have to be isolated and dealt with in a manner different from the contributions of the other modes. Therefore it will be useful to express the graviton propagator in a form which explicitly separates the terms which propagator the different modes. I will do this by expressing the graviton propagator as a linear combination of projectors onto the various modes which compose a generic symmetric rank-two tensor, namely a transverse-traceless tensor, a transverse vector, and two scalar modes. This scheme resembles the one followed in e.g. [64].[11] However in this thesis I take special care to isolate the "physical" scalar mode which actually appears in the Einstein-Hilbert action from the scalar and vector modes which appear only as a result of the Faddeev-Popov gauge-fixing. I also decompose the spin-one ghost propagator into its transverse vector and scalar modes. This is not as likely to be practically useful in a continuation scheme, but it does not introduce a great deal of complication, and it serves as a helpful warmup for the more complicated spin-two procedure.

[11] I would like to offer special thanks to Marc Schiffer for many incredibly helpful conversations on this topic.

3.1 GAUGE-FIXING

3.1.1 In general

Consider a generic gauge theory with gauge field ϕ and classical action $S_{\mathrm{cl}}[\phi]$. Denote by $\mathcal{V}$ the space of configurations of ϕ and by $\mathcal{W}$ the space of gauge degrees of freedom λ. (For example in electromagnetism ϕ is a one-form, so $\mathcal{V}$ is the space of sections of the cotangent bundle, and gauge transformations are parametrized by smooth functions, so $\mathcal{W} = \mathrm{C}^\infty(M)$.) The Faddeev-Popov procedure yields the gauge-fixed path integral

$$Z = \int \mathscr{D}\phi \,\mathrm{Det}\left(\frac{\delta C^a(x)}{\delta \lambda^b(y)}[\phi]\right)\delta_{\mathcal{W}}\big(C[\phi]\big)\mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}[\phi]}, \tag{3.2}$$

in which we write an arbitrary $\lambda \in \mathcal{W}$ in components as $\lambda^a(x)$, $\delta_{\mathcal{W}}$ is the Dirac delta on $\mathcal{W}$, and C is some arbitrary[12] functional $C : \mathcal{V} \to \mathcal{W}$.

The Faddeev-Popov determinant can be evaluated perturbatively in terms of a pair of ghost fields c and $\bar{c}$, where c is in the Grassmann-valued version of $\mathcal{W}$ and $\bar{c}$ is in the dual space thereof:

$$\mathrm{Det}\left(\frac{\delta C^a(x)}{\delta \lambda^b(y)}[\phi]\right) \propto \int \mathscr{D}c\,\mathscr{D}\bar{c}\exp\big(\theta\tilde{S}_{\mathrm{gh}}[c,\bar{c},\phi]\big), \quad \tilde{S}_{\mathrm{gh}}[c,\bar{c},\phi] = \int \mathrm{d}^d x \int \mathrm{d}^d y\,\bar{c}_a(x)\frac{\delta C^a(x)}{\delta \lambda^b(y)}c^b(y). \tag{3.3}$$

In the first equation above θ is an arbitrary phase (the choice of which we discuss at the end of this section), which is the only quantity on which the proportionality constant in the first of eqs. (3.3) depends. Hence this proportionality constant cancels out of all observables and can be entirely ignored. Note also that for any local $C[\phi]$ the derivative will be proportional to $\delta^d(x-y)$, so the two spacetime integrals in the second of eqs. (3.3) collapse into one (or in other words, any local $C[\phi]$ will result in a local ghost action).

The $\delta_{\mathcal{W}}$ can be eliminated as follows. Choose our C to be of the form $C[\phi] = F[\phi] - \omega$, where

[12] Well, not exactly arbitrary - we require that for every value of the physical degrees of freedom there exists a unique root of $C[\phi]$, which fixes the gauge degrees of freedom.

$\omega \in \mathcal{W}$ is independent of ϕ and F is our gauge-fixing functional. By construction Z is independent of C, meaning that it is also independent of ω. Hence if we multiply Z by some functional of ω and integrate over it we will at most change its value by an overall constant (i.e. the value of the integral of that functional), which will (as before) drop out of all observables and can therefore be ignored. The standard choice of functional is $\exp\left(\varphi \frac{1}{2\alpha} \int \mathrm{d}^d x \, \mathbf{t}\big(\omega(x), \omega(x)\big)\right)$, where φ is another arbitrary phase, α is an unfixed parameter, and $\mathbf{t}$ is the metric on the fibers of $\mathcal{W}$.[13] (I'm keeping the phases explicit in order to later make clear certain signs related only to conventions.) When we integrate over ω the $\delta_{\mathcal{W}}$ then sets $\omega = F[\phi]$ in the integrand, yielding[14]

$$Z = \int \mathscr{D}\phi \, \mathscr{D}c \, \mathscr{D}\bar{c} \exp\Big(\mathrm{i}S_{\mathrm{cl}}[\phi] + \theta \tilde{S}_{\mathrm{gh}}[c, \bar{c}, \phi] + \frac{\varphi}{2\alpha} \tilde{S}_{\mathrm{gf}}[\phi]\Big), \quad \tilde{S}_{\mathrm{gf}}[\phi] = \int \mathrm{d}^d x \, \mathbf{t}\big(F(x), F(x)\big), \tag{3.4}$$

for our final gauge-fixed path integral.

Let's briefly discuss the phases. Conventional choices of F include at least one term which is linear in ϕ, meaning that (a) the ghost integrals will include at least one term which is quadratic in the ghosts and contains no other fields, i.e. a standard kinetic term for the ghosts, and (b) the F integral will include at least one term which modifies the kinetic terms of ϕ. We therefore choose θ in order to give the ghosts a properly normalized kinetic term (with respect to the signature of the metric), and we choose φ in order for the ϕ propagator to have the desired functional dependence on α. Similarly I denote the ghost and gauge-fixed actions with tildes because if we choose θ or φ to be not equal to i then the actual corresponding action picks up a phase (given by θ/i or φ/i) relative to the ones defined above.

However to avoid this overabundance of notation I will instead simply write

$$Z = \int \mathscr{D}\phi \, \mathscr{D}c \, \mathscr{D}\bar{c} \exp\big(\mathrm{i}S[c, \bar{c}, \phi]\big), \quad S[c, \bar{c}, \phi] = S_{\mathrm{cl}}[\phi] + S_{\mathrm{gh}}[c, \bar{c}, \phi] + S_{\mathrm{gf}}[\phi], \tag{3.5}$$

[13] So for example in a general Yang-Mills theory $\omega \in \mathrm{Lie}(\mathcal{G}) \otimes \mathrm{C}^\infty(M)$ is a $\mathrm{Lie}(\mathcal{G})$-valued smooth function (with $\mathcal{G}$ the gauge group) so the metric on the fibers is would be $\mathrm{tr}(\omega(x)\omega(x)) = \frac{1}{2}\delta_{ab}\omega^a(x)\omega^b(x)$; and in gravity, which we will soon focus on, $\omega \in \mathcal{X}^*(M)$ is a one-form, so the metric on the fibers is given by the spacetime metric as $g^{\mu\nu}\omega_\mu(x)\omega_\nu(x)$.

[14] (ignoring those previously-noted overall factors)

in which the ghost and gauge-fixing actions are provisionally defined by the gauge-fixing function F as

$$S_{\text{gf}}[\phi] = \frac{1}{2\alpha}\int \mathrm{d}^d x\, \mathbf{t}\big(F(x), F(x)\big), \quad S_{\text{gh}}[c,\bar{c},\phi] = \int \mathrm{d}^d x \int \mathrm{d}^d y\, \bar{c}_a(x)\frac{\delta F^a(x)}{\delta \lambda^b(y)} c^b(y) \tag{3.6}$$

with the understanding that we're free to fiddle with the normalizations and signs of the ghost and gauge-fixing actions.

3.1.2 Example: Yang-Mills theory in Minkowski space

As a relatively simple example (before diving into Einstein-Hilbert gravity in sec. 3.1.3) let's consider a generic Yang-Mills theory with gauge group $\mathcal{G}$.[15] In such a theory the gauge field A is a Lie-algebra-valued one-form, $A \in \mathfrak{g}\otimes\mathcal{X}^*(M)$, and the gauge transformations are parametrized by a Lie-algebra-valued function, $\gamma = \gamma^a T_a \in \mathfrak{g}\otimes \mathrm{C}^\infty(M)$:

$$A \mapsto UAU^\dagger + \frac{\mathrm{i}}{\lambda} U\nabla U^\dagger, \quad U(x) = \exp\big(-\mathrm{i}\lambda\gamma(x)\big) = \exp\big(-\mathrm{i}\lambda\gamma^a(x)T_a\big), \tag{3.7}$$

where $\{T_a\}$ is some basis of $\mathfrak{g}$ and λ is the coupling of the theory.

Our classical action is

$$S_{\text{cl}}[A] = \int \mathrm{d}^d x \left\{ -\tfrac{1}{2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big)\right\}, \tag{3.8}$$

where $\mathcal{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - \mathrm{i}\lambda\big[A_\mu, A_\nu\big]$ is the field strength of A and tr is the trace (i.e. metric) on $\mathfrak{g}$, with respect to which we assume $\{T_a\}$ has been normalized to $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$.

Since we are here interested only in the propagators of the theory we need only consider the kinetic terms in $S_{\text{cl}}[A]$:

$$S_{\text{cl,kin}}[A] = -\int \mathrm{d}^d x \left\{ \tfrac{1}{2}\big(\nabla_\mu A^a_\nu \nabla^\mu A^{a\nu} - \nabla_\nu A^a_\mu \nabla^\mu A^{a\nu}\big)\right\}. \tag{3.9}$$

Fourier-transforming with $A_\mu(x) = \int_k \mathrm{e}^{\mathrm{i}kx} A_\mu(k)$ yields the momentum-space representation of

[15] In my mathematical description of Yang-Mills theory I follow most closely Nakahara's text [65].

the kinetic terms:

$$S_{\mathrm{cl,kin}}[A] = -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^2 A^a_\mu(k) P^{ab\mu\nu}(k) A^b_\nu(-k), \quad P^{ab\mu\nu}(k) = \delta^{ab}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right). \tag{3.10}$$

The gauge-fixing and ghost actions are fixed by our choice of gauge-fixing function F, which (by definition) takes as its argument an generic field configuration $A \in \mathfrak{g} \otimes \mathcal{X}^*(M)$ and returns an element of the space of gauge degrees of freedom $\mathfrak{g} \otimes C^\infty(M)$. We choose the gauge-fixing function appropriate to R_α gauge,

$$F[A] = \nabla \cdot A, \tag{3.11}$$

from which we obtain

$$S_{\mathrm{gf}}[A] = -\frac{1}{\alpha}\int \mathrm{d}^d x \, \mathrm{tr}\big(\nabla^\mu A_\mu \nabla^\nu A_\nu\big) = -\frac{1}{2\alpha}\int \mathrm{d}^d x \, \nabla^\mu A^a_\mu \nabla^\nu A^a_\nu, \tag{3.12}$$

making use of our freedom to rescale the gauge-fixing action. Fourier-transforming S_{gf} as we did $S_{\mathrm{cl,kin}}$ gives

$$S_{\mathrm{gf}}[A] = -\frac{1}{2\alpha}\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^\mu k^\nu A^a_\mu(k) A^a_\nu(-k). \tag{3.13}$$

To obtain the ghost action we need to compute the functional derivative of F with respect to the gauge degree of freedom γ. To do so we first need the infinitesimal variation of A, which we obtain as follows. Recall the general finite gauge transformation:

$$A \mapsto A' = UAU^\dagger + \frac{\mathrm{i}}{\lambda} U \nabla U^\dagger. \tag{3.14}$$

Writing $U = \exp\big(-\mathrm{i}\lambda\gamma(x)\big)$ and expanding to first order in λ yields the infinitesimal variation of A:

$$A \mapsto A' = A + \mathrm{i}\lambda\big[A, \gamma\big] - \nabla\gamma + \mathcal{O}(\lambda^2) = A - D\gamma + \mathcal{O}(\lambda^2), \tag{3.15}$$

where $D\gamma = \nabla\gamma - \mathrm{i}\lambda\big[A,\gamma\big]$ is the gauge covariant derivative in the adjoint representation. The resulting functional derivative of F with respect to γ is

$$\frac{\delta F^a(x)}{\delta \gamma^b(y)} = -\Big(\delta^a{}_b \nabla^2 - \lambda f_{cb}{}^a A^c_\mu(y) \nabla^\mu\Big) \delta^d(x-y), \tag{3.16}$$

in which the derivatives are with respect to y and the structure constants $f_{ab}{}^c$ are defined by the Lie bracket of the generators, $[T_a, T_b] = \mathrm{i} f_{ab}{}^c T_c$. The above yields the ghost action

$$S_{\mathrm{gh}}[c,\bar{c},A] = -\int \mathrm{d}^d x\, \bar{c}_a(x) \int \mathrm{d}^d y\, \frac{\delta F^a(x)}{\delta \gamma^b(y)} c^b(y) = \int \mathrm{d}^d x\, \bar{c}_a \nabla \cdot D c^a, \tag{3.17}$$

using our freedom to rescale S_{gh} by rescaling F, which in particular contains the kinetic term

$$S_{\mathrm{gh,kin}}[c,\bar{c}] = \int \mathrm{d}^d x\, \bar{c}_a \nabla^2 c^a. \tag{3.18}$$

And this kinetic term is straightforward to Fourier transform:

$$S_{\mathrm{gh,kin}}[c,\bar{c}] = -\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^2 \bar{c}_a(k) c^a(-k). \tag{3.19}$$

3.1.3 Einstein-Hilbert gravity: the classical action

Now we specialize to the case of interest: Einstein-Hilbert gravity in Minkowski space with a space-negative metric. In this theory the gauge transformations are coordinate transformations, and hence the gauge degrees of freedom are parametrized by the generators $\xi \in \mathfrak{X}(M)$ of these coordinate transformations. In particular under an infinitesimal coordinate transformation $\mathrm{x}^\mu \mapsto \mathrm{x}^\mu - \kappa \xi^\mu$ the metric $\boldsymbol{g}$ transforms as $\boldsymbol{g} \mapsto \boldsymbol{g} + \kappa \mathcal{L}_\xi \boldsymbol{g}$.

In this signature the classical action is

$$S_{\mathrm{cl}}[\boldsymbol{g}] = -\frac{2}{\kappa^2} \int \mathrm{d}^d x\, \sqrt{-g} R. \tag{3.20}$$

(In $d = 4$ the coupling κ is given in terms of Newton's constant by $\kappa^2 = 32\pi G$.) Since we are here interested in only the propagators of the theory let's rewrite $S_{\text{cl}}[\boldsymbol{g}]$ in terms of a perturbation about flat space, i.e. $\boldsymbol{g} = \bar{\boldsymbol{g}} + \kappa h$ with $\bar{\boldsymbol{g}}$ flat, and ignore all terms other than those quadratic in h. We find

$$\begin{aligned} -\frac{2}{\kappa^2}\sqrt{-g}R = 4h^{\mu\nu}\nabla_\nu\nabla_\rho h_\mu{}^\rho + 2\nabla_\mu h^{\mu\nu}\nabla_\rho h_\nu{}^\rho - 2h^{\mu\nu}\nabla^2 h_{\mu\nu} + \nabla_\nu h_{\mu\rho}\nabla^\rho h^{\mu\nu} - \frac{3}{2}\nabla_\rho h_{\mu\nu}\nabla^\rho h^{\mu\nu} \\ - 2h^{\mu\nu}\nabla_\mu\nabla_\nu h - 2\nabla^\nu h\nabla_\rho h_\nu{}^\rho - h\nabla_\nu\nabla_\rho h^{\nu\rho} + \frac{1}{2}\nabla_\nu h\nabla^\nu h + h\nabla^2 h \\ + \mathcal{O}(\kappa), \end{aligned} \tag{3.21}$$

in which ∇ is the covariant derivative with respect to $\bar{\boldsymbol{g}}$ (which is also the metric with which we raise and lower indices), and $h = h^\mu{}_\mu$. After integrating by parts (writing $\nabla_\nu h^{\nu\mu} \equiv \nabla h^\mu$ for shorthand) the classical action becomes

$$\begin{aligned} S_{\text{cl}}[\boldsymbol{g}] \mapsto S_{\text{cl,kin}}[h] &= \int \mathrm{d}^d x \bigg\{ -4\nabla h^\mu \nabla h_\mu + 2\nabla h^\nu \nabla h_\nu - 2h^{\mu\nu}\nabla^2 h_{\mu\nu} + \nabla h_\mu \nabla h^\mu + \frac{3}{2} h_{\mu\nu}\nabla^2 h^{\mu\nu} \\ &\qquad + 2\nabla h^\nu \nabla_\nu h - 2\nabla^\nu h \nabla h_\nu + \nabla h^\nu \nabla_\nu h - \frac{1}{2} h\nabla^2 h + h\nabla^2 h \bigg\} \\ &= \int \mathrm{d}^d x \bigg\{ -\nabla h^\mu \nabla h_\mu - \frac{1}{2} h^{\mu\nu}\nabla^2 h_{\mu\nu} + \nabla h^\nu \nabla_\nu h + \frac{1}{2} h\nabla^2 h \bigg\}. \end{aligned} \tag{3.22}$$

3.1.4 The gauge-fixing action

The gauge-fixing and ghost actions are fixed by our choice of gauge-fixing function. Let's choose the gauge-fixing function appropriate to generalized harmonic gauge,

$$F_\mu = \nabla h_\mu - \frac{1+\beta}{d}\nabla_\mu h, \tag{3.23}$$

where β is an arbitrary parameter, in terms of which we obtain linearized harmonic gauge[16] by setting $\beta = \frac{d}{2} - 1$. The gauge-fixing action is then

$$S_{\mathrm{gf}}[h] = \frac{1}{2\alpha}\int \mathrm{d}^d x \left(\nabla h^\mu - \frac{1+\beta}{d}\nabla^\mu h\right)\left(\nabla h_\mu - \frac{1+\beta}{d}\nabla_\mu h\right). \tag{3.24}$$

3.1.5 The ghost action

To obtain the ghost action we need the functional derivative of F with respect to the gauge degree of freedom ξ, to compute which we require the variation of $\boldsymbol{h}$:

$$h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) + \kappa\left(\xi^\rho \nabla_\rho h_{\mu\nu} + h_{\rho\nu}\nabla_\mu \xi^\rho + h_{\rho\mu}\nabla_\nu \xi^\rho\right). \tag{3.25}$$

The resulting functional derivative yields the ghost action

$$\tilde{S}_{\mathrm{gh}}[c,\bar{c},h] = \tilde{S}_{\mathrm{gh,kin}}[c,\bar{c}] + \tilde{S}_{c\bar{c}h}[c,\bar{c},h] = \int \mathrm{d}^d x\, \tilde{\mathcal{L}}_{\mathrm{gh,kin}}[c,\bar{c}] + \int \mathrm{d}^d x\, \tilde{\mathcal{L}}_{c\bar{c}h}[c,\bar{c},h], \tag{3.26}$$

where the kinetic terms are given by

$$\tilde{\mathcal{L}}_{\mathrm{gh,kin}}[c,\bar{c}] = \bar{c}^\mu \nabla^2 c_\mu + \left(1 - \frac{2(1+\beta)}{d}\right)\bar{c}^\mu \nabla_\mu \nabla^\nu c_\nu \tag{3.27}$$

and the interactions with the graviton are given by

$$\begin{aligned}\tilde{\mathcal{L}}_{c\bar{c}h}[c,\bar{c},h] = \kappa\Bigg\{&\left(1 - \frac{2(1+\beta)}{d}\right)\bar{c}^\mu h_{\nu\rho}\nabla^\rho \nabla_\mu c^\nu - \frac{1+\beta}{d}\left(\bar{c}^\mu \nabla_\mu c^\nu \nabla_\nu h + \bar{c}^\mu c^\nu \nabla_\mu \nabla_\nu h\right) - \frac{2(1+\beta)}{d}\bar{c}^\mu \nabla_\mu h_{\nu\rho}\nabla^\rho c^\nu \\ &+ \bar{c}^\mu \nabla_\mu c^\nu \nabla_\rho h_\nu{}^\rho + \bar{c}^\mu c^\nu \nabla_\nu \nabla_\rho h_\nu{}^\rho + \bar{c}^\mu h_{\mu\nu}\nabla^2 c^\nu + \bar{c}^\mu \nabla_\nu h_{\mu\rho}\nabla^\rho c^\nu + \bar{c}^\mu \nabla_\rho h_{\mu\nu}\nabla^\rho c^\nu\Bigg\}.\end{aligned} \tag{3.28}$$

Further we'll immediately drop the tilde on S_{gh}, i.e. set $S_{\mathrm{gh}} = \tilde{S}_{\mathrm{gh}}$, and set the corresponding phase in the path integral to $\theta = \mathrm{i}$. We make this choice since from $\tilde{\mathcal{L}}_{\mathrm{gh,kin}}$ we see that if we choose our gauge by $\beta = \frac{d}{2} - 1$ in order to cancel the second term then the ghost kinetic term

[16] I define linearized harmonic gauge by $\nabla h_\mu = \frac{1}{2}\nabla_\mu h$ for all d, since this is the condition obtained by linearizing the nonlinear harmonic gauge condition $g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = 0$ for all d.

becomes the very simple $\bar{c}^\mu \nabla^2 c_\mu$, which we recognize as the usual Lorenz-gauge spin-one kinetic term, correct sign and all. Setting $\beta = (d/2) - 1$ yields the ghost kinetic terms and interaction vertex commonly found in the literature, e.g. [66].

3.1.6 Fourier-transforming the various kinetic terms

Now let's find the Fourier transformation of the graviton and ghost kinetic terms. The ghost is easy:

$$S_{\text{gh,kin}}[c,\bar{c}] = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \bar{c}^\mu(k) \left\{ -\delta_\mu{}^\nu k^2 - \left(1 - \frac{2(1+\beta)}{d}\right) k_\mu k^\nu \right\} c_\nu(-k). \tag{3.29}$$

However the graviton terms are fairly long and unilluminating when expressed directly in terms of k and $g_{\mu\nu}$. So let us instead introduce a shorthand. First let's define the momentum-space kinetic matrices for the gauge-invariant classical kinetic terms and the gauge-fixing terms:

$$S_{\text{cl,kin}}[h] = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} h_{\mu\nu}(k) P^{\mu\nu\rho\sigma}_{\text{cl,kin}}(k) h_{\rho\sigma}(-k), \quad S_{\text{gf}}[h] = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} h_{\mu\nu}(k) P^{\mu\nu\rho\sigma}_{\text{gf}}(k) h_{\rho\sigma}(-k). \tag{3.30}$$

In practice these matrices are most easily computed in MATHEMATICA, in which it's easy to verify that $P_{\text{cl,kin}}$ and P_{gf} are symmetric. We can therefore parametrize the P's in terms of the five independent symmetric rank-four tensor structures that we can compose out of k and $g_{\mu\nu}$:

$$\begin{aligned}
T_{1\mu\nu\rho\sigma} &= G_{\mu\nu\rho\sigma} = \tfrac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right), \\
T_{2\mu\nu\rho\sigma} &= \frac{1}{d} g_{\mu\nu}g_{\rho\sigma} \equiv \mathrm{tr}_{\mu\nu\rho\sigma}, \\
T_{3\mu\nu\rho\sigma} &= \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4} \equiv A_{\mu\nu\rho\sigma}, \\
T_{4\mu\nu\rho\sigma} &= \frac{1}{2p^2}\left(g_{\mu\nu}p_\rho p_\sigma + g_{\rho\sigma}p_\mu p_\nu\right) \equiv B_{\mu\nu\rho\sigma}, \\
T_{5\mu\nu\rho\sigma} &= \frac{1}{4p^2}\left(g_{\mu\rho}p_\nu p_\sigma + g_{\mu\sigma}p_\nu p_\rho + g_{\nu\rho}p_\mu p_\sigma + g_{\nu\sigma}p_\mu p_\rho\right) \equiv C_{\mu\nu\rho\sigma}.
\end{aligned} \tag{3.31}$$

We will see in sec. 3.3.1 that T_1 is the natural metric induced by $\boldsymbol{g}$ on the space of symmetric rank-two tensors, which is why I dub it '**G**'. Note as well that T_2 can be trivially verified to be precisely the projector onto the trace mode, which is why I identify it as such. The final three tensors are labeled in ascending alphabetical order of the number of terms comprising them. We can obtain an expression for $P_{\mathrm{cl,kin}}$ and P_{gf} in terms of these structures as follows:

$$P_{\mathrm{cl,kin}} = p^2\Big(\mathbf{G} - d\,\mathrm{tr} + 2(B - C)\Big), \quad P_{\mathrm{gf}} = \frac{p^2}{\alpha}\left\{-\frac{(1+\beta)^2}{d}\,\mathrm{tr} - \frac{2(1+\beta)}{d}B + C\right\}. \tag{3.32}$$

3.1.7 Summarizing

In sum the gauge-fixed path integral is given by

$$Z = \int \mathscr{D}h\,\mathscr{D}c\,\mathscr{D}\bar{c}\,\exp\!\Big(\mathrm{i}\big(S_{\mathrm{cl}}[h] + S_{\mathrm{gf}}[h] + S_{\mathrm{gh}}[h]\big)\Big), \tag{3.33}$$

where the classical action is given by

$$S_{\mathrm{cl}}[\boldsymbol{g}] = -\frac{2}{\kappa^2}\int \mathrm{d}^d x\,\sqrt{-g}R \tag{3.34}$$

and the gauge-fixing function $F_\mu = \nabla h_\mu - \frac{1+\beta}{d}\nabla_\mu h$ determines the gauge-fixing action,

$$S_{\mathrm{gf}}[h] = \frac{1}{2\alpha}\int \mathrm{d}^d x\left(\nabla h^\mu - \frac{1+\beta}{d}\nabla^\mu h\right)\left(\nabla h_\mu - \frac{1+\beta}{d}\nabla_\mu h\right), \tag{3.35}$$

and the ghost action, which we can write as $S_{\mathrm{gh}}[c,\bar{c},h] = \int \mathrm{d}^d x\left(\mathcal{L}_{\mathrm{gh,kin}}[c,\bar{c}] + \mathcal{L}_{c\bar{c}h}[c,\bar{c},h]\right)$ with

$$\begin{aligned}
\mathcal{L}_{\mathrm{gh,kin}}[c,\bar{c}] &= \bar{c}^\mu\nabla^2 c_\mu + \left(1 - \frac{2(1+\beta)}{d}\right)\bar{c}^\mu\nabla_\mu\nabla^\nu c_\nu,\\
\mathcal{L}_{c\bar{c}h}[c,\bar{c},h] &= \kappa\left\{\left(1 - \frac{2(1+\beta)}{d}\right)\bar{c}^\mu h_{\nu\rho}\nabla^\rho\nabla_\mu c^\nu - \frac{1+\beta}{d}\left(\bar{c}^\mu\nabla_\mu c^\nu\nabla_\nu h + \bar{c}^\mu c^\nu\nabla_\mu\nabla_\nu h\right) - \frac{2(1+\beta)}{d}\bar{c}^\mu\nabla_\mu h_{\nu\rho}\nabla^\rho c^\nu\right.\\
&\quad\left. + \bar{c}^\mu\nabla_\mu c^\nu\nabla_\rho h_\nu{}^\rho + \bar{c}^\mu c^\nu\nabla_\nu\nabla_\rho h_\nu{}^\rho + \bar{c}^\mu h_{\mu\nu}\nabla^2 c^\nu + \bar{c}^\mu\nabla_\nu h_{\mu\rho}\nabla^\rho c^\nu + \bar{c}^\mu\nabla_\rho h_{\mu\nu}\nabla^\rho c^\nu\right\}.
\end{aligned} \tag{3.36}$$

Fourier-transforming the ghost action yields

$$S_{\mathrm{gh,kin}}[c,\bar{c}] = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\bar{c}^\mu(k)\left\{-\delta_\mu{}^\nu k^2 - \left(1 - \frac{2(1+\beta)}{d}\right)k_\mu k^\nu\right\}c_\nu(-k), \tag{3.37}$$

and, defining the kinetic matrices of the graviton terms by

$$S_A[h] = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} h_{\mu\nu}(k) P_A^{\mu\nu\rho\sigma}(k) h_{\rho\sigma}(-k), \tag{3.38}$$

we can write the P's in terms of the tensor structures of sec. 3.1.6 as

$$P_{\text{cl,kin}} = p^2\Big(\mathbf{G} - d\,\mathrm{tr} + 2(B-C)\Big), \quad P_{\text{gf}} = \frac{p^2}{\alpha}\left\{-\frac{(1+\beta)^2}{d}\,\mathrm{tr} - \frac{2(1+\beta)}{d}B + C\right\}. \tag{3.39}$$

3.2 Decomposing the ghost action

Here I will construct the standard decomposition of a generic vector into a transverse vector and scalar part, obtain the projectors onto these modes, and use these projectors to efficiently obtain the ghost propagator in this basis. While this decomposition is not likely to be necessary for a continuation of correlators to Euclidean space it serves as a clean and simple example of the logic that will used in decomposing the graviton, without the attendant complications and extensive equations.

3.2.1 Definitions

- Let's write $\mathcal{V} = \mathbb{C}^{1,3}$, so that $\mathcal{V}^*$ is the space in which (any given Fourier component of) an arbitrary one-form field lives. Then given any basis e_μ for $\mathcal{V}$ and its dual basis ϑ^μ for $\mathcal{V}$ we'd write e.g. $A \in \mathcal{V}^*$ in components as $A = A_\mu \vartheta^\mu$.

- Let's also denote by $\mathcal{T}$ and $\mathcal{T}^*$ the space of covariant and contravariant tensors on $\mathcal{V}$ (i.e. the space of maps $\mathcal{V} \to \mathcal{V}^*$ and $\mathcal{V}^* \to \mathcal{V}$ respectively). In components we would then write any $T \in \mathcal{T}$ as $T = T_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$ and any $\tilde{T} \in \mathcal{T}^*$ as $\tilde{T} = \tilde{T}^{\mu\nu} e_\mu \otimes e_\nu$.

- Finally let's assume that we are given some symmetric and invertible $\boldsymbol{g} \in \mathcal{T}$ (i.e. the metric), in terms of which we define $X \cdot X = \boldsymbol{g}(X, X')$ for $X, X' \in \mathcal{V}$ and $A \cdot A' \equiv \boldsymbol{g}^{-1}(A, A')$ for $A, A' \in \mathcal{V}^*$, and some distinguished $p \in \mathcal{V}^*$ (i.e. the momentum of the Fourier component under consideration), which we assume to be real.

With respect to the metric we use the usual convention in which $\boldsymbol{g} = g_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$ and $\boldsymbol{g}^{-1} = g^{\mu\nu} e_\mu \otimes e_\nu$, and we also adopt the usual convention in which the isomorphisms $\mathcal{V} \leftrightarrow \mathcal{V}^*$ and $\mathcal{T} \leftrightarrow \mathcal{T}^* \leftrightarrow \ldots$ (with the dots standing for the spaces of mixed tensors) induced by $\boldsymbol{g}$ are implicit, so that from here on out I'll only refer to $\mathcal{V}$ and $\mathcal{T}$ (with explicit reference to index position when necessary).

3.2.2 The decomposition

First observe that to any $A \in \mathcal{V}$ we can associate a unique $A_\perp \in \mathcal{V}$ such that $p \cdot A_\perp = 0$ via

$$A_{\perp\mu} = A_\mu - \frac{p \cdot A}{p^2} p_\mu. \tag{3.40}$$

We call $A_\perp$ the *transverse part* of A. Defining also a unique *scalar part* $S \in \mathbb{C}$ of A via

$$S = -\mathrm{i}\frac{p \cdot A}{p^2} \tag{3.41}$$

we can therefore decompose A uniquely into transverse and scalar parts as

$$A_\mu = A_{\perp\mu} + \mathrm{i}p_\mu S. \tag{3.42}$$

(We introduce the i into the scalar part so that the position-space version of this decomposition is $A = A_\perp + \nabla S$.)

3.2.3 Obtaining the projectors onto the transverse and scalar parts

Our goal is to obtain the projectors $\Pi_\perp{}^{\mu\nu}$ and $\Pi_\mathrm{S}{}^{\mu\nu}$ such that $\Pi_\perp{}^{\mu\nu} A_\nu = A_{\perp\mu}$ and $\Pi_\mathrm{S}{}^{\mu\nu} A_\nu = \mathrm{i}p_\mu S$, i.e. such that $\Pi_\perp$ projects out the transverse part of A and Π_S the scalar part. We do this as follows.

- First consider any projector $\Pi \in \mathcal{T}$ such that $A_0 \cdot A_1 = 0$ for all $A_0 \in \ker \Pi$ and $A_1 \in \operatorname{Im} \Pi$.

Since we can write any $A \in \mathcal{V}$ as $A = A_0 + A_1$ in terms of some such A_0 and A_1 it follows that

$$A \cdot (\Pi A') = (A_0 + A_1) \cdot A'_1 = A_1 \cdot A'_1 = A_1 \cdot (A'_0 + A'_1) = (\Pi A) \cdot A' \tag{3.43}$$

for all $A, A' \in \mathcal{V}$, i.e. that any such Π is self-adjoint with respect to the inner product $A \cdot A' = \boldsymbol{g}^{-1}(A, A')$. In more prosaic terms: any such Π is symmetric.

Our desired $\Pi_\perp$ and Π_S both certainly satisfy this property, which we can quickly check as follows: every $A \in \ker \Pi_\perp$ is of the form $A_\mu = \mathrm{i} p_\mu S$ and every $A' \in \operatorname{Im} \Pi_\perp$ is of the form $A' = A'_{\perp\mu}$, and the resulting inner product is $A \cdot A' = \mathrm{i} p^\mu S A'_{\perp\mu} = 0$ since $p^\mu A'_{\perp\mu}$ by definition. Further $\ker \Pi_\perp = \operatorname{Im} \Pi_\mathrm{S}$ and $\operatorname{Im} \Pi_\perp = \ker \Pi_\perp$, so the same holds for Π_S. Hence $\Pi_\perp$ and Π_S are both symmetric:

$$\Pi_\perp^{\mu\nu} = \Pi_\perp^{\nu\mu}, \quad \Pi_\mathrm{S}^{\mu\nu} = \Pi_\mathrm{S}^{\nu\mu}. \tag{3.44}$$

- Since $\Pi_\perp$ and Π_S should apply to arbitrary $A \in \mathcal{V}$ they should not depend on the A on which they act. Hence the only tensors on which they can depend are the ones we are given in our setup, namely the metric $\boldsymbol{g}$ and the distinguished $p \in \mathcal{V}$ which labels the Fourier component under consideration. That $\Pi_\perp$ and Π_S are symmetric implies further that they can only depend on p in the combination $p_\mu p_\nu$:

$$\Pi^{\mu\nu} = C^1 g^{\mu\nu} + C^2 \frac{p^\mu p^\nu}{p^2}, \tag{3.45}$$

  including a factor of $1/p^2$ in the $p^\mu p^\nu$ term so that C^1 and C^2 have the same dimension. We can therefore find C^1 and C^2 for each of $\Pi_\perp$ and Π_S by acting this ansatz for Π on an arbitrary decomposed $A \in \mathcal{V}$ and insisting that it select the desired component.

So let's turn the crank. Acting a generic $\Pi^{\mu\nu} = C^1 g^{\mu\nu} + C^2 p^\mu p^\nu / p^2$ on $A_\mu = A_{\perp\mu} + \mathrm{i} p_\mu S$ yields

$$(\Pi A)_\mu = \left(C^1 \delta_\mu{}^\nu + C^2 \frac{p_\mu p^\nu}{p^2} \right) \left(A_{\perp\nu} + \mathrm{i} p_\nu S \right) = C^1 A_{\perp\mu} + \mathrm{i} \left(C^1 + C^2 \right) p_\mu S. \tag{3.46}$$

Hence we can set the result equal to $A_{\perp\mu}$ by choosing $C^1 = 1$ and $C^2 = -C^1 = -1$,

$$\Pi_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \tag{3.47}$$

and we can set the result equal to $\mathrm{i}p_\mu S$ by choosing $C^1 = 0$ and $C^2 = 1$,

$$\Pi_{\mathrm{S}}^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}. \tag{3.48}$$

Note that $\Pi_{\perp}^{\mu\nu} + \Pi_{\mathrm{S}}^{\mu\nu} = g^{\mu\nu}$, as must be the case.

3.2.4 Decomposing the ghost kinetic terms

Now recall from eq. (3.37) that the ghost kinetic matrix is given by

$$P_{\mathrm{gh}}^{\mu\nu} = -p^2 g^{\mu\nu} - \left(1 - \frac{2(1+\beta)}{d}\right) p^\mu p^\nu. \tag{3.49}$$

This matrix can be written in terms of the transverse and scalar projectors (3.47) and (3.48) as follows. Since $\Pi_\perp$ is the only one of the two to contain $g^{\mu\nu}$ its coefficient is fixed:

$$P_{\mathrm{gh}} = -p^2 \Pi_\perp + C\,\Pi_{\mathrm{S}}, \tag{3.50}$$

with C a constant to be determined. Using our expressions for P_{gh} and the Π's gives

$$-p^2 g^{\mu\nu} - \left(1 - \frac{2(1+\beta)}{d}\right) p^\mu p^\nu = -p^2 g^{\mu\nu} + p^\mu p^\nu \left(1 + \frac{C}{p^2}\right), \tag{3.51}$$

from which it follows that

$$C = \frac{2(1+\beta-d)}{d} p^2. \tag{3.52}$$

And thus we find that the ghost kinetic matrix may be written in terms of the transverse and scalar vector projectors as

$$P_{\mathrm{gh}} = -p^2\left(\Pi_\perp + \frac{2(d-1-\beta)}{d}\Pi_{\mathrm{S}}\right). \tag{3.53}$$

3.2.5 The ghost propagator

The ghost propagator $S_{\mu\nu}(p)$ is the inverse of the ghost kinetic matrix P_{gh}. A major advantage of the projector formalism is that it streamlines the process of inverting this kinetic matrix: since we know how to write P_{gh} entirely in terms of $\Pi_\perp$ and Π_{S}, and these satisfy $\Pi_A \cdot \Pi_A = \delta_{AB}\Pi_B$ (no sum), the ghost propagator is found by simply inverting the coefficients of the projectors:

$$S(p) = -\frac{1}{p^2}\left(\Pi_\perp + \frac{d}{2(d-1-\beta)}\Pi_{\mathrm{S}}\right), \tag{3.54}$$

or explicitly

$$S_{\mu\nu}(p) = \frac{1}{p^2}\left\{\left(1 - \frac{d}{2(d-1-\beta)}\right)p_\mu p_\nu - g_{\mu\nu}\right\}. \tag{3.55}$$

3.3 The York decomposition of the graviton kinetic terms

3.3.1 Definitions

The transition from the spin-one construction of the previous section to the spin-two construction is as follows. The vector space in which our field lives is now the space $\mathcal{V} \equiv \mathrm{Sym}^2(\mathbb{C}^{1,3})$ of symmetric rank-two tensors on Minkowski space (i.e. the symmetric subspace of the $\mathcal{T}$ from the previous section), and the tensor space $\mathcal{T}$ is the space of rank-two tensors on $\mathcal{V}$ (so rank-four tensors on $\mathbb{C}^{1,3}$). Given a dual basis ϑ^μ of $\mathbb{C}^{1,3}$ we would then write any $h \in \mathcal{V}$ as $h = h_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$ with the added condition that $h_{\mu\nu} = h_{\nu\mu}$, and any $T \in \mathcal{T}$ as $T = T^{\mu\nu\rho\sigma} e_\mu \otimes e_\nu \otimes e_\rho \otimes e_\sigma$, with the added condition that $T^{\mu\nu\rho\sigma} = T^{\mu\nu\sigma\rho} = T^{\nu\mu\rho\sigma}$. We also retain the assumption of a given metric $g_{\mu\nu}$, with which we raise and lower indices, and a given $p \in \mathbb{R}^{1,3}$, which labels the Fourier mode under consideration.

