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On homogeneous closed gradient Laplacian solitons and the modified conformal Hessian

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Abstract

Laplacian solitons are self-similar solutions to a geometric flow of G_2-structures $\varphi \in \Omega^3(M)$ on smooth 7-manifolds M called the Laplacian flow. Recently, Laplacian solitons on homogeneous spaces have received increased interest and many new examples have been found by Fernandez-Raffero, Lauret-Nicolini, and others (see, e.g., [FR20, Lau17a, Lau17b, LN20, Nic18], and [Nic22]). Though there has been recent work on gradient Laplacian solitons in the nonhomogeneous setting due to Haskins and his collaborators (see, e.g., [HN21, HKP22]), very little is known about gradient solitons of a closed Laplacian flow on homogeneous spaces.

In this thesis, we investigate homogeneous closed gradient Laplacian solitons. We prove a Structure Theorem for homogeneous closed gradient Laplacian solitons. We then use the Structure Theorem to "eliminate" closed gradient Laplacian solitons. That is, we use the Structure Theorem to show that some closed Laplacian solitons or closed G_2-structures cannot be made gradient. We also use the Structure Theorem to obtain the structure of almost abelian solvmanifolds admitting closed gradient Laplacian solitons.

We then study weighted sectional curvature of Riemannian manifolds with density. In particular, we study how weighted sectional curvature bounds give us control over the modified conformal hessian. We use this to prove an inequality resembling the law of cosines, which we call a "warped law of cosines".

On homogeneous closed gradient Laplacian solitons and the modified conformal Hessian

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Contents

List of Figures

List of Tables

1 | Introduction

A major theme in mathematics is classifying mathematical objects. That is, given a mathematical object, it is of interest to know how to classify them from the properties they possess. Results of this type are called *classification theorems*. An example of a classification theorem from plane geometry is that two circles are congruent if and only if they have the same radius. Another example is the uniformization theorem for compact surfaces, which roughly states that depending on the sign of the *Gaussian curvature* K, that every compact surface is either "bowl-shaped", "saddle-like", or "flat". This is, roughly speaking, a classification of compact surfaces. Yet another classification is that every compact orientable surface, i.e., every surface that is connected and "closes up" (does not extend infinitely in any direction), is diffeomorphic to a genus g surface, where g can be thought of as the number of "holes" in the surface (e.g., the outside of a donut is a genus 1 surface). In higher dimensions, classification of objects analogous to surfaces require more sophisticated tools, e.g., more general notions of curvature, to obtain.

Riemannian geometry is the study of *Riemannian manifolds* M, which we can think of as "higher dimensional smooth surfaces", with a "smooth" *Riemannian metric* g, a "ruler" for which at each point of the manifold allows us to measure angles, lengths, area, and volume. That the metric is "smooth" means that small changes in the points on which we take these measurements affects small changes in those measurements. Any Riemannian metric yields a generalized notion of the "derivative" of vector fields on a manifold called the *Levi-Civita connection* ∇. This in turn gives us a notion of *curvature* of Riemannian manifolds, which plays an important role in Riemannian geometry. A well known classification is that Riemannian manifolds with constant sectional curvature, a generalization of constant Gaussian curvature, are either sphere-like, flat, or hyperbolic (see Fact 1.2.5). Another important classification theorem is Thurston's geometrization conjecture, which states that a closed (i.e., compact without boundary) 3-manifold can be decomposed into two parts, each of which can be classified as one of eight possible geometric structures. This result is often thought of as the 3-manifold analog of the uniformization theorem for surfaces (2-manifolds). Classification theorems in higher dimensions are much harder to obtain. One way to approach them is by studying geometric and topological consequences of Riemannian manifolds admitting certain structures. A prominently studied structure is the metric g as it determines the curvature of a manifold. By studying how Riemannian metrics $g(t)$ evolve over time t, one can glean information about the manifold it is defined on and how it evolves. An equation that describes how the metric evolves is an instance of the more general concept of geometric flows.

A *geometric flow* is a partial differential equation describing how a *geometric structure* (e.g., a metric $\gamma = g$) evolves in time. One line of inquiry is to study which manifolds equipped with a geometric structure behave in the manner described by a given flow. A manifold along with a geometric structure that behaves in the manner described by a given flow (i.e., satisfies a geometric flow) is called a *solution* to the flow. A *self-similar solution*, i.e., a solution that changes in time only by diffeomorphism or scaling, is called a *geometric soliton (or soliton geometric structure)*. Solitons are important in the study of geometric flows as they model singularities of the flow; we can think of solitons as “fixed points” or equilibrium solutions to a differential equation. Moreover, solitons can often tell us how solutions close to them behave in time. An example of a well-studied geometric flow is the *Ricci flow*. In this setting the geometric structure is the Riemannian metric g and the flow describes that the metric evolves in a manner proportional to the (Ricci) curvature of the manifold. Self-similar solutions to the Ricci flow are called *Ricci solitons*. A famous example of how flows can help in classifying manifolds is Perelman's use of the Ricci flow in his proof of Thurston's geometrization conjecture (see [MT07]).

In addition to helping with classifying manifolds, curvature can often give us information about their overall topology. A fundamental example of this is the Gauss-Bonnet theorem, which states that a compact Riemannian 2-manifold M must satisfy the equation

$$\int_M K dA = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic. Note that the local geometric property of (Gaussian) curvature K can give us information about the global topological invariant of the manifold $\chi(M)$ via the preceding equation. Thus results like these are often referred to as "local-to-global" results. The Bonnet-Myers theorem is another example of a "local-to-global" theorem (see Fact 1.2.6). In 2015, Wylie introduced the notion of weighted sectional curvature, a generalization of sectional curvature, for manifolds with density. Wylie and his collaborators have since generalized many classical "local-to-global" results with sectional curvature hypotheses to the weighted sectional curvature setting.

This thesis consists of two parts. The first part concerns the solitons of the natural geometric flow of G_2-structures φ called the Laplacian flow, which was originally introduced by Bryant and his collaborators in 1992 to aid in obtaining metrics with holonomy in G_2. More precisely, we study the gradient solitons of this flow on homogeneous 7-manifolds M and obtain a structure theorem for them, i.e., a classification of the possible structures of M admitting such gradient solitons. The second part of this thesis concerns manifolds with density. More specifically, we study how the local geometric property of weighted sectional curvature affects the hessian of modified distance functions on manifolds with density.

1.1 Summary of results

The first two chapters cover background needed for the results of this thesis. The rest of Chapter 1 is a brief (informal) review of basic Riemannian geometry. Chapter 2 covers some foundational material on G_2-structures and the Laplacian flow.

We prove the main result of this thesis, the Structure Theorem for homogeneous closed gradient Laplacian solitons, in Chapter 3. We define the notion of an "orthogonally nice basis", i.e., we say that a basis $(e_i)_i$ for a Lie algebra $\mathfrak{g}$ is *orthogonally nice* if $[e_i, e_j] = ce_k$ and $e_i, e_j \perp e_k$. The motivation for this definition is due to it being a sufficient condition for *diagonally trivial derivatives*, i.e., $\nabla_{e_i} e_i = 0$ for all i, provided $(e_i)_i$ is orthonormal. We also obtain a "Key Lemma", that $g_\varphi(\mathrm{Ric}_\varphi(\nabla f), \cdot) = -2^{-1} \operatorname{div} \tau_\varphi^2(\cdot)$ on homogeneous spaces, which has been very useful. We obtain a few corollaries of the Structure Theorem using these observations.

We then use the Structure Theorem to "eliminate" gradient Laplacian solitons. More specifically, in Chapter 4, we show that the Laplacian solitons on nilpotent Lie groups found by Nicolini in [Nic18] are not gradient up to homothetic G_2-structures except for N_1, where f must be a Gaussian. We also show that the closed G_2-structure φ_{12} on N_{12} constructed in [FFM16] cannot be a gradient soliton. In eliminating gradient solitons, we observe a distinguishing feature of Laplacian solitons which is that the corresponding G-invariant symmetric 2-tensor $q = q(\tau_\varphi^2)$ is not always divergence-free whereas they are always divergence-free for Ricci and Bach solitons. A related question of whether product metrics $N \times \mathbb{R}^k$ admit closed G_2-structures is studied in the last section of Chapter 4. The takeaway from this section is that to find closed G_2-structures on product metrics $N \times \mathbb{R}^k$, one should consider $\dim N \geq 4$. We further use the Structure Theorem to show that closed non-torsion-free gradient Laplacian solitons on almost abelian solvmanifolds are isometric to products $N \times \mathbb{R}^k$ with f constant on N in Chapter 5. We generalize matrix formulas of Arroyo and Lauret (see [Arr13] and [Lau17a]) from the almost abelian case to the non-almost abelian case in the process of obtaining these results.

In Chapter 6, we study weighted sectional curvature of Riemannian manifolds (M, g) with density φ introduced by Wylie in [Wyl15]. Let $\tilde{g} = e^{-2\varphi} g$ be a conformal metric and $\mathrm{Hess}_{\tilde{g}}$ denote the Hessian in $\tilde{g}$. Kennard-Wylie-Yeroshkin in [KWY19] studied a lower ordered perturbation of $\mathrm{Hess}_{\tilde{g}}\, u$ called the modified conformal hessian of smooth (often distance) function u, which we denote by $\mathrm{MCHess}\, u$. We show that assumptions of nonnegative weighted sectional curvature and bounded density φ yields upper bounds on $\mathrm{MCHess}\, u$, where u is a modified distance function. We then use these bounds to obtain inequalities resembling the law of cosines, which we call a "warped law of cosines".

1.2 Basic Riemannian geometry

This section is a brief review of some basic Riemannian geometry needed for this thesis. We discuss Riemannian manifolds, the Levi-Civita connection, notions of curvature, and homogeneous spaces. We also review basic properties and identities involving the Hodge star operator which is needed for Chapter 5. For detailed definitions, explanations, and proofs of these concepts, we refer the reader to [Lee18, Pet16], and [dC92].

1.2.1 Riemannian manifolds and curvature

Recall that a smooth manifold, roughly speaking, is a space that "locally" looks like Euclidean space $\mathbb{R}^n$. A *Riemannian manifold*, denoted (M,g), is a smooth manifold M with a smoothly varying inner product $g_p = \langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}$ called a Riemannian metric; T_pM denotes the tangent space to M at p. As noted in the introduction, a Riemannian metric g can be thought of as a "ruler" that allows us to measure geometric quantities at each point $p \in M$. In particular, g allows us to measure lengths of vectors $X_p \in T_pM$ and the angles between them at each point $p \in M$. Since g varies smoothly from point to point, the measurements of these quantities also varies smoothly.

A vector field X on a manifold M is an assignment of a vector to each point $p \in M$ and a smooth vector field is one for which small changes in the points of M affects small changes in the assignment of vectors. In order to compute changes in vector fields on a Riemannian manifold, one generalizes the notion of the directional derivative of functions between Euclidean spaces to a geometric object called an *affine connection*, denoted ∇.

Definition 1.2.1. Let $\mathfrak{X}(M)$ is the space of all smooth vector fields on M. An *affine connection (or covariant derivative)* is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $(X,Y) \mapsto \nabla_X Y$ such that

1. For any $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$, $\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y$.

2. For any $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$, $\nabla_X(a_1Y_1 + a_2Y_2) = a_1\nabla_XY_1 + a_2\nabla_XY_2$.

3. (Leibniz rule) For any $f \in C^\infty(M)$, $\nabla_X(fY) = f\nabla_XY + X(f)Y$.

We can think of $\nabla_X Y$ as the derivative of vector field Y in the direction X, or equivalently, the directional derivative of Y along X. An affine connection ∇ on a manifold allows us to study generalized notions of "straight lines" called geodesics (i.e., smooth curves on M that are locally length-minimizing with zero acceleration) and parallelism, e.g., parallel vector fields and parallel transports.

Given a Riemannian metric g on M, there is a "nice" choice of connection called the Levi-Civita connection chosen to satisfy "nice" properties.

Theorem 1.2.2 (Fundamental theorem of Riemannian geometry)**.** *Let (M,g) be a Riemannian manifold. The Levi-Civita connection $\nabla = \nabla^g$ is the unique affine connection that is*

1. *metric (i.e., compatible with the metric g): $D_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for any vectors $X,Y,Z \in TM$; and*

2. *torsion-free (or symmetric): $[X,Y] = \nabla_X Y - \nabla_Y X$, where $[X,Y]$ is the Lie bracket of two vectors $X,Y \in TM$.*

The Levi-Civita connection ∇ (or covariant derivative) allows us to define curvature of Riemannian manifolds. Moreover, it is desirable that the notion of curvature is a local isometry invariant, i.e., smooth maps between Riemannian manifolds that preserves distances also preserves curvature. As will be seen below, since curvature is defined in terms of the Levi-Civita connection which is a local isometry invariant, it follows that curvature is also a local isometry invariant. From this point forward, ∇ denotes the Levi-Civita connection corresponding to the metric g.

We first recall that the curvature κ for a smooth curve γ on a plane corresponds to the size of an approximating osculating or "kissing" or "touching (at one point $p \in \gamma$)" circle to the curve via

$$\kappa_p = \frac{1}{r_p}.$$

The main idea is that the bigger the curvature, the smaller the osculating circle (the smaller the radius r_p) and the smaller the curvature, the bigger the osculating circle (the bigger the radius r_p).

Another way to think about curvature is that it is a quantity computed at a point of the curve γ that tells us how far away the curve is from being flat (a straight line) at that point. Equivalently, curvature is a measure that tells us how non-flat a curve is at that point. It turns out that these ideas carry over to curvature of Riemannian manifolds. Roughly speaking, we have that the larger the curvature, the smaller the manifold and the smaller the curvature, the larger the manifold. Furthermore, the curvature of a Riemannian manifold at a point tells us how far away it is from being Euclidean space $\mathbb{R}^n$, a flat space. Equivalently, curvature tells us how non-flat a manifold is.

We now introduce the notion of curvature for Riemannian manifolds.

Definition 1.2.3. Let (M,g) be a Riemannian manifold. The map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the *Riemann curvature endomorphism* or $(1,3)$-*curvature tensor*. [Note: R is a $(1,3)$-tensor field as it is multilinear over $C^\infty(M)$.]

The $(1,3)$-curvature tensor R is the building block that gives us the following notions of curvature of Riemannian manifolds:

- The $(0,4)$-*Riemann curvature tensor Rm* is given by

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W).$$

- *Sectional curvature* sec of a pair of vectors (U,V) is

$$\sec(U,V) = \frac{g(R(V,U)U,V)}{g(U,U)g(V,V) - g(U,V)^2}.$$

  Note that sec depends only on the plane $\Pi = \operatorname{span}\{U,V\}$. Sectional curvature is often referred to as just curvature as it tells us how much a surface "curves". As noted in the introduction, sectional curvature is a generalization of Gaussian curvature of surfaces (i.e., they are equal when $n = 2$). Recall that Gaussian curvature describes how a normal vector N to a surface changes or "turns" as we move in different directions on the surface. The larger the Gaussian curvature, the faster N turns and the smaller the Gaussian curvature, the slower N turns. In higher dimensions, the larger sec is, the more curved (less flat) the space is and the smaller sec is, the less curved (more flat) the space is.

- *Ricci curvature* Ric is a symmetric bilinear form given by

$$\mathrm{Ric}(U,V) = \mathrm{tr}(X \mapsto R(X,U)V) = \sum_{i=1}^{n} g(R(U,E_i)E_i,V),$$

where $(E_i)_{i=1}^n$ is an orthonormal basis for T_pM. It can also be defined as a symmetric $(1,1)$-tensor $\mathrm{Ric}(U) = \sum_i R(U,E_i)E_i$. The Ricci curvature is related to the sectional curvature by $\mathrm{Ric}(U,U) = \sum_{i=2}^n \sec(U,E_i)$, where $\{U,E_2,...,E_n\}$ is the completion of U to an orthonormal basis for T_pM. We can think of Ricci curvature as the average of sectional curvatures.

Remark 1.2.4. *Einstein metrics* are metrics g such that $\mathrm{Ric} = \lambda g$ for some constant λ. These metrics play an important role in Riemannian geometry.

- *Scalar curvature* scal is the function $\mathrm{scal} : M \to \mathbb{R}$ given by the trace of the Ricci curvature

$$\mathrm{scal} = \mathrm{tr}\,\mathrm{Ric} = \sum_{j=1}^{n} g(\mathrm{Ric}(E_j),E_j),$$

where $(E_j)_{j=1}^n$ is an orthonormal basis for T_pM. We can think of scalar curvature as the average of Ricci curvatures.

We will be interested in deducing geometric and topological information and consequences from sectional, Ricci, or scalar curvature bounds. For example, we have the following classical results.

Fact 1.2.5 (Classification theorem for constant sec manifolds)**.** *If M is a complete Riemannian manifold with constant sectional curvature, then M is a quotient of either a Euclidean space $\mathbb{R}^n$, a round sphere S^n, or hypoerbolic space $\mathbb{H}^n$.*

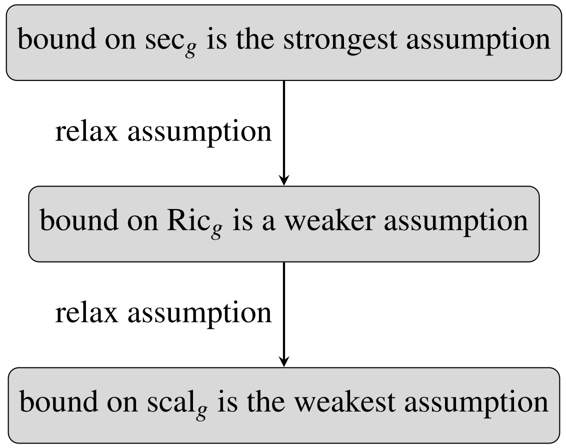

Figure 1.1: Curvature bound assumptions.

Fact 1.2.6 (Bonnet-Myers theorem)**.** *If M is a complete Riemannian manifold and all* Ric *are bounded below by a positive constant, then M is compact and the fundamental group of M is finite.*

Other important consequences coming from hypotheses on curvature are estimates on geometric quantities, e.g., estimates on the diameter of a manifold, or comparisons of geometric quantities, e.g., comparisons between volumes of manifolds. The study of such results is called *comparison geometry*. Lee's and Petersen's texts [Lee18] and [Pet16] contain many of these classical comparison results in Riemannian geometry. For more details on curvature tensors, we refer the reader to [[Lee18], Chapter 7] and [[Pet16], Chapter 3]. For a review of tensors and tensor fields, we refer the reader to [[Lee13], Chapter 12]. We close this section with the following diagram on curvature bound assumptions.

1.2.2 Homogeneous spaces

In this subsection we review some definitions and properties of smooth G-spaces, homogeneous G-spaces, and homogeneoeus manifolds. We include definitions and relevant facts needed to discuss these spaces for completeness.

First, recall that an *isometry* is a diffeomorphism $f : (M,g) \to (\tilde{M},\tilde{g})$ such that

$$f^*\tilde{g} = g.$$

In other words, f is an isometry if it is a smooth bijection and each $df_p : T_pM \to T_{\varphi(p)}M$ is a *linear isometry*, i.e.,

$$g_p(U,V) = \tilde{g}_{f(p)}(df_p(U), df_p(V)), \;\; \forall\, U,V \in T_pM.$$

It is easy to see that isometries are distance-preserving and angle-preserving maps. The space of all isometries of M, denoted $\mathrm{Iso}(M,g) := \{\varphi : M \to M \mid \varphi \text{ is an isometry}\}$, is a group under composition called the *isometry group* of M.

A *left action (or G-action)* of a Lie group $(G, \circ)$ on M, denoted $G \curvearrowright M$, is a map $\theta : G \times M \to M$ defined by $(g, p) \mapsto g \cdot p =: \theta_g(p) = \theta(g, p)$ such that

(a) $g_1 \cdot (g_2 \cdot p) = (g_1 \circ g_2) \cdot p$; and

(b) $e \cdot p = p \;\forall\, g_1, g_2 \in G$, $p \in M$; $e \in G$ is the identity and $\theta_g : M \to M$.

It is said to be *continuous or smooth* if the defining map θ is continuous or smooth.

We now define G-spaces and list basic properties regarding G-actions and G-spaces.

Definition 1.2.7 ((topological) G-space)**.** A manifold M with a (continuous) G-action, denoted (G, M, θ), is called a *(topological) G-space*. If M is smooth and the action is smooth, we call M a *smooth G-space*.

1. For all $g \in G$, the map $\theta_g : M \to M$ is a homeomorphism with inverse $\theta_g^{-1} : M \to M$. In particular, if the action is smooth, then θ_g is a diffeomorphism on M.

2. The *orbit (space)* of $p \in M$ under the G-action is $G \cdot p := \{g \cdot p \mid g \in G\} \leq M$. The quotient M/G is the set of all orbits

$$M/G = \{G \cdot p \mid p \in M\} = \{[p] \mid p \in M\}.$$

Note, M/G is the set of right cosets of G; these cosets viewed as elements of the quotient M/G are equivalence classes $[p]$ determined by equivalences $p \sim q$ if and only if there exists a $g \in G$ such that $g \cdot p = q$. Also note that M/G is not necessarily a group as G may not be a subset or subgroup of M. And even if it is, G is not assumed to be normal.

3. The *isotropy (stabilizer) subgroup* of $p \in M$ is $G_p := \{g \in G \mid g \cdot p = p\} \leq G$.

4. The action of $G \curvearrowright M$ is *free* if $G_p = \{e\} \;\; \forall\, p \in M$.

5. A continuous action $G \curvearrowright M$ is said to be *proper* if $G \times M \to M \times M$ defined by $(g, p) \mapsto (g \cdot p, p)$ is a *proper map*, i.e., it is a map between topological spaces such that the preimage of any compact subset in the codomain is compact in the domain. Below are two equivalent characterizations of continuous proper actions.

   (a) If $(p_i)_i \subset M$, $(g_i)_i \subset G$ such that $(p_i)_i$ and $(g_i \cdot p_i)_i$ converge in M, then a subsequence of $(g_i)_i$ converges in G.

   (b) For any compact $K \subset M$, the set $G_K = \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is compact.

6. The action $G \curvearrowright M$ is said to be *transitive* if for any $p, q \in M$, there exists a $g \in G$ such that $g \cdot p = q$.

7. A map $F : M \to N$ between manifolds M and N is said to be *equivariant* if $F(g \cdot p) = g \cdot F(p)$ for all $g \in G$ and for all $p \in M$, i.e., if the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g = g\cdot"'] & N \arrow[d, "\psi_g = g\cdot"] \\ M \arrow[r, "F"] & N \end{tikzcd} .$$

   Note both M and N are G-spaces.

Definition 1.2.8 (Homogeneous G-space)**.** A *Homogeneous G-space (or Homogeneous space)* is a smooth manifold M endowed with a smooth transtive action by a Lie group.

Fact 1.2.9. *By the Myers-Steenrod theorem,* $\mathrm{Iso}(M, g)$ *is a Lie group and acts smoothly on* M*. The action* $\mathrm{Iso}(M, g) \times M \to M$ *is given by* $(\varphi, p) \mapsto \varphi(p)$.

Definition 1.2.10 (Homogeneous manifold)**.** We say that (M, g) is a *homogeneous Riemannian manifold* if $\mathrm{Iso}(M, g)$ acts transitively on M, i.e., $\forall\, p, q \in M$, there exists an isometry $\varphi \in \mathrm{Iso}(M, g)$ s.t. $\varphi(p) = q$.

