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# Numerical invariants of Cohen-Macaulay local and graded rings

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# Abstract

We characterize different classes of Cohen-Macaulay local rings $(R, \mathfrak{m}, k)$ with positive Krull dimension in terms of MCM approximations of finitely-generated $R$-modules.

Assume $R$ has a canonical module. For each finitely-generated $R$-module $M$, Auslander's $\delta$-invariant $\delta_R(M)$ equals the rank of a maximal free direct summand of the minimal MCM approximation $X_M$ of $M$. We have $\delta_R(R/\mathfrak{m}) = 1$ if and only if $R$ is a regular local ring. Auslander defined the index of $R$, denoted $\operatorname{index}(R)$, as the infimum of positive integers $n$ such that $\delta_R(R/\mathfrak{m}^n) = 1$.

When $R$ is Gorenstein, we have $\operatorname{index}(R) \leq \mathrm{g}\ell\ell(R) < \infty$, where $\mathrm{g}\ell\ell(R)$ denotes the generalized Loewy length of $R$, the smallest positive integer $n$ such that $\mathfrak{m}^n \subseteq \mathbf{x}R$ for some system of parameters $\mathbf{x}$ for $R$. We call such a system of parameters a witness to the generalized Loewy length of $R$.

In Chapter 3, we generalize a theorem of Ding, who proved that if $R$ is Gorenstein with infinite residue field $k$ and Cohen-Macaulay associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$, then $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)$. We prove that if $R$ is a one-dimensional Cohen-Macaulay local ring with finite index and nonzerodivisor $x$ of order $t$ with $\mathrm{gr}_{\mathfrak{m}}(R)$-regular initial form $x^*$, then $\mathrm{g}\ell\ell(R) \leq \operatorname{index}(R) + t - 1$.

We use this estimate to derive a formula for the generalized Loewy length of a one-dimensional hypersurface $R = k[\![x,y]\!]/(f)$. If $z$ is a witness to $\mathrm{g}\ell\ell(R)$ such that $z^*$ is $\mathrm{gr}_{\mathfrak{m}}(R)$-regular, then $\mathrm{g}\ell\ell(R) = \operatorname{ord}_R(z) + e(R) - 1$, where $e(R)$ denotes the Hilbert-Samuel multiplicity of $R$. We compute the generalized Loewy lengths of several families $\{R_n\}_{n=1}^{\infty}$ of one-dimensional hypersurfaces over finite and infinite fields such that $\mathrm{g}\ell\ell(R_n) = \operatorname{index}(R_n)$ for all $n \geq 1$ or $\mathrm{g}\ell\ell(R_n) = \operatorname{index}(R_n) + 1$ for all $n \geq 1$. Lastly, we study a graded version of the generalized Loewy length of a Noetherian local ring for Noetherian $k$-algebras $(R, \mathfrak{m}, k)$, where $k$ is an arbitrary field and $\mathfrak{m}$ is the irrelevant

ideal of $R$. This invariant is called the generalized graded length of $R$ and denoted $\mathbf{ggl}(R)$. After determining bounds for $\mathbf{ggl}(R)$ in terms of $\mathrm{g}\ell\ell(R)$ and the degrees of generators for $R$, we compute the generalized graded length of numerical semigroup rings. We also characterize witnesses to the generalized graded length of numerical semigroup rings for semigroups with two generators.

In Chapter 4, we study criteria for when an MCM module over a Gorenstein complete local ring $R$ is stably isomorphic to an MCM approximation of a finitely-generated $R$-module of some fixed positive codimension $r$. If this condition holds for an MCM $R$-module $M$, we say with Kato that $M$ satisfies the $\mathrm{SC}_r$-condition. If this condition holds for every MCM $R$-module, we say that $R$ satisfies the $\mathrm{SC}_r$-condition.

Only the $\mathrm{SC}_1$- and $\mathrm{SC}_2$- conditions have been characterized for Gorenstein complete local rings $R$. Kato proved that $R$ satisfies the $\mathrm{SC}_1$-condition if and only if $R$ is a domain, and $R$ satisfies the $\mathrm{SC}_2$-condition if and only if $R$ is a UFD. For rings of dimension $d \geq 3$ and $3 \leq r \leq d$, we prove an inductive criterion for when an MCM $R$-module satisfies the $\mathrm{SC}_r$-condition when its first syzygy module $\Omega^1_R(M)$ satisfies the $\mathrm{SC}_{r-1}$-condition. We use this criterion to prove the equivalence of the $\mathrm{SC}_d$- and $\mathrm{SC}_{d-1}$-conditions for Gorenstein complete local rings of dimension $d \geq 3$ that remain UFDs when factoring out certain regular sequences of length $d-2$.

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# Contents

# 1 | Introduction

The unifying theme of this dissertation is the theory of maximal Cohen-Macaulay (MCM) approximations, originally constructed by Auslander and Buchweitz in the context of abelian categories [1]. We apply this theory to the category of finitely-generated modules over a Cohen-Macaulay local ring $(R,\mathfrak{m},k)$ with canonical module. Every module $M$ has a unique short exact sequence

$$0 \longrightarrow Y_M \xrightarrow{\iota} X_M \xrightarrow{\pi} M \longrightarrow 0$$

where $X_M$ is an MCM $R$-module, $Y_M$ has finite injective dimension over $R$, and the modules $X_M$ and $Y_M$ have no direct summand in common via $\iota$, called the minimal MCM approximation of $M$. The module $X_M$ is called the minimal MCM approximation of $M$ as well.

The construction of MCM approximations is motivated in part by the orthogonality relation between MCM modules and modules of finite injective dimension, described in Proposition 2.2.3. In particular, for each integer $i > 0$, we have $\operatorname{Ext}^i_R(Z,W) = 0$ for each MCM $R$-module $Z$ and each finitely-generated $R$-module $W$ of finite injective dimension. Moreover, for each MCM $R$-module $Z$, every $R$-map $Z \longrightarrow M$ factors through $X_M$. We are able to characterize different classes of Cohen-Macaulay rings in terms of minimal MCM approximations.

Assume $R$ has positive dimension $d$ and let $0 \le r \le d$. With Kato, we say that an MCM $R$-module $X$ satisfies the $\mathrm{SC}_r$-condition if $X$ is stably isomorphic to the minimal MCM approximation of a finitely-generated $R$-module of codimension $r$. We say that $R$ satisfies the $\mathrm{SC}_r$-condition if every MCM $R$-module is stably isomorphic to the minimal MCM approximation of a finitely-generated $R$-module of codimension $r$ [15]. Every Cohen-Macaulay local ring with canonical module satisfies the $SC_0$-condition, since every MCM module is its own minimal MCM approximation.

In their 2000 paper, Yoshino and Isogawa proved that Gorenstein complete local domains satisfy the $SC_1$-condition. They also proved that a normal Gorenstein complete local ring of dimension two satisfies the $SC_2$-condition if and only if it is a UFD [25].

In her 2007 paper, Kato proved that a Gorenstein complete local ring satisfies the $SC_1$-condition if and only if it is a domain, and satisfies the $SC_2$-condition if and only if it is a UFD [15]. More generally, Leuschke and Wiegand proved that a Cohen-Macaulay local ring with canonical module satisfies the $SC_1$-condition if and only if it is a domain [17].

For Gorenstein complete local rings $R$ of dimension $d$, the $SC_r$-conditions for $3 \le r \le d$ have not yet been characterized. Since the $SC_{r+1}$-condition implies the $SC_r$-condition, rings that satisfy the $SC_r$-condition for $3 \le r \le d$ must be UFDs. In chapter 4, we give a criterion for when an MCM $R$-module satisfies the $SC_r$-condition when its first syzygy module satisfies the $SC_{r-1}$-condition. With this criterion, we prove the equivalence of the $SC_d$- and $SC_{d-1}$-conditions for rings that remain UFDs when we factor out certain regular sequences of length $d-2$.

Let $R$ be a Cohen-Macaulay local ring with canonical module. In chapter 3, we study a ring invariant that can be computed using minimal MCM approximations. For each finitely-generated $R$-module $M$, *Auslander's $\delta$-invariant*, denoted $\delta_R(M)$, is the rank of a maximal free direct summand of $X_M$. Consider the $\delta$-invariants of the quotients $\{R/\mathfrak{m}^n\}_{n=1}^{\infty}$. For $n \ge 1$, we have

$$0 \le \delta_R(R/\mathfrak{m}^n) \le \delta_R(R/\mathfrak{m}^{n+1}) \le 1.$$

Auslander defined the *index* of $R$, denoted $\operatorname{index}(R)$, as the infimum of positive integers $n$ for which $\delta_R(R/\mathfrak{m}^n) = 1$. By a result of Auslander, a Cohen-Macaulay local ring is a regular local ring if and only if $\operatorname{index}(R) = 1$ [17, Proposition 11.37].

The index of a Gorenstein local ring $R$ is bounded above by the generalized Loewy length $g\ell\ell(R)$. The generalized Loewy length is more elementary than the index, being defined for a

Noetherian local ring $(R,\mathfrak{m})$ as the smallest positive integer $n$ such that $\mathfrak{m}^n \subseteq \mathbf{x}R$ for some system of parameters $\mathbf{x}$ of $R$. We call such a system of parameters a *witness* to the generalized Loewy length of $R$.

Ding proved that if $R$ is a Gorenstein local ring with infinite residue field and Cohen-Macaulay associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$, then $\mathrm{index}(R) = \mathrm{g}\ell\ell(R)$ [8, Theorem 2.1].

It is natural to ask about the relation between the index and generalized Loewy length of $R$ when either its residue field is finite or the associated graded ring is not Cohen-Macaulay. In [4], De Stefani gave examples of one-dimensional Gorenstein rings $R$ with infinite residue field and non-Cohen-Macaulay associated graded ring for which $\mathrm{g}\ell\ell(R) = \mathrm{index}(R) + 1$.

All hypersurfaces have Cohen-Macaulay associated graded ring. Moreover, the index of a hypersurface is easy to compute, since it equals the Hilbert-Samuel multiplicity $e(R)$. However, we do not obtain equality of the index and generalized Loewy length for hypersurfaces over finite fields. In [11], Hashimoto and Shida proved that for $R = \mathbb{F}_2[\![x,y]\!]/(xy(x+y))$, $\mathrm{index}(R) = 3$ and $\mathrm{g}\ell\ell(R) = 4$. The inequality between the index and generalized Loewy length results from the fact that $R$ has no homogeneous linear nonzerodivisors.

Since we have $\mathrm{index}(R) \leq \mathrm{g}\ell\ell(R)$ for any Gorenstein ring $R$, it is natural to seek an upper bound for the generalized Loewy length in terms of the index that holds when the residue field is either finite or infinite. In [8, Theorem 2.1], Ding uses the hypothesis that $k$ is infinite to obtain a maximal regular sequence consisting of linear forms in $(\mathrm{gr}_{\mathfrak{m}}(R))_1$. When $k$ is finite and $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, a maximal regular sequence in $\mathrm{gr}_{\mathfrak{m}}(R)$ may consist of homogeneous elements of degrees greater than one, which cannot be used in Ding's argument to prove $\mathrm{index}(R) = \mathrm{g}\ell\ell(R)$. However, if $R$ is a one-dimensional Cohen-Macaulay local ring with finite index and Cohen-Macaulay associated graded ring, then we can use a homogeneous $\mathrm{gr}_{\mathfrak{m}}(R)$-regular element of minimal degree to obtain an upper bound for $\mathrm{g}\ell\ell(R)$ in terms of $\mathrm{index}(R)$. In Theorem 3.2.4 we prove that if $\mathrm{gr}_{\mathfrak{m}}(R)$ has a regular homogeneous element of degree $t$, then $\mathrm{g}\ell\ell(R) \leq \mathrm{index}(R) + t - 1$. Using this estimate, we obtain the following formula for the generalized Loewy length of a one-dimensional hypersurface.

**Proposition.** *Let $k$ be a field and $S = k[\![x,y]\!]$. Let $R = k[\![x,y]\!]/(f)$ with $\operatorname{ord}_S(f) = e$ and $z \in R$ such that the initial form $z^*$ is $\operatorname{gr}_{\mathfrak{m}}(R)$-regular. If $z$ is a witness to $\mathrm{g}\ell\ell(R)$, then*

$$\mathrm{g}\ell\ell(R) = \operatorname{ord}_R(z) + e - 1.$$

We then compute the generalized Loewy length of infinite families of hypersurfaces over finite fields, generalizing Hashimoto and Shida's example of a hypersurface $R$ for which $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)+1$. We also give examples of hypersurfaces of the form $R = k[\![x,y]\!]/(xy(x^n+y^n))$ where $k$ is a finite field and $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)$. Lastly, we study an invariant for Noetherian algebras over an arbitrary field $k$ that is a graded version of the generalized Loewy length of a Noetherian local ring. Let $(R,\mathfrak{m})$ denote a Noetherian $k$-algebra $R = \bigoplus_{i=0}^{\infty} R_i$ with irrelevant ideal $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$. For $n \geq 1$, define $\mathfrak{m}_n := \bigoplus_{i=n}^{\infty} R_i$. The *generalized graded length* of $R$, denoted $\mathbf{ggl}(R)$, is the smallest positive integer $n$ such that $\mathfrak{m}_n \subseteq \mathbf{x}R$, where $\mathbf{x}$ is a homogeneous system of parameters for $R$.

In this setting, it is natural to define the generalized Loewy length $\mathrm{g}\ell\ell(R)$ of $R$ as the smallest positive integer $n$ such that $\mathfrak{m}^n \subseteq \mathbf{x}R$, where $\mathbf{x}$ is a homogeneous system of parameters for $R$. Since $\mathfrak{m}^n \subseteq \mathfrak{m}_n$ for each $n$, we have $\mathrm{g}\ell\ell(R) \leq \mathbf{ggl}(R)$. Let $x_1,\ldots,x_n \in \mathfrak{m}$ be homogeneous elements generating $R$. In Proposition 3.4.3, we prove that

$$a(\mathrm{g}\ell\ell(R)) - (a-1)^2 \leq \mathbf{ggl}(R) \leq b(\mathrm{g}\ell\ell(R)) - b + 1,$$

where $\min\{\deg(x_i)\}_{i=1}^n = a \leq b = \max\{\deg(x_i)\}_{i=1}^n$. In particular, we have $\mathbf{ggl}(R) = \mathrm{g}\ell\ell(R)$ if $a = b = 1$. We compute the generalized graded length of numerical semigroup rings $k[H] := k[t^{a_1},\ldots,t^{a_n}] \subseteq k[t]$, where $|t^a| = a$ and $H = \langle a_1,\ldots,a_n\rangle$ is the numerical semigroup generated by positive integers $a_1 < \cdots < a_n$ with $\gcd(a_1,\ldots,a_n) = 1$. For $R = k[H]$, we have $\mathbf{ggl}(R) = C + a_1$, where $C$ denotes the conductor of $H$. When $n = 2$, the conductor of $\langle a,b\rangle$ with $a < b$ coprime is $ba - a - b + 1$, so $\mathbf{ggl}(R) = ba - b + 1$.

A homogeneous system of parameters $\mathbf{x}$ for a Noetherian $k$-algebra $(R,\mathfrak{m})$ is a witness to the generalized graded length of $R$ if $\mathfrak{m}_g \subseteq \mathbf{x}R$, where $g = \mathbf{ggl}(R)$. We prove that a witness $z$ to the generalized graded length of $k[t^a, t^b]$ satisfies $(z) = (t^{ia})$, where $1 \leq i \leq b-a+1$.