In general when comparing this to the spin-one case it's useful to think of each pair of indices as a single multi-index, e.g. $h^{(\mu\nu)}$ and $T^{(\mu\nu)(\rho\sigma)}$, so that the action of T on h is $(Th)_{\mu\nu} = T_{\mu\nu\rho\sigma}h^{\rho\sigma}$. From this perspective the symmetry of $T^{\mu\nu\rho\sigma}$ under $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$ is, in a sense, trivial, and T being actually 'symmetric', i.e. self-adjoint, would manifest as a symmetry under $(\mu,\nu) \leftrightarrow (\rho,\sigma)$, i.e. $T^{\mu\nu\rho\sigma} = T^{\rho\sigma\mu\nu}$. Of course if we want to refer to T as 'self-adjoint' it must be with respect to a particular inner product on $\mathcal{W}$, which is given by the **G** mentioned in sec. 3.1.6:

$$\mathbf{G}(h,h') = \tfrac{1}{2}\big(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\big)h^{\mu\nu}h'^{\rho\sigma} = \tfrac{1}{2}\big(h_{\rho\sigma}h'^{\rho\sigma} + h_{\sigma\rho}h'^{\rho\sigma}\big) = h_{\mu\nu}h^{\mu\nu}. \tag{3.56}$$

This last form for **G** demonstrates that it is the *unique* rank-four tensor we can construct out of the provided metric $\boldsymbol{g}$ such that $G_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} = h_{\mu\nu}$, and is in this sense the natural choice of an inner product on $\mathcal{V}$ given the metric $\boldsymbol{g}$, as promised. And indeed if we act T on h',

$$\mathbf{G}(h,Th') = h_{\mu\nu}T^{\mu\nu}{}_{\rho\sigma}h^{\rho\sigma} = h_{\mu\nu}h_{\rho\sigma}T^{\mu\nu\rho\sigma}, \tag{3.57}$$

we see that T is self-adjoint if and only if it is symmetric in the sense described above, $T^{\mu\nu\rho\sigma} = T^{\rho\sigma\mu\nu}$.

3.3.2 The decomposition

Our goal now is to decompose $h_{\mu\nu}$ as far as possible, namely into a transverse and traceless (TT) part $h_{\perp\mu\nu}$ and some combination of one or more scalars and/or transverse vectors. This decomposition will, as is probably expected, be more involved than the spin-one case, which was done essentially by inspection.

So let's get after it. First observe that we can associate a unique traceless $\hat{h}_{\mu\nu}$ with each $h_{\mu\nu}$ via

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{d}g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma}, \tag{3.58}$$

i.e.

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d} g_{\mu\nu}\phi \tag{3.59}$$

with the *trace mode* defined by $\phi = g^{\mu\nu} h_{\mu\nu}$.

It remains to associate a unique transverse and traceless $h_{\perp\mu\nu}$ with our arbitrary traceless $\hat{h}_{\mu\nu}$. In other words we want to find some $\Delta_{\mu\nu}(p, \hat{h})$ such that

$$h_{\perp\mu\nu} = \hat{h}_{\mu\nu} - \Delta_{\mu\nu}, \tag{3.60}$$

and the requirement that $g^{\mu\nu} h_{\perp\mu\nu} = 0 = p^\mu h_{\perp\mu\nu}$ implies that we need $g^{\mu\nu}\Delta_{\mu\nu} = 0$ and $p^\mu\Delta_{\mu\nu} = p^\mu \hat{h}_{\mu\nu}$. We will proceed by first writing down a general traceless $\Delta_{\mu\nu}$ in terms of p and an arbitrary vector ξ, and then use the condition $p^\mu\Delta_{\mu\nu} = p^\mu \hat{h}_{\mu\nu}$ to obtain an expression for (the transverse and scalar components of) ξ in terms of $\hat{h}$.

Let's begin by justifying our expectation that $\Delta_{\mu\nu}$ can be determined in terms of a vector ξ. We want to obtain $h_{\perp\mu\nu}$ from $\hat{h}_{\mu\nu}$ by imposing the additional constraint $p^\mu h_{\perp\mu\nu} = 0$, which has d component equations. In other words $h_\perp$ should have d fewer independent components than $\hat{h}$, which is precisely the number of independent components of ξ.

Now let's consider the ways in which ξ can enter into $\Delta_{\mu\nu}$, given only as extra ingredients p_μ and $g_{\mu\nu}$. For simplicity we'll consider only terms linear in ξ.[17] There are three:

$$p_\mu\xi_\nu + p_\nu\xi_\mu, \quad \frac{p_\mu p_\nu}{p^2} p\cdot\xi, \quad g_{\mu\nu} p\cdot\xi. \tag{3.61}$$

The latter two terms contain only the scalar part of ξ: if we write $\xi_\mu = V_\mu + \mathrm{i} p_\mu S$ then $p\cdot\xi = \mathrm{i}p^2 S$. It therefore makes sense to decompose ξ in the first as well:

$$p_\mu\xi_\nu + p_\nu\xi_\mu = p_\mu V_\nu + p_\nu V_\mu + 2\mathrm{i} p_\mu p_\nu S. \tag{3.62}$$

[17] This is justified by hindsight because we know this is where we want to end up, and also because we don't need to obtain *every* way to decompose $h_{\mu\nu}$, only one in particular.

So we can equivalently phrase the three options above in terms of the transverse vector V and the scalar S:

$$p_\mu V_\nu + p_\nu V_\mu, \quad p_\mu p_\nu S, \quad g_{\mu\nu} p^2 S. \tag{3.63}$$

Let's write $T_{\mu\nu}$ as an arbitrary linear combination of the three:

$$T_{\mu\nu} = \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right) + \left(A p_\mu p_\nu + B g_{\mu\nu} p^2\right) S. \tag{3.64}$$

Note that we can without loss of generality choose the coefficient in front of the first term to be i by rescaling V.[18] Let's impose that $T_{\mu\nu}$ is traceless:

$$g^{\mu\nu} T_{\mu\nu} = \left(A + Bd\right) p^2 S, \tag{3.65}$$

using the fact that $p \cdot V = 0$. It follows that we must have $B = -A/d$:

$$T_{\mu\nu} = \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right) + A\left(p_\mu p_\nu - \frac{1}{d} g_{\mu\nu} p^2\right) S = \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right) - \left(p_\mu p_\nu - \frac{1}{d} g_{\mu\nu} p^2\right) S, \tag{3.66}$$

choosing $A = -1$ by rescaling S.[19]

Using the above we arrive at our ansatz: we aim to decompose a generic symmetric $h_{\mu\nu}$ into a TT rank-two tensor $h_\perp$, a transverse vector V, and two scalars S and ϕ, in the form

$$h_{\mu\nu} = h_{\perp\mu\nu} + \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right) - \left(p_\mu p_\nu - \frac{1}{d} g_{\mu\nu} p^2\right) S + \frac{1}{d} g_{\mu\nu} \phi. \tag{3.67}$$

We have already obtained $\phi = g^{\mu\nu} h_{\mu\nu}$. It remains only to express V and S in terms of h, since once we do so we also arrive at an implicit equation for $h_\perp$ in terms of h (and hence at an unambiguous, completely defined decomposition of h into $h_\perp$, V, S, and ϕ). To keep some of the

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[18] *Q.* What if the correct coefficient is zero? Then you can't rescale it away! *A.* The coefficient can't be zero because that would eliminate the $d - 1$ independent components of V, leaving us without enough independent components to decompose an arbitrary $h_{\mu\nu}$ in full generality. Note that we can't yet apply the same logic to rescale away either of A or B, since we're not a priori guaranteed that neither is zero, although we do know that they can't both be, because if they were then we'd lose the independent component S.

[19] And now using the fact that we must have $A \neq 0$ in order to retain the independent component S.

expressions manageable let's recall our notation for the traceless part of h,

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \phi, \tag{3.68}$$

and further introduce its *scalar-less part*,

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \phi + \left(p_\mu p_\nu - \frac{1}{d} g_{\mu\nu} p^2 \right) S. \tag{3.69}$$

Since we already have an expression for ϕ in terms of h the traceless part $\hat{h}$ is defined unambiguously for any h. Similarly once we obtain an expression for S in terms of h the above will provide an unambiguous definition of h' for all h.

To solve for S from $\hat{h}_{\mu\nu}$ let's consider $p^\mu p^\nu \hat{h}_{\mu\nu}$. This immediately eliminates the $h_\perp$ and V terms, leaving us with

$$p^\mu p^\nu \hat{h}_{\mu\nu} = -\left(p^4 - \frac{1}{d} p^4 \right) S = -\frac{d-1}{d} p^4 S \implies S = -\frac{d}{d-1} \frac{p^\mu p^\nu}{p^4} \hat{h}_{\mu\nu}. \tag{3.70}$$

Writing $\hat{h}$ in terms of h and ϕ[20] yields our final (and unambiguous) expression for S in terms of h:

$$S = -\frac{d}{d-1} \frac{p^\mu p^\nu}{p^4} \left(h_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \phi \right) = -\frac{d}{d-1} \frac{p^\mu p^\nu}{p^4} h_{\mu\nu} + \frac{1}{d-1} \frac{1}{p^2} \phi. \tag{3.71}$$

Since S is the scalar part of the vector ξ, i.e. the only part which doesn't vanish under $p \cdot \xi$, we'll call it the *longitudinal scalar part* of h.

It only remains to find V, which also follows straightforwardly. Contracting the scalar-less part of h with p eliminates the $h_\perp$ term and leaves only V:

$$p^\nu h'_{\mu\nu} = \mathrm{i} p^2 V_\mu \implies V_\mu = -\frac{\mathrm{i} p^\nu}{p^2} h'_{\mu\nu}. \tag{3.72}$$

Expressing h' in terms of h and ϕ is a little more involved, but after a little algebra things sim-

[20] I leave ϕ, instead of writing $g^{\mu\nu} h_{\mu\nu}$, as notational shorthand. Mostly I just think 'ϕ' looks better than '$\mathrm{tr}\, h$' or '$h^\mu{}_\mu$' or whatever.

plify nicely:

$$V_\mu = -\frac{\mathrm{i}}{p^2}\left(p^\nu h_{\mu\nu} - \frac{1}{d}p_\mu\phi + \frac{d-1}{d}p_\mu p^2 S\right) = -\frac{\mathrm{i}}{p^2}\left(p^\nu h_{\mu\nu} - \frac{p_\mu p^\rho p^\sigma}{p^2}h_{\rho\sigma}\right) \tag{3.73}$$

And now that we have explicit expressions for S and V on top of ϕ we can in principle write down an explicit expression for $h_\perp$. I say 'in principle' because the resulting expression is long and unilluminating, so I won't write it down here. The point is rather that we have now demonstrated that the decomposition we have written down,

$$h_{\mu\nu} = h_{\perp\mu\nu} + \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right) - \left(p_\mu p_\nu - \frac{1}{d}g_{\mu\nu}p^2\right)S + \frac{1}{d}g_{\mu\nu}\phi, \tag{3.74}$$

uniquely defines the TT part $h_\perp$, the transverse vector part V, and the scalar parts S and ϕ. This decomposition is called the *York decomposition* [67].

3.3.3 The York projectors

Our goal in this section is to express the projectors $\Pi_\perp$, Π_V, Π_S, and Π_ϕ which project onto the York modes defined above, i.e.

$$\begin{aligned} \Pi_{\perp\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} &= h_{\perp\mu\nu}, & \Pi_{V\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} &= \mathrm{i}\left(p_\mu V_\nu + p_\nu V_\mu\right), \\ \Pi_{S\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} &= -\left(p_\mu p_\nu - \frac{1}{d}g_{\mu\nu}p^2\right)S, & \Pi_{\phi\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} &= \mathrm{i}g_{\mu\nu}\phi. \end{aligned} \tag{3.75}$$

By definition these Π's sum to the identity,

$$\sum_n \Pi_{n\mu\nu}{}^{\rho\sigma} = G_{\mu\nu}{}^{\rho\sigma} = \tfrac{1}{2}\left(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho\right), \tag{3.76}$$

and it is straightforward to verify that each is orthogonal and hence symmetric.[21] We can therefore write down a completely general ansatz in terms of the symmetric tensor structures

[21] Recall that each multi-index (μ,ν) is by definition symmetric - the nontrivial symmetry referred to above is between multi-indices, i.e. $\Pi_{\mu\nu\rho\sigma} = \Pi_{\rho\sigma\mu\nu}$.

of sec. 3.1.6:

$$\Pi_n = \sum_i X_n^i T_i, \tag{3.77}$$

where $n \in \{\perp, V, S, \phi\}$ and the tensor structures are as given in eq. (3.31),

$$\begin{aligned}
&T_{1\mu\nu\rho\sigma} = G_{\mu\nu\rho\sigma} = \tfrac{1}{2}\big(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\big), \quad T_{2\mu\nu\rho\sigma} = \frac{1}{d}g_{\mu\nu}g_{\rho\sigma} \equiv \mathrm{tr}_{\mu\nu\rho\sigma}, \quad T_{3\mu\nu\rho\sigma} = \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4} \equiv A_{\mu\nu\rho\sigma}, \\
&T_{4\mu\nu\rho\sigma} = \frac{1}{2p^2}\big(g_{\mu\nu}p_\rho p_\sigma + \ldots\big) \equiv B_{\mu\nu\rho\sigma}, \quad T_{5\mu\nu\rho\sigma} = \frac{1}{4p^2}\big(g_{\mu\rho}p_\nu p_\sigma + \ldots\big) \equiv C_{\mu\nu\rho\sigma}.
\end{aligned} \tag{3.78}$$

Each of the four desired projectors (i.e. the coefficients X_n^i for the desired n) are entirely determined by the definitions (3.75). This calculation is most efficiently done in MATHEMATICA, which yields the following. For the trace projector we find, as mentioned above,

$$\Pi_\phi = \mathrm{tr}. \tag{3.79}$$

The projectors onto the longitudinal scalar and the vector are a bit more complicated:

$$\Pi_S = \frac{1}{d-1}\mathrm{tr} + \frac{d}{d-1}A - \frac{2}{d-1}B, \quad \Pi_V = 2\big(C - A\big). \tag{3.80}$$

And finally we have the projector onto the TT mode:

$$\Pi_\perp = G - \frac{d}{d-1}\mathrm{tr} + \frac{d-2}{d-1}A + \frac{2}{d-1}B - 2C. \tag{3.81}$$

As a check on these results it is straightforward to verify[22] that these four projectors do indeed sum to the identity.

3.3.4 The scalar mixing pseudoprojector

Let's take stock for a moment. The symmetric tensor h which we've been concerned with decomposing is (by definition) an element of the vector space $\mathcal{V} = \mathrm{Sym}^2(\mathbb{C}^{1,3})$, and we can think

[22] In the shorthand we've been using - when the explicit form is written out, in all its rank-four glory, it's still quite a lengthy sum.

of our decomposition as a choice of (a class of) basis for $\mathcal{V}$.[23] Let's write this as a column vector:

$$h = \begin{pmatrix} h_\perp \\ V \\ S \\ \phi \end{pmatrix}. \tag{3.82}$$

In the above we can think of $h_\perp$ as a column vector with $\mathrm{dof}[h_\perp] = \frac{1}{2}(d+1)(d-2)$ independent components[24] and V as a column vector with $d-1$ components. In this picture the York projectors can be represented as the usual diagonal projectors:

$$\Pi_\perp = \begin{pmatrix} 1_\perp & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \Pi_V = \begin{pmatrix} 0 & & & \\ & 1_V & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \Pi_S = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad \Pi_\phi = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \tag{3.83}$$

in which $1_\perp$ and 1_V are the (respectively $(d+1)(d-2)/2$- and $(d-1)$-dimensional) identity operators in the TT and vector subspaces.

Now, our eventual goal is to decompose the gauge-fixed graviton kinetic terms into a sum of projectors onto the various parts of h. The York projectors are insufficient for this purpose, which we can see as follows. If the gauge-fixed graviton kinetic matrix $P_{\text{kin}} = P_{\text{cl,kin}} + P_{\text{gf}}$ (where $P_{\text{cl,kin}}$ and P_{gf} are given in sec. 3.1.6 in terms of the tensor structures above) is a sum of the York projectors then the gauge-fixed graviton kinetic terms $h_{\mu\nu}(p)P_{\text{kin}}^{\mu\nu\rho\sigma}h_{\rho\sigma}(-p)$ should decompose into terms quadratic in each of the York components, with no cross-terms. However

[23] I say 'a class of' because, while we have decomposed $\mathcal{V}$ into orthogonal subspaces, we haven't chosen a basis within those subspaces.

[24] (i.e. the $\sum_{k=1}^{d} k = \frac{1}{2}d(d+1)$ independent components of a generic symmetric rank-two tensor in d dimensions, minus d components from the d constraints $p^\mu h_{\perp\mu\nu} = 0$ and another component from the constraint $g^{\mu\nu}h_{\perp\mu\nu} = 0$)

a direct evaluation reveals that this product contains a cross term in ϕ and S:[25]

$$h_{\mu\nu} P_{\text{kin}}^{\mu\nu\rho\sigma} h_{\rho\sigma} \sim h_\perp^2 + V^2 + S^2 + \phi^2 + \phi S. \tag{3.84}$$

Hence P_{kin} must contain a term $\Pi_{\phi S}$ which mixes ϕ and S. We'll call $\Pi_{\phi S}$ the *(scalar) mixing pseudoprojector*,[26] and we can think of it as being analogous to

$$\Pi_{\phi S} \sim \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}. \tag{3.85}$$

We'll obtain $\Pi_{\phi S}$ as follows. Let's define the 'raw' mixing pseudoprojector $\tilde{\Pi}_{\phi S}$ in the most naive way possible,

$$\left(\tilde{\Pi}_{\phi S} h\right)_{\mu\nu} = -\left(\frac{p_\mu p_\nu}{p^2} - \frac{1}{d} g_{\mu\nu}\right)\phi + \frac{1}{d} g_{\mu\nu} p^2 S, \tag{3.86}$$

i.e. by requiring that it project onto the scalar subspace and swap the longitudinal scalar and trace parts, $\phi \leftrightarrow p^2 S$. (The factors of p^2 are there because the mass dimension of S is two less than the mass dimension of ϕ.) The $\tilde{\Pi}_{\phi S}$ defined in this way is not symmetric,[27] which we can see either by attempting to solve for it as a linear combination of the symmetric tensor structures enumerated above (which attempt fails) or by directly comparing $\mathbf{G}\left(h, \tilde{\Pi}_{\phi S} h'\right)$ and $\mathbf{G}\left(\tilde{\Pi}_{\phi S} h, h'\right)$. However since our goal is to write the graviton kinetic terms, and the corresponding propagator, in terms of the Π's, and the kinetic terms are symmetric, it follows that the only way in which the kinetic terms can depend on $\Pi_{\phi S}$ is through its symmetric part $\Pi_{\phi S} = \frac{1}{2}\left(\tilde{\Pi}_{\phi S} + \tilde{\Pi}_{\phi S}^{\mathrm{T}}\right)$.

[25] N.B. the vector contribution is pure gauge, but the longitudinal and mixing contributions are not. I will return to this point in the next section and in sec. 3.4.1.

[26] Really it's in no sense a projector, so 'pseudoprojector' is kind of overselling it. However we'll be using it along with the other Π's as a (partial) basis for the space of tensors on the space of h's, so it's useful to give it a similar name.

[27] Note that this holds even though even though its image and kernel are orthogonal, since it's not a projector - the proof that $\ker \Pi \perp \operatorname{Im} \Pi \implies \Pi = \Pi^{\mathrm{T}}$ requires that $\Pi^2 = 1$.

So, to work. Since $\tilde{\Pi}_{\phi S}$ is not symmetric we need to include new antisymmetric terms in our ansatz. It turns out to be sufficient to include the antisymmetric version of B:

$$\tilde{B}_{\mu\nu\rho\sigma} \equiv \frac{1}{2p^2}\left(g_{\mu\nu}p_\rho p_\sigma - g_{\rho\sigma}p_\mu p_\nu\right). \tag{3.87}$$

Imposing that our ansatz satisfy eq. (3.86) is sufficient to determine it uniquely:

$$\tilde{\Pi}_{\phi S} = \frac{d}{d-1}\,\mathrm{tr} - \frac{d}{d-1}B + \frac{d-2}{d-1}\tilde{B}. \tag{3.88}$$

And we can in this form (since tr and B are symmetric and $\tilde{B}$ is antisymmetric) read off the symmetrized scalar mixing pseudoprojector:

$$\Pi_{\phi S} = \frac{d}{d-1}\left(\mathrm{tr} - B\right). \tag{3.89}$$

The symmetrized pseudoprojector acts on h in essentially the same way as the raw pseudoprojector. The only difference is some extra numerical factors:

$$\left(\Pi_\phi \Pi_{\phi S} h\right)_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} p^2 S, \quad \left(\Pi_S \Pi_{\phi S} h\right)_{\mu\nu} = -\frac{d}{2(d-1)}\left(\frac{p_\mu p_\nu}{p^2} - \frac{1}{d} g_{\mu\nu}\right)\phi. \tag{3.90}$$

Or in other words, writing $\Pi_{\phi S} h = h'$,

$$\phi' = \frac{d}{2} p^2 S, \quad S' = \frac{d}{2(d-1)} \frac{\phi}{p^2}. \tag{3.91}$$

As an aside: that the "raw" mixing pseudoprojector is not symmetric is a direct consequence of the fact that the scalar modes it is defined to swap are not normalized with respect to the inner product defined by $\mathbf{G}$. I'll return to this point in sec. 3.4.1.

3.3.5 Decomposing the graviton kinetic terms

Now that we have the projectors $\Pi_\perp$, Π_V, Π_S, and Π_ϕ from eqs. (3.79), (3.80), and (3.81), along with the pseudoprojector $\Pi_{\phi S}$ from eq. (3.89), we are equipped to reexpress the graviton kinetic terms in terms of the York decomposition. Recall from eq. (3.39) that the classical and gauge-fixing kinetic matrices are given in terms of the tensor structures (3.31) by

$$P_{\text{cl,kin}} = p^2\Big(\mathbf{G} - d\,\text{tr} + 2(B-C)\Big), \quad P_{\text{gf}} = \frac{p^2}{\alpha}\left\{\frac{(1+\beta)^2}{d}\text{tr} - \frac{2(1+\beta)}{d}B + C\right\}. \tag{3.92}$$

Both of these may be straightforwardly reexpressed in terms of the Π's by matching the coefficients of the tensor structures, in a manner analogous to but more lengthy than that of sec. 3.2.4. Doing so yields for the classical terms

$$P_{\text{cl,kin}} = p^2\left\{\Pi_\perp - \frac{2-3d+d^2}{d}\Pi_\phi - \frac{d-2}{d}\Pi_S - \frac{2(2-3d+d^2)}{d^2}\Pi_{\phi S}\right\}, \tag{3.93}$$

and for the gauge-fixing terms

$$P_{\text{gf}} = \frac{p^2}{\alpha}\left\{\frac{1}{2}\Pi_V + \frac{\beta^2}{d}\Pi_\phi + \frac{d-1}{d}\Pi_S + \frac{2\beta(d-1)}{d^2}\Pi_{\phi S}\right\}. \tag{3.94}$$

N.B. eq. (3.93) explicitly demonstrates that none of the scalar pieces (trace, longitudinal, and mixing) are pure gauge, while the vector part is: the Einstein-Hilbert action propagates a TT tensor and (as we will see in sec. 3.4.1) precisely one scalar mode, which is a mixture of the trace and the longitudinal scalar.

3.3.6 The graviton propagator in the York decomposition

The graviton propagator $\Delta_{\mu\nu\rho\sigma}(p)$ is the inverse of the gauge-fixed kinetic matrix $P_{\text{kin}} = P_{\text{cl,kin}} + P_{\text{gf}}$. The TT and vector parts of this inverse may be found as we did for the ghost by simply inverting the coefficients. The presence of the mixing term in the scalar sector makes inverting that piece a little less trivial, but it may still be done in an essentially algebraic way by using the straightforwardly-verified facts that

$$\Pi_{\phi S}\cdot\Pi_{\phi}=\Pi_S\cdot\Pi_{\phi S},\quad \Pi_{\phi S}\cdot\Pi_S=\Pi_{\phi}\cdot\Pi_{\phi S},\quad \Pi_{\phi S}\cdot\Pi_{\phi S}=\frac{d}{4(d-1)}\big(\Pi_{\phi}+\Pi_S\big), \tag{3.95}$$

along with the orthonormality of Π_ϕ and Π_S. Doing so yields

$$\begin{aligned}\Delta(p)=\frac{1}{p^2}\Bigg\{&\Pi_\perp+2\alpha\Pi_V+\frac{d(1-d-2\alpha+d\alpha)}{(d-2)(d-1-\beta)^2}\Pi_\phi\\&+\frac{d(2\alpha-3d\alpha+d^2\alpha-\beta^2)}{(d-2)(d-1-\beta)^2}\Pi_S-\frac{2(2\alpha-3d\alpha+d^2\alpha+\beta-d\beta)}{(d-2)(d-1-\beta)^2}\Pi_{\phi S}\Bigg\}.\end{aligned} \tag{3.96}$$

3.4 Diagonalizing the scalar sector of the Einstein-Hilbert kinetic terms

3.4.1 The Einstein-Hilbert action only propagates one scalar mode

Let's take a look at the scalar sector of the Einstein-Hilbert kinetic terms (3.93):

$$P_{\text{cl,scalar}}=-p^2\left\{\frac{2-3d+d^2}{d}\Pi_\phi+\frac{d-2}{d}\Pi_S+\frac{2(2-3d+d^2)}{d^2}\Pi_{\phi S}\right\}. \tag{3.97}$$

Now, recall from sec. 3.3.4 that the generic $h_{\mu\nu}$ which we are concerned with decomposing is an element of the vector space $\mathcal{V}=\mathrm{Sym}^2(\mathbb{C}^{1,3})$, and the longitudinal and trace modes span a two-dimensional subspace of $\mathcal{V}$, say $\mathcal{V}_{\text{scalar}}$. Then the Einstein-Hilbert kinetic terms constitute a symmetric operator on $\mathcal{V}_{\text{scalar}}$, and the presence of the mixing term indicates that this operator is not diagonalized in the (ϕ,S) basis. In this section I will diagonalize this operator and in the process demonstrate that the Einstein-Hilbert kinetic terms only propagate a single scalar mode.

To start let's look at the structure of $\mathcal{V}_{\text{scalar}}$. The York decomposition of a generic element of $\mathcal{V}$ is

$$h_{\mu\nu}=h_{\perp\mu\nu}+\mathrm{i}\big(p_\mu V_\nu+p_\nu V_\mu\big)-\left(p_\mu p_\nu-\frac{1}{d}g_{\mu\nu}p^2\right)S+\frac{1}{d}g_{\mu\nu}\phi, \tag{3.98}$$

meaning that a generic element of $\mathcal{V}_{\text{scalar}}$ is an arbitrary linear combination of

$$\mathbf{u}_{\mu\nu} \equiv \frac{1}{d} g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \quad \mathbf{v}_{\mu\nu} \equiv \frac{1}{d} g_{\mu\nu}, \tag{3.99}$$

using vector notation to emphasize that the above form a basis for a vector space on which we are considering operators (even though they are themselves matrices). In terms of $\mathbf{u}$ and $\mathbf{v}$ an arbitrary element of $\mathcal{V}_{\text{scalar}}$ is

$$\boldsymbol{h} = p^2 S\mathbf{u} + \phi\mathbf{v}. \tag{3.100}$$

However $\mathbf{u}$ and $\mathbf{v}$ are not normalized with respect to the inner product $\mathbf{G}_{\mu\nu\rho\sigma} = (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})/2$ on $\mathcal{V}$:

$$\mathbf{G}(\mathbf{u}, \mathbf{u}) = \frac{d-1}{d}, \quad \mathbf{G}(\mathbf{v}, \mathbf{v}) = \frac{1}{d}. \tag{3.101}$$

As a sidenote, this is why the "raw" mixing pseudoprojector $\tilde{\Pi}_{\phi S}$ of sec. 3.3.4 is not symmetric: it was defined to directly swap two vectors of different norms, namely $\mathbf{u}$ and $\mathbf{v}$.

Now let's define the normalized forms of $\mathbf{u}$ and $\mathbf{v}$:

$$\hat{\mathbf{u}} = \sqrt{\frac{d}{d-1}}\,\mathbf{u}, \quad \hat{\mathbf{v}} = \sqrt{d}\,\mathbf{v}. \tag{3.102}$$

Since $\mathcal{V}_{\text{scalar}}$ is a two-dimensional vector space its elements may be represented by column vectors, and since $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are normalized we may therefore represent them as

$$\hat{\mathbf{u}} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{v}} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.103}$$

This picture makes clear another way to find the projectors onto the longitudinal and trace modes: simply take the outer products $\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}$ and $\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}$! Doing so and turning the crank reveals that these are precisely the projectors we've already been working with, with no extra numerical factors:

$$\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} = \Pi_S, \quad \hat{\mathbf{v}} \otimes \hat{\mathbf{v}} = \Pi_\phi. \tag{3.104}$$

These projectors may therefore be represented by the matrices

$$\Pi_S \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_\phi \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.105}$$

Similarly we may find the matrix representation of $\Pi_{\phi S}$ by acting it on $\mathbf{u}$ and $\mathbf{v}$, which reveals

$$\Pi_{\phi S}\hat{\mathbf{u}} = \frac{d}{2\sqrt{d-1}}\hat{\mathbf{v}}, \quad \Pi_{\phi S}\hat{\mathbf{v}} = \frac{d}{2\sqrt{d-1}}\hat{\mathbf{u}}, \tag{3.106}$$

so the mixing pseudoprojector is given by

$$\Pi_{\phi S} = \frac{d}{2\sqrt{d-1}}\left(\hat{\mathbf{u}} \otimes \hat{\mathbf{v}} + \hat{\mathbf{v}} \otimes \hat{\mathbf{u}}\right) \mapsto \frac{d}{2\sqrt{d-1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.107}$$

Using the above representations it follows that the Einstein-Hilbert kinetic term may be represented in this basis by

$$P_{\text{cl,scalar}} \mapsto -p^2 \begin{pmatrix} \frac{d-2}{d} & \frac{2-3d+d^2}{d\sqrt{d-1}} \\ \frac{2-3d+d^2}{d\sqrt{d-1}} & \frac{2-3d+d^2}{d} \end{pmatrix} \equiv -p^2\mathbf{M}. \tag{3.108}$$

To diagonlize the scalar sector of the Einstein-Hilbert kinetic terms we therefore wish to diagonalize the matrix $\mathbf{M}$. Doing so yields the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = d-2$, with corresponding normalized eigenvectors

$$\hat{\mathbf{e}}_0 = -\sqrt{\frac{d-1}{d}}\hat{\mathbf{u}} + \frac{\hat{\mathbf{v}}}{\sqrt{d}}, \quad \hat{\mathbf{e}}_1 = \frac{\hat{\mathbf{u}}}{\sqrt{d}} + \sqrt{\frac{d-1}{d}}\hat{\mathbf{v}}. \tag{3.109}$$

These eigenvectors can be reexpressed straightforwardly in terms of the metric and the momentum using eqs. (3.99) and (3.102):

$$
\begin{aligned}
(\hat{\mathbf{e}}_0)_{\mu\nu} &= (-\mathbf{u}+\mathbf{v})_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}, \\
(\hat{\mathbf{e}}_1)_{\mu\nu} &= \left(\frac{1}{\sqrt{d-1}}\mathbf{u}+\sqrt{d-1}\mathbf{v}\right)_{\mu\nu} = \frac{1}{\sqrt{d-1}}\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right).
\end{aligned}
\tag{3.110}
$$

That $P_{\text{cl,scalar}}$ has a zero eigenvalue is critical: it tells us that the scalar part of the Einstein-Hilbert kinetic term is in fact a projector in its own right (or at least proportional to one), specifically onto the subspace spanned by $\hat{\mathbf{e}}_1$, and eliminates the subspace spanned by $\hat{\mathbf{e}}_0$. In other words, the only scalar mode propagated by the Einstein-Hilbert kinetic term is the mode corresponding to $\hat{\mathbf{e}}_1$, say Φ, with any appearance of the mode corresponding to $\hat{\mathbf{e}}_0$, say Σ, in the action being pure gauge. For this reason I will call Φ the *physical* scalar mode and Σ the *gauge* scalar mode, and this basis the *physical basis*.

3.4.2 The physical and gauge scalars in terms of the longitudinal and trace modes

Let's now find the new scalars Φ and Σ in terms of the longitudinal and trace modes S and ϕ.

To start let's define the transformation matrix from the York basis $\{\hat{\mathbf{u}},\hat{\mathbf{v}}\}$ to the normalized physical basis $\{\hat{\mathbf{e}}_0,\hat{\mathbf{e}}_1\}$ in the usual way, by arranging the eigenvectors as the rows of the matrix:

$$
\mathbf{U} = \frac{1}{\sqrt{d}}\begin{pmatrix} -\sqrt{d-1} & 1 \\ 1 & \sqrt{d-1} \end{pmatrix}. \tag{3.111}
$$

Since both bases are normalized this matrix is orthogonal, so $\mathbf{M}$ is diagonalized by $\tilde{\mathbf{M}} = \mathbf{U}\mathbf{M}\mathbf{U}^T$. This transformation matrix also tells us how to find physical and gauge modes in terms of the York modes:

$$
\tilde{\Sigma}\hat{\mathbf{e}}_0+\tilde{\Phi}\hat{\mathbf{e}}_1 = p^2 S\mathbf{u}+\phi\mathbf{v} = p^2 S\sqrt{\frac{d-1}{d}}\hat{\mathbf{u}}+\frac{\phi}{\sqrt{d}}\hat{\mathbf{v}} \implies \begin{pmatrix}\tilde{\Sigma}\\ \tilde{\Phi}\end{pmatrix} = \frac{1}{\sqrt{d}}\mathbf{U}\begin{pmatrix} p^2 S\sqrt{d-1} \\ \phi\end{pmatrix}, \tag{3.112}
$$

from which we find

$$\tilde{\Sigma} = \frac{1}{d}\Big(\phi - (d-1)p^2 S\Big), \quad \tilde{\Phi} = \frac{\sqrt{d-1}}{d}(\phi + p^2 S). \tag{3.113}$$

I use tildes in the above since I will now adjust the normalization. Let's pull the overall d-dependent factors out of the scalars,

$$\Sigma = \phi - (d-1)p^2 S = d\tilde{\Sigma}, \quad \Phi = \phi + p^2 S = \frac{d}{\sqrt{d-1}}\tilde{\Phi}, \tag{3.114}$$

and shunt them into the basis vectors,

$$\mathbf{e}_0 = \frac{1}{d}\hat{\mathbf{e}}_0, \quad \mathbf{e}_1 = \frac{\sqrt{d-1}}{d}\hat{\mathbf{e}}_1. \tag{3.115}$$

Then the scalar sector of a generic $\boldsymbol{h}$ may be written

$$p^2 S\mathbf{u} + \phi\mathbf{v} = \Sigma\mathbf{e}_0 + \Phi\mathbf{e}_1, \tag{3.116}$$

or explicitly in terms of the tensor structures

$$\frac{1}{d}g_{\mu\nu}\phi + \left(p_\mu p_\nu - \frac{1}{d}g_{\mu\nu}p^2\right)S = \frac{p_\mu p_\nu}{dp^2}\Sigma + \frac{1}{d}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Phi. \tag{3.117}$$

3.4.3 The projectors onto the physical basis

The projectors onto the Φ and Σ modes may be found by taking the outer products of the corresponding normalized basis vectors:

$$\Pi_\Sigma = \hat{\mathbf{e}}_0 \otimes \hat{\mathbf{e}}_0, \quad \Pi_\Phi = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1. \tag{3.118}$$

(N.B. we still use the normalized basis vectors to construct the projectors, even though we've changed the normalization of the modes themselves, since it is only by using the normalized basis vectors that we retain the property that $\Pi\cdot\Pi = \Pi$.) Recalling that $\hat{\mathbf{u}}\otimes\hat{\mathbf{u}} = \Pi_S$ and $\hat{\mathbf{v}}\otimes\hat{\mathbf{v}} = \Pi_\phi$, along with $\hat{\mathbf{u}}\otimes\hat{\mathbf{v}}+\hat{\mathbf{v}}\otimes\hat{\mathbf{u}} \propto \Pi_{\phi S}$ (with the constant of proportionality given in eq. (3.107)), we then straightforwardly find that the projectors onto the Σ and Φ modes may be written in terms of the York scalar projectors as

$$\Pi_\Sigma = \frac{1}{d}\Pi_\phi + \frac{d-1}{d}\Pi_S - \frac{2(d-1)}{d^2}\Pi_{\phi S}, \quad \Pi_\Phi = \frac{d-1}{d}\Pi_\phi + \frac{1}{d}\Pi_S + \frac{2(d-1)}{d^2}\Pi_{\phi S}. \tag{3.119}$$

Similarly we may construct the symmetric (Φ,Σ) mixing pseudoprojector as[28]

$$\Pi_{\Phi\Sigma} = \hat{\mathbf{e}}_0\otimes\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_0 = \frac{2\sqrt{d-1}}{d}\Pi_\phi - \frac{2\sqrt{d-1}}{d}\Pi_S - \frac{2(d-2)\sqrt{d-1}}{d^2}\Pi_{\phi S}. \tag{3.120}$$

The above may also be written in terms of the symmetric tensor structures (3.31), using the expressions (3.79), (3.80), (3.81), and (3.89) for the York projectors:

$$\Pi_\Sigma = A, \quad \Pi_\Phi = \frac{d}{d-1}\mathrm{tr} + \frac{1}{d-1}A - \frac{2}{d-1}B, \quad \Pi_{\Phi\Sigma} = -\frac{2}{\sqrt{d-1}}A + \frac{2}{\sqrt{d-1}}B. \tag{3.121}$$

3.4.4 Decomposing the graviton kinetic terms

Now that we have physical basis for the scalar sector of the graviton we may reexpress the graviton kinetic terms in terms of the projectors onto these modes.

The classical kinetic terms are immediate. Recall that our initial motivation in finding the Φ and Σ modes was to diagonalize the Einstein-Hilbert kinetic terms and thus to find the physical scalar mode which it propagates, and that in doing so we found in eqs. (3.108) and (3.109) that the scalar sector of the Einstein-Hilbert kinetic matrix may be written $-p^2\mathbf{M}$, where the matrix $\mathbf{M}$ has eigenvalues 0 and $d-2$, corresponding respectively to the Σ and Φ modes. It follows that $\mathbf{M}$ may be written in terms of the projectors onto these modes as $\mathbf{M} = 0\cdot\Pi_\Sigma + (d-$

[28]This pseudoprojector differs qualitatively from the York mixing pseudoprojector $\Pi_{\phi S}$ in that in the York case there is an overall numerical factor in front of the symmetrized outer product, which is just an artifact of its construction from the nonnormalized modes. In principle one could make the two cases exactly analogous, either by normalizing $\Pi_{\phi S}$ or by including some sagacious overall factor in the definition of $\Pi_{\Phi\Sigma}$, but this point doesn't actually matter in any of the calculations, so I will leave it as is.

$2)\Pi_\Phi$, so that the Einstein-Hilbert kinetic matrix may in total be written

$$P_{\mathrm{cl,kin}} = p^2\Big(\Pi_\perp - (d-2)\Pi_\Phi\Big). \tag{3.122}$$

The form (3.122) for the classical kinetic terms makes manifest the claim that the Einstein-Hilbert action propagates only the TT and Φ modes. This form may also be found more directly by using the expression (3.81) for $\Pi_\perp$ in terms of the symmetric tensor structures to eliminate $\mathbf{G}$ and C from the expression (3.39) for $P_{\mathrm{cl,kin}}$ in favor of $\Pi_\perp$ and comparing the result to eq. (3.121).

For the gauge-fixing terms we have no similarly pretty argument available, and so must perform the direct calculation. Doing so yields

$$P_{\mathrm{gf}} = \frac{p^2}{\alpha}\left\{\frac{1}{2}\Pi_V + \frac{(d-1)(1+\beta)^2}{d^2}\Pi_\Phi + \frac{(d-1-\beta)^2}{d^2}\Pi_\Sigma - \frac{(d-1-\beta)(1+\beta)\sqrt{d-1}}{d^2}\Pi_{\Phi\Sigma}\right\}. \tag{3.123}$$

3.4.5 The graviton propagator in the physical basis

To find the graviton propagator in the physical basis one may either use the relations (3.119) and (3.120) between the York and physical basis projectors to reexpress the scalar sector of the graviton propagator (3.96) in the physical basis, or directly invert the total kinetic terms $P_{\mathrm{cl,kin}} + P_{\mathrm{gf}}$ in the form given by eqs. (3.122) and (3.123). Either way one obtains the result

$$\Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d^2\alpha(d-2) - (d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}. \tag{3.124}$$

N.B. the contribution of Π_Φ to the graviton propagator is gauge-independent, as one would expect from the fact that it is the physical scalar mode propagated by the Einstein-Hilbert kinetic term.