The simplest examples of homogeneous spaces are Lie groups with left-invariant metrics. Recall that for a finite dimensional $\mathbb{R}$-vector space V, $\mathrm{GL}(V)$ is the Lie group of all invertible linear maps from V to itself. For $G \leq \mathrm{GL}(V)$, an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is *G-invariant* if $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y \in V$ and for all $g \in G$. The following lemma gives a necessary and sufficient condition for the existence of G-invariant inner products on V.

Lemma 1.2.11 ([Lee18], Lemma 3.13)**.** *Suppose V is a finite-dimensional $\mathbb{R}$-vector space and $G \leq \mathrm{GL}(V)$. Then there exists a G-invariant inner product on V if and only if G has compact closure in* $\mathrm{GL}(V)$.

We now discuss what G-invariance means for a Riemannian metric g on M. We say that a metric g is *G-invariant* if the inner product $g_p = \langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}$ at each $p \in M$ is invariant under the isotropy representation (at p) $\Theta_p : G_p \to \mathrm{GL}(T_pM)$ defined by $\Theta_p(\varphi) = d\varphi_p$. That is,

$$g_p(U,V) = g_{\varphi(p)}(d\varphi_p(U), d\varphi_p(V)) = g_p(d\varphi_p(U), d\varphi_p(V)), \ \ \forall\, U, V \in T_pM;\ \forall\, \varphi \in G_p.$$

Note $\varphi(p) = p$ as $\varphi \in G_p$. Lemma 1.2.11 and the definition of G-invariant metrics gives the following necessary and sufficient condition for G-invariant Riemannian metrics.

Theorem 1.2.12 ([Lee18], Theorem 3.17)**.** *Suppose G is a Lie group and M is a homogeneous G-space. Fix $p \in M$ and let $\Theta_p : G_p \to \mathrm{GL}(T_pM)$ be the isotropy representation at p. Then there exists a G-invariant Riemannian metric on M if and only if $\Theta_p(G_p)$ has compact closure in* $\mathrm{GL}(T_pM)$.

Corollary 1.2.13. *If a Lie group G acts smoothly and transitively on a smooth manifold M with compact isotropy subgroups G_p, then there exists a G-invariant Riemannian metric on M.*

Remark 1.2.14. If G is a Lie group and M a homogeneous G-space that admits at least one G-invariant metric, then for each $p \in M$, the map $g \mapsto g_p$ gives a bijection between G-invariant metrics on M and $\Theta_p(G_p)$-invariant inner products on T_pM (see [[Lee18], Exercise 3.19]).

The next theorem is an important characterization of homogeneous spaces. It is the main way we will view homogeneous spaces throughout this thesis.

Theorem 1.2.15. *Let G be a Lie group. Let M be a homogeneous G-space and fix a $p \in M$. Then isotropy group G_p is a closed subgroup of G and the map $F : G/G_p \to M$ defined by $G_p \cdot g \mapsto g \cdot p$ is an equivariant diffeomorphism. Thus we have the identification $M \equiv G/G_p$.*

The takeaway for this subsection is that homogeneous Riemannian manifolds M are geometrically the same at each point $p \in M$. In particular, the curvatures are the same at each point. Prototypical examples of homogeneous manifolds are the constant sectional curvature spaces $\mathbb{R}^n$, S^n, and $\mathbb{H}^n$. We will use that the scalar curvature scal_g is constant on homogeneous spaces frequently throughout this thesis.

1.2.3 Geometric flows

A *geometric q-flow* of a metric g is a partial differential equation given by

$$\begin{cases} \partial_t g(t) = q \\ g(0) = g, \end{cases}$$

where q is some 2-tensor involving the curvature of the manifold and g is the initial metric at time $t = 0$. This system describes that the metric evolves in accordance with the curvature and possibly other tensors in q. By studying how metrics evolve in time under a given flow, one hopes to gain insight on how the manifold and its geometric information obtained from metric, e.g., curvature, evolves under the flow.

Recall that an important part of studying differential equations involves finding its fixed point (or equilibrium) solutions and classifying them. In the context of a geometric flow, this amounts to studying its *solitons* (or *self-similar solutions*), i.e., solutions that change in time only by diffeomorphism or scaling. In mathematical terms, the solitons of a q-flow, called *q-solitons*, are metrics g such that $g(t) = c(t)f(t)^*g$ where $c(t) \in \mathbb{R}^*$ and $f(t) \in \mathrm{Diff}(M)$. It is well known that an equivalent condition for metric g being a soliton is that it satisfies the *q-soliton equation*

$$\frac{1}{2}\mathscr{L}_X g = cg + \frac{1}{2}q.$$

More generally, consider the geometric flow of geometric structures γ,

$$\begin{cases} \partial_t \gamma(t) = q(\gamma(t)) \\ \gamma(0) = \gamma \end{cases},$$

where $\gamma(t)$ is a one-parameter family of tensor fields and $q(\gamma)$ is a tensor field of the same type typically involving the curvature, a Laplacian, or gradient field of a geometric functional. The solitons of this flow, called *soliton geometric structures*, are geometric structures γ such that the solutions $\gamma(t) = c(t) f(t)^* \gamma$ for $c(t) \in \mathbb{R}^*$ and $f(t) \in \mathrm{Diff}(M)$. This is equivalent to γ satisfying the *soliton equation*

$$q(\gamma) = c\gamma + \mathscr{L}_X \gamma.$$

We refer to the triple (γ, X, c) as the *soliton (geometric structure)* satisfying the soliton equation. The conventions may vary, but depending on the sign of c (or $-c$), we say that the soliton is *steady, shrinking, or expanding* if $c = 0$, $c < 0$, or $c > 0$, respectively. Furthermore, we say that a soliton is *gradient* if X is a gradient field.

As stated in the introduction, solitons can be thought of as fixed points (or equilibrium) solutions to the flow. They model singularities of the flow and can often tell us how solutions close to them behave under the flow. Moreover, given a geometric flow, it is often desirable to study existence and uniqueness of solutions, as well as convergence of solutions to the given flow.

In this thesis, we study gradient solitons of a geometric flow of 3-forms called the Laplacian flow introduced Chapter 2. The main idea in our approach is to use results known for q-flows to study the solitons of the corresponding flow of metric g obtained from the flow of 3-forms.

2 | Background

2.1 G_2-geometry

2.1.1 The Hodge star operator

Let V be an n-dimensional vector space over $\mathbb{R}$. Recall that an *inner product on V* is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for any $u, v, w \in V$ and $a \in \mathbb{R}$, we have

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle au, v \rangle = a \langle u, v \rangle$ and $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle u, u \rangle \geq 0$ and is 0 if and only if $u = 0$.

Let $f : V \to V^*$ be defined by $u \mapsto \langle u, \cdot \rangle$ for any $u \in V$. The *natural (or induced) inner product on V^** with respect to the inner product on V is

$$\langle u^*, v^* \rangle = \left\langle f^{-1}(u^*), f^{-1}(v^*) \right\rangle.$$

Then the space of *alternating k-tensors on V*, denoted by $\Lambda^k(V)$, has the natural inner product given by

$$\langle u_1^* \wedge \cdots \wedge u_k^*, v_1^* \wedge \cdots \wedge v_k^* \rangle = \det\left(\langle u_i^*, v_j^* \rangle \right),$$

where u_i's and v_j's are 1-forms.

Let $(e_i)_{i=1}^n$ be an oriented orthonormal basis for $(V, \langle\cdot,\cdot\rangle)$ such that $(e^i)_{i=1}^n$ is the dual basis for V^*. Then the basis for $\Lambda^k(V^*)$ is $\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \cdots < i_k\}$ and is of dimension $\binom{n}{k}$. The *Hodge star operator* $* : \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)$ is given by

$$e^{i_1} \wedge \cdots \wedge e^{i_k} \mapsto e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$$

such that $e_{i_1}, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_{n-k}}$ is an oriented basis for V. We list some properties of $*$:

1. Let $(u_i)_{i=1}^n$ be a basis for V. Then

   $$*1 = e^1 \wedge \cdots \wedge e^n = \sqrt{\det\left(\langle u_i, u_j \rangle\right)}\, u^1 \wedge \cdots \wedge u^n$$

   and

   $$*(e^1 \wedge \cdots \wedge e^n) = 1.$$

2. $*e^j = (-1)^{j-1} e^1 \wedge \cdots \wedge \widehat{e^j} \wedge \cdots \wedge e^n$ where $\widehat{e^j}$ denotes that the e^j factor in the wedge product is left out.

3. $** : \Lambda^k(V^*) \to \Lambda^k(V^*)$ is $** = (-1)^{k(n-k)}$.

4. For $n = 7$, we have $e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{7-k}} = \pm e^1 \wedge \cdots \wedge e^7$ and $*^2 = ** = 1$, the identity map on k-forms.

5. For any $u, v \in \Lambda^k(V^*)$,

   $$\langle u, v \rangle = *(u \wedge *v) = *(v \wedge *u).$$

Let (M, g) be a an oriented Riemannian n-manifold and $\Omega^k(M)$ be the *space of all smooth (or differential) k-forms.* Recall that $\Omega^k(M)$ is the smooth section of the bundle $\Lambda^k(T^*M)$, i.e., any $\omega \in \Omega^k(M)$ is a smooth map from $M \to \Lambda^k(T^*M)$ such that $\omega_p = \omega(p) \in \Lambda^k(T^*M)$ is an alternating k-tensor for any point $p \in M$. One can define the Hodge star operator on smooth k-forms.

Definition 2.1.1. The *Hodge star operator* on smooth k-forms $*: \Omega^k(M) \to \Omega^{n-k}(M)$ maps $\omega \in \Omega^k(M)$ to $*\omega \in \Omega^{n-k}(M)$ such that for any $\alpha \in \Omega^k(M)$, we have

$$\alpha \wedge *\omega = \langle \alpha, \omega \rangle_g \operatorname{vol}_g,$$

where vol_g is the volume form. Note that the Hodge star operator is determined by the metric and orientation on M.

Let $(e_i)_{i=1}^n$ be a local basis of TM. We have the following properties of the Hodge star operator on k-forms that follow from its definition and preceding properties for alternating k-forms:

1. $*1 = \operatorname{vol}_g = \sqrt{\det g} e^1 \wedge \cdots \wedge e^n$
2. $* \operatorname{vol}_g = 1$
3. $\alpha \wedge *\beta = \beta \wedge *\alpha$
4. For any $\alpha \in \Omega^k(M)$, $* * \alpha = (-1)^{k(n-k)} \omega$ and $* * \alpha = \alpha$ when $n = 7$.
5. $\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$ for any $\alpha, \beta \in \Omega^k(M)$, i.e., $*$ is a linear isometry.

Let $d: \Omega^{k-1}(M) \to \Omega^k(M)$ be the differential (exterior derivative) of k-forms. The *codifferential (or coderivative)* is a map $d^*: \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$d^* \alpha = (-1)^{n(k+1)+1} * d * \alpha, \quad \alpha \in \Omega^k(M).$$

The codifferential is sometimes denoted δ. When $n = 7$, we get $d^* \alpha = (-1)^k * d * \alpha$.

Remark 2.1.2. For M closed (i.e., compact without boundary), one can define an L^2 inner product on $\Omega^k(M)$ by $(\alpha, \beta) = \int_M \alpha \wedge *\beta$. It follows from Stoke's Theorem that $(\alpha, d\beta) = (d^* \alpha, \beta)$ for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Hence the codifferential d^* is often referred to as the *formal adjoint* of d.

Definition 2.1.3. The *Hodge-Laplacian operator* on k-forms is the map $\Delta : \Omega^k(M) \to \Omega^k(M)$ defined by

$$\Delta = dd^* + d^*d.$$

A useful property is that Hodge star operator commutes with the Hodge-Laplacian, i.e.,

$$*\Delta = \Delta * .$$

This follows directly from definitions. When $k = 0$, $\Delta = d^*d$ is the *Laplace-Beltrami operator*.

Proposition 2.1.4. *For closed M, we have:*

(i) $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$*, i.e., Δ is symmetric with respect to the L^2 inner product;*

(ii) $(\Delta\alpha, \alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2 \geq 0$*;*

(iii) $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $d^*\alpha = 0$*.*

Proof. Item (i) follows from Remark 2.1.2, i.e., d^* being the formal adjoint of d with respect to the L^2 inner product:

$$\begin{aligned}
(\Delta\alpha, \beta) = ((dd^* + d^*d)\alpha, \beta) &= (dd^*\alpha, \beta) + (d^*d\alpha, \beta) \\
&= (d^*\alpha, d^*\beta) + (d\alpha, d\beta) \\
&= (\alpha, dd^*\beta) + (\alpha, d^*d\beta) = (\alpha, (dd^* + d^*d)\beta) = (\alpha, \Delta\beta).
\end{aligned}$$

Item (ii) follows immediately from substituting $\beta = \alpha$ in the above string to get $(\Delta\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2 \geq 0$. The forwards implication of (iii) follows from (ii) and the backwards implication follows from the definition of $\Delta\alpha$. $\square$

We say that a k-form α is *harmonic* if $\Delta\alpha = 0$. One can further study the space of harmonic k-forms, $\mathscr{H}^k(M)$, on closed manifolds and the Hodge Decomposition Theorem, which enables one to compute cohomology groups with real coefficients and via Poincaré duality, compute homology groups and Betti numbers of closed manifolds. We refer the interested reader to [[Pet16], Chapter 9].

2.1.2 The Lie group G_2

We first give some definitions needed to discuss the group G_2.

Definition 2.1.5. A *normed division algebra* is an algebra $\mathbb{A} \cong \mathbb{R}^n$ over $\mathbb{R}$ with multiplicative identity $1 \neq 0$ such that $\|ab\| = \|a\|\|b\|$ for any $a, b \in \mathbb{A}$; $\|a\|^2 = \langle a, a\rangle$ is the Euclidean norm on $\mathbb{R}^n$.

Definition 2.1.6. Let $\mathbb{A}$ be a normed division algebra over $\mathbb{R}$ such that $\operatorname{Re}\mathbb{A} = \langle 1\rangle$ and $\operatorname{Im}\mathbb{A} = (\operatorname{Re}\mathbb{A})^{\perp}$. A *(2-fold) vector cross product on* $\operatorname{Im}\mathbb{A}$ *induced by the algebraic structure on* $\mathbb{A}$ is a bilinear map $\times : \mathbb{A}^2 \to \mathbb{A}$ defined by $a \times b := \operatorname{Im}(ab)$ for any $a, b \in \operatorname{Im}\mathbb{A}$. That is, the vector cross product is the projection of ab, the product of a and b in the algebra, onto its imaginary part.

The vector cross product on $\operatorname{Im}\mathbb{A}$ satisfies the following properties:

1. $a \times b = -b \times a$ (i.e., $\times$ is skew-symmetric)
2. $\langle a \times b, a\rangle = 0$ (i.e., $a \times b \perp a$ and $a \times b \perp b$)
3. $\operatorname{Re}(ab) = -\langle a, b\rangle\, 1$.
4. $\|a \times b\|^2 = \|a\|^2\|b\|^2 - \langle a, b\rangle^2 = \|a \wedge b\|^2$
5. $a \times (b \times c) = -\langle a, b\rangle\, c + \langle a, c\rangle\, b - \frac{1}{2}[a, b, c]$ where $[a, b, c] = (ab)c - a(bc)$ is the *associator* of $\mathbb{A}$.

We note that $\mathbb{V} \cong \mathbb{R}^m$ equipped with a Euclidean inner product $\langle \cdot, \cdot\rangle$ has a cross product if there is a skew-symmetric bilinear map $\times : \mathbb{V}^2 \to \mathbb{V}$ such that for any $a, b \in \mathbb{V}$, properties (2) and (4) hold.

In 1898, Hurwitz showed that there are only four normed division algebras over $\mathbb{R}$ up to isomorphism. They are the real numbers, the complex numbers, the quaternions, and the octonions, denoted $\mathbb{R}$, $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$, and $\mathbb{O} \cong \mathbb{R}^8$, respectively. There is a one-to-one correspondence between normed division algebras and spaces admitting vector cross products (see [Kar20] for a proof). Hence there are exactly four spaces up to isomorphism that admit vector cross products, namely, $\operatorname{Im}\mathbb{R} \cong \{0\}$, $\operatorname{Im}\mathbb{C} \cong \mathbb{R}$, $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$, and $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$.

We consider the octonion algebra $\mathbb{O}$. Let $g_0 = \langle \cdot, \cdot \rangle$ be the restriction of the standard Euclidean metric on $\mathbb{O}$ to $\operatorname{Im}\mathbb{O}$ and let $(e_i)_{i=1}^7$ be the standard orthonormal basis for $\operatorname{Im}\mathbb{O}$ with respect to g_0. Let $\mu_0 = e^1 \wedge \cdots \wedge e^7$ be the standard volume form where $(e^i)_{i=1}^7$ is the basis for $(\operatorname{Im}\mathbb{O})^*$. We denote the cross product on the imaginary part of the octionions $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ by $\times_0$. From this setup, we can define a 3-form φ_0 on $\operatorname{Im}\mathbb{O}$ by

$$\varphi_0(a,b,c) := \langle a \times_0 b, c \rangle = \langle ab, c \rangle, \quad \forall\, a,b,c \in \operatorname{Im}\mathbb{O}.$$

By octonion multiplication, we can write

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{2.1}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. This 3-form φ_0 is referred to as the *associative 3-form*. Now the metric g_0 and orientation from μ_0 determines a unique Hodge star operator $* : \Lambda^k(\operatorname{Im}\mathbb{O})^* \to \Lambda^{7-k}(\operatorname{Im}\mathbb{O})^*$, from which we obtain the *coassociative 4-form* $\psi_0 = *\varphi_0$. We call the tuple $(g_0, \mu_0, \varphi_0, \psi_0, \times_0)$ the standard *G_2-package*.

Definition 2.1.7. The group G_2 is the subgroup of $\mathrm{GL}(7,\mathbb{R})$ that preserves the standard G_2-package $(g_0, \mu_0, \varphi_0, \psi_0, \times_0)$. This is equivalent to Bryant's theorem, which states that $G_2 = \{A \in \mathrm{GL}(7,\mathbb{R}) \mid A^*\varphi_0 = \varphi_0\} = \mathrm{Stab}(\varphi_0)$. [Formally, G_2 is isomorphic to the stabilizer subgroup of φ_0 in $\mathrm{GL}(7,\mathbb{R})$.] Moreover, $G_2 \cong \mathrm{Aut}(\mathbb{O})$. The latter two statements are often taken as definitions of G_2.

It is well known that G_2 is a simply connected compact 14-dimensional simple Lie group and is a subgroup of $\mathrm{SO}(7)$. In fact, G_2 is one of the *exceptional Riemannian holonomy groups* in Berger's classification theorem (1955) with the other exceptional holonomy group being the Lie group Spin(7). Both of these groups are related to the stucture of the octonions $\mathbb{O}$. These groups are of interest as manifolds with holonomy G_2 or Spin(7) are necessarily Ricci-flat. An important fact is that the orbits of the $\mathrm{GL}(7,\mathbb{R})$-action on φ are open in $\Lambda^3((\mathbb{R}^7)^*)$. This follows from the fact that $\dim(\mathrm{GL}(7,\mathbb{R})/\mathrm{Stab}(\varphi)) = \dim \mathrm{GL}(7,\mathbb{R}) - \dim G_2 = 49 - 14 = 35 = \binom{7}{3} = \dim \Lambda^3((\mathbb{R}^7)^*)$.

Remark 2.1.8. There are many descriptions and properties of the Lie group G_2 and its Lie algebra $\mathfrak{g}_2$ which we will not discuss in this thesis. We refer the reader to [Bry06, FG82, Kar09], and [Kar20] for more details.

2.1.3 G_2-structures

Given a *non-degenerate* 3-form $\varphi \in \Lambda^3(\mathfrak{p}^*)$ where $\mathfrak{p} \cong \mathbb{R}^7$ is any real 7-dimensional vector space, we can associate a symmetric bilinear form b_φ and a volume form Ω_φ on $\mathfrak{p}$ by

$$b_\varphi(X,Y)\Omega_\varphi = \frac{1}{6}\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi. \tag{2.2}$$

It can be shown that for a non-vanishing volume form Ω_φ, b_φ is non-degenerate, and so is a metric. Hitchin showed that there are exactly two open $\mathrm{GL}(\mathfrak{p}^*)$-orbits in $\Lambda^3(\mathfrak{p}^*)$, namely, $\Lambda^3_+(\mathfrak{p}^*) = \{\varphi \in \Lambda^3(\mathfrak{p}^*) \mid b_\varphi \text{ is positive definite}\}$ and $\Lambda^3_-(\mathfrak{p}^*) = \{\varphi \in \Lambda^3(\mathfrak{p}^*) \mid b_\varphi \text{ is indefinite}\}$. The convention is to consider $\varphi \in \Lambda^3_+(\mathfrak{p}^*)$ as such φ defines G_2-structures. The use of the term "positive" in the definitions to follow comes from the this choice of orbit $\Lambda^3_+(\mathfrak{p}^*)$.

We say that $\varphi \in \Lambda^3(\mathfrak{p}^*)$ is a *fixed positive* 3*-form* if it can be written as in (2.1); it is often denoted by φ_0. A fixed positive 3-form is sometimes referred to as a *model* or *fundamental* 3*-form*. The natural left $\mathrm{GL}(\mathfrak{p})$-action on 3-forms is given by $h\cdot\psi = (h^{-1})^*\psi = \psi(h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot)$. A 3-form $\psi \in \Lambda^3(\mathfrak{p}^*)$ is said to be *positive* if there is an $h \in \mathrm{GL}(\mathfrak{p})$ such that $h\cdot\varphi_0 = \psi$, i.e., ψ is positive if it is in the $\mathrm{GL}(\mathfrak{p})$-orbit of φ_0. By the discussion in the preceding section, any positive 3-form ψ induces a unique inner product and orientation via (2.2), which together determines a unique Hodge star operator.

Definition 2.1.9. Let M be a smooth 7-manifold. A 3-form $\varphi \in \Omega^3(M)$ is a G_2*-structure* if at each point $p \in M$, φ_p is *positive*, i.e., there exists a basis $(e_i)_{i=1}^7$ of T_pM such that

$$\varphi_p = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $(e^i)_i$ is the dual basis to $(e_i)_i$. Any such 3-form φ induces a Riemannian metric g_φ and an orientation vol_φ from (2.2), which in turn determines a *Hodge star operator* $*_\varphi : \Omega^k(M) \to \Omega^{7-k}(M)$. It is a well known fact that a smooth 7-manifold admits a G_2-structure φ if and only if it is orientable and spin, which is equivalent to the vanishing of the first two Stiefel-Whitney classes (see [[Kar20], Proposition 4.18]). Moreover, M admitting a G_2-structure φ is equivalent to M having a subbundle of the $\mathrm{GL}(7,\mathbb{R})$-frame bundle with structure group $G_2 \subset \mathrm{SO}(7)$. In other words, there exists local frames such that all transition functions are in G_2. We write (M,φ) to denote a smooth 7-manifold admitting G_2-structure φ.