# 2 | Background

In this chapter, we give definitions and preliminary results required for the results in chapters 3 and 4. Throughout, $(R,\mathfrak{m},k)$ is a Noetherian local ring $R$ with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

## 2.1 Cohen-Macaulay rings and modules

### 2.1.1 Krull dimension

Given a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$, the *height* of $\mathfrak{p}$, denoted $\operatorname{height}\mathfrak{p}$ or $\operatorname{ht}\mathfrak{p}$, is the supremum of integers $t$ such that there exists a chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_t,$$

where $\mathfrak{p}_i$ is a prime ideal for $i = 0,...,t$. The *Krull dimension* of $R$, or simply the dimension of $R$, denoted $\dim R$, is the supremum of the heights of prime ideals of $R$.

$$\dim R := \sup\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \subseteq \mathfrak{m} \text{ is prime}\}$$

Since $R$ is Noetherian local, the dimension of $R$ is equal to the height of $\mathfrak{m}$, and therefore finite. If $M$ is a finitely-generated $R$-module, then the Krull dimension of $M$, denoted $\dim_R M$, is defined as

$$\dim_R M := \dim\left(R/\operatorname{Ann}_R(M)\right),$$

where $\mathrm{Ann}_R(M)$ denotes the annihilator of $M$ in $R$ [19, Chapter 2].

### 2.1.2 System of parameters

Let $(R,\mathfrak{m})$ be a $d$-dimensional Noetherian local ring, where $d > 0$. A sequence of $d$ elements $\mathbf{x} = x_1,\dots,x_d$ in $R$ is a *system of parameters* (s.o.p.) for $R$ if $\sqrt{\mathbf{x}R} = \mathfrak{m}$. Every Noetherian local ring has a system of parameters [19, Chapter 5, Theorem 13.4].

### 2.1.3 Depth

**Definition 2.1.1.** [2, Chapter 1] Let $M$ be a finitely-generated $R$-module. An element $x \in R$ is a *nonzerodivisor* on $M$ if $xm \neq 0$ for all nonzero $m \in M$. If also $xM \neq M$, then we say $x$ is *$M$-regular*.

**Definition 2.1.2.** [2, Definition 1.1.1] Let $t$ be a positive integer. A sequence of $t$ elements $\mathbf{x} = x_1,\dots,x_t$ in $R$ is an *$M$-regular sequence* if the following two conditions hold.

(1) $x_1$ is $M$-regular.

(2) $x_i$ is regular on $M/(x_1,\dots,x_{i-1})M$ for each $i = 2,\dots,t$.

When $M = R$, we simply say that $\mathbf{x}$ is a regular sequence. Since $xM \neq M$ for an $M$-regular element $x$, we have $\mathbf{x} \in \mathfrak{m}$ for every $M$-regular sequence $\mathbf{x}$ by Nakayama's lemma [19, Chapter 1, Theorem 2.2]. Since every regular sequence is part of a system of parameters, we obtain the following.

**Proposition 2.1.3.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely-generated $R$-module. If $\mathbf{x} = x_1,\dots,x_i \in \mathfrak{m}$ is an $M$-regular sequence, then*

$$\dim_R(M/\mathbf{x}M) = \dim_R(M) - i.$$

**Definition 2.1.4.** [2, Theorem 1.2.5] An $M$-regular sequence is *maximal* if it cannot be extended to a longer $M$-regular sequence. Since $R$ is Noetherian, all maximal $M$-regular sequences have the same finite length, which is

$$\min\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

This length is called the *depth* of $M$, and denoted $\operatorname{depth}_R M$. When $M = R$, we write $\operatorname{depth} R$ for the depth of the ring $R$ as an $R$-module.

**Lemma 2.1.5.** [2, Proposition 1.2.9] *Given an exact sequence of finitely-generated R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*we have the following inequalities.*

*(1)* $\operatorname{depth}_R(M) \geq \min\{\operatorname{depth}_R(M'), \operatorname{depth}_R(M'')\}$

*(2)* $\operatorname{depth}_R(M') \geq \min\{\operatorname{depth}_R(M), \operatorname{depth}_R(M'') + 1\}$

*(3)* $\operatorname{depth}_R(M'') \geq \min\{\operatorname{depth}_R(M') - 1, \operatorname{depth}_R(M)\}$

### 2.1.4 Projective dimension

**Definition 2.1.6.** [2, Chapter 1] Let $M$ be a finitely-generated $R$-module. A *projective resolution* of $M$ is an exact sequence of the form

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{2.1}$$

where $P_n$ is a projective $R$-module for each $n \geq 0$. The *projective dimension* of $M$, denoted $\operatorname{pd}_R M$, is the infimum of lengths of projective resolutions of $M$.

Since $R$ is local, projective $R$-modules and free $R$-modules are the same, and we also refer to (2.1) as a free resolution of $M$.

**Definition 2.1.7.** [2, Chapter 1] Let $M$ be a finitely-generated $R$-module. The projective resolution (2.1) is *minimal* if $d_n(P_n) \subseteq \mathfrak{m}P_{n-1}$ for all $n \geq 1$. Every finitely-generated $R$-module has a minimal projective resolution which is unique up to isomorphism of chain complexes of $R$-modules.

**Definition 2.1.8.** [2, Chapter 1] Let $M$ be a finitely-generated $R$-module. For $n \geq 1$, the $n^{\text{th}}$ *syzygy* of $M$, denoted $\Omega_R^n(M)$, is the kernel of the $(n-1)$st chain map in the minimal projective resolution of $M$.

$$\Omega_R^n(M) := \ker d_{n-1} \text{ for } n \geq 1.$$

**Theorem 2.1.9.** [2, Theorem 1.3.3 (Auslander-Buchsbaum)] *Let $R$ be a Noetherian local ring, and $M$ a non-zero finitely-generated $R$-module with finite projective dimension. Then*

$$\operatorname{pd}_R M + \operatorname{depth}_R M = \operatorname{depth} R.$$

### 2.1.5 The Cohen-Macaulay property

**Definition 2.1.10.** [2, Definition 2.1.1] Let $R$ be a Noetherian local ring and $M$ a finitely-generated $R$-module. If $\operatorname{depth}_R M = \dim_R M$, we say that $M$ is a *Cohen-Macaulay* $R$-module. If $\dim R = \operatorname{depth} R$, we say that $R$ is a Cohen-Macaulay local ring, or simply that $R$ is a Cohen-Macaulay ring. A finitely-generated $R$-module is a *maximal Cohen-Macaulay* $R$-module (MCM $R$-module) if $\operatorname{depth}_R M = \dim_R M = \dim R$.

Equality of the depth and dimension of a Cohen-Macaulay module is preserved when factoring out a regular sequence.

**Proposition 2.1.11.** [2, Theorem 2.1.3] *Let $R$ be a Noetherian ring, and $M$ a finitely-generated $R$-module. Suppose $\mathbf{x}$ is an $M$-regular sequence. Then $M$ is Cohen-Macaulay if and only if $M/\mathbf{x}M$ is Cohen-Macaulay (over $R$ or $R/\mathbf{x}R$).*

The Cohen-Macaulay property makes ring and ideal invariants more tractable. For example, if $R$ is a Noetherian local ring and $\mathfrak{p}$ is a prime ideal of $R$, then

$$\operatorname{height} \mathfrak{p} + \dim R/\mathfrak{p} \leq \dim R$$

[19, Chapter 2]. The Cohen-Macaulay property is sufficient for equality to hold.

**Proposition 2.1.12.** [2, Corollary 2.1.4] *Let $R$ be a Cohen-Macaulay local ring and let $\mathfrak{p}$ be a prime ideal of $R$. Then*

$$\operatorname{height}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R.$$

For an $R$-module $M$, we let $\operatorname{Ass}_R(M)$ or $\operatorname{Ass}(M)$ denote the set of associated primes of $M$.

**Proposition 2.1.13.** [2, Corollary 2.1.4] *Let $R$ be a Noetherian local ring and $M$ a finitely-generated $R$-module. If $M$ is Cohen-Macaulay, then for each $\mathfrak{p} \in \operatorname{Ass}_R(M)$, we have*

$$\dim R/\mathfrak{p} = \dim_R(M) = \operatorname{depth}_R(M).$$

## 2.2 MCM approximations and the canonical module

### 2.2.1 Cohen-Macaulay local rings with canonical module

**Definition 2.2.1.** [2, Definition 3.1.7] Let $M$ be a finitely-generated $R$-module. An *injective resolution* of $M$ is an exact sequence of the form

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \cdots$$

where $I_n$ is an injective $R$-module for each $n \geq 0$.

It is well-known that every $R$-module has an injective resolution.

**Definition 2.2.2.** [2, Chapter 3] Let $M$ be an $R$-module. The *injective dimension* of $M$, denoted $\operatorname{id}_R M$, is the infimum of lengths of injective resolutions of $M$.

When $R$ is Cohen-Macaulay, we can characterize maximal Cohen-Macaulay modules and modules of finite injective dimension in terms of the vanishing of Ext modules in positive homological dimension.

**Proposition 2.2.3.** [17, Theorem 11.2] *Let $R$ be a Cohen-Macaulay local ring and let $M$ and $N$ be non-zero finitely-generated $R$-modules. The following statements hold.*

*(1) $M$ is MCM if and only if $\mathrm{Ext}_R^i(M,Y)=0$ for all $i>0$ and all finitely-generated $R$-modules $Y$ of finite injective dimension.*

*(2) $N$ has finite injective dimension if and only if $\mathrm{Ext}_R^i(X,N)=0$ for all $i>0$ and all MCM $R$-modules $X$.*

Informally, we can think of MCM modules and modules of finite injective dimension over a Cohen-Macaulay local ring as orthogonal sets, with the Ext functor in positive homological degree as an inner product.

**Definition 2.2.4.** [17, Definition 11.4] Let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay local ring. A *canonical module* is a finitely-generated $R$-module $\omega$ with the following properties.

(1) $\omega$ is MCM.

(2) $\omega$ has finite injective dimension over $R$.

(3) $\dim_k \mathrm{Ext}_R^d(k,\omega)=1$.

**Theorem 2.2.5.** [10, 20] *A Cohen-Macaulay local ring $R$ has a canonical module if and only if $R$ is a homomorphic image of a Gorenstein local ring.*

**Proposition 2.2.6.** [17, Theorem 11.5] *Let $R$ be a Cohen-Macaulay local ring. If a Cohen-Macaulay local ring $R$ has a canonical module, then it is unique up to isomorphism.*

The canonical module of a Cohen-Macaulay local ring has the following properties.

**Theorem 2.2.7.** [17, Theorem 11.5] *Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$. Let $M$ be a Cohen-Macaulay $R$-module of codepth $t$ and let $M^\vee := \mathrm{Ext}_R^t(M,\omega)$. The following properties hold.*

*(1)* $\mathrm{End}_R(\omega)\cong R$.

*(2) $M^\vee$ is also Cohen-Macaulay of codepth $t$.*

*(3)* $M^{\vee\vee}$ *is naturally isomorphic to* $M$.

*(4)* $\omega$ *is well-behaved with respect to factoring out a regular sequence, completion, and localization.*

The only finitely-generated modules over a Cohen-Macaulay local ring with canonical module that are both MCM and of finite injective dimension are finite direct sums of copies of the canonical module [17, Proposition 11.7]. We now define the Cohen-Macaulay rings that we study in this thesis.

**Definition 2.2.8.** [2, Definition 3.1.18] A Noetherian local ring $R$ is *Gorenstein* if $R$ has finite injective dimension as an $R$-module.

The canonical module of a Gorenstein local ring $R$ is the ring itself as an $R$-module. In fact, a Cohen-Macaulay local ring with canonical module $\omega$ is Gorenstein if and only if $\omega \cong R$ [17, Theorem 11.5]. One advantage of working with a Gorenstein ring $R$ is that an $R$-module has finite injective dimension if and only if it has finite projective dimension.

Our first proper class of Gorenstein local rings is the class of complete intersections. Recall that a Noetherian local ring $(R, \mathfrak{m})$ is a *regular local ring* if $\mathfrak{m}$ is generated by a system of parameters, and such a system of parameters is called a *regular system of parameters*. For a nonzero finitely-generated $R$-module $M$, the minimal number of generators of $M$ is denoted $\mu_R(M)$ and defined as follows.

$$\mu_R(M) := \dim_k(M/\mathfrak{m}M)$$

The number $\mu_R(\mathfrak{m})$ is called the *embedding dimension* of $R$. Since $\dim R \leq \mu_R(\mathfrak{m})$, we have $R$ is a regular local ring if and only if $\dim R = \mu_R(\mathfrak{m})$ [2, Chapter 2].

**Theorem 2.2.9.** [2, Theorem 2.2.7] *Let* $(R, \mathfrak{m}, k)$ *be a Noetherian local ring. The following statements are equivalent.*

*(1)* $R$ *is a regular local ring.*

*(2) k has finite projective dimension over R.*

*(3) Every finitely-generated R-module has finite projective dimension over R.*

We restrict our attention to a particular class of regular local rings in this thesis-rings of power series $k[\![x_1,\dots,x_m]\!]$ in finitely-many variables $x_1,\dots,x_m$ over a field $k$.

**Definition 2.2.10.** [2, Definition 2.3.1] A Noetherian local ring $(R,\mathfrak{m})$ is a *complete intersection* if its $\mathfrak{m}$-adic completion $\hat{R}$ is isomorphic to the quotient of a regular local ring by an ideal generated by a regular sequence. That is, for some regular local ring $S$ and a regular sequence $\mathbf{x} = x_1,\dots,x_n \in S$, we have $\hat{R} \cong S/\mathbf{x}S$.

We focus on complete intersections of the form $R = k[\![x_1,\dots,x_m]\!]/(f_1,\dots,f_n)$, where $k$ is a field and $f_1,\dots,f_n \in k[\![x_1,\dots,x_m]\!]$ is a regular sequence. A complete intersection of the form $k[\![x_1,\dots,x_m]\!]/(f)$, where $f \in (x_1,\dots,x_m)$ is nonzero, is called a *hypersurface ring*, or hypersurface.

In summary, we have the following nested classes of Cohen-Macaulay local rings [2, Proposition 3.1.20]. Below, we use the abbreviations "CM local rings" for Cohen-Macaulay local rings and "local CI rings" for local complete intersection rings.

$$\boxed{\text{CM local rings}} \supsetneq \boxed{\text{Gorenstein local rings}} \supsetneq \boxed{\text{local CI rings}} \supsetneq \boxed{\text{local hypersurface rings}}$$

### 2.2.2 MCM approximations

Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$. We consider an exact sequence construction that arises from the orthogonality relation between MCM modules and modules of finite injective dimension. This construction is called a *maximal Cohen-Macaulay approximation*, or MCM approximation, and is defined for each finitely-generated $R$-module. In chapter 3, we study a ring invariant defined in terms of MCM approximations of the sequence of quotient modules $\{R/\mathfrak{m}^n\}_{n=1}^{\infty}$.

**Definition 2.2.11.** [17, Definition 11.8] Let $M$ be a finitely-generated $R$-module. An *MCM approximation* of $M$ is an exact sequence of $R$-modules

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

where $X$ is an MCM $R$-module and $Y$ has finite injective dimension over $R$.

The following construction is dual to the MCM approximation.

**Definition 2.2.12.** [17, Definition 11.8] Let $M$ be a finitely-generated $R$-module. A *hull of finite injective dimension*, or FID hull, of $M$ is an exact sequence of $R$-modules

$$0 \longrightarrow M \longrightarrow Y' \longrightarrow X' \longrightarrow 0$$

where $Y'$ has finite injective dimension over $R$ and either $X'$ is an MCM $R$-module or $X' = 0$.

The vanishing of $\operatorname{Ext}^1_R(X,Y)$ when $X$ is MCM and $Y$ has finite injective dimension yields the following lifting properties for MCM approximations and FID hulls.