3.5 GAUGE CHOICES

Let me first collect the most important of the above results in one place. Any vector field A may be decomposed into a transverse vector $A_\perp$ and a scalar S via

$$A_\mu = A_{\perp\mu} + \mathrm{i} p_\mu S. \tag{3.125}$$

The projectors onto these parts are given by

$$\Pi_\perp^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad \Pi_\mathrm{S}^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}, \tag{3.126}$$

and in terms of these projectors the ghost propagator is given by

$$S = -\frac{1}{p^2}\left(\Pi_\perp + \frac{d}{2(d-1-\beta)}\Pi_\mathrm{S}\right). \tag{3.127}$$

Similarly, any rank-two tensor $\boldsymbol{h}$ may be written in terms of a transverse-traceless tensor $h_\perp$, a transverse vector V, a physical scalar Φ (in the sense that this scalar is the one propagated by the Einstein-Hilbert action), and a pure-gauge scalar Σ via

$$h_{\mu\nu} = h_{\perp\mu\nu} + \mathrm{i}(p_\mu V_\nu + p_\nu V_\mu) + \frac{1}{d}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Phi + \frac{1}{d}\frac{p_\mu p_\nu}{p^2}\Sigma. \tag{3.128}$$

In terms of the symmetric tensor structures

$$\begin{aligned} &G_{\mu\nu\rho\sigma} = \tfrac{1}{2}\big(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\big), \quad \mathrm{tr}_{\mu\nu\rho\sigma} = \frac{1}{d}g_{\mu\nu}g_{\rho\sigma}, \quad A_{\mu\nu\rho\sigma} = \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}, \\ &B_{\mu\nu\rho\sigma} = \frac{1}{2p^2}\big(g_{\mu\nu}p_\rho p_\sigma + \ldots\big), \quad C_{\mu\nu\rho\sigma} = \frac{1}{4p^2}\big(g_{\mu\rho}p_\nu p_\sigma + \ldots\big), \end{aligned} \tag{3.129}$$

in which the dots stand for "all other terms necessary for the expression to be symmetric", the projectors onto these modes are given by

$$\begin{aligned}\Pi_\perp &= \mathbf{G} - \frac{d}{d-1}\mathrm{tr} + \frac{d-2}{d-1}A + \frac{2}{d-1}B - 2C, &\quad \Pi_V &= 2(C-A),\\ \Pi_\Phi &= \frac{d}{d-1}\mathrm{tr} + \frac{1}{d-1}A - \frac{2}{d-1}B, & \Pi_\Sigma &= A,\end{aligned} \tag{3.130}$$

along with the normalized scalar mixing pseudoprojector

$$\Pi_{\Phi\Sigma} = -\frac{2}{\sqrt{d-1}}A + \frac{2}{\sqrt{d-1}}B, \tag{3.131}$$

and in terms of these projectors the graviton propagator is given by

$$\Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d^2\alpha(d-2)-(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}. \tag{3.132}$$

The parameters α and β describe the gauge-fixing: β determines the classical gauge being imposed, given by the gauge-fixing function

$$F_\mu = \partial h_\mu - \frac{1+\beta}{d}\partial_\mu h, \tag{3.133}$$

while α determines how strictly the gauge is imposed, since it determines the width of the Gaussian $\mathrm{e}^{\mathrm{i}S_{\mathrm{gf}}}$. In this section I will discuss three families of choices for these parameters: *harmonic gauge*, in which $\beta = (d/2) - 1$; *diagonal gauge*, in which $\beta = -1$; and the *Landau limit*, in which $\alpha \to 0$.

3.5.1 The gauge condition at the classical level

To begin let's consider the gauge condition $F_\mu = 0$ at the classical level. In momentum space the gauge condition is

$$p^\alpha h_{\alpha\mu} - \frac{1+\beta}{d}p_\mu h^\alpha{}_\alpha = 0. \tag{3.134}$$

Using the decomposition (3.128) in the above we find that the TT mode vanishes entirely, leaving

$$0 = \mathrm{i}p^2 V_\mu + \frac{d-1-\beta}{d^2} p_\mu \Sigma - \frac{(d-1)(1+\beta)}{d^2} p_\mu \Phi. \tag{3.135}$$

Since V is subject to the constraint $p \cdot V = 0$ it has $d-1$ independent components, so its components plus the gauge scalar Σ constitute the d gauge degrees of freedom. The d equations (3.135) are therefore precisely enough to constrain the gauge components V_μ and Σ, as they must be in order to properly fix the gauge.

In harmonic gauge the gauge condition does not simplify a great deal in this basis: setting $\beta = (d/2) - 1$ yields

$$0 = \mathrm{i}p^2 V_\mu + \frac{1}{2d} p_\mu \Sigma - \frac{d-1}{2d} p_\mu \Phi. \tag{3.136}$$

As we will see in sec. 3.5.3 the popularity of harmonic gauge instead comes from the simple form the graviton propagator takes in that gauge. By contrast in diagonal gauge $\beta = -1$ the physical scalar is eliminated entirely from the gauge condition, yielding

$$0 = \mathrm{i}p^2 V_\mu + \frac{1}{d} p_\mu \Sigma. \tag{3.137}$$

In fact in this gauge V and Σ are constrained to actually vanish, which may be seen as followed. Taking the momentum p as given, let's choose our spatial axes so that $(p_\mu) = (p_0, 0, \ldots, 0, p_z)$ (calling the last spatial axis the z-axis). With i referring to any spatial axis other than z the i^{th} component of the gauge condition then implies that $V_i = 0$, while the condition $p \cdot V = 0$ implies $V_z = p_0 V_0 / p_z$. The $\mu = 0$ and $\mu = z$ components of the gauge equation then yield

$$\Sigma = -\frac{\mathrm{i}d p^2 V_0}{p_0}, \quad \Sigma = -\frac{\mathrm{i}d p^2 V_z}{p_z} = -\frac{\mathrm{i}d p^2 p_0 V_0}{p_z^2}, \tag{3.138}$$

which can only be simultaneously satisfied for arbitrary p if $V_0 = 0$, immediately implying that $V_z = 0$ and $\Sigma = 0$ as well.

That the gauge condition with $\beta = -1$ does not mix the physical and gauge scalars also manifests in the action in that with this value for β the gauge-fixing action simplifies dramatically to

$$P_{\text{gf}} = \frac{p^2}{\alpha}\left(\frac{1}{2}\Pi_V + \Pi_\Sigma\right), \tag{3.139}$$

and thus the full gauge-fixed graviton kinetic matrix is

$$P_{\text{kin}} = p^2\left(\Pi_\perp - (d-2)\Pi_\Phi + \frac{1}{\alpha}\Pi_\Sigma + \frac{1}{2\alpha}\Pi_V\right). \tag{3.140}$$

So with $\beta = -1$ the graviton kinetic matrix is diagonalized (hence the name for the gauge), with no mixing between the physical and gauge scalars.

3.5.2 The Landau limit

The Landau limit $\alpha \to 0$ is the strictest possible imposition of the gauge condition in the path integral, since it is the limit in which the gauge-fixing Gaussian $\mathrm{e}^{\mathrm{i}S_{\text{gf}}}$ is sent back to the Dirac delta function it was introduced to eliminate.

This limit does *not* constitute an actual choice of gauge in the classical sense, since it says nothing about the gauge-fixing function itself. In our generalized harmonic gauge this limit therefore leaves all the β-dependence in the propagators. It follows that the ghost propagator doesn't simplify at all in this limit:

$$S = -\frac{1}{p^2}\left(\Pi_\perp + \frac{d}{2(d-1-\beta)}\Pi_S\right). \tag{3.141}$$

The graviton propagator in this limit does undergo the slight simplification that the unphysical vector mode is eliminated:

$$\Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp - \frac{1}{d-2}\Pi_\Phi - \frac{(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}. \tag{3.142}$$

However the gauge scalar Σ, and its mixing with the physical scalar Φ, still appears unless a judicious choice of β is made.

3.5.3 Harmonic gauge

Linearized harmonic gauge, which I will just call "harmonic gauge", is given by the condition that[29]

$$\partial h_\mu - \tfrac{1}{2}\partial_\mu h = 0, \quad \text{i.e.} \quad \beta = \frac{d}{2} - 1. \tag{3.143}$$

This gauge choice is common in the literature (see e.g. [68, 69]) because it makes the ghost and graviton propagators particularly simple. The ghost propagator is α-independent, so it is entirely determined by the choice of β:

$$S = -\frac{1}{p^2}\Big(\Pi_\perp + \Pi_\mathrm{S}\Big) \implies S_{\mu\nu} = -\frac{\eta_{\mu\nu}}{p^2}, \tag{3.144}$$

which is just the standard vector propagator in Feynman gauge. With α arbitrary the graviton propagator is most conveniently expressed in terms of the symmetric tensor structures:

$$\Delta = \frac{1}{p^2}\bigg(\mathbf{G} - \frac{d}{d-2}\,\mathrm{tr} + (4\alpha - 2)C\bigg), \tag{3.145}$$

from which we can see that the graviton propagator can be brought into its simplest form by setting $\alpha = 1/2$,

$$\Delta = \frac{1}{p^2}\bigg(\mathbf{G} - \frac{d}{d-2}\,\mathrm{tr}\bigg). \tag{3.146}$$

In $d = 4$ dimensions simplifies to the common form [68, 69]

$$\Delta_{\mu\nu\rho\sigma} = \frac{1}{2p^2}\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big). \tag{3.147}$$

By contrast taking the Landau limit $\alpha \to 0$ leaves the momentum-dependent tensor structure C,

$$\Delta = \frac{1}{p^2}\bigg(\mathbf{G} - \frac{d}{d-2}\,\mathrm{tr} - 2C\bigg). \tag{3.148}$$

[29] As a sidenote on conventions, I define harmonic gauge with a coefficient of $1/2$ in all dimensions, as opposed to the also-common $2/d$ [64], because this is the linearization of the nonperturbative harmonic gauge condition $\Gamma^\mu_{\alpha\beta} g^{\alpha\beta} = 0$.

In terms of the physical decomposition of the graviton this gauge is less simple: setting $\beta = (d/2) - 1$ but leaving α and the dimension d arbitrary gives

$$\Delta = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{4\alpha(d-2)-d+1}{d-2}\Pi_\Sigma - \frac{\sqrt{d-1}}{d-2}\Pi_{\Phi\Sigma}\right\}. \tag{3.149}$$

Thus in this gauge there is no strength of gauge-fixing which eliminates the mixing between the physical and gauge scalars in the propagator, since α only affects the vector and pure Σ contribution. Setting $\alpha = 1/2$ in this form gives

$$\Delta = \frac{1}{p^2}\left\{\Pi_\perp + \Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d-3}{d-2}\Pi_\Sigma - \frac{\sqrt{d-1}}{d-2}\Pi_{\Phi\Sigma}\right\}, \tag{3.150}$$

or in $d = 4$

$$\Delta = \frac{1}{p^2}\left\{\Pi_\perp + \Pi_V + \frac{1}{2}\left(\Pi_\Sigma - \Pi_\Phi - \sqrt{3}\Pi_{\Phi\Sigma}\right)\right\}. \tag{3.151}$$

Taking the Landau limit instead gives

$$\Delta = \frac{1}{p^2}\left\{\Pi_\perp - \frac{1}{d-2}\Pi_\Phi - \frac{d-1}{d-2}\Pi_\Sigma - \frac{\sqrt{d-1}}{d-2}\Pi_{\Phi\Sigma}\right\}. \tag{3.152}$$

3.5.4 Diagonal gauge

In the diagonal gauge $\beta = -1$ the gauge condition reduces to

$$\partial^\alpha h_{\alpha\mu} = 0, \tag{3.153}$$

or in momentum space

$$p^\alpha h_{\alpha\mu} = 0. \tag{3.154}$$

As foreshadowed previously this is the gauge in which the gauge-fixed graviton kinetic matrix is diagonalized,

$$P_{\text{kin}} = p^2\left(\Pi_\perp - (d-2)\Pi_\Phi + \frac{1}{\alpha}\Pi_\Sigma + \frac{1}{2\alpha}\Pi_V\right), \tag{3.155}$$

and hence the graviton propagator is as well:

$$\Delta = \frac{1}{p^2}\left(\Pi_\perp - \frac{1}{d-2}\Pi_\Phi + \alpha\Pi_\Sigma + 2\alpha\Pi_V\right). \tag{3.156}$$

In particular taking the Landau limit in this gauge yields the naive inverse of the original Einstein-Hilbert kinetic matrix (3.122),

$$\Delta = \frac{1}{p^2}\left(\Pi_\perp - \frac{1}{d-2}\Pi_\Phi\right), \tag{3.157}$$

as we would expect, since in this gauge the vector and gauge scalar vanish at the classical level and the Landau limit corresponds to the strictest possible imposition of this gauge constraint.

4 Feynman rules

In this section I provide the Feynman rules for a massive real scalar minimally coupled to Einstein-Hilbert gravity, which is the theory in which I will compute the correlators of sec. 5. The action for this theory is

$$S = S_{\text{EH}} + S_{\text{gf}} + S_{\text{gh}} + S_\phi, \tag{4.1}$$

where the Einstein-Hilbert action is

$$S_{\text{EH}} = -\frac{2}{\kappa^2}\int \mathrm{d}^d x\,\sqrt{-g}R; \tag{4.2}$$

the gauge-fixing action and ghost actions are obtained from the gauge-fixing function

$$F_\mu = \partial h_\mu - \frac{1+\beta}{d}\partial_\mu h \tag{4.3}$$

(obtained explicitly in sec. 3.1, and restated below where relevant); and the scalar action is just its kinetic term,

$$S_\phi = \frac{1}{2}\int \mathrm{d}^d x\,\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right). \tag{4.4}$$

4.1 Expansion pieces

If one wishes to obtain the propagators and vertices below by hand one first needs the expansions of the inverse metric, the determinant, and the scalar curvature.

4.1.1 The inverse metric

To obtain the perturbative expansion of the inverse metric let us define its expansion coefficients as

$$g^{\mu\nu} = \sum_n \kappa^n \tilde{g}_n^{\mu\nu}. \tag{4.5}$$

The $\tilde{g}_n^{\mu\nu}$'s can be obtained order-by-order by imposing the definition $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$ with $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. At zeroth order we immediately find $\tilde{g}_0^{\mu\nu} = \eta^{\mu\nu}$. At first order we then have

$$\delta^\mu_\nu = \left(\eta^{\mu\rho} + \kappa \tilde{g}_1^{\mu\rho}\right)\left(\eta_{\rho\nu} + \kappa h_{\rho\nu}\right) + \mathcal{O}(\kappa^2) = \delta^\mu_\nu + \kappa\left(\tilde{g}_1{}^\mu{}_\nu + h^\mu{}_\nu\right) + \mathcal{O}(\kappa^2), \tag{4.6}$$

from which we find $\tilde{g}_1^{\mu\nu} = -h^{\mu\nu}$. At second order we then find

$$\delta^\mu_\nu = \left(\eta^{\mu\rho} - \kappa h^{\mu\rho} + \kappa^2 \tilde{g}_2^{\mu\rho}\right)\left(\eta_{\rho\nu} + \kappa h_{\rho\nu}\right) + \mathcal{O}(\kappa^3) = \delta^\mu_\nu + \kappa^2\left(\tilde{g}_2{}^\mu{}_\nu - h^{\mu\rho} h_{\rho\nu}\right) + \mathcal{O}(\kappa^3), \tag{4.7}$$

which yields $\tilde{g}_2^{\mu\nu} = h^{\mu\rho} h_\rho{}^\nu$, and proceeding in this way we find

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha{}^\nu - \kappa^3 h^{\mu\alpha} h^{\nu\beta} h_{\alpha\beta} + \kappa^4 h^{\mu\alpha} h^{\nu\beta} h_\alpha{}^\gamma h_{\beta\gamma} + \mathcal{O}(\kappa^5). \tag{4.8}$$

4.1.2 The volume factor

This proceeds from the matrix identities

$$\ln \circ \det A = \operatorname{tr} \circ \ln A, \quad \det(AB) = \det A \det B, \tag{4.9}$$

and the Taylor expansions

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \mathcal{O}(x^3), \quad \sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \mathcal{O}(x^3). \tag{4.10}$$

Ignoring the square root to start and using $\det\boldsymbol{\eta} = -1$ we have

$$-\det\boldsymbol{g} = -\det\left(\boldsymbol{\eta} + \kappa\boldsymbol{h}\right) = \det\left(1 + \kappa\boldsymbol{\eta}^{-1}\boldsymbol{h}\right) = \exp\circ\operatorname{tr}\circ\ln\left(1 + \kappa\boldsymbol{\eta}^{-1}\boldsymbol{h}\right). \tag{4.11}$$

To be clear on the notation here, by $\boldsymbol{\eta}^{-1}\boldsymbol{h}$ I mean the matrix with elements $\eta^{\mu\alpha}h_{\alpha\nu}$, and by 1 I mean the identity matrix. Now let's expand this expression from the inside out. Expanding the logarithm and taking the trace yields

$$\operatorname{tr}\circ\ln\left(1 + \kappa\boldsymbol{\eta}^{-1}\boldsymbol{h}\right) = \operatorname{tr}\left\{\kappa\boldsymbol{\eta}^{-1}\boldsymbol{h} - \tfrac{1}{2}\kappa^2\left(\boldsymbol{\eta}^{-1}\boldsymbol{h}\right)^2 + \mathcal{O}(\kappa^3)\right\} = \kappa\operatorname{tr}\boldsymbol{h} - \tfrac{1}{2}\kappa^2\operatorname{tr}\left(\boldsymbol{h}^2\right) + \mathcal{O}(\kappa^3). \tag{4.12}$$

Then expanding the exponentiation of the above yields

$$-\det\boldsymbol{g} = 1 + \kappa\operatorname{tr}\boldsymbol{h} + \tfrac{1}{2}\kappa^2\left(\left(\operatorname{tr}\boldsymbol{h}\right)^2 - \operatorname{tr}\left(\boldsymbol{h}^2\right)\right) + \mathcal{O}(\kappa^3). \tag{4.13}$$

Finally taking the square root and expanding we obtain

$$\sqrt{-\det\boldsymbol{g}} = 1 + \tfrac{1}{2}\kappa\operatorname{tr}\boldsymbol{h} + \kappa^2\left(\tfrac{1}{8}\left(\operatorname{tr}\boldsymbol{h}\right)^2 - \tfrac{1}{4}\operatorname{tr}(\boldsymbol{h})^2\right) + \mathcal{O}(\kappa^3), \tag{4.14}$$

or in components and using the more standard notation $h = \operatorname{tr}\boldsymbol{h}$ and $\det\boldsymbol{g} = g$,

$$\sqrt{-g} = 1 + \tfrac{1}{2}\kappa h + \kappa^2\left(\tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(\kappa^3). \tag{4.15}$$

One may proceed in the manner above to arbitrarily high order. Doing so introduces no conceptual wrinkles but quickly becomes algebraically overwhelming, meaning that the details are best left to a computer program, e.g. xAct. The result is

$$\sqrt{-g} = 1 + \tfrac{1}{2}\kappa h + \kappa^2\Big(\tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\Big) + \kappa^3\Big(\tfrac{1}{6}h^{\mu\nu}h_\mu{}^\alpha h_{\alpha\nu} - \tfrac{1}{8}hh_{\mu\nu}h^{\mu\nu} + \tfrac{1}{48}h^3\Big)$$
$$+ \kappa^4\Big(-\tfrac{1}{8}h^{\mu\nu}h_\mu{}^\alpha h_\nu{}^\beta h_{\alpha\beta} + \tfrac{1}{12}hh^{\mu\nu}h_\mu{}^\alpha h_{\alpha\nu} + \tfrac{1}{32}(h_{\mu\nu}h^{\mu\nu})^2 - \tfrac{1}{32}h^2h_{\mu\nu}h^{\mu\nu} + \tfrac{1}{384}h^4\Big) + \mathcal{O}(\kappa^5). \tag{4.16}$$

For later reference let's denote these expansion terms as

$$\sqrt{-g} = \sum_n \kappa^n \gamma_n. \tag{4.17}$$

4.1.3 The scalar curvature

The expansion of the scalar curvature follows from its definition in terms of the Christoffel symbols:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}\Big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\Big), \quad R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}, \quad R = g^{\mu\nu}R^\alpha{}_{\mu\alpha\nu}. \tag{4.18}$$

Beyond first order the expansion gets quite lengthy, so to aid in readability I will organize the presentation of the result as follows. Define its expansion coefficients as

$$R = \sum_n \kappa^n R_n. \tag{4.19}$$

The zeroth and first order terms can be found by hand as follows. To first order the Christoffel symbols are

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\kappa\Big(\partial_\mu h^\alpha{}_\nu + \partial_\nu h^\alpha{}_\mu - \partial^\alpha h_{\mu\nu}\Big) + \mathcal{O}(\kappa^2), \tag{4.20}$$

N.B. there are no zeroth order terms since every term in the definition (4.18) of the Christoffel symbols involves a partial derivative of the metric, and the partial derivatives of our flat background metric vanish. Following through the definitions (4.18) it immediately follows that the

Riemann tensor, and hence the scalar curvature, vanish at zeroth order:

$$R^{\alpha}{}_{\beta\mu\nu} = 0 + \mathcal{O}(\kappa) \implies R_0 = 0. \tag{4.21}$$

It also follows that to first order in $R = g^{\mu\nu} R^{\alpha}{}_{\mu\alpha\nu}$ we need only keep $g^{\mu\nu} = \eta^{\mu\nu} + \mathcal{O}(\kappa)$ and the $\partial\Gamma$ terms in the Riemann tensor, since the Γ^2 terms start at $\mathcal{O}(\kappa^2)$. Working through the perturbations yields for the Riemann tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \tfrac{1}{2}\kappa\Big(\partial_\mu\partial_\beta h^{\alpha}{}_{\nu} - \partial_\nu\partial_\beta h^{\alpha}{}_{\mu} - \partial_\mu\partial^\alpha h_{\nu\beta} + \partial_\nu\partial^\alpha h_{\mu\beta}\Big) + \mathcal{O}(\kappa^2), \tag{4.22}$$

from which we find the Ricci tensor and hence the scalar curvature

$$\begin{aligned} R_{\mu\nu} &= \tfrac{1}{2}\kappa\Big(\partial_\mu\partial_\alpha h^{\alpha}{}_{\nu} - \partial_\nu\partial_\mu h - \partial^2 h_{\mu\nu} + \partial_\nu\partial_\alpha h^{\alpha}{}_{\mu}\Big) + \mathcal{O}(\kappa^2) \\ \implies R &= \eta^{\mu\nu} R_{\mu\nu} = \kappa\big(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h_{\mu\nu}\big) + \mathcal{O}(\kappa^2), \end{aligned} \tag{4.23}$$

i.e. $R_1 = \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h_{\mu\nu}$.

At higher orders the algebra quickly becomes frightening, and so for them I defer to MATHEMATICA. The second-order term is

$$\begin{aligned} R_2 = h^{\mu\nu}\partial_\mu\partial_\nu h - \tfrac{1}{4}\partial_\mu h\partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\alpha h_\nu{}^\alpha + \partial_\mu h\partial_\nu h^{\mu\nu} - 2h^{\mu\nu}\partial_\nu\partial_\alpha h_\mu{}^\alpha \\ + h^{\mu\nu}\partial^2 h_{\mu\nu} - \tfrac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\mu h^{\nu\alpha} + \tfrac{3}{4}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu}; \end{aligned} \tag{4.24}$$

at third order we have

$$\begin{aligned} R_3 = -\tfrac{3}{4}h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + \tfrac{1}{4}h^{\mu\nu}\partial_\mu h\partial_\nu h - h^{\mu\nu}\partial_\mu h\partial_\alpha h_\nu{}^\alpha - h^{\mu\nu}\partial_\mu h_\nu{}^\alpha\partial_\alpha h - h^{\mu\nu}h_\mu{}^\alpha\partial_\nu\partial_\alpha h \\ + \tfrac{1}{2}h^{\mu\nu}\partial_\alpha h_{\mu\nu}\partial^\alpha h + h^{\mu\nu}\partial_\alpha h_\mu{}^\alpha\partial_\beta h_\nu{}^\beta + 2h^{\mu\nu}\partial_\nu h_\mu{}^\alpha\partial_\beta h_\alpha{}^\beta - h^{\mu\nu}\partial_\alpha h_{\mu\nu}\partial_\beta h^{\alpha\beta} + h^{\mu\nu}h^{\alpha\beta}\partial_\mu\partial_\alpha h_{\nu\beta} \\ - h^{\mu\nu}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + 2h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha\partial_\beta h_\nu{}^\beta - h^{\mu\nu}h_\mu{}^\alpha\partial^2 h_{\nu\alpha} + h^{\mu\nu}\partial_\nu h_\alpha{}^\beta\partial_\beta h_\mu{}^\alpha + \tfrac{1}{2}h^{\mu\nu}\partial_\alpha h_\mu{}^\beta\partial_\beta h_\nu{}^\alpha \\ - \tfrac{3}{2}h^{\mu\nu}\partial_\alpha h_{\mu\beta}\partial^\alpha h_\nu{}^\beta; \end{aligned} \tag{4.25}$$

and at fourth order

$$\begin{aligned}
R_4 = &-\tfrac{1}{2}h^{\mu\nu}h^{\alpha\beta}\partial_\nu h_{\beta\lambda}\partial_\alpha h_\mu{}^\lambda + \tfrac{3}{4}h^{\mu\nu}h_\mu{}^\alpha\partial_\nu h^{\beta\lambda}\partial_\alpha h_{\beta\lambda} - \tfrac{1}{4}h^{\mu\nu}h_\mu{}^\alpha\partial_\nu h\partial_\alpha h + h^{\mu\nu}h_\mu{}^\alpha h^{\beta\lambda}\partial_\nu\partial_\alpha h_{\beta\lambda}\\
&+ h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha h\partial_\beta h_\nu{}^\beta + \tfrac{3}{2}h^{\mu\nu}h^{\alpha\beta}\partial_\alpha h_\mu{}^\lambda\partial_\beta h_{\nu\lambda} - h^{\mu\nu}h^{\alpha\beta}\partial_\nu h_\mu{}^\lambda\partial_\beta h_{\alpha\lambda} + h^{\mu\nu}h^{\alpha\beta}\partial_\nu h_{\mu\alpha}\partial_\beta h\\
&- \tfrac{1}{2}h^{\mu\nu}h^{\alpha\beta}\partial_\alpha h_{\mu\nu}\partial_\beta h + h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha h_\nu{}^\beta\partial_\beta h + h^{\mu\nu}h_\mu{}^\alpha h_\nu{}^\beta\partial_\alpha\partial_\beta h - \tfrac{1}{2}h^{\mu\nu}h_\mu{}^\alpha\partial_\beta h\partial^\beta h_{\nu\alpha}\\
&- h^{\mu\nu}h_\mu{}^\alpha\partial_\beta h_\nu{}^\beta\partial_\lambda h_\alpha{}^\lambda - 2h^{\mu\nu}h^{\alpha\beta}\partial_\nu h_{\mu\alpha}\partial_\lambda h_\beta{}^\lambda + h^{\mu\nu}h^{\alpha\beta}\partial_\alpha h_{\mu\nu}\partial_\lambda h_\beta{}^\lambda - 2h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha h_\nu{}^\beta\partial_\lambda h_\beta{}^\lambda\\
&+ h^{\mu\nu}h_\mu{}^\alpha\partial_\beta h_{\nu\alpha}\partial_\lambda h^{\beta\lambda} - 2h^{\mu\nu}h_\mu{}^\alpha h^{\beta\lambda}\partial_\alpha\partial_\lambda h_{\nu\beta} + h^{\mu\nu}h_\mu{}^\alpha h^{\beta\lambda}\partial_\beta\partial_\lambda h_{\nu\alpha} - 2h^{\mu\nu}h_\mu{}^\alpha h_\nu{}^\beta\partial_\beta\partial_\lambda h_\alpha{}^\lambda\\
&+ h^{\mu\nu}h_\mu{}^\alpha h_\nu{}^\beta\partial^2 h_{\alpha\beta} + h^{\mu\nu}h^{\alpha\beta}\partial_\alpha h_\beta{}^\lambda\partial_\lambda h_{\mu\nu} - \tfrac{1}{4}h^{\mu\nu}h^{\alpha\beta}\partial_\lambda h_{\mu\nu}\partial^\lambda h_{\alpha\beta} - h^{\mu\nu}h^{\alpha\beta}\partial_\alpha h_\mu{}^\lambda\partial_\lambda h_{\nu\beta}\\
&+ \tfrac{3}{4}h^{\mu\nu}h^{\alpha\beta}\partial_\lambda h_{\mu\alpha}\partial^\lambda h_{\nu\beta} - h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha h_{\beta\lambda}\partial^\lambda h_\nu{}^\beta - \tfrac{1}{2}h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha h_{\beta\lambda}\partial^\lambda h_\nu{}^\beta + \tfrac{3}{2}h^{\mu\nu}h_\mu{}^\alpha\partial_\lambda h_{\alpha\beta}\partial^\lambda h_\nu{}^\beta.
\end{aligned} \tag{4.26}$$

4.2 The propagators

4.2.1 The scalar propagator

Since the scalar field kinetic matrix in position space is just $-\partial^2 - m^2$ the scalar field propagator is

$$\overset{p}{\overline{\quad\longrightarrow\quad}} = \mathrm{i}D(p) = \frac{\mathrm{i}}{p^2 - m^2}. \tag{4.27}$$

The vector ghost propagator was found in sec. 3.2. The kinetic terms are

$$\mathcal{L}_{\text{gh,kin}} = \bar{c}^\mu\partial^2 c_\mu + \left(1 - \frac{2(1+\beta)}{d}\right)\bar{c}^\mu\partial_\mu\partial^\nu c_\nu, \tag{4.28}$$

yielding the propagator

$$\mu \overset{p}{\dashrightarrow} \nu = \mathrm{i}S_{\mu\nu}(p) = \frac{\mathrm{i}}{p^2}\left\{\left(1 - \frac{d}{2(d-1-\beta)}\right)p_\mu p_\nu - g_{\mu\nu}\right\}. \tag{4.29}$$

In terms of the operators

$$\Pi_\perp^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad \Pi_\mathrm{S} = \frac{p^\mu p^\nu}{p^2} \tag{4.30}$$

which project an arbitrary vector onto respectively its transverse and scalar mode, the ghost propagator may be written

$$S(p) = -\frac{1}{p^2}\left(\Pi_\perp + \frac{d}{2(d-1-\beta)}\Pi_S\right). \tag{4.31}$$

The graviton's kinetic terms come from the second-order expansion of the Einstein-Hilbert action,

$$\mathcal{L}_{\text{EH,kin}} = -\partial h^\mu \partial h_\mu - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \partial h^\nu \partial_\nu h + \frac{1}{2}h\partial^2 h, \tag{4.32}$$

along with the gauge-fixing action

$$\mathcal{L}_{\text{gf}} = \frac{1}{2\alpha}\left(\partial h^\mu - \frac{1+\beta}{d}\partial^\mu h\right)\left(\partial h_\mu - \frac{1+\beta}{d}\partial_\mu h\right). \tag{4.33}$$

The resulting propagator is most conveniently given as follows. As discussed in sec. 3.4, any rank-two tensor may be decomposed into a transverse-traceless tensor $h_\perp$, a transverse vector V, a physical scalar Φ, and a pure-gauge scalar Σ. Define the symmetric tensor structures

$$\begin{aligned} G_{\mu\nu\rho\sigma} &= \tfrac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right), \quad \text{tr}_{\mu\nu\rho\sigma} = \frac{1}{d}g_{\mu\nu}g_{\rho\sigma}, \quad A_{\mu\nu\rho\sigma} = \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}, \\ B_{\mu\nu\rho\sigma} &= \frac{1}{2p^2}\left(g_{\mu\nu}p_\rho p_\sigma + \ldots\right), \quad C_{\mu\nu\rho\sigma} = \frac{1}{4p^2}\left(g_{\mu\rho}p_\nu p_\sigma + \ldots\right), \end{aligned} \tag{4.34}$$

in which the dots stand for "all other terms necessary for the expression to be symmetric". Then the projectors onto these modes are

$$\begin{aligned} \Pi_\perp &= \mathbf{G} - \frac{d}{d-1}\text{tr} + \frac{d-2}{d-1}A + \frac{2}{d-1}B - 2C, \quad && \Pi_V = 2(C-A), \\ \Pi_\Phi &= \frac{d}{d-1}\text{tr} + \frac{1}{d-1}A - \frac{2}{d-1}B, && \Pi_\Sigma = A. \end{aligned} \tag{4.35}$$

Also necessary is the scalar mixing pseudoprojector

$$\Pi_{\Phi\Sigma} = -\frac{2}{\sqrt{d-1}}A + \frac{2}{\sqrt{d-1}}B. \tag{4.36}$$

In terms of these projectors the graviton propagator is given by

$$\Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d^2\alpha(d-2)-(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}, \tag{4.37}$$

which I represent the graviton propagator diagrammatically as

$$\mu\nu \; \underset{\sim\!\sim\!\sim\!\sim\!\sim}{\xrightarrow{\;p\;}} \; \rho\sigma \;\; = \mathrm{i}\Delta_{\mu\nu\rho\sigma}(p). \tag{4.38}$$

4.2.2 The Fourier transform convention and derivative interactions

In each case above the given propagator relates to the corresponding free position-space two-point function via a Fourier transform in the usual way, e.g. for the scalar

$$\langle\phi(x)\phi(y)\rangle_0 = \mathrm{i}D(x,y) = \mathrm{i}\int\frac{\mathrm{d}^d p}{(2\pi)^d}D(p)\mathrm{e}^{\mathrm{i}p(x-y)}. \tag{4.39}$$

Now, observe that in eq. (4.39) we are free to choose the sign of the momentum p in the exponent. This choice relates to the sign of the momentum in the Feynman rule for a vertex as follows. Consider for example the tree-level contribution to the two-point function $\langle\phi(x)\partial_\mu\phi(y)\rangle$. Proceeding as above and using the Fourier expansion of the propagator yields

$$\langle\phi(x)\partial_\mu\phi(y)\rangle = \mathrm{i}\frac{\partial}{\partial y^\mu}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2-m^2} = \int\frac{\mathrm{d}^d k}{(2\pi)^d}(-\mathrm{i}k_\mu)\frac{\mathrm{i}}{k^2-m^2}\mathrm{e}^{\mathrm{i}k(x-y)}. \tag{4.40}$$

Now suppose we instead wished to arrive at this result from a diagrammatic route. The single contributing diagram (in a free theory) is a line carrying the momentum k from x and y, corresponding to the momentum-space propagator $\mathrm{i}/(k^2-m^2)$. The field at x has the trivial external vertex factor of 1, while the derivative at ϕ yields a momentum factor whose sign convention must be chosen. I choose the convention that *all outgoing momenta are positive*, meaning that if the momentum k points from x to y then the vertex factor at y is $-\mathrm{i}k_\mu$:

$$= (-\mathrm{i}k_\mu)\frac{\mathrm{i}}{k^2 - m^2}. \tag{4.41}$$

Comparing to the previous result we can see that my chosen convention does indeed correspond to the exponential sign choice $\mathrm{e}^{\mathrm{i}k(x-y)}$, whereas the opposite choice $\mathrm{e}^{-\mathrm{i}k(x-y)}$ corresponds to writing the *incoming* momenta as positive.

4.3 Scalar field external insertions

Given a scalar field ϕ its invariantized form is given by eq. (2.77),

$$\hat{\phi} = \phi - \kappa \mathsf{X}_1^\alpha \partial_\alpha \phi + \kappa^2 \left(\tfrac{1}{2} \mathsf{X}_1^\alpha \mathsf{X}_1^\beta \partial_\alpha \partial_\beta \phi + \mathsf{X}_1^\alpha \partial_\alpha \mathsf{X}_1^\beta \partial_\beta \phi - \mathsf{X}_2^\alpha \partial_\alpha \phi \right) + \mathcal{O}(\kappa^3), \tag{4.42}$$

where X_1 and X_2 are given by eqs. (2.20) and (2.23),

$$\mathsf{X}_1^\mu(x) = \int \mathrm{d}^d x'\, G(x,x') J_1^\mu(x'), \quad \mathsf{X}_2^\mu(x) = \int \mathrm{d}^d x'\, G(x,x') \Big(J_2^\mu(x') + K_1 \mathsf{X}_1^\mu(x') \Big), \tag{4.43}$$

where $G(x,x')$ is a Green's function of the D'alembertian $\Box$ and the J's and K's are given in turn by eqs. (2.17) and (2.22),

$$J_1^\mu = \partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h, \quad K_1 = h^{\alpha\beta}\partial_\alpha\partial_\beta + J_1^\alpha \partial_\alpha, \quad J_2^\mu = \frac{1}{2}\Big(h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + h^{\alpha\mu}\partial_\alpha h \Big) - \partial_\alpha \Big(h^{\alpha\beta} h_\beta{}^\mu \Big). \tag{4.44}$$

When calculating a correlator which includes the invariantized scalar field $\hat{\phi}(X)$ it therefore follows that at X we will have not only the "standard" external vertex factor (which for a scalar field is trivial) but also an infinite series of external vertices arising from the invariantization. I'll call these latter *coordinate corrections*. For a scalar field we see from eq. (4.42) that at $\mathcal{O}(\kappa^0)$ we have only the standard trivial factor, while at $\mathcal{O}(\kappa)$ and up we have only the coordinate corrections. However for other observables, e.g. the volume factor (sec. 4.4) and the scalar curvature (sec. 4.5), we will see that it is perfectly possible to have standard insertions and coordinate corrections at the same order.

Before proceeding we will find it useful to rewrite the above expressions for the X's more explicitly. Our first step is to repackage the information in J_1 and J_2 as constant tensors acting on the single objects $\partial_\alpha h_{\mu\nu}$ and $h_{\mu\nu}\partial_\alpha h_{\rho\sigma}$ respectively. For J_1 we have

$$J_1^\mu = \mathcal{J}_1^{\mu\alpha\rho\sigma}\partial_\alpha h_{\rho\sigma}, \quad \mathcal{J}_1^{\mu\alpha\rho\sigma} = \eta^{\mu\rho}\eta^{\alpha\sigma} - \tfrac{1}{2}\eta^{\mu\alpha}\eta^{\rho\sigma}, \tag{4.45}$$

while for J_2

$$J_2^\mu = \mathcal{J}_2^{\mu\alpha\rho\sigma\lambda\tau}h_{\rho\sigma}\partial_\alpha h_{\lambda\tau}, \quad \mathcal{J}_2^{\mu\alpha\rho\sigma\lambda\tau} = \tfrac{1}{2}\eta^{\mu\alpha}\eta^{\rho\lambda}\eta^{\sigma\tau} + \tfrac{1}{2}\eta^{\mu\sigma}\eta^{\alpha\rho}\eta^{\lambda\tau} - \eta^{\mu\sigma}\eta^{\alpha\lambda}\eta^{\rho\tau} - \eta^{\mu\tau}\eta^{\alpha\rho}\eta^{\sigma\lambda}. \tag{4.46}$$

Writing the Green function as

$$G(x,x') = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\left(-\frac{1}{p^2}\right)\mathrm{e}^{\mathrm{i}p(x-x')}, \tag{4.47}$$

it follows that we can write X_1 as

$$\mathsf{X}_1^\mu(x) = \mathcal{J}_1^{\mu\alpha\rho\sigma}\int_{x',p}\left(-\frac{1}{p^2}\right)\partial_\alpha h_{\rho\sigma}(x')\mathrm{e}^{\mathrm{i}p(x-x')}, \tag{4.48}$$

introducing the shorthand $\int_x = \int \mathrm{d}^d x$ and $\int_p = \int \mathrm{d}^d p/(2\pi)^d$ respectively.

In principle similar logic could be applied to X_2. However this is much messier, and X_2 will only contribute to a one-scalar two-graviton external vertex, of which I will not make explicit use in this thesis, so I omit it here.