Let Ω^k_ℓ denote the space of smooth k-forms with pointwise dimension ℓ. Fernández-Gray showed in [FG82] that any G_2-structure φ determines g_φ-orthogonal decompositions of k-forms on M into irreducible G_2-representations. There are the two decompositions

$$\Omega^2(M) = \Omega^2_7 \oplus \Omega^2_{14}, \qquad \Omega^3(M) = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where the summands are:

- $\Omega^2_7 = \{v \lrcorner \varphi \mid v \in \Gamma(TM)\} = \{\alpha \in \Omega^2(M) \mid *(\alpha \wedge \varphi) = 2\alpha\} \cong \Omega^1_7 = T^*M \cong TM$
- $\Omega^2_{14} = \{\alpha \in \Omega^2(M) \mid \alpha \wedge *\varphi = 0\} = \{\alpha \in \Omega^2(M) \mid *(\alpha \wedge \varphi) = -\alpha\}; \Lambda^2_{14} \cong \mathfrak{g}_2(\varphi) \subset \mathfrak{so}(TM) \cong \Lambda^2(M)$
- $\Omega^3_1 = \{f\varphi \mid f \in C^\infty(M)\}$; $\Lambda^1_3 \cong \mathbb{R}\varphi$
- $\Omega^3_7 = \{v \lrcorner * \varphi \mid v \in \Gamma(TM)\} = \{*(\beta \wedge \varphi) \mid \beta \in T^*M\} \cong \Omega^1_7 = T^*M \cong TM$
- $\Omega^3_{27} = \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0, \gamma \wedge *\varphi = 0\} \cong S^2_0(T^*M)$, where $S^2_0(T^*M)$ denotes the space of traceless symmetric 2-tensors on M.

These decompositions are obtained from the splitting of the bundles $\Lambda^k(T^*M)$. Decompositions for Ω^4 and Ω^5 are obtained by taking the Hodge star of Ω^3 and Ω^2, respectively. Details regarding the identifications (isomorphisms) listed above can be found in [Bry06, FG82, Kar09], and [Kar20].

Remark 2.1.10. The orientation convention and coefficients in conditions $*(\alpha\wedge\varphi) = 2\alpha$ and $*(\alpha\wedge\varphi) = -\alpha$ for Ω^2_7 and Ω^2_{14}, respectively, may be different depending on the author.

The unique *torsion forms* $\tau_i \in \Omega^i(M)$, $i = 0,1,2,3$ of a G_2-structure φ are the independent components of the *intrinsic torsion* $\nabla\varphi$. They can be defined as the unique i-forms such that

$$d\varphi = \tau_0\psi + 3\tau_1\wedge\varphi + *\tau_3, \quad d\psi = 4\tau_1\wedge\psi + \tau_2\wedge\varphi.$$

These equations are obtained from the decompositions of $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$ and $\Omega^5 = \Omega^5_7 \oplus \Omega^5_{14}$; $\tau_0\psi \in \Omega^4_1$, $3\tau_1\wedge\varphi \in \Omega^4_7$, and $*\tau_3 \in \Omega^4_{27}$; $4\tau_1\wedge\psi \in \Omega^5_7$ and $\tau_2\wedge\varphi \in \Omega^5_{14}$.

Remark 2.1.11. The defining equations for τ_i and subsequent conventions in this thesis are consistent with those of Bryant, Lauret, et al (see, e.g., [Lau17a]). The constant coefficients in the defining equations for $d\varphi$ and $d\psi$ are chosen for convenience.

Remark 2.1.12. Note $\tau_2 \in \Omega^2_{14}$ and so the term $\tau_2\wedge\varphi$ in $d\psi$ is equivalent to $-*\tau_2$ in Lauret's convention whereas in [Kar09], the term is $+*\tau_2$.

Torsion-free (or *parallel*) G_2-structures, i.e., φ such that $\nabla\varphi = 0$, have the property that its induced metric g_φ has holonomy in G_2. Existence of such metrics were suggested by Berger's classification theorem in 1955 and first examples were constructed by Bryant-Salamon in the 1980s. Metrics with holonomy in G_2 are hard to find, yet are desirable as they are necessarily Ricci-flat and are important in string theory. We note that many examples of such metrics have since been constructed (see, e.g., references in [[Kar20], Section 6.2]). Smooth 7-manifolds admitting torsion-free G_2-structures are called *G_2-manifolds*.

Theorem 2.1.13 ([FG82], Fernández-Gray)**.** *φ is torsion-free if and only if $d\varphi = 0$ and $d*\varphi = 0$ (i.e., φ is both closed and coclosed).*

Remark 2.1.14. It is not hard to see from the defining equations for τ_i that φ is torsion-free if and only if $\tau_i = 0$ for all $i = 0,1,2,3$. That is, φ is torsion-free if and only if all torsion forms vanish.

There are 16 distinct classes of G_2-structures (see [Kar20] for a table of the possible classes). We consider the class of closed G_2-structures in this thesis. We say that φ is a *closed* (or *calibrated*) G_2-structure if φ_p is positive for each $p \in M$ and $d\varphi = 0$. This class of G_2-structures are of interest as they are "close" to being torsion-free in the sense that τ_2 is the only surviving torsion form. Hence it is common to write τ_φ or, simply, τ, as it is understood that they denote the surviving torsion form determined by a closed G_2-structure. There are many results on closed G_2-structures. We include some relevant properties regarding them below. For more on torsion forms and detailed treatments of the following results, see [Bry06, FG82, Kar09, Kar20, Lau17a], and [LW17].

In order to discuss an important symmetric operator Q_φ associated with a closed G_2-structure φ, we need to discuss several maps and how they relate to each other. We follow the exposition in [Lau17a]. Let $\theta : \mathfrak{gl}(\mathfrak{p}) \to \mathrm{End}(\Lambda^3\mathfrak{p}^*)$ be the representation

$$\theta(A)\psi = -\psi(A\cdot,\cdot,\cdot) - \psi(\cdot,A\cdot,\cdot) - \psi(\cdot,\cdot,A\cdot) \quad A \in \mathfrak{gl}(\mathfrak{p}),\ \psi \in \Lambda^3\mathfrak{p}^*$$

obtained as the derivative of the natural left $\mathrm{GL}(\mathfrak{p})$-action on 3-forms:

$$(h,\psi) \mapsto h\cdot\psi = (h^{-1})^*\psi = \psi(h^{-1}\cdot,h^{-1}\cdot,h^{-1}\cdot).$$

That is, $\theta(A)\psi = \frac{d}{dt}\,|_0\, e^{tA}\cdot\psi$. For a fixed positive 3-form φ on $\mathfrak{p}$, we have that $\theta(\mathfrak{gl}(\mathfrak{p}))\varphi = \Lambda^3\mathfrak{p}^*$ since the orbit $\mathrm{GL}(\mathfrak{p})\cdot\varphi$ is open in $\Lambda^3\mathfrak{p}^*$. Note that $\mathfrak{g}_2(\varphi) = \{A \in \mathfrak{gl}(\mathfrak{p}) \mid \theta(A)\varphi = 0\} \cong \mathfrak{g}_2$ is the Lie algebra of the stabilizer subgroup $G_2(\varphi) := \mathrm{GL}(\mathfrak{p})_\varphi \cong G_2$. Then we have $G_2(\varphi)$-invariant decompositions $\mathfrak{gl}(\mathfrak{p}) = \mathfrak{g}_2(\varphi) \oplus \mathfrak{q}(\varphi)$ and $\mathfrak{q}(\varphi) = \mathfrak{q}_1(\varphi) \oplus \mathfrak{q}_7(\varphi) \oplus \mathfrak{q}_{27}(\varphi)$, where $\mathfrak{q}(\varphi)$ is the orthogonal complement of $\mathfrak{g}_2(\varphi)$ in $\mathfrak{gl}(\mathfrak{p})$ with respect to $\langle\cdot,\cdot\rangle_\varphi$ and $\mathfrak{q}_1(\varphi) \oplus \mathfrak{q}_7(\varphi) \oplus \mathfrak{q}_{27}(\varphi)$ is the decomposition corresponding to the splitting of the bundle $\Lambda^3\mathfrak{p}^* = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ from which we get the decomposition of $\Omega^3(M)$ above. From the decomposition of $\mathfrak{gl}(\mathfrak{p})$ we get $\theta(\mathfrak{q}(\varphi))\varphi = \Lambda^3\mathfrak{p}^*$. Thus for any $\psi \in \Lambda^3\mathfrak{p}^*$, there is a unique operater $Q_\psi \in \mathfrak{q}(\varphi)$ such that $\theta(Q_\psi)\varphi = \psi$. The following is [[Lau17a], Proposition 2.2] along with the formula for the Ricci tensor Ric_φ coming from g_φ.

Proposition 2.1.15. *For any closed G_2-structure φ, there is a unique symmetric operator $Q_\varphi \in$ sym(TM) such that $\theta(Q_\varphi)\varphi = \Delta_\varphi \varphi$ and*

$$Q_\varphi = \mathrm{Ric}_\varphi - \frac{1}{12}\mathrm{tr}(\tau_\varphi^2)I + \frac{1}{2}\tau_\varphi^2, \tag{2.3}$$

where Ric_φ is the Ricci operator of (M, g_φ) and $\tau_\varphi \in \mathfrak{so}(TM)$ is the skew-symmetric operator corresponding to the torsion 2-form $\tau_\varphi = g_\varphi(\tau_\varphi \cdot, \cdot)$ for closed G_2-structures. We also have

1. $|\tau_\varphi|^2 = -\frac{1}{2}\,\mathrm{tr}\,\tau_\varphi^2$;
2. $\mathrm{scal}_\varphi = -\frac{1}{2}|\tau_\varphi|^2 = \frac{1}{4}\,\mathrm{tr}\,\tau_\varphi^2 = \frac{3}{2}\,\mathrm{tr}\,Q_\varphi$;
3. $\mathrm{scal}_\varphi \leq 0$ *and is equal to* 0 *if and only if* φ *is torsion-free;*
4. $\mathrm{Ric}_\varphi = \frac{1}{4}|\tau_\varphi|^2 g - \frac{1}{4} j_\varphi(d\tau_\varphi - \frac{1}{2} * (\tau_\varphi \wedge \tau_\varphi))$ *where the surjective map* $j_\varphi : \Lambda^3(T^*M) \to S^2(T^*M)$ *is defined by* $j_\varphi(\gamma)(v,w) = *(v \lrcorner \varphi \wedge w \lrcorner \varphi \wedge \gamma)$.

Proof. Since $\Delta_\varphi \varphi \in \Omega^3$, by the preceding discussion there is a unique operator $Q_\varphi \in \mathfrak{q}(\varphi) \subset \mathrm{End}(TM)$ such that $\theta(Q_\varphi)\varphi = \Delta_\varphi\varphi$. That $Q_\varphi \in \mathrm{sym}(TM)$ follows from the fact that it coincides with the symmetric 2-tensor $-h$ from [LW17], i.e., $-h = q_\varphi$ where $q_\varphi := g(Q_\varphi \cdot, \cdot)$ is the symmetric bilinear 2-form corresponding to Q_φ. The formula for Q_φ is obtained from the formula for $-h$ (see [[LW17], equation (3.4)]). The expression for scalar curvature, $\mathrm{scal}_\varphi = -\frac{1}{2}|\tau_\varphi|^2$, from item (2) is the statement of [[LW17], Corollary 2.5] where $T = -\frac{1}{2}\tau_\varphi$. We note this expression for scalar curvature is also obtained in [Bry06]. Item (1) follows from [[LW17], Corollary 2.5]. The rest of the identities in item (2) are obtained as follows. For the third identity in item (2), observe $-\frac{1}{2}|\tau_\varphi|^2 = -\frac{1}{2}(-\frac{1}{2}\,\mathrm{tr}\,\tau_\varphi^2) = \frac{1}{4}\,\mathrm{tr}\,\tau_\varphi^2$, where we used (1) in the second equality. For the last identity in item (2), observe that the trace of (2.3) is $\mathrm{tr}\,Q_\varphi = \mathrm{scal}_\varphi - \frac{7}{12}\,\mathrm{tr}\,\tau_\varphi^2 + \frac{1}{2}\,\mathrm{tr}\,\tau_\varphi^2 = (\frac{1}{4} - \frac{7}{12} + \frac{1}{2})\,\mathrm{tr}\,\tau_\varphi^2 = \frac{1}{6}\,\mathrm{tr}\,\tau_\varphi^2$. Thus $\frac{3}{2}\,\mathrm{tr}\,Q_\varphi = \frac{1}{4}\,\mathrm{tr}\,\tau_\varphi^2$. Item (3) follows from item (2) as $|\tau_\varphi|^2 = 0$ if and only if $\tau_\varphi = 0$, i.e.,

φ is torsion-free. Item (4) is derived in [Bry06] and its formula in local coordinates is [[LW17], formula (2.26)]. $\square$

Remark 2.1.16. The full torsion $T \in \mathrm{End}(TM) \cong \Omega^2 \oplus S^2 \cong \Omega^2_7 \oplus \Omega^2_{14} \oplus S^2_0 \oplus C^\infty(M)g$ from [LW17] is

$$T = \frac{\tau_0}{4} g_\varphi - \tau_1^\# \lrcorner \varphi - \frac{1}{2}\tau_2 - j_\varphi(\tau_3),$$

where $\frac{\tau_0}{4} g \in C^\infty(M) g_\varphi$, $\tau_1^\# \lrcorner \varphi \in \Omega^2_7$, $-\frac{1}{2}\tau_2 \in \Omega^2_{14}$, and $j_\varphi(\tau_3) \in S^2_0$. When φ is closed, $T = -\frac{1}{2}\tau_2 = -\frac{1}{2}\tau_\varphi$. For the derivation of T, we refer the reader to [[Kar09], Theorem 2.27]. We note that the formulas for scal_φ and Ric_φ in Proposition 2.1.15 were obtained earlier by Cleyton-Ivanov in [CI07] and Bryant in [Bry06].

We give a brief explanation of the map j_φ in the formula for the Ricci tensor Ric_φ and the full torsion tensor T. We first introduce a map that is initimately related to j_φ and is important in the theory. Let $i_\varphi : S^2(T^*M) \to \Lambda^3(T^*M)$ be defined by

$$i_\varphi(\alpha \circ \beta) = \alpha \wedge *(\beta \wedge *\varphi) + \beta \wedge *(\alpha \wedge *\varphi),$$

where $\circ$ is the symmetric product of composable elements α and β; $S^2(T^*M)$ is the space of symmetric bilinear 2-forms. In local coordinates, there are two conventions for i_φ:

1. $i_\varphi(\eta) = \frac{1}{2}\eta_i^l \varphi_{ljk} dx^i \wedge dx^j \wedge dx^k$; in particular $i_\varphi(g_\varphi) = 3\varphi$ ([Kar09, LW17]);
2. $i_\varphi(\eta) = \eta_i^l \varphi_{ljk} dx^i \wedge dx^j \wedge dx^k$; in particular $i_\varphi(g_\varphi) = 6\varphi$ ([Bry06, Lau17a]).

The map i_φ is linear, injective, and isomorphic to its image $i_\varphi(S^2(T^*M)) = \Lambda^3_1 \oplus \Lambda^3_{27}$ with the linear isomorphism given by

$$i_\varphi(A) = -2\theta(A)\varphi, \quad \text{so that} \quad i_\varphi(Q_\psi) = -2\theta(Q_\psi)\varphi = -2\psi.$$

In particular, $i_\varphi(Q_\varphi) = -2\theta(Q_\varphi)\varphi = -2\Delta_\varphi \varphi$, which is used in obtaining the Laplacian soliton equation (2.8) (see Appendix B.3). [Note $S^2(T^*M)$ has G_2-irreducible decomposition $S^2(T^*M) =$

$\mathbb{R}g \oplus S_0^2(T^*M)$, where $S_0^2(T^*M)$ is the space of all traceless symmetric bilinear 2-forms; and $S_0^2(T^*M) \cong \Lambda^3_{27}$ since $i_\varphi(S_0^2(T^*M)) = \Lambda^3_{27}$ (see [Bry06]).]

Remark 2.1.17. Formally, $Q_\varphi \in \mathrm{sym}(TM)$, and so when we write $i_\varphi(Q_\varphi)$, we mean evaluation of i_φ at $q_\varphi := g_\varphi(Q_\varphi \cdot, \cdot)$, the symmetric bilinear 2-form corresponding to Q_φ. The formulas for $i_\varphi(A)$ are as in [Bry06] and [Lau17a]. We note that $i_\varphi(h) = \theta(h)\varphi = \Delta_\varphi \varphi$ in [Kar09] and [LW17]; the difference is due to the differing conventions in the factor of $1/2$ in the definition of i_φ mentioned above along with the fact that $-h = q_\varphi$.

As stated in Proposition 2.1.15, the map $j_\varphi : \Lambda^3(T^*M) \to S^2(T^*M)$ is defined by

$$j_\varphi(\gamma)(v,w) = *(v \lrcorner \varphi \wedge w \lrcorner \varphi \wedge \gamma).$$

Recall that $\Lambda^3(T^*M)$ has G_2-irreducible decomposition $\Lambda^3(T^*M) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$. The image $j_\varphi(\Lambda^3_7) = 0$ and so j_φ gives an isomorphism between $\Lambda^3_1 \oplus \Lambda^3_{27}$ and $S^2(T^*M)$. Furthermore, j_φ is related to i_φ by

$$j_\varphi(i_\varphi(h)) = 8h + 4(\mathrm{tr}_g(h))g \quad \forall\, h \in S^2(T^*M).$$

[In [Kar09] and [LW17], $j_\varphi(i_\varphi(h)) = -4h - 2(\mathrm{tr}_g(h))g$.] The maps i_φ and j_φ are in a sense "inverses" of one another and allow us to go from symmetric bilinear 2-forms to 3-forms and back again.

Notation: For closed G_2-structure φ, it is understood that $* = *_\varphi$, $g = g_\varphi$, and $\tau = \tau_\varphi$ unless stated otherwise. We will write τ_φ and g_φ to stress the torsion form and metric, respectively, are determined by φ.

We now include a few well known facts regarding closed G_2-structures and their proofs to illustrate how these types of are arguments are made. Many foundational results require using the conditions from the G_2-irreducible decompositions of k-forms in which the relevant k-forms reside along with fundamental identities involving $\lrcorner, *, d, \varphi$, and ψ. These fundamental identities can be found in the appendices of [Kar09] and [Kar20] as well as in the background section in [LW17].

Fact 2.1.18. *For closed G_2-structures φ, the torsion 2-form τ_φ satisfies*

$$\tau_\varphi = -*_\varphi d *_\varphi \varphi \ \ \textit{and} \ \ \Delta_\varphi \varphi = d\tau_\varphi = -d *_\varphi d *_\varphi \varphi. \tag{2.4}$$

Proof. Recall that the induced metric g_φ and orientation vol_φ on M determines the unique Hodge star operator $*_\varphi$ and the Hodge-Laplacian $\Delta_\varphi = d^*d + dd^*$. The defining equation for τ_φ is $d\psi = d*\varphi = 4\tau_1 \wedge \psi + \tau_\varphi \wedge \varphi$. Since φ is closed, the only surviving torsion form is τ_φ and so $\tau_1 = 0$, from which we get $d\psi = \tau_\varphi \wedge \varphi$. Note $\tau_\varphi \in \Omega^2_{14}$ means that $*(\tau_\varphi \wedge \varphi) = -\tau_\varphi$. Taking the Hodge star of both sides and using $*^2 = \mathrm{id}$ yields $\tau_\varphi \wedge \varphi = -*\tau_\varphi$. Putting this together gives $d*\varphi = -*\tau_\varphi$, from which it follows that $\tau_\varphi = -*d*\varphi$ by taking the Hodge star of both sides again and multiplying through by -1. Since the dimension of M is $n = 7$, $d^* = (-1)^7 *d* = -*d*$. We get

$$\Delta_\varphi \varphi = (d^*d + d^*d)\varphi = d^* \underbrace{d\varphi}_{=0} + dd^*\varphi = -d*d*\varphi = d(-*d*\varphi) = d\tau_\varphi.$$

$\square$

From this fact, we see that τ_φ is determined by the closed G_2-structure φ along with the structure equations as it involves the exterior derivative d. Both of these are in turn obtained with respect to an orthonormal basis $(e_i)_i$ for T_pM and its dual basis $(e^i)_i$ for T_p^*M.

In Riemmanian geometry, it is desirable to find Einstein metrics as they are "optimal" metrics in the sense that they are the critical points for the *total scalar curvature functional*:

$$\mathscr{S}(g) = \int_M \mathrm{scal}_g \, dV_g.$$

In the context of 7-manifolds admitting closed G_2-structures, one might ask what conditions on φ gives rise to Einstein metrics. The following proposition gives a necessary and sufficient condition for a closed G_2-structure to induce an Einstein metric.

Proposition 2.1.19 (Einstein condition)**.** *For φ a closed G_2-structure, g_φ is Einstein if and only if*

$$d\tau_\varphi = \frac{3}{14}|\tau_\varphi|^2\varphi + \frac{1}{2}*(\tau_\varphi \wedge \tau_\varphi).$$

Proof. Applying i_φ to the expression of for Ric_φ from Proposition 2.1.15 (4), we get

$$d\tau_\varphi = \frac{3}{14}|\tau_\varphi|^2\varphi + \frac{1}{2}*(\tau_\varphi \wedge \tau_\varphi) - \frac{1}{2}i_\varphi(\mathrm{Ric}^0_\varphi),$$

where $\mathrm{Ric}^0_\varphi = \mathrm{Ric}_\varphi - \frac{\mathrm{scal}_\varphi}{7}g_\varphi$ is the traceless Ricci tensor. So if g_φ is Einstein, $\mathrm{Ric}^0_\varphi = 0$ and so $i_\varphi(\mathrm{Ric}^0_\varphi) = 0$ as i_φ is linear. On the other hand, if the equation above holds, then $i_\varphi(\mathrm{Ric}^0_\varphi) = 0$. But since i_φ is injective, $\mathrm{Ric}^0_\varphi = 0$ and so g_φ is Einstein. □

The Einstein condition is used to obtain the following result in the compact case.

Theorem 2.1.20. *On compact M, the induced metric g_φ from a closed G_2-structure φ is Einstein if and only if φ is torsion-free.*

Bryant's Proof. We use shorthand notation $\tau = \tau_\varphi$ and $\tau^n = \tau \wedge \cdots \wedge \tau$ n-times. First note that since $\tau \in \Omega^2_{14}$, we have $\tau \wedge \varphi = -*\tau$. Then

$$\tau \wedge \tau \wedge \varphi = \tau \wedge (\tau \wedge \varphi) = \tau_\varphi \wedge (-*\tau) = -\langle \tau, \tau \rangle \mathrm{vol}_\varphi = -|\tau|^2 \mathrm{vol}_\varphi .$$

By Proposition 2.1.19, g is Einstein if and only if $d\tau = \frac{3}{14}|\tau|^2\varphi + \frac{1}{2}*(\tau \wedge \tau)$. Observe that

$$\begin{aligned}
d(\frac{1}{3}\tau^3) &= \frac{1}{3}(d\tau \wedge \tau^2 + \tau \wedge d\tau^2) = \frac{1}{3}(d\tau \wedge \tau^2 + \tau \wedge (d\tau \wedge \tau - \tau \wedge d\tau)) \\
&= \tau^2 \wedge d\tau \\
&= \tau^2 \wedge \left(\frac{3}{14}|\tau|^2\varphi + \frac{1}{2}*(\tau \wedge \tau)\right) \\
&= \frac{3}{14}|\tau|^2(\tau \wedge \tau \wedge \varphi) + \frac{1}{2}(\tau \wedge \tau) \wedge *(\tau \wedge \tau) \\
&= -\frac{3}{14}|\tau|^4 \mathrm{vol}_\varphi + \frac{1}{2}|\tau \wedge \tau|^2 \mathrm{vol}_\varphi = \frac{2}{7}|\tau|^4 \mathrm{vol}_\varphi,
\end{aligned}$$

where we used $\tau \wedge \tau \wedge \varphi = -|\tau|^2 \operatorname{vol}_\varphi$ obtained above and the identity $|\tau^2|^2 = |\tau|^4$ for any $\tau \in \Omega^2_{14}$ (see [Bry06]) in the last equality. Integrating both sides over compact M and applying Stoke's theorem yields

$$0 = \int_M d(\frac{1}{3}\tau^3) = \frac{2}{7}\int_M |\tau|^4 \operatorname{vol}_\varphi,$$

which holds if and only if $\tau = 0$. □

So when M is compact, a closed G_2-structure induces an Einstein metric if and only if it is also coclosed. Fernández-Fino-Manero obtain a similar statement in the non-compact case when M is a simply connected solvable Lie group with left-invariant closed G_2-structure.