**Proposition 2.2.13.** [17, Proposition 11.9] *Let $M$ be a finitely-generated $R$-module with MCM approximation*

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} M \longrightarrow 0.$$

*Let $Z \xrightarrow{\phi} M$ be an $R$-map, where $Z$ is an MCM $R$-module. Then there exists a lifting $Z \xrightarrow{\psi} X$ such that $\pi\psi = \phi$; i.e., the following diagram commutes.*

$$\begin{array}{ccc} & & Z \\ & \swarrow^{\psi} & \downarrow^{\phi} \\ X & \xrightarrow{\pi} & M \end{array}$$

*Any two liftings of $\phi$ are homotopic, i.e. their difference factors through $Y$.*

**Proposition 2.2.14.** [17] *Let $M$ be a finitely-generated $R$-module with FID hull*

$$0 \longrightarrow M \xrightarrow{\iota} Y' \xrightarrow{\pi} X' \longrightarrow 0.$$

*Let $M \xrightarrow{\phi} Z$ be an $R$-map, where $Z$ is an $R$-module of finite injective dimension. Then there exists a lifting $Y' \xrightarrow{\psi} Z$ such that $\psi\iota = \phi$; i.e., the following diagram commutes.*

$$\begin{array}{ccc} Z & & \\ \phi \uparrow & \nwarrow \psi & \\ M & \xrightarrow{\iota} & Y' \end{array}$$

*Any two liftings of $\phi$ are homotopic, i.e. their difference factors through $X'$.*

Although each finitely-generated module has an MCM approximation and FID hull, these are not unique. We define minimality conditions that give us a unique MCM approximation and FID hull from which all others are built.

**Definition 2.2.15.** [17, Definition 11.10] Let $s : 0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} M \longrightarrow 0$ be an MCM approximation of a non-zero, finitely-generated $R$-module $M$. We say that $s$ is *minimal* if, given any direct-sum decomposition $X = X_0 \oplus X_1$ with $X_0 \subseteq \operatorname{Im} \iota$, we have $X_0 = 0$.

**Definition 2.2.16.** [14] Let $s' : 0 \longrightarrow M \xrightarrow{\iota} Y' \xrightarrow{\pi} X' \longrightarrow 0$ be an FID hull of a non-zero, finitely-generated $R$-module $M$. We say that $s'$ is *minimal* if, given any direct-sum decomposition $Y' = Y_0 \oplus Y_1$ such that $\pi(Y_0)$ is a direct summand of $X'$, we have $Y_0 = 0$.

**Theorem 2.2.17.** [17, Proposition 11.13, Theorem 11.17] *Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$, and let $M$ be a finitely-generated $R$-module. Then $M$ has a minimal MCM approximation and minimal FID hull that are unique up to isomorphism of exact sequences inducing the identity on $M$. We denote the minimal MCM approximation $s$ and the minimal FID hull $s'$ of $M$ as follows.*

$$s : 0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0$$

$$s' : 0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0$$

We sometimes refer to the module $X_M$ as the minimal MCM approximation of $M$. Each MCM approximation of $M$ can be written as follows for some non-negative integer $m$ [5, Proposition 1.5].

$$0 \longrightarrow \omega^m \oplus Y_M \longrightarrow \omega^m \oplus X_M \longrightarrow M \longrightarrow 0$$

Likewise, each FID hull of $M$ can be written as follows for some non-negative integer $n$ [5, Proposition 1.6].

$$0 \longrightarrow M \longrightarrow \omega^n \oplus Y^M \longrightarrow \omega^n \oplus X^M \longrightarrow 0$$

**Definition 2.2.18.** [15] Two finitely-generated $R$-modules $M$ and $N$ are *stably isomorphic* if there are projective (i.e. free) $R$-modules $P$ and $Q$ such that $M \oplus P \cong N \oplus Q$. This is denoted $M \overset{st}{\cong} N$.

**Definition 2.2.19.** [25] Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$. Let $D_R(-) := \mathrm{Hom}_R(-,\omega)$ and let $M$ be an MCM $R$-module. For each integer $i < 0$, we define the $R$-module $\Omega_R^i(M)$ by $\Omega_R^i(M) := D_R(\Omega_R^{-i}(D_R(M)))$.

**Lemma 2.2.20.** [25] *Let $R$ be a $d$-dimensional Cohen-Macaulay local ring with canonical module. Let $M$ be a finitely-generated $R$-module. Then $X_M \overset{st}{\cong} \Omega_R^{-d}(\Omega_R^d(M))$.*

**Proposition 2.2.21.** [17, Proposition 11.19] *Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$. Let $M$ be a finitely-generated $R$-module. Up to adding or deleting direct summands isomorphic to $\omega$, we have the following.*

*(i)* $Y_M \cong Y^{\Omega_R^1(M)}$

*(ii)* $X^M \cong X^{X_M}$

*(iii) There is an exact sequence* $0 \longrightarrow F \longrightarrow X_M \longrightarrow X^{\Omega_R^1(M)} \longrightarrow 0$ *where $F$ is free.*

*(iv) If $R$ is Gorenstein, then we also have the following.*

- $X_M \overset{st}{\cong} X^{\Omega_R^1(M)}$

- $X_M \overset{st}{\cong} \Omega^1_R(X^M)$
- $Y_M \overset{st}{\cong} \Omega^1_R(Y^M)$

In chapter 4, we study an existence property of minimal MCM approximations motivated by a uniqueness property of minimal FID hulls [15, Theorem 1.2]. If $R$ is Gorenstein and a finitely-generated $R$-module $M$ has finite projective dimension, then the minimal MCM approximation of $M$ is obtained by truncating its minimal projective resolution.

**Proposition 2.2.22.** [17, Proposition 11.20] *Let $R$ be a Gorenstein local ring and $M$ a finitely-generated $R$-module of finite projective dimension. The minimal MCM approximation of $M$ is*

$$0 \longrightarrow \Omega^1_R(M) \longrightarrow F \longrightarrow M \longrightarrow 0$$

*where $F$ is a free module of minimal rank mapping onto $M$.*

### 2.2.3 Auslander's $\delta$-invariant

**Definition 2.2.23.** [17, Chapter 11] Let $Z$ be a finitely-generated $R$-module. The *free rank* of $Z$, denoted f-rank$Z$, is the rank of a maximal free direct summand of $Z$. In other words, $Z \cong \underline{Z} \oplus R^{\text{f-rank}Z}$ with $\underline{Z}$ stable, i.e. having no non-trivial free direct summands.

**Definition 2.2.24.** Let $R$ be a Cohen-Macaulay local ring and $M$ a finitely-generated $R$-module. We define $\delta_R(M)$ as the minimum free rank of all MCM $R$-modules $Z$ for which there exists a surjective $R$-map $Z \longrightarrow M$. We refer to $\delta_R(-)$ as *Auslander's $\delta$-invariant*.

If $R$ has a canonical module, then Auslander's $\delta$-invariant is the free rank of the minimal MCM approximation of a module.

**Proposition 2.2.25.** [17, Definition 11.24, Proposition 11.27] *Let $R$ be a Cohen-Macaulay local ring with canonical module and $M$ a finitely-generated $R$-module. Then $\delta_R(M) =$ f-rank $X_M$.*

**Proposition 2.2.26.** [17, Corollary 11.28] *For finitely-generated $R$-modules $M$ and $N$, Auslander's $\delta$-invariant satisfies the following properties.*

*(1)* $\delta_R(M \oplus N) = \delta_R(M) + \delta_R(N)$

*(2)* $\delta_R(N) \leq \delta_R(M)$ *if there is a surjective R-map* $M \longrightarrow N$

*(3)* $\delta_R(M) \leq \mu_R(M)$

Consider the sequence of non-negative integers $\{\delta_R(R/\mathfrak{m}^n)\}_{n=1}^{\infty}$. For $n \geq 1$, we have the expansion of cosets map $R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n$. By (2), we have $\delta_R(R/\mathfrak{m}^n) \leq \delta_R(R/\mathfrak{m}^{n+1})$. By (3), we have $\delta_R(R/\mathfrak{m}^n) \leq 1$. Therefore, if $\delta_R(R/\mathfrak{m}^{n_o}) = 1$ for some $n_o$, then $\delta_R(R/\mathfrak{m}^n) = 1$ for all $n \geq n_o$. By a result of Auslander, regular local rings are the Cohen-Macaulay local rings for which $\delta_R(R/\mathfrak{m}^n) = 1$ for all $n \geq 1$. To prove this, we require the following well-known application of Theorem 2.2.9.

**Lemma 2.2.27.** *Let R be a d-dimensional Cohen-Macaulay local ring. If every MCM R-module is free, then R is regular.*

*Proof.* Suppose every MCM $R$-module is free. Let $M$ be a finitely-generated $R$-module and let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of $M$. By successive applications of the Depth Lemma, we see that $\Omega_R^d(M)$ is MCM, and therefore free. Therefore, the exact sequence

$$0 \longrightarrow \Omega_R^d(M) \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a projective resolution of $M$ of finite length. So $M$ has finite projective dimension. By Theorem 2.2.9, $R$ is regular. □

**Proposition 2.2.28.** [17, Proposition 11.37] *Let* $(R, \mathfrak{m}, k)$ *be a Cohen-Macaulay local ring. The following are equivalent.*

*(1)* $R$ *is a regular local ring*

*(2)* $\delta_R(k) = 1$

*Proof.* Suppose $R$ is regular. By Theorem 2.2.9, the residue field $k$ has finite projective dimension over $R$. Since $R$ is a regular local ring, it is Gorenstein, and by Proposition 2.2.22, the minimal MCM approximation of $k$ is

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0.$$

Since $\text{f-rank}\, X_k = \text{f-rank}\, R = 1$, we have $\delta_R(k) = 1$. Now suppose $R$ is not regular. By Lemma 2.2.27, there is an MCM $R$-module $M$ that is not free. Write $M = N \oplus F$, where $F$ is a free $R$-module and $N$ has no non-trivial free direct summand. Since $N$ is a nonzero direct summand of an MCM $R$-module, it is also MCM. Let $n = \mu_R(N)$. Then $N/\mathfrak{m}N \cong k^n$ as $k$-vector spaces and as $R$-modules. the following composition of surjective maps gives us a surjection $N \longrightarrow k$.

$$N \longrightarrow N/\mathfrak{m}N \longrightarrow k^n \longrightarrow k$$

Since $\text{f-rank}\, N = 0$, we have $\delta_R(k) = 0$. □

In light of this result, it is natural to ask when the sequence $\{\delta_R(R/\mathfrak{m}^n)\}_{n=1}^{\infty}$ stabilizes at one for different classes of Cohen-Macaulay rings. The smallest positive integer $n$ for which $\delta_R(R/\mathfrak{m}^n) = 1$ is the following invariant defined by Auslander.

**Definition 2.2.29.** [17, Chapter 11] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. The *index* of $R$, denoted $\text{index}(R)$, is defined as follows.

$$\text{index}(R) := \inf\{n \geq 1 \mid \delta_R(R/\mathfrak{m}^n) = 1\}$$

If $\delta_R(R/\mathfrak{m}^n) = 0$ for all $n \geq 1$, we say that $\text{index}(R) = \infty$.

**Proposition 2.2.30.** [7, Theorem 1.1] *Let $R$ be a Cohen-Macaulay local ring with canonical module. Then the index of $R$ is finite if and only if $R_{\mathfrak{p}}$ is Gorenstein for each non-maximal prime ideal $\mathfrak{p}$ of $R$.*

**Definition 2.2.31.** [17, Lemma 11.41] Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$. The *trace of* $\omega$ *in* $R$*, denoted* $\tau_\omega(R)$*, is the ideal of* $R$ *generated by all* $R$*-homomorphic images of* $\omega$ *in* $R$*.*

We let $e(R)$ denote the Hilbert-Samuel multiplicity of a ring $R$.

**Proposition 2.2.32.** [7, Proposition 2.3] *Let* $R$ *be a* $d$*-dimensional Cohen-Macaulay local ring with canonical module* $\omega$ *such that* $\mathfrak{m} \subseteq \tau_\omega(R)$*. Then* $\mu_R(\mathfrak{m}) \leq e(R) + d - \operatorname{index}(R) + 1$ *and*

$$\operatorname{index}(R) \leq e(R).$$

**Theorem 2.2.33.** [6, Theorem 3.3] *If* $R$ *is a hypersurface, then* $\operatorname{index}(R) = e(R)$.

# 3 | Generalized Loewy length of Cohen-Macaulay local and graded rings

We generalize a theorem of Ding relating the generalized Loewy length $\mathrm{g}\ell\ell(R)$ and index of a one-dimensional Cohen-Macaulay local ring $(R,\mathfrak{m},k)$. Ding proved that if $R$ is Gorenstein, the associated graded ring is Cohen-Macaulay, and $k$ is infinite, then the generalized Loewy length and index of $R$ are equal. However, if $k$ is finite, equality may not hold. We prove that if the index of a one-dimensional Cohen-Macaulay local ring is finite and the associated graded ring has a homogeneous nonzerodivisor of degree $t$, then $\mathrm{g}\ell\ell(R) \leq \mathrm{index}(R)+t-1$.

Next we prove that if $R$ is a one-dimensional hypersurface ring with a witness to the generalized Loewy length that induces a regular initial form on the associated graded ring, then the generalized Loewy length achieves this upper bound. We then compute the generalized Loewy lengths of several families of examples of one-dimensional hypersurface rings over finite fields. Finally, we study a graded version of the generalized Loewy length and determine its value for numerical semigroup rings.

## 3.1 Ding's conjecture

Throughout this section, $(R,\mathfrak{m},k)$ is a Cohen-Macaulay local ring. Recall that the maximal ideal of an Artinian local ring is nilpotent.

**Definition 3.1.1.** Let $(R,\mathfrak{m})$ be an Artinian local ring. The *Loewy length* of $R$, denoted $\ell\ell(R)$, is the smallest positive integer $n$ such that $\mathfrak{m}^n = 0$.

Suppose $R$ be a $d$-dimensional Noetherian local ring. If $d = 0$, then $R$ is Artinian. If $d > 0$, then $R$ has a system of parameters $\mathbf{x}$ and the ring $\overline{R} = R/\mathbf{x}R$ is Artinian. Let $\overline{\mathfrak{m}} = \mathfrak{m}/\mathbf{x}R$ and suppose $\ell\ell(\overline{R}) = n$. Then $\overline{\mathfrak{m}}^n = 0$, or equivalently, $\mathfrak{m}^n \subseteq \mathbf{x}R$.

**Definition 3.1.2.** [17, Chapter 11] Let $(R, \mathfrak{m})$ be a Noetherian local ring. The *generalized Loewy length* of $R$, denoted $\mathrm{g}\ell\ell(R)$, is the smallest positive integer $n$ such that $\mathfrak{m}^n$ is contained in an ideal generated by a system of parameters for $R$.

$$\mathrm{g}\ell\ell(R) := \min\{n \geq 1 \mid \mathfrak{m}^n \subseteq \mathbf{x}R \text{ for some s.o.p. } \mathbf{x} \in R\}$$

**Proposition 3.1.3.** [17, Chapter 11] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring. Then*

$$\mathrm{index}(R) \leq \mathrm{g}\ell\ell(R).$$

*Proof.* Let $g = \mathrm{g}\ell\ell(R)$. Then there is a system of parameters $\mathbf{x} \in \mathfrak{m}$ such that $\mathfrak{m}^g \subseteq \mathbf{x}R$. Since $R$ is Cohen-Macaulay, $\mathbf{x}$ is a regular sequence. Consider the expansion of cosets map

$$R/\mathfrak{m}^g \longrightarrow R/\mathbf{x}R. \tag{3.1}$$

The $R$-module $R/\mathbf{x}R$ has finite projective dimension, with truncated minimal projective resolution

$$0 \longrightarrow \mathbf{x}R \longrightarrow R \longrightarrow R/\mathbf{x}R \longrightarrow 0. \tag{3.2}$$

Since $R$ is Gorenstein, it follows from Proposition 2.2.22 that (3.2) is the minimal MCM approximation of $R/\mathbf{x}R$. Therefore, $\delta_R(R/\mathbf{x}R) = 1$. By Proposition 2.2.26 and (3.1), we have $\delta_R(R/\mathfrak{m}^g) = 1$. Therefore, $\mathrm{index}(R) \leq g$. □

Ding proved the following generalization of Proposition 3.1.3.