To find the vertex factor corresponding to the $\mathcal{O}(\kappa)$ term in eq. (4.42), $-\kappa\mathsf{X}_1^\alpha\partial_\alpha\phi$, we can use the above repackaging (4.48) of X_1 as follows. Consider the three-point function

$$\left\langle\left(-\kappa\mathsf{X}_1^\alpha\partial_\alpha\phi(x)\right)\phi(y)h_{\rho\sigma}(z)\right\rangle = \kappa\mathcal{J}_1^{\alpha\beta\mu\nu}\int_{x',p}\frac{1}{p^2}\mathrm{e}^{\mathrm{i}p(x-x')}\left\langle\partial_\alpha\phi(x)\partial_\beta h_{\mu\nu}(x')\phi(y)h_{\rho\sigma}(z)\right\rangle. \tag{4.49}$$

There is only one Wick contraction, so

$$\left\langle\left(-\kappa \mathsf{X}_1^\alpha \partial_\alpha \phi(x)\right)\phi(y) h_{\rho\sigma}(z)\right\rangle = \kappa \mathcal{J}_1^{\alpha\beta\mu\nu}\int_{x',p}\frac{1}{p^2}\mathrm{e}^{\mathrm{i}p(x-x')}\left\langle\partial_\alpha\phi(x)\phi(y)\right\rangle\left\langle\partial_\beta h_{\mu\nu}(x')h_{\rho\sigma}(z)\right\rangle. \tag{4.50}$$

These two-point functions can be written in terms of the momentum-space propagators as

$$\begin{aligned}\left\langle\partial_\alpha\phi(x)\phi(y)\right\rangle &= \frac{\partial}{\partial x^\alpha}\int_k\left[\mathrm{i}D(k)\right]\mathrm{e}^{\mathrm{i}k(x-y)} = \int_k\left[\mathrm{i}D(k)\right]\left[\mathrm{i}k_\alpha\right]\mathrm{e}^{\mathrm{i}k(x-y)},\\ \left\langle\partial_\beta h_{\mu\nu}(x')h_{\rho\sigma}(z)\right\rangle &= \frac{\partial}{\partial x'^\beta}\int_{p'}\left[\mathrm{i}\Delta_{\mu\nu\rho\sigma}(p')\right]\mathrm{e}^{\mathrm{i}p'(x'-z)} = \int_{p'}\left[\mathrm{i}\Delta_{\mu\nu\rho\sigma}(p')\right]\left[\mathrm{i}p'_\beta\right]\mathrm{e}^{\mathrm{i}p'(x'-z)},\end{aligned} \tag{4.51}$$

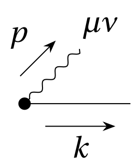
so the three-point function under consideration becomes

$$\left\langle\left(-\kappa \mathsf{X}_1^\alpha \partial_\alpha \phi(x)\right)\phi(y) h_{\rho\sigma}(z)\right\rangle = -\kappa \mathcal{J}_1^{\alpha\beta\mu\nu}\int_{x',p,k,p'}\frac{1}{p^2}\mathrm{e}^{\mathrm{i}p(x-x')}\mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{e}^{\mathrm{i}p'(x'-z)}k_\alpha p'_\beta\left[\mathrm{i}D(k)\right]\left[\mathrm{i}\Delta_{\mu\nu\rho\sigma}(p')\right]. \tag{4.52}$$

The x' integral then produces a Dirac delta function which sets $p = p'$,

$$\left\langle\left(-\kappa \mathsf{X}_1^\alpha \partial_\alpha \phi(x)\right)\phi(y) h_{\rho\sigma}(z)\right\rangle = -\kappa \mathcal{J}_1^{\alpha\beta\mu\nu}\int_{p,k}\frac{1}{p^2}k_\alpha p_\beta\left[\mathrm{i}D(k)\right]\left[\mathrm{i}\Delta_{\mu\nu\rho\sigma}(p)\right]\mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{e}^{\mathrm{i}p(x-z)}, \tag{4.53}$$

from which we can read off the one-scalar one-graviton external vertex



$$= E_{\phi h}^{\mu\nu}(k,p) = -\kappa\frac{1}{p^2}\mathcal{J}_1^{\alpha\beta\mu\nu}k_\alpha p_\beta = -\kappa\frac{1}{p^2}\left(k^\mu p^\nu - \tfrac{1}{2}(k\cdot p)\eta^{\mu\nu}\right). \tag{4.54}$$

4.4 Volume external insertions

From eq. (2.91) we have the expansion of the invariantized volume factor,

$$\begin{aligned}\sqrt{-\det\mathcal{G}} = 1 &+ \kappa\left(\tfrac{1}{2}h - \partial_\mu \mathsf{X}_1^\mu\right) + \kappa^2\Big(\tfrac{1}{2}\partial_\mu \mathsf{X}_1^\mu\partial_\nu \mathsf{X}_1^\nu + \mathsf{X}_1^\mu\partial_\mu\partial_\nu \mathsf{X}_1^\nu + \tfrac{1}{2}\partial_\mu \mathsf{X}_{1\nu}\partial^\nu \mathsf{X}_1^\mu - \partial_\mu \mathsf{X}_2^\mu\\ &- \tfrac{1}{2}\mathsf{X}_1^\mu\partial_\mu h - \tfrac{1}{2}h\partial_\mu \mathsf{X}_1^\mu + \tfrac{1}{8}h^2 - \tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}\Big) + \mathcal{O}(\kappa^3).\end{aligned} \tag{4.55}$$

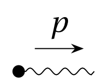
Just as for the invariantized scalar field this expansion yields both standard external vertices and coordinate corrections, but unlike the invariantized scalar field there are both types at all orders.

4.4.1 One-point: standard

The standard term at $\mathcal{O}(\kappa)$, $\kappa h/2$, yields a one-point external vertex, which we can find straightforwardly by considering the two-point function

$$\left\langle \tfrac{1}{2}\kappa h(x) h_{\rho\sigma}(y)\right\rangle = \tfrac{1}{2}\kappa\eta^{\mu\nu}\left\langle h_{\mu\nu}(x) h_{\rho\sigma}(y)\right\rangle. \tag{4.56}$$

Thus the external vertex factor for this term is



$$= \mathrm{i}\Delta_{\mu\nu\rho\sigma}(p) = \tfrac{1}{2}\eta^{\mu\nu}. \tag{4.57}$$

4.4.2 One-point: coordinate corrections

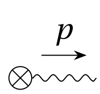
The coordinate correction term at $\mathcal{O}(\kappa)$, $-\kappa\partial_\mu \mathrm{X}_1^\mu$, also yields a one-point external vertex. Using eq. (4.48) we can obtain this vertex also from a two-point function:

$$\begin{aligned}\left\langle\Big(-\kappa\partial_\mu \mathrm{X}_1^\mu(x)\Big) h_{\rho\sigma}(y)\right\rangle &= \kappa\mathcal{J}_1^{\alpha\beta\mu\nu}\int_{x',p}\frac{1}{p^2}\left\langle\partial_\alpha\partial_\beta h_{\mu\nu}(x') h_{\rho\sigma}(y)\right\rangle \mathrm{e}^{\mathrm{i}p(x-x')}\\ &= -\kappa\mathcal{J}_1^{\alpha\beta\mu\nu}\int_{x',p,p'}\frac{1}{p^2}p'_\alpha p'_\beta\Delta_{\mu\nu\rho\sigma}(p')\mathrm{e}^{\mathrm{i}p'(x'-y)}\mathrm{e}^{\mathrm{i}p(x-x')}.\end{aligned} \tag{4.58}$$

The x' integral sets $p = p'$,

$$\left\langle\Big(-\kappa\partial_\mu \mathrm{X}_1^\mu(x)\Big) h_{\rho\sigma}(y)\right\rangle = -\kappa\mathcal{J}_1^{\alpha\beta\mu\nu}\int_p \frac{1}{p^2}p_\alpha p_\beta \Delta_{\mu\nu\rho\sigma}(p)\mathrm{e}^{\mathrm{i}p(x-y)}, \tag{4.59}$$

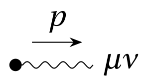
from which we can read off the external vertex



$$= -\kappa\frac{1}{p^2}\mathcal{J}_1^{\alpha\beta\mu\nu}p_\alpha p_\beta = -\kappa\frac{1}{p^2}\Big(p^\mu p^\nu - \tfrac{1}{2}p^2\eta^{\mu\nu}\Big). \tag{4.60}$$

4.5 Scalar curvature external insertions

The standard external vertices come from the expansion terms found in sec. 4.1.3. The linear term in the expansion of R is $R_1 = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h$, yielding the external vertex factor

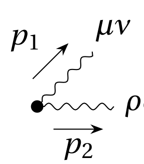


$$= E_h^{\mu\nu}(p) = \kappa\big(p^2 \eta^{\mu\nu} - p^\mu p^\nu\big). \tag{4.61}$$

Similarly the quadratic term in the expansion of R is

$$\begin{aligned} R_2 = h^{\mu\nu}\partial_\mu\partial_\nu h - \tfrac{1}{4}\partial_\mu h \partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\alpha h_\nu{}^\alpha + \partial_\mu h \partial_\nu h^{\mu\nu} \\ - 2h^{\mu\nu}\partial_\nu\partial_\alpha h_\mu{}^\alpha + h^{\mu\nu}\partial^2 h_{\mu\nu} - \tfrac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\mu h^{\nu\alpha} + \tfrac{3}{4}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu}. \end{aligned} \tag{4.62}$$

I denote the resulting external vertex by



$$= E_{h^2}^{\mu\nu\rho\sigma}(p_1, p_2), \tag{4.63}$$

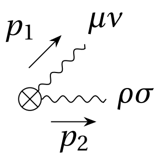
and it is given explicitly by

$$\begin{aligned} E_{h^2}^{\mu\nu\rho\sigma}(p_1,p_2) = \kappa^2\Big(& - p_2^\mu p_2^\nu \eta^{\rho\sigma} - p_1^\rho p_1^\sigma \eta^{\mu\nu} + \tfrac{1}{2}(p_1\cdot p_2)\eta^{\mu\nu}\eta^{\rho\sigma} + 2p_1^\mu p_2^\rho \eta^{\nu\sigma} - p_1^\rho p_2^\sigma \eta^{\mu\nu} - p_1^\mu p_2^\nu \eta^{\rho\sigma} \\ & + 2p_2^\mu p_2^\rho \eta^{\nu\sigma} + 2p_1^\mu p_1^\rho \eta^{\nu\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}\big(p_1^2 + p_2^2\big) + p_1^\rho p_2^\mu \eta^{\nu\sigma} - \tfrac{3}{2}(p_1\cdot p_2)\eta^{\mu\rho}\eta^{\nu\sigma}\Big). \end{aligned} \tag{4.64}$$

The invariantized scalar curvature also receives a coordinate correction at $\mathcal{O}(\kappa^2)$, given by (2.97):

$$-\kappa^2 \mathsf{X}_1^\alpha \partial_\alpha R_1 = -\kappa^2 \partial_\beta\big(\partial_\mu\partial_\nu h^{\mu\nu}(x) - \partial^2 h(x)\big)\int \mathrm{d}^d x'\, G(x,x')\Big(\partial_\alpha h^{\alpha\beta}(x') - \tfrac{1}{2}\partial^\beta h(x')\Big). \tag{4.65}$$

This yields an external vertex



$$= \tilde{E}^{\mu\nu\rho\sigma}_{h^2}(p_1, p_2) = \frac{\kappa^2}{2}\left\{ \frac{1}{p_1^2}\big(-\eta^{\mu\nu}(p_1 \cdot p_2) + p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\big)\big(p_2^\rho p_2^\sigma - p_2^2 \eta^{\rho\sigma}\big) \right.$$
$$\left. + \frac{1}{p_2^2}\big(-\eta^{\rho\sigma}(p_1 \cdot p_2) + p_1^\rho p_2^\sigma + p_1^\sigma p_2^\rho\big)\big(p_1^\mu p_1^\nu - p_1^2 \eta^{\mu\nu}\big)\right\}. \tag{4.66}$$

4.6 The graviton self-interactions

Expanding the Einstein-Hilbert Lagrangian $(-2/\kappa^2)\sqrt{-g}R$ in $\boldsymbol{h}$ yields an infinite series of graviton self-interactions. For the purposes of this thesis we need only the three- and four-graviton vertices, which are nevertheless quite lengthy (136 and 1118 terms respectively!). In this section I will explicitly provide the corresponding terms in the Lagrangian (which are 28 and 66 terms respectively), from which the resulting vertices may be obtained by the standard Wick contraction algorithm.

4.6.1 Cubic

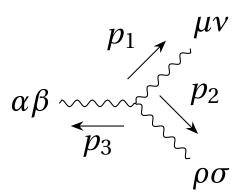
In terms of the expansion coefficients of $\sqrt{-g}$ and R given in secs. 4.1.2 and 4.1.3 the $\mathcal{O}(\kappa)$ terms in the Einstein-Hilbert Lagrangian are

$$\mathcal{L}_{h^3} = -2\kappa\Big(R_3 + \gamma_1 R_2 + \gamma_2 R_1\Big), \tag{4.67}$$

using the fact that $\gamma_0 = 1$ and $R_0 = 0$. In terms of $\boldsymbol{h}$ this is

$$\begin{aligned}
\mathcal{L}_{h^3} = \kappa\Big\{ &\tfrac{3}{2} h^{\mu\nu}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} - \tfrac{1}{2} h^{\mu\nu}\partial_\mu h\partial_\nu h + 2h^{\mu\nu}\partial_\mu h\partial_\alpha h_\nu{}^\alpha + 2h^{\mu\nu}\partial_\mu h_\nu{}^\alpha\partial_\alpha h + 2h^{\mu\nu}h_\mu{}^\alpha\partial_\nu\partial_\alpha h \\
&- hh^{\mu\nu}\partial_\mu\partial_\nu h - h^{\mu\nu}\partial_\alpha h\partial^\alpha h_{\mu\nu} + \tfrac{1}{4} h\partial_\mu h\partial^\mu h - 2h^{\mu\nu}\partial_\alpha h_\mu{}^\alpha\partial_\beta h_\nu{}^\beta - 4h^{\mu\nu}\partial_\mu h_\nu{}^\alpha\partial_\beta h_\alpha{}^\beta \\
&+ h\partial_\mu h^{\mu\nu}\partial_\alpha h_\nu{}^\alpha + 2h^{\mu\nu}\partial^\alpha h_{\mu\nu}\partial_\beta h_\alpha{}^\beta - h\partial_\mu h\partial_\nu h^{\mu\nu} - 2h^{\mu\nu}h^{\alpha\beta}\partial_\mu\partial_\alpha h_{\nu\beta} + 2h^{\mu\nu}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} \\
&- 4h^{\mu\nu}h_\mu{}^\alpha\partial_\alpha\partial_\beta h_\nu{}^\beta + 2hh^{\mu\nu}\partial_\mu\partial_\alpha h_\nu{}^\alpha + \tfrac{1}{2} h_{\mu\nu}h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \tfrac{1}{4} h^2\partial_\mu\partial_\nu h^{\mu\nu} + 2h^{\mu\nu}h_\mu{}^\alpha\partial^2 h_{\nu\alpha} \\
&- hh^{\mu\nu}\partial^2 h_{\mu\nu} - \tfrac{1}{2} h_{\mu\nu}h^{\mu\nu}\partial^2 h + \tfrac{1}{4} h^2\partial^2 h - 2h^{\mu\nu}\partial_\mu h_\alpha{}^\beta\partial_\beta h_\nu{}^\alpha - h^{\mu\nu}\partial_\alpha h_\mu{}^\beta\partial_\beta h_\nu{}^\alpha \\
&\qquad + 3h^{\mu\nu}\partial_\alpha h_{\nu\beta}\partial^\alpha h_\mu{}^\beta + \tfrac{1}{2} h\partial_\mu h_\nu{}^\alpha\partial_\alpha h^{\mu\nu} - \tfrac{3}{4} h\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu}\Big\}.
\end{aligned} \tag{4.68}$$

I denote the resulting vertex as



$$= V_{h^3}^{\mu\nu\rho\sigma\alpha\beta}(p_1, p_2, p_3). \tag{4.69}$$

4.6.2 Quartic

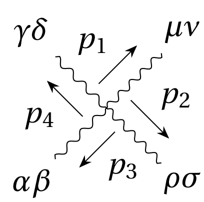
In terms of the expansion coefficients of $\sqrt{-g}$ and R given in sec. 4.1 the $\mathcal{O}(\kappa^2)$ terms in the Einstein-Hilbert Lagrangian are

$$\mathcal{L}_{h^4} = -2\kappa^2\Big(R_4 + \gamma_1 R_3 + \gamma_2 R_2 + \gamma_3 R_1\Big), \tag{4.70}$$

using the fact that $\gamma_0 = 1$ and $R_0 = 0$. In terms of $\boldsymbol{h}$ this is

$$
\begin{aligned}
\mathcal{L}_{h^4} = \kappa^2 \Big\{ & h^{\mu\nu} h^{\alpha\beta} \partial_\nu h_{\beta\lambda} \partial_\alpha h_\mu{}^\lambda - \tfrac{3}{2} h^{\mu\nu} h_\mu{}^\alpha \partial_\nu h^{\beta\lambda} \partial_\alpha h_{\beta\lambda} + \tfrac{3}{4} h h^{\mu\nu} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} + \tfrac{1}{2} h^{\mu\nu} h_\mu{}^\alpha \partial_\nu h \partial_\alpha h \\
& - \tfrac{1}{4} h h^{\mu\nu} \partial_\mu h \partial_\nu h - 2 h^{\mu\nu} h_\mu{}^\alpha h^{\beta\lambda} \partial_\nu \partial_\alpha h_{\beta\lambda} - 2 h^{\mu\nu} h_\mu{}^\alpha \partial_\alpha h \partial_\beta h_\nu{}^\beta + h h^{\mu\nu} \partial_\mu h \partial_\alpha h_\nu{}^\alpha \\
& - 3 h^{\mu\nu} h^{\alpha\beta} \partial_\alpha h_\mu{}^\lambda \partial_\beta h_{\nu\lambda} + 2 h^{\mu\nu} h^{\alpha\beta} \partial_\mu h_\nu{}^\lambda \partial_\alpha h_{\beta\lambda} - 2 h^{\mu\nu} h^{\alpha\beta} \partial_\mu h_{\nu\alpha} \partial_\beta h + h^{\mu\nu} h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h \\
& - 2 h^{\mu\nu} h_\mu{}^\alpha \partial_\alpha h_\nu{}^\beta \partial_\beta h + h h^{\mu\nu} \partial_\mu h_\nu{}^\alpha \partial_\alpha h - 2 h^{\mu\nu} h_\mu{}^\alpha h_\nu{}^\beta \partial_\alpha \partial_\beta h + h h^{\mu\nu} h_\mu{}^\alpha \partial_\nu \partial_\alpha h \\
& + \tfrac{1}{2} h_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} \partial_\alpha \partial_\beta h - \tfrac{1}{4} h^2 h^{\mu\nu} \partial_\mu \partial_\nu h + h^{\mu\nu} h_\mu{}^\alpha \partial_\beta h \partial^\beta h_{\nu\alpha} - \tfrac{1}{2} h h^{\mu\nu} \partial_\alpha h \partial^\alpha h_{\mu\nu} \\
& - \tfrac{1}{8} h_{\mu\nu} h^{\mu\nu} \partial_\alpha h \partial^\alpha h + \tfrac{1}{16} h^2 \partial_\alpha h \partial^\alpha h + 2 h^{\mu\nu} h_\mu{}^\alpha \partial_\beta h_\nu{}^\beta \partial_\lambda h_\alpha{}^\lambda - h h^{\mu\nu} \partial_\alpha h_\mu{}^\alpha \partial_\beta h_\nu{}^\beta \\
& + 4 h^{\mu\nu} h^{\alpha\beta} \partial_\mu h_{\nu\alpha} \partial_\lambda h_\beta{}^\lambda - 2 h^{\mu\nu} h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\lambda h_\beta{}^\lambda + 4 h^{\mu\nu} h_\mu{}^\alpha \partial_\alpha h_\nu{}^\beta \partial_\lambda h_\beta{}^\lambda - 2 h h^{\mu\nu} \partial_\mu h_\nu{}^\alpha \partial_\beta h_\alpha{}^\beta \\
& - \tfrac{1}{2} h_{\mu\nu} h^{\mu\nu} \partial_\alpha h^{\alpha\beta} \partial_\lambda h_\beta{}^\lambda + \tfrac{1}{4} h^2 \partial_\mu h^{\mu\nu} \partial_\alpha h_\nu{}^\alpha - 2 h^{\mu\nu} h_\mu{}^\alpha \partial_\beta h_{\nu\alpha} \partial_\lambda h^{\beta\lambda} + h h^{\mu\nu} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\alpha\beta} \\
& + \tfrac{1}{2} h_{\mu\nu} h^{\mu\nu} \partial_\alpha h \partial_\beta h^{\alpha\beta} - \tfrac{1}{4} h^2 \partial_\mu h \partial_\nu h^{\mu\nu} + 4 h^{\mu\nu} h_\mu{}^\alpha h^{\beta\lambda} \partial_\alpha \partial_\lambda h_{\nu\beta} - h h^{\mu\nu} h^{\alpha\beta} \partial_\mu \partial_\alpha h_{\nu\beta} \\
& - 2 h^{\mu\nu} h_\mu{}^\alpha h^{\beta\lambda} \partial_\beta \partial_\lambda h_{\nu\alpha} + h h^{\mu\nu} h^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} + 4 h^{\mu\nu} h_\mu{}^\alpha h_\nu{}^\beta \partial_\alpha \partial_\lambda h_\beta{}^\lambda - 2 h h^{\mu\nu} h_\mu{}^\alpha \partial_\alpha \partial_\beta h_\nu{}^\beta \\
& - h_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} \partial_\beta \partial_\lambda h_\alpha{}^\lambda + \tfrac{1}{2} h^2 h^{\mu\nu} \partial_\mu \partial_\alpha h_\nu{}^\alpha - \tfrac{1}{3} h^{\mu\nu} h_\mu{}^\alpha h_{\nu\alpha} \partial_\beta \partial_\lambda h^{\beta\lambda} + \tfrac{1}{4} h h_{\mu\nu} h^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} \\
& - \tfrac{1}{24} h^3 \partial_\mu \partial_\nu h^{\mu\nu} - 2 h^{\mu\nu} h_\mu{}^\alpha h_\nu{}^\beta \partial^2 h_{\alpha\beta} + h h^{\mu\nu} h_\mu{}^\alpha \partial^2 h_{\nu\alpha} + \tfrac{1}{2} h_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} \partial^2 h_{\alpha\beta} \\
& - \tfrac{1}{4} h^2 h^{\mu\nu} \partial^2 h_{\mu\nu} + \tfrac{1}{3} h^{\mu\nu} h_\mu{}^\alpha h_{\nu\alpha} \partial^2 h - \tfrac{1}{4} h h_{\mu\nu} h^{\mu\nu} \partial^2 h + \tfrac{1}{24} h^3 \partial^2 h \\
& - 2 h^{\mu\nu} h^{\alpha\beta} \partial_\lambda h_{\mu\nu} \partial_\alpha h_\beta{}^\lambda + \tfrac{1}{2} h^{\mu\nu} h^{\alpha\beta} \partial_\lambda h_{\mu\nu} \partial^\lambda h_{\alpha\beta} + 2 h^{\mu\nu} h^{\alpha\beta} \partial_\mu h_\alpha{}^\lambda \partial^\lambda h_{\nu\beta} - \tfrac{3}{2} h^{\mu\nu} h^{\alpha\beta} \partial_\lambda h_{\mu\alpha} \partial^\lambda h_{\nu\beta} \\
& + 2 h^{\mu\nu} h_\mu{}^\alpha \partial_\alpha h^{\beta\lambda} \partial_\lambda h_{\nu\beta} - h h^{\mu\nu} \partial_\mu h^{\alpha\beta} \partial_\beta h_{\nu\alpha} + h^{\mu\nu} h_\mu{}^\alpha \partial_\beta h_{\alpha\lambda} \partial^\lambda h_\nu{}^\beta - \tfrac{1}{2} h h^{\mu\nu} \partial_\alpha h_{\nu\beta} \partial^\beta h_\mu{}^\alpha \\
& - 3 h^{\mu\nu} h_\mu{}^\alpha \partial_\lambda h_{\alpha\beta} \partial_\lambda h_\nu{}^\beta + \tfrac{3}{2} h h^{\mu\nu} \partial_\alpha h_{\nu\beta} \partial^\alpha h_\mu{}^\beta - \tfrac{1}{4} h_{\mu\nu} h^{\mu\nu} \partial_\lambda h_\alpha{}^\beta \partial_\beta h^{\alpha\lambda} + \tfrac{1}{8} h^2 \partial_\mu h_\nu{}^\alpha \partial_\alpha h_{\mu\nu} \\
& \qquad + \tfrac{3}{8} h_{\mu\nu} h^{\mu\nu} \partial_\lambda h_{\alpha\beta} \partial^\lambda h^{\alpha\beta} - \tfrac{3}{16} h^2 \partial_\lambda h_{\alpha\beta} \partial^\lambda h^{\alpha\beta} \Big\}.
\end{aligned}
\tag{4.71}
$$

I denote the resulting vertex diagrammatically as



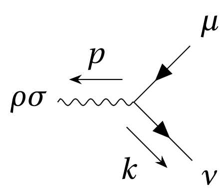
$$
= V_{h^4}^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4). \tag{4.72}
$$

4.7 The ghost-graviton vertex

The ghost-graviton interaction terms are given in eq. (3.36),

$$\mathcal{L}_{c\bar{c}h} = \kappa\Big\{\Big(1-\frac{2(1+\beta)}{d}\Big)\bar{c}^\mu h_{\nu\rho}\nabla^\rho\nabla_\mu c^\nu - \frac{1+\beta}{d}\big(\bar{c}^\mu\nabla_\mu c^\nu\nabla_\nu h + \bar{c}^\mu c^\nu\nabla_\mu\nabla_\nu h\big) - \frac{2(1+\beta)}{d}\bar{c}^\mu\nabla_\mu h_{\nu\rho}\nabla^\rho c^\nu$$
$$+ \bar{c}^\mu\nabla_\mu c^\nu\nabla_\rho h_\nu{}^\rho + \bar{c}^\mu c^\nu\nabla_\nu\nabla_\rho h_\nu{}^\rho + \bar{c}^\mu h_{\mu\nu}\nabla^2 c^\nu + \bar{c}^\mu\nabla_\nu h_{\mu\rho}\nabla^\rho c^\nu + \bar{c}^\mu\nabla_\rho h_{\mu\nu}\nabla^\rho c^\nu\Big\}. \tag{4.73}$$

This yields a three-point vertex, which I will denote



$$= V^{\mu\nu\rho\sigma}_{\bar{c}ch}(k,p), \tag{4.74}$$

giving no name to the antighost momentum because it doesn't appear in the vertex (since no derivatives act on $\bar{c}$ in the Lagrangian). Explicitly this vertex is

$$V^{\mu\nu\rho\sigma}_{\bar{c}ch}(k,p) = -\mathrm{i}\kappa\Big\{\Big(1-\frac{2(1+\beta)}{d}\Big)k^\mu k^\rho\eta^{\nu\sigma} - \frac{1+\beta}{d}\big(k^\mu p^\nu\eta^{\rho\sigma} + p^\mu p^\nu\eta^{\rho\sigma}\big) - \frac{2(1+\beta)}{d}p^\mu k^\rho\eta^{\nu\sigma}$$
$$+ k^\mu p^\rho\eta^{\nu\sigma} + p^\nu p^\rho\eta^{\mu\sigma} + k^2\eta^{\mu\rho}\eta^{\nu\sigma} + k^\rho p^\nu\eta^{\mu\sigma} + (k\cdot p)\eta^{\mu\rho}\eta^{\nu\sigma}\Big\}. \tag{4.75}$$

4.8 The scalar-graviton interactions

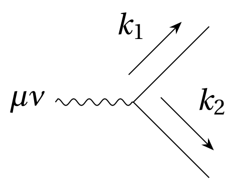
The expansions of the metric determinant and inverse metric in the scalar action yields an infinite series of two-scalar n-graviton interactions. For the purposes of this thesis we need only the cases $n \in \{1,2\}$.

4.8.1 Two-scalar one-graviton

The two-scalar one-graviton interaction terms are

$$\mathcal{L}_{\phi^2 h} = \tfrac{1}{2}\kappa\Big(\gamma_1\big[(\partial\phi)^2 - m^2\phi^2\big] + \tilde{g}_1^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big) = \kappa\Big(\tfrac{1}{4}h\big[(\partial\phi)^2 - m^2\phi^2\big] - \tfrac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big), \tag{4.76}$$

and yield the vertex



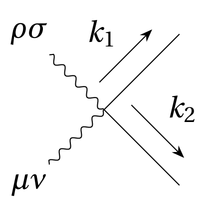
$$= V^{\mu\nu}_{\phi^2 h}(k_1, k_2) = \tfrac{1}{2}\mathrm{i}\kappa\Big(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu}(k_1\cdot k_2 + m^2)\Big). \tag{4.77}$$

4.8.2 Two-scalar two-graviton

The two-scalar two-graviton interaction terms are

$$\begin{aligned}\mathcal{L}_{\phi^2 h^2} &= \tfrac{1}{2}\kappa^2\Big(\gamma_2\big[(\partial\phi)^2 - m^2\phi^2\big] + \gamma_1\tilde{g}_1^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tilde{g}_2^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)\\ &= \kappa^2\Big\{\Big(\tfrac{1}{16}h^2 - \tfrac{1}{8}h_{\mu\nu}h^{\mu\nu}\Big)\big[(\partial\phi)^2 - m^2\phi^2\big] - \tfrac{1}{4}hh^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}h^{\mu\alpha}h_\alpha{}^\nu\partial_\mu\phi\partial_\nu\phi\Big\}.\end{aligned} \tag{4.78}$$

I denote the resulting vertex by



$$= V^{\mu\nu\rho\sigma}_{\phi^2 h^2}(k_1, k_2), \tag{4.79}$$

and it is given explicitly by

$$\begin{aligned}V^{\mu\nu\rho\sigma}_{\phi^2 h^2}(k_1, k_2) = \tfrac{1}{4}\mathrm{i}\kappa^2\Big\{ &-\eta^{\mu\nu}\eta^{\rho\sigma}(k_1\cdot k_2 + m^2) + \eta^{\mu\rho}\eta^{\nu\sigma}(k_1\cdot k_2 + m^2) + \eta^{\mu\sigma}\eta^{\nu\rho}(k_1\cdot k_2 + m^2)\\ &+ \eta^{\mu\nu}\big(k_1^\rho k_2^\sigma + k_1^\sigma k_2^\rho\big) + \eta^{\rho\sigma}\big(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu\big) - \eta^{\mu\rho}\big(k_1^\nu k_2^\sigma + k_1^\sigma k_2^\nu\big)\\ &- \eta^{\mu\sigma}\big(k_1^\nu k_2^\rho + k_1^\rho k_2^\nu\big) - \eta^{\nu\rho}\big(k_1^\mu k_2^\sigma + k_1^\sigma k_2^\rho\big) - \eta^{\nu\sigma}\big(k_1^\mu k_2^\rho + k_1^\rho k_2^\mu\big)\Big\}.\end{aligned} \tag{4.80}$$

5 Correlators

We now, finally, come to the entire purpose of all of the above-developed machinery: the computation of gauge-invariant correlation functions. In particular in this section I calculate the tree-level two-point function of the invariantized volume factor in sec. 5.1; the one-loop gravitational correction to the scalar field mass in sec. 5.2; and the one-loop two-point function of the invariantized scalar field in sec. 5.3. The first and third of these feature the invariantized observables constructed in sec. 2, and in all three I perform the calculations in the general (α, β)-parametrized gauge of sec. 3. In all three cases the dependence on α and β fully cancels out, supporting the claimed gauge invariance of this formalism.

Before proceeding I will note that, although in secs. 5.2 and 5.3 I do perform a pair of one-loop calculations, I do *not* address any renormalization in these calculations, for two reasons. First, neither calculation requires it of me – in both cases all divergences cancel without introducing any counterterms. Second, recall from sec. 1 that the long-term goal of this program is to obtain gauge-invariant correlators in position space whose scaling with distance can be compared to corresponding observables on the lattice. In other words I am interested in the long-ranged behavior of e.g. $\langle \mathscr{R}(x)\mathscr{R}(y)\rangle$ as a function of $(x-y)^2$. In momentum space this means that I am interested in the nonanalytic contributions to the correlators, since it is those contributions which govern the long-ranged behavior of the position-space correlator (see e.g. [16] in which the same logic was used to extract the one-loop correction to the Newtonian potential at long range). But these nonanalytic contributions are not affected by renormalization: the counterterm vertices themselves are local, since they arise from renormalization of the couplings in the effective gravitational action

$$\sqrt{-g}\left(\frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \dots\right), \tag{5.1}$$

and at one loop, i.e. at $\mathcal{O}(\kappa^2)$, a diagram which features a counterterm vertex cannot also feature a loop integration, meaning that no nonanalytic behavior can arise in such a diagram. In other words, even if there were surviving divergences in the one-loop calculations below, those divergences would not affect the long-ranged behavior of the corresponding position-space observables, and would therefore be irrelevant to my purposes.

5.1 THE VOLUME TWO-POINT FUNCTION

I'll begin with the two-point function of the volume factor $\sqrt{-g}$, which I will do at tree level. Let's recall from sec. 4.1.2 that the standard expansion of the volume factor is

$$\sqrt{-g} = 1 + \tfrac{1}{2}\kappa h + \mathcal{O}(\kappa^2), \tag{5.2}$$

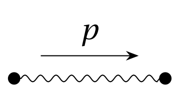
which is augmented in the invariantized volume factor by a coordinate correction term,

$$\sqrt{-\det\mathcal{G}} = 1 + \kappa\left(\tfrac{1}{2}h - \partial_\mu X_1^\mu\right). \tag{5.3}$$

The two-point function of the invariantized volume factor therefore receives three contributions at tree level, corresponding to both external vertices being standard; one being a coordinate correction and the other standard; and both being a coordinate correction:

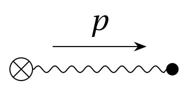
$$\left\langle \sqrt{-\det\mathcal{G}(x)}\sqrt{-\det\mathcal{G}(y)} \right\rangle = 1 + \kappa^2\left\{ \left\langle \left(\tfrac{1}{2}h(x)\right)\left(\tfrac{1}{2}h(y)\right)\right\rangle - 2\left\langle \left(\partial_\mu X_1^\mu(x)\right)\left(\tfrac{1}{2}h(y)\right)\right\rangle + \left\langle \left(\partial_\mu X_1^\mu(x)\right)\left(\partial_\mu X_1^\mu(y)\right)\right\rangle \right\}. \tag{5.4}$$

Note that by Lorentz invariance the two cross-terms must be equal. In momentum space we therefore find three diagrams at this order. With no coordinate corrections we have



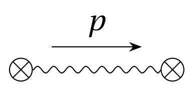
$$= i\mathcal{A}_0 = \tfrac{1}{4}i\kappa^2\eta^{\mu\nu}\eta^{\rho\sigma}\Delta_{\mu\nu\rho\sigma}(p). \tag{5.5}$$

With one, and including the factor of 2 to account for the coordinate correction being on either end, we have



$$= \mathrm{i}\mathcal{A}_1 = -\mathrm{i}\kappa^2 \tfrac{1}{p^2} \mathcal{J}_1^{\alpha\beta\mu\nu} p_\alpha p_\beta \eta^{\rho\sigma} \Delta_{\mu\nu\rho\sigma}(p). \tag{5.6}$$

And finally, with coordinate corrections on both ends,



$$= \mathrm{i}\mathcal{A}_2 = \mathrm{i}\kappa^2 \tfrac{1}{p^4} \mathcal{J}_1^{\alpha\beta\mu\nu} \mathcal{J}_1^{\gamma\delta\rho\sigma} p_\alpha p_\beta p_\gamma p_\delta \Delta_{\mu\nu\rho\sigma}(p). \tag{5.7}$$

N.B. without accounting for coordinate corrections we would only have the diagram $\mathcal{A}_0$, which we will see below is not sufficient to achieve a gauge-invariant result. (This is probably not surprising in itself, since $\sqrt{-g}$ is certainly not a gauge-invariant observable anyway.)

5.1.1 In a simple gauge

Before calculating the above diagrams for general values of the gauge parameters (α, β) let's work them out in a simple gauge, say harmonic gauge $\alpha = 1/2$, $\beta = (d/2) - 1$. Further since we're only working at tree level we can safely set $d = 4$. In this gauge the propagator becomes

$$\Delta_{\mu\nu\rho\sigma}(p) = \frac{1}{2p^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right). \tag{5.8}$$

The standard diagram then becomes

$$\mathrm{i}\mathcal{A}_0 = \frac{1}{8p^2}\mathrm{i}\kappa^2 \eta^{\mu\nu}\eta^{\rho\sigma}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) = -\frac{\mathrm{i}\kappa^2}{p^2}. \tag{5.9}$$

We happen to find the same value for the first coordinate correction diagram,

$$\mathrm{i}\mathcal{A}_1 = -\mathrm{i}\kappa^2 \frac{1}{2p^4}\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\right) p_\alpha p_\beta \eta^{\rho\sigma}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) = -\frac{\mathrm{i}\kappa^2}{p^2}, \tag{5.10}$$

while for the second we find

$$\mathrm{i}\mathcal{A}_2 = \mathrm{i}\kappa^2 \frac{1}{2p^6}\big(\eta^{\alpha\mu}\eta^{\beta\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\big)\big(\eta^{\gamma\rho}\eta^{\delta\sigma} - \tfrac{1}{2}\eta^{\gamma\delta}\eta^{\rho\sigma}\big)\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big) = \frac{\mathrm{i}\kappa^2}{2p^2}. \tag{5.11}$$

And therefore we find in this gauge that the tree-level two-point function of the invariantized volume factor is

$$\mathrm{i}\mathcal{A}_0 + \mathrm{i}\mathcal{A}_1 + \mathrm{i}\mathcal{A}_2 = -\frac{3\mathrm{i}\kappa^2}{2p^2}. \tag{5.12}$$

5.1.2 In a general gauge

Now let's subject our machinery to its first real test: if we leave α and β arbitrary, do we still find a gauge-invariant result for the two-point function of the invariantized volume factor?

Here we still have the three expressions (5.5), (5.6), and (5.7) for the three relevant tree-level diagrams, but we have the much more complicated form (3.132) for the graviton propagator,

$$\Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d^2\alpha(d-2)-(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}, \tag{5.13}$$

where the projectors are given by eqs. (3.130) and (3.131). Since we're working at tree level we can here set $d = 4$, which does slightly simplify the propagator – for example, in terms of the symmetric tensor structures (3.129), the propagator simplifies to

$$\Delta(p) = \frac{1}{p^2}\left\{\mathbf{G} - 2\,\mathrm{tr} - \frac{2(\beta-1)(3+2\alpha(\beta-5)+\beta)}{(\beta-3)^2}A + \frac{2(\beta-1)}{\beta-3}B + (4\alpha-2)C\right\}. \tag{5.14}$$

However this is does still massively complicate even the simplest of calculations. Accordingly I perform these calculations in xAct.

For the standard diagram we find

$$\mathrm{i}\mathcal{A}_0 = \tfrac{1}{4}\mathrm{i}\kappa^2\eta^{\mu\nu}\eta^{\rho\sigma}\Delta_{\mu\nu\rho\sigma} = \frac{2(1+4(\alpha-1)-2\alpha)}{(\beta-3)^2}\frac{\mathrm{i}\kappa^2}{p^2}. \tag{5.15}$$

Note that this diagram is *gauge-dependent!* Firstly, this is what we expect – this is the only diagram that occurs at tree level in $\langle\sqrt{-g(x)}\sqrt{-g(y)}\rangle$ without coordinate corrections, and $\sqrt{-g}$ is not gauge-invariant, so its two-point function shouldn't be either. Secondly this means that, if $\langle\sqrt{-\det\mathscr{G}(x)}\sqrt{-\det\mathscr{G}(y)}\rangle$ is to be gauge-invariant, the gauge parameters must cancel in a nontrivial manner.

Indeed, this is exactly what we find. For the first coordinate correction diagram we have

$$\mathrm{i}\mathcal{A}_1 = -\frac{16(1+\alpha)+2(1+\beta)-4(3+2\alpha+2\beta)}{(\beta-3)^2}\frac{\mathrm{i}\kappa^2}{p^2}, \tag{5.16}$$

and for the second

$$\mathrm{i}\mathcal{A}_2 = \frac{64(\alpha-1)+4(\beta+1)^2+16(5-2\alpha+4\beta)-16(2+3\beta+\beta^2)}{8(\beta-3)^2}\frac{\mathrm{i}\kappa^2}{p^2}. \tag{5.17}$$

Since all three prefactors share a common denominator we can focus on the sum of the numerators, which simplifies nicely:

$$\begin{aligned} &\Big[2(1+4(\alpha-1)-2\alpha)\Big]-\Big[16(1+\alpha)+2(1+\beta)-4(3+2\alpha+2\beta)\Big] \\ &\qquad\qquad+\tfrac{1}{8}\Big[64(\alpha-1)+4(\beta+1)^2+16(5-2\alpha+4\beta)-16(2+3\beta+\beta^2)\Big] \\ &= -\tfrac{3}{2}(\beta-3)^2. \end{aligned} \tag{5.18}$$

This then cancels the shared denominator, as it must, leaving precisely our prior result:

$$\mathrm{i}\mathcal{A}_0+\mathrm{i}\mathcal{A}_1+\mathrm{i}\mathcal{A}_2 = -\frac{3\mathrm{i}\kappa^2}{2p^2}. \tag{5.19}$$

We therefore see that, even though none of the individual diagrams were gauge-invariant, their sum *is* – indicating that our construction of a gauge-invariant volume correlator was successful!