Theorem 2.1.21 ([FFM16], Fernández-Fino-Manero)**.** *A 7-dimensional simply connected solvable Lie group cannot admit left-invariant closed G_2-structures such that g_φ is Einstein unless φ is torsion-free.*

Proof. This is one of the main results of [FFM16]. □

Remark 2.1.22. We note that homogeneous Einstein torsion-free metrics are flat since by Proposition 2.1.15 φ is torsion-free if and only if $\operatorname{scal}_\varphi = 0$ and so if $\operatorname{Ric}_\varphi = \lambda g$, then $\operatorname{scal}_\varphi = \operatorname{tr}\operatorname{Ric}_\varphi = 7\lambda = 0$ if and only if $\lambda = 0$. Thus $\operatorname{Ric}_\varphi = 0$, from which it follows that g_φ is flat as Ricci-flat homogeneous spaces are flat by a result of Alekseevskiĭ-Kimel′fel′d in [AK75]. An open question posed by Bryant is whether or not there exists not necessarily complete closed G_2-structures φ such that g_φ is Einstein and non-Ricci-flat (see [[Bry06], Remark 12]).

Bryant observed that in the compact case, the formula for $d\tau_\varphi$ in the proof of Proposition 2.1.19 is a special case which can be obtained from a more general Ricci pinching condition [[Bry06], Corollary 3]. From the Ricci pinching condition, Bryant obtains the following on compact manifolds.

Theorem 2.1.23 (Bryant)**.** *For φ a closed G_2-structure on compact smooth 7-manifold M,*

$$\int_M \operatorname{scal}_\varphi^2 \operatorname{vol}_\varphi \leq 3\int_M |\operatorname{Ric}_\varphi|^2 \operatorname{vol}_\varphi,$$

and equality holds if and only if $d\tau_\varphi = \frac{|\tau_\varphi|^2}{6}\varphi + \frac{1}{6} * (\tau_\varphi \wedge \tau_\varphi)$.

Proof. We refer the reader to [[Bry06], Corollary 3 and Remark 13] for the proof. □

Remark 2.1.24. This inequality does not hold in the non-compact homogeneous case. Lauret exhibited some examples in [Lau17b].

Definition 2.1.25. A closed G_2-structure is said to be *extremely Ricci pinched* (ERP) if $d\tau_\varphi = \frac{|\tau_\varphi|^2}{6}\varphi + \frac{1}{6} * (\tau_\varphi \wedge \tau_\varphi)$. Bryant suggested that such G_2-structures may be of interest as they are the most "extremely Ricci pinched" a G_2-structure can get on a compact manifold.

Remark 2.1.26. Closed G_2-structures satisfying either the Einstein condition or the ERP condition are special cases of a general class of G_2-structures called *quadratic closed G_2-structures*, i.e., closed G_2-structures where $d\tau_\varphi$ depends quadratically on τ_φ. More concretely, a closed G_2-structure is λ*-quadratic* if $d\tau = \frac{1}{7}|\tau|^2\varphi + \lambda(\frac{1}{7}|\tau|^2\varphi + *(\tau \wedge \tau))$. Quadratic closed G_2-structures have recently been studied by authors like Ball, Lauret-Nicolini, Fino-Raffero, et al.

2.2 The Laplacian flow

2.2.1 The Laplacian flow and its solitons

Bryant introduced a natural geometric flow of G_2-structures called the *Laplacian flow* given by

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)}\varphi(t) \\ \varphi(0) = \varphi \end{cases},$$

where φ is the initial 3-form and $\Delta_{\varphi(t)} = *_\varphi d *_\varphi d - d *_\varphi d *_\varphi$ is the *Hodge Laplacian operator* on 3-forms.

Fact 2.2.1. $\varphi(t)$ *is closed for all* t.

Proof. Note $\partial_t(d\varphi(t)) = d(\partial_t\varphi(t)) = d(\Delta_\varphi\varphi) = d(d\tau_\varphi) = 0$ and so $d\varphi(t)$ does not change in t. Since the initial G_2-structure $\varphi(0) = \varphi$ is closed, i.e., $d\varphi(0) = d\varphi = 0$, so is $\varphi(t)$. □

Fact 2.2.2. *The stationary points of the Laplacian flow are torsion-free G_2-structures.*

Proof. From (2.4), $\Delta_\varphi \varphi = d\tau_\varphi \in \Omega^3(M) = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$ so we can write

$$\Delta_\varphi \varphi = d\tau_\varphi = f\varphi + *(\beta \wedge \varphi) + \gamma,$$

where $f\varphi \in \Omega^3_1$ for some $f \in C^\infty(M)$, $*(\beta \wedge \varphi) \in \Omega^3_7$ for some 1-form $\beta \in T^*M$, and $\gamma \in \Omega^3_{27}$. Note that

$$d\tau_\varphi \wedge \varphi = (f\varphi) \wedge \varphi + *(\beta \wedge \varphi) \wedge \varphi + \gamma \wedge \varphi = *(\beta \wedge \varphi) \wedge \varphi$$

since $(f\varphi) \wedge \varphi = f(\varphi \wedge \varphi) = 0$ as φ is a 3-form [$\omega \wedge \omega = 0$ for any odd k-form ω] and $\gamma \wedge \varphi = 0$ since $\gamma \in \Omega^3_{27}$. On the other hand

$$d\tau_\varphi \wedge \varphi = d\tau_\varphi \wedge \varphi + (-1)^3 \tau_\varphi \wedge d\varphi = d(\tau_\varphi \wedge \varphi) = d(d * \varphi) = 0,$$

where the second to last equality follows from the defining equation for torsion forms $d\psi = d * \varphi = \tau_\varphi \wedge \varphi$ when φ is closed. Thus $\beta = 0$. By taking the inner product of φ with $\Delta_\varphi \varphi$ (see details in [[LW17], Section 2.2]), one gets $f = \frac{|\tau_\varphi|^2}{7}$ and so

$$\Delta_\varphi \varphi = d\tau_\varphi = \frac{|\tau_\varphi|^2}{7}\varphi + \gamma.$$

It follows that $\partial_t \varphi = \Delta_\varphi \varphi = d\tau_\varphi = 0$ if and only if $\gamma = 0$ and $\tau_\varphi = 0$. In particular, $\Delta_\varphi \varphi = 0$ if and only if $\tau_\varphi = 0$. $\square$

Remark 2.2.3. When M is compact, there is a unique solution $\varphi(t)$ of closed G_2-structures satisfying the Laplacian flow on some time interval $0 < T \leq \infty$ (see remarks of Bryant on the methods of DeTurck and Hamilton in [Bry06]).

Since torsion-free G_2-structures yield metrics with holonomy in G_2, Bryant and his collaborators investigated the conditions under which a closed G_2-structure converges to a torsion-free one under the flow, along with the possible obstructions to such convergence. More generally, it is of interest to study the long time behavior, long time existence, uniqueness, and convergence of solutions to the closed Laplacian flow. Short-time existence and uniqueness of solutions to the flow was proven by Bryant-Xu in [BX11]. Lotay-Wei obtained long-time existence criteria for Laplacian flow solutions based on torsion estimates along the flow in [LW17]. Another reason to study the Laplacian flow is due to its relationship with the volume functional. It is known that the Laplacian flow is the upward gradient flow for Hitchin's volume functional:

$$\varphi \in [\varphi_0] \mapsto \mathrm{vol}(\varphi) = 7^{-1} \int_M \varphi \wedge *\varphi.$$

This functional is monotonically increasing along the flow. Moreover, its critical points are torsion-free G_2-structures and are local maxima of $\mathrm{vol}(\varphi)$ in a fixed cohomology class $[\varphi_0]$. Details regarding this functional can be found in [Hit00].

It is known that a Laplacian flow solution $\varphi(t)$ starting at φ is *self-similar*, i.e.,

$$\varphi(t) = c(t) f(t)^* \varphi \quad c(t) \in \mathbb{R}^*, \ \ f(t) \in \mathrm{Diff}(M)$$

if and only if

$$\Delta_\varphi \varphi = \lambda \varphi + \mathscr{L}_X \varphi, \quad \lambda \in \mathbb{R}, \, X \in \mathfrak{X}(M). \tag{2.5}$$

Hence any triple (φ, X, λ) satisfying (2.5), the *Laplacian soliton equation*, is a *Laplacian soliton*. Laplacian solitons are said to be *steady, shrinking, or expanding* if $\lambda = 0$, $\lambda < 0$, or $\lambda > 0$, respectively. A Laplacian soliton is *gradient* if X is a gradient field, i.e., $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$. In the last decade, Laplacian solitons on homogeneous spaces have received increased interest and many new examples have been found (see, e.g., [FR20,Lau17a,Lau17b,LN20,Nic18], and [Nic22]). These solitons are of interest as they model singularities of the flow.

In this thesis, we consider only Laplacian flows with closed initial G_2-structure:

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)} \varphi(t) \\ \varphi(0) = \varphi \\ d\varphi = 0 \end{cases}.$$

In the notation of [LW17], the associated metric g_φ for a closed Laplacian soliton φ satisfies

$$\begin{cases} \partial_t g_\varphi(t) = 2h(t) \\ g_\varphi(0) = g_\varphi \end{cases}, \tag{2.6}$$

where the 2-form h in local coordinates is

$$h_{ij} = -R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i^k T_{kj};$$

R_{ij} is the Ricci tensor; $T = -\frac{1}{2}\tau_\varphi$ is the full torsion tensor for closed G_2-structures.

Remark 2.2.4. The evolution of the metric g_φ from (2.6) can also be written

$$\partial_t g = -2\,\mathrm{Ric}_\varphi + \frac{|\tau_\varphi|^2}{6} g + \frac{1}{4} j_{\varphi(t)}(*(\tau_\varphi \wedge \tau_\varphi)).$$

Note that this flow is a perturbation of the Ricci flow by "quadratic" torsion terms.

Lotay-Wei uses (2.5) and injectivity of i_φ to show that the associated metric must also satisfy

$$-R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i^k T_{kj} = \frac{1}{3}\lambda g_{ij} + \frac{1}{2}(\mathscr{L}_X g)_{ij},$$

or equivalently

$$h - \frac{1}{3}\lambda g - \frac{1}{2}\mathscr{L}_X g = 0. \tag{2.7}$$

We refer the reader to [[LW17], Proposition 9.4] or [[Kar09], Corollary 3.2] for more details.

Remark 2.2.5. The evolution of the volume form is

$$\partial_t \operatorname{vol}_\varphi = \frac{|\tau_\varphi|^2}{3} \operatorname{vol}_\varphi,$$

which is pointwise non-decreasing along the Laplacian flow. We refer the reader to [Bry06, Kar09], and [LW17] for a detailed study of these evolution equations and the evolution equations of torsion forms τ_i.

We now discuss a few well known facts about Laplacian solitons and compare them to solitons of the well studied Ricci flow. We first remark that Laplacian solitons of the form $(\varphi, 0, \lambda)$ are called *eigenforms*. From (2.5), we see that eigenforms φ satisfy $\Delta_\varphi \varphi = \lambda \varphi$ and can be viewed as the 3-form analogue of Einstein metrics g satisfying $\operatorname{Ric}_g = \lambda g$. In this thesis, we will refer to Laplacian solitons of the form (φ, X, λ) where $X \neq 0$ as *non-trivial* solitons and solitons where $X = 0$ as *trivial* solitons.

Lin showed in [Lin13] that there are no compact shrinking Laplacian solitons and that compact steady Laplacian solitons are torsion-free. Furthermore, it is known that stationary points of the Laplacian flow on compact manifolds are always torsion-free as harmonic forms, i.e., φ such that $\Delta_\varphi \varphi = 0$, are always closed and coclosed. Lotay-Wei show that this is true for any 7-manifold (not necessarily compact). More precisely, Lotay-Wei showed that any Laplacian soliton of the form $(\varphi, 0, \lambda)$ is either expanding, i.e., $\lambda > 0$, or is torsion-free. Thus stationary points of the flow have solitons of the form $(\varphi, 0, 0)$ and by the preceding result φ must be torsion-free. Lotay-Wei also obtained a non-existence result that there are no compact Laplacian solitons of the form $(\varphi, 0, \lambda)$ unless φ is torsion-free. This together with Lin's result shows that non-torsion-free compact Laplacian solitons must be expanding and $X \neq 0$.

Now recall the *Ricci flow* is given by

$$\begin{cases} \partial_t g = -2 \operatorname{Ric}_g \\ g(0) = g_0, \end{cases}$$

and *Ricci solitons* (g, X, λ) satisfy the *Ricci soliton equation*

$$\mathrm{Ric}_g + \frac{1}{2}\mathscr{L}_X g = \lambda g.$$

By Lotay-Wei's result, there are no compact Laplacian solitons of the form $(\varphi, 0, \lambda)$ unless φ is torsion-free while there exists compact Ricci solitons of the form $(g, 0, \lambda)$, i.e., compact Einstein metrics $\mathrm{Ric}_g = \lambda g$, where the soliton can be steady, shrinking, or expanding. There are examples of homogeneous Laplacian solitons on noncompact manifolds with $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ constructed by Fino-Raffero, Lauret-Nicolini, et al.

2.2.2 Gradient Laplacian solitons

We are primarily interested in equation (2.7) cast in the notation of Lauret's papers [Lau17a] and [Lau17b]. By reconciling differing conventions by Lotay-Wei and Lauret, one sees that the unique symmetric operator Q_φ from (2.3) coincides with $-h$ in (2.6). Setting $q_\varphi := g_\varphi(Q_\varphi \cdot, \cdot)$ as in Remark 2.1.17, we get $q_\varphi = -h$ and so from (2.7), we obtain the following equation

$$\frac{1}{2}\mathscr{L}_X g_\varphi = -q_\varphi - \frac{1}{3}\lambda g_\varphi, \quad \lambda \in \mathbb{R},\ X \in \mathfrak{X}(M). \tag{2.8}$$

We also call (2.8) the Laplacian soliton equation (in terms of the induced metric g_φ). Note that (2.8) corresponds to a *geometric q-flow* of the metric g_φ with $c = -(1/3)\lambda$ and *q-soliton* $-2q_\varphi$. When $X = \nabla f$ is a gradient field for some smooth function $f : M \to \mathbb{R}$, (2.8) becomes the *closed gradient Laplacian soliton equation*

$$\mathrm{Hess}\, f = -q_\varphi - \frac{1}{3}\lambda g_\varphi \tag{2.9}$$

and we call the triple $(\varphi, \nabla f, \lambda)$ satisfying (2.9) a *closed gradient Laplacian soliton*. The function f is commonly referred to as the *potential function*. We elaborate on the derivation of (2.8) in Appendix B.3.

Remark 2.2.6. Petersen-Wylie showed that all homogeneous gradient Ricci solitons are rigid, i.e., isometric to a quotient of $N \times \mathbb{R}^k$ where N is Einstein and potential $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor. In other words, there are no non-trivial homogeneous gradient Ricci solitons aside from ones that are rigid with f a Gaussian on the Euclidean factor. This also means that non-trivial Ricci solitons that are not rigid must be non-gradient. For Laplacian solitons, there are non-trivial gradient Laplacian solitons on homogeneous spaces, e.g., the solitons where potential f is either a Gaussian or an affine function and φ is torsion-free (see, e.g., case $(\mathfrak{n}_1, \varphi_1)$ in Chapter 4). For more results on rigidity of gradient Ricci solitons, we refer the reader to [PW09a, PW09b], and [PW10].

Remark 2.2.7. It is well known that every compact Ricci soliton is gradient by a result of Perel'man (see [MT07]). An open question is whether there exists compact Laplacian solitons that are gradient. In fact, the existence of non-trivial Laplacian solitons on compact manifolds is still open.

For more foundational material on G_2-structures and the Laplacian flow, we refer the reader to [Bry06, FG82, Kar09, Kar20], and [LW17]. We note that Haskins-Nordström investigate cohomogeneity-one steady gradient Laplacian solitons with symmetry groups $\mathrm{Sp}(2)$ and $\mathrm{SU}(3)$ in [HN21]. Examples of gradient Laplacian solitons in the non-homogeneous setting are referenced in [HN21]. A recent preprint [HKP22] by Haskins-Kahn-Payne shows uniqueness of asymptotically conical gradient Laplacian solitons. We also mention that Garrone in [Gar22] studies closed G_2-structures in the setting of isometric flow, where the critical points are G_2-structures with divergence-free full torsion tensor.

Remark 2.2.8. Henceforth, all G_2-structures and (gradient) Laplacian solitons we consider are assumed to be closed. So whenever we refer to (gradient) Laplacian solitons, we mean closed (gradient) Laplacian solitons.

2.2.3 G_2-structures on homogeneous spaces

When (M, φ) is homogeneous, M has a presentation $M = G/K$ for some transitive Lie subgroup $G \subseteq \mathrm{Aut}(M, \varphi) = \{f \in \mathrm{Diff}(M) \mid f^*\varphi = \varphi\} \subset \mathrm{Iso}(M, \varphi)$ and isotropy subgroup $K \subset G$. In this setting, we have the following.

(i) φ is a G-invariant G_2-structure on M.

(ii) When $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a *reductive deomposition*, i.e., $\mathrm{Ad}(K)\mathfrak{p} \subset \mathfrak{p}$, we can identify $\mathfrak{p}$ with T_pM such that any G-invariant G_2-structure φ is determined by a fixed positive 3-form on $\mathfrak{p}$.

Remark 2.2.9. There is a one-to-one correspondence between left-invariant G_2-structures on simply-connected Lie groups and G_2-structures on its associated Lie algebra (see, e.g., [Fre13] or [Lau17a]). Thus it is common in the literature to identify Lie groups $G = G_\mu$ admitting a (closed) left-invariant G_2-structure (similarly, Laplacian, algebraic, or semi-algebraic soliton) φ with its corresponding Lie algebra $(\mathfrak{g}, \mu = [\cdot,\cdot])$ admitting the fixed positive 3-form φ. We refer to both (G,φ) and $(\mathfrak{g},\varphi)$ as G_2-structures and further, as Laplacian solitons if φ satisfies (2.8) for some $\lambda \in \mathbb{R}$.

3 | The Structure Theorem

3.1 Structure theorem for homogeneous closed gradient Laplacian solitons

Let (M,φ) be homogeneous, i.e., $M = G/K$ for some transitive Lie subgroup $G \subseteq \mathrm{Aut}(M,\varphi)$ and K an isotropy subgroup of G at some point of M. If (M,φ) admits a closed gradient Laplacian soliton $(\varphi,\nabla f,\lambda)$, then the triple satisfies the gradient Laplacian soliton equation (2.9). To be consistent with notation of [PW22], we set

$$q := -q_\varphi - \frac{1}{3}\lambda g_\varphi, \tag{3.1}$$

where q is a G-invariant symmetric 2-tensor. Observe that the gradient Laplacian soliton equation being satisfied by $(\varphi,\nabla f,\lambda)$ where $\nabla f \neq 0$ is equivalent to there being a non-constant $f \in F(q) = \{\mathrm{Hess}\, f = q\}$ as studied in [PW22]. Petersen-Wylie's motivation for studying the solution space $F(q)$ is due to the equation $\mathrm{Hess}\, f = q$ arising naturally from gradient solitons for geometric flows where the tensor q involves the curvature of the manifold. It is clear that q involves the curvature of the manifold as q_φ does. We remark that if f is constant (or trivial), then $X = \nabla f = 0$ and so such gradient solitons would correspond to eigenforms. We call gradient solitons where $\nabla f \neq 0$ *non-trivial* gradient solitons and consider only non-trivial gradient solitons in this thesis. We now state the main result of this section.

Theorem 3.1.1 (Structure Theorem)**.** *Let M be a 7-dimensional homogeneous space admitting a* *closed gradient Laplacian soliton* $(\varphi, \nabla f, \lambda)$ *where* f *is non-constant.*

1. *The square of the intrinsic torsion* τ_φ^2 *is divergence-free if and only if* (M, g_φ) *is isometric to a product* $N \times \mathbb{R}^k$ *where* f *is constant on* N.

2. *If the square of the intrinsic torsion* τ_φ^2 *is not divergence-free, then either*

   (a) (M, g_φ) *is a one-dimensional extension;* $g_\varphi = dr^2 + g_r$*; and* $f(x,y) = ar + b$*; or*

   (b) (M, g_φ) *is isometric to a product* $N \times \mathbb{R}^k$ *where* N *is a one-dimensional extension and* $f(x,y) = ar(x) + v(y)$*, where* v *is a function on* $\mathbb{R}^k$ *and* r *is a distance function on* N.

To obtain the structure theorem for closed gradient Laplacian solitons on homogeneous spaces, we recall [[PW22], Theorem 3.6] of Petersen-Wylie.

Theorem 3.1.2 (Structure Theorem of Petersen-Wylie)**.** *Let* (M, g) *be a* G*-homogeneous manifold and let* q *be a* G*-invariant symmetric* 2*-tensor. If* $f \in F(q)$ *is a non-constant function, then either*

1. (M, g) *is isometric to a product* $N \times \mathbb{R}^k$ *where* f *is constant on* N

2. (M, g) *is a* 1*-dimensional extension,* $g = dr^2 + g_r$*, and* $f(x,y) = ar + b$*,*

3. (M, g) *is isometric to a product* $N \times \mathbb{R}^k$ *where* N *is a* 1*-dimensional extension and* $f(x,y) = ar(x) + v(y)$*, where* v *is a function on* $\mathbb{R}^k$ *and* r *is a distance function on* N.

The three possible structures depend on the divergence of the G-invariant symmetric 2-tensor q. That is, if q is divergence-free, then we are in case (1) of [[PW22], Theorem 1.1]. The converse also holds: if the product $N \times \mathbb{R}^k$ has a non-constant function $f \in F(q)$ that is constant on N, then q is divergence-free. To see this, we use the Bochner formula $\operatorname{div}(\nabla\nabla f) = \operatorname{Ric}(\nabla f) + \nabla \Delta f$. Since f is a function on the Euclidean factor only, $\nabla f \in T_p\mathbb{R}^k$. It follows that $\nabla f \in \ker \operatorname{Ric}$, hence $\operatorname{Ric}(\nabla f) = 0$. Moreover, $\Delta f = \operatorname{tr} \nabla\nabla f = \operatorname{tr} \operatorname{Hess} f = \operatorname{tr} q$ is constant as M is homogeneous and q is G-invariant. Thus $\operatorname{div} q = \operatorname{div} \operatorname{Hess} f = \operatorname{div}(\nabla\nabla f) = 0$. If q is not divergence-free, then the structure of M can be as in either case (2) or (3). So to apply Theorem 3.1.2, we must compute the divergence of $q = -q_\varphi - \frac{1}{3}\lambda g_\varphi$.