**Proposition 3.1.4.** [7, Proposition 2.4] *Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega$ such that $\mathfrak{m} \subseteq \tau_\omega(R)$. Then* $\operatorname{index}(R) \leq \operatorname{g\ell\ell}(R)$.

If $R$ is Gorenstein with infinite residue field and Cohen-Macaulay associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$, then we have equality in Proposition 3.1.4.

**Theorem 3.1.5.** [8, Theorem 2.1] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. If $k$ is infinite and* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay, then* $\operatorname{index}(R) = \operatorname{g\ell\ell}(R)$.

In general, if $R$ is a Cohen-Macaulay local ring that satisfies the equality in Theorem 3.1.5, we say that $R$ satisfies Ding's conjecture. Below, we study how the finiteness of the residue field can cause Ding's conjecture to fail. In particular, we prove that there are infinitely-many hypersurfaces with Cohen-Macaulay associated graded ring and finite residue field that do not satisfy Ding's conjecture. Each of our families of hypersurfaces generalizes the following example from Hashimoto and Shida.

**Example 3.1.6.** [11, Example 3.2] Let $R = \mathbb{F}_2[\![x,y]\!]/(xy(x+y))$. Then $\operatorname{index}(R) = 3$ and $\operatorname{g\ell\ell}(R) = 4$. Refer to Proposition 3.3.5 for proofs of these claims.

When $k$ is finite, the assumption that $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay does not guarantee the existence of a homogeneous system of parameters of degree one $x_1^*, \ldots, x_d^*$ in $(\operatorname{gr}_{\mathfrak{m}}(R))_1$. If a homogeneous system of parameters in $\operatorname{gr}(R)$ does not consist of linear elements, it cannot be used in Ding's argument to prove that $\operatorname{index}(R) = \operatorname{g\ell\ell}(R)$. However, if $R$ is a one-dimensional Cohen-Macaulay local ring with finite index and $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then we can use a homogeneous $\operatorname{gr}_{\mathfrak{m}}(R)$-regular element of minimal degree to obtain an upper bound for $\operatorname{g\ell\ell}(R)$ in terms of $\operatorname{index}(R)$. In Theorem 2.3, we prove that if $R$ is one-dimensional Cohen-Macaulay and $\operatorname{gr}_{\mathfrak{m}}(R)$ has a homogeneous nonzerodivisor $z^*$, where $z \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$, then $\operatorname{g\ell\ell}(R) \leq \operatorname{index}(R) + t - 1$. If $R$ is Gorenstein, then $\operatorname{index}(R) \leq \operatorname{g\ell\ell}(R) \leq \operatorname{index}(R) + t - 1$.

By Theorem 2.2.33, $\operatorname{index}(R) = e(R)$ when $R$ is a hypersurface. Therefore, the index of hypersurface rings is easy to compute: if $R = k[\![x_1, \ldots, x_n]\!]/(f)$, $\mathfrak{m} = (x_1, \ldots, x_n)R$, and $f \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$, then $\operatorname{index}(R) = e(R) = r$. In section 3, we prove that if $R$ is a one-dimensional hypersurface with a witness $z$ to its generalized Loewy length that induces a regular initial form on $\operatorname{gr}_{\mathfrak{m}}(R)$, then

$$\mathrm{g}\ell\ell(R) = \operatorname{ord}_R(z) + e(R) - 1.$$

We then compute the generalized Loewy lengths of families of examples of one-dimensional hypersurface rings with finite residue field and Cohen-Macaulay associated graded ring. These examples illustrate differences between hypersurface rings $R$ with finite residue field and Cohen-Macaulay associated graded ring for which $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)$ and $\mathrm{g}\ell\ell(R) = \operatorname{index}(R) + 1$. In [4], De Stefani gave examples of one-dimensional Gorenstein local rings with infinite residue field for which $\mathrm{g}\ell\ell(R) = \operatorname{index}(R) + 1$.

In section 4, we let $R$ be a positively-graded Noetherian $k$-algebra, where $k$ is an arbitrary field. We show that several families of one-dimensional standard graded hypersurfaces attain the graded version of the upper bound for the generalized Loewy length from Theorem 2.3. We then study a graded version of the generalized Loewy length: the *generalized graded length* of $R$, denoted $\mathbf{ggl}(R)$. After determining bounds for $\mathbf{ggl}(R)$ in terms of $\mathrm{g}\ell\ell(R)$ and the minimum and maximum degrees of generators of $R$, we compute the generalized graded length of numerical semigroup rings. For $R = k[t^a, t^b]$, where $a < b$, we prove that $\mathbf{ggl}(R) = ba - b + 1$ and if $z$ is a witness to $\mathbf{ggl}(R)$, then $(z) = (t^{ia})$ for some $1 \le i \le 1 + b - a$.

## 3.2 Estimating the generalized Loewy length of one-dimensional Cohen-Macaulay rings

Throughout this section, $(R, \mathfrak{m}, k)$ is a Noetherian local ring. We assume that $R$ has a nonzerodivisor $x$ of order $t$ such that multiplication by $x$ is injective on graded components of the associated graded ring in degrees less than $\operatorname{index}(R)$. Generalizing [8, Lemma 2.3] to this context, we prove that if $R$ is a one-dimensional Cohen-Macaulay local ring with finite index, then

$\mathrm{g}\ell\ell(R) \le \operatorname{index}(R) + t - 1$.

**Lemma 3.2.1.** *Let $s$ and $t$ be positive integers and $x \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ an $R$-regular element. Suppose the induced map $\overline{x} : \mathfrak{m}^{i-1}/\mathfrak{m}^i \longrightarrow \mathfrak{m}^{i+t-1}/\mathfrak{m}^{i+t}$ is injective for $1 \le i \le s$. Then*

$$(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s \cong R/\mathfrak{m}^s \oplus (\mathfrak{m}^{s+t-1}, x)/xR.$$

*Proof.* Let $I = xR \cap \mathfrak{m}^{s+t-1}$ and $W = (I + \mathfrak{m}^{s+t})/\mathfrak{m}^{s+t}$. Since $W$ is a $k$-subspace of $\mathfrak{m}^{s+t-1}/\mathfrak{m}^{s+t}$, there is a direct sum decomposition

$$\mathfrak{m}^{s+t-1}/\mathfrak{m}^{s+t} = W \oplus V$$

for some subspace $V \subseteq \mathfrak{m}^{s+t-1}/\mathfrak{m}^{s+t}$. Let $e_1, ..., e_n$ be a $k$-basis for $V$. For each $i$, let $e_i = \overline{y_i}$, where $y_i \in \mathfrak{m}^{s+t-1}$. Let $B$ denote the $R$-submodule of $(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s$ generated by $[y_1], ..., [y_n] \in (\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s$. We will prove that $(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s = A \oplus B$, where $A = xR/x\mathfrak{m}^s$. First we show that

$$A + B = (\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s.$$

Choose $r_1, ..., r_\alpha \in R$ such that $I = (r_1x, ..., r_\alpha x)$. Then $\mathfrak{m}^{s+t-1}/\mathfrak{m}^{s+t}$ is generated as a vector space by $\{\overline{r_ix}\}_{i=1}^{\alpha} \cup \{\overline{y}_j\}_{j=1}^{n}$, and by Nakayama's lemma, $\mathfrak{m}^{s+t-1}$ is generated as an $R$-module by $\{r_ix\}_{i=1}^{\alpha} \cup \{y_j\}_{j=1}^{n}$. Let $[z] \in (\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s$. Then $[z] = r[x] + r'[v]$, where $r, r' \in R$, $v \in \mathfrak{m}^{s+t-1}$, and $v = r''x + \sum_{i=1}^n \rho_i y_i$, where $r'', \rho_i \in R$. So

$$[z] = (r + r'r'')[x] + \sum_{i=1}^{n} r'\rho_i[y_i] \in A + B.$$

Now we show that $A \cap B = 0$. Let $[z] \in A \cap B$. Then $[z] = a[x] = \sum_{i=1}^n a_i[y_i]$, where $a, a_i \in R$, and $ax - \sum_{i=1}^n a_iy_i \in x\mathfrak{m}^s$. Let $ax - \sum_{i=1}^n a_iy_i = xy$, where $y \in \mathfrak{m}^s$. Then $\sum_{i=1}^n a_iy_i = (a-y)x \in I$, so

$$\overline{(a-y)x} = \overline{0} \in \mathfrak{m}^{s+t-1}/\mathfrak{m}^{s+t}.$$

If $a = y$ we are done, so assume $a - y \neq 0$. Then there is a nonnegative integer $l$ such that $a - y \in \mathfrak{m}^l \setminus \mathfrak{m}^{l+1}$. Suppose $0 \leq l < s$. Since $\overline{(a-y)x} = \overline{0}$ in $\mathfrak{m}^{l+t}/\mathfrak{m}^{l+t+1}$, it follows from the injectivity of the induced map $\overline{x}$ that $a - y \in \mathfrak{m}^{l+1}$, a contradiction. Therefore, $a - y \in \mathfrak{m}^s$, and $ax - xy \in x\mathfrak{m}^s$. Since $xy \in x\mathfrak{m}^s$, $ax \in x\mathfrak{m}^s$, and $[z] = a[x] = [0]$. It follows that $(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s = xR/x\mathfrak{m}^s \oplus B$ and $B \cong (\mathfrak{m}^{s+t-1}, x)/xR$. Since $x$ is $R$-regular, it follows that $(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s \cong R/\mathfrak{m}^s \oplus (\mathfrak{m}^{s+t-1}, x)/xR$. □

**Lemma 3.2.2.** *Let $(R, \mathfrak{m})$ be a local ring, $I \subseteq R$ an ideal, and $x, y \in \mathfrak{m}$ such that $(x, I) = (y)$. If $I$ is not a principal ideal, then $(x) = (y)$.*

*Proof.* Let $a, b \in R$ and $z \in I$ such that $y = ax + bz$. Let $c \in R$ such that $x = cy$. Then $y = acy + bz$ and $(1 - ac)y = bz$. Suppose $c \in \mathfrak{m}$. Then $1 - ac$ is invertible and $y = (1 - ac)^{-1}bz \in I$, so $(y) = I$, which is false. Therefore $c$ is invertible and $(x) = (y)$. □

**Lemma 3.2.3.** [16, Lemma 2.5] *Let $R$ be a one-dimensional Cohen-Macaulay local ring and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. If $\delta_R(I) > 0$, then $I$ is generated by a regular element.*

**Theorem 3.2.4.** *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring for which* $\operatorname{index}(R)$ *is finite. Let $s = \operatorname{index}(R)$ and $x \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ a nonzerodivisor, where $t \geq 1$. If the induced map*

$$\overline{x} : \mathfrak{m}^{i-1}/\mathfrak{m}^i \longrightarrow \mathfrak{m}^{i+t-1}/\mathfrak{m}^{i+t}$$

*is injective for $1 \leq i \leq s$, then*

$$\mathrm{g}\ell\ell(R) \leq \operatorname{index}(R) + t - 1.$$

*If $\mathfrak{m}^{s+t-1}$ is not a principal ideal, then $\mathfrak{m}^{s+t-1} \subseteq (x)$.*

*Proof.* By Lemma 3.2.1, $(\mathfrak{m}^{s+t-1}, x)/x\mathfrak{m}^s \cong R/\mathfrak{m}^s \oplus (\mathfrak{m}^{s+t-1}, x)/xR$, so there is a surjection

$$(\mathfrak{m}^{s+t-1}, x) \longrightarrow R/\mathfrak{m}^s.$$

Therefore, $\delta_R((\mathfrak{m}^{s+t-1}, x)) > 0$. By Lemma 3.2.3, $(\mathfrak{m}^{s+t-1}, x)$ is a parameter ideal of $R$. Let

$(\mathfrak{m}^{s+t-1}, x) = (y)$, where $y \in \mathfrak{m}$ is a regular element. Since $\mathfrak{m}^{s+t-1} \subseteq (y)$, we have $\mathrm{g}\ell\ell(R) \leq s+t-1$. If $\mathfrak{m}^{s+t-1}$ is not a principal ideal, then by Lemma 3.2.2 we have $\mathfrak{m}^{s+t-1} \subseteq (x)$. □

**Corollary 3.2.5.** *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring with canonical module $\omega$ such that $\mathfrak{m} \subseteq \tau_\omega(R)$. Let $x \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ such that $x^* \in \mathrm{gr}_{\mathfrak{m}}(R)$ is a regular element. Then*

$$\mathrm{index}(R) \leq \mathrm{g}\ell\ell(R) \leq \mathrm{index}(R) + t - 1.$$

*Proof.* This follows from Proposition 3.1.4 and Theorem 3.2.4 □

## 3.3 Examples

In this section we derive a formula for the generalized Loewy length of one-dimensional hypersurface rings and compute the generalized Loewy lengths of several families of examples of one-dimensional hypersurfaces. The associated graded ring of each of these hypersurface rings has a homogeneous nonzerodivisor of degree one or two, so the index and generalized Loewy length differ by at most one.

Using techniques from the proof of [11, Example 3.2], we prove that for several families of hypersurfaces $\{R_n\}_{n=1}^{\infty}$,

$$\mathrm{g}\ell\ell(R_n) - \mathrm{index}(R_n) = 1$$

for $n \geq 1$. This difference is positive for each $n$ because of the absence of a regular linear form in certain one-dimensional hypersurface rings over finite fields.

Throughout this section, $S = k[\![x, y]\!]$, where $k$ is a field and $\mathfrak{n} = (x, y)S$. We say that the *order* of an element $f \in S$ is $r$ if $f \in \mathfrak{n}^r \setminus \mathfrak{n}^{r+1}$, and write $\mathrm{ord}_S(f) = r$. Let $R = S/(f)$, where $f \in \mathfrak{n}$. Let $\mathfrak{m} = (x, y)R$. The order of an element $z \in R$ is $r$ if $z \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$, and we write $\mathrm{ord}_R(z) = r$. Recall that $\mathrm{index}(R) = e(R)$. Finally, if $(R, \mathfrak{m})$ is any local ring of embedding dimension $n$, then $\mu_R(\mathfrak{m}^r) \leq \binom{n+r-1}{r}$.

**Lemma 3.3.1.** *Let* $R = k[\![x,y]\!]/(f)$*, where* $\operatorname{ord}_S(f) = e$ *and* $g = \mathrm{g}\ell\ell(R)$*. Let* $z \in \mathfrak{m}$ *such that* $\mathfrak{m}^g \subseteq (z)$ *and* $i \geq 0$*. If* $\mathrm{g}\ell\ell(R) \leq e+i$*, then* $\operatorname{ord}_R(z) \leq i+1$.