5.2 The scalar field mass correction

As another test of the machinery thus far developed let's consider the one-loop gravitational correction to the mass m of a scalar ϕ. This was calculated in the appendix to [70]. I will provide two extensions to their analysis. First, the calculation in [70] was done in harmonic gauge, while I will perform the calculation in our arbitrary (α, β)-parametrized gauge. Second, I claim that their calculation suffers from a subtle error involving the dimension-dependence of the graviton propagator.

Before proceeding I will first note that this calculation does not involve any of the gauge-invariant observables introduced in sec. 2. Rather, as I review in sec. 5.2.1, in this section I calculate the sum of the one-loop gravitational 1PI diagrams with two external legs in the limit $p^2 \to m^2$, where p is the momentum of the amputated external legs. However the gauge invariance of my result is still a valuable check of the validity of the propagators and vertices I'm working with.

5.2.1 The pole mass and self-energy

Before getting into the perturbative calculation I'll quickly review the logic which identifies the pole in the two-point function of a field with that field's physical mass, i.e. with the energy eigenvalue of a zero-momentum single-particle state. (This derivation is standard quantum field theory fare – for a textbook treatment see e.g. any of [68, 71, 72]. In my treatment here I follow [71].)

Consider our massive scalar field ϕ. In the canonical formalism its two-point function is the expectation value of its time-ordered product,

$$\left\langle \phi(x)\phi(y) \right\rangle = \langle \Omega | \, T\phi(x)\phi(y) \, | \Omega \rangle \,, \tag{5.20}$$

where $|\Omega\rangle$ is the vacuum of the full (nonperturbative) theory, governed by some Hamiltonian $\hat{H}$. Let's also denote by $\left|\mu, 0\right\rangle$ the state which is an eigenstate both of $\hat{H}$ with eigenvalue μ and of $\hat{\boldsymbol{p}}$ with eigenvalue 0:

$$\hat{H}\left|\mu,0\right\rangle = \mu\left|\mu,0\right\rangle, \quad \hat{\boldsymbol{p}}\left|\mu,0\right\rangle = 0. \tag{5.21}$$

Then the energy eigenvalue μ of $\left|\mu,0\right\rangle$ is its total mass, whatever that may be. N.B. the $\left|\mu,0\right\rangle$'s include not just the single-particle state with zero momentum but also a plethora of many-particle states with zero *total* momentum (but whose individual particles may have arbitrarily high momentum). We can span the Hilbert space with the boosts of these states, say $\left|\mu,\boldsymbol{p}\right\rangle$ with

$$\hat{\boldsymbol{p}}\left|\mu,\boldsymbol{p}\right\rangle = \boldsymbol{p}\left|\mu,\boldsymbol{p}\right\rangle, \quad \hat{H}\left|\mu,\boldsymbol{p}\right\rangle = E(\mu,\boldsymbol{p})\left|\mu,\boldsymbol{p}\right\rangle, \quad E(\mu,\boldsymbol{p}) = \sqrt{\mu^2+\boldsymbol{p}^2}, \tag{5.22}$$

which have total mass μ and total momentum $\boldsymbol{p}$.

The lovely thing about the argument to come, which yields the Källen-Lehmann spectral representation and hence the identification of the physical mass as the pole of the two-point function, is that the actual details of the theory don't matter at all. All that matters is that our scalar field ϕ is governed by some Hamiltonian $\hat{H}$, with eigenstates $\left|\mu,\boldsymbol{p}\right\rangle$ so defined which span the Hilbert space. With this being the case the identity operator can then be written

$$1 = |\Omega\rangle\langle\Omega| + \sum_\mu \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E(\mu,\boldsymbol{p})}\left|\mu,\boldsymbol{p}\right\rangle\left\langle\mu,\boldsymbol{p}\right|, \tag{5.23}$$

where the sum is over all eigenvalues μ of $\hat{H}$ and in which we make use of the Lorentz-invariant measure $\mathrm{d}^3\boldsymbol{p}/E(\mu,\boldsymbol{p})$ with its standard normalization. One can then insert the identity in this form into the two-point function to obtain the Källen-Lehmann spectral representation [73, 74]

$$\left\langle\phi(x)\phi(y)\right\rangle = \int_0^\infty \frac{\mathrm{d}(M^2)}{2\pi}\rho(M^2)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-M^2+\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}p(x-y)}, \tag{5.24}$$

in which the spectral density function is

$$\rho(M^2) = \sum_\mu (2\pi)\delta(M^2-\mu^2)\left|\langle\Omega|\phi(0)\left|\mu,0\right\rangle\right|^2. \tag{5.25}$$

Note the appearance of the Feynman propagator of ϕ from x to y, with the mass integration variable M in place of the actual mass of the field.

Let's unpack the above. The sums are taken over all the values of μ, i.e. over all the mass eigenvalues of $\hat{H}$ – i.e. over the energies of all the energy eigenstates which also have zero momentum. These eigenstates can feature one or many particles. A one-particle zero-momentum eigenstate of $\hat{H}$ just describes a single ϕ particle at rest somewhere in the universe, and its eigenvalue is then the *physical* mass of the ϕ field, including all quantum corrections, which may well differ from the bare mass which appears in the Lagrangian. The many-particle zero-momentum eigenstates can be bound or unbound. The bound states will have some discrete spectrum of eigenvalues above the physical mass but below the minimum possible total mass M_{min} of an unbound state, above which any energy eigenvalue is achieved by some zero-momentum state (since for any energy above this minimum there are the very least exists an unbound state of two particles which share the energy equally and are heading off in opposite directions from each other).[30]

From eq. (5.24) we can therefore see that the discrete mass eigenvalues – the energies of the one-particle state and the bound states – contribute a set of isolated poles to the momentum-space two-point function, all at $p^2 < M^2_{\text{min}}$. The continuous mass eigenvalues yield a branch cut which starts at $p^2 = M^2_{\text{min}}$ and continues out to infinity. Further the physical mass of the field appears as the first pole in the full two-point function – hence its common name, the *pole mass.*

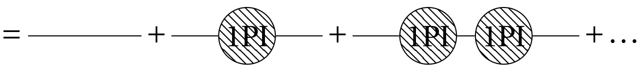
The above is a fully nonperturbative argument which identifies the first pole in the full two-point function with the physical mass of the field. It remains to actually calculate the quantum corrections to this pole. Thankfully this is also standard textbook fare![31] The key observation is that *any* diagram which contributes to the two-point function, at *any* loop order, can be written in terms of one-particle-irreducible (1PI) diagrams – every diagram features a

[30] So the symbol $\sum_\mu$ really stands for "sum over the discrete part of the mass spectrum and then integrate over the continuous part".

[31] I again refer to [68, 71, 72].

finite number of lines, and is therefore n-particle reducible for some integer n, and is therefore the result of joining $n+1$ 1PI diagrams with n propagators! Diagramatically, and representing by $-\mathrm{i}\Sigma$ the sum of all 1PI diagrams with two amputated external legs, it follows that the full momentum-space two-point function is a geometric series in Σ,

$$\int \mathrm{d}^4x \left\langle \phi(x)\phi(y)\right\rangle \mathrm{e}^{-\mathrm{i}p(x-y)} =$$



$$\begin{aligned} &= \mathrm{i}D(p) + \big(\mathrm{i}D(p)\big)\big(-\mathrm{i}\Sigma(p)\big)\big(\mathrm{i}D(p)\big) + \big(\mathrm{i}D(p)\big)\big(-\mathrm{i}\Sigma(p)\big)\big(\mathrm{i}D(p)\big)\big(-\mathrm{i}\Sigma(p)\big)\big(\mathrm{i}D(p)\big) + \ldots \\ &= \mathrm{i}D(p)\sum_{n=0}^{\infty}\big(\Sigma D(p)\big)^n = \frac{\mathrm{i}D(p)}{1-\Sigma D(p)} = \frac{\mathrm{i}}{D(p)^{-1}-\Sigma}. \end{aligned} \tag{5.26}$$

(The extra negative sign in the definition of $-\mathrm{i}\Sigma$ is just a convention which allows the denominator to end up as it does.)

The pole mass $\bar{m}$ is therefore the value of p^2 at which the denominator is zero:

$$0 = \big(D(p)^{-1} - \Sigma\big)_{p^2=\bar{m}^2} = \big(p^2 - m^2 - \Sigma\big)_{p^2=\bar{m}^2}. \tag{5.27}$$

To be clear, I denote the pole mass with a bar and the "bare" mass without, as opposed to denoting the bare mass with an explicit subscripted 0 as is more conventional and denoting the physical mass by m. I do this because in the calculation to come I am explicitly considering the *gravitational* corrections to the scalar mass, meaning that I am assuming that this whole process, and the accompanying mass renormalization, has already been performed in the nongravitational sector, so that in the absence of gravity the unbarred m would already be the full physical mass of ϕ to any relevant loop order.

To actually compute $\bar{m}$ in terms of m let's expand Σ in powers of the coupling. (I'll use the gravitational coupling κ here, but the same argument applies no matter the specific theory under consideration.) The sum Σ of all 1PI diagrams receives contributions from all orders, starting at κ^2:

$$\Sigma(p^2) = \kappa^2\Sigma_2(p^2) + \mathcal{O}(\kappa^3). \tag{5.28}$$

Inserting $p^2 = \bar{m}^2$ into the condition (5.27) we then have

$$\bar{m}^2 = m^2 + \kappa^2 \Sigma_2(p^2 = \bar{m}^2) + \mathcal{O}(\kappa^3). \tag{5.29}$$

It follows that $\bar{m} = m$ at $\mathcal{O}(\kappa^0)$ (as we would certainly hope!), meaning that $\Sigma_2(\bar{m}^2) = \Sigma_2(m^2) + \mathcal{O}(\kappa)$, and hence we can drop the bar on the right hand side:

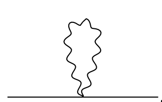
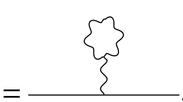
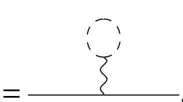
$$\bar{m}^2 = m^2 + \kappa^2 \Sigma_2(p^2 = m^2). \tag{5.30}$$

We therefore arrive at the well-known result that the first-order quantum corrections to the mass of a scalar field are given by the sum of all 1PI diagrams with two amputated external legs, evaluated at $p^2 = m^2$.

I'll now perform this calculation to obtain the gravitational correction to the scalar mass in three different ways: first, following [70], by using the "standard" dimension-independent harmonic-gauge graviton propagator; second, using the same harmonic-gauge propagator, but with the dimension dependence restored; and third, using the arbitrary parametrized gauge.

5.2.2 Harmonic gauge with a dimension-independent graviton propagator

Given the couplings (4.77) and (4.80) between the scalar and graviton, there are four diagrams we can draw:

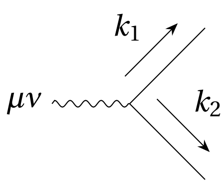
$-\mathrm{i}\Sigma_{\mathrm{A}} =$ , $\quad -\mathrm{i}\Sigma_{\mathrm{B}} =$ , $\quad -\mathrm{i}\Sigma_{\mathrm{C}} =$ , $\quad -\mathrm{i}\Sigma_{\mathrm{D}} =$ . (5.31)

I'll call these the "self-energy", "graviton bubble", "graviton tadpole", and "ghost tadpole" diagrams respectively. The latter three feature tadpole-like momentum structures, and therefore naively ought to vanish in dimensional regularization [75–77]. This does turn out to be the case, although Σ_{C} and Σ_{D} require an IR regularization of the graviton to verify this, since they also feature a zero-momentum graviton leg attached to the tadpole. I'll leave the actual analysis of these diagrams for the general treatment in sec. 5.2.4 and focus on Σ_A for now.

As foreshadowed, I will in this section replicate the calculation in [70], using the d-independent harmonic gauge graviton propagator

$$\Delta_{\mu\nu\rho\sigma}(p) = \frac{1}{2p^2}\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big) \equiv \frac{1}{p^2}T_{\mu\nu\rho\sigma}, \tag{5.32}$$

introducing the tensor structure $T_{\mu\nu\rho\sigma}$ for shorthand. We also need the two-scalar one-graviton vertex (4.77),



$$= V^{\mu\nu}_{\phi^2 h}(k_1,k_2) = \tfrac{1}{2}\mathrm{i}\kappa\Big(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu}(k_1\cdot k_2)\Big) \equiv \tfrac{1}{2}\mathrm{i}\kappa\tilde{V}^{\mu\nu}(k_1,k_2), \tag{5.33}$$

again introducing a shorthand. In terms of these ingredients the self-energy diagram is

$$\mathrm{i}\Sigma_\mathrm{A} = -\int_\ell \big(\mathrm{i}\Delta_{\mu\nu\rho\sigma}(\ell)\big)\big(\mathrm{i}D(p-\ell)\big)V^{\mu\nu}(-p,p-\ell)V^{\rho\sigma}(-p+\ell,p) = -\tfrac{1}{4}\kappa^2\int_\ell \frac{T_{\mu\nu\rho\sigma}\tilde{V}^{\mu\nu}(-p,p-\ell)\tilde{V}^{\rho\sigma}(-p+\ell,p)}{(\ell^2+\mathrm{i}\varepsilon)((p-\ell)^2-m^2+\mathrm{i}\varepsilon)}. \tag{5.34}$$

Introducing a Feynman parameter via

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\,\big(xA + (1-x)B\big)^{-2} \tag{5.35}$$

and defining $q = \ell - xp$ and $f(x) = -x(1-x)p^2 + xm^2$ the above can be rearranged to

$$\mathrm{i}\Sigma_\mathrm{A} = -\tfrac{1}{4}\kappa^2\int_0^1 \mathrm{d}x\int_q \frac{\tilde{V}^{\mu\nu}(-p,p-(q+xp))\tilde{V}^{\rho\sigma}(-p+q+xp,p)T_{\mu\nu\rho\sigma}}{(q^2-f(x)+\mathrm{i}\varepsilon)^2}. \tag{5.36}$$

Performing the tensor contractions in the numerator, throwing out the terms that are odd in the integration momentum q, and using the identity (see e.g. [68])

$$\int_q \frac{q^\mu q^\nu}{(q^2-f)^n} = \frac{1}{d}\eta^{\mu\nu}\int_q \frac{q^2}{(q^2-f)^n} \tag{5.37}$$

to replace all appearances of $(q\cdot p)^2$ with q^2p^2/d, we find

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\tfrac{1}{4}\kappa^2\int_0^1 \mathrm{d}x \int_q \frac{a_1 q^2 p^2 + a_2 p^4(1-x)^2 + a_3 m^2 p^2(1-x) + a_4 m^4}{(q^2 - f(x) + \mathrm{i}\varepsilon)^2}, \tag{5.38}$$

where

$$a_1 = 2 - \frac{(d-4)(d-2)}{2d}, \quad a_2 = 2 - \frac{(d-4)(d-2)}{2}, \quad a_3 = (d-2)^2, \quad a_4 = -\frac{d(d-2)}{2}. \tag{5.39}$$

Note that factors of d now abound!

I use this form in part to facilitate comparison with [70]. My values of a_1 and a_2 agree with theirs, while my values of a_3 and a_4 differ by factors of -4 and $+4$ respectively. However as I follow my analysis through I obtain the same result here that they did for this diagram, so these appear to be mere typographical errors.

As further shorthand let me define

$$b_1 = a_1 p^2, \quad b_2 = a_2 p^4(1-x)^2 + a_3 m^2 p^2(1-x) + a_4 m^4. \tag{5.40}$$

Then the self-energy diagram becomes

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\tfrac{1}{4}\kappa^2\int_0^1 \mathrm{d}x \int_q \left\{ b_1 \frac{q^2}{(q^2-f+\mathrm{i}\varepsilon)^2} + b_2 \frac{1}{(q^2-f+\mathrm{i}\varepsilon)^2}\right\}. \tag{5.41}$$

The two q-integrals have the standard pole structure and can therefore be evaluated in the usual way via a rotation of the contour:

$$\int_q \frac{q^2}{(q^2-f+\mathrm{i}\varepsilon)^2} = -\mathrm{i}\int_{\bar q} \frac{\bar q^2}{(\bar q^2+f)^2}, \quad \int_q \frac{1}{(q^2-f+\mathrm{i}\varepsilon)^2} = \mathrm{i}\int_{\bar q} \frac{1}{(\bar q^2+f)^2}, \tag{5.42}$$

where $\bar{q}$ is the Euclidean momentum $q^0 = \mathrm{i}\bar{q}^0$, $q^i = \bar{q}^i$. Now let's recall the general result (see e.g. [72]) that

$$\int_{\bar{q}} \frac{\bar{q}^{2a}}{(\bar{q}^2 + f)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)}. \tag{5.43}$$

Defining $d = 4-2\epsilon$, and attaching the usual mass scale $\tilde{\mu}^{2\epsilon}$ to the integration measure to keep fixed the dimensions of these integrals, we have

$$\int_q \frac{q^2}{(q^2 - f + \mathrm{i}\varepsilon)^2} = -\frac{\mathrm{i}f(x)}{(4\pi\tilde{\mu}^2)^2}\left(\frac{4\pi}{f(x)}\right)^{\epsilon}(2-\epsilon)\Gamma(-1+\epsilon), \quad \int_q \frac{1}{(q^2 - f + \mathrm{i}\varepsilon)^2} = \frac{\mathrm{i}}{(4\pi\tilde{\mu}^2)^2}\left(\frac{4\pi}{f(x)}\right)^{\epsilon}(-1+\epsilon)\Gamma(-1+\epsilon). \tag{5.44}$$

Using these results in eq. (5.41) gives

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\tfrac{1}{4}\kappa^2 \int_0^1 \mathrm{d}x \left\{ \frac{\mathrm{i}}{(4\pi)^2}\left(\frac{4\pi\tilde{\mu}^2}{f(x)}\right)^{\epsilon} \Gamma(-1+\epsilon)\Big(-b_1 f(x)(2-\epsilon) + b_2(-1+\epsilon)\Big)\right\}. \tag{5.45}$$

Recall from eqs. (5.40) that the b's depend on the momentum p of the amputated external legs, and are themselves defined in terms of the a's, which depend on the dimension d. Recall also that our goal is not to calculate Σ_{A} for an arbitrary value of p, but specifically for $p^2 = m^2$, since it is at this value that we obtain the first-order gravitational correction to the scalar mass. The next step is therefore to set $p^2 = m^2$ and $d = 4-2\epsilon$ in the b's, and expand the whole result in ϵ. Doing so is rather horrible, but eventually yields

$$\begin{aligned}&\left(\frac{4\pi\tilde{\mu}^2}{f(x)}\right)^{\epsilon}\Gamma(-1+\epsilon)\Big(-b_1 f(x)(2-\epsilon) + b_2(-1+\epsilon)\Big)\\ &\qquad = \frac{1}{\epsilon}\Big[2m^4(3x-1)(x-1)\Big] + m^4\left[2(3x-1)(x-1)\ln\left(\frac{4\pi\tilde{\mu}^2 \mathrm{e}^{-\gamma}}{m^2x^2}\right) + x(5x+4)\right] + \mathcal{O}(\epsilon).\end{aligned} \tag{5.46}$$

The integral of the $1/\epsilon$ coefficient over $0 \le x \le 1$ vanishes, while the finite piece integrates to 5, yielding the final result that

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\frac{5\mathrm{i}\kappa^2 m^4}{4(4\pi)^2} = -\frac{5\mathrm{i}Gm^4}{2\pi}, \tag{5.47}$$

using $\kappa = \sqrt{32\pi G}$ in the last step. This is the result obtained in [70], and yields for the first-order correction to the scalar mass

$$\bar{m}^2 = m^2 - \frac{5Gm^4}{2\pi}. \tag{5.48}$$

5.2.3 Harmonic gauge with a dimension-dependent propagator

On its face there is nothing wrong with the calculation above – we put together the diagram from the well-known forms of the scalar and graviton propagators, along with the two-scalar one-graviton vertex which is well known in the literature (e.g. [14–16]), and used the tried and tested methods of Feynman parametrization and dimensional regularization to evaluate the resulting integral, obtaining a result already obtained in [70]. In fact, starting from the integral (5.36), every subsequent step is perfectly valid: the final result *is* the correct evaluation *of that integral.* However I claim that this integral itself is an incorrect starting point!

The problem is that the way in which we evaluate the integral (5.36) is by considering it in an arbitrary dimension $d < 4$, holding $d < 4$ while we do our various manipulations, and only at the very end – once the $1/\epsilon$ divergence has obligingly cancelled out of eq. (5.46) – do we send d back up to 4. However we built the integral (5.36) out of, among other things, the graviton propagator, and the form of the graviton propagator that we used *is only valid in four dimensions!* Indeed, recall from sec. 3.5.3 that in harmonic gauge, $\beta = (d/2) - 1$ and $\alpha = 1/2$, the graviton propagator is

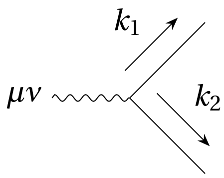
$$\Delta_{\mu\nu\rho\sigma} = \frac{1}{p^2}\left(\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{1}{2}\eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right), \tag{5.49}$$

and only in $d = 4$ dimensions does it obtain its more commonly quoted form (5.32). Thus the analytic continuation to d dimensions above is incomplete: we have taken a physical quantity, the gravitational correction to the scalar mass; found a mathematical expression whose result we hope to identify with this physical quantity; and continued, in an essentially arbitrary fashion, only a *piece* of this mathematical expression from $d = 4$ to $d = 4 - 2\epsilon$ dimensions, leaving the the rest (the graviton propagator) in $d = 4$ the whole time.

Seen in this light, the problem is clear – if dimensional regularization is to be a coherent procedure, it must be applied to entire physical quantities, instead of selectively to arbitrarily-selected pieces of a by no means unique mathematical representation of those physical quantities. Stated more broadly, we should think of the whole process of dimensional regularization as being a deformation of *the entire path integral of the theory* from $d = 4$ to $d = 4 - 2\epsilon$, and hence as a deformation of all physical quantities calculated from that path integral. If we then attempt to perturbatively evaluate some such physical quantity we will eventually, through the usual Taylor expansions and Wick contractions, find ourselves confronted with a graviton propagator, which, being obtained from the arbitrary-d path integral, will itself be in the arbitrary-d form (5.49). And this d-dependence in the propagator can, and as we will soon see *does*, have a nontrivial effect on final results for these physical quantities.

Before proceeding I should note that this subtlety seems to be unique to gravity. In particular the propagators in theories which provide one-loop results which have actually been compared to experiment – by which I mean the scalar, fermion, and vector propagators – do not pick up any d-dependence when evaluated in arbitrary dimensions, and therefore this problem has not reared its head in those fields. I should also note the somewhat disappointing fact that checking against a general gauge does not discriminate between the two forms of the graviton propagator – if the general-gauge calculation of sec. 5.2.4 is carried with $d = 4$ in the graviton propagator, but otherwise maintaining the arbitrary (α, β) gauge, then the gauge parameters do still cancel and yield the result (5.48). Thus I cannot make the claim that maintaining $d = 4$ in the propagator is *demonstrably* wrong; my argument instead rests on the somewhat philosophical points raised above.

With all that being said, let's now see what happens to the calculation of sec. 5.2.2 with the d-dependence restored to the graviton propagator. We use the form (5.49) of the graviton propagator in place of (5.32), but still maintain the two-scalar one-graviton vertex (5.33),



$$= V^{\mu\nu}_{\phi^2 h}(k_1, k_2) = \tfrac{1}{2}\mathrm{i}\kappa\Big(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu}(k_1\cdot k_2)\Big) \equiv \tfrac{1}{2}\mathrm{i}\kappa\,\tilde{V}^{\mu\nu}(k_1, k_2), \tag{5.50}$$

and the usual scalar propagator $D(p) = 1/(p^2 - m^2 + \mathrm{i}\varepsilon)$. These go together to form an integral which looks just like (5.34),

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\tfrac{1}{4}\kappa^2 \int_\ell \frac{T_{\mu\nu\rho\sigma}\tilde{V}^{\mu\nu}(-p, p-\ell)\tilde{V}^{\rho\sigma}(-p+\ell, p)}{(\ell^2 + \mathrm{i}\varepsilon)((p-\ell)^2 - m^2 + \mathrm{i}\varepsilon)}, \tag{5.51}$$

but with the graviton propagator tensor structure $T_{\mu\nu\rho\sigma}$ now carrying the restored d-dependence. We proceed identically as before, introducing a Feynman parameter and shifting the integration momentum, and again obtain a form of the self-energy of the form (5.38),

$$\mathrm{i}\Sigma_{\mathrm{A}} = -\tfrac{1}{4}\kappa^2 \int_0^1 \mathrm{d}x \int_q \frac{a_1 q^2 p^2 + a_2 p^4 (1-x)^2 + a_3 m^2 p^2 (1-x) + a_4 m^4}{(q^2 - f(x) + \mathrm{i}\varepsilon)^2}. \tag{5.52}$$

However, the d-dependence in the propagator rears its head here in the actual values of the a's! Where previously these coefficients were given by eq. (5.39), the new graviton propagator tensor structure actually yields a simpler set of results:

$$a_1 = a_2 = 2, \quad a_3 = 4, \quad a_4 = -\frac{2d}{d-2}. \tag{5.53}$$

This is in fact the *only* material change brought about by restoring the d-dependence to the propagator, but it is a significant one. We can proceed in a formally identical manner from eq. (5.40), in which we define the b's in terms of the a's, to eq. (5.45), in which the momentum integral has been carried out and Σ_{A} waits on the precipice of its expansion in ϵ. It is at this point that we use $p^2 = m^2$ and substitute back in for the b's in terms of the a's, and then substitute in for the a's in terms of d, and then finally expand in $d = 4 - 2\epsilon$. And since the a's now depend on d in a different manner, this expansion yields a different result! As opposed to eq. (5.46), we now find

$$\left(\frac{4\pi\tilde{\mu}^2}{f(x)}\right)^{\epsilon}\Gamma(-1+\epsilon)\Big(-b_1 f(x)(2-\epsilon)+b_2(-1+\epsilon)\Big)$$
$$=\frac{1}{\epsilon}\Big[2m^4(3x-1)(x-1)\Big]+2m^4(x-1)\left[(3x-1)\ln\left(\frac{4\pi\tilde{\mu}^2 \mathrm{e}^{-\gamma}}{m^2x^2}\right)+x+1\right]+\mathcal{O}(\epsilon). \tag{5.54}$$

The $1/\epsilon$ coefficient is unchanged, and therefore still vanishes upon performing the x-integral, which is relieving. But the finite piece is different! In fact, we find the somewhat curious result that the finite piece *also* integrates to zero, meaning that the self-energy diagram vanishes on-shell:

$$\Sigma_{\mathrm{A}}(p^2=m^2)=0. \tag{5.55}$$

And, since as I will show soon in sec. (5.2.4) all the previously-dismissed tadpole diagrams do in fact also vanish, I conclude that the one-loop gravitational correction to the scalar mass vanishes entirely:

$$\bar{m}^2=m^2+0+\mathcal{O}(\kappa^3). \tag{5.56}$$

5.2.4 General gauge

Let's now confirm all of the above calculations by performing them in the parametrized (α,β) gauge and tackling the three diagrams $\Sigma_{\mathrm{B-D}}$ which I initially postponed.

First I will again address the self-energy diagram. The setup is again formally identical to before: we have a graviton propagator and a scalar propagator joined at both ends by the two-scalar one-graviton vertex. The major complication now is that instead of using the simple three-term harmonic gauge propagator for the graviton, we use the full graviton propagator (3.124),

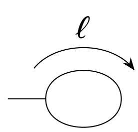
$$\Delta(p)=\frac{1}{p^2}\left\{\Pi_{\perp}+2\alpha\Pi_V-\frac{1}{d-2}\Pi_{\Phi}+\frac{d^2\alpha(d-2)-(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_{\Sigma}-\frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}, \tag{5.57}$$

which there is no simple way to write. For this reason I perform every calculation in this section in MATHEMATICA [50], using the FEYNCALC [59–61] package for the tensor contractions and Passarino-Veltman reduction [78, 79]. Doing so and going on-shell yields

$$\mathrm{i}\Sigma_{\mathrm{A}} = \mathrm{i}\pi^2\kappa^2\left(-\tfrac{1}{2}m^2 A_0(m^2) + \frac{d-3}{d-2}m^4 B_0(m^2,0,m^2)\right), \tag{5.58}$$

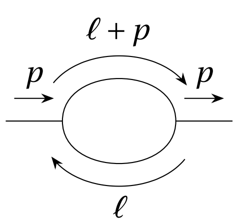
where A_0 and B_0 are Passarino-Veltman scalar integrals, namely the tadpole and bubble respectively. Note that even in just this one diagram all of the gauge parameters cancel! This is not the case *before* going on-shell, which one might expect, since before going on-shell this diagram contributes to the very much not gauge-invariant two-point function of ϕ, but setting $p^2 = m^2$ kills all of the dependence on α and β.

In the result (5.58) I have not set $d = 4-2\epsilon$ and investigated the limit $\epsilon \to 0$. To do so I need actual expressions for the Passarino-Veltman integrals involved, since these also contain poles in ϵ which may (and do!) combine nontrivially with the explicit factors of d. The tadpole is defined by



$$= \mathrm{i}\pi^2 A_0(m^2) = \tilde{\mu}^{2\epsilon}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2-m^2}, \tag{5.59}$$

while the bubble is defined by



$$= \mathrm{i}\pi^2 B_0(p^2,m_1^2,m_2^2) = \tilde{\mu}^{2\epsilon}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2-m_1^2}\frac{1}{(\ell+p)^2-m_2^2}. \tag{5.60}$$

The tadpole can be straightforwardly found to be

$$A_0(m^2) = -\frac{\tilde{\mu}^{2\epsilon}}{\pi^2}\Gamma(\epsilon-1)\frac{(m^2)^{1-\epsilon}}{(4\pi)^{2-\epsilon}} = \frac{m^2}{16\pi^4}\left\{\frac{1}{\epsilon}+1+\ln\left(\frac{\mu^2}{m^2}\right)+\mathcal{O}(\epsilon)\right\}, \tag{5.61}$$

where $\mu^2 = 4\pi\tilde{\mu}^2\mathrm{e}^{-\gamma}$ with γ the Euler-Mascheroni constant, while with our particular combination of arguments the bubble of interest is

$$B_0(m^2, 0, m^2) = \frac{1}{16\pi^4}\left\{\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 2\right\}. \tag{5.62}$$

Also expanding the explicitly d-dependent factor in ϵ yields

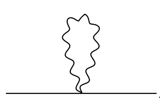
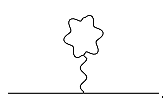
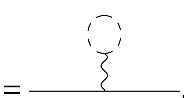
$$\frac{d-3}{d-2} = \frac{1}{2} - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.63}$$

Combining all of the above confirms the previous result that the self-energy diagram vanishes,

$$\Sigma_{\mathrm{A}} = 0, \tag{5.64}$$

this time with the added confidence that the calculation was done without fixing a particular gauge.

Now let's consider the three other diagrams,

$-\mathrm{i}\Sigma_{\mathrm{B}} =$ , $-\mathrm{i}\Sigma_{\mathrm{C}} =$ , $-\mathrm{i}\Sigma_{\mathrm{D}} =$ . (5.65)

I claimed before that these diagrams all vanish. I will now demonstrate this, again with the copious aid of FeynCalc.

First let's look at Σ_{B}. This diagram is in fact totally nonproblematic: the closed graviton loop contracts with the two-scalar two-graviton vertex to give a (lengthy) series of massless tadpoles, all of which vanish upon loop integration in dimensional regularization. To be as careful as possible would could insert a small mass in the graviton propagator (5.57),

$$\Delta(p) = \frac{1}{p^2 - \lambda^2}\left\{\cdots\right\}, \tag{5.66}$$

and investigate the result as $\lambda \to 0$. This makes a lengthy calculation even longer, but the end result – performing the loop integration but leaving the external momentum p arbitrary – is of the form

$$\Sigma_\mathrm{B} = \kappa^2 A_0(\lambda^2)\Big\{\cdots\Big\}, \tag{5.67}$$

where the dots stand for a ~ 60 term sum which depends only on the gauge parameters α and β, the scalar mass m, the dimension d, and the external momentum p. In particular the graviton mass λ appears only in $A_0(\lambda^2)$, and since $A_0(\lambda^2 \to 0) \to 0$ this confirms that the bubble diagram Σ_B vanishes independent of gauge.

The remaining two diagrams proceed similarly. In both cases we do have a closed massless loop, since should vanish on its own by the exact logic above. However also in both cases that massless loop is attached to a zero-momentum massless tail, which, in the absence of an IR regulator, yields the undefined result $0/0$. For these diagrams we are therefore *required* to insert a small graviton mass, whereas we only chose to do so for the bubble out of an abundance of caution. Doing so for the graviton tadpole yields a result of the same form as (5.67), although this result is much shorter and can therefore be written out explicitly:

$$-\mathrm{i}\Sigma_\mathrm{C} = \mathrm{i}\pi^2\kappa^2 A_0(\lambda^2)\left\{p^2\left(\frac{d^2}{8} - \frac{7d}{8} + \frac{3}{4}\right) + m^2\left(-\frac{d^3}{8(d-2)} + \frac{7d^2}{8(d-2)} - \frac{3d}{4(d-2)}\right)\right\}. \tag{5.68}$$

So since the only λ dependence remaining is in $A_0(\lambda^2)$ we can again safely take $\lambda \to 0$ to confirm that $\Sigma_\mathrm{C} = 0$.

Meanwhile doing so for the ghost tadpole is even easier: the small graviton mass allows us to write down an expression for the diagram without dividing by zero, but giving the graviton mass doesn't give the vector ghost a mass, and therefore as soon as we write down the IR-regulated diagram we come across a massless tadpole, which vanishes immediately.

To summarize, we have now shown that all four of the one-loop diagrams which contribute to Σ vanish in an explicitly gauge-invariant manner, and therefore at one loop the gravitational correction to the scalar mass vanishes as well.

5.3 The scalar field two-point function

The final calculation I will present is the two-point function of the invariantized scalar field, evaluated at one loop. This calculation was first attempted in [47]. However I take a different calculational approach, leaning on the computational efficiency of FeynCalc [59–61] and obtain a different result than [47].[32] Also, as discussed in the introduction to this section, I will not worry about any renormalization in this calculation, both because no uncancelled divergences appear for me and because they are irrelevant to my long-term goal of obtaining the long-ranged position-space behavior of these gauge-invariant correlators.

5.3.1 Relevant diagrams

The invariantized scalar field $\hat{\phi}$ is an infinite series in the "plain" scalar field ϕ and the graviton $h_{\mu\nu}$, with the $\mathcal{O}(\kappa^n)$ terms containing one power of ϕ and $n-1$ powers of $h_{\mu\nu}$. To $\mathcal{O}(\kappa^2)$ the invariantized scalar field is given by eq. (2.77),

$$\hat{\phi} = \phi \circ \mathsf{X}^{-1} = \phi - \kappa \mathsf{X}_1^\alpha \partial_\alpha \phi + \kappa^2 \Big(\tfrac{1}{2} \mathsf{X}_1^\alpha \mathsf{X}_1^\beta \partial_\alpha \partial_\beta \phi + \mathsf{X}_1^\alpha \partial_\alpha \mathsf{X}_1^\beta \partial_\beta \phi - \mathsf{X}_2^\alpha \partial_\alpha \phi \Big) + \mathcal{O}(\kappa^3), \tag{5.69}$$

where X_1 and X_2 are given by eqs. (2.20) and (2.23),

$$\mathsf{X}_1^\mu(x) = \int \mathrm{d}^d x'\, G(x,x') J_1^\mu(x'), \quad \mathsf{X}_2^\mu(x) = \int \mathrm{d}^d x'\, G(x,x') \Big(J_2^\mu(x) + K_1 \mathsf{X}_1^\mu(x) \Big), \tag{5.70}$$

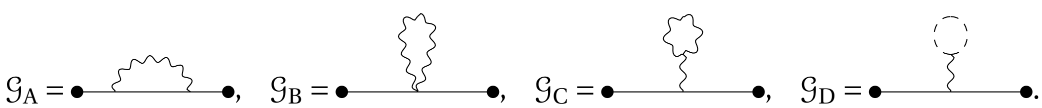
[32] Having worked through [47] in their notation I have tracked our disagreement to a single factor of 2 in the tensor structure they call $I^{4,1}$. Since my calculation is entirely automated and I have rebuilt that automation machinery multiple times in multiple ways to guard against bugs, and I have obtained the same results every time, I am therefore inclined to trust my result. (I have also checked that, if I insert that factor of 2 by hand and then apply my machinery to their work, I do indeed otherwise replicate their result, whereas if I remove that factor of 2 and do the same I obtain mine, which I take both as further evidence that my machinery is bug-free and as my reasoning that this lone factor is the source of our disagreement.)

where $G(x,x')$ is a Green's function of the D'alembertian $\Box = \partial^2$ and the J's and K's are given in turn by eqs. (2.17) and (2.22),

$$J_1^\mu = \partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h, \quad K_1 = h^{\alpha\beta}\partial_\alpha\partial_\beta + J_1^\alpha\partial_\alpha, \quad J_2^\mu = \frac{1}{2}\Big(h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + h^{\alpha\mu}\partial_\alpha h\Big) - \partial_\alpha\Big(h^{\alpha\beta}h_\beta{}^\mu\Big). \tag{5.71}$$

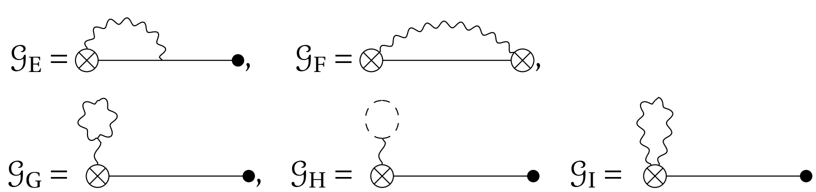
As a result the diagrammatic expansion of $\langle\hat\phi(x)\hat\phi(y)\rangle$ contains not only the "internal" diagrams whose amputated on-shell values contributed to the mass correction of sec. 5.2, but also diagrams containing the external vertices of sec. 4.3.

To completely enumerate the diagrams which could in principle contribute to $\langle\hat\phi(x)\hat\phi(y)\rangle$ we therefore must account for internal vertices which connect two scalars to one and two gravitons; the three-graviton vertex, and the ghost-antighost graviton vertex; and the one-, two- and three-point external insertions arising from eq. (5.69), which connect one scalar to zero, one, and two gravitons respectively. At one loop the possible diagrams are then the unamputated and off-shell versions of the diagrams from eq. (5.2),



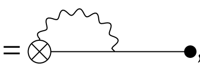
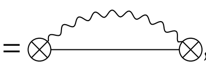
(5.72)

as well as a variety of topologies with coordinate corrections at one or both external points,



(5.73)

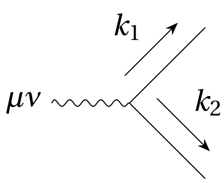
In sec. 5.2.4 I found that the amputated forms of $\mathcal{G}_{\mathrm{B-D}}$, appropriately IR-regulated, vanish in the limit of vanishing graviton mass, before sending the amputated momentum on-shell. Thus the diagrams $\mathcal{G}_{\mathrm{B-D}}$ vanish as well, since the one-point external scalar vertex is trivial and the only new terms are the no-longer-amputated scalar propagators. The diagrams $\mathcal{G}_{\mathrm{G-I}}$ feature identical tadpole structures but now attached to the external vertices; I will in this section presume these diagrams to vanish as well.[33] In this section I will therefore concern myself only with the remaining diagrams, which I'll relabel as

$$\mathcal{G}_1 = $$  $$, \quad \mathcal{G}_2 = $$  $$, \quad \mathcal{G}_3 = $$  $$, \tag{5.74}$$

and which I dub the "sunset", "thorn",[34] and "circle" respectively. The relevant pieces are the scalar and graviton propagators (4.27) and (4.37),

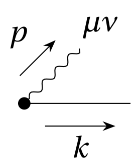
$$D(p) = \frac{1}{p^2 - m^2}, \quad \Delta(p) = \frac{1}{p^2}\left\{\Pi_\perp + 2\alpha\Pi_V - \frac{1}{d-2}\Pi_\Phi + \frac{d^2\alpha(d-2)-(d-1)(1+\beta)^2}{(d-2)(d-1-\beta)^2}\Pi_\Sigma - \frac{(1+\beta)\sqrt{d-1}}{(d-2)(d-1-\beta)}\Pi_{\Phi\Sigma}\right\}; \tag{5.75}$$

the two-scalar one-graviton vertex (4.77),



$$= V^{\mu\nu}_{\phi^2 h}(k_1, k_2) = \tfrac{1}{2}\mathrm{i}\kappa\left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu}(k_1 \cdot k_2)\right). \tag{5.76}$$

and the one-scalar one-graviton external vertex (4.54),



$$= E^{\mu\nu}_{\phi h}(k,p) = -\kappa\frac{1}{p^2}\left(k^\mu p^\nu - \tfrac{1}{2}(k\cdot p)\eta^{\mu\nu}\right). \tag{5.77}$$

5.3.2 Harmonic gauge

As before I will first perform this calculation in harmonic gauge, in which

$$\Delta_{\mu\nu\rho\sigma}(p) = \frac{1}{p^2}\left(\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{1}{2}\eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right). \tag{5.78}$$

[33] I address this more fully in sec. 6.