Proof of Theorem 3.1.1. Let (M,φ) be a closed homogeneous G_2-structure. By Proposition 2.1.15,

$$\mathrm{scal}_\varphi = -\frac{1}{2}|\tau_\varphi|^2 \quad \text{and} \quad |\tau_\varphi|^2 = -\frac{1}{2}\operatorname{tr}\tau_\varphi^2.$$

Putting these two equations together yields

$$-\frac{1}{3}\mathrm{scal}_\varphi = -\frac{1}{12}\operatorname{tr}\tau_\varphi^2.$$

The expression for Q_φ from (2.3) can be written

$$Q_\varphi = \mathrm{Ric}_\varphi - \frac{1}{3}\mathrm{scal}_\varphi I + \frac{1}{2}\tau_\varphi^2. \tag{3.2}$$

Taking the divergence gives

$$\operatorname{div} Q_\varphi = \frac{1}{2}\operatorname{div}\tau_\varphi^2, \tag{3.3}$$

where we used the 2nd contracted Bianchi identity, $\operatorname{div}\mathrm{Ric}_\varphi = \frac{1}{2}D\,\mathrm{scal}_\varphi$, along with the fact that scal_φ is constant on homogeneous spaces. Performing a type change of equation (3.1) to $(1,1)$-tensors with respect to g_φ, we get $Q = -Q_\varphi - \frac{1}{3}\lambda I$. Then $\operatorname{div} Q = -\frac{1}{2}\operatorname{div}\tau_\varphi^2$, or equivalently,

$$\operatorname{div} q = -\frac{1}{2}\operatorname{div}\tau_\varphi^2, \tag{3.4}$$

where $\tau_\varphi^2 = g_\varphi(\tau_\varphi^2\cdot,\cdot)$. So to compute $\operatorname{div} q$, it suffices to compute $\operatorname{div}\tau_\varphi^2$.

Suppose $(\varphi,\nabla f,\lambda)$ is a gradient Laplacian soliton where $\nabla f \neq 0$. Then equation (2.9) is satisfied and implies that there is a non-constant $f \in \{\operatorname{Hess} f = q\}$. It is clear that q is a symmetric 2-tensor as both q_φ and $(1/3)\lambda g_\varphi$ are symmetric. That q is G-invariant follows from φ being G-invariant. More precisely, for $\gamma \in G$, we have $\gamma^*\varphi = \varphi$ and $\gamma^* g_\varphi = g_\varphi$. It follows from isometry invariance of the Ricci tensor that $\gamma^*\mathrm{Ric}(g_\varphi) = \mathrm{Ric}(\gamma^* g_\varphi) = \mathrm{Ric}(g_\varphi)$ as $\gamma \in G \subset \mathrm{Aut}(M,\varphi) \subset \mathrm{Iso}(M,g_\varphi)$. Moreover, since the torsion 2-form τ_φ is determined by φ, $\gamma^*\tau_\varphi = \tau_{\gamma^*\varphi} = \tau_\varphi$. Thus q is G-invariant. Combining these observations with the preceding discussion, we apply Theorem

3.1.2 to obtain the possible structures determined by whether τ_φ^2 is divergence-free or not. $\square$

3.2 Computing $\operatorname{div}\tau_\varphi^2$

The following are several useful lemmas for computing $\operatorname{div}\tau_\varphi^2$. We first state a lemma regarding the divergence of general 2-tensors.

Lemma 3.2.1. *Let (M,g) be a Riemannian manifold and $(e_i)_i$ an orthonormal basis on T_pM. For any $(0,2)$-tensor of the form $T(\cdot,\cdot)=g(A\cdot,\cdot)$, where A is its $(1,1)$-dual tensor with respect to g, we have*

$$(\operatorname{div}T)(U)=\sum_{i=1}^{7}g(\nabla_{e_i}(A(e_i))-A(\nabla_{e_i}e_i),U). \tag{3.5}$$

Proof. Observe that

$$\begin{aligned}(\operatorname{div}T)(U)&=\sum_{i=1}^{7}(\nabla_{e_i}T)(e_i,U)\\&=\sum_{i=1}^{7}[(\nabla_{e_i}(T(e_i,U))-T(\nabla_{e_i}e_i,U)-T(e_i,\nabla_{e_i}U)]\\&=\sum_{i=1}^{7}[(\nabla_{e_i}(g(A(e_i),U))-g(A(\nabla_{e_i}e_i),U)-g(A(e_i),\nabla_{e_i}U)]\\&=\sum_{i=1}^{7}[g(\nabla_{e_i}(A(e_i)),U)+g(A(e_i),\nabla_{e_i}U)\\&\qquad\qquad-g(A(\nabla_{e_i}e_i),U)-g(A(e_i),\nabla_{e_i}U)]\\&=\sum_{i=1}^{7}g(\nabla_{e_i}(A(e_i))-A(\nabla_{e_i}e_i),U).\end{aligned}$$

$\square$

We will refer to the sums $\sum_i g(\nabla_{e_i}(A(e_i)),\cdot)$ and $\sum_i g(A(\nabla_{e_i}e_i),\cdot)$ from formula (3.5) as (3.5a) and (3.5b), respectively.

Remark 3.2.2. Note $\operatorname{div}q$ is a $(0,1)$-tensor. When A is symmetric (hence T is symmetric) it is not hard to show $(\nabla_{e_i}T)(e_i,U)=(\nabla_{e_i}T)(U,e_i)$ for any vector U. Also, (3.5b)$=0$ whenever $\nabla_{e_i}e_i=0$

$\forall\, i$.

We make a definition that will be useful in proofs.

Definition 3.2.3. We say that a basis $(e_i)_i$ for $\mathfrak{g}$ is *orthogonally nice* if

$$[e_i, e_j] = ce_k \ \ \& \ e_i, e_j \perp e_k.$$

If $(e_i)_i$ is an orthonormal basis, then this condition is equivalent to

$$[e_i, e_j] = ce_k \ \ \& \ e_i, e_j \neq e_k.$$

The motivation for defining such a basis is due to it being a sufficient condition for *diagonally trivial derivatives*, i.e.,

$$\nabla_{e_i} e_i = 0 \ \ \forall\, i,$$

provided $(e_i)_i$ is orthonormal (see Lemma 3.2.5 (4)).

Remark 3.2.4. Our definition of an "orthogonally nice" basis differs from the notion of a "nice" basis as defined by Lauret-Will in [LW13]: a basis of a Lie algebra is *nice* if $[e_i, e_j]$ is always a scalar multiple of some element in the basis and $[e_i, e_j]$, $[e_r, e_s]$ can be a nonzero multiple of the same e_k only if $\{i, j\} \cap \{r, s\} = \emptyset$. All of the bases $(e_i)_i$ for $(\mathfrak{n}_i, \varphi_i)$ for $i = 1, \ldots, 7$ are nice. Moreover, they are orthogonally nice, hence the structure equations for $\mathfrak{n}_i$ yields diagonally trivial derivatives. The characterization of nice bases is used by Lauret-Will as well as others referenced in [LW13] to study nilsolitons on nilmanifolds and stably Ricci-diagonal metrics. A basis for a Lie algebra is *stably Ricci-diagonal* if any diagonal left-invariant metric has diagonal Ricci tensor (see [LW13] and [Kri21]). One relevant fact in the nilpotent case is the following: a basis of a nilpotent Lie algebra is stably Ricci-diagonal if and only if it is nice [[LW13], Theorem 1.1]. We note that some of the results to follow may hold with the hypotheses of nice bases on nilpotent Lie groups. We also note that Krishnan studies nice bases and diagonality of the Ricci tensor in a more general setting in [Kri21].

Lemma 3.2.5 (Consequences of the Koszul Formula)**.** *For any orthonormal basis $(e_i)_i$,*

1. $g(\nabla_{e_i} e_i, e_j) = -g([e_i, e_j], e_i) = g([e_j, e_i], e_i)$
2. $g(\nabla_{e_i} e_j, e_k) = \frac{1}{2}[g([e_i, e_j], e_k) - g([e_i, e_k], e_j) - g([e_j, e_k], e_i)]$
3. $\sum_i g(\nabla_{e_i} e_i, e_j) = \operatorname{tr}(\operatorname{ad}_{e_j})$
4. *If $(e_i)_i$ is orthogonally nice, then $\nabla_{e_i} e_i = 0 \ \ \forall i$.*

Proof. (1) and (2) follow from the Koszul formula, $(e_i)_i$ being an orthonormal basis, and skew-symmetry of the Lie bracket. (3) follows from (1), $\operatorname{ad}_{e_j}(e_i) = [e_j, e_i]$, and the definition of trace. (4) follows from (1) and $(e_i)_i$ being orthogonally nice. □

Proposition 3.2.6. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be the Lie algebra of a Lie group G with closed G_2-structure φ. If τ_φ^2 is diagonal with respect to an orthogonally nice orthonormal basis $(e_i)_i$, then τ_φ^2 is divergence-free, hence Q_φ is divergence-free. Moreover, if $\operatorname{Ric}_\varphi$ is also diagonal with respect to $(e_i)_i$, then Q_φ is diagonal if and only if τ_φ^2 is.*

Proof. Since $(e_i)_i$ is an orthogonally nice orthonormal basis, by Lemma 3.2.5 (4) we have $\nabla_{e_i} e_i = 0$ $\forall\, i$. So if τ_φ^2 is diagonal, then $\tau_\varphi^2(e_i) = a_i e_i$ and we get

$$\nabla_{e_i}(\tau_\varphi^2(e_i)) = \nabla_{e_i}(a_i e_i) = a_i \nabla_{e_i} e_i = 0 \ \ \forall\, i.$$

Hence the sum (3.5a) = 0. Moreover, diagonally trivial derivatives implies the sum (3.5b) = 0. Thus $\operatorname{div} \tau_\varphi^2 = 0$. The last statement follows easily from (3.2). □

Remark 3.2.7. The converse of Proposition 3.2.6 is not true. In the case of $\mathfrak{n}_4$, $\operatorname{div} \tau_{\varphi_4}^2 = 0$ while $\tau_{\varphi_4}^2$ is not diagonal (see Chapter 4).

We now state a key lemma used in the proof of the non-divergence-free cases of Theorem 4.1.1. This key lemma is an instance of [[Gri21], Proposition 3.1]. We also include [[Gri21], Corollary 3.2] as it will be used to prove some cases of Theorem 4.1.1.

Lemma 3.2.8 (Key Lemma)**.** *Let (M,φ) be a closed G_2-structure. For any gradient Laplacian soliton $(\varphi,\nabla f,\lambda)$, we have*

$$g(\mathrm{Ric}(\nabla f),\cdot) = -\frac{1}{2}\operatorname{div}\tau_\varphi^2 + \nabla \operatorname{tr} q_\varphi. \tag{3.6}$$

If in addition $\operatorname{tr} q_\varphi$ *is constant (e.g., when M is homogeneous) then*

$$g(\mathrm{Ric}(\nabla f),\cdot) = -\frac{1}{2}\operatorname{div}\tau_\varphi^2. \tag{3.7}$$

Proof. The gradient Laplacian soliton equation (2.9) type changed to $(1,1)$-tensors is

$$\nabla\nabla f = -Q_\varphi - \frac{1}{3}\lambda I. \tag{3.8}$$

Taking the divergence of (3.8) and using the Bochner formula, $\operatorname{div}\nabla\nabla f = \mathrm{Ric}(\nabla f) + \nabla\Delta f$, yields

$$\mathrm{Ric}(\nabla f) + \nabla\Delta f = -\operatorname{div} Q_\varphi. \tag{3.9}$$

On the other hand, taking the trace of (3.8) yields

$$\Delta f = -\operatorname{tr} Q_\varphi - \frac{7}{3}\lambda. \tag{3.10}$$

Substituting (3.10) into (3.9) yields

$$\mathrm{Ric}(\nabla f) = -\operatorname{div} Q_\varphi + \nabla \operatorname{tr} Q_\varphi. \tag{3.11}$$

Combining (3.3), $\operatorname{div} q_\varphi = \frac{1}{2}\operatorname{div}\tau_\varphi^2$, and 3.11 yields (3.6). If $\operatorname{tr} q_\varphi$ is constant, $\nabla \operatorname{tr} q_\varphi = 0$ and we get (3.7). The fact that $\operatorname{tr} q_\varphi$ is constant on homogeneous spaces follows from observing that it is a constant multiple of scal_φ, which is constant on homogeneous spaces (or one can simply note that q_φ is G-invariant to get $\operatorname{tr} q_\varphi = 0$). $\square$

Lemma 3.2.9 ([[Gri21], Corollary 3.2])**.** *For any constant trace, divergence-free 2-tensor q, the gradient solitons of its flow has the property that* $\mathrm{Ric}(\nabla f) = 0$.

Definition 3.2.10. A homogeneous G_2-structure (M, φ) is *Laplacian flow diagonal* if the $\mathrm{Aut}(M, \varphi)$-invariant Laplacian flow solution $\varphi(t)$ starting at φ satisfies the following property: at some $p \in M$, there is an orthonormal basis β with respect to $\langle \cdot, \cdot \rangle_\varphi$ at T_pM such that $Q_\varphi(t)$ is diagonal with respect to β for all t.

Remark 3.2.11. For homogeneous Laplacian solitons, $(M = G/K, \varphi)$ being Laplacian flow diagonal is equivalent to it being an algebraic soliton (see [[Lau17a], Theorem 4.10]).

Corollary 3.2.12. *Let G be a Lie group with closed G_2-structure φ that is Laplacian flow diagonal with respect to an orthogonally nice orthonormal basis $(e_i)_i$. Suppose* Ric_φ *is diagonal with respect to $(e_i)_i$.*

1. *If $(\varphi, \nabla f, \lambda)$ is a gradient Laplacian soliton, then G must be a product metric $\mathbb{R}^k \times N$ with f constant on N.*
2. *If in addition the kernel of the Ricci tensor is trivial, then φ is not a gradient soliton.*

Proof. By the last statement of Proposition 3.2.6, τ_φ^2 is diagonal. Since $(e_i)_i$ is an orthogonally nice orthonormal basis, we get $\operatorname{div} \tau_\varphi^2 = 0$ by Proposition 3.2.6. Thus (1) follows from the Structure Theorem. To show (2), note that the Key Lemma gives that $\mathrm{Ric}_\varphi(\nabla f) = 0$. Since $\ker \mathrm{Ric}_\varphi = 0$, it must be that $\nabla f = 0$. Hence f is constant, a contradiction. $\square$

Remark 3.2.13. Corollary 3.2.12 can be useful in determining the structure of a homogeneous closed gradient Laplacian soliton without having to compute $\operatorname{div} \tau_\varphi^2$ explicitly.

3.3 Some related consequences of the gradient Laplacian soliton equation

Definition 3.3.1. G_2-structures $(\mathfrak{g}_1, \psi_1)$ and $(\mathfrak{g}_2, \psi_2)$ are said to be *equivalent*, denoted $(\mathfrak{g}_1, \psi_1) \simeq (\mathfrak{g}_2, \psi_2)$, if there is a Lie algebra isomorphism $h : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $h \cdot \psi_1 = \psi_2$. Moreover, we say that G_2 structures are *homothetic* if there is a $c \in \mathbb{R}^*$ such that $(\mathfrak{g}_1, \psi_1) \simeq (\mathfrak{g}_2, c\psi_2)$.

We show that if two G_2-structures on the same Lie algebra are equivalent or homothetic, then one is a gradient Laplacian soliton if and only if the other is. This is needed for Theorem 4.1.1.

Proposition 3.3.2. *If ψ_1, ψ_2 are positive and if either $(\mathfrak{g}, \psi_1) \simeq (\mathfrak{g}, \psi_2)$ or $(\mathfrak{g}, \psi_1) \simeq (\mathfrak{g}, c\psi_2)$ for some $c \in \mathbb{R}^*$, then ψ_1 is a gradient Laplacian soliton if and only if ψ_2 is.*

Proof. For any diffeomorphism $\varphi \in \mathrm{Diff}(M)$, tensor T, and vector field X, we have

$$\varphi^*(\mathscr{L}_X T) = \mathscr{L}_{\varphi^* X}(\varphi^* T)$$

(see exercise 1.23 in [CLN06]). Also, if $f : M \to \mathbb{R}$, we have

$$\varphi^*(\nabla^g f) = \nabla^{\varphi^* g}(f \circ \varphi).$$

Suppose $(\mathfrak{g}, \psi_1) \simeq (\mathfrak{g}, \psi_2)$ and ψ_2 is a gradient Laplacian soliton, i.e., $\Delta_{\psi_2}\psi_2 = \lambda\psi_2 + \mathscr{L}_{\nabla^{g_{\psi_2}} f}\psi_2$ for some potential function f. Since $(\mathfrak{g}, \psi_1) \simeq (\mathfrak{g}, \psi_2)$, there is a Lie algebra isomorphism $h : \mathfrak{g} \to \mathfrak{g}$ in $\mathrm{Aut}(\mathfrak{g})$ such that $h \cdot \psi_2 = \psi_1$ [Note: $h \in \mathrm{Diff}(\mathfrak{g})$ as any linear isomorphism of vector spaces is smooth]. We know for any geometric structure γ, $h \cdot \gamma = (h^{-1})^*\gamma$. Moreover, [[Nic18], Lemma 2.2

(ii)(a)] states that for any $h \in \mathrm{Aut}(\mathfrak{g})$, $\Delta_{h\cdot\psi} h\cdot\psi = h\cdot\Delta_\psi\psi$. Putting these together, we get

$$\begin{aligned}
\Delta_{\psi_1}\psi_1 &= \Delta_{h\cdot\psi_2}(h\cdot\psi_2) = h\cdot\Delta_{\psi_2}\psi_2 = h\cdot(\lambda\psi_2 + \mathscr{L}_{\nabla^{g_{\psi_2}} f}\psi_2)\\
&= \lambda(h\cdot\psi_2) + h\cdot(\mathscr{L}_{\nabla^{g_{\psi_2}} f}\psi_2) = \lambda\psi_1 + (h^{-1})^*(\mathscr{L}_{\nabla^{g_{\psi_2}} f}\psi_2)\\
&= \lambda\psi_1 + \mathscr{L}_{(h^{-1})^*(\nabla^{g_{\psi_2}} f)}((h^{-1})^*\psi_2) = \lambda\psi_1 + \mathscr{L}_{\nabla^{(h^{-1})^* g_{\psi_2}}(f\circ h^{-1})}(h\cdot\psi_2)\\
&= \lambda\psi_1 + \mathscr{L}_{\nabla^{g_{\psi_1}}(f\circ h^{-1})}\psi_1,
\end{aligned}$$

where in the last equality we used $(h^{-1})^* g_{\psi_2} = h\cdot g_{\psi_2} = g_{h\cdot\psi_2} = g_{\psi_1}$ for any $h \in \mathrm{GL}(\mathfrak{g})$. Thus ψ_1 is also a gradient Laplacian soliton. If instead ψ_1 is a gradient Laplacian soliton, the same argument with h^{-1} in place of h gives that ψ_2 is also a gradient soliton.

Suppose $(\mathfrak{g},\psi_1) \simeq (\mathfrak{g}, c\psi_2)$ for some $c \in \mathbb{R}^*$ and that ψ_1 is a gradient Laplacian soliton. Since $(\mathfrak{g},\psi_1) \simeq (\mathfrak{g}, c\psi_2)$, there is some Lie algebra isomorphism $h : \mathfrak{g} \to \mathfrak{g}$ in $\mathrm{Aut}(\mathfrak{g})$ such that $h\cdot\psi_1 = c\psi_2$. By [[Nic18], Lemma 2.2 (ii)(b)], $\Delta_{c\psi} c\psi = c^{\frac{1}{3}}\Delta_\psi\psi$. Then

$$\begin{aligned}
c^{\frac{1}{3}}\Delta_{\psi_2}\psi_2 &= \Delta_{c\psi_2}(c\psi_2) = \Delta_{h\cdot\psi_1}(h\cdot\psi_1) = h\cdot(\Delta_{\psi_1}\psi_1) = h\cdot(\lambda\psi_1 + \mathscr{L}_{\nabla^{g_{\psi_1}} f}\psi_1)\\
&= \lambda(h\cdot\psi_1) + (h^{-1})^*(\mathscr{L}_{\nabla^{g_{\psi_1}} f}\psi_1) = c\lambda\psi_2 + \mathscr{L}_{(h^{-1})^*(\nabla^{g_{\psi_1}} f)}((h^{-1})^*\psi_1)\\
&= c\lambda\psi_2 + \mathscr{L}_{\nabla^{(h^{-1})^* g_{\psi_1}}(f\circ h^{-1})}(h\cdot\psi_1) = c\lambda\psi_2 + \mathscr{L}_{\nabla^{c^{2/3} g_{\psi_2}}(f\circ h^{-1})}(c\psi_2)\\
&= c\lambda\psi_2 + c\mathscr{L}_{\nabla^{g_{\psi_2}}(f\circ h^{-1})}\psi_2,
\end{aligned}$$

where we used [[Nic18], Lemma 2.1(iii)]

$$(h^{-1})^* g_{\psi_1} = h\cdot g_{\psi_1} = g_{h\cdot\psi_1} = g_{c\psi_2} = c^{2/3} g_{\psi_2}$$

in the second to last equality and $\nabla^{c^{2/3} g_{\psi_2}} = \nabla^{g_{\psi_2}}$ as $c^{2/3} > 0$ in the last. So

$$c^{\frac{1}{3}}\Delta_{\psi_2}\psi_2 = c\lambda\psi_2 + c\mathscr{L}_{\nabla^{g_{\psi_2}}(f\circ h^{-1})}\psi_2$$

if and only if

$$\Delta_{\psi_2}\psi_2 = c^{\frac{2}{3}}\lambda\,\psi_2 + \mathscr{L}_{\nabla^{g_{\psi_2}}(c^{\frac{2}{3}} f\circ h^{-1})}\psi_2.$$

Thus $(\psi_2, \nabla^{g_{\psi_2}} c^{\frac{2}{3}}(f\circ h^{-1}), c^{\frac{2}{3}}\lambda)$ is a gradient Laplacian soliton. Similar arguments show if ψ_2 is a gradient soliton, then so is ψ_1.

$\square$

We include for completeness some consequences of the gradient Laplacian soliton equation (2.9) on closed G_2-structures (see [Bry06, LW17], and [HN21] for more details). Note these results are immediate consequences of formulas in Section 9 of [LW17].

Lemma 3.3.3. *Let (M,φ) be a closed G_2-structure. If $(\varphi, \nabla f, \lambda)$ is a gradient Laplacian soliton, then*

1. $\Delta f = -\frac{7}{3}\lambda - \frac{2}{3}\operatorname{scal}_\varphi$
2. $\nabla\Delta f = -\frac{2}{3}\nabla\operatorname{scal}_\varphi$
3. $\nabla f \lrcorner T = 0$ *where* $T = -\frac{1}{2}\tau_\varphi$.