*Proof.* Let $\operatorname{ord}_R(z) = r$ and $\zeta \in \mathfrak{n}^r \setminus \mathfrak{n}^{r+1}$ such that $\overline{\zeta} = z$. Then $\mathfrak{n}^g \subseteq (f,\zeta)$. Let $M$ be the $k$-vector space of leading forms of degree $g$ of elements of $(f,\zeta)$. Since $\operatorname{ord}_S(\zeta) = r$, we obtain leading forms of degree $g$ from this element by multiplying $\zeta$ by generators of $\mathfrak{n}^{g-r}$. Similarly, we multiply $f$ by generators of $\mathfrak{n}^{g-e}$ to obtain leading forms of degree $g$. Therefore,

$$\dim_k M \leq \binom{2+(g-e)-1}{g-e} + \binom{2+(g-r)-1}{g-r} = 2g-(e+r)+2.$$

On the other hand, the vector space of forms of degree $g$ in $\mathfrak{n}^g$ has dimension $g+1$. Therefore, $g+1 \leq 2g-(e+r)+2$ and $e+r \leq g+1$. The result follows from this inequality. □

**Definition 3.3.2.** Let $(R,\mathfrak{m})$ be a $d$-dimensional local ring. A system of parameters $\mathbf{x} = x_1,\ldots,x_d \in \mathfrak{m}$ is a *witness to* $g = \mathrm{g}\ell\ell(R)$ if $\mathfrak{m}^g \subseteq (\mathbf{x})$.

If $R$ is a one-dimensional hypersurface with a witness $z$ to $\mathrm{g}\ell\ell(R)$ that induces a regular initial form on $\operatorname{gr}_{\mathfrak{m}}(R)$, then we can compute $\mathrm{g}\ell\ell(R)$ using the following formula. We see that the order of $z$ is uniquely determined by $\mathrm{g}\ell\ell(R)$ and $e(R)$.

**Proposition 3.3.3.** *Let* $R = k[\![x,y]\!]/(f)$*, where* $\operatorname{ord}_S(f) = e$ *and* $z \in \mathfrak{m}$ *such that* $z^*$ *is* $\operatorname{gr}_{\mathfrak{m}}(R)$*-regular. If* $z$ *is a witness to* $\mathrm{g}\ell\ell(R)$*, then*

$$\mathrm{g}\ell\ell(R) = \operatorname{ord}_R(z) + e - 1.$$

*Proof.* Recall that $\operatorname{index}(R) = e$. Let $g = \mathrm{g}\ell\ell(R)$ and $n = g-e$. Then $g = e+n$ and by Lemma 3.3.1, $\operatorname{ord}_R(z) \leq n+1$. By Theorem 3.2.4, $g \leq e + \operatorname{ord}_R(z) - 1 \leq e+n = g$. □

If we cannot find an element of a one-dimensional hypersurface that is a witness to $\mathrm{g}\ell\ell(R)$ and induces a regular form on $\operatorname{gr}(R)$, then we can use the following lemma to estimate the generalized Loewy length.

**Lemma 3.3.4.** *Let* $R = k[\![x,y]\!]/(f)$*, where* $\operatorname{ord}_S(f) = e \geq 2$*. If* $R$ *has no nonzerodivisors of the form* $\alpha x + \beta y$*, where* $\alpha, \beta \in k$*, then* $\mathrm{g}\ell\ell(R) > e$.

*Proof.* Since $\operatorname{index}(R) = e$, we have $e \leq \mathrm{g}\ell\ell(R)$ by Proposition 3.1.4. Suppose $\mathrm{g}\ell\ell(R) = e$. Let $z \in \mathfrak{m}$ such that $\mathfrak{m}^e \subseteq (z)$. By Lemma 3.3.1, we have $\operatorname{ord}_R(z) = 1$. Let $\zeta \in \mathfrak{n} \setminus \mathfrak{n}^2$ be a preimage of $z$. Note that for each invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$, the map $\phi : S \longrightarrow S$ defined by $\phi(x) = ax + by$ and $\phi(y) = cx + dy$ is a $k$-algebra automorphism. Letting an appropriate invertible matrix in $\mathrm{GL}_2(k)$ act on $S$, we may assume without loss of generality that $\zeta = x - h$ for some nonzero element $h \in S$ with $\operatorname{ord}_S(h) \geq 2$. Since $x$ is a zerodivisor on $R$, there is an element $g \in \mathfrak{n}^{e-1}$ such that $f = xg$.

Let $R' = S/(\zeta)$. Since $S$ is a regular local ring and $\operatorname{ord}_S(\zeta) = 1$, it follows that $R'$ is a one-dimensional regular local ring, and thus a discrete valuation ring. Let $\overline{f}$ denote the image of $f$ in $R'$. Then

$$R/(z) \cong R'/(\overline{f}).$$

Since $\overline{g} \in (x,y)^{e-1}R'$ and $\overline{x} = \overline{h} \in (x,y)^2R'$, it follows that $\overline{f} \in (x,y)^{e+1}R'$, so $l_{R'}(R'/(\overline{f})) = \operatorname{ord}_{R'}(\overline{f}) \geq e+1$ and $l_R(R/(z)) \geq e+1$. Now let $R_1 := R/(z)$ and $\mathfrak{m}_1 := \mathfrak{m}/(z)$. Then

$$0 = \mathfrak{m}_1^e \subseteq \mathfrak{m}_1^{e-1} \subseteq \cdots \subseteq \mathfrak{m}_1 \subseteq R_1$$

is a composition series for $R_1$, so $l_R(R/(z)) = e$. This is a contradiction. □

If $(R, \mathfrak{m}, k)$ is a one-dimensional local ring with Cohen-Macaulay associated graded ring and infinite residue field, then $\mathrm{gr}_{\mathfrak{m}}(R)$ has a homogeneous linear nonzerodivisor. We now consider one-dimensional hypersurface rings with finite residue field such that the associated graded ring does not have a homogeneous linear nonzerodivisor. If the associated graded ring has a homogeneous quadratic nonzerodivisor, then it follows from Theorem 3.2.4 and Lemma 3.3.4 that the difference between the generalized Loewy length and index is one.

**Proposition 3.3.5.** *Let $k$ be a finite field and $R = k[\![x,y]\!]/y(\prod_{\alpha \in k}(x + \alpha y))$. Then*

$$\mathrm{g}\ell\ell(R) = \operatorname{index}(R) + 1 = |k| + 2.$$

*Proof.* We construct a homogeneous nonzerodivisor of degree 2 in $\operatorname{gr}_{\mathfrak{m}}(R)$. Let $f \in k[x]$ be a degree 2 irreducible polynomial. Define $g(x,y) \in k[x,y]$ *by* $g(x,y) := y^2 f(\frac{x}{y})$. We claim that the element

$$\overline{g} = g(\overline{x},\overline{y}) \in \operatorname{gr}_{\mathfrak{m}}(R) = k[x,y]/y(\prod_{\alpha \in k}(x+\alpha y))$$

is $\operatorname{gr}_{\mathfrak{m}}(R)$-regular. Let $h \in k[x,y]$ such that $\overline{g}\overline{h} = \overline{0}$. Then there exists a polynomial $p(x,y) \in k[x,y]$ such that

$$gh = py(\prod_{\alpha \in k}(x+\alpha y)).$$

Let $\alpha \in k$. Suppose $(x+\alpha y) \mid g$ and $q(x,y) \in k[x,y]$ such that $(x+\alpha y)q(x,y) = g(x,y)$. Then $(x+\alpha)q(x,1) = g(x,1) = f(x)$. This contradicts the irreducibility of $f$. It follows that $(x+\alpha y) \mid h$. Clearly $y \nmid g$, so $y \mid h$ as well, and $y(\prod_{\alpha \in k}(x+\alpha y)) \mid h$. Therefore we have $\overline{h} = \overline{0}$, and $\overline{g}$ is $\operatorname{gr}_{\mathfrak{m}}(R)$-regular. By Theorem 3.2.4 and Lemma 3.3.4, we have $g\ell\ell(R) = \operatorname{index}(R) + 1$. □

*Remark* 3.3.6. When $k = \mathbb{F}_2$, Proposition 3.3.5 is Hashimoto and Shida's counterexample to Ding's conjecture: $\mathbb{F}_2[\![x,y]\!]/(xy(x+y))$. In the following propositions, we compute the generalized Loewy lengths of families of one-dimensional hypersurface rings of the form $k[\![x,y]\!]/(xy(x^n+y^n))$, where $k$ is a finite field and $n$ is a positive integer.

**Proposition 3.3.7.** *Let $n \geq 1$ and $k$ a field such that $\operatorname{char} k \neq 2$ and $\operatorname{char} k \nmid 1+(-2)^n$. Let $R = k[\![x,y]\!]/(xy(x^n+y^n))$. Then $\mathfrak{m}^{n+2} = (x+2y)\mathfrak{m}^{n+1}$ and*

$$g\ell\ell(R) = \operatorname{index}(R) = n+2.$$

*Proof.* Since $\mathfrak{m}^{n+1}$ is generated by $\{x^{n+1-i}y^i\}_{i=0}^{n+1}$, it follows that $(x+2y)\mathfrak{m}^{n+1}$ is generated by $\{x^{n+2-i}y^i + 2x^{n+1-i}y^{i+1}\}_{i=0}^{n+1}$. Let

$$z_i = x^{n+2-i}y^i + 2x^{n+1-i}y^{i+1} \text{ for } 0 \leq i \leq n+1.$$

Since $xy^{n+1} = -x^{n+1}y$, we have

$$\sum_{i=1}^{n} (-2)^{i-1} z_i = x^{n+1}y + 2(-2)^{n-1}xy^{n+1}$$

$$= x^{n+1}y - 2(-2)^{n-1}x^{n+1}y$$

$$= (1 + (-2)^n)x^{n+1}y.$$

Since $z_i \in (x+2y)\mathfrak{m}^{n+1}$ for $0 \le i \le n+1$, we have $x^{n+1}y \in (x+2y)\mathfrak{m}^{n+1}$ and $\mathfrak{m}^{n+2} \subseteq (x+2y)\mathfrak{m}^{n+1}$. Therefore, $\mathrm{g}\ell\ell(R) \le n+2 = \mathrm{index}(R) \le \mathrm{g}\ell\ell(R)$. □

**Corollary 3.3.8.** *Let $k$ be a field of characteristic $p > 2$ and $R = k[\![x,y]\!]/(xy(x^{p^n} + y^{p^n}))$, where $n \ge 0$. Then $\mathfrak{m}^{p^n+2} = (x+2y)\mathfrak{m}^{p^n+1}$, and*

$$\mathrm{g}\ell\ell(R) = \mathrm{index}(R) = p^n + 2.$$

*Proof.* Suppose $p \mid 1 + (-2)^{p^n}$. Since $1 + (-2)^{p^n} = 1 - 2^{p^n}$, we have $2^{p^n} = 1 \bmod p$. Since $2^{p^n} = 2 \bmod p$, it follows that $2 = 1 \bmod p$, which is false. Therefore, $p \nmid 1 + (-2)^{p^n}$. The result now follows from Proposition 3.3.7. □

If we let $p = 2$ in Corollary 3.3.8, then the generalized Loewy length and index of $R$ differ by one. This is a special case of Proposition 3.3.12. To prove Proposition 3.3.12, we require the following results about the reducibility of cyclotomic polynomials modulo prime integers and primitive roots of powers of prime integers.

**Lemma 3.3.9.** [18, Theorem 2.47] *Let $K = \mathbb{F}_q$, where $q$ is prime and $q \nmid n$. Let $\varphi$ denote Euler's totient function and $d$ the least positive integer such that $q^d = 1 \bmod n$. Then the $n^{th}$ cyclotomic polynomial $\Phi_n$ factors into $\varphi(n)/d$ distinct monic irreducible polynomials in $K[x]$ of degree $d$.*

**Lemma 3.3.10.** [18, Example 2.46] *Let $p$ be prime and $m \in \mathbb{N}$. Then the $p^m$th cyclotomic polynomial $\Phi_{p^m}$ equals*

$$1 + x^{p^{m-1}} + x^{2p^{m-1}} + \cdots + x^{(p-1)p^{m-1}}.$$

**Lemma 3.3.11.** [3, Proposition 3.4.1] *Let $p$ be a prime and $g$ a positive integer. Then the following three assertions are equivalent:*

*(1) $g$ is a primitive root modulo $p$ and $g^{p-1} \neq 1$ mod $p$;*

*(2) $g$ is a primitive root modulo $p^2$;*

*(3) For every $i \geq 2$, $g$ is a primitive root modulo $p^i$.*

**Proposition 3.3.12.** *Let $R = \mathbb{F}_2[\![x,y]\!]/(xy(x^{2^n p^m} + y^{2^n p^m}))$, where $m,n \geq 0$ and $p > 3$ is a prime such that $2$ is a primitive root modulo $p^2$. Then*

$$\mathrm{g}\ell\ell(R) = \mathrm{index}(R) + 1 = 2^n p^m + 3 \ \ \textit{and}$$

$$\mathfrak{m}^{2^n p^m + 3} \subseteq (x^2 + xy + y^2).$$

*If $m = 1$, then we need only assume that $2$ is a primitive root modulo $p$.*

*Proof.* First assume $m > 0$. We show that $x^2 + xy + y^2$ is $\mathrm{gr}_{\mathfrak{m}}(R)$-regular. Suppose $f, g \in \mathbb{F}_2[x,y]$ such that

$$(x^2 + xy + y^2)f = g(xy(x^{2^n p^m} + y^{2^n p^m})) = g(xy(x^{p^m} + y^{p^m})^{2^n}). \tag{3.3}$$

By Lemmas 3.3.9 through 3.3.11, $\Phi_{p^i}(x)$ is an irreducible polynomial over $\mathbb{F}_2$ of degree $p^i - p^{i-1}$ for $1 \leq i \leq m$. We obtain the following factorization of $x^{p^m} + 1$ into irreducible polynomials over $\mathbb{F}_2$.

$$x^{p^m} + 1 = (x+1)\prod_{i=1}^{m} \Phi_{p^i}(x).$$

Let $h_i(x,y) := y^{p^i - p^{i-1}} \Phi_{p^i}(x/y)$ for $i = 1, \ldots, m$. Then $h_i(x,y)$ is a homogeneous polynomial of degree $p^i - p^{i-1}$, and

$$x^{p^m} + y^{p^m} = (x+y)\prod_{i=1}^{m} h_i(x,y). \tag{3.4}$$

We claim that each $h_i(x,y)$ is irreducible over $\mathbb{F}_2$. Suppose $p, q \in \mathbb{F}_2[x,y]$ such that

$$h_i(x,y) = p(x,y)q(x,y).$$

Since $h_i$ is homogeneous, $p$ and $q$ are homogeneous. Let $y = 1$ in the above equation. Then

$$\Phi_{p^i}(x) = h_i(x,1) = p(x,1)q(x,1).$$

Since $\Phi_{p^i}$ is irreducible over $\mathbb{F}_2$, $p(x,1) = \Phi_{p^i}(x)$ or $q(x,1) = \Phi_{p^i}(x)$. Assume $p(x,1) = \Phi_{p^i}(x)$. Then $p(x,y) = h_i(x,y)$, so $h_i(x,y)$ is irreducible. By equations (3.3) and (3.4), we have

$$h_i \mid (x^2+xy+y^2) \text{ or } h_i \mid f.$$

Since the degree of $h_i$ is

$$p^i - p^{i-1} = p^{i-1}(p-1) \geq p^{i-1}3,$$

it follows that $h_i \mid f$. It is clear that $x$, $y$, and $x+y$ divide $f$ as well, so $f \in (xy(x^{p^m}+y^{p^m})^{2^n})$, and $x^2+xy+y^2$ is a nonzerodivisor on $\operatorname{gr}_{\mathfrak{m}}(R)$. By Theorem 3.2.4 and Lemma 3.3.4, $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)+1$ and $\mathfrak{m}^{2^n p^m+3} \subseteq (x^2+xy+y^2)$. If $m = 0$, then (3.3) becomes

$$(x^2+xy+y^2)f = g(xy(x^{2^n}+y^{2^n})) = g(xy(x+y)^{2^n}).$$

It follows that $f \in xy(x^{2^n}+y^{2^n})$, so $x^2+xy+y^2$ is a nonzerodivisor on $\operatorname{gr}_{\mathfrak{m}}(R)$. Therefore, $\mathfrak{m}^{2^n+3} \subseteq (x^2+xy+y^2)$ and $\mathrm{g}\ell\ell(R) = \operatorname{index}(R)+1$. □

*Remark* 3.3.13. Whether there are infinitely many primes $p$ such that 2 is a primitive root modulo $p$ is an open question. This is a special case of Artin's conjecture on primitive roots [2, p.66]. A list of the first primes $p$ for which 2 is a primitive root modulo $p$ is sequence A001122 in the OEIS.