[34] (for its resemblance to the Icelandic letter of the same name)

Even in this gauge the calculations can become fairly lengthy – for example, the sunset diagram features a $\Delta_{\mu\nu\rho\sigma}$ sandwiched between two $V_{\phi^2 h}$'s, and since each of these has three terms apiece the resulting product has twenty-seven terms. But contracting and simplifying, again using the substitution (5.37), eventually reduces this diagram down to a single scaleless integral,

$$\mathcal{G}_1 = -\frac{\kappa^2}{2p^2}\int_\ell \frac{1}{\ell^2} = 0. \tag{5.79}$$

Similarly the thorn and circle yield

$$\mathcal{G}_2 = \frac{\kappa^2}{2p^2}\int_\ell \frac{(\ell\cdot p)}{\ell^4} = 0, \quad \mathcal{G}_3 = -\frac{\kappa^2}{2}\int_\ell \frac{1}{\ell^4} = 0. \tag{5.80}$$

Thus in this gauge all three diagrams individually vanish, meaning that at one loop the invariantized scalar correlator vanishes.

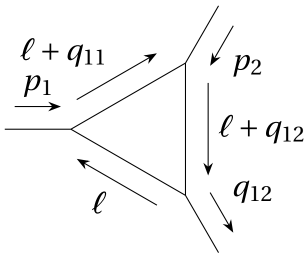
5.3.3 General gauge

Let's now confirm the prior result that the invariantized scalar field two-point function vanishes by performing the calculation in the general parametrized (α,β) gauge. With $\beta \neq (d/2)-1$ the individual diagrams no longer vanish, and in fact themselves become quite lengthy.

To organize the results I'll first observe that each of these diagrams turns out to depend on the same three Passarino-Veltman integrals, namely

$$F_B(p) \equiv B_0(p^2,0,0), \quad F_C(p) \equiv p^2 C_0(0,p^2,p^2,0,0,0), \quad F_D(p) \equiv p^4 D_0(0,0,p^2,p^2,0,p^2,0,0,0,0). \tag{5.81}$$

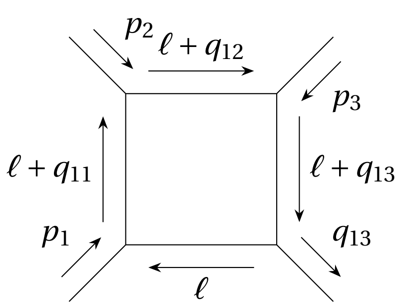
The bubble B_0 is given by (5.60) as before, while the triangle C_0 and box D_0 are given by



$$= \mathrm{i}\pi^2 C_0\left(p_1^2, p_2^2 \middle| q_{12}^2 \middle| m_1^2, m_2^2, m_3^2\right)$$

$$= \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 - m_1^1} \frac{1}{(\ell + q_{11})^2 - m_2^2} \frac{1}{(\ell + q_{12})^2 - m_3^2}, \tag{5.82}$$

and



$$= \mathrm{i}\pi^2 D_0\left(p_1^2, p_2^2, p_3^2 \middle| q_{13}^2, q_{12}^2, q_{23}^2 \middle| m_1^2, m_2^2, m_3^2, m_4^2\right)$$

$$= \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 - m_1^2} \frac{1}{(\ell + q_{11})^2 - m_2^2} \frac{1}{(\ell + q_{12})^2 - m_3^2} \frac{1}{(\ell + q_{13})^2 - m_4^2}, \tag{5.83}$$

with $q_{mn} = \sum_{i=m}^{n} p_i$.

With this shorthand we can compactly provide the diagram results as

$$\mathcal{G}_i(p) = \mathrm{i}\pi^2\kappa^2 \sum_a N_i^a F_a(p), \tag{5.84}$$

where the coefficients N_i^a depend only (and nontrivially) on the gauge parameters. In fact every N_i^a turns out to share a common β- and d-dependent factor as well, which we can factor out:

$$\mathcal{G}_i(p) = \mathrm{i}\pi^2\kappa^2 \frac{d - 2(\beta + 1)}{(d-2)(\beta - d + 1)^2} \sum_a N_i^a F_a(p). \tag{5.85}$$

N.B. this prefactor vanishes when $\beta = (d/2) - 1$, so just the fact that we can write the two-point function as (5.85) (and that, as we will see in the moment, the N's are not singular) demonstrates consistency with the prior result that the two-point function vanishes in harmonic gauge. In fact it also demonstrates that I could have left α arbitrary in the previous calculation and still found that the diagrams all individually vanish.

However it remains to show that the two-point function vanishes for any value of α and β. To see this we need the N's themselves. For the sunset diagram these are

$$\begin{aligned} N_1^B &= \frac{1}{16}\Big\{2(2\alpha-5)(\beta+1)+(3\alpha-5)d^2+d(-2\alpha(\beta+4)+4\beta+15)\Big\}, \\ N_1^C &= -\frac{1}{8}\Big\{4(\alpha-1)(\beta+1)+(3\alpha-2)d^2+d(-2\alpha(\beta+4)+\beta+6)\Big\}, \\ N_1^D &= \frac{1}{16}\Big\{2(2\alpha-1)(\beta+1)+(3\alpha-1)d^2+d(3-2\alpha(\beta+4))\Big\}. \end{aligned} \tag{5.86}$$

For the thorn we have

$$\begin{aligned} N_2^B &= \frac{1}{8}\Big\{-4(\alpha-1)(\beta+1)+(2-3\alpha)d^2+d(2\alpha(\beta+4)-\beta-6)\Big\}, \\ N_2^C &= \frac{1}{8}\Big\{2(4\alpha-1)(\beta+1)-(1-6\alpha)d^2-d(4\alpha(\beta+4)+\beta-3)\Big\}, \\ N_2^D &= \frac{1}{8}\Big\{\beta d-\alpha(d-2)(3d-2(\beta+1))\Big\}. \end{aligned} \tag{5.87}$$

Note that the thorn diagram is multiplied by an extra factor of 2 to account for the fact that there are two such diagrams, corresponding to putting the external insertion at each side. (The two diagrams are guaranteed to be equal by Lorentz invariance.) Finally for the circle the coefficients are

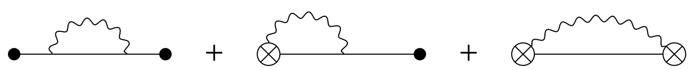
$$\begin{aligned} N_3^B &= \frac{1}{16}\Big\{2(2\alpha+1)(\beta+1)+(3\alpha+1)d^2-d(2\alpha(\beta+4)+2\beta+3)\Big\}, \\ N_3^C &= -\frac{1}{8}\Big\{2(2\alpha+1)(\beta+1)+(3\alpha+1)d^2-d(2\alpha(\beta+4)+2\beta+3)\Big\}, \\ N_3^D &= \frac{1}{16}\Big\{2(2\alpha+1)(\beta+1)+(3\alpha+1)d^2-d(2\alpha(\beta+4)+2\beta+3)\Big\}. \end{aligned} \tag{5.88}$$

So in a general gauge these diagrams certainly do not vanish! However their sum, and hence the invariantized scalar two-point function, does still vanish, independent of the values of α

and β. To see this we don't even need to actually evaluate the Passarino-Veltman integrals – we need only observe that each integral's coefficients all sum to zero separately:

$$\sum_i N_i^a = 0 \quad \text{for all } a. \tag{5.89}$$

Thus we have obtained again my prior result that the invariantized scalar two-point function vanishes, and confirmed that this result is gauge-independent:



$$= \quad 0. \tag{5.90}$$

6 Conclusion

I began this thesis with the goal of obtaining gauge-invariant long-range predictions of general relativity, treated as an effective quantum field theory, to compare with corresponding lattice observables. In sec. 2 I demonstrated how to construct gauge-invariant observables in quantum gravity via the relational approach. In sec. 3 I carefully obtained the propagators relevant to a scalar minimally coupled to Einstein-Hilbert gravity, using a parametrized gauge in order to test the gauge invariance of my results, and in sec. 4 I obtained the rest of the necessary Feynman rules, including the external vertices arising from expressing the invariant observables as perturbative expansions of their original versions. Finally in sec. 5 I applied this machinery to three correlation functions, namely the tree-level two-point function of the volume factor, the one-loop gravitational correction to the mass of a scalar field, and the one-loop gravitational correction to the two-point function of a massless scalar field. In all three cases I found explicit cancellation of the gauge parameters, supporting the viability of the relational scheme for obtaining gauge-invariant correlation functions in quantum gravity.

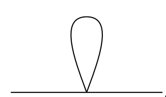
However none of these correlators are the actual object of interest. Rather, as stated in the introduction, it is the two-point function of the Regge curvature which is calculated on the lattice, and therefore it is the two-point function of the invariantized scalar curvature to which we would hope to apply the machinery developed in this thesis. And unfortunately I must report that all such attempts to date have been unsuccessful: while the gauge-fixing prefactor α does cancel out of my results, I have been unable to eliminate the β-dependence.

Out of an abundance of caution I will first and foremost allow for the possibility that there exist bugs or errors of some form in my calculational machinery: the contributing diagrams feature, for example, the three-graviton internal vertex, which is extremely long. However I have automated the calculations of the lengthiest relevant vertices and have done so in multiple ways and obtained the same answers by each route. Further I have compared my vertices to the literature (e.g. [66]) where possible and, while a by-hand comparison of 136-term-long polynomials can only be done with so much confidence, I have not found any disagreements. I have also successfully applied my results to the calculation of the one-loop corrections to the Newtonian potential in harmonic gauge and found agreement with [16], which gives me some further confidence that my machinery is free of bugs. Rather I conjecture that the missing piece is something more subtle than a typo somewhere in a MATHEMATICA notebook.

I conjecture this for two reasons, with which I will conclude my thesis.

Subtleties in the IR regulation of tadpoles attached to coordinate correction vertices

First, in my attempts at $\langle \mathscr{R}(x)\mathscr{R}(y)\rangle$ I have assumed that all tadpole diagrams vanish. This is certainly true for the internal bubble,



(6.1)

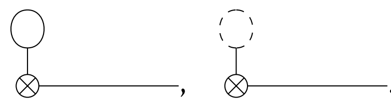
while the internal tadpoles

(6.2)

can be IR regulated in the usual way to deform from the apparent $0/0$ before sending the IR regulator to zero. Similarly the external bubbles and tadpoles

(6.3)

should vanish in the usual way when the external vertices are the local ones which arise from the standard expansion of the scalar curvature. However it is unclear what to do about the external tadpoles when the external vertex is the coordinate correction:



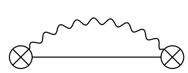
(6.4)

In both cases we would certainly want to again insert an IR regulator λ to move away from the undefined $0/0$ behavior before returning to the massless limit $\lambda \to 0$. However the wrinkle is that the coordinate correction vertices (4.66) themselves contain inverse powers of the momentum, coming directly from the Green function in (2.20) which inverts the Laplacian in (2.16). These factors of $1/p^2$ take the momentum of the graviton leg which attaches to the vertex, which for the external tadpoles certainly have momentum zero.[35] However these factors of $1/p^2$ are not themselves propagators of the graviton, and there is therefore no a priori reason to insert any fictitious graviton mass in these factors, even when the identical-looking factors which do arise from the graviton propagators do get such a mass. Altering these Green functions would instead constitute a deformation of the defining condition (2.10) of the coordinate scalars themselves, and it is not at all obvious that this deformation is well-defined. In particular if we insert a graviton mass in this factor then we are altering the background Laplacian which appears in eq. (2.16). But this background Laplacian is not the only piece of the defining condition (2.10) which appears in this vertex – the tensor structure of the external vertex is due to the structure of J_1^μ (2.17), and it is not at all obvious either that altering the

[35]The external bubble avoids this issue since the graviton legs attached to the vertex have nonzero momenta $\pm\ell$.

one without the other maintains the coordinate invariance of the definition (2.10), or if one chooses to also alter J_1^μ, how such an alteration should be done. Whatever the resolution to this dilemma it is certainly a possibility that these diagrams do not all simply vanish on their own, and that the remaining gauge dependence in the diagrams I have evaluated and summed is cancelled in this way.

In fact I should note that this difficulty also arises even in the scalar field two-point function when considering the bottom three diagrams of eq. (5.73). In sec. 5 I simply set all such diagrams to zero, and maintained a level of confidence in doing so due to the eventual gauge-invariance of my result. However a more careful analysis, featuring IR regulation of the zero-momentum graviton lines, would certainly be desirable, and it is not obvious how this should be performed. In fact there is strange IR behavior even in one of the diagrams I explicitly addressed above. If one attempts to calculate the three diagrams in (5.74) in harmonic gauge but with an IR regulator λ, the first two (the sunset and the thorn) can be successfully taken to zero, in agreement with the unregulated results in (5.79) and the first of (5.80). However, in disagreement with the second of (5.80), the IR regulated circle diagram does not return to zero in the limit $\lambda \to 0$: instead, one obtains upon tensor reduction



$$\sim \frac{A_0(\lambda^2)}{\lambda^2}, \tag{6.5}$$

which diverges. The origin of this strange IR behavior is not clear, but its resolution likely lies in a nontrivial IR cancellation with one of the other diagrams whose subtleties I have discussed.

The measure and the Euclidean continuation

Even if I didn't care about IR regulation for the reasons outlined above, I would likely have to tackle it if I hoped to implement the Euclidean contination. As previously mentioned the Euclidean continuation of Einstein-Hilbert gravity is made subtle by the wrong-sign kinetic term of the scalar mode Φ, and any attempt to make contact between my Minkowski-space calculations and the Euclidean lattice will have to surmount this difficulty. It is likely that any such technique will involve analytically continuing the Φ propagators which appear in loop integrals differently than the loop propagators of other fields. One possibility for distinguishing Φ's from the TT and gauge modes in loops is to give a small mass to only the Φ piece of the graviton propagator, since such a mass would "follow" the Φ propagator through any tensor reduction process, allowing one to automate the reduction while keeping track of the Φ contributions for later continuation. However if such a procedure is well-defined it should also yield the already-established Minkowski-space results if one does not apply any experimental new continuation but simply inserts the Φ mass at the start of the calculation and then sends that mass to zero at the end. That the IR regulation of the invariantized scalar two-point function is not yet understood therefore presents an obstacle to any such attempt at a Euclidean continuation.

The question of the Eucliean continuation of the conformal mode was addressed in [39], in which it was proposed that the resolution to the problem lies in a nonlocal field redefinition of the conformal mode by $\chi = \sqrt{-\nabla^2}\Phi$; by analytically continuing χ, instead of Φ, the Euclidean path integral is rendered convergent. This field redefinition arises from explicitly integrating out the gauge degrees of freedom from the path integral, which leaves behind a Jacobean factor in the gravitational measure whose effect is to convert the functional integral over Φ into a functional integral over χ. Not only does this result indicate that a correct Euclidean continuation, and hence a correct identification of predictions of the low-energy effective theory with lattice calculations, must take into account this Jacobean factor – it may in fact be the case that to perform correct gravitational loop calculations, *even in Minkowski space*, one must incorporate this Jacobean factor into the measure, since, to quote the authors, "the correct Euclidean continuation depends on the correct functional measure".

To see why this subtlety in the Euclidean continuation could affect Minkowski-space results let's recall (see e.g. [71]) that, in the path integral formulation, the pole structure of the standard propagator arises from the issue of actually defining the path integral. In Minkowski space and with a purely real action the free path integral is a divergent purely imaginary Gaussian,

$$Z \sim \int \mathscr{D}\phi \mathrm{e}^{\mathrm{i}\int_k k^2\phi^2}. \tag{6.6}$$

To make the path integral convergent the action is given a small imaginary part,

$$Z \sim \int \mathscr{D}\phi \mathrm{e}^{\mathrm{i}\int_k (k^2+\mathrm{i}\varepsilon)\phi^2} \sim \int \mathscr{D}\phi \mathrm{e}^{\mathrm{i}\int_k k^2\phi^2} \mathrm{e}^{-\varepsilon\int_k \phi^2}, \tag{6.7}$$

which if $\varepsilon > 0$ provides a real Gaussian factor with which the path integral converges. One then calculates whatever quantity one wishes in perturbation theory and sends $\varepsilon \to 0$ at the end, obtaining well-defined results. The key thing is that this iε is precisely the same iε which appears in the propagator,

$$D(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}, \tag{6.8}$$

which in turn determines the location of the poles of D. And the location of these poles governs the actual calculation of $D(k)$, and of all the scattering amplitudes, correlation functions, loop integrals, etc. constructed out of it.

The above indicates the problem posed by a wrong-sign kinetic term: in order to define a convergent path integral with a wrong-sign kinetic term, the action must be deformed in the opposite direction, i.e. via $k^2 \mapsto k^2 - \mathrm{i}\varepsilon$. It follows that, in fact, any loop calculation featuring a field with a wrong-sign kinetic term should not be performed under the assumption that all propagators have a standard pole structure, since the propagator of such a field should have a denominator of the form $k^2 - \mathrm{i}\varepsilon$ instead of $k^2 + \mathrm{i}\varepsilon$. This issue seems to be generally ignored in the literature, and I have followed suit in my work, in no small part because I do not know of any tools by which to implement this nonstandard pole structure in an automated loop calculation. However it seems at the very least possible that the challenges of obtaining sensible gauge-invariant loop results for gravitational correlators stems from this very oversight.

That this is the source of my difficulties is also supported by the pattern of my successes and failures to date. Thus far I have successfully calculated the three correlators discussed in sec. 5, and I have also had success in a variety of harmonic gauge calculations with which I have tested my machinery against the literature. By contrast the two calculations attempted thus far in a general gauge and which feature potentially nonvanishing ghost loops have failed.[36] In other words I have so far been challenged by precisely those calculations in which the gravitational measure is manifestly relevant in the form of ghosts, and as argued in [39], incorrectly continuing the wrong-sign field amounts to a misidentification of the gravitational measure itself. Thus it might be hoped that, if one incorporates the results of [39] to correctly define the gravitational measure, then the loop calculations performed and attempted to date might yield different and more well-behaved results, and further that the correct method of comparison of those results to the Euclidean lattice will become more apparent. However significant challenges remain in this direction, including but not limited to the actual calculation of the Passarino-Veltman integrals with this altered pole structure and the automation of gravitational loop calculations featuring these nonstandard propagators.

[36] The other is the one-loop contribution to $\langle \sqrt{-\det \mathscr{G}(x)} \sqrt{-\det \mathscr{G}(y)} \rangle$. This calculation involves the three-point coordinate correction to $\sqrt{-\det \mathscr{G}}$, which features X_2, which is both lengthy and resistant to my attempts at automation, and so I am less confident that my attempts at this result are bug-free.

A Coordinate transformations

This appendix arose out of a set of personal notes which I maintained to keep clear the precise nature of the geometric objects involved in the relational approach. I include it here so that I can reference it where necessary in the main text, and to provide a detailed exposition of the often very dense notation I employ. In compiling these notes I mostly referenced [63, 65] for mathematical exposition.

A.1 Differentials, pushforwards, and pullbacks

In this section I review some machinery which will be useful for our discussion of coordinate transformations, wfhich will begin in earnest in sec. A.2.

Throughout this section we will consider two manifolds M and M', with a generic smooth map between them denoted by $F : M \to M'$. All objects on M' are denoted with a prime in order to keep it notationally clear where each objects lives. Otherwise I abide by the conventions laid out in sec. 1.

A.1.1 Pointwise operations: the differential and the pointwise pullback

Let $F : M \to M'$ be a smooth map between smooth manifolds. Given any $p \in M$ the map F defines a linear map between the tangent spaces T_pM and $T_{F(p)}M'$, called the *differential* or *pointwise pushforward*, as follows:

$$\mathrm{d}F_p : T_pM \to T_{F(p)}M', \quad \left(\mathrm{d}F_p\, v\right) f' = v(f' \circ F) \tag{A.1}$$

for any $f' \in C^\infty(M')$. Keep in mind that the differential maps a single vector at $p \in M$ to another single vector at $F(p) \in M'$ – we're not saying anything about vector fields yet.

Since the differential of $F : M \to M'$ at p is a linear map between vector spaces it certainly has a dual,[37] called the *pointwise pullback*:

$$\mathrm{d}F_p^* : T_{F(p)}M' \to T_pM, \quad \left(\mathrm{d}F_p^*\,\omega'\right) v = \omega'\left(\mathrm{d}F_p\, v\right). \tag{A.2}$$

So the differential $\mathrm{d}F_p$ moves vectors at $p \in M$ to $F(p) \in M'$, while the pointwise pullback $\mathrm{d}F_p^*$ moves one-forms from $F(p) \in M'$ to $p \in M$.[38]

This definition of the pointwise pullback generalizes straightforwardly to the pointwise pullback of covariant tensors of arbitrary rank:

$$B' \in \left(T_k\right)_{F(p)} M' \implies \left(\mathrm{d}F_p^*\, B'\right)(u, \ldots, v) = B'\left(\mathrm{d}F_p\, u, \ldots, \mathrm{d}F_p\, v\right). \tag{A.3}$$

To see the corresponding generalization of the pushforward observe first that we can obtain a relation for the pointwise pushforward which mirrors the definition (A.2) of the pointwise pullback as follows. Recall that, given any vector v and one-form ω, the vector v may be defined to act on the one-form ω by the action of ω on v, i.e. $v\omega = \omega v$. Acting $\mathrm{d}F_p\, v \in T_{F(p)}M'$ on $\omega' \in T^*_{F(p)}M'$ in this way then yields

$$\left(\mathrm{d}F_p\, v\right)\omega' = \omega'\left(\mathrm{d}F_p\, v\right) = \left(\mathrm{d}F_p^*\,\omega'\right) v = v\left(\mathrm{d}F_p^*\,\omega'\right). \tag{A.4}$$

This last equality, $(\mathrm{d}F_p\, v)\omega' = v(\mathrm{d}F_p\,\omega')$, then generalizes to contravariant tensors straightforwardly:

$$A \in \left(T^k\right)_p M \implies \left(\mathrm{d}F_p\, A\right)\left(\omega', \ldots, \eta'\right) = A\left(\mathrm{d}F_p^*\,\omega', \ldots, \mathrm{d}F_p^*\,\eta'\right). \tag{A.5}$$

To keep clear the logical progression here: we first define the pointwise pushforward of vectors via the explicit action of vectors on functions; we then obtain the pointwise pullback of one-forms from the pointwise pushforward of vectors; and then one we know how to push and pull

[37] (the dual of a linear map $A : V \to V'$ being the map $A^* : V'^* \to V^*$ given by $(A^*\omega')v = \omega'(Av)$)

[38] When F is invertible I'll sometimes refer to the pullback of a one-form by F^{-1} as its "pushforward" by F. This will mostly come up when we're simultaneously pushing forward vectors by F and pulling back one-forms by F^{-1}, just as a vague gesture in the direction of concision.

vectors and one-forms around we can do the same for arbitrary-rank tensors by pushing and pulling their arguments. Further, if F is a diffeomorphism, we can define the pullback[39] of a mixed-rank tensor $C' \in \left(T^k_\ell\right)_{F(p)} M'$:

$$\left(\mathrm{d}F^*_p\, C'\right)(\omega,\ldots,\eta \,|\, u,\ldots,v) = C'\Big(\mathrm{d}\big(F^{-1}\big)^*_{F(p)}\,\omega,\ldots,\mathrm{d}\big(F^{-1}\big)^*_{F(p)}\,\eta \,\Big|\, \mathrm{d}F_p\, u,\ldots,\mathrm{d}F_p\, v\Big). \tag{A.6}$$

In other words, the pullback of C' by F acts on a collection of one-forms and vectors on M in the same way that C' acts on the pullbacks of the one-forms by F^{-1} and the pushforwards of the vectors by F.

Finally I will note that all the pushforwards and pullbacks defined above may all be straightforwardly shown to be (multi)linear, considering every tensor space acted on above as a vector space over the real numbers.

A.1.2 Maps of fields: the pushforward and the pullback

Now suppose that, instead of a single vector at a point in M, we have a vector field $v \in \mathfrak{X}(M)$. Then we could certainly pick any point $p \in M$, evaluate v_p, and apply $\mathrm{d}F_p$ to v_p to obtain a vector in $T_{F(p)}M'$. However we can *not* necessarily in this way obtain a vector field on M'. To see this note that if F is not surjective then there exists a point in M' to which F assigns no point in M, and hence which this process does not map to any vector in its tangent space; and if F is not injective then there exists a point in M' to which F assigns multiple points in M, and hence which this process may map to multiple vectors in its tangent space.

However if F is a diffeomorphism then we *are* guaranteed that the differential of F maps a vector field on M to a vector field on M', which we can define by

$$p' \in M' \mapsto \mathrm{d}F_{F^{-1}(p')}\, v_{F^{-1}(p)}. \tag{A.7}$$

[39]Note that there isn't any distinct notion of the pushforward of a mixed tensor – we could just as easily refer to the about-to-be-defined operator as the pushforward of C by F^{-1} (which is indeed what the definition reduces to in the case where C is fully contravariant).

In other words, for each $p' \in M'$ we travel back to $F^{-1}(p') \in M$, evaluate v at that point, and use the differential of F to push that vector to M'. We can therefore use F to define a map not just between the individual tangent spaces of M and M', but between the spaces of vector fields:

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(M'), \quad (F_* v)_{p'} = \mathrm{d}F_{F^{-1}(p')}\, v_{F^{-1}(p')}. \tag{A.8}$$

We call $F_* v$ the *pushforward* of v by F.

The pullback is a little less subtle, in that F need not be a diffeomorphism: for any smooth $F : M \to M'$ we can use the pointwise pullback to define a map of one-form fields by

$$F^* : \mathfrak{X}^*(M') \to \mathfrak{X}^*(M), \quad \left(F^* \omega'\right)_p = \mathrm{d}F_p^*\, \omega'_{F(p)}. \tag{A.9}$$

We call $F^* \omega'$ the *pullback* of ω' by F.

Finally we can in a precisely analogous way define the pullbacks and pushforwards of tensor fields of arbitrary rank:

$$\begin{aligned} A \in \Gamma^k M &\implies \left(F_* A\right)_{p'} = \mathrm{d}F_{F^{-1}(p')}\, A_{F^{-1}(p')}, \\ B' \in \Gamma_\ell M' &\implies \left(F^* B'\right)_p = \mathrm{d}F_p^*\, B'_{F(p)}, \\ C' \in \Gamma_\ell^k M' &\implies \left(F^* C'\right)_p = \mathrm{d}F_p^*\, C'_{F(p)}. \end{aligned} \tag{A.10}$$

As before the pushforward is defined only when F is as diffeomorphism (as can be seen by the explicit reference to F^{-1} in the definitions), while the pullback is defined for any smooth F. Also as before every map defined in this section is (multi)linear, this time considering the relevant section space $\Gamma_\ell^k M$ to be a module over the ring of smooth functions $C^\infty(M)$ (or identically for M' in the case of the pullback).

A.1.3 Basis frames and coframes

A nice example of the use of the above machinery is in the abstract definition of the frame and coframe defined by a coordinate system, namely that the frame defined by a coordinate system x is the pushforward by x^{-1} of the canonical coordinate frame on Euclidean space and the coframe defined by x is simply the differentials of the component functions x^μ.[40]

Let's start with the coordinate frame. Consider some coordinate system $\mathsf{x} : U \subseteq M \to \mathbb{R}^d$. For any $x \in \mathbb{R}^d$ the tangent space to $\mathbb{R}^d$ at x is isomorphic to $\mathbb{R}^d$, and the canonical basis for this tangent space is the set of partial derivatives $\partial/\partial x^\mu\,|_x$ at this point with respect to the canonical coordinates $\{x^1, \ldots, x^d\}$ on $\mathbb{R}^d$. Further, the coordinate system x is by definition a diffeomorphism between $U \subseteq M$ and its image $\mathsf{x}(U) \subseteq \mathbb{R}^d$, and therefore at any point $p \in U$ the differential $\mathrm{d}\mathsf{x}_p$ is an isomorphism between the tangent spaces T_pM and $T_{\mathsf{x}(p)}\mathbb{R}^d \cong \mathbb{R}^d$. Thus the preimages of the basis vectors $\partial/\partial x^\mu\,|_{\mathsf{x}(p)}$ for $T_{\mathsf{x}(p)}\mathbb{R}^d$ under $\mathrm{d}\mathsf{x}_p$ provide a basis for T_pM. This is the familiar coordinate basis[41]

$$\partial_\mu\big|_p = \mathrm{d}\big(\mathsf{x}^{-1}\big)_{\mathsf{x}(p)} \frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{x}(p)}, \tag{A.11}$$

and it defines the coordinate frame

$$\partial_\mu = \big(\mathsf{x}^{-1}\big)_* \frac{\partial}{\partial x^\mu}. \tag{A.12}$$

To see that this object is in fact the familiar coordinate partial derivative consider acting it on some $f \in C^\infty(M)$ at some $p \in U$, whose coordinate we will for convenience denote by $\mathsf{x}(p) = x$. Then we have

$$(\partial_\mu f)_p = \left(\mathrm{d}\big(\mathsf{x}^{-1}\big)_x \frac{\partial}{\partial x^\mu}\bigg|_x\right) f = \frac{\partial\big(f \circ \mathsf{x}^{-1}\big)}{\partial x^\mu}\bigg|_x. \tag{A.13}$$

And the composition being differentiated, $f \circ \mathsf{x}^{-1}$, is just the coordinate representation of f in

[40] Proofs of the claims I make in this section may be found in most any text on differential geometry, e.g. [63, 65].

[41] That $(\mathrm{d}\mathsf{x}_p)^{-1} = \mathrm{d}(\mathsf{x}^{-1})_{\mathsf{x}(p)}$ follows directly from Lemma A.2 in the next section.

the coordinate system x: while f eats a point $p \in M$ and returns a number $f(p)$, the composition $f \circ \mathsf{x}^{-1}$ eats the coordinates $x = \mathsf{x}(p)$ and returns the number $f \circ \mathsf{x}^{-1}(x) = f(p)$. Thus acting $\partial_\mu\big|_p$ on f yields the partial derivative with respect to x^μ of the coordinate representation of f, which is precisely what it should do.

The pushforward may also be used to define the coordinate coframe. To see this consider some function $f \in C^\infty(M)$, which is itself a smooth map between manifolds $f : M \to \mathbb{R}$. The differential $\mathrm{d}f_p$ at some $p \in M$ is then a linear map from T_pM to $T_{f(p)}\mathbb{R} \cong \mathbb{R}$, and is therefore a one-form at p:

$$f \in C^\infty(M) \implies \mathrm{d}f_p \in T_p^*M \text{ for all } p \in M. \tag{A.14}$$

This holds in particular for the component functions $\mathsf{x}^\mu : U \subseteq M \to \mathbb{R}$ of our coordinate system x, and it may be shown that $\mathrm{d}\mathsf{x}^\mu\big|_p$ is the basis for T_p^*M dual to the basis $\partial_\mu\big|_p$ for T_pM, in terms of which the differential of any other function may be written

$$\mathrm{d}f_p = \left(\partial_\mu f\right)_p \mathrm{d}\mathsf{x}^\mu\big|_p. \tag{A.15}$$

The above also translates directly to the standard statements about one-form fields, namely that the map $\mathrm{d}f : p \mapsto \mathrm{d}f_p$ is a one-form field,

$$\mathrm{d}f \in \mathfrak{X}^*(M), \tag{A.16}$$

and the maps $\mathrm{d}\mathsf{x}^\mu \in \mathfrak{X}^*(M)$ form the coframe dual to ∂_μ, in terms of which

$$\mathrm{d}f = \left(\partial_\mu f\right) \mathrm{d}\mathsf{x}^\mu. \tag{A.17}$$

A.1.4 The tensor transformation law

In this section I provide a derivation and statement of the standard rule for the transformation of the components of a tensor under a diffeomorphism. In the interest of readability I'll first summarize the results.

- In Lemma A.1 I write the linearity of the pushforward and pullback in terms of functions,

$$F_*(fv) = fF_*v \quad \text{and} \quad F^*(f'\omega') = (f' \circ F)F^*\omega', \tag{A.18}$$

and in Lemma A.2 I show their composition rules,

$$(F' \circ F)_* = F'_* \circ F_* \quad \text{and} \quad (F' \circ F)^* = F^* \circ F'^*. \tag{A.19}$$

- In Theorems A.1 and A.2 I demonstrate that the pushforward of a vector and the pullback of a one-form by a diffeomorphism $F : M \to M'$ can both be written in coordinates in terms of the matrix of partial derivatives of F:

$$(F_* v)^\mu = \left(\partial_\nu F^\mu v^\nu\right) \circ F^{-1}, \quad \text{and} \quad (F^*\omega')_\mu \quad = \left(\partial_\mu F^\nu\right)(\omega'_\nu \circ F) \tag{A.20}$$

- Finally in Theorem A.3 I apply the previous theorems to obtain the rule by which the pullback of a generic mixed-rank tensor may be written in coordinates in terms of that same matrix of partial derivatives:

$$\left(\mathrm{d}F^*_p C'\right)^\mu{}_\nu = \left(\partial'_\alpha \left(F^{-1}\right)^\mu\right)_{F(p)} \left(\partial_\nu F^\beta\right)_p C'^\alpha{}_\beta. \tag{A.21}$$

This final theorem is, in a sense, the only point of this subsection, in that it is the precise statement of the familiar tensor transformation law

$$C^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\nu} C'^\alpha{}_\beta \tag{A.22}$$

with which we will be concerned in our coming discussion of passive and active transformations.

Lemma A.1. LINEARITY OF PUSHFORWARDS AND PULLBACKS.

Let $F : M \to M'$ *be a diffeomorphism;* $v \in \mathfrak{X}(M)$ *and* $\omega' \in \mathfrak{X}^*(M')$*; and* $f \in C^\infty(M)$ *and* $f' \in C^\infty(M')$*. Then* $F_*(fv) = fF_*v$ *and* $F^*(f'\omega') = (f' \circ F)F^*\omega'$*, the latter also holding if* F *is smooth but not a diffeomorphism.*

Proof. First note that the linearity of the pointwise pushforward follows from its definition: $\mathrm{d}F_p(av)f = av(f \circ F) = a\,\mathrm{d}F_p\,v$ for any $v \in T_pM$, $f \in C^\infty(M)$, and $a \in \mathbb{R}$; and the linearity of the pullback is then guaranteed by construction, since the dual of any linear map is also linear.

The given forms, $F_*(fv) = fF_*v$ and $F^*(f'\omega') = (f' \circ F)F^*\omega'$, for the linearity of the push-forwards and pullbacks of fields now follow straightforwardly from pointwise evaluations. For the pushforward we have

$$\big(F_*(fv)\big)_p = \mathrm{d}F_p\big(f(p)v_p\big) = f(p)\,\mathrm{d}F_p\,v_p = \big(fF_*v\big)_p. \tag{A.23}$$

The first step uses the fact that the scalar multiplication $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which provides the module structure of $\mathfrak{X}(M)$ is defined pointwise, $(fv)_p = f(p)v_p$, and the second uses the fact that for any $f \in C^\infty(M)$ the value $f(p)$ is just a real number and may therefore be pulled out of the linear map $\mathrm{d}F_p$. Similarly for the pullback

$$\big(F^*(f'\omega')\big)_p = \mathrm{d}F_p^*\Big(f'\big(F(p)\big)\,\omega'_{F(p)}\Big) = f'\big(F(p)\big)\,\mathrm{d}F_p^*\,\omega'_{F(p)} = \big((f' \circ F)F^*\omega'\big)_p, \tag{A.24}$$

by the exact same logic. □

Lemma A.2. COMPOSITION OF PUSHFORWARDS AND PULLBACKS.

Let $F : M \to M'$ *and* $F' : M' \to M''$ *be smooth. Then for any* $p \in M$ *we have* $\mathrm{d}(F' \circ F)_p = \mathrm{d}F'_{F(p)} \circ \mathrm{d}F_p$ *and* $\mathrm{d}(F' \circ F)_p^* = \mathrm{d}F_p^* \circ \mathrm{d}F'^*_{F(p)}$*. These imply* $(F' \circ F)_* = F'_* \circ F_*$ *and* $(F' \circ F)^* = F^* \circ F'^*$*, the former holding only if* F *and* F' *are diffeomorphisms.*

Proof. We need prove only the pointwise statements, since the others follow immediately.

Let's begin with the pushforward. Given any $v \in T_pM$ the quantity $\mathrm{d}(F' \circ F)_p\,v$ is a vector on M'' at $F' \circ F(p)$ and may therefore act on any $f'' \in C^\infty(M'')$. Unraveling the definition of the pushforward we find

$$\big(\mathrm{d}(F' \circ F)_p\, v\big)\big(f''\big) = v\big(f'' \circ F' \circ F\big) = \big(\mathrm{d}F_p\, v\big)\big(f'' \circ F'\big) = \big(\mathrm{d}F'_{F(p)} \circ \mathrm{d}F_p\, v\big)\big(f''\big). \tag{A.25}$$

Similarly for the pullback, given any $\omega'' \in T_{F' \circ F(p)} M''$ the quantity $\mathrm{d}(F' \circ F)^*_p\, \omega''$ is a one-form on M at p and may therefore act on any $v \in T_p M$:

$$\big(\mathrm{d}(F' \circ F)^*_p\, \omega''\big)\, v = \omega''\big(\mathrm{d}(F' \circ F)_p\, v\big) = \omega''\Big(\mathrm{d}F'_{F(p)} \circ \mathrm{d}F_p\, v\Big) = \mathrm{d}F'^*_{F(p)}\, \omega''\big(\mathrm{d}F_p\, v\big) = \big(\mathrm{d}F^*_p \circ \mathrm{d}F'^*_{F(p)}\, \omega''\big)\, v. \tag{A.26}$$

And in the above v, ω'' and f'' are all arbitrary, so these hold as identities of pointwise pushforwards and pullbacks respectively, as desired. □

Theorem A.1. THE MATRIX REPRESENTATION OF THE PUSHFORWARD OF A VECTOR.

Let $F : M \to M'$ be smooth and let $\mathsf{x} : M \to \mathbb{R}^d$ and $\mathsf{x}' : M' \to \mathbb{R}^{d'}$ be coordinate systems on the domain and codomain with coordinate frames ∂_μ and ∂'_μ respectively. Then for any $p \in M$ and $v \in T_p M$ we can expand $\mathrm{d}F_p\, v$ in the coordinate frame of x' as $\big(\mathrm{d}F_p\, v\big)^\mu = (\partial_\nu F^\mu)_p\, v^\nu$, where $F^\mu = \mathsf{x}'^\mu \circ F$. If instead $v \in \mathfrak{X}(M)$ and F is a diffeomorphism then we have $(F_* v)^\mu = (\partial_\nu F^\mu v^\nu) \circ F^{-1}$.

Proof. We will prove the above by demonstrating that

$$\mathrm{d}F_p\, \big(\partial_\mu|_p\big) = \big(\partial_\mu F^\nu\big)_p \big(\partial'_\nu|_{F(p)}\big), \tag{A.27}$$

since given eq. (A.27) the first claim follows immediately from expanding v as $v = v^\mu \partial_\mu|_p$, and the second claim is just the first in the case of a vector field. Further we will demonstrate eq. (A.27) in steps as follows.

- THE DIFFERENTIAL OF THE COORDINATE REPRESENTATION OF THE MAP. Denote by $\hat{F} = \mathsf{x}' \circ F \circ \mathsf{x}^{-1} : \mathsf{x}(M) \to \mathsf{x}'(M')$ the coordinate representation of F. In other words, if F eats a point $p \in M$ and returns $F(p) \in M'$, then $\hat{F}$ eats the coordinates of p in x and returns the coordinates of $F(p)$ in x'.