Proof. Taking the trace of (2.9) yields

$$\Delta f = -\operatorname{scal}_\varphi + \frac{7}{3}\operatorname{scal}_\varphi - \frac{1}{2}\operatorname{tr}\tau_\varphi^2 - \frac{7}{3}\lambda.$$

By Proposition 2.1.15,

$$-\frac{1}{2}\operatorname{tr}\tau_\varphi^2 = |\tau_\varphi|^2 = -2\operatorname{scal}_\varphi.$$

Substituting this back into the preceding equation and collecting the scalar curvature terms yields (1). Taking the derivative of (1) yields (2). (3) follows from the discussion in Section 9 of [LW17].

$\square$

Corollary 3.3.4. *If* $(\varphi, \nabla f, \lambda)$ *is a homogeneous closed gradient Laplacian soliton and* τ_φ^2 *is divergence-free, then*

$$\frac{1}{2} D_X \|\nabla f\|^2 = \frac{1}{3}(\mathrm{scal}_\varphi - \lambda) g(\nabla f, X) \quad \forall X \in TM.$$

If in addition $\|\nabla f\| =$ *constant, then* $\lambda = \mathrm{scal}_\varphi$. *Since* $\mathrm{scal}_\varphi \leq 0$ *for closed* G_2*-structures, the soliton is either shrinking or steady.*

Proof. The gradient Laplacian soliton equation (2.9) yields

$$\mathrm{Hess}\, f(\nabla f, X) = -\mathrm{Ric}_\varphi(\nabla f, X) - \frac{1}{2}\tau_\varphi^2(\nabla f, X) + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda) g(\nabla f, X).$$

Since τ_φ^2 is divergence-free, by the Key Lemma we have $\mathrm{Ric}_\varphi(\nabla f) = -\frac{1}{2} \operatorname{div} \tau_\varphi^2 = 0$. By Lemma 3.3.3 (3), we have $\nabla f \lrcorner \tau_\varphi = 0$ and so $\tau_\varphi^2(\nabla f, X) = g(\tau_\varphi^2(\nabla f), X) = -g(\tau_\varphi(\nabla f), \tau_\varphi(X)) = 0$. By [[Pet16], Proposition 3.2.1 (3)], $\mathrm{Hess}\, f(\nabla f, X) = \frac{1}{2} D_X \|\nabla f\|^2$ for all $X \in TM$. Putting these items together in the soliton equation gives the desired formula. If $\|\nabla f\| =$ constant, then the left-hand side of the formula is zero while the right-hand side is $\frac{1}{3}(\mathrm{scal}_\varphi - \lambda)\|\nabla f\|^2$. Since $\|\nabla f\|^2 > 0$ as f is non-constant, it follows that $\lambda = \mathrm{scal}_\varphi$. □

Remark 3.3.5. Without the homogeneous assumption, the formula in Corollary 3.3.4 is $2^{-1} D_X \|\nabla f\|^2 = -g(\nabla \operatorname{tr} q_\varphi, X) + 3^{-1}(\mathrm{scal}_\varphi - \lambda) g(\nabla f, X)$.

4 | Eliminating Gradient Solitons

4.1 Eliminating gradient Laplacian solitons on nilpotent Lie groups

A first application of the Structure Theorem for homogeneous closed gradient Laplacian solitons is in "eliminating gradient solitons" as discussed in the introduction. We prove the following.

Theorem 4.1.1. *The closed Laplacian solitons φ_i on N_i for $i = 2,3,4,5,6,7$ found by Nicolini in* [Nic18] *are not gradient up to homethetic G_2-structures. If N_1 does admit a gradient Laplacian soliton, then it must be a Gaussian.*

Tables consisting of relevant data for each nilpotent Lie algebra $(\mathfrak{n}_i, \varphi_i)$ are provided. Note that τ_{φ_i}, hence $\tau^2_{\varphi_i}$, are obtained with respect to bases and corresponding structure equations from the tables in [Nic18]. We first compute the divergence of $\tau^2_{\varphi_i}$. We then consider divergence-free and non-divergence-free cases separately in the proof of Theorem 4.1.1. Lastly, we show the closed G_2-structure $(\mathfrak{n}_{12}, \varphi_{12})$ constructed in [FFM16] is not gradient.

Notation: N is as in the structure theorem while N with a subscript, N_i, denotes the nilpotent Lie group with corresponding nilpotent Lie algebra $\mathfrak{n}_i$.

Table 4.1: $(\mathfrak{n}_2(1,1),\varphi_2)$ & $(\mathfrak{n}_3(1,1-c,c),\varphi_3)$

	$(\mathfrak{n}_2(1,1),\varphi_2)$	$(\mathfrak{n}_3(1,1-c,c),\varphi_3),\ 0<c<1/2$
Ric_{φ_i}	$-\mathrm{Diag}(1,\frac{1}{2},\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},0)$	$\frac{1}{2}\mathrm{Diag}(-2+2c-c^2,-1-c^2,-1+2-2c^2,1,(-1+c)^2,c^2,0)$
τ_{φ_i}	$-e^{35}+e^{26}$	$-ce^{16}+(1-c)e^{25}-e^{34}$
$\tau^2_{\varphi_i}$	$\mathrm{Diag}(0,-1,-1,0,-1,-1,0)$	$\mathrm{Diag}(-c^2,-(1-c)^2,-1,-1,-(1-c)^2,-c^2,0)$
Q_{φ_i}	$\frac{1}{3}\mathrm{Diag}(-2,-2,-2,1,1,1,1)$	$\frac{1-c+c^2}{3}\mathrm{Diag}(-2,-2,-2,1,1,1,1)$
λ_i	5	$5(1-c+c^2)$

Table 4.2: $(\mathfrak{n}_4(\sqrt{2},1,\sqrt{2},1),\varphi_4)$ & $(\mathfrak{n}_6(\sqrt{2},\sqrt{2},1,1),\varphi_6)$

	$(\mathfrak{n}_4(\sqrt{2},1,\sqrt{2},1),\varphi_4)$	$(\mathfrak{n}_6(\sqrt{2},\sqrt{2},1,1),\varphi_6)$
Ric_{φ_i}	$\mathrm{Diag}(-2,-2,\frac{1}{2},-1,-\frac{1}{2},\frac{3}{2},\frac{1}{2})$	$\mathrm{Diag}(-3,-1,-1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$
τ_{φ_i}	$-\sqrt{2}e^{34}+\sqrt{2}e^{16}-e^{56}+e^{37}$	$-\sqrt{2}e^{34}+\sqrt{2}e^{25}-e^{56}+e^{47}$
$\tau^2_{\varphi_i}$	$\begin{pmatrix} -2 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & -1 \end{pmatrix}$
Q_{φ_i}	$\begin{pmatrix} -2 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 1 \end{pmatrix}$
λ_i	9	9

Table 4.3: $(\mathfrak{n}_5(\sqrt{2},1,1,\sqrt{2}),\varphi_5)$ & $(\mathfrak{n}_7(-4,2,2,\sqrt{6},\sqrt{6}),\varphi_7)$

	$(\mathfrak{n}_5(\sqrt{2},1,1,\sqrt{2}),\varphi_5)$	$(\mathfrak{n}_7(-4,2,2,\sqrt{6},\sqrt{6}),\varphi_7)$
Ric_{φ_i}	$\mathrm{Diag}(-2,-2,\frac{1}{2},-\frac{1}{2},-1,\frac{1}{2},\frac{3}{2})$	$\mathrm{Diag}(-10,-10,3,11,-1,-1,-10)$
τ_{φ_i}	$\tau_{\varphi_5}=-e^{46}+e^{37}-\sqrt{2}e^{35}+\sqrt{2}e^{17}$	$\tau_{\varphi_7}=-2e^{15}+2e^{26}-\sqrt{6}e^{36}+\sqrt{6}e^{45}-4e^{47}$,
$\tau^2_{\varphi_i}$	$\begin{pmatrix} -2 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -3 \end{pmatrix}$	$\begin{pmatrix} -4 & 0 & 0 & 2\sqrt{6} & 0 & 0 & 0 \\ 0 & -4 & 2\sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{6} & -6 & 0 & 0 & 0 & 0 \\ 2\sqrt{6} & 0 & 0 & -22 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 & 4\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 4\sqrt{6} & 0 & -16 \end{pmatrix}$
Q_{φ_i}	$\begin{pmatrix} -2 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 0 & -\frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & 0 & 17 & \frac{\sqrt{6}}{2} & 0 & -2 \\ 1 & 0 & 0 & -\frac{\sqrt{6}}{2} & 5 & 0 & 0 \\ 0 & -1 & \frac{\sqrt{6}}{2} & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -4 \end{pmatrix}$
λ_i	9	54

4.1.1 Computing $\operatorname{div}\tau^2_{\varphi_i}$ for $(\mathfrak{n}_i,\varphi_i)$, $i=1,\ldots,7$

Proposition 4.1.2. *Let $(\mathfrak{n}_i,\varphi_i)$, $i=1,\ldots,7$ be the nilpotent Lie algebras admitting closed Laplacian solitons φ_i found in* [Nic18]. *The square of the torsion 2-form $\tau^2_{\varphi_i}$ is divergence-free for $i=1,2,3,4,6$ and not divergence-free for $i=5,7$.*

Proof. The torsion 2-form $\tau_{\varphi_1}=0$. More precisely, the exterior derivatives obtained from trivial brackets are all 0, hence $\tau_{\varphi_1}=-*d*\varphi_1=0$ regardless of what φ_1 is. It follows that $\tau^2_{\varphi_1}=0$, hence its divergence is 0.

The torsion 2-forms τ_{φ_i} for all other cases can be obtained via $\tau_{\varphi_i}=-*d*\varphi_i$ (see [Nic18]). We obtain $\tau^2_{\varphi_i}$ from the skew-symmetric matrix representation of τ_{φ_i} with respect to $(e^j)_j$. We claim that when $A=\tau^2_{\varphi_i}$, the sum (3.5b) $=0$ for each $i=1,\ldots,7$.

Proof (3.5b) = 0. Unimodular Lie groups can be characterized by the property that there is a basis $(X_j)_j$ such that $\mathrm{tr}(\mathrm{ad}_X)=\sum_j g(\mathrm{ad}_X(X_j),X_j)=0$ for any X. As nilpotent Lie groups are unimodular, it follows that ad_X is trace-free in all cases $\mathfrak{n}_i$. Moreover, the Lie brackets for $\mathfrak{n}_i$, $i=1,\ldots,7$, are all orthogonally nice. Thus either of these two conditions imply the sum (3.5b) $= \sum_j g(A(\nabla_{e_j} e_j), \cdot) = 0$ whenever A is symmetric. To see this, first note by symmetry of A we have $g(A(\nabla_{e_j} e_j), \cdot) = g(\nabla_{e_j} e_j, A(\cdot))$ as $A^* = A^t = A$ over $\mathbb{R}$. Then for any $U = \sum_k U^k e_k$,

$$\begin{aligned}
\sum_j g(\nabla_{e_j} e_j, A(U)) = \sum_j g(\nabla_{e_j} e_j, A(\sum_k U^k e_k)) &= \sum_j \sum_k U^k g(\nabla_{e_j} e_j, A(e_k)) \\
&= \sum_k U^k \left(\sum_j g(\nabla_{e_j} e_j, A(e_k)) \right) \\
&= \sum_k U^k \left(\sum_j g(\nabla_{e_j} e_j, \sum_\ell a_{\ell k} e^k_\ell) \right) \\
&= \sum_k U^k \left(\sum_\ell \sum_j a_{\ell k} g(\nabla_{e_j} e_j, e^k_\ell) \right) = \sum_k U^k \sum_\ell a_{\ell k} \operatorname{tr}(\operatorname{ad}_{e^k_\ell}),
\end{aligned}$$

where the last expression is 0 as $\operatorname{tr}(\operatorname{ad}_{e^k_\ell}) = 0 \ \forall\, k, \ell$. On the other hand, whenever $(e_j)_j$ is orthogonally nice, $\nabla_{e_j} e_j = 0$ for all j and so $\sum_j g(A(\nabla_{e_j} e_j), U) = \sum_j g(\nabla_{e_j} e_j, A(U)) = 0$. $\square$

It remains to compute the sum (3.5a) when $A = \tau^2_{\varphi_i}$ for $i = 2, \ldots, 7$. Computing (3.5a) when $A = \tau^2_{\varphi_i}$ amounts to computing the terms $\nabla_{e_j}(\tau^2_\varphi(e_j))$. This depends on both the matrix representation of $\tau^2_{\varphi_i}$ with respect to the bases $(e_j)_j$ as well as the derivatives $\nabla_{e_j} e_k$.

Since $\tau^2_{\varphi_2}$, $\tau^2_{\varphi_3}$ are diagonal and the corresponding bases are orthogonally nice, by Proposition 3.2.6 both $\operatorname{div} \tau^2_{\varphi_2} = 0$ and $\operatorname{div} \tau^2_{\varphi_3} = 0$. We include computation of $\operatorname{div} \tau^2_{\varphi_5}$. That $\operatorname{div} \tau^2_{\varphi_i} = 0$ for $i = 4, 6$ and $\operatorname{div} \tau^2_{\varphi_7}(U, V) = -16\sqrt{6} g(e_2, U)$ follows from similar computations as for $\operatorname{div} \tau^2_{\varphi_5}$. We include tables of derivatives for $\mathfrak{n}_i$ and computations of $\operatorname{div} \tau_{\varphi_i}$ in Appendix B.1.

Remark 4.1.3. The derivatives for each case $\mathfrak{n}_i$ are obtained from the Koszul formula and the structure equations as in the tables of [Nic18]. [[Nic18], Lemma 3.10] states that for $\mathfrak{n}_5(a, b, c, d)$, where a, b, c, d are the structure constants, φ_5 is closed if and only if $a = d$ and $b = c$. The lemma further states that if $a^2 = 2b^2$, then $(\mathfrak{n}_5(a, b, b, a), \varphi_5)$ is a semi-algebraic soliton, hence is a Laplacian soliton. We prove the result for $(b = 1, a = \sqrt{2})$ and note that it holds for general (a, b) where $a^2 = 2b^2$ by scaling. We do the same for all other cases $\mathfrak{n}_i$.

Table of derivatives for $\mathfrak{n}_5(\sqrt{2}, 1, 1, \sqrt{2})$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{\sqrt{2}}{2}e_3$	$\frac{\sqrt{2}}{2}e_2-\frac{1}{2}e_6$	$-\frac{1}{2}e_7$	0	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$
2	$\frac{\sqrt{2}}{2}e_3$	0	$-\frac{\sqrt{2}}{2}e_1$	0	$-\frac{\sqrt{2}}{2}e_7$	0	$\frac{\sqrt{2}}{2}e_5$
3	$\frac{\sqrt{2}}{2}e_2+\frac{1}{2}e_6$	$-\frac{\sqrt{2}}{2}e_1$	0	0	0	$-\frac{1}{2}e_1$	0
4	$\frac{1}{2}e_7$	0	0	0	0	0	$-\frac{1}{2}e_1$
5	0	$\frac{\sqrt{2}}{2}e_7$	0	0	0	0	$-\frac{\sqrt{2}}{2}e_2$
6	$\frac{1}{2}e_3$	0	$-\frac{1}{2}e_1$	0	0	0	0
7	$\frac{1}{2}e_4$	$\frac{\sqrt{2}}{2}e_5$	0	$-\frac{1}{2}e_1$	$-\frac{\sqrt{2}}{2}e_2$	0	0

Case $(\mathfrak{n}_5, \varphi_5)$. We compute each term of the sum (3.5a):

$$\nabla_{e_1}(\tau^2_{\varphi_5}(e_1)) = \nabla_{e_1}(-2e_1-\sqrt{2}e_3) = -\sqrt{2}(\frac{\sqrt{2}}{2}e_2-\frac{1}{2}e_6) = -e_2+\frac{\sqrt{2}}{2}e_6$$
$$\nabla_{e_2}(\tau^2_{\varphi_5}(e_2)) = \nabla_{e_2}(0) = 0$$
$$\nabla_{e_3}(\tau^2_{\varphi_5}(e_3)) = \nabla_{e_3}(-\sqrt{2}e_1-3e_3) = -\sqrt{2}(\frac{\sqrt{2}}{2}e_2+\frac{1}{2}e_6) = -e_2-\frac{\sqrt{2}}{2}e_6$$
$$\nabla_{e_4}(\tau^2_{\varphi_5}(e_4)) = \nabla_{e_4}(-e_4) = 0$$
$$\nabla_{e_5}(\tau^2_{\varphi_5}(e_5)) = \nabla_{e_5}(-2e_5+\sqrt{2}e_7) = \sqrt{2}(-\frac{\sqrt{2}}{2}e_2) = -e_2$$
$$\nabla_{e_6}(\tau^2_{\varphi_5}(e_6)) = \nabla_{e_6}(-e_6) = 0$$
$$\nabla_{e_7}(\tau^2_{\varphi_5}(e_7)) = \nabla_{e_7}(\sqrt{2}e_5-3e_7) = \sqrt{2}(-\frac{\sqrt{2}}{2}e_2) = -e_2$$

Thus

$$\operatorname{div}\tau^2_{\varphi_5}(U,V) = \sum_{i=1}^{7} g(\nabla_{e_i}(\tau^2_{\varphi_5}(e_i)),U) = g(-4e_2,U) = -4g(e_2,U),$$

which is nonzero whenever the e_2 component of U is nonzero.

□

Remark 4.1.4. Recall that the Ricci soliton equation is $\frac{1}{2}\mathscr{L}_X g = -\operatorname{Ric}_g + \lambda g$ where the G-invariant symmetric 2-tensor as in [PW22] is $q = -\operatorname{Ric}_g + \lambda g$. On homogeneous spaces, q is always divergence-free as $\operatorname{div}\operatorname{Ric}_g = \frac{1}{2}D\operatorname{scal}_g = 0$. Griffin in [Gri21] studies homogeneous Bach solitons, which also have that the corresponding G-invariant symmetric 2-tensor q is divergence-free. What makes homogeneous Laplacian solitons and the Laplacian flow interesting is the fact that it is the first setting we have encountered in which the corresponding G-invariant symmetric 2-tensor $q = q(\tau^2_\varphi)$ is not always divergence-free, e.g., $\operatorname{div}\tau^2_{\varphi_i} \neq 0$ for $i = 5,7$ as in Proposition 4.1.2. Al-though we show these solitons are not gradient in the next section, the fact that their corresponding q is not divergence-free leaves open the possibility that there may be gradient Laplacian solitons with $\operatorname{div}\tau_\varphi^2 \neq 0$.

4.1.2 Eliminating gradient solitons on $(\mathfrak{n}_i, \varphi_i)$, $i = 1, \ldots, 7$

We now prove Theorem 4.1.1.

Divergence-free cases: $\operatorname{div}\tau_{\varphi_i}^2 = 0$**.**

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_1, \varphi_1)$. The Lie brackets $[\cdot,\cdot]$ with respect to orthonormal basis $(e_i)_{i=1}^7$ for $\mathfrak{n}_1$ are trivial, hence the covariant derivatives $\nabla_{e_i}e_j$ are trivial. So for some closed G_2-structure φ_1, Ric_{φ_1}, τ_{φ_1}, and $\mathrm{scal}_{\varphi_1}$ are 0. Since $\mathrm{Ric}_{\varphi_1} = 0$ and N_1 is homogeneous, it follows the space is flat. Suppose $(\varphi_1, \nabla f, \lambda_1)$ is a gradient Laplacian soliton. Since $\operatorname{div}\tau_{\varphi_1}^2 = 0$, the Structure Theorem yields $N_1 = N \times \mathbb{R}^k$ where f is constant on N. Note $\nabla f \in T_p\mathbb{R}^k \subseteq \ker(\mathrm{Ric}_{\varphi_1}) = \mathrm{Span}(e_i)_{i=1}^7 = T_p\mathbb{R}^7$, i.e., ∇f can be written as a linear combination of elements from $(e_i)_{i=1}^7$ and $k \leq 7$. The gradient Laplacian soliton equation

$$\operatorname{Hess} f = -\frac{1}{3}\lambda_1 g$$

is diagonal with respect to basis $(e_i)_{i=1}^7$ and so $\operatorname{Hess} f$ must also be diagonal, i.e., $\nabla_i\nabla_j f = 0$ whenever $i \neq j$. Equating matrix entries, we get $\nabla_i\nabla_i f = -\dfrac{\lambda_1}{3}$ for each i and so the potential function f must be of the form

$$\begin{aligned} f(x,y,z,s,u,v,w) = &-\frac{\lambda_1}{6}(x^2+y^2+z^2+s^2+u^2+v^2+w^2) \\ &-(\alpha_1 x+\alpha_2 y+\alpha_3 z+\alpha_4 s+\alpha_5 u+\alpha_6 v+\alpha_7 w)-\beta \end{aligned}$$

which is a Gaussian soliton; (x,y,z,s,u,v,w) are coordinates with respect to $(e_i)_{i=1}^7$.