## 3.4 Generalized Loewy length of graded algebras

We now consider positively-graded Noetherian $k$-algebras and a graded analogue of the generalized Loewy length of a local ring. Throughout this section, $k$ is an arbitrary field.

**Definition 3.4.1.** Let $R = \bigoplus_{i \geq 0} R_i$ be a positively-graded Noetherian $k$-algebra, where $R_0 = k$ and $\mathfrak{m} = \bigoplus_{i \geq 1} R_i$ is the irrelevant ideal. For $n \geq 0$, let $\mathfrak{m}_n := \bigoplus_{i \geq n} R_i$. The *generalized graded length* of $R$, denoted $\mathbf{ggl}(R)$, is the smallest positive integer $n$ for which $\mathfrak{m}_n$ is contained in the ideal generated by a homogeneous system of parameters.

In this context, the generalized Loewy length, $\mathrm{g}\ell\ell(R)$, is the smallest positive integer $n$ for which $\mathfrak{m}^n$ is contained in the ideal generated by a homogeneous system of parameters. With Herzog, we note that all of the above definitions can be transferred accordingly to standard graded Gorenstein $k$-algebras [12, page 98]. For $R = k[x,y]/(f)$, one can prove that $\mathrm{index}(R) = \deg(f)$ by using Ding's arguments in [6] to prove the standard graded version of [6, Theorem 3.3]. By the standard graded version of Proposition 3.3.3, the generalized Loewy length of $k[x,y]/(f)$ is one less than the sum of the degree of $f$ and the degree of a witness to $\mathrm{g}\ell\ell(R)$.

**Proposition 3.4.2.** *Let $R = k[x,y]/(f)$ be standard graded, where $f \in k[x,y]$ is a form of degree $e$. Let $z \in (x,y)R$ be a witness to $\mathrm{g}\ell\ell(R)$. Then $\mathrm{g}\ell\ell(R) = deg_R(z) + e - 1$.*

Let $(R,\mathfrak{m})$ be a positively-graded Noetherian $k$-algebra. It is clear that for each $n \geq 1$, we have $\mathfrak{m}^n \subseteq \mathfrak{m}_n$, so $\mathrm{g}\ell\ell(R) \leq \mathbf{ggl}(R)$. We now determine upper and lower bounds for $\mathbf{ggl}(R)$ in terms of $\mathrm{g}\ell\ell(R)$ and the minimum and maximum degrees of generators of $R$.

**Proposition 3.4.3.** *Let $(R,\mathfrak{m})$ be a positively-graded Noetherian $k$-algebra, where $R_0 = k$ and $\mathfrak{m}$ is the irrelevant ideal. Suppose $x_1,\ldots,x_n \in \mathfrak{m}$ are homogeneous elements such that $R = k[x_1,\ldots,x_n]$. Let*

$$min\{deg(x_i)\}_{i=1}^{n} = a \leq b = max\{deg(x_i)\}_{i=1}^{n}.$$

*Then*

$$a(\mathrm{g}\ell\ell(R)) - (a-1)^2 \leq \mathbf{ggl}(R) \leq b(\mathrm{g}\ell\ell(R)) - b + 1.$$

*If $a = b = 1$, then $\mathbf{ggl}(R) = \mathrm{g}\ell\ell(R)$.*

*Proof.* We claim that for $n \geq 0$, $\mathfrak{m}_{nb+1} \subseteq \mathfrak{m}^{n+1}$. This is trivial when $n = 0$. Suppose the inclusion holds for some $n \geq 0$. Let $x \in \mathfrak{m}_{(n+1)b+1}$ be homogeneous, and suppose $x = \sum_{i=1}^{n} s_i x_i$, where each $s_i \in R$ is homogeneous. Then $\deg(s_i) \geq (n+1)b+1-\deg(x_i) \geq nb+1$. Therefore, $s_i \in \mathfrak{m}_{nb+1} \subseteq \mathfrak{m}^{n+1}$, and $x \in \mathfrak{m}^{n+2}$. This proves the claim. Let $n = \mathrm{g}\ell\ell(R)-1$. Then by the above inclusion, $\mathbf{ggl}(R) \leq b(\mathrm{g}\ell\ell(R)-1)+1$.

Let $m = \mathbf{ggl}(R)$. There exists an integer $c \geq 0$ and an integer $0 \leq l < a$ such that $m = ac+l$. It is clear that $\mathfrak{m}^i \subseteq \mathfrak{m}_{ia}$ for $i \geq 0$. We claim that $\mathfrak{m}^{i+j} \subseteq \mathfrak{m}_{ia+j}$ for $i, j \geq 0$. Fix $i$. If the inclusion holds for some $j \geq 0$, then

$$\mathfrak{m}^{i+j+1} = \mathfrak{m}\cdot\mathfrak{m}^{i+j} \subseteq \mathfrak{m}\cdot\mathfrak{m}_{ia+j} \subseteq \mathfrak{m}_{ia+j+1}.$$

It follows that $\mathfrak{m}^{c+l} \subseteq \mathfrak{m}_m$ and $c+l \geq n = \mathrm{g}\ell\ell(R)$. Since $ac+al \geq an$, we have

$$\mathbf{ggl}(R) \geq a(\mathrm{g}\ell\ell(R)) - (a-1)l \quad \text{and}$$

$$\mathbf{ggl}(R) \geq a(\mathrm{g}\ell\ell(R)) - (a-1)^2.$$

□

Let $H = \langle a_1, ..., a_n\rangle$ be the numerical semigroup with unique minimal generating set $0 < a_1 < a_2 < \cdots < a_n$, where $\gcd(a_1, ..., a_n) = 1$. Let $C$ denote the *conductor* of $H$, the smallest integer $n \in H$ for which every integer larger than $n$ is also in $H$. Define $k[H] := k[t^{a_1}, ..., t^{a_n}] \subseteq k[t]$, where $k[H]$ is positively-graded via $|t^a| = a$.

**Proposition 3.4.4.** *Let $R = k[H]$, where $H = \langle a_1, ..., a_n\rangle$. Then* $\mathbf{ggl}(R) = C + a_1$.

*Proof.* Let $\mathfrak{m} = (t^{a_1}, ..., t^{a_n})$. It is clear that $\mathfrak{m}_{C+a_1} \subseteq (t^{a_1})$, so $\mathbf{ggl}(R) \leq C+a_1$. Let $n, d \geq 0$ and suppose $\mathfrak{m}_{C+n} \subseteq (t^d)$. This inclusion holds if and only if $t^{C+n+i} \in (t^d)$ for all $i \geq 0$, which is true if and only if $C+n+i-d \in H$ for all $i \geq 0$. This is equivalent to the inequality $C+n-d \geq C$, or $n \geq d$. Therefore, $\mathfrak{m}_{C+a_1-1} \not\subseteq (t^d)$ for all $d \in H \setminus \{0\}$. It follows that $\mathbf{ggl}(R) = C+a_1$. □

**Corollary 3.4.5.** *Let $R = k[t^a, t^b]$, where $a < b$. Then* $\mathbf{ggl}(R) = ba - b + 1$.

*Proof.* The conductor of $\langle a,b\rangle$ is $ba-a-b+1$ [21, page 201]. □

Veliche notes that for $R = k[\![t^a,t^b]\!]$, where $a<b$ and $k$ is infinite, we have $\mathrm{g}\ell\ell(R) = \mathrm{index}(R) = a$ [24, page 3]. She then determines formulas for the generalized Loewy lengths of Gorenstein local numerical semigroup rings of embedding dimension at least three over infinite fields. [24, Corollary 2.4, Corollary 3.3, Proposition 3.9]. If we know the conductor of the semigroup that determines one of these rings, then the generalized graded length of the corresponding graded ring is easier to compute than the generalized Loewy length of this local ring.

**Proposition 3.4.6.** *Let $R = k[H]$, where $H = \langle a,b\rangle$ and $a<b$. Suppose $z$ is a witness to* $\mathbf{ggl}(R)$*. Then $(z) = (t^{ia})$ for some $1 \le i \le 1+b-a$.*

*Proof.* We have $R \cong k[x,y]/(x^b-y^a) = k[\bar{x},\bar{y}]$, where $\deg(\bar{x}) = a$, $\deg(\bar{y}) = b$, and $\mathfrak{m} = (\bar{x},\bar{y})$. Let $z \in \mathfrak{m}$ be a witness to $\mathbf{ggl}(R)$. Suppose $z \in (\bar{y})$. By Proposition 3.4.4, $\bar{x}$ is also a witness to $\mathbf{ggl}(R)$, so $\mathfrak{m}_{ab-(b-1)} \subseteq (\bar{x}) \cap (\bar{y})$. Since $a<b$ are coprime, we have $b = as+r$ for some $s>0$ and $0<r<b$, so

$$ab-(b-1) = a(as+r)-(as+r-1) = a((a-1)s+r)-(r-1).$$

It follows that $\bar{x}^{(a-1)s+r} \in (\bar{x}) \cap (\bar{y}) = (\overline{xy}, \bar{x}^b)$. Since $(a-1)s+r<b$, we have $\bar{x}^{(a-1)s+r} = \overline{fxy}$ for some $f \in k[x,y]$ and $x^{(a-1)s+r} - fxy = gx^b - gy^a$ for some $g \in k[x,y]$. It follows that $fxy - gy^a = x^{(a-1)s+r} - gx^b$. Therefore, $y \mid (x^{(a-1)s+r} - gx^b)$. Since $(a-1)s+r<b$, this is false. We conclude that $z \notin (\bar{y})$. Therefore, $z = \bar{x}^i$ for some $1 \le i < b$. For each $n \ge 1$, we have $\mathfrak{m}^n \subseteq \mathfrak{m}_{na}$, so

$$\mathfrak{m}^b \subseteq \mathfrak{m}_{ab} \subseteq \mathfrak{m}_{ab-(b-1)}.$$

It follows that $\mathfrak{n}^b \subseteq (x^i, x^b-y^a)S = (x^i,y^a)S$, where $\mathfrak{n} = (x,y)S$, and $S$ denotes $k[x,y]$ with the standard grading. Let $M$ denote the $k$-vector space generated by monomials of degree $b$ in $(x^i,y^a)S$. Then

$$\dim_k M \le \binom{2+(b-i)-1}{b-i} + \binom{2+(b-a)-1}{b-a} = 2+2b-(a+i).$$

On the other hand, the $k$-vector space generated by monomials of degree $b$ in $\mathfrak{n}^b$ has dimension $b+1$. It follows that $b+1 \leq 2+2b-(a+i)$ and $i \leq 1+b-a$. □

**Definition 3.4.7.** Let $(R,\mathfrak{m})$ be a positively-graded Noetherian $k$-algebra, where $R_0 = k$ and $\mathfrak{m}$ is the irrelevant ideal. Let $I \subseteq R$ be a graded ideal. We say that $I$ is a *graded reduction* of $\mathfrak{m}$ of degree $d$ if there is a positive integer $i$ such that $I\mathfrak{m}_i = \mathfrak{m}_{i+d}$.

It is clear that for a numerical semigroup ring $k[t^{a_1},\ldots,t^{a_n}]$, the ideal $(t^{a_1})$ is a graded reduction of $\mathfrak{m}$ of degree $a_1$. We therefore ask the following questions, which parallel a question asked by De Stefani [4, Questions 4.5 (ii)].

**Questions 3.4.8.** Suppose $R$ is a positively-graded Noetherian $k$-algebra. Is there a witness to $\mathbf{ggl}(R)$ that generates a graded reduction of $\mathfrak{m}$? What can be said about the degree of such a graded reduction?

# 4 | SC$_r$-conditions

We study criteria for when MCM modules are MCM approximations of finitely-generated modules of some fixed codimension. Let $R$ be a $d$-dimensional Cohen-Macaulay local ring with canonical module and let $M$ be an MCM $R$-module. For $0 \leq r \leq d$, we say that $M$ satisfies the SC$_r$*-condition* if $M$ is stably isomorphic to the minimal MCM approximation of a finitely-generated $R$-module of codimension $r$. If each MCM $R$-module satisfies the SC$_r$-condition, we say $R$ satisfies the SC$_r$-condition.

Yoshino, Isogawa, and Kato determined the classes of rings which satisfy the SC$_1$- and SC$_2$-conditions. For $d \geq 3$ and $3 \leq r \leq d$, we prove a criterion for when an MCM $R$-module $M$ satisfies the SC$_r$-condition when $\Omega_R^1(M)$ satisfies the SC$_{r-1}$-condition. We use this criterion to prove the equivalence of the SC$_d$- and SC$_{d-1}$-conditions for Gorenstein complete local rings of dimension $d \geq 3$ that remain UFDs when factoring out certain regular sequences of length $d-2$. Throughout this chapter, $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with canonical module.

## 4.1 The SC$_1$- and SC$_2$-conditions

The minimal MCM approximation and minimal FID hull of a finitely-generated $R$-module are unique up to isomorphism by Theorem 2.2.17. By restricting our attention to modules of positive codimension, we obtain the following uniqueness result for minimal FID hulls.

**Theorem 4.1.1.** [15, Theorem 1.2] *Let $R$ be a Gorenstein complete local ring and $M$ a finitely-generated $R$-module with positive codimension. If $N$ is a finitely-generated $R$-module such that*

*$X^N \cong X^M$ and $Y^N \cong Y^M$, then $M \cong N$.*

Since MCM approximations and FID hulls are dual constructions, it is natural to ask if the map $M \mapsto X_M$ from finitely-generated modules of positive codimension to isomorphism classes of MCM modules is surjective. This leads to the following definition.

**Definition 4.1.2.** [15, Definition 2.1] Let $R$ be a $d$-dimensional Cohen-Macaulay local ring with canonical module and let $0 \leq r \leq d$. An MCM $R$-module $X$ satisfies the $\mathrm{SC}_r$-condition if there is a finitely-generated $R$-module $M$ of codimension $r$ such that $X_M \overset{st}{\cong} X$. If every MCM $R$-module satisfies the $\mathrm{SC}_r$-condition, we say that $R$ satisfies the $\mathrm{SC}_r$-condition.

The following result also motivates our study of the $\mathrm{SC}_r$-condition.

**Proposition 4.1.3.** [15, Proposition 2.5] *Let $R$ be a Gorenstein complete local ring and let $r$ be a positive integer. If $R$ satisfies the $\mathrm{SC}_r$-condition, then $R_{\mathfrak{p}}$ is regular for each prime ideal $\mathfrak{p}$ of $R$ with $\operatorname{height} \mathfrak{p} < r$.*

In the definition of the $\mathrm{SC}_r$-condition, it suffices to consider Cohen-Macaulay modules of codimension $r$, instead of all finitely-generated $R$-modules of codimension $r$.