Applying the differential of $\hat{F}$ to a canonical basis vector on $\mathbb{R}^d$ and applying the chain rule of multivariable calculus we have

$$\mathrm{d}\hat{F}_x\left(\frac{\partial}{\partial x^\mu}\bigg|_x\right)f' = \frac{\partial\big(f'\circ\hat{F}\big)}{\partial x^\mu}(x) = \frac{\partial f'}{\partial x'^\nu}\big(\hat{F}(x)\big)\frac{\partial \hat{F}^\nu}{\partial x^\mu}(x), \tag{A.28}$$

with $f' \in C^\infty\big(\mathbb{R}^{d'}\big)$ and denoting by x and x' the canonical coordinates on $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively (and, in what could pedantically be considered a slight abuse of notation, also using x to denote an arbitrary point in $\mathbb{R}^d$). Thus the differential of the coordinate representation of F is represented by the matrix $\partial\hat{F}^\nu/\partial x^\mu$:

$$\mathrm{d}\hat{F}_x\left(\frac{\partial}{\partial x^\mu}\bigg|_x\right) = \frac{\partial \hat{F}^\nu}{\partial x^\mu}(x)\frac{\partial}{\partial x'^\nu}\bigg|_{\hat{F}(x)}. \tag{A.29}$$

- Simplifying the matrix. Suppose that instead of just evaluating the matrix of partial derivatives $\partial\hat{F}/\partial x^\mu$ at some $x \in \mathbb{R}^d$ we choose a point $p \in M$ and evaluate the matrix at its coordinates $\mathsf{x}(p)$. We then have, recalling the definition of the coordinate frame and that $F^\nu = \mathsf{x}'^\nu \circ F$,

$$\frac{\partial \hat{F}^\nu}{\partial x^\mu}\big(\mathsf{x}(p)\big) = \frac{\partial\big(\mathsf{x}'^\nu\circ F\circ \mathsf{x}^{-1}\big)}{\partial x^\mu}\big(\mathsf{x}(p)\big) = \mathrm{d}\big(\mathsf{x}^{-1}\big)_{\mathsf{x}(p)}\left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{x}(p)}\right)F^\nu = \big(\partial_\mu F^\nu\big)_p. \tag{A.30}$$

  So we can rewrite our previous result in this instance as

$$\mathrm{d}\hat{F}_{\mathsf{x}(p)}\left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{x}(p)}\right) = \big(\partial_\mu F^\nu\big)_p\frac{\partial}{\partial x'^\nu}\bigg|_{\mathsf{x}'\circ F(p)}. \tag{A.31}$$

- The differential of the map itself. Now consider the actual object of interest. Using the definition of the coordinate frame and Lemma A.2 we have

$$\mathrm{d}F_p\left(\partial_\mu|_p\right) = \mathrm{d}\left(F\circ \mathrm{x}^{-1}\right)_{\mathrm{x}(p)}\left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathrm{x}(p)}\right). \tag{A.32}$$

Since the coordinate representation of F is defined by $\hat{F} = \mathrm{x}'\circ F\circ \mathrm{x}^{-1}$ we have $F\circ \mathrm{x}^{-1} = \mathrm{x}'^{-1}\circ \hat{F}$, so from our previous result and again using Lemma A.2 this becomes

$$\mathrm{d}F_p\left(\partial_\mu|_p\right) = \left(\partial_\mu F^\nu\right)_p \mathrm{d}\left(\mathrm{x}'^{-1}\right)_{\mathrm{x}'\circ F(p)}\left(\frac{\partial}{\partial x'^\nu}\bigg|_{\mathrm{x}'\circ F(p)}\right) = \left(\partial_\mu F^\nu\right)_p \partial'_\nu|_{F(p)}, \tag{A.33}$$

as desired.

And now, as promised, consider some $v\in T_pM$. We can expand in x as $v = v^\mu\partial_\mu|_p$, and thus

$$\begin{aligned}\mathrm{d}F_p\, v &= \left(\mathrm{d}F_p\, v\right)^\mu \partial'_\mu|_{F(p)},\\ &= \mathrm{d}F_p\left(v^\mu\partial_\mu|_p\right) = v^\mu\left(\partial_\mu F^\nu\right)_p\partial'_\nu|_{F(p)},\end{aligned} \tag{A.34}$$

and hence

$$\left(\mathrm{d}F_p\, v\right)^\mu = \left(\partial_\nu F^\mu\right)_p v^\nu, \tag{A.35}$$

as desired. □

Theorem A.2. THE MATRIX REPRESENTATION OF THE PULLBACK OF A ONE-FORM.

Let $F: M\to M'$ be smooth and let $\mathrm{x}: M\to\mathbb{R}^d$ and $\mathrm{x}': M'\to\mathbb{R}^{d'}$ be coordinate systems on the domain and codomain. Then $F^*\omega' = (\partial_\mu F^\nu)(\omega'_\nu\circ F)\,\mathrm{d}\mathrm{x}^\mu$ for any $\omega'\in\mathfrak{X}^*(M')$, where $F^\mu = \mathrm{x}'^\mu\circ F$.

Proof. Begin by expanding ω' in components as $\omega' = \omega'_\mu\,\mathrm{d}\mathrm{x}'^\mu$. Then we have

$$F^*\omega' = F^*\left(\omega'_\mu\,\mathrm{d}\mathrm{x}'^\mu\right) = \left(\omega'_\mu\circ F\right)F^*\,\mathrm{d}\mathrm{x}'^\mu = \left(\omega'_\mu\circ F\right)\mathrm{d}\left(\mathrm{x}'^\mu\circ F\right) = \left(\omega'_\mu\circ F\right)\mathrm{d}F^\mu. \tag{A.36}$$

using Lemma A.1 for pullbacks in the second step and Lemma A.2 for pushforwards in the third.[42] And the map $F^\mu = \mathrm{x}'^\mu\circ F: M\to\mathbb{R}$ is just another smooth function in $C^\infty(M)$, and it

[42]In a little more detail: acting $F^*\,\mathrm{d}\mathrm{x}'^\mu$ on some $v\in T_pM$ gives $(F^*\,\mathrm{d}\mathrm{x}'^\mu)_p\, v = \mathrm{d}\mathrm{x}'^\mu\big|_{F(p)}\circ\mathrm{d}F_p\, v$ at which point we

may therefore be expanded in the coordinate coframe in the usual way as $\mathrm{d}F^\mu = (\partial_\nu F^\mu)\,\mathrm{dx}^\nu$, yielding

$$F^*\omega' = \big(\omega'_\mu \circ F\big)\big(\partial_\nu F^\mu\big)\,\mathrm{dx}^\nu \tag{A.37}$$

as desired. □

Theorem A.3. THE MATRIX REPRESENTATION OF THE PULLBACK OF AN ARBITRARY TENSOR.

Let $F: M \to M'$ be a diffeomorphism and let $\mathrm{x}: M \to \mathbb{R}^d$ and $\mathrm{x}': M' \to \mathbb{R}^{d'}$ be coordinate systems on the domain and codomain. Also let $C' \in \Gamma^k_\ell M'$. The components of F^*C' in x are related to those of C' in x' in the usual way (given in eq. (A.41) below).

Proof. This result follows directly from applying Theorems A.1 and A.2 to the frame and coframe of $\mathrm{d}F_p^*C'$. By definition we have (considering a rank $\binom{1}{1}$ tensor for notational clarity)

$$\big(\mathrm{d}F_p^*C'\big)^\mu{}_\nu = \mathrm{d}F_p^*C'\Big(\mathrm{dx}^\mu\big|_p, \partial_\nu\big|_p\Big) = C'\Big(\mathrm{d}\big(F^{-1}\big)^*_{F(p)}\,\mathrm{dx}^\mu\big|_p, \mathrm{d}F_p\,\partial_\nu\big|_p\Big). \tag{A.38}$$

(Keep in mind that $\mathrm{d}(F^{-1})^*_{F(p)}\,\mathrm{dx}^\mu\big|_p$ refers to the pullback of the one-form dx^μ from $p \in M$ to $F(p) \in M'$ via $F^{-1}: M' \to M$.) Applying Theorem A.1 to the vector argument gives

$$\mathrm{d}F_p\,\partial_\nu\big|_p = \big(\partial_\nu F^\beta\big)_p \partial'_\beta\big|_{F(p)}, \tag{A.39}$$

and applying Theorem A.2 to the one-form argument gives

$$\mathrm{d}\big(F^{-1}\big)^*_{F(p)}\,\mathrm{dx}^\mu\big|_p = \Big(\partial'_\alpha\big(F^{-1}\big)^\mu\Big)_{F(p)}\,\mathrm{dx}'^\alpha\big|_{F(p)}, \tag{A.40}$$

from which we obtain the desired result:

$$\big(\mathrm{d}F_p^*C'\big)^\mu{}_\nu = \Big(\partial'_\alpha\big(F^{-1}\big)^\mu\Big)_{F(p)}\big(\partial_\nu F^\beta\big)_p C'^\alpha{}_\beta. \tag{A.41}$$

In the interest of clarity let's express the above matrices in terms of the coordinate representa-

apply Lemma A.2 to obtain the expression above.

tion of F:

$$\hat{F} = \mathsf{x}' \circ F \circ \mathsf{x}^{-1}, \quad \hat{F}^{-1} = \mathsf{x} \circ F^{-1} \circ \mathsf{x}'^{-1}. \tag{A.42}$$

Then we find

$$\begin{aligned} \left(\partial_\mu F^\nu\right)_p &= \mathrm{d}\left(\mathsf{x}^{-1}\right)_{\mathsf{x}(p)} \left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{x}(p)}\right) F^\nu = \frac{\partial \hat{F}^\nu}{\partial x^\mu}\bigg|_{\mathsf{x}(p)}, \\ \left(\partial'_\mu \left(F^{-1}\right)^\nu\right)_{F(p)} &= \mathrm{d}\left(\mathsf{x}'^{-1}\right)_{\mathsf{x}'\circ F(p)} \left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathsf{x}'\circ F(p)}\right) \left(F^{-1}\right)^\nu = \frac{\partial \left(\hat{F}^{-1}\right)^\nu}{\partial x^\mu}\bigg|_{\mathsf{x}'\circ F(p)}. \end{aligned} \tag{A.43}$$

We can think of $\hat{F}$ as a rule which sends the "old coordinate" value $x \in \mathbb{R}^d$ to a "new coordinate" value $x' \in \mathbb{R}^{d'}$, so in more conventional notation we wouldn't give the map $\hat{F} : x \mapsto \hat{F}(x) \equiv x'$ a special symbol at all and would just write $x'(x)$, and similarly we would write $x(x')$ in place of $\hat{F}^{-1}$. Thus in more conventional notation we would write the above as

$$\partial_\mu F^\nu \to \frac{\partial x'^\nu}{\partial x^\mu}, \quad \partial'_\mu \left(F^{-1}\right)^\nu \to \frac{\partial x^\nu}{\partial x'^\mu}, \tag{A.44}$$

and in this form the transformation law (A.41) becomes the one with which a physicist is likely already intimately familiar. □

A.2 Passive transformations in general

Now we turn to the main topic of discussion, namely coordinate transformations. Let M be a smooth manifold and $\mathsf{x} : M \to \mathbb{R}^d$ a coordinate system on M.[43] The component functions of x are x^μ, i.e. $\mathsf{x}(p) = \left(\mathsf{x}^\mu(p)\right) \in \mathbb{R}^d$. The more conventional italic symbols x^μ are reserved for the canonical Cartesian coordinates on $\mathbb{R}^d$ itself, i.e. $\mathsf{x}(p) = x = (x^\mu)$.

A.2.1 Definition and the transition map

Suppose that we also have a second coordinate system $\tilde{\mathsf{x}} : M \to \mathbb{R}^d$. This is a *passive* coordinate transformation: we aren't actually transforming any tensors, we're just exchanging their

[43] Here as in the main text I ignore the fact that coordinate systems are generally defined only on subsets of M.

component expansions in the old coordinates for the new, e.g. for a rank $\binom{1}{1}$ tensor

$$C = C^{\mu}{}_{\nu}\partial_{\mu} \otimes \mathrm{d}\mathsf{x}^{\nu} = \tilde{C}^{\mu}{}_{\nu}\tilde{\partial}_{\mu} \otimes \mathrm{d}\tilde{\mathsf{x}}^{\nu}\,, \tag{A.45}$$

where $\tilde{\partial}_{\mu}$ and $\mathrm{d}\tilde{\mathsf{x}}^{\mu}$ are the frame and coframe of the new coordinates.

N.B. the frame $\tilde{\partial}_{\mu}$ of the new coordinates is given by the pushforward of the *same* canonical frame $\partial/\partial x^{\mu}$ on $\mathbb{R}^d$ by the inverse of the *new* coordinate map, $\tilde{\partial}_{\mu} = (\tilde{\mathsf{x}}^{-1})_*(\partial/\partial x^{\mu})$. This is a point at which the distinction between the coordinate maps $\mathsf{x}, \tilde{\mathsf{x}} : M \to \mathbb{R}^d$ and the coordinate values $x \in \mathbb{R}^d$ is important, since if we were conflating the two it would be tempting to also include a tilde in the denominator of the $\partial/\partial x^{\mu}$ that we're pushing forward, and this would incorrectly imply that we need an extra factor of $\partial\tilde{x}^{\mu}/\partial x^{\nu}$ to relate the coordinate frames.

Since by definition a coordinate system is an injection it follows that we can construct a *transition map* from the original coordinates to the new:

$$\mathsf{T} \equiv \tilde{\mathsf{x}} \circ \mathsf{x}^{-1} : \mathsf{x}(M) \subseteq \mathbb{R}^d \to \tilde{\mathsf{x}}(M) \subseteq \mathbb{R}^d. \tag{A.46}$$

Note that $\mathsf{T} : \mathsf{x}(M) \to \tilde{\mathsf{x}}(M)$ is what is commonly denoted as $\tilde{x}(x)$: it's the value of the new coordinates at the point whose old coordinates were x. Similarly its inverse $\mathsf{T}^{-1} : \tilde{\mathsf{x}}(M) \to \mathsf{x}(M)$ is what is more commonly denoted as $x(\tilde{x})$.

In secs. A.2.2 and A.2.3 we will come across the matrices $\partial_{\mu}\tilde{\mathsf{x}}^{\nu}$ and $\tilde{\partial}_{\mu}\mathsf{x}^{\nu}$, so I will here show how to write these matrices explicitly in terms of the transition map. Let's evaluate the first at a point $p \in M$:

$$\left(\partial_{\mu}\tilde{\mathsf{x}}^{\nu}\right)_p = \left(\mathsf{x}^{-1}\right)_* \left(\left.\frac{\partial}{\partial x^{\mu}}\right|_{\mathsf{x}(p)}\right)\tilde{\mathsf{x}}^{\nu} = \frac{\partial\left(\tilde{\mathsf{x}}^{\nu} \circ \mathsf{x}^{-1}\right)}{\partial x^{\mu}}\left(\mathsf{x}(p)\right). \tag{A.47}$$

Using the fact that $\tilde{\mathsf{x}}^{\nu} \circ \mathsf{x}^{-1}$ is just the ν^{th} component function of the transition map:

$$\left(\partial_{\mu}\tilde{\mathsf{x}}^{\nu}\right)_p = \frac{\partial\mathsf{T}^{\nu}}{\partial x^{\mu}}\left(\mathsf{x}(p)\right). \tag{A.48}$$

In other words (as I'm sure you already knew) we can write the matrix $\left(\partial_{\mu}\tilde{\mathsf{x}}^{\nu}\right)_p$, which is constructed in terms of geometrical objects on M and evaluated at a point $p \in M$, entirely in terms of a standard partial derivative of a real-valued function of a vector variable, $\mathsf{T}^\nu : \mathbb{R}^d \to \mathbb{R}$, evaluated at the image of p in $\mathbb{R}^d$ under the old coordinates x.

An exactly analogous argument also yields

$$\left(\tilde{\partial}_\mu \mathsf{x}^\nu\right)_p = \frac{\partial\left(\mathsf{T}^{-1}\right)^\nu}{\partial x^\mu}\left(\tilde{\mathsf{x}}(p)\right). \tag{A.49}$$

The results (A.48) and (A.49) are more commonly written as $\partial_\mu \tilde{\mathsf{x}}^\nu = \partial \tilde{x}^\nu / \partial x^\mu$ and $\tilde{\partial}_\mu x^\nu = \partial x^\nu / \partial \tilde{x}^\mu$ respectively. However note that I have *not* dropped a tilde in the denominator of the right-hand side of eq. (A.49): $\left(\mathsf{T}^{-1}\right)^\nu$ is a map $\mathbb{R}^d \to \mathbb{R}$ like any other, and we are applying to it the *same* canonical coordinate frame vector $\frac{\partial}{\partial x^\mu}$ on $\mathbb{R}^d$ that appears in eq. (A.48). What the $\tilde{x}^\mu$ in the denominator of the more standard expression actually denotes is that we *evaluate* this derivative at a different point in $\mathbb{R}^d$, namely the image of p under the new coordinates $\tilde{\mathsf{x}}$.

A.2.2 Relating the frames via the transition map

To obtain the standard frame transformation law let's recall that each frame is the pushforward of the canonical frame on $\mathbb{R}^d$ by the corresponding coordinate system:

$$\partial_\mu = \left(\mathsf{x}^{-1}\right)_* \left(\frac{\partial}{\partial x^\mu}\right), \quad \tilde{\partial}_\mu = \left(\tilde{\mathsf{x}}^{-1}\right)_* \left(\frac{\partial}{\partial x^\mu}\right). \tag{A.50}$$

To write ∂_μ in terms of $\tilde{\partial}_\mu$ we use the fact that the definition $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1}$ of the transition map implies that $\mathsf{x}^{-1} = \tilde{\mathsf{x}}^{-1} \circ \mathsf{T}$. Then from Lemma A.2, specifically the result $(F' \circ F)_* = F'_* \circ F_*$, it follows that

$$\partial_\mu = \left(\tilde{\mathsf{x}}^{-1} \circ \mathsf{T}\right)_* \left(\frac{\partial}{\partial x^\mu}\right) = \left(\tilde{\mathsf{x}}^{-1}\right)_* \left(\mathsf{T}_* \frac{\partial}{\partial x^\mu}\right) \tag{A.51}$$

For clarity let's consider the above at some $p \in M$:

$$\partial_\mu\big|_p = \mathrm{d}\left(\tilde{\mathsf{x}}^{-1}\right)_{\tilde{\mathsf{x}}(p)} \circ \mathrm{d}\mathsf{T}_{\mathsf{x}(p)} \left.\frac{\partial}{\partial x^\mu}\right|_{\mathsf{x}(p)} \tag{A.52}$$

Now let's apply Theorem A.1 to represent the action of $\mathrm{dT}_{\mathrm{x}(p)}$ on $\frac{\partial}{\partial x^\mu}$. In fact we need only the intermediate step eq. (A.29), with T playing the role of $\hat{F}$ and $\mathrm{x}(p)$ playing the role of x:

$$\mathrm{dT}_{\mathrm{x}(p)} \frac{\partial}{\partial x^\mu}\bigg|_{\mathrm{x}(p)} = \frac{\partial \mathrm{T}^\nu}{\partial x^\mu}\big(\mathrm{x}(p)\big) \frac{\partial}{\partial x^\nu}\bigg|_{\tilde{\mathrm{x}}(p)}, \tag{A.53}$$

using the fact that $\mathrm{T} \circ \mathrm{x} = \tilde{\mathrm{x}}$. Using this in the previous expression we then find

$$\partial_\mu\big|_p = \frac{\partial \mathrm{T}^\nu}{\partial x^\mu}\big(\mathrm{x}(p)\big)\, \mathrm{d}\big(\tilde{\mathrm{x}}^{-1}\big)_{\tilde{\mathrm{x}}(p)} \frac{\partial}{\partial x^\nu}\bigg|_{\tilde{\mathrm{x}}(p)} = \frac{\partial \mathrm{T}^\nu}{\partial x^\mu}\big(\mathrm{x}(p)\big)\tilde{\partial}_\nu\big|_p, \tag{A.54}$$

And this is the standard result: to obtain the old frame at a point p in terms of the new at that same point, take the function T which gives the new coordinates in terms of the old and evaluate its matrix of partial derivatives at the old coordinates of p. Via eq. (A.48) this also be written in its more compact and conventional form:

$$\partial_\mu = \big(\partial_\mu \tilde{\mathrm{x}}^\nu\big)\tilde{\partial}_\nu. \tag{A.55}$$

A.2.3 Relating the coframes via the transition map

The coframe transformation law follows straightforwardly from the frame transformation. Consider the action of the old coframe on a generic $v \in \mathfrak{X}(M)$. Expanding v in the new coordinates as $v = \tilde{v}^\mu \tilde{\partial}_\mu$ we have

$$\mathrm{dx}^\mu(v) = \tilde{v}^\nu\, \mathrm{dx}^\mu(\tilde{\partial}_\nu). \tag{A.56}$$

Using the frame transformation rule (A.55) with $\mathrm{x} \leftrightarrow \tilde{\mathrm{x}}$ and the fact that $\tilde{v}^\nu = \mathrm{d}\tilde{\mathrm{x}}^\nu(v)$ this becomes

$$\mathrm{dx}^\mu(v) = \big(\tilde{\partial}_\nu \mathrm{x}^\rho\big)\tilde{v}^\nu\, \mathrm{dx}^\mu(\partial_\rho) = \big(\tilde{\partial}_\nu \mathrm{x}^\mu\big)\, \mathrm{d}\tilde{\mathrm{x}}^\nu(v). \tag{A.57}$$

And since this holds for all $v \in \mathfrak{X}(M)$ we can drop the v entirely to obtain the relation between the coframes:

$$\mathrm{dx}^\mu = \big(\tilde{\partial}_\nu \mathrm{x}^\mu\big)\, \mathrm{d}\tilde{\mathrm{x}}^\nu. \tag{A.58}$$

Again evaluating at $p \in M$ and expressing in terms of the transition map for clarity:

$$\mathrm{dx}^\mu\big|_p = \frac{\partial(\mathsf{T}^{-1})^\mu}{\partial x^\nu}\big(\tilde{\mathrm{x}}(p)\big)\,\mathrm{d}\tilde{\mathrm{x}}^\nu\big|_p. \tag{A.59}$$

So whereas the old frame is given in terms of the new at a point p by the matrix of partial derivatives of T at the old coordinates $\mathrm{x}(p)$, the old coframe is given in terms of the new at p by the matrix of partial derivatives of T^{-1} at the new coordinates $\tilde{\mathrm{x}}(p)$.

A.2.4 The transformation of the components of a generic tensor field

The transformation rule for the components of a generic tensor $C \in \Gamma^k_\ell M$ follow immediately from the frame and coframe transformation rules. In the new coordinates we have

$$C = \tilde{C}^\mu{}_\nu \tilde{\partial}_\mu \otimes \mathrm{d}\tilde{\mathrm{x}}^\nu \tag{A.60}$$

and in the old coordinates

$$C = C^\mu{}_\nu \partial_\mu \otimes \mathrm{dx}^\nu = C^\alpha{}_\beta\big(\partial_\alpha \tilde{\mathrm{x}}^\mu \tilde{\partial}_\mu\big) \otimes \big(\tilde{\partial}_\nu \mathrm{x}^\beta\,\mathrm{d}\tilde{\mathrm{x}}^\nu\big), \tag{A.61}$$

so we find

$$\tilde{C}^\mu{}_\nu = C^\alpha{}_\beta \partial_\alpha \tilde{\mathrm{x}}^\mu \tilde{\partial}_\nu \mathrm{x}^\beta, \tag{A.62}$$

i.e. the normal rule: raised indices transform with $\partial_\mu \tilde{\mathrm{x}}^\nu$, and lowered indices transform with $\tilde{\partial}_\mu \mathrm{x}^\nu$.

A.3 Induced passive transformations

In the previous section we considered a generic passive coordinate transformation. We now restrict ourselves to what I will call *induced* passive transformations, i.e. those in which the coordinate systems are related by an automorphism $F : M \to M$ of the manifold under consideration:[44]

$$\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}. \tag{A.63}$$

In other words the new coordinates of a point p are given by the old coordinates of the point $F^{-1}(p)$ which is mapped to p by F.

A.3.1 The transition map and the inducing diffeomorphism

A first, very important point is that induced passive transformations are a *subset* of all passive transformations: given an arbitrary passive transformation there is no guarantee that there exists any diffeomorphism (not even a local diffeomorphism!) satisfying the definition (A.63). To see this let's (for this section only) take into account the fact that coordinate systems are generally only locally defined by writing $\mathsf{x}, \tilde{\mathsf{x}} : U, \tilde{U} \subseteq M \to \mathbb{R}^d$, with the assumption that $U \cap \tilde{U} \neq \emptyset$.

Even with this additional wrinkle the transition map $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1}$ is always defined on $U \cap \tilde{U}$: no matter what, we can pick some $x \in \mathsf{x}(U \cap \tilde{U})$, follow x^{-1} back to $U \cap \tilde{U}$, and then follow $\tilde{\mathsf{x}}$ into $\tilde{\mathsf{x}}(U \cap \tilde{U})$.

Now let's solve for F from eq. (A.63):

$$F = \tilde{\mathsf{x}}^{-1} \circ \mathsf{x}. \tag{A.64}$$

We can feed x any $p \in U$ and end up in $\mathsf{x}(\tilde{U})$, but we can only feed $\mathsf{x}(p)$ into $\tilde{\mathsf{x}}^{-1}$ if $\mathsf{x}(p) \in \tilde{\mathsf{x}}(U)$ as well. In other words F is only defined on a nonempty set if $\mathsf{x}(U) \cap \tilde{\mathsf{x}}(\tilde{U}) \neq \emptyset$, i.e. if there is some common ground between the ranges of coordinate values mapped into by the old and new coordinate systems, and this is *not* guaranteed, even though we do assume that $U \cap \tilde{U} \neq \emptyset$.

To emphasize the difference between T and F let's contrast their actions. We can write the

[44]The term "induced" what I'm choosing to call it for the sake of conciseness. I'm unaware of broadly agreed-upon terminology to make this distinction.

new coordinate system in terms of the old and the transition map as

$$\tilde{\mathsf{x}} = \mathsf{T} \circ \mathsf{x} : M \xrightarrow{\mathsf{x}} \mathbb{R}^d \xrightarrow{\mathsf{T}} \mathbb{R}^d, \tag{A.65}$$

or in terms of the old and the inducing diffeomorphism as

$$\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1} : M \xrightarrow{F^{-1}} M \xrightarrow{\mathsf{x}} \mathbb{R}^d. \tag{A.66}$$

In other words: in the former x takes us from M into $\mathbb{R}^d$, and then T moves us around in $\mathbb{R}^d$; while in the latter F^{-1} first moves us around in M, and then x takes us from M to $\mathbb{R}^d$. And we are guaranteed to be able to do the former so long as $U \cap \tilde{U}$ is nonempty (and this is a trivial condition to impose in this context, because if $U \cap \tilde{U}$ is empty then there's no coordinate transformation of which to speak), while this is not sufficient to guarantee that we can do the latter.

All that being said: we are in this section restricting our considerations to those transformations for which F does exist, since technically speaking it's diffeomorphisms of spacetime, not coordinate transformations, which are the gauge transformations of general relativity. We will also from here on out return to ignoring the locality of coordinate systems and just write $\mathsf{x}, \tilde{\mathsf{x}} : M \to \mathbb{R}^d$ for simplicity.

Finally observe that, since by definition $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1}$ and $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}$, when F does exist the transition map from x to $\tilde{\mathsf{x}}$ is the coordinate representation of F^{-1} in x:

$$\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1} = \mathsf{x} \circ F^{-1} \circ \mathsf{x}^{-1}. \tag{A.67}$$

And by exactly analogous logic, since $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1} \implies \mathsf{x} = \tilde{\mathsf{x}} \circ F$, the transition map is *also* the coordinate representation of F^{-1} in $\tilde{\mathsf{x}}$:

$$\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1} = \tilde{\mathsf{x}} \circ \left(\tilde{\mathsf{x}} \circ F\right)^{-1} = \tilde{\mathsf{x}} \circ F^{-1} \circ \tilde{\mathsf{x}}^{-1}. \tag{A.68}$$

A.3.2 Relating the frames and the coframes via the inducing diffeomorphism

For an induced passive transformation we can straightforwardly rewrite the relationships between the frames and coframes in terms of the inducing diffeomorphism. Let's start with the frame. Since $\tilde{\mathrm{x}} = \mathrm{x} \circ F^{-1}$ implies that $\tilde{\mathrm{x}}^{-1} = F \circ \mathrm{x}^{-1}$ we have via lemma A.2

$$\tilde{\partial}_\mu = \left(\tilde{\mathrm{x}}^{-1}\right)_* \frac{\partial}{\partial x^\mu} = \left(F \circ \mathrm{x}^{-1}\right)_* \frac{\partial}{\partial x^\mu} = F_* \left(\left(\mathrm{x}^{-1}\right)_* \frac{\partial}{\partial x^\mu}\right) = F_* \partial_\mu. \tag{A.69}$$

To relate the coframes via F we the same lemma:

$$\mathrm{d}\tilde{\mathrm{x}}^\mu = \mathrm{d}\left(\mathrm{x}^\mu \circ F^{-1}\right) = \mathrm{d}\mathrm{x}^\mu \circ \mathrm{d}\left(F^{-1}\right). \tag{A.70}$$

And this we can recognize as the pullback by F^{-1}:

$$\mathrm{d}\tilde{\mathrm{x}}^\mu = \left(F^{-1}\right)^* \mathrm{d}\mathrm{x}^\mu. \tag{A.71}$$

So, in short: the new frame is obtained from the old via the pushforward by F, and the new coframe is obtained from the old via the pullback by F^{-1}. Expressed pointwise the above are

$$\tilde{\partial}_\mu\big|_p = \mathrm{d}F_{F^{-1}(p)}\, \partial_\mu\big|_{F^{-1}(p)}, \quad \mathrm{d}\tilde{\mathrm{x}}^\mu\big|_p = \mathrm{d}\left(F^{-1}\right)^*_p \mathrm{d}\mathrm{x}^\mu\big|_{F^{-1}(p)}. \tag{A.72}$$

A.3.3 Relating the frame and coframe transformation rules of sec. A.3.2 to those of secs. A.2.2 and A.2.3

In sec. A.3.2 we wrote the new frame and coframe in terms of the old and the inducing diffeomorphism:

$$\tilde{\partial}_\mu = F_* \partial_\mu, \quad \mathrm{d}\tilde{\mathrm{x}}^\mu = \left(F^{-1}\right)^* \mathrm{d}\mathrm{x}^\mu, \tag{A.73}$$

while in secs. A.2.2 and A.2.3 we related the frames and coframes via the transition map $\mathsf{T} = \tilde{\mathsf{x}} \circ \mathsf{x}^{-1}$,

$$\begin{aligned} \partial_\mu &= \partial_\mu \tilde{\mathsf{x}}^\nu \tilde{\partial}_\nu \quad &\text{with} \quad \left(\partial_\mu \tilde{\mathsf{x}}^\nu\right)_p = \frac{\partial \mathsf{T}^\nu}{\partial x^\mu}\big(\mathsf{x}(p)\big), \\ \mathrm{d}\mathsf{x}^\mu &= \tilde{\partial}_\nu \mathsf{x}^\mu \,\mathrm{d}\tilde{\mathsf{x}}^\nu \quad &\text{with} \quad \left(\tilde{\partial}_\nu \mathsf{x}^\mu\right)_p = \frac{\partial \left(\mathsf{T}^{-1}\right)^\mu}{\partial x^\nu}\big(\tilde{\mathsf{x}}(p)\big). \end{aligned} \tag{A.74}$$

It follows that in the case of an induced passive transformation eq. (A.73) ought to imply eq. (A.74). And indeed it does!

To see this let's start with the inverses of the first:

$$\partial_\mu = \left(F^{-1}\right)_* \tilde{\partial}_\mu, \quad \mathrm{d}\mathsf{x}^\mu = F^* \,\mathrm{d}\tilde{\mathsf{x}}^\mu . \tag{A.75}$$

Evaluating at a point $p \in M$ for clarity these become

$$\partial_\mu\big|_p = \mathrm{d}\left(F^{-1}\right)_{F(p)} \tilde{\partial}_\mu\big|_{F(p)}, \quad \mathrm{d}\mathsf{x}^\mu\big|_p = \mathrm{d}F_p^* \,\mathrm{d}\tilde{\mathsf{x}}^\mu\big|_{F(p)}. \tag{A.76}$$

Now let's restate the matrix representations of the pushforward and pullback from Theorems (A.1) and (A.2), which are

$$\mathrm{d}G_p\, v = \left(\partial_\nu G^\mu\right)_p v^\nu \partial'_\mu\big|_{G(p)}, \quad \mathrm{d}G_p^* \,\omega' = \left(\partial_\mu G^\nu\right)_p \omega'_\nu \,\mathrm{d}\mathsf{x}^\mu\big|_p, \tag{A.77}$$

recalling that in the notation of those theorems x and x' are coordinates on the domain and codomain of some smooth $G : M \to M'$ and $v \in T_p M$ and $\omega' \in T_{G(p)} M'$ are a single vector and one-form respectively (as opposed to fields). Using the coordinate representation $\hat{G} = \mathsf{x}' \circ G \circ \mathsf{x}^{-1}$ of G, whose component functions are related to the G^μ's as $\hat{G}^\mu = G^\mu \circ \mathsf{x}^{-1}$, we can rewrite the matrix in the above as

$$\left(\partial_\nu G^\mu\right)_p = \frac{\partial \hat{G}^\mu}{\partial x^\nu}\big(\mathsf{x}(p)\big). \tag{A.78}$$

Now let's apply the first of eqs. (A.77) to the first of eqs. (A.76), using the representation

(A.78). We wish to express ∂_μ as a sum of the $\tilde{\partial}_\mu$'s, so since it's the x' of the theorems' notation which determines the frame which is summed over in eq. (A.77) it follows that we should identify the theorems' x' with our new coordinates $\tilde{\mathsf{x}}$, so that e.g. $G^\mu = \mathsf{x}'^\mu \circ G \mapsto \tilde{\mathsf{x}}^\mu \circ F^{-1} = \left(F^{-1}\right)^\mu$. We also need to expand the vector v of the theorem's notation in components, and the coordinate frame in which we expand v is the same frame which will act on $G^\mu \mapsto \left(F^{-1}\right)^\mu$ in the matrix. Since the whole point of this exercise is that we don't already know how to expand the new frame in terms of the old, and since the role of the v of the theorem is being played by $\tilde{\partial}_\mu$, it follows that all we can do is also use $\tilde{\mathsf{x}}$ for this expansion, meaning that $\tilde{\mathsf{x}}$ is playing the role of *both* of the theorem's coordinate systems. It follows that the coordinate representation of F^{-1} which will appear when we apply eq. (A.78) is the transition map, $\hat{G} = \mathsf{x}' \circ G \circ \mathsf{x}^{-1} \to \tilde{\mathsf{x}} \circ F^{-1} \circ \tilde{\mathsf{x}}^{-1} = \mathsf{T}$, and thus as promised we do indeed obtain the frame transformation law in terms of the transition map,

$$\partial_\mu\big|_p = \mathrm{d}\left(F^{-1}\right)_{F(p)} \tilde{\partial}_\mu\big|_{F(p)} = \left(\tilde{\partial}_\rho \left(F^{-1}\right)^\nu\right)_{F(p)} \delta^\rho_\mu \tilde{\partial}_\nu\big|_p = \frac{\partial \mathsf{T}^\nu}{\partial x^\mu}\left(\mathsf{x}(p)\right)\tilde{\partial}_\nu\big|_p, \tag{A.79}$$

using also the fact that $\tilde{\mathsf{x}} \circ F = \mathsf{x}$.

The second of eqs. (A.76) goes through analogously. We wish to expand $\mathrm{d}\mathsf{x}^\mu\big|_p$ in terms of $\mathrm{d}\tilde{\mathsf{x}}^\mu\big|_p$, so we again identify the x' of the theorems with $\tilde{\mathsf{x}}$, and we need to expand the $\mathrm{d}\tilde{\mathsf{x}}^\mu$ on which $\mathrm{d}F^*_p$ acts in eq. (A.76) in components, and again our only option is to also choose $\tilde{\mathsf{x}}$ for those coordinates. It follows that here we end up with the coordinate representation of F when we apply eq. (A.78), i.e. T^{-1}. Thus, again as promised, we obtain the coframe transformation law in terms of the transition map:

$$\mathrm{d}\mathsf{x}^\mu\big|_p = \mathrm{d}F^*_p\, \mathrm{d}\tilde{\mathsf{x}}^\mu\big|_{F(p)} = \left(\tilde{\partial}_\rho F^\nu\right)_p \delta^\mu_\nu\, \mathrm{d}\tilde{\mathsf{x}}^\rho\big|_p = \frac{\partial\left(\mathsf{T}^{-1}\right)^\mu}{\partial x^\nu}\left(\tilde{\mathsf{x}}(p)\right)\mathrm{d}\tilde{\mathsf{x}}^\nu\big|_p. \tag{A.80}$$

A.3.4 The transformation of the components of a generic tensor field

We now come, after an exorbitant amount of notation, to the point of this section: the rule for the transformation of the components of a generic tensor field $C \in \Gamma^k_\ell M$, in terms of the inducing diffeomorphism F. Let's start with the components in the new coordinates:

$$\left(\tilde{C}^\mu{}_\nu\right)_p = C_p\left(\mathrm{d}\tilde{\mathrm{x}}^\mu\big|_p, \tilde{\partial}_\nu\big|_p\right) = C_p\left(\mathrm{d}\left(F^{-1}\right)^*_p \mathrm{d}\mathrm{x}^\mu\big|_{F^{-1}(p)}, \mathrm{d}F_{F^{-1}(p)}\,\partial_\nu\big|_{F^{-1}(p)}\right). \tag{A.81}$$

And from eq. (A.6) we recognize this last as the pullback of the generic mixed tensor C by the inducing diffeomorphism F from p to $F^{-1}(p)$:

$$\left(\tilde{C}^\mu{}_\nu\right)_p = \left(\mathrm{d}F^*_{F^{-1}(p)}\,C_p\right)\left(\mathrm{d}\mathrm{x}^\mu\big|_{F^{-1}(p)}, \partial_\nu\big|_{F^{-1}(p)}\right) = \left(\left(F^*C\right)^\mu{}_\nu\right)_{F^{-1}(p)}, \tag{A.82}$$

or without reference to the point p

$$\tilde{C}^\mu{}_\nu = \left(F^*C\right)^\mu{}_\nu \circ F^{-1}. \tag{A.83}$$

In other words the components of C at p in the new coordinates are the components of F^*C at $F^{-1}(p)$ in the old coordinates.

From eq. (A.82) we can use Theorem A.3 to obtain the more familiar form of the transformation law in terms of the matrix of partial derivatives of F:

$$\left(\tilde{C}^\mu{}_\nu\right)_p = \left(\mathrm{d}F^*_{F^{-1}(p)}\,C_p\right)^\mu{}_\nu = \left(\partial_\alpha\left(F^{-1}\right)^\mu\right)_p \left(\partial_\nu F^\beta\right)_{F^{-1}(p)} \left(C^\alpha{}_\beta\right)_p. \tag{A.84}$$

Note that the derivative on F^{-1} is the frame of the *old* coordinates, and therefore the same as the derivative on F, since by evaluating the components of F^*C in the old coordinates we're thinking of F^*C as the pullback of C from M with coordinates x to M with the *same* coordinates x. In other words the old coordinates x are playing the role of *both* coordinate systems in the notation of Theorem A.3. That being said, observe that when we write the matrices explicitly in terms of the transition map (i.e. the coordinate representation of F) we obtain

$$\begin{aligned}
\left(\partial_\mu F^\nu\right)_{F^{-1}(p)} &= \frac{\partial\left(F^\nu \circ \mathsf{x}^{-1}\right)}{\partial x^\mu}\left(\mathsf{x}\circ F^1(p)\right) = \frac{\partial\left(\mathsf{T}^{-1}\right)^\nu}{\partial x^\mu}\left(\tilde{\mathsf{x}}(p)\right),\\
\left(\partial_\mu\left(F^{-1}\right)^\nu\right)_p &= \frac{\partial\left((F^{-1})^\nu \circ \mathsf{x}^{-1}\right)}{\partial x^\mu}\left(\mathsf{x}(p)\right) = \frac{\partial \mathsf{T}^\nu}{\partial x^\mu}\left(\mathsf{x}(p)\right),
\end{aligned} \tag{A.85}$$

i.e. precisely the matrices which are more typically written $\partial x^\nu/\partial\tilde{x}^\mu$ and $\partial\tilde{x}^\nu/\partial x^\mu$, so we do recover the familiar result that (in standard physics parlance) raised indices transform with $\partial\tilde{x}^\nu/\partial x^\mu$ and lowered indices transform with its inverse $\partial x^\nu/\partial\tilde{x}^\mu$.