□

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_2(1,1), \varphi_2)$. By Proposition 4.1.2 $\operatorname{div}\tau_{\varphi_2}^2 = 0$. If $(\varphi_2, \nabla f, \lambda_2)$ is a gradient Laplacian soliton, then by the Structure Theorem $N_2 = N \times \mathbb{R}^k$ where f is constant on N. Note $\nabla f \in T_p\mathbb{R}^k \subset \ker \mathrm{Ric}_{\varphi_2} = \mathrm{Span}\{e_4, e_7\}$ and so $k \leq 2$. In an appropriate basis $\mathscr{B}$, $\mathrm{Hess}\, f\big|_N = 0$ and so the restriction of the gradient Laplacian soliton equation to N with respect to $\mathscr{B}$ becomes $q_{\varphi_2}\big|_N = -\frac{1}{3}\lambda_2 g_N$. But this means $-\frac{2}{3} = q_{\varphi_2}\big|_N(e_1,e_1) = q_{\varphi_2}\big|_N(e_6,e_6) = \frac{1}{3}$, a contradiction. Thus $(\varphi_2, X, \lambda_2)$ cannot be gradient Laplacian soliton. $\square$

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_3(1,1-c,c),\varphi_3)$. By Proposition 4.1.2 $\mathrm{div}\,\tau^2_{\varphi_3} = 0$. If $(\varphi_3, \nabla f, \lambda_3)$ is a gradient Laplacian soliton, then by the Structure Theorem $N_3 = N \times \mathbb{R}^k$ where f is constant on N. Note $\nabla f \in T_p\mathbb{R}^k \subset \ker \mathrm{Ric}_{\varphi_3} = \mathrm{Span}\{e_7\}$ and so $k = 1$. In an appropriate basis $\mathscr{B}$, $\mathrm{Hess}\, f\big|_N = 0$ and so the restriction of the gradient Laplacian soliton equation to N with respect to $\mathscr{B}$ becomes $q_{\varphi_3}\big|_N = -\frac{1}{3}\lambda_3 g_N$. But this means $-\frac{2(1-c+c^2)}{3} = q_{\varphi_2}\big|_N(e_1,e_1) = q_{\varphi_2}\big|_N(e_6,e_6) = \frac{1-c+c^2}{3}$, a contradiction. Thus $(\varphi_3, X, \lambda_3)$ cannot be gradient Laplacian soliton. $\square$

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_4(\sqrt{2},1,\sqrt{2},1),\varphi_4)$. Suppose $(\varphi_4, \nabla f, \lambda_4)$ is a gradient Laplacian soliton. By Proposition 4.1.2 $\mathrm{div}\,\tau^2_{\varphi_4} = 0$. In the context of a $(-2q_{\varphi_4})$-flow, we get $-2q_{\varphi_4}$ is also divergence-free. Furthermore, $\mathrm{tr}(-2q_{\varphi_4})$ is constant as N_4 is homogeneous. We apply Lemma 3.2.9 to the $(-2q_{\varphi_4})$-flow to get the potential function f satisfies $\mathrm{Ric}_{\varphi_4}(\nabla f) = 0$. But Ric_{φ_4} has trivial kernel and so $\nabla f = 0$. Thus f is constant, a contradiction. Therefore $(\mathfrak{n}_4, X, \lambda_4)$ cannot be a gradient Laplacian soliton. $\square$

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_6(\sqrt{2},\sqrt{2},1,1),\varphi_6)$. Suppose $(\varphi_6, \nabla f, \lambda_6)$ is a gradient Laplacian soliton. By Proposition 4.1.2 $\mathrm{div}\,\tau^2_{\varphi_6} = 0$. In the context of a $(-2q_{\varphi_6})$-flow, we get $-2q_{\varphi_6}$ is also divergence-free. Furthermore, $\mathrm{tr}(-2q_{\varphi_6})$ is constant as N_6 is homogeneous. We apply Lemma 3.2.9 to the $(-2q_{\varphi_6})$-flow to get the potential function f satisfies $\mathrm{Ric}_{\varphi_6}(\nabla f) = 0$. But Ric_{φ_6} has trivial kernel and so $\nabla f = 0$. Thus f is constant, a contradiction. Therefore $(\mathfrak{n}_6, X, \lambda_6)$ cannot be a gradient Laplacian soliton. $\square$

Non-divergence-free cases: $\mathrm{div}\,\tau^2_{\varphi_i} \neq 0$**.**

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_5,\varphi_5)$. Suppose $(\varphi_5, \nabla f, \lambda_5)$ is a gradient Laplacian soliton. By Proposition 4.1.2 $\mathrm{div}\,\tau^2_{\varphi_5} \neq 0$ and so by the Structure Theorem (N_5, φ_5) has either structure 2(a) or

2(b). As Ric_{φ_5} has trivial kernel, N_5 cannot split as a product and so the structure must be as in 2(a).

Suppose (N_5, φ_5) is a one-dimensional extension where $f = ar + b$. By the Key Lemma the potential function f satisfies

$$g(\mathrm{Ric}(\nabla f), \cdot) = -\frac{1}{2}\operatorname{div}\tau^2_{\varphi_5}(\cdot) = 2g(e_2, \cdot),$$

where the last equality follows from $\operatorname{div}\tau^2_{\varphi_5}(\cdot) = -4g(e_2, \cdot)$ as computed in the proof of Proposition 4.1.2. Note $\mathrm{Ric}_{\varphi_5}(\nabla f) = 2e_2$. Since Ric_{φ_5} is diagonal with respect to $(e_i)_i$ with nonzero diagonal entries, $\nabla f = c_2 e_2$. Substituting $\mathrm{Ric}_{\varphi_5}(\nabla f) = -2c_2 e_2$ in the Key Lemma yields $c_2 = -1$, and so $\nabla f = -e_2$. Since $f = ar + b$, it follows that $e_2 = \pm\nabla r$.

Assume $\nabla r = e_2$. Applying the (1,1)-tensor version of the gradient Laplacian soliton equation (2.9) to $\nabla r = e_2$ and noting that $\operatorname{Hess} f(\nabla r) = a \operatorname{Hess} r(\nabla r) = 0$, we get

$$0 = -\mathrm{Ric}_{\varphi_5}(e_2) - \frac{1}{2}\tau^2_{\varphi_5}(e_2) + \frac{1}{3}(\mathrm{scal}_{\varphi_5} - \lambda_5)I(e_2). \tag{4.1}$$

Since $\tau^2_{\varphi_5}(e_2) = 0$, (4.1) becomes

$$\mathrm{Ric}_{\varphi_5}(e_2) = -\frac{1}{3}(\mathrm{scal}_{\varphi_5} - \lambda_5)I(e_2).$$

Substituting $\mathrm{scal}_{\varphi_5} = -3$ and $\lambda_5 = 9$ yields

$$-2e_2 = \mathrm{Ric}_{\varphi_5}(e_2) = -4I(e_2) = -4e_2,$$

from which it follows that $-2 = -4$, a contradiction. By similar arguments, we arrive at a contradiction when $\nabla r = -e_2$. [Note: There cannot be two distinct contraction constants satisfying the soliton equation for if $(\varphi, X_1, \lambda_1)$ and $(\varphi, X_2, \lambda_2)$ both satisfy (2.8) and $\lambda_1 \neq \lambda_2$, then $L_X g_\varphi = L_{X_2 - X_1} g_\varphi = 2(\lambda_2 - \lambda_1) g_\varphi = c g_\varphi$ for some nonzero constant $c \in \mathbb{R}$ and non-trivial vector field X, which would imply the space is flat.] $\square$

Proof of Theorem 4.1.1 Case $(\mathfrak{n}_7, \varphi_7)$. Suppose $(\varphi_7, \nabla f, \lambda_7)$ is a gradient Laplacian soliton. By Proposition 4.1.2 $\operatorname{div} \tau^2_{\varphi_7} \neq 0$ and so by the Structure Theorem (N_7, φ_7) has either structure 2(a) or 2(b). As Ric_{φ_7} has trivial kernel, N_7 cannot split as a product and so the structure must be as in 2(a).

Suppose (N_7, φ_7) is a one-dimensional extension where $f = ar + b$. By the Key Lemma, the potential function f satisfies

$$g(\mathrm{Ric}_{\varphi_7}(\nabla f), \cdot) = -\frac{1}{2} \operatorname{div} \tau^2_{\varphi_7} = -\frac{1}{2}(-16\sqrt{6} g(e_2, \cdot)) = 8\sqrt{6} g(e_2, \cdot),$$

where we used $\operatorname{div} \tau^2_{\varphi_7}(\cdot) = -16\sqrt{6} g(e_2, \cdot)$. Note $\mathrm{Ric}_{\varphi_7}(\nabla f) = 8\sqrt{6} e_2$. Since Ric_{φ_7} is diagonal with respect to $(e_i)_i$ with all nonzero diagonal entries, it must be that $\nabla f = c_2 e_2$. Solving

$$-10 c_2 = g(\mathrm{Ric}_{\varphi_7}(c_2 e_2), e_2) = g(\mathrm{Ric}_{\varphi_7}(\nabla f), e_2) = 8\sqrt{6}$$

yields $c_2 = -\frac{4\sqrt{6}}{5}$, i.e., $\nabla f = -\frac{4\sqrt{6}}{5} e_2$. Since $f = ar + b$, we have $a \nabla r = \nabla f = -\frac{4\sqrt{6}}{5} e_2$ and so taking the norms shows that $a = \pm \frac{4\sqrt{6}}{5}$.

Assume $a = \frac{4\sqrt{6}}{5}$ so that $\nabla r = -e_2$. Applying the (1,1)-tensor version of the gradient Laplacian soliton equation (2.9) to $\nabla r = -e_2$ and noting that $\operatorname{Hess} f(\nabla r) = a \operatorname{Hess} r(\nabla r) = 0$, we get

$$0 = -\mathrm{Ric}_{\varphi_7}(-e_2) - \frac{1}{2} \tau^2_{\varphi_7}(-e_2) + \frac{1}{3}(\mathrm{scal}_{\varphi_7} - \lambda_7) I(-e_2),$$

from which we get

$$\mathrm{Ric}_{\varphi_7}(e_2) = -\frac{1}{2} \tau^2_{\varphi_7}(e_2) + \frac{1}{3}(\mathrm{scal}_{\varphi_7} - \lambda_7) e_2.$$

Substituting $\mathrm{scal}_{\varphi_7} = -18$ and $\lambda = 54$ yields

$$-10 e_2 = -\frac{1}{2}(-4 e_2 + 2\sqrt{6} e_3) - 24 e_2,$$

from which we get $-10 = -22$, a contradiction. By similar arguments, we arrive at a contradiction when $a = -\frac{4\sqrt{6}}{5}$. $\square$

Some final remarks on the Proof of Theorem 4.1.1. When f is constant, the possible gradient Laplacian solitons are of the form $(\varphi, 0, \lambda)$. In these cases, $\operatorname{Hess} f = 0$ and the gradient soliton equation of type $(1,1)$ is equivalent to $Q_\varphi = -3^{-1}\operatorname{Diag}(\lambda, \ldots, \lambda)$. None of the matrix expressions Q_{φ_i} for $i = 2, \ldots, 7$ satisfy this equality and thus such gradient Laplacian solitons cannot occur on $\mathfrak{n}_i$ for $i = 2, \ldots, 7$. For $i = 1$, $Q_\varphi = 0$ and so we must have $\lambda = 0$, i.e., the soliton is steady; φ is torsion-free as $\mathfrak{n}_1$ has trivial structure. Thus the only non-trivial gradient solitons on $\mathfrak{n}_1$ are Guassian as shown above. Lastly, the result is up to homothetic G_2-structures by Proposition 3.3.2. $\square$

4.1.3 $(\mathfrak{n}_{12}, \varphi_{12})$

Proposition 4.1.5. *The closed G_2-structure φ_{12} on N_{12} as constructed in* [FFM16] *is not gradient up to homothetic G_2-structures.*

Proof. Let $(e_i)_i$ be the basis with structure equations

$$\mathfrak{n}_{12} = (0,0,0,\frac{3}{6}e^{12}, \frac{1}{4}e^{23} + \frac{\sqrt{3}}{12}e^{13}, -\frac{\sqrt{3}}{12}e^{23} - \frac{1}{4}e^{13},\\ -\frac{\sqrt{3}}{6}e^{34} + \frac{\sqrt{3}}{12}e^{25} + \frac{1}{4}e^{26} + \frac{\sqrt{3}}{12}e^{16} - \frac{1}{4}e^{15})$$

and closed G_2 structure given by

$$\varphi_{12} = -e^{124} + e^{135} + e^{167} - e^{236} + e^{257} + e^{347} - e^{456}$$

as in [FFM16]. This basis and its corresponding structure equations are obtained from the canonical one for $\mathfrak{n}_{12}$ (see [[FFM16], Theorem 3.1] or [[Nic18], Table 1]). The structure constants and exterior derivatives are:

$$[e_1, e_2] = -\frac{\sqrt{3}}{6}e_4, [e_2, e_3] = \frac{1}{4}e_5, [e_1, e_3] = -\frac{\sqrt{3}}{12}e_5, [e_2, e_3] = \frac{\sqrt{3}}{12}e_6,$$

$$[e_1,e_3]=\frac{1}{4}e_6,[e_3,e_4]=\frac{\sqrt{3}}{6}e_7,[e_2,e_5]=-\frac{\sqrt{3}}{12}e_7,[e_2,e_6]=-\frac{1}{4}e_7,$$

$$[e_1,e_6]=-\frac{\sqrt{3}}{12}e_7,[e_1,e_5]=\frac{1}{4}e_7.$$

and

$$de^1=de^2=de^3=0,\;\; de^4=\frac{\sqrt{3}}{6}e^{12},de^5=-\frac{1}{4}e^{23}+\frac{\sqrt{3}}{12}e^{13},de^6=-\frac{\sqrt{3}}{12}e^{23}-\frac{1}{4}e^{13}$$

$$de^7=-\frac{\sqrt{3}}{6}e^{34}+\frac{\sqrt{3}}{12}e^{25}+\frac{1}{4}e^{26}+\frac{\sqrt{3}}{12}e^{16}-\frac{1}{4}e^{15}.$$

As shown in [FFM16], the basis is orthonormal with respect to the associated metric $g_{\varphi_{12}}$ and the Ricci tensor is given by

$$\mathrm{Ric}_{\varphi_{12}}=\mathrm{Diag}\left(-\frac{1}{8},-\frac{1}{8},-\frac{1}{8},0,0,0,\frac{1}{8}\right)=-\frac{1}{4}I+\frac{1}{8}D,$$

where $D=\mathrm{Diag}(1,1,1,2,2,2,3)$, i.e., $g_{\varphi_{12}}$ is a nilsoliton. In [[FFM16], Section 4], it is shown that $\mathfrak{n}_{12}$ is Laplacian flow diagonal with respect to $(e_i)_i$ and at $t=0$, $\varphi_{12}(0)=\varphi_{12}$. In other words $Q_{\varphi_{12}}(t)$ is diagonal along the Laplacian flow in the time interval stated in [FFM16]. In particular, $Q_{\varphi_{12}}$ is diagonal with respect to $(e_i)_i$ at $t=0$. Hence $\tau^2_{\varphi_{12}}$ is diagonal by Proposition 3.2.6. The basis $(e_i)_i$ is orthogonally nice. So if $(\varphi_{12},\nabla f,\lambda_{12})$ is a gradient Laplacian soliton, then by Corollary 3.2.12 (1) (N_{12},φ_{12}) must be a product metric $N\times\mathbb{R}^k$ where f is constant on N. But since $\ker\mathrm{Ric}_{\varphi_{12}}\neq\{0\}$, we cannot use Corollary 3.2.12 (2). We compute $\tau^2_{\varphi_{12}}$:

$$\begin{aligned}
\varphi_{12} &= -e^{124}+e^{135}+e^{167}-e^{236}+e^{257}+e^{347}-e^{456}\\
*\varphi_{12} &= e^{3567}-e^{2467}+e^{2345}+e^{1457}+e^{1346}+e^{1256}+e^{1237}\\
d*\varphi_{12} &= -\frac{1}{2}e^{12347}-\frac{1}{2}e^{12456}\\
d\varphi_{12} &= -\frac{1}{2}e^{56}+\frac{1}{2}e^{37}\\
\tau_{\varphi_{12}} &= -*d*\varphi_{12} = \frac{1}{2}e^{56}-\frac{1}{2}e^{37}
\end{aligned}$$

Then

$$\tau^2_{\varphi_{12}} = \mathrm{Diag}\left(0,0,-\frac{1}{4},0,-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right).$$

Since f is a function on $\mathbb{R}^k$, $\nabla f \in T_p\mathbb{R}^k$. So $\mathrm{Ric}_{\varphi_{12}}(\nabla f) = 0$, i.e., $\nabla f \in \ker(\mathrm{Ric}_{\varphi_{12}})$, which is contained in $\mathrm{Span}\{e_4, e_5, e_6\}$. So $k \leq 3$. We obtain

$$Q_{\varphi_{12}} = \frac{1}{24}\mathrm{Diag}(-1,-1,-4,2,-1,-1,1)$$

with respect to $(e_i)_i$. Since f is constant on N, $\mathrm{Hess} f\big|_N = 0$. The gradient Laplacian soliton equation becomes $q_{\varphi_{12}}\big|_N = -\frac{1}{3}\lambda_{12} g_{\varphi_{12}}\big|_N$. But this implies $-1 = q_{\varphi_{12}}\big|_N(e_1,e_1) = q_{\varphi_{12}}\big|_N(e_3,e_3) = -4$, a contradiction. $\square$

4.2 Observations on product metrics $N^{7-k} \times \mathbb{R}^k$

We collect some immediate observations from the soliton equation in the product case, i.e., the case when $\mathrm{div}\,\tau^2 = 0$. Recall that for products, $(T_{(p,q)}(N^{7-k} \times \mathbb{R}^k), g) = (T_pN^{7-k} \oplus T_q\mathbb{R}^k, g = g_N + g_{\mathbb{R}^k})$.

Proposition 4.2.1. *If $(\varphi, \nabla f, \lambda)$ is a homogeneous closed gradient Laplacian soliton with τ^2 divergence-free, i.e., $M = N^{7-k} \times \mathbb{R}^k$, and f is a function only on $\mathbb{R}^k$, we get the following:*

1. $0 = (-\mathrm{Ric}_{g_N} - \frac{1}{2}\tau^2 + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda)g)(X_i, X_j)$ *for any* $X_i, X_j \in T_pN$.

2. $\tau^2(X,Y) = 0$ *for any* $X \in T_pN$ *and* $Y \in T_q\mathbb{R}^k$.

3. $\mathrm{Hess}\, f(Y_i,Y_j) = -\frac{1}{2}\tau^2(Y_i,Y_j) + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda)g(Y_i,Y_j)$ *for any* $Y_i, Y_j \in T_q\mathbb{R}^k$. *If in addition* τ^2 *is a multiple of the metric, then* f *is a Gaussian.*

4. $2^{-1}D_X\|\nabla f\|^2 = cg(\nabla f, X)$ *where* $c = 3^{-1}(\mathrm{scal}_\varphi - \lambda)$ *is constant. Hence* f *is an isoperimetric function as* $\|\nabla f\|^2 = \phi(f)$.

Proof. For (1), note that since f is constant on N, $\mathrm{Hess}\, f(X_i,X_j) = 0$ for any $X_i,X_j \in T_pN$. Furthermore, since $T_q\mathbb{R}^k \subset \ker \mathrm{Ric}_\varphi$, $\mathrm{Ric}_\varphi(X_i,X_j) = \mathrm{Ric}_{g_N}(X_i,X_j)$ for any $X_i,X_j \in T_pN$. Putting this together in equation (2.9) gives (1). For (2), note $\mathrm{Hess}\, f(X,Y) = g(\nabla_X \nabla f, Y) = 0$ since f is constant on N and $\frac{1}{3}(\mathrm{scal}_\varphi - \lambda)g(X,Y) = 0$ since $X \perp Y$. Also, $\mathrm{Ric}_\varphi(X,Y) = g(\mathrm{Ric}_\varphi(X),Y) = g(X, \mathrm{Ric}_\varphi(Y)) = 0$ since $Y \in T_q\mathbb{R}^k \subset \ker \mathrm{Ric}_\varphi$. Putting this together in equation (2.9) yields $-\frac{1}{2}\tau^2(X,Y) = 0$, from which we get $\tau^2(X,Y) = 0$. The equation in item (3) is a direct application of (2.9) with $\mathrm{Ric}_\varphi(Y_i,Y_j) = 0$ since $Y_i,Y_j \in T_q\mathbb{R}^k \subset \ker \mathrm{Ric}_\varphi$. If τ^2 is a multiple of the metric, i.e., $\tau^2 = cg$, then from the equation in item (3), we have

$$\mathrm{Hess}\, f = -\frac{1}{2}cg + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda)g = (-\frac{1}{2}c + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda))g = Kg$$

on $\mathbb{R}^k$, where $K = -\frac{1}{2}c + \frac{1}{3}(\mathrm{scal}_\varphi - \lambda)$ is constant. Thus $f = \frac{K}{2}|x|^2$ on $\mathbb{R}^k$, i.e., f is a Gaussian. Item (4) is Corollary 3.3.4. □

We make some further observations when f is a Gaussian.

Corollary 4.2.2. *Suppose* $(\varphi, \nabla f, \lambda)$ *is a homogeneous closed gradient Laplacian soliton with* τ^2 *divergence-free, i.e.,* $M = N^{7-k} \times \mathbb{R}^k$, *and* f *is a Gaussian. Then*

1. $\mathrm{Hess}\, f(Y_i,Y_j) = cg(Y_i,Y_j)$ *where constant* $c = 3^{-1}(\mathrm{scal}_\varphi - \lambda)$ *for any* $Y_i,Y_j \in T_q\mathbb{R}^k$.

2. $\tau^2(Y_i,Y_j) = 0$ *for any* $Y_i,Y_j \in T_q\mathbb{R}^k$.

3. *$\tau^2(X_i,X_j) = (-2\,\mathrm{Ric}_{g_N} + 2cg)(X_i,X_j)$ for any $X_i, X_j \in T_pN$ and*

$$\tau^2 = \begin{pmatrix} -2\,\mathrm{Ric}_{g_N} + 2cI_{7-k} & \\ & 0_{k\times k} \end{pmatrix}$$

with respect to basis $(X_1,\ldots,X_{7-k},Y_1,\ldots,Y_k)$.

4. $$\lambda = -\left(\frac{2+k}{7-k}\right)\mathrm{scal}_{g_N} = -\left(\frac{2+k}{7-k}\right)\mathrm{scal}_{\varphi}\,.$$

Hence the gradient soliton is either steady or expanding; N^{7-k} must have constant nonpositive scalar curvature; and if φ is closed non-torsion-free, then N^{7-k} must have constant negative scalar curvature. It also follows from (3) and (4) that τ^2 is determined by $\dim N$ *and g_N.*

Proof. Since f is a Gaussian on the Euclidean factor only, we have $\mathrm{Hess}\, f(Y_i,Y_j) = cg(Y_i,Y_j)$ for some constant c and $Y_i, Y_j \in T_q\mathbb{R}^k$. Setting $Y_i = Y_j = \nabla f$ and noting that $\nabla f \lrcorner \tau = 0$, by Proposition 4.2.1 (3) we get $c\|\nabla f\|^2 = 3^{-1}(\mathrm{scal}_\varphi - \lambda)\|\nabla f\|^2$. It follows that $c = 3^{-1}(\mathrm{scal}_\varphi - \lambda)$ since $\|\nabla f\| > 0$ as f is assumed to be non-constant. Using Proposition 4.2.1 (3) again yields $\tau^2(Y_i,Y_j) = 0$ for any $Y_i, Y_j \in T_q\mathbb{R}^k$. Furthermore, substituting c in Proposition 4.2.1 (1) yields

$$\tau^2(X_i,X_j) = (-2\,\mathrm{Ric}_{g_N} + 2cg)(X_i,X_j) \;\; \forall\, X_i, X_j \in T_pN.$$

Thus τ^2 has the matrix representation as in (3) with respect to the basis $(X_1,\ldots,X_{7-k},Y_1,\ldots,Y_k)$.

Taking the trace yields

$$\mathrm{tr}\,\tau^2 = -2\,\mathrm{scal}_{g_N} + \frac{2}{3}(7-k)(\mathrm{scal}_\varphi + \lambda).$$

Recall $-\frac{1}{2}\,\mathrm{tr}\,\tau^2 = -2\,\mathrm{scal}_\varphi$ and so $\mathrm{tr}\,\tau^2 = 4\,\mathrm{scal}_\varphi$. Putting this together with $\mathrm{scal}_\varphi = \mathrm{scal}_{g_N} + \mathrm{scal}_{g_{\mathbb{R}^k}} = \mathrm{scal}_{g_N}$ gives

$$4\,\mathrm{scal}_{g_N} = -2\,\mathrm{scal}_{g_N} + \frac{2}{3}(7-k)(\mathrm{scal}_{g_N} - \lambda),$$

from which we get $\lambda = -\left(\frac{2+k}{7-k}\right) \mathrm{scal}_{g_N} = -\left(\frac{2+k}{7-k}\right) \mathrm{scal}_\varphi$. Since $\mathrm{scal}_\varphi \leq 0$ for closed G_2-structures, it follows that $\lambda \geq 0$, i.e., the soliton is either steady or expanding. From the expression for λ, the fact that $\mathrm{scal}_\varphi \leq 0$ also shows $\mathrm{scal}_N \leq 0$. $\square$

Remark 4.2.3. The $0_{k\times k}$ block of τ^2 in Corollary 4.2.2 (3) may be nonzero when f is not Gaussian.