**Proposition 4.1.4.** [15, Proposition 2.2] *Let $R$ be a Gorenstein complete local ring and let $X$ be an MCM $R$-module. Let $r$ be a positive integer. The following are equivalent.*

*(1) $X$ satisfies the $\mathrm{SC}_r$-condition; there is a finitely-generated $R$-module $M$ of codimension $r$ such that $X_M \overset{st}{\cong} X$.*

*(2) There is a Cohen-Macaulay $R$-module $C$ of codimension $r$ such that $X_C \overset{st}{\cong} X$.*

Let $R$ be a Gorenstein complete local ring and let $X$ be an MCM $R$-module that satisfies the $\mathrm{SC}_r$-condition. By Proposition 4.1.4, we have $X \overset{st}{\cong} X_C$, where $C$ is a Cohen-Macaulay $R$-module of codimension $r$. Let $C^\vee = \mathrm{Ext}^r_R(C,R)$. Then $X_C \cong \mathrm{Hom}_R(\Omega^r_R(C^\vee),R)$ and $X \overset{st}{\cong} \mathrm{Hom}_R(\Omega^r_R(C^\vee),R)$ [17, Proposition 11.15].

Since every MCM module is its own minimal MCM approximation, every Cohen-Macaulay local ring with canonical module satisfies the $\mathrm{SC}_0$-condition. We next quote a characterization of the $\mathrm{SC}_1$-condition. Recall that a ring $R$ is *generically Gorenstein* if the ring $R_{\mathfrak{p}}$ is Gorenstein for each minimal prime $\mathfrak{p}$ of $R$ [17, page 177].

**Proposition 4.1.5.** [17, Corollary 11.23] *Let $R$ be a Cohen-Macaulay local ring with canonical module. Assume $R$ is generically Gorenstein. Then the following statements are equivalent.*

*(1) $R$ satisfies the $\mathrm{SC}_1$-condition; that is, every MCM $R$-module is stably isomorphic to the minimal MCM approximation of a Cohen-Macaulay $R$-module of codimension 1.*

*(2) $R$ is a domain*

Yoshino and Isogawa proved that a Gorenstein complete local ring satisfies the $\mathrm{SC}_1$-condition if it is a domain [25, Section 2]. Kato then proved that these conditions are equivalent for a Gorenstein complete local ring [15]. Throughout the rest of this chapter, $R$ is a Gorenstein complete local ring. We first note that the $\mathrm{SC}_r$-condition implies the $\mathrm{SC}_i$-condition for all $i < r$.

**Proposition 4.1.6.** [15, Proposition 2.5] *Let $R$ be a Gorenstein complete local ring and let $X$ be an MCM $R$-module. Let $r > 0$. If $X$ satisfies the $\mathrm{SC}_{r+1}$-condition, then $X$ satisfies the $\mathrm{SC}_r$-condition. Therefore, if $R$ satisfies the $\mathrm{SC}_{r+1}$-condition, then $R$ satisfies the $\mathrm{SC}_r$-condition.*

Suppose $R$ satisfies the $\mathrm{SC}_2$-condition. By Proposition 4.1.6, $R$ also satisfies the $\mathrm{SC}_1$-condition. Therefore, $R$ is a domain by Proposition 4.1.4. In [15], Kato proved that $R$ satisfies the $\mathrm{SC}_2$-condition if and only if $R$ is a UFD. Yoshino and Isogawa first proved that the following statements are equivalent for a normal Gorenstein complete local ring $R$ of dimension two.

(1) $R$ is a UFD.

(2) For any MCM $R$-module, there is an $R$-module $L$ of finite length (hence a Cohen-Macaulay $R$-module of codimension 2) such that $M \overset{st}{\cong} \Omega_R^2(L)$.

(3) $R$ satisfies the $\mathrm{SC}_2$-condition.

**Theorem 4.1.7.** [15, Theorem 2.9] *A Gorenstein complete local ring R satisfies the* $\mathrm{SC}_2$*-condition if and only if R is a UFD.*

## 4.2 $\mathrm{SC}_r$-conditions for MCM modules

In this section, $R$ is a Gorenstein complete local ring. Using arguments from the proof of Theorem 4.1.7, we prove an inductive criterion for determining when an MCM module satisfies the $\mathrm{SC}_r$-condition. For $r > 0$, we let $\mathrm{CM}^r(R)$ denote the class of all Cohen-Macaulay $R$-modules of codimension $r$. We let $\mathrm{CM}(R)$ denote the class of MCM $R$-modules.

**Lemma 4.2.1.** [25, Theorem 1.4] *Let* $\mathbf{x}$ *be a regular sequence in R, and let M be a finitely-generated* $R/\mathbf{x}R$*-module. For* $n \geq 0$*, we have* $\Omega_R^{n+1}(M) \overset{st}{\cong} \Omega_R^n(\Omega^1_{R/\mathbf{x}R}(M))$.

**Proposition 4.2.2.** *Let R be a Gorenstein complete local ring of dimension* $d \geq 3$ *and* $3 \leq r \leq d$*. Let M be an MCM R-module and suppose* $\Omega_R^1(M)$ *satisfies the* $\mathrm{SC}_{r-1}$*-condition. Let* $L \in \mathrm{CM}^{r-1}(R)$ *such that* $X_L \overset{st}{\cong} \Omega_R^1(M)$*. If there is a regular sequence* $\mathbf{x} \in \mathrm{Ann}_R(L)$ *of length* $r-2$ *such that* $R/\mathbf{x}R$ *is a UFD, then M satisfies the* $\mathrm{SC}_r$*-condition.*

*Proof.* We first prove that $\Omega_R^{r-1}(L) \overset{st}{\cong} \Omega_R^r(M)$. Taking minimal projective resolutions of $L$ and $Y_L$, we apply the Horseshoe Lemma to the minimal MCM approximation of $L$ and obtain the following diagram with exact rows, and columns that are truncated projective resolutions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_R^{r-1}(Y_L) & \longrightarrow & Z_{r-1} & \longrightarrow & \Omega_R^{r-1}(L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_{r-1} & \longrightarrow & P_{r-1} & \longrightarrow & P''_{r-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_0 & \longrightarrow & P_0 & \longrightarrow & P''_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_L & \longrightarrow & X_L & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the Depth Lemma (Lemma 2.1.5) we have $\operatorname{depth}_R(Y_L) \geq \min\{\operatorname{depth}_R(X_L), \operatorname{depth}_R(L)+1\}$. Since $X_L$ is MCM and $\operatorname{depth}_R(L) = d-(r-1)$, we have $\operatorname{depth}_R(Y_L) > d-(r-1)$. By successively applying the Depth Lemma, we see that $\Omega_R^{r-1}(Y_L)$ is an MCM $R$-module of finite projective dimension. Therefore, $\Omega_R^{r-1}(Y_L)$ is free by [17, Proposition 11.7]. Likewise, $\Omega_R^{r-1}(L)$ is an MCM $R$-module. Applying $\operatorname{Hom}_R(\Omega_R^{r-1}(L), -)$ to the top row in the diagram, we obtain the following exact sequence.

$$\operatorname{Hom}_R(\Omega_R^{r-1}(L), Z_{r-1}) \longrightarrow \operatorname{Hom}_R(\Omega_R^{r-1}(L), \Omega_R^{r-1}(L)) \longrightarrow \operatorname{Ext}_R^1(\Omega_R^{r-1}(L), \Omega_R^{r-1}(Y_L))$$

Since $\Omega_R^{r-1}(L)$ is MCM and $\Omega_R^{r-1}(Y_L)$ has finite projective dimension, it follows from Proposition 2.2.3 that

$$\operatorname{Ext}_R^1(\Omega_R^{r-1}(L), \Omega_R^{r-1}(Y_L)) = 0.$$

Therefore, the top row of the diagram splits and we have $\Omega_R^{r-1}(X_L) \overset{st}{\cong} Z_{r-1} \overset{st}{\cong} \Omega_R^{r-1}(L)$. Since $X_L \overset{st}{\cong} \Omega_R^1(M)$, it follows that $\Omega_R^{r-1}(L) \overset{st}{\cong} \Omega_R^r(M)$.

Let $S = R/\mathbf{x}R$. Then $L \in \mathrm{CM}^1(S)$. Let $\mathrm{Ass}(L) = \{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$ and let $\Gamma := S - \bigcup_{i=1}^{m} \mathfrak{p}_i$. Since $L$ is a Cohen-Macaulay $S$-module of codimension 1, we have $\operatorname{ht} \mathfrak{p}_i = 1$ for each $i$ by Proposition 2.1.12 and Proposition 2.1.13. Since $S$ is a UFD, $\mathfrak{p}_i$ is a principal ideal for each $i$ [2, Lemma 2.2.17]. Write $\mathfrak{p}_i = (p_i)$, where $p_i \in S$. Let $\mathfrak{q}$ be a nonzero prime ideal of $\Gamma^{-1}S$ and let $h : S \longrightarrow \Gamma^{-1}S$ be the localization map. Then $h^{-1}(\mathfrak{q}) \subseteq \bigcup_{i=1}^{m} (p_i)$ is a nonzero prime ideal, and by prime avoidance, $h^{-1}(\mathfrak{q}) \subseteq (p_i)$ for some $(p_i)$. Therefore, $h^{-1}(\mathfrak{q}) = (p_i)$ and $\mathfrak{q} = p_i\Gamma^{-1}S$. We conclude that every prime ideal of $\Gamma^{-1}S$ is principal. Therefore, $\Gamma^{-1}S$ is a PID. Since $\Gamma^{-1}L$ is a finitely generated $\Gamma^{-1}S$-module, there are elements $a_1, ..., a_t \in \Gamma^{-1}S$ such that $\Gamma^{-1}L \cong \bigoplus_{k=1}^{t} \Gamma^{-1}S/a_k\Gamma^{-1}S$ as $\Gamma^{-1}S$-modules. Let $\phi : \Gamma^{-1}L \longrightarrow \bigoplus_{k=1}^{t} \Gamma^{-1}S/a_k\Gamma^{-1}S$ be the corresponding isomorphism. Since $L \in \mathrm{CM}^1(S)$, it follows that $L$ is a torsion $S$-module, and $\Gamma^{-1}L$ is a torsion $\Gamma^{-1}S$-module. Therefore, $a_k \neq 0$ for all $k$.

Fix $k$. We claim that each associated prime ideal of $I = h^{-1}(a_k\Gamma^{-1}S)$ has height one. Since $I \neq 0$, the associated primes of $I$ have height at least one. Suppose $I$ has an associated prime $\mathfrak{q}$ of height greater than one. Then $\Gamma \cap \mathfrak{q} \neq \emptyset$. Let $s \in \Gamma \cap \mathfrak{q}$. Since $\mathfrak{q}$ is an associated prime of $I$, there exists an element $x \in S \setminus I$ such that $\mathfrak{q} = \mathrm{Ann}_S(\bar{x})$, with $\bar{x} \in S/I$. Therefore, $\mathfrak{q}x \subseteq I$ and $sx \in I$. It follows that $\frac{sx}{1} \in a_k\Gamma^{-1}S$, so $\frac{x}{1} \in a_k\Gamma^{-1}S$ and $x \in I$, which is a contradiction. We conclude that every associated prime of $I$ has height one, and is therefore principal. Let $\mathrm{Ass}(I) = \{(q_1), ..., (q_m)\}$. Take an irredundant primary decomposition of $I$. Then

$$I = Q_1 \cap Q_2 \cap ... \cap Q_v,$$

where for each $l$, $\sqrt{Q_l} = (q_l)$. We claim that each $Q_l$ is a principal ideal. Fix $1 \leq l \leq v$. We have $Q_l \subseteq (q_l)$. Since $\bigcap_{\alpha \geq 0} (q_l^{\alpha}) = 0$, there is a positive integer $\alpha$ such that $Q_l \subseteq (q_l^{\alpha})$ and $Q_l \not\subseteq (q_l^{\alpha+1})$. Since $S$ is Noetherian, $Q_l$ is finitely-generated. Write $Q_l = (y_1, ..., y_r)$, where $y_1, ..., y_r \in S$. Since $Q_l \subseteq (q_l^{\alpha})$, there are elements $c_1, ..., c_r$ in $S$ such that $y_j = c_j q_l^{\alpha}$ for $j = 1, ..., r$. So $Q_l = (c_1 q_l^{\alpha}, ..., c_r q_l^{\alpha})$. Since $Q_l \not\subseteq (q_l^{\alpha+1})$, there is an index $j$ such that $c_j \notin (q_l)$. Since $c_j q_l^{\alpha} \in Q_l$

and $Q_l$ is a primary ideal, $q_l^\alpha \in Q_l$ or $c_j^\beta \in Q_l$ for some $\beta > 0$. If $c_j^\beta \in Q_l$ for some $\beta > 0$, then $c_j \in \sqrt{Q_l} = (q_l)$, which is false. Therefore, $q_l^\alpha \in Q_l \subseteq (q_l^\alpha)$, whence $Q_l = (q_l^\alpha)$. We conclude that $Q_l$ is a principal ideal for $l = 1, ..., v$. Therefore, $I$ is an intersection of principal ideals, and since $S$ is a UFD, $I$ is also a principal ideal.

Let $b_k \in S$ such that $I = (b_k)$. Then $a_k\Gamma^{-1}S = b_k\Gamma^{-1}S$. We claim that $\operatorname{Ass}(b_k) = \operatorname{Ass}(I) \subseteq \operatorname{Ass}(L) = \{(p_1), ..., (p_m)\}$. Suppose $\mathfrak{q} \in \operatorname{Ass}(I) - \operatorname{Ass}(L)$. Since $\mathfrak{q}$ has height 1, $\mathfrak{q} \cap \Gamma \neq \emptyset$. Let $s \in \mathfrak{q} \cap \Gamma$, and let $x \in S - I$ such that $\mathfrak{q} = \operatorname{Ann}_S(\bar{x})$, with $\bar{x} \in S/I$. Then $sx \in I$, so $\frac{sx}{1} \in a_k\Gamma^{-1}S$ and $x \in I$, a contradiction. We conclude that $\operatorname{Ass}(b_k) \subseteq \operatorname{Ass}(L)$. We may therefore assume that $a_k \in S$ and that every associated prime of $a_k$ is an associated prime of $L$. Since there is an isomorphism of $\Gamma^{-1}S-$modules

$$\Gamma^{-1}\operatorname{Hom}_S(L, \bigoplus_{k=1}^{t} S/a_kS) \cong \operatorname{Hom}_{\Gamma^{-1}S}(\Gamma^{-1}L, \bigoplus_{k=1}^{t} \Gamma^{-1}S/a_k\Gamma^{-1}S),$$

there is an $S$-map $f : L \longrightarrow \bigoplus_{k=1}^{t} S/a_kS$ such that $\Gamma^{-1}f = \phi$. We claim that $f$ is a monomorphism. Suppose not. Then $\ker f \neq 0$, and $\operatorname{Ass}(\ker f) \neq \emptyset$. Let $\mathfrak{p} \in \operatorname{Ass}(\ker f)$. Since $\operatorname{Ass}(\ker f) \subseteq \operatorname{Supp}(\ker f)$, we have $(\ker f)_\mathfrak{p} \neq 0$. Also, since $\ker f \subseteq L$, we have $\mathfrak{p} \in \operatorname{Ass} L = \{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$. Let $T = S - \mathfrak{p}$ and let $T'$ be the image of $T$ in $\Gamma^{-1}S$. Since $\Gamma = S - \bigcup_{i=1}^{m} \mathfrak{p}_i$, we have $\Gamma \subseteq T$. Starting with our $\Gamma^{-1}S$-isomorphism $\Gamma^{-1}f : \Gamma^{-1}L \longrightarrow \bigoplus_{k=1}^{t} \Gamma^{-1}S/a_k\Gamma^{-1}S$, we localize at $T'$, obtaining the $T^{-1}S$-isomorphism

$$T^{-1}f : T^{-1}L \longrightarrow \bigoplus_{k=1}^{t} T^{-1}S/a_kT^{-1}S.$$

But this map is just $f_\mathfrak{p} : L_\mathfrak{p} \to \bigoplus_{k=1}^{t} S_\mathfrak{p}/a_kS_\mathfrak{p}$. Therefore, we have $0 = \ker(f_\mathfrak{p}) = (\ker f)_\mathfrak{p} \neq 0$, a contradiction. Therefore, $\ker f = 0$ and we have an exact sequence of $S$-modules

$$0 \longrightarrow L \xrightarrow{f} \bigoplus_{k=1}^{t} S/a_kS \longrightarrow L' \longrightarrow 0. \tag{4.1}$$