Note finally that the point $F^{-1}(p)$ at which we evaluate the right hand side of eq. (A.82) is precisely the point whose old coordinates (under x) are the same as the new coordinates of p (under $\tilde{\mathsf{x}}$), which we can also see by noting that eq. (A.83) implies that

$$\tilde{C}^\mu{}_\nu \circ \tilde{\mathsf{x}}^{-1} = \left(F_* C\right)^\mu{}_\nu \circ \mathsf{x}^{-1}. \tag{A.86}$$

Thus, in short: if you want to know the new components of C at some $p \in M$, go to the point $F^{-1}(p)$ whose old coordinates agree with the new coordinates of p, and find the old components of $F^* C$ there.

A.4 Active transformations

In the previous sections we considered passive coordinate transformations. In other words we left the tensor fields themselves unchanged and only swapped out the coordinates in which we expressed them. In sec. A.2 we considered a general such transformation, only exchanging one coordinate system x for another $\tilde{\mathsf{x}}$, while in sec. A.3 we considered the case in which $\tilde{\mathsf{x}}$ is obtained from x via a diffeomorphism F of the manifold.

In this section we consider instead *active transformations*: as in sec. A.3 we begin with a coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ and a diffeomorphism $F : M \to M$, but we now leave the coordinate system alone and instead think of F as acting on the various tensor fields of interest themselves,

$$C \in \Gamma^k_\ell M \mapsto \tilde{C} \equiv F^* C. \tag{A.87}$$

In other words instead of being interested in the relationship between the components of a single tensor field $C \in \Gamma^k_\ell M$ in two different coordinate systems, we're now interested in the relationship between the components of the *distinct* tensor fields $C, F^* C \in \Gamma^k_\ell M$ in the *same* coordinate system x. However it's immediately apparent that this relationship is (almost) precisely the same, formally speaking, as the passive transformation law (A.83):

$$\tilde{C}^{\mu\ldots\nu}{}_{\rho\ldots\sigma} = \left(F^* C\right)^{\mu\ldots\nu}{}_{\rho\ldots\sigma}. \tag{A.88}$$

The only difference between this and the passive transformation rule is the lack of a "$\circ F^{-1}$" on the end of the active version. But the similarity becomes even more apparent when we consider the coordinate version and compare to eq. (A.86):

$$\tilde{C}^{\mu\ldots\nu}{}_{\rho\ldots\sigma} \circ \mathsf{x}^{-1} = \left(F^* C\right)^{\mu\ldots\nu}{}_{\rho\ldots\sigma} \circ \mathsf{x}^{-1}. \tag{A.89}$$

The only difference here is the lack of a tilde on the left hand side, which just comes down to the fact that we are here relating the components of two different tensor fields in the same coordinate system, whereas in the passive case we are relating the components of the same tensor field in distinct coordinate systems.

Finally observe that, through Theorem A.3, we can immediately obtain the transformation law for the active case, which I will again give in the case of a $\binom{1}{1}$ tensor for notational brevity:

$$\left(\tilde{C}^\mu{}_\nu\right)_p = \left(\mathrm{d}F^*_p C_{F(p)}\right)^\mu{}_\nu = \left(\partial_\alpha \left(F^{-1}\right)^\mu\right)_{F(p)} \left(\partial_\nu F^\beta\right)_p \left(C^\alpha{}_\beta\right)_{F(p)}, \tag{A.90}$$

i.e. precisely the passive transformation rule (A.84), just with $p \mapsto F(p)$, which manifests in the explicit coordinate expressions as

$$\left(\partial_\mu F^\nu\right)_p = \frac{\partial\left(\mathsf{T}^{-1}\right)^\nu}{\partial x^\mu}\left(\tilde{\mathsf{x}} \circ F(p)\right), \quad \left(\partial_\mu\left(F^{-1}\right)^\nu\right)_{F(p)} = \frac{\partial \mathsf{T}^\nu}{\partial x^\mu}\left(\mathsf{x} \circ F(p)\right). \tag{A.91}$$

In other words, the passive and active transformation rules are nearly identical in their coordinate expressions, with the sole difference being that both sides of the passive transformation rule are evaluated at the same point p, while the right hand side of the active transformation rule is evaluated at $F(p)$. And this is, once again, emblematic of the sole difference between active and passive transformations: in a passive transformation we consider F to act on the observer, whereas in an active transformation we consider it to act on the fields themselves.

A.5 Infinitesimal transformations

In secs. A.3 and A.4 we showed that, given any coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ and any automorphism $F : M \to M$, we can define a passive transformation by constructing a new coordinate system via $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}$, and equivalently we can define an active transformation by pushing forward all tensor fields via $C \mapsto F_* C$. In the former we think of the tensor fields as staying the same while we push the frame and coframe[45] forward by F, while in the latter we think of the observer as staying the same while we pull all tensor fields back by F.

In this section I will provide a rigorous discussion of *infinitesimal* coordinate transformations by considering those automorphisms which are parametrized as flows. I will then define the infinitesimal variation of a tensor under such an infinitesimal coordinate transformation in terms of the Lie derivative, and interpret this infinitesimal variation in the passive and active pictures.

[45](the "pushforward" of the coframe by F being its pullback by F^{-1})

A.5.1 Flows

Requisite for the definition of infinitesimal coordinate transformations is the notion of *flows*, which I will here quickly review in bullet points.

- Given a smooth manifold M a *flow domain* on M is an open subset $D_M \subseteq \mathbb{R} \times M$ such that each $D_p \equiv \{t \in \mathbb{R} \,|\, (t,p) \in D\}$ is an open interval in $\mathbb{R}$ containing 0.

- A *flow* on M is a smooth map $F : D_M \to M$ for some flow domain D_M on M which satisfies the following:

  - For all $p \in M$ we have $F(0,p) = p$.
  - For all $t \in D_p$ and $t' \in D_{F(t,p)}$ – i.e. for all $t \in \mathbb{R}$ such that $F(t,p)$ exists, and for all $t' \in \mathbb{R}$ such that $F(t', F(t,p))$ exists – we have $F(t', F(t,p)) = F(t+t', p)$. In other words, thinking of the $\mathbb{R}$-valued argument of F as time, flowing from p for a time t and then from that point for a further t' is equivalent to flowing from p for a time $t+t'$.

  A flow is *maximal* if it cannot be extended to a flow on a larger flow domain, and a flow is *global* if its domain is $\mathbb{R} \times M$.

- Given a flow $F : D_M \to M$ and a point $p \in M$ define the curve $F_p : D_p \subseteq \mathbb{R} \to M$ by $F_p(t) = F(t,p)$. Then the *generator* of F is the vector field $p \mapsto \dot{F}_p(0)$.[46]

- The above assigns to each flow a unique smooth vector field as its generator. However it does not guarantee that for every smooth vector field there exists a unique flow. This is a nontrivial result, known as the *fundamental theorem of flows* (e.g. theorem 9.12 of [**?**]): given any smooth vector field $v \in \mathfrak{X}(M)$ there exists a unique maximal flow $F : D_M \to M$ whose generator is v and which satisfies the following properties.

[46]For the heck of it I'll here remind the reader that the tangent $\dot{\gamma}$ to a curve $\gamma : I \subseteq \mathbb{R} \to M$ is given at each $t \in \mathbb{R}$ by $\dot{\gamma}(t) = \mathrm{d}\gamma_t \left(\mathrm{d}/\mathrm{d}t \,\big|_t \right) \in T_{\gamma(t)}M$, and that writing $\gamma^\mu = \mathrm{x}^\mu \circ \gamma : \mathbb{R} \to \mathbb{R}$ in a coordinate system $\mathrm{x} : M \to \mathbb{R}^d$ and unraveling this definition gives the familiar coordinate expression $\dot{\gamma}^\mu = \mathrm{d}\gamma^\mu / \mathrm{d}t$.

- For each $p \in M$ the curve $F_p : D_p \to M$ is the unique maximal integral curve of v starting at p.
- If $t \in D_p$ then $D_{F(t,p)} = D_p - t$. In other words suppose that $D_p = (-a, b)$ for some $0 \le a, b \in \mathbb{R}$, i.e. that at p we can flow forward in time by an amount b or backward in time by an amount a. Then if we flow forward by an amount t to $F(t,p)$ it follows that $D_{F(t,p)} = (-a-t, b-t)$, i.e. from $F(t,p)$ we can flow forward by $b-t$ or backward by $a+t$. Even more colloquially: if we start with a past of length a and a future of length b, and then a time t passes, our past is now longer and our future shorter, both by that amount of time t.
- Let $M_t = \left\{p \in M \,\middle|\, (t,p) \in D_M\right\} \subseteq M$ (i.e. the slice of D_M defined by t, analogous to the slice D_p of D_M defined by p). Then for each $t \in \mathbb{R}$ the map[47] $F_t : M_t \to M_{-t}$ is a diffeomorphism with inverse $F_{-t} : M_{-t} \to M_t$.

Note in particular that the final bullet point tells us that if $M_t = M$ then F_t is an automorphism of M.

A.5.2 Infinitesimal coordinate transformations

The point of the above is that flows provide us with a concrete notion of *parametrized* diffeomorphisms. In previous sections we considered the passive and active transformations defined by individual diffeomorphisms F. By now considering those diffeomorphisms which are embedded in flows we can consider transformations which are not just individual instantaneous transformations between configurations, but rather continuous evolutions which begin at the identity and smoothly reconfigure the coordinates/tensors (in the passive/active pictures) as the flow parameter evolves. For this reason I will also refer to a flow on M as a *continuous transformation* of M, and if there exists a neighborhood $(-\varepsilon, \varepsilon)$ of 0 such that $M_t = M$ for

[47]To make sense of the domain and codomain observe first that $p \in M_t$ implies that $t \in D_p$, and recall from the previous bullet point that $D_{F(t,p)} = D_p - t$. Then in particular $-t \in D_{F(t,p)}$ for all p, since $0 \in D_p$ by definition, and thus $F(t,p) \in M_{-t}$. In other words: the domain M_t is the set of all points in M which F is capable of pushing forward by an amount t, i.e. the set of all points in M on which $F_t : p \mapsto F(t,p)$ is defined, and F_{-t} is therefore defined on every point in $F_t(M_t)$, meaning that F_t sends points in M_t to points in M_{-t}.

all $t \in (-\varepsilon, \varepsilon)$ then any such $F_t : M \to M$ is a *continuous diffeomorphism* on M.

In particular we can now sensibly consider *infinitesimal* diffeomorphisms, which are those diffeomorphisms defined by evaluating a continuous diffeomorphism F on M at infinitesimal values of the flow parameter, i.e. the resulting diffeomorphisms $F_\varepsilon : M \to M$ with $\varepsilon \ll 1$.

These infinitesimal diffeomorphisms yield infinitesimal passive and active transformations as follows. Let $\mathsf{x} : U \subseteq M \to \mathbb{R}^d$ be a coordinate system and $F : D_M \to M$ a flow on M.

- Passive. For all $t \in \mathbb{R}$ such that $U \subseteq M_t$ (i.e. for all t such that F_t is defined on U) define $\tilde{U}_t = F_t(U)$. Then since by the fundamental theorem of flows F_t is a diffeomorphism $M_t \to M_{-t}$ it follows that the map $\tilde{\mathsf{x}}_t : \tilde{U}_t \to \mathbb{R}^d$ defined by $\tilde{\mathsf{x}}_t = \mathsf{x} \circ F_{-t}$ is a coordinate system on $\tilde{U}_t$. The "new" coordinates of some $p \in U \cap \tilde{U}_t$ can then be written in terms of the "old" coordinate representation of the flow as

$$\tilde{\mathsf{x}}^\mu_t(p) = \mathsf{x}^\mu \circ F_{-t}(p) = F^\mu_p(-t), \tag{A.92}$$

merrily swapping between notations $F(t,p) = F_t(p) = F_p(t)$. Now, each $F^\mu_p : \mathbb{R} \to \mathbb{R}$ is a smooth function and therefore admits a Taylor expansion. Evaluating at infinitesimal parameter values gives

$$F^\mu_p(-\varepsilon) \approx F^\mu_p(0) - \varepsilon \frac{\mathrm{d}F^\mu_p}{\mathrm{d}t}(0). \tag{A.93}$$

The first term is just the "old" coordinates $F^\mu_p(0) = \mathsf{x}^\mu(p)$ of the point p, while the second is the components of the generator $v \subseteq \mathfrak{X}(M)$, $v : p \mapsto v_p = \dot{F}_p(0)$ of the flow:

$$\tilde{\mathsf{x}}^\mu_\varepsilon(p) \approx \mathsf{x}^\mu(p) - \varepsilon v^\mu_p. \tag{A.94}$$

- Active. In the active perspective we think of the flow as acting on the manifold itself, not on the coordinate system. Thus instead of defining a new coordinate system via $\tilde{\mathsf{x}} = \mathsf{x} \circ F_{-\varepsilon}$ we define a new *point* via $\tilde{p} = F_{-\varepsilon}(p)$, and compute the coordinates of this point in the *same* system x. However these coordinates are just $\mathsf{x}(\tilde{p}) = \mathsf{x} \circ F_{-\varepsilon}(p)$, and so turning the crank yields the exact same steps and the exact same result as before:

$$\mathsf{x}^\mu_\varepsilon(\tilde{p}) \approx \mathsf{x}^\mu(p) - \varepsilon v^\mu_p, \tag{A.95}$$

the only difference being the location of the tilde on the left hand side, which is again reflective of the difference between the active and passive pictures: in the passive picture we think of $\mathsf{x}^\mu - \varepsilon v^\mu$ as the new coordinates of the same point p, which is given by the old coordinates of the point $\tilde{p}$ which is mapped to p by the diff; while in the active picture we think of it as the coordinates of $\tilde{p}$ itself in the same unchanged coordinate system. And these perspectives are quite evidently equivalent.

A.5.3 The infinitesimal variation of a tensor

We now come to the point of this section, which to obtain the infinitesimal variation of a tensor field under the (active or passive) transformation defined by a continuous transformation $F : D \to M$. As before this variation is in practice equivalent from either perspective and differs only in its interpretation.

Let's begin by reiterating the (certainly familiar) expression for the variation of a smooth function $f : \mathbb{R} \to \mathbb{R}$ under an infinitesimal change $x \mapsto x + \varepsilon$ of its argument:

$$\delta f(x) \equiv f(x+\varepsilon) - f(x) \approx \varepsilon \frac{\mathrm{d}f}{\mathrm{d}x}(x). \tag{A.96}$$

This expression is typically thought of as a truncation of the Taylor series of f, but may in fact be considered to be a direct result of the limit definition of the derivative, evaluated at small but nonzero ε. This latter perspective is more useful for generalization, since for a generic tensor field $C \in \Gamma^k_\ell M$ on a smooth manifold we do not have an intrinsic Taylor series expansion available to us.[48] But we do have the *Lie derivative*, whose limit definition I'll here recall:

$$\mathcal{L}_v C = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\big(F_\varepsilon\big)^* C - C \Big), \tag{A.97}$$

where v is the generator of the continuous transformation F. Considering the difference quotient at small but nonzero ε then gives us our infinitesimal tensor variation:

$$\delta C \equiv \big(F_\varepsilon\big)^* C - C \approx \varepsilon\, \mathcal{L}_v C. \tag{A.98}$$

Let's interpret this in the active and passive pictures.

- ACTIVE. In the active picture the flow F induces a transformation of the tensor field C by the pullback $C \mapsto \tilde{C} = \big(F_\varepsilon\big)^* C$. The definition above is therefore straightforward: the infinitesimal variation δC is the difference between the old and new values of the tensor field at each point,

$$(\delta C)_p = \tilde{C}_p - C_p, \tag{A.99}$$

  where the new value at p is found by pulling back the old value at $F_\varepsilon(p)$ by F_ε. Given a coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ we can write this in components and as a function of the coordinates as

$$(\delta C)^\mu{}_\nu \circ \mathsf{x}^{-1} = \tilde{C}^\mu{}_\nu \circ \mathsf{x}^{-1} - C^\mu{}_\nu \circ \mathsf{x}^{-1}. \tag{A.100}$$

- PASSIVE. In the passive picture the interpretation is a little more subtle. Recall from sec. A.3 that the coordinate transformation $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}$ induced by a diffeomorphism F results in a transformation of the components of C as $\tilde{C}^\mu{}_\nu = (F^* C)^\mu{}_\nu \circ F^{-1}$. It follows that in the passive picture the quantity $(F_\varepsilon)^* C$, evaluated at any point p, gives the components of C in the new coordinates $\tilde{\mathsf{x}}_\varepsilon = \mathsf{x} \circ F_{-\varepsilon}$ at the *different* point $F_\varepsilon(p)$, and hence the variation $(\delta C)_p$ compares the components of the *same* tensor field C in different coordinates

[48] We could of course treat each component in some coordinate system as a real-valued function of some number of real variables and use the multivariable Taylor expansion, but the route we text in the main text is much more transparent and geometrically meaningful.

systems and at *different* points. To make this clear let's consider the components of δC:

$$(\delta C)^{\mu}{}_{\nu} = \tilde{C}^{\mu}{}_{\nu} \circ F_{\varepsilon} - C^{\mu}{}_{\nu}. \tag{A.101}$$

To express this as a function of the coordinates let's compose both sides of the above with x^{-1}. Using the fact that $\tilde{\mathsf{x}}_{\varepsilon} = \mathsf{x} \circ F_{-\varepsilon}$ we then find

$$(\delta C)^{\mu}{}_{\nu} \circ \mathsf{x}^{-1} = \tilde{C}^{\mu}{}_{\nu} \circ \tilde{\mathsf{x}}_{\varepsilon}^{-1} - C^{\mu}{}_{\nu} \circ \mathsf{x}^{-1}. \tag{A.102}$$

In other words, to compute δC at the point $p \in M$ which corresponds to $x \in \mathbb{R}^{d}$ via the old coordinates x: calculate the components of C in x at $p = \mathsf{x}^{-1}(x)$; calculate the components of C in the new coordinates $\tilde{\mathsf{x}}$ at the point $\tilde{p} = \tilde{\mathsf{x}}_{\varepsilon}^{-1}(x)$ which corresponds to the *same* coordinate value x via the *new* coordinate system $\tilde{\mathsf{x}}_{\varepsilon}$; and subtract the former from the latter.

To emphasize the equivalence of the above perspectives I will note that, in the standard notations in which all fields are expressed in components as a function of the coordinates and the explicit maps between spacetime points and coordinate values are suppressed, the active and passive coordinate representations (A.100) and (A.102) become identical,

$$\big(\delta C\big)^{\mu}{}_{\nu}(x) = \tilde{C}^{\mu}{}_{\nu}(x) - C^{\mu}{}_{\nu}(x). \tag{A.103}$$

I will also note that the first of eqs. (A.98) is in fact not specific to infinitesimal variations: given any diffeomorphism $F : M \to M$ we may define the (not necessarily infinitesimal) variation of the tensor C by

$$\delta C = F^{*} C - C, \tag{A.104}$$

with the same interpretations as above. It is only in writing $\delta C \approx \varepsilon\, \mathcal{L}_{v} C$ that we use the assumption that F is infinitesimal.

A.5.4 Infinitesimal variations in practice

Here I'll mention two points which are relevant in applying the above to physics.

- THE GENERATOR OF THE COORDINATE TRANSFORMATION. In this appendix I have denoted the generator of a continuous transformation $F : D_M \to M$ as $v \in \mathfrak{X}(M)$, in terms of which an infinitesimal coordinate transformation is given by

$$\tilde{\mathrm{x}}^\mu(p) \approx \mathrm{x}^\mu(p) - \varepsilon v_p^\mu \tag{A.105}$$

with $|\varepsilon| \ll 1$ an infinitesimal value of the flow parameter. In the physics literature it is standard to absorb the infinitesimal flow parameter into the generator, $\xi \equiv \varepsilon v$, so that

$$\tilde{\mathrm{x}}^\mu(p) = \mathrm{x}^\mu(p) - \xi_p^\mu. \tag{A.106}$$

This is essentially just cosmetic. However it's useful to keep in mind because in many contexts, such as the main text of this thesis, it is also common to pull a factor of the gravitational coupling κ out of ξ:

$$\tilde{\mathrm{x}}^\mu(p) = \mathrm{x}^\mu(p) - \kappa\xi_p^\mu. \tag{A.107}$$

Eqs. (A.105) and (A.107) are formally identical, and it's standard (and perfectly legal) practice to therefore treat κ as the parameter controlling the transformation. I just bring this up to point out that, if one wants to imagine such a transformation as actually occurring in a physical world like ours in which κ has a single value, then as the continuous transformation smoothly departs the identity what's really happening is that the generator ξ^μ in eq. (A.107) is smoothly growing from zero, due to having an infinitesimal ε buried in its definition as $\xi = \varepsilon v$ (with v the actual fixed generator of the transformation).

- Calculating the Lie derivative. When it comes to actually calculating the Lie derivative of a tensor field we do not use the limit definition any more than we do for the derivative of normal functions $\mathbb{R} \to \mathbb{R}$. Rather we use the following properties, all of which can be found in any standard text on differential geometry (e.g. [**?**] or [**?**]).

  - Let $u, v \in \mathfrak{X}(M)$. Then $\mathcal{L}_u v = [u, v]$. Given a coordinate frame ∂_μ this becomes

$$\mathcal{L}_u v = [u, v] = \left(u^\alpha \partial_\alpha v^\mu - v^\alpha \partial_\alpha u^\mu\right)\partial_\mu. \tag{A.108}$$

  - Let additionally $f \in C^\infty(M)$. Then

$$\mathcal{L}_v f = v(f), \quad \mathcal{L}_v \mathrm{d}f = \mathrm{d}\left(\mathcal{L}_v f\right) = \mathrm{d}\left(v(f)\right). \tag{A.109}$$

  - Finally let A and A' be tensor fields on M of the same (covariant, contravariant, or mixed) type, and B another tensor field on M (of not necessarily the same type as A and A'). Then

$$\mathcal{L}_v\left(A + A'\right) = \mathcal{L}_v A + \mathcal{L}_v A', \quad \mathcal{L}_v\left(A \otimes B\right) = \left(\mathcal{L}_v A\right) \otimes B + A \otimes \left(\mathcal{L}_v B\right). \tag{A.110}$$

  From these properties the Lie derivative of any tensor field of any type may be obtained. For example if $\boldsymbol{g} \in \Gamma_2 M$ then it may be shown that, given some coordinate coframe $\mathrm{d}x^\mu$,

$$\left(\mathcal{L}_v \boldsymbol{g}\right)_{\mu\nu} = v^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu v^\alpha + g_{\mu\alpha}\partial_\nu v^\alpha. \tag{A.111}$$

A.6 Coordinate transformations in perturbation theory

In this final section I adapt the ideas developed in Mukhanov, Feldman, and Brandenberger [80] to the notation and perspective taken so far in this appendix.

A.6.1 The passive perspective

In the passive perspective of perturbation theory we, per usual, suppose that we have one spacetime manifold M and multiple coordinate systems $\mathsf{x}, \tilde{\mathsf{x}} : M \to \mathbb{R}^d$. The passive perspective is defined by associating with every tensor field $C \in \Gamma^k_\ell M$ a *background value* $\bar{C} \in \Gamma^k_\ell \mathbb{R}^d$, i.e. a rank $\binom{k}{\ell}$ tensor field on $\mathbb{R}^d$.

N.B. this background value is *not* a geometrical object on M: in order to assign a background value of C to any $p \in M$ we first need a way to assign a value of $x \in \mathbb{R}^d$ to $p \in M$, i.e. a coordinate system $\mathsf{x} : M \to \mathbb{R}^d$. Given any such coordinate system we may assign a background value of C to each $p \in M$ by pulling $\bar{C}$ back by x, and the *perturbation* of C in this coordinate system is then defined to be

$$\Delta_{\mathsf{x}} C_p = C_p - \mathrm{d}\mathsf{x}^*_p \, \bar{C}_{\mathsf{x}(p)}. \tag{A.112}$$

In other words an observer with these coordinates would pick a point $p \in M$, use x to pull $\bar{C}$ back from $\mathsf{x}(p) \in \mathbb{R}^d$ to p, and then subtract this value from the actual value of C at p. In another coordinate system $\tilde{\mathsf{x}}$ the perturbation is defined identically, $\Delta_{\tilde{\mathsf{x}}} C_p = C_p - \mathrm{d}\tilde{\mathsf{x}}^*_p \, \bar{C}_{\tilde{\mathsf{x}}(p)}$, and the passive transformation from x to $\tilde{\mathsf{x}}$ manifests as a transformation of the perturbations, $\Delta_{\mathsf{x}} C \mapsto \Delta_{\tilde{\mathsf{x}}} C$.

Before proceeding to the explicit transformation rule for the perturbation I will note that our definition, $p \mapsto \left(\mathsf{x}^* \bar{C}\right)_p = \mathrm{d}\mathsf{x}^*_p \, \bar{C}_{\mathsf{x}(p)}$, of the assignment of a background value of C to the spacetime point p does indeed reproduce the more familiar statement that the background value of a tensor field has the same components at the same coordinate values in any coordinate system. To see this let's consider calculating the components of $\mathsf{x}^* \bar{C}$:

$$\left(\mathsf{x}^* \bar{C}\right)^\mu{}_\nu(p) = \mathrm{d}\mathsf{x}^*_p \, \bar{C}_{\mathsf{x}(p)}\left(\mathrm{d}\mathsf{x}^\mu\big|_p, \partial_\nu\big|_p\right) = \bar{C}_{\mathsf{x}(p)}\left(\mathrm{d}\left(\mathsf{x}^{-1}\right)^*_{\mathsf{x}(p)} \mathrm{d}\mathsf{x}^\mu\big|_p, \mathrm{d}\mathsf{x}_p \, \partial_\nu\big|_p\right). \tag{A.113}$$

To evaluate the arguments we use the facts that[49]

$$\partial_\mu\big|_p = \mathrm{d}\big(\mathsf{x}^{-1}\big)_{\mathsf{x}(p)} \left.\frac{\partial}{\partial x^\mu}\right|_{\mathsf{x}(p)}, \quad \mathrm{d}\mathsf{x}^\mu\big|_p = \mathrm{d}\mathsf{x}_p^*\, \mathrm{d}x^\mu\big|_{\mathsf{x}(p)}, \tag{A.114}$$

along with the composition rules (Lemma A.2) for the pushforward and pullback, to find

$$\big(\mathsf{x}^*\bar{C}\big)^\mu{}_\nu(p) = \bar{C}_{\mathsf{x}(p)}\left(\mathrm{d}x^\mu\big|_{\mathsf{x}(p)}, \left.\frac{\partial}{\partial x^\nu}\right|_{\mathsf{x}(p)}\right) = \bar{C}^\mu{}_\nu \circ \mathsf{x}(p), \tag{A.115}$$

where $\bar{C}^\mu{}_\nu$ refers to the components of $\bar{C} \in \Gamma^k_\ell \mathbb{R}^d$ in the canonical coordinate system of $\mathbb{R}^d$. Written as a function of the coordinates the above is

$$\big(\mathsf{x}^*\bar{C}\big)^\mu{}_\nu \circ \mathsf{x}^{-1} = \bar{C}^\mu{}_\nu, \tag{A.116}$$

meaning that, if two *different* observers calculate the components of $\bar{C}$ in their respective coordinate systems x and $\tilde{\mathsf{x}}$ at the same coordinate value x (and hence at different points $p = \mathsf{x}^{-1}(x)$ and $\tilde{p} = \tilde{\mathsf{x}}^{-1}(x)$), they'll find the *same* components, namely the components of $\bar{C}$ in the canonical coordinate system on $\mathbb{R}^d$.

In fact, the perturbation $\Delta_{\mathsf{x}}C$ of C as given in eq. (A.112) is a perfectly well-defined tensor field in its own right, and we may therefore describe its variation under a change in coordinates $\tilde{\mathsf{x}} = \mathsf{x} \circ F^{-1}$ via eq. (A.98) as we would for any other tensor field:

$$\delta\big(\Delta C\big)_p = \mathrm{d}F_p^*\left(\Delta_{\tilde{\mathsf{x}}}C\right)_{F(p)} - \left(\Delta_{\mathsf{x}}C\right)_p. \tag{A.117}$$

(Keep in mind that a capital Δ refers to the perturbation of a tensor in a particular coordinate system, while a lowercase δ refers to the variation of a tensor under a change of coordinates.)

A convenient feature of this formulation is that the variation of ΔC is identical to the vari-

[49] To see the latter note that, thinking of the right hand side as a map from T_pM to $\mathbb{R}$, we have $\mathrm{d}\mathsf{x}_p^*\, \mathrm{d}x^\mu\big|_{\mathsf{x}(p)} = \mathrm{d}x^\mu\big|_{\mathsf{x}(p)} \circ \mathrm{d}\mathsf{x}_p = \mathrm{d}\big(x^\mu \circ \mathsf{x}\big)_p = \mathrm{d}\mathsf{x}^\mu\big|_p$, since $x^\mu \circ \mathsf{x} = \mathsf{x}^\mu$ (thinking of x^μ as the map which eats a point in $\mathbb{R}^d$ and returns the μ^{th} component of its canonical component value).

ation of C itself, since the background value (by construction) doesn't change. To see this let's evaluate the first term on the right hand side of eq. (A.117):

$$\mathrm{d}F_p^*\left(\Delta_{\tilde{\mathsf{x}}}C\right)_{F(p)} = \mathrm{d}F_p^*\,C_{F(p)} - \mathrm{d}F_p^*\circ\mathrm{d}\tilde{\mathsf{x}}_{F(p)}^*\,\bar{C}_{\tilde{\mathsf{x}}\circ F(p)}. \tag{A.118}$$

Since $\tilde{\mathsf{x}}\circ F = \mathsf{x}$ this reduces to

$$\mathrm{d}F_p^*\left(\Delta_{\tilde{\mathsf{x}}}C\right)_{F(p)} = \mathrm{d}F_p^*\,C_{F(p)} - \mathrm{d}\mathsf{x}_p\,\bar{C}_{\mathsf{x}(p)}, \tag{A.119}$$

and hence the variation of the perturbation becomes

$$\delta\big(\Delta C\big)_p = \mathrm{d}F_p^*\,C_{F(p)} - C_p = \big(\delta C\big)_p, \tag{A.120}$$

as promised. And, via eq. (A.98), this further yields the useful result that the variation of a perturbation under an infinitesimal coordinate transformation is given by the Lie derivative of the full field with respect to the generator of that transformation:

$$\delta\big(\Delta C\big) = \delta C \approx \varepsilon\,\mathcal{L}_v C. \tag{A.121}$$

A.6.2 The active perspective

In the active perspective we consider only one coordinate system $\mathsf{x} : M \to \mathbb{R}^d$ and think of diffeomorphisms $F : M \to M$ as moving around the tensor fields themselves via $C \mapsto \tilde{C} = F^*C$ (as discussed above). From the active perspective the degrees of freedom of a diffeomorphism-invariant theory are then in fact *equivalence classes* of tensor fields, say $[C] = \{F^*C\,|\,F : M \to M \text{ is a diff}\}$ for $C \in \Gamma^k_\ell M$, and with each such equivalence class we associate a background value $\bar{C} \in \Gamma^k_\ell M$.

N.B. unlike the passive perspective we here seem to have constructed the background value as an actual geometric object on M. However in this case the "non-geometric-ness" of

$\bar{C}$ is in fact preserved, and manifests here in the fact that, while the tensor fields themselves transform under diffeomorphisms as $C \mapsto F^*C$, the background values do not, $\bar{C} \mapsto \bar{C}$. In other words, the physical content of a set of tensor fields is in the equivalence classes to which they belong, and we are therefore free to change the actual tensor fields at hand by any diffeomorphism we please. However we have associated a single background value to each *equivalence class*, not to each individual tensor field value, and thus any diffeomorphism by which we change our physical tensor field does *not* change the background value of that field.

Note also that, since in this perspective the background value doesn't change under diffeomorphisms and neither does the coordinate system, we also preserve (in a much simpler way) the statement that under diffeomorphisms the components of the background value are unchanged at a given coordinate value.

In the active perspective we define the perturbation of C by

$$\Delta C = C - \bar{C}. \tag{A.122}$$

Under a diffeomorphism $F : M \to M$ the actual field C changes as $C \mapsto \tilde{C} = F^*C$ while the background value is unchanged, so the perturbation also changes:

$$\Delta\tilde{C} = \tilde{C} - \bar{C} = F^*C - \bar{C}. \tag{A.123}$$

The variation of the perturbation under this diffeomorphism is therefore identical to the variation of the full field, just as we found in eq. (A.120) for the passive case:

$$\delta\left(\Delta C\right) = F^*C - C = \delta C. \tag{A.124}$$

And by the same logic the variation under an infinitesimal active transformation is given by the Lie derivative with respect to the generator of that transformation,

$$\delta\big(\Delta C\big) = \delta C \approx \varepsilon\, \mathcal{L}_v C, \tag{A.125}$$

as in eq. (A.121).

A.6.3 The infinitesimal variation of the metric perturbation

The above construction of a background value and perturbation for any tensor field applies to any smooth manifold M, and in particular makes no reference to any metric on M. Thus if we do have a metric $\boldsymbol{g}$ on M then the whole construction applies to $\boldsymbol{g}$ as it would for any other rank $\binom{0}{2}$ tensor field on M, so we can write (in e.g. the active perspective)

$$\boldsymbol{g} = \bar{\boldsymbol{g}} + \Delta\boldsymbol{g} \equiv \bar{\boldsymbol{g}} + \kappa\boldsymbol{h}, \tag{A.126}$$

pulling out a factor of the gravitational coupling κ to give the metric perturbation $\boldsymbol{h}$ its canonical mass dimension of one.

In the main text we'll be interested in an expression for the variation of $\boldsymbol{h}$ under an infinitesimal transformation $\tilde{\mathsf{x}}^\mu = \mathsf{x}^\mu - \kappa\xi^\mu$ (in the language of sec. A.5.4) in terms of $\boldsymbol{h}$ itself (as opposed to the full metric $\boldsymbol{g}$) in the particular case where the background metric is flat, $\bar{\boldsymbol{g}} = \boldsymbol{\eta}$. In terms of the Lie derivative this is

$$\delta\boldsymbol{h} = \frac{1}{\kappa}\delta\big(\Delta\boldsymbol{g}\big) = \mathcal{L}_\xi\boldsymbol{g}. \tag{A.127}$$

This can be evaluated straightforwardly from eq. (A.111), which tells us the Lie derivative of the full metric:

$$\big(\mathcal{L}_\xi\boldsymbol{g}\big)_{\mu\nu} = \xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\xi^\alpha + g_{\mu\alpha}\partial_\nu\xi^\alpha. \tag{A.128}$$

Using $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ we then find[50]

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \kappa\Big(\xi^\alpha \partial_\alpha h_{\mu\nu} + h_{\alpha\nu}\partial_\mu \xi^\alpha + h_{\mu\alpha}\partial_\nu \xi^\alpha\Big). \tag{A.129}$$

In fact an identical expression holds for an arbitrary background metric with the partial derivatives replaced by the covariant derivative with respect to the background. We can see this by keeping track of the partial derivatives of the background metric in the Lie derivative of the full expanded metric:

$$\begin{aligned}\delta h_{\mu\nu} &= \xi^\rho \partial_\rho(\bar{g}_{\mu\nu} + \kappa h_{\mu\nu}) + \Big((\bar{g}_{\rho\nu} + \kappa h_{\rho\nu})\partial_\mu + (\bar{g}_{\rho\mu} + \kappa h_{\rho\mu})\partial_\nu\Big)\xi^\rho \\ &= \xi^\rho \partial_\rho \bar{g}_{\mu\nu} + \bar{g}_{\rho\nu}\partial_\mu \xi^\rho + \bar{g}_{\rho\mu}\partial_\nu \xi^\rho + \kappa\Big(\xi^\rho \partial_\rho h_{\mu\nu} + h_{\rho\nu}\partial_\mu \xi^\rho + h_{\rho\mu}\partial_\nu \xi^\rho\Big).\end{aligned} \tag{A.130}$$

Now recall that the covariant derivative $\bar{\nabla}$ with respect to the background metric acts on ξ and $\boldsymbol{h}$ as

$$\bar{\nabla}_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma^\beta_{\alpha\gamma}\xi^\gamma, \quad \bar{\nabla}_\alpha h_{\beta\gamma} = \partial_\alpha h_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} h_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} h_{\beta\delta}. \tag{A.131}$$

Hence if we write down the κ coefficient with $\bar{\nabla}$'s in place of ∂'s and expand we find

$$\begin{aligned}\xi^\rho \bar{\nabla}_\rho h_{\mu\nu} + h_{\rho\nu}\bar{\nabla}_\mu \xi^\rho + h_{\rho\mu}\bar{\nabla}_\nu \xi^\rho &= \Big(\xi^\rho \partial_\rho h_{\mu\nu} - \xi^\rho \Gamma^\alpha_{\rho\mu} h_{\alpha\nu} - \xi^\rho \Gamma^\alpha_{\rho\nu} h_{\mu\alpha}\Big) \\ &\quad + \Big(h_{\rho\nu}\partial_\mu \xi^\rho + h_{\rho\nu}\Gamma^\rho_{\mu\alpha}\xi^\alpha\Big) + \Big(h_{\rho\mu}\partial_\nu \xi^\rho + h_{\rho\mu}\Gamma^\rho_{\nu\alpha}\xi^\alpha\Big) \\ &= \xi^\rho \partial_\rho h_{\mu\nu} + h_{\rho\nu}\partial_\mu \xi^\rho + h_{\rho\mu}\partial_\nu \xi^\rho\end{aligned} \tag{A.132}$$

with the second term in the first set of brackets cancelling with the second term in the second, and the third in the first cancelling with the second in the third. So in other words we recover the κ coefficient.

To rewrite the κ^0 coefficient in terms of $\bar{\nabla}$ we need to dig into the actual definition of the Christoffel symbol. Considering the Christoffel symbol part of the $\partial \mapsto \bar{\nabla}$ version of the last

[50] Keep in mind that $\delta h_{\mu\nu}$ are the components of the variation $\delta\boldsymbol{h}$ of $\boldsymbol{h}$, not the variation of the components $h_{\mu\nu}$ of $\boldsymbol{h}$.

term in that coefficient, we have

$$\bar{g}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma}\xi^{\sigma} = \tfrac{1}{2}\bar{g}_{\mu\rho}g^{\rho\alpha}\big(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\nu\sigma}\big)\xi^{\sigma} = \tfrac{1}{2}\xi^{\rho}\big(\partial_\nu g_{\mu\rho} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\big). \tag{A.133}$$

If we symmetrize in $\mu \leftrightarrow \nu$ the antisymmetric part (the first and third terms) vanishes, yielding

$$\bar{g}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma}\xi^{\sigma} + \bar{g}_{\rho\mu}\Gamma^{\rho}_{\nu\sigma}\xi^{\sigma} = \xi^{\rho}\partial_\rho g_{\mu\nu}, \tag{A.134}$$

i.e. the first term in the κ^0 coefficient. And since the partial derivative parts of $\bar{g}_{\rho\nu}\bar{\nabla}_\mu\xi^{\rho}+\bar{g}_{\rho\mu}\bar{\nabla}_\nu\xi^{\rho}$ are precisely the second and third terms in this coefficient it follows that this coefficient is given in terms of the covariant derivative by

$$\xi^{\rho}\partial_\rho\bar{g}_{\mu\nu} + \bar{g}_{\rho\nu}\partial_\mu\xi^{\rho} + \bar{g}_{\rho\mu}\partial_\nu\xi^{\rho} = \bar{g}_{\rho\nu}\bar{\nabla}_\mu\xi^{\rho} + \bar{g}_{\rho\mu}\bar{\nabla}_\nu\xi^{\rho} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu. \tag{A.135}$$

So in sum the gauge transformation of the metric perturbation is given by

$$h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + \Big(\bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu\Big) + \kappa\Big(\xi^{\rho}\bar{\nabla}_\rho h_{\mu\nu} + h_{\rho\nu}\bar{\nabla}_\mu\xi^{\rho} + h_{\rho\mu}\bar{\nabla}_\nu\xi^{\rho}\Big), \tag{A.136}$$

as promised.

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