Taking minimal projective resolutions of $L$ and $L'$, the Horseshoe Lemma gives us the following diagram with exact rows, and columns that are truncated projective resolutions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_S^1(L) & \longrightarrow & Z_1 & \longrightarrow & \Omega_S^1(L') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \bigoplus P_0' & \longrightarrow & P_0' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \bigoplus_{k=1}^{t} S/a_kS & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $\operatorname{pd}_S S/a_kS = 1$ for each $k$, we have $\Omega_S^1(S/a_kS) \overset{st}{\cong} S$, and therefore $Z_1$ is stably isomorphic to a free $S$-module. Therefore, $\Omega_S^2(L') \overset{st}{\cong} \Omega_S^1(L)$. Since $\Omega_R^{r-1}(L) \overset{st}{\cong} \Omega_R^r(M)$, by Lemma 4.2.1 we have

$$\Omega_R^{r-2}(\Omega_S^2(L')) \overset{st}{\cong} \Omega_R^{r-2}(\Omega_S^1(L)) \overset{st}{\cong} \Omega_R^{r-1}(L) \overset{st}{\cong} \Omega_R^r(M).$$

On the other hand,

$$\Omega_R^{r-2}(\Omega_S^2(L')) \overset{st}{\cong} \Omega_R^{r-1}(\Omega_S^1(L')) \overset{st}{\cong} \Omega_R^r(L').$$

Therefore, $\Omega_R^r(L') \overset{st}{\cong} \Omega_R^r(M)$. We claim that $L'$ is Cohen-Macaulay. We have

$$\operatorname{Supp}(L') \subseteq \operatorname{Supp}(\bigoplus_{k=1}^{t} S/a_kS) \subseteq \bigcup_{k=1}^{t} \operatorname{Supp}(S/a_kS).$$

Applying the Depth Lemma to 4.1, we have $\operatorname{depth}_S(L') \geq \operatorname{depth}(S) - 2 = d - r$. Since

$$\dim_S(\bigoplus_{k=1}^{t} S/a_kS) = \dim(S) - 1 = d - r + 1,$$

we have $\dim_S(L') \leq d - r + 1$. Suppose $\dim_S(L') = d - r + 1$. Then there is a chain of prime ideals $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_{d-r+1}$ in $\operatorname{Supp}(L')$. Since $\mathfrak{q}_0 \in \operatorname{Supp}(L')$, there exists an index $k$ such that

$\mathfrak{q}_0 \in \operatorname{Supp}(S/a_kS)$. Therefore, $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_{d-r+1}$ is a chain of prime ideals in $\operatorname{Supp}(S/a_kS)$. Since the Krull dimension of $\operatorname{Supp}(S/a_kS)$ is $d-r+1$, it follows that $\mathfrak{q}_0$ is a minimal prime in $\operatorname{Supp}(S/a_kS)$. Therefore, $\mathfrak{q}_0 = \mathfrak{p}_i \in \operatorname{Ass}(L)$ for some $1 \le i \le m$. Since $f_{\mathfrak{p}_i} : L_{\mathfrak{p}_i} \longrightarrow \bigoplus_{k=1}^{t} S_{\mathfrak{p}_i}/a_kS_{\mathfrak{p}_i}$ is an isomorphism, we have $0 \neq L'_{\mathfrak{q}_0} = L'_{\mathfrak{p}_i} = 0$, a contradiction. We conclude that $\dim_S(L') = d-r$ and

$$L' \in \mathrm{CM}^2(S) \subseteq \mathrm{CM}^r(R).$$

Since $\Omega_R^r(L') \overset{st}{\cong} \Omega_R^r(M)$, by Lemma 2.2.20, we have

$$X_{L'} \overset{st}{\cong} \Omega_R^{-r}(\Omega_R^r(L')) \overset{st}{\cong} \Omega_R^{-r}(\Omega_R^r(M)) \overset{st}{\cong} M.$$

. □

Let $\operatorname{Spec}(R)$ denote the set of prime ideals of $R$ and let $U_R := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ denote the punctured spectrum of $R$. Let $\operatorname{Pic}(U_R)$ denote the Picard group of $U_R$ [9, Chapter 5].

**Definition 4.2.3.** [9, Chapter 5] A Noetherian local ring $R$ is *parafactorial* if $\operatorname{depth}(R) \ge 2$ and $\operatorname{Pic}(U_R) = 0$.

This weaker notion of factoriality gives us the following criterion for determining when a ring is a UFD. We use this criterion and Kato's result on regular localizations of rings satisfying the $\mathrm{SC}_r$-condition to study the relation between the $\mathrm{SC}_r$-condition and UFDs obtained by factoring out a regular sequence.

**Proposition 4.2.4.** [9, Corollary 18.11] *Suppose $R$ is a Noetherian local ring such that $\dim(R) \ge 2$. Then $R$ is a UFD if and only if $R$ is parafactorial and $R_{\mathfrak{p}}$ is a UFD for all $\mathfrak{p} \in U_R$.*

**Corollary 4.2.5.** *Let $(R, \mathfrak{m})$ be a Gorenstein complete local ring of dimension $d \ge 3$. The following are equivalent.*

*(1) $R$ satisfies the $\mathrm{SC}_{d-1}$-condition and for each $M \in \mathrm{CM}(R)$, there is a module $L \in \mathrm{CM}^{d-1}(R)$ such that $X_L \overset{st}{\cong} \Omega_R^1(M)$ and $\operatorname{Ann}_R(L)$ contains a regular sequence $\mathbf{x}$ of length $d-2$ such that $R/\mathbf{x}R$ is a UFD.*

*(2) $R$ satisfies the* $\mathrm{SC}_d$*-condition and for each* $M \in \mathrm{CM}(R)$*, there is a module* $L \in \mathrm{CM}^{d-1}(R)$ *such that* $X_L \overset{st}{\cong} \Omega^1_R(M)$ *and* $\mathrm{Ann}_R(L)$ *contains a regular sequence* $\mathbf{x}$ *of length* $d-2$ *that satisfies the following.*

*(i)* $\mathbf{x}$ *is a subset of a regular system of parameters for* $R_{\mathfrak{p}}$ *for all prime ideals* $\mathfrak{p} \in U_R$ *containing* $\mathbf{x}$

*(ii)* $\mathrm{Pic}(U_{R/\mathbf{x}R}) = 0$

*Proof.* $(1) \Rightarrow (2)$. By Proposition 4.2.2, $R$ satisfies the $\mathrm{SC}_d$-condition. Let $M \in \mathrm{CM}(R)$, and let $L$ and $\mathbf{x}$ be as in $(1)$. Let $\mathfrak{p} \in U_R$ be a prime ideal that contains $\mathbf{x}$. Let $\pi : R \longrightarrow R/\mathbf{x}R$ be the quotient map, and let $\mathfrak{P} = \pi(\mathfrak{p})$. Then $\mathfrak{P}$ is a prime ideal and $\mathrm{height}(\mathfrak{P}) < 2$. Since $\mathrm{height}(\mathfrak{p}) < d$ and $R$ satisfies the $\mathrm{SC}_d$-condition, the localization $R_{\mathfrak{p}}$ is regular by Proposition 4.1.3. Since $R/\mathbf{x}R$ is a Gorenstein complete local UFD, it satisfies the $\mathrm{SC}_2$-condition by Theorem 4.1.7. By Proposition 4.1.3, we have $(R/\mathbf{x}R)_{\mathfrak{P}} \cong R_{\mathfrak{p}}/\mathbf{x}R_{\mathfrak{p}}$ is a regular local ring. Thus, $\mathbf{x}$ is a subset of a regular system of parameters for $R_{\mathfrak{p}}$. Finally, we have $\mathrm{Pic}(U_{R/\mathbf{x}R}) = 0$ by Proposition 4.2.4.

$(2) \Rightarrow (1)$ Since $R$ satisfies the $\mathrm{SC}_d$-condition, $R$ satisfies the $\mathrm{SC}_{d-1}$-condition by Proposition 4.1.6. Let $M \in \mathrm{CM}(R)$, and let $L$ and $\mathbf{x}$ be as in $(2)$. We prove that $R/\mathbf{x}R$ is a UFD. Let $\mathfrak{P} \in U_{R/\mathbf{x}R}$ and let $\mathfrak{p} = \pi^{-1}(\mathfrak{P})$. Then $\mathfrak{p}$ is a prime ideal containing $\mathbf{x}$ and $\mathrm{height}(\mathfrak{p}) < d$. Since $R$ satisfies the $\mathrm{SC}_d$-condition, $R_{\mathfrak{p}}$ is a regular local ring. Since $\mathbf{x}$ is a subset of a regular system of parameters for $R_{\mathfrak{p}}$, we have $R_{\mathfrak{p}}/\mathbf{x}R_{\mathfrak{p}} \cong (R/\mathbf{x}R)_{\mathfrak{P}}$ is a regular local ring, and therefore a UFD [2, Theorem 2.2.19]. Therefore, $R/\mathbf{x}R$ is a UFD by Proposition 4.2.4. □

**Corollary 4.2.6.** *Let $R$ be a Gorenstein complete local ring of dimension* 3*. Assume that $R$ is a UFD and for each* $M \in \mathrm{CM}(R)$ *there exists* $L \in \mathrm{CM}^2(R)$ *such that* $X_L \overset{st}{\cong} \Omega^1_R(M)$ *and a nonzerodivisor* $x \in \mathrm{Ann}_R(L)$ *such that* $R/(x)$ *is also a UFD. Then $R$ satisfies the* $\mathrm{SC}_3$*-condition.*

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## Education

- **Syracuse University** — **Syracuse, NY**
  *PhD-Mathematics* — *Expected May 2024*
  - PhD Advisor: Graham Leuschke
- **Syracuse University** — **Syracuse, NY**
  *MS-Mathematics* — *May 2019*
- **West Chester University of Pennsylvania** — **West Chester, PA**
  *BA-Mathematics* — *January 2017*
  - *summa cum laude*

## Research

### Research Interests

Cohen-Macaulay rings, graded Noetherian algebras, maximal Cohen-Macaulay approximations, Auslander's delta invariant, generalized Loewy length, Ding's conjecture.

### Papers

*Generalized Loewy length of Cohen-Macaulay local and graded rings* (submitted) arXiv version

### Talks and Posters

- **Route 81 Conference on Commutative Algebra and Algebraic Geometry** — **Syracuse University**
  *Generalized Lowey length of Cohen-Macaulay local and graded rings* — *November 2023*
- **Algebra Days at Arizona State (poster)** — **Arizona State**
  *Generalized Lowey length of Cohen-Macaulay local and graded rings* — *November 2023*
- **Syracuse University Algebra Seminar** — **Syracuse University**
  *Generalized Lowey length of Cohen-Macaulay local and graded rings* — *September 2023*
- **KUMUNU 2023 (poster)** — **University of Missouri**
  *Generalized Loewy length of Cohen-Macaulay local and graded rings* — *September 2023*
- **Commutative Algebra Regional Expository Seminar (CARES)** — **Virtual**
  *Maximal Cohen-Macaulay approximations part 2* — *April 25, 2023*
- **Commutative Algebra Regional Expository Seminar (CARES)** — **Virtual**
  *Maximal Cohen-Macaulay approximations part 1* — *April 17, 2023*

- **Working Algebra Knowledge Seminar (WALKS)** — **Syracuse University**
  *Generalized graded length of positively-graded algebras* — *October 2023*
- **Working Algebra Knowledge Seminar (WALKS)** — **Syracuse University**
  *Generalized Lowey length of Cohen-Macaulay local and graded rings* — *September 2023*
- **Working Algebra Knowledge Seminar (WALKS)** — **Syracuse University**
  *Characterizing regular local rings in terms of Auslander's $\delta$-invariant* — *March 2023*
- **Working Algebra Knowledge Seminar (WALKS)** — **Syracuse University**
  *Existence and uniqueness of maximal Cohen-Macaulay approximations* — *February 2023*
- **Working Algebra Knowledge Seminar (WALKS)** — **Syracuse University**
  *Integral extensions* — *January 2023*
- **Mathematics Graduate Organization Colloquium** — **Syracuse University**
  *Cohen-Macaulay approximations and the index of Cohen-Macaulay local rings* — *October 2022*
- **Mathematics Graduate Organization Colloquium** — **Syracuse University**
  *Indecomposable decompositions of projective and injective modules* — *April 2021*

## Conferences and Workshops Attended

- **Conference on Unexpected and Asymptotic Properties of Algebraic Varieties**
  *University of Nebraska-Lincoln* — *August 11, 2023*
- **Commutative Algebra Market Preparation Workshop**
  *University of Nebraska-Lincoln* — *August 8-10, 2023*
- **48th Annual New York State Regional Graduate Mathematics Conference**
  *Syracuse University* — *April 1, 2023*
- **GTA Philadelphia Mathematics Conference**
  *Temple University* — *May 20-22 2022*
- **47th Annual New York State Regional Graduate Mathematics Conference**
  *Syracuse University* — *April 1, 2022*

## Awards and Fellowships

- **Academic Fellowship**
  *Syracuse University* — *2017-18, 2020-21, Summer 2023*

## Teaching Experience

### Instructor of Record

- **MAT 397-Calculus III**
  *Syracuse University* — *Summer 2022, 2021, and 2020*

- **MAT 286-Calculus for the Life Sciences II**
  *Syracuse University* — *Fall and Spring 2022*

- **MAT 295-Calculus I**
  *Syracuse University* — *Spring 2024 and Fall 2023*

### Recitation Instructor

- **MAT 397-Calculus III**
  *Syracuse University* — *Spring 2023*

- **MAT 296-Calculus II**
  *Syracuse University* — *Spring 2019*

- **MAT 295-Calculus I**
  *Syracuse University* — *Fall 2019 and 2018*

## Service and Mentorship

- **WALKS Founder and Coordinator** — **WALKS website**
  *Syracuse University* — *Spring 2024, Fall and Spring 2023*

- **Directed Reading Program Mentor** — **DRP website**
  *Syracuse University* — *Spring 2024, Fall and Spring 2023, Spring 2021*
  Spring 2024-Hilbert space theory
  Fall 2023-Category theory
  Spring 2023-Commutative algebra
  Spring 2021-Real analysis

## Tutoring Experience

- **Undergraduate Math Tutor**
  *Syracuse University* — *Spring 2024, Fall and Spring 2022, Spring 2021*
  I worked as a private tutor for the following courses.
  - MAT 517-Partial Differential Equations and Fourier Series
  - MAT 485-Differential Equations and Matrix Algebra for Engineers
  - MAT 414-Ordinary Differential Equations
  - MAT 412-Introduction to Real Analysis I
  - MAT 397-Calculus III
  - MAT 296-Calculus II
  - MAT 295-Calculus I
  - MAT 286-Calculus for the Life Sciences II
  - MAT 285-Calculus for the Life Sciences I
  - MAT 284-Business Calculus

- **Math and English Tutor** — **Doylestown, PA**
  *C2 Education* — *Fall 2021*
  I tutored students in Calculus, Pre-Calculus, the SAT and ACT.

- **Math Clinic Tutor**
  *Syracuse University* — *Summer and Spring 2019, Fall and Summer 2018*

## Professional Memberships

- **American Mathematical Society**
- **Mathematical Association of America**