We rule that τ^2 cannot be a constant multiple of the metric. If so, then it would follow from Proposition 4.2.1 (3) that f is a Gaussian on $\mathbb{R}^k$ and that g_N is an Einstein metric. But if $\tau^2 = cg$, for some nonzero $c \in \mathbb{R}$, then $\tau^2(\nabla f, \nabla f) = c\|\nabla f\|^2$. Since $\tau(\nabla f, \nabla f) = 0$ by Lemma 3.3.3 (3), it would follow that $\nabla f = 0$, a contradiction as we are considering non-trivial gradient solitons.

An open question remains whether there are any homogeneous closed gradient Laplacian solitons on products other than the Gaussian. If such non-trivial examples do exist, it would be desirable to obtain a classification of homogeneous closed gradient solitons on products. A more fundamental question arises of whether there are homogeneous closed G_2-structures on product metrics $N^{7-k} \times \mathbb{R}^k$. [There are known examples outside the homogeneous setting (see [HN21] and [HKP22]).] We investigate this question for our choice of model (fundamental) 3-form φ from Chapter 2. The main observation is that to find closed G_2-structures on product metrics $N \times \mathbb{R}^k$, one should consider $\dim N \geq 4$.

Case $N^1 \times \mathbb{R}^6$: We assume $(e_i)_{i=1}^7$ is an basis such that $\{e_1\}$ is the basis for T_pN^1 and such that the 3-form is the model form $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$ with respect $(e_i)_i$. Note that on product $N^1 \times \mathbb{R}^6$, the structure is given by $de^i = 0$ for all i. It is easy to see that $d\varphi = 0$ and so $N^1 = S^1$ or $\mathbb{R}$, i.e., the space is flat.

Case $N^2 \times \mathbb{R}^5$: Let $(e_i)_{i=1}^7$ be a basis for $T_{(p,q)}(N^2 \times \mathbb{R}^5)$ where $\{e_1, e_2\}$ and $\{e_3, e_4, e_5, e_6, e_7\}$ are bases for T_pN^2 and $T_q\mathbb{R}^5$, respectively. We have $\{e^{12}\}$ is a basis for $\Lambda^2(T_p^*N^2)$ and that the structure is given by

$$de^1 = ae^{12}, \;\; de^2 = be^{12}, \; a, b \in \mathbb{R} \;\text{ and }\; de^i = 0 \,\forall\, i \neq 1, 2.$$

Suppose the model 3-form φ is with respect to the basis $(e_i)_i$. Then

$$d\varphi = ae^{1234} - ae^{1247} - be^{1236} - be^{1245} = 0$$

if and only if $a = b = 0$. Thus in order for φ to be closed in this basis, the space must be flat.

Case $N^3 \times \mathbb{R}^4$: Let $(e_i)_{i=1}^7$ be a basis for $T_{(p,q)}(N^3 \times \mathbb{R}^k)$ where $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6, e_7\}$ are bases for T_pN^3 and $T_q\mathbb{R}^4$, respectively. We have $\{e^{12}, e^{13}, e^{23}\}$ is a basis for $\Lambda^2(T_p^*N^3)$ and that the structure is given by

$$\begin{cases} de^1 = a_{11}e^{12} + a_{12}e^{13} + a_{13}e^{23} \\ de^2 = a_{21}e^{12} + a_{22}e^{13} + a_{23}e^{23} \\ de^3 = a_{31}e^{12} + a_{32}e^{13} + a_{33}e^{23} \\ de^4 = de^5 = de^6 = de^7 = 0. \end{cases}$$

By straightforward computations,

$$\begin{aligned} d\varphi = {} & (-a_{12} - a_{23})e^{1237} + a_{31}e^{1247} + a_{32}e^{1347} + a_{33}e^{2347} \\ & + (a_{11} - a_{33})e^{1235} - a_{11}e^{1246} - a_{12}e^{1346} - a_{13}e^{2346} \\ & + (-a_{21} - a_{32})e^{1236} - a_{21}e^{1245} - a_{22}e^{1345} - a_{23}e^{2345} = 0 \end{aligned}$$

if and only if $a_{ij} = 0$ for all $i, j = 1, 2, 3$. We obtain again that in order for φ to be closed, the space must be flat.

Case $N^4 \times \mathbb{R}^3$: Let $(e_i)_{i=1}^7$ be a basis for $T_{(p,q)}(N^4 \times \mathbb{R}^3)$ where $\{e_1, e_2, e_3, e_4\}$ and $\{e_5, e_6, e_7\}$ are bases for T_pN^4 and $T_q\mathbb{R}^3$, respectively. We have $\{e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34}\}$ is a basis for $\Lambda^2(T_p^*N^4)$

and that the structure is given by

$$\begin{cases} de^1 = a_{11}e^{12} + a_{12}e^{13} + a_{13}e^{14} + a_{14}e^{23} + a_{15}e^{24} + a_{16}e^{34} \\ de^2 = a_{21}e^{12} + a_{22}e^{13} + a_{23}e^{14} + a_{24}e^{23} + a_{25}e^{24} + a_{26}e^{34} \\ de^3 = a_{31}e^{12} + a_{32}e^{13} + a_{33}e^{14} + a_{34}e^{23} + a_{35}e^{24} + a_{36}e^{34} \\ de^4 = a_{41}e^{12} + a_{42}e^{13} + a_{43}e^{14} + a_{44}e^{23} + a_{45}e^{24} + a_{46}e^{34} \\ de^5 = de^6 = de^7 = 0. \end{cases}$$

Then by straightforward computations we get

$$\begin{aligned} d\varphi &= (-a_{13} - a_{25} + a_{31})e^{1247} + (-a_{12} - a_{24} - a_{41})e^{1237} + (a_{16} + a_{34} + a_{45})e^{2347} \\ &\quad + (-a_{26} + a_{32} + a_{43})e^{1347} + (a_{11} - a_{34} - a_{42})e^{1235} + (-a_{13} - a_{36} - a_{22})e^{1345} \\ &\quad + (-a_{15} - a_{24} + a_{46})e^{2345} + (-a_{35} - a_{21} - a_{43})e^{1245} + (-a_{11} + a_{45} - a_{33})e^{1246} \\ &\quad + (-a_{12} + a_{46} + a_{23})e^{1346} + (-a_{14} + a_{25} + a_{36})e^{2346} + (a_{44} - a_{21} - a_{32})e^{1236} = 0, \end{aligned}$$

which yields an undetermined system of 12 linear equations in 24 unknowns. We do not know if $N^4 \times \mathbb{R}^3$ can admit closed G_2-structures.

Case $N^5 \times \mathbb{R}^2$: A non-trivial example of a homogeneous product of the form $N^5 \times \mathbb{R}^2$ admitting a closed G_2-structure is the space $K^7 = H(1,2) \times \mathbb{R}^2$ constructed in [Fer87] where

$$H(1,2) = \left\{ \begin{pmatrix} I_2 & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| X = (x_1, x_2)^t, Z = (z_1, z_2)^t, x_i, z_j, y \in \mathbb{R} \right\}$$

is the generalized Heisenberg group. It is known that K^7 is a connected nilpotent Lie group. We show there is a Lie algebra isomorphism taking the dual basis $(f_j)_j$ to the basis $(e_i)_i$ for $(\mathfrak{n}_2, \varphi_2)$ in

[Nic18]. We label the left-invariant 1-forms on K^7:

$$f^1 = dx_1, f^1 = dx_2, f^3 = dy, f^4 = dz_1 - x_1 dy, f^5 = dz_2 - x_2 dy, f^6 = du_1, f^7 = du_2.$$

The structure on K^7 is

$$df^4 = -f^{13}, \;\; df^5 = -f^{23}, \;\; \text{and } df^j = 0 \; \forall \; j \neq 4,5,$$

or equivalently, $[f_1, f_3] = f_4, [f_2, f_3] = f_5$, and $[f_s, f_t] = 0$ for all other s,t. The metric is given by $\sum_j (f^j)^2$. Let $(e_i)_i$ be the basis for $(\mathfrak{n}_2, \varphi_2)$ which has structure $[e_1, e_2] = -e_5$ and $[e_1, e_3] = -e_6$. Then the Lie algebra isomorphism $h : (\mathfrak{n}_2, \varphi_2) \to (K^7, \varphi_{K^7})$ taking

$$e_1 \mapsto f_3, e_2 \mapsto f_1, e_3 \mapsto f_2, e_4 \mapsto f_7, e_5 \mapsto f_4, e_6 \mapsto f_5, e_7 \mapsto f_6$$

satisfies $h \cdot \varphi_2 = \varphi_{K^7}$ where $\varphi_{K^7} = -f^{147} + f^{257} + f^{156} + f^{246} + f^{345} + f^{123} - f^{367}$ is the closed G_2 structure on K^7. We do not know whether K^7 admits gradient Laplacian solitons.

Case $N^6 \times \mathbb{R}$: This is a special case of the construction of one-dimensional extensions discussed in the next section (see Remark 5.2.2).

5 | Almost abelian solvmanifolds admitting gradient solitons

5.1 One-dimensional extensions

A G-homogeneous space $(M = G/G_x, g)$ is a *one-dimensional extension* if there is a closed subgroup $H \subset G$ contaning G_x such that there is a surjective Lie group homomorphism $G \to (\mathbb{R}, +)$ with kernel H. The simplest case is when H is abelian. In this section, we study one-dimensional extensions admitting closed G_2-structures. We first recall the setup for one-dimensional extensions in more detail.

Let H be a Lie group and $(M = H/K, g)$ an H-homogeneous space. Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H and K, respectively. The family of automorphisms $\{\Phi_t\}_t \subset \mathrm{Aut}(H)$ such that $\Phi_t(K) = K$ induces a well defined family of diffeomorphisms $\{\phi_t\}_t \subset \mathrm{Diff}(H/K)$ given by

$$\phi_t(hK) = \Phi_t(h)K \quad \forall\, h \in H.$$

We fix an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{h} = \mathfrak{p} \oplus \mathfrak{k}$. We can identify $\mathfrak{p} \equiv T_xM$ via the orthogonal projection $\mathfrak{h} \to \mathfrak{p}$.

Now suppose H is a Lie group with $(N = H/K, h)$ a H-homogeneous space. Fix a derivation $D \in \mathrm{Der}(\mathfrak{h})$ that preserves K, an isotropy subgroup at some point $x \in N$. To obtain a one-dimensional extension of (N,h), we consider the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$$

with Lie bracket given by

$$\mathrm{ad}_\xi(X) = D(X) \;\text{ and }\; \mathrm{ad}_Y(X) = \mathrm{ad}^{\mathfrak{h}}_Y(X) \;\; \forall\, X,Y \in \mathfrak{h}.$$

Let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then

(i) $G \supset H$, a codimension one normal subgroup of G as $\mathrm{ad}_\xi(X) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$;

(ii) $G = H \ltimes \mathbb{R}$;

(iii) any $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{h} = \mathfrak{p} \oplus \mathfrak{k}$ yields a corresponding $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{k} = (\mathfrak{p} \oplus \mathbb{R}\xi) \oplus \mathfrak{k}$;

(iv) G-invariant metrics are identified with restrictions of $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{g}$ to $\mathfrak{q}$.

The G-homoegeneous space $(M = G/K, g)$ where the metric satisfies $g\big|_{\mathfrak{p}} = h$, $g(\xi, X) = 0$ for all $X \in \mathfrak{p}$, and $g(\xi,\xi) = 1$ is the one-dimensional extension of (N,h). The one-dimensional extensions obtained in this way are equivalent to the ones described at beginning of this section (see [PW22]).

The main result of this chapter is the following.

Theorem 5.1.1. *If $(\varphi, \nabla f, \lambda)$ is a closed non-torsion-free gradient Laplacian soliton on Lie group G_D with Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ and $\mathfrak{h}$ is a codimension-one abelian ideal, then it must be a product $N \times \mathbb{R}^k$ and f is constant on N.*

Proof Outline. Suppose G_D admits a gradient Laplacian soliton $(\varphi, \nabla f, \lambda)$. If G_D is not a product metric $N \times \mathbb{R}^k$ with f constant on N, then by the Structure Theorem the potential function is either of the form $f = ar + b$ or $f(x,y) = ar(x) + v(y)$ and either $\nabla r = \pm e_7$ or $\nabla r \neq \pm e_7$. If $\nabla r = \pm e_7$, then by Theorem 5.3.4, the space is flat, contradicting φ being closed non-torsion-free. If $\nabla r \neq \pm e_7$, then by Theorem 5.4.1, the space is also flat, contradicting φ being closed non-torsion-free. Thus by the Structure Theorem, G_D must be a product $N \times \mathbb{R}^k$ with f constant on N. $\square$

We now include some facts and set some notation needed for subsequent results of this chapter. Connections between Ricci solitons and Einstein metrics on such homogeneous spaces have been studied by He-Petersen-Wylie in [HPW15]. We will need [[HPW15], Lemma 2.9].

Lemma 5.1.2 ([HPW15], Lemma 2.9)**.** *The Ricci tensor of one-dimensional extensions (M,g) with Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ is given by*

1. $\mathrm{Ric}(\xi,\xi) = -\mathrm{tr}(S^2)$
2. $\mathrm{Ric}(X,\xi) = -\mathrm{div}(S)$
3. $\mathrm{Ric}(X,X) = \mathrm{Ric}^N(X,X) - (\mathrm{tr}\, S)h(S(X),X) - h([S,A](X),X),$

where $S = (D+D^t)/2$ and $A = (D-D^t)/2$, the symmetric and skew-symmetric parts of D, respectively.

Recall that an *almost-Hermitian structure* on a complex vector space $\mathfrak{h}$ is a pair (g,J) where J is an almost-complex structure on $\mathfrak{h}$, i.e., $J^2 = -\mathrm{id}$, and g is a metric such that $g(JX,JY) = g(X,Y)$ for any $X,Y \in \mathfrak{h}$. An $\mathrm{SU}(n)$-*structure* on a Lie algebra $\mathfrak{h}$ of dimension $2n$ is a triple (g,J,Ψ) such that (g,J) is an almost-Hermitian structure on $\mathfrak{h}$ and $\Psi = \rho^+ + i\rho^-$ is a complex volume $(n,0)$-form such that $(-1)^{n(n-1)/2}\left(\frac{i}{2}\right)^n \Psi \wedge \overline{\Psi} = \frac{1}{n!}\omega^n$, where $\overline{\Psi}$ is the conjugate of Ψ and ω is the Kähler 2-form corresponding to (g,J). It is known that if $\mathfrak{h}$ has an $\mathrm{SU}(3)$-structure, then there is an orthonormal basis $\{e_1,\ldots,e_6\}$ for $\mathfrak{h}$ such that the $\mathrm{SU}(3)$-structure is characterized by the pair of forms $(\omega,\rho^+) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ where

$$\omega = e^{12} + e^{34} + e^{56} \quad \text{and} \quad \rho^+ = e^{135} - e^{146} - e^{236} - e^{245}.$$

We note that the complex volume $(3,0)$-form $\Psi = \rho^+ + i\rho^-$ can be written as $\Psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$ and that its complex part is $\rho^- = -e^{246} + e^{235} + e^{145} + e^{136}$. In fact, classes of

SU(3)-structures can be defined in terms of (ω, ρ^+, ρ^-). We are interested in *symplectic half-flat* SU(3)-structures, i.e., the class of SU(3)-structures where (ω, ρ^+) are closed.

Now let $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ be the Lie algebra of Lie group M. If $\mathfrak{h}$ has an SU(3)-structure, then $\varphi = \omega \wedge e^7 + \rho^+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$ is a G_2-structure on M. Manero showed in [Man20] that $\mathfrak{h} \subset \mathfrak{g}$ admitting symplectic half-flat SU(3)-structure is equivalent to $\varphi = \omega \wedge e^7 + \rho^+$ being closed whenever D is the real representation of some $A \in \mathfrak{sl}(3, \mathbb{C})$. [Manero uses the classification of symplectic half-flat SU(3)-structures on solvable Lie algebra $\mathfrak{h}$ to construct new examples of closed G_2-structures in [Man20]].

If in addition $\mathfrak{h}$ is an abelian ideal, we call $\mathfrak{g}$ *almost abelian* and M an *almost abelian solvmanifold*. The Lie algebras for almost abelian solvmanifolds are completely determined by derivation $D : \mathfrak{h} \to \mathfrak{h}$ defined by

$$D(e_i) = \mathrm{ad}_{e_7}\big|_{\mathfrak{h}}(e_i) = [e_7, e_i\big|_{\mathfrak{h}}].$$

[Note: D coincides with A in [Lau17a].] For almost abelian solvmanifolds, $\varphi = \omega \wedge e^7 + \rho^+$ is closed if and only if the derivation D is the real representation of some element $A \in \mathfrak{sl}(3, \mathbb{C})$ (see [Fre13] and [Lau17a]).

Notation: We write (G_D, g) to denote the Lie group with Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ and $\mu_D = [\cdot, \cdot]_D$ to denote the Lie bracket of $\mathfrak{g}$ obtained from D. We write Ric_D and scal_D for the Ricci and scalar curvatures from the metric g, respectively, where g is the extension of the metric h on N. We also write τ_D and Q_D for the torsion form and unique symmetric operator from 3.2 corresponding to (G_D, φ) when φ is closed.

We now collect some facts regarding $d_{\mathfrak{g}} : \Lambda^\ell \mathfrak{g}^* \to \Lambda^{\ell+1} \mathfrak{g}^*$ and $d_{\mathfrak{h}} : \Lambda^k \mathfrak{h}^* \to \Lambda^{k+1} \mathfrak{h}^*$, the exterior derivatives (*Chevalley-Eilenberg differentials*) on $\mathfrak{g}$ and $\mathfrak{h}$, respectively, which will be used in later computations. These facts are included or deduced from [[Lau17a], Lemma 5.12]. It is known that if ω is an invariant k-form on a Lie group, then $\omega(X_1, ..., X_k)$ is a constant. In particular, if γ is a 1-form, then by [[Lee13], Proposition 2.19]

$$d\gamma(X, Y) = X\gamma(Y) - Y\gamma(X) - \gamma([X, Y]),$$

and $d\gamma(X,Y) = -\gamma([X,Y])$ if γ is invariant. Let $(e_i)_i$ be a basis of left-invariant vector fields and $(e^i)_i$ be its co-basis of left-invariant 1-forms. Since e^i is an invariant 1-form,

$$de^i(e_j, e_k) = -e^i([e_j, e_k]).$$

Then the structure equations are

$$[e_j, e_k] = c^{\ell}_{jk} e_{\ell},$$

where c^{ℓ}_{jk} are the structure constants. It follows that

$$de^i = -c^i_{jk} e^{jk}.$$

Also recall the map $\theta : \mathfrak{gl}(\mathfrak{h}) \to \mathrm{End}(\Lambda^k \mathfrak{h}^*)$ is the representation obtained as the derivative of the natural $\mathrm{GL}(\mathfrak{h})$-action $h \cdot \gamma = (h^{-1})^* \gamma$:

$$\theta(D)\gamma = -\gamma(D\cdot, \ldots, \cdot) - \cdots - \gamma(\cdot, \ldots, D\cdot) \;\; \forall\, \gamma \in \Lambda^k \mathfrak{h}^*.$$

Lemma 5.1.3. *Given Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$, where derivation $D : \mathfrak{h} \to \mathfrak{h}$ defined by $D(X) = [e_7, X]$ for all $X \in \mathfrak{h}$ determines the structure equations, the following holds:*

(i) $d_{\mathfrak{g}} e^7 = 0$.

(ii) $d_{\mathfrak{g}} \gamma = d_{\mathfrak{h}} \gamma + (-1)^k (\theta(D)(\gamma)) \wedge e^7$ *for any* $\gamma \in \Lambda^k \mathfrak{h}^*$.

(iii) $d_{\mathfrak{g}}(\gamma \wedge e^7) = d_{\mathfrak{h}} \gamma \wedge e^7$ *for any* $\gamma \in \Lambda^k \mathfrak{h}^*$.

(iv) For $\varphi = \omega \wedge e^7 + \rho^+$,

$$d_{\mathfrak{g}} \varphi = d_{\mathfrak{h}} \omega \wedge e^7 + (d_{\mathfrak{h}} \rho^+ - (\theta(D)\rho^+) \wedge e^7).$$

Thus φ is closed if and only if

$$d_{\mathfrak{h}}\omega = 0, d_{\mathfrak{h}}\rho^{+} = 0, \text{ and } \theta(D)\rho^{+} = 0.$$

(v) $\theta(D)\rho^{+} = 0$ *if and only if* $\theta(D)\rho^{-} = 0$*, if and only if* $D \in \mathfrak{sl}(3,\mathbb{C})$.

(vi) If $\mathrm{tr}(D) = 0$*, then* $\theta(D) *_{\mathfrak{h}} = - *_{\mathfrak{h}} \theta(D^t)$ *on* $\Lambda\mathfrak{h}^*$.

Proof. Statement (i) follows from the fact that $[e_i, e_j] \in \mathfrak{h}$ for all i, j; (ii) follows from [[Lee13], Proposition 2.19] and the fact that $\mathfrak{h}$ is not assumed to be abelian, hence the term $d_{\mathfrak{h}}\gamma$ appears; (iii) follows from (ii); (iv) follows from (i)-(iii). Statements (v) and (vi) are [[Lau17a], Lemma 5.12 (iv) and (v)]. □

Remark 5.1.4. Note the second term in (ii) is $d_A\gamma$ in [[Lau17a], Lemma 5.12 (ii)].

5.2 Matrix formulas for Q_φ and related operators

The rest of this chapter consists of observations culminating in the propositions used to prove Theorem 5.1.1. We fix some notation for the statements to follow. We consider only Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ where the derivation $D : \mathfrak{h} \to \mathfrak{h}$ given by $D(X) = [e_7, X]$ for all $X \in \mathfrak{h}$ is the real representation of some $A \in \mathfrak{sl}(3,\mathbb{C})$. Let S be the symmetrization of D, i.e., $S = (D + D^t)/2$. The hypothesis that the simply-connected Lie group (G_D, φ) is a closed G_2-structure in the following statements can be replaced by $(\mathfrak{h}, \omega, \rho^{+})$ being a symplectic half-flat $\mathrm{SU}(3)$-structure by the result of Manero. We first obtain general matrix formulas for the operators in 3.2 in the case of one-dimensional extensions. These matrix formulas generalize matrix formulas in the almost abelian case found in [Lau17a] to the not almost abelian case.

Proposition 5.2.1. *Suppose G_D has Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ and admits closed G_2-structure φ. Then with respect to an orthonormal basis $(e_i)_{i=1}^{7}$ where $\mathfrak{h} = \mathrm{Span}\{e_1, \ldots, e_6\}$, we have the following:*

1. $\tau_D^2 = \begin{pmatrix} -(D+D^t)^2 + J(D+D^t)B + BJ(D+D^t) + B^2 & \\ & 0 \end{pmatrix}$ *where* $B = *_{\mathfrak{h}} d_{\mathfrak{h}} \rho^-$.

2. *The matrix representation for* Ric_D *is*

$$\mathrm{Ric}_D = \begin{pmatrix} \mathrm{Ric}^H & \\ & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}[D,D^t] & \\ & -\frac{1}{4}\operatorname{tr}((D+D^t)^2) \end{pmatrix} - P,$$

$$\textit{where } P = \begin{pmatrix} 0 & \cdots & 0 & \operatorname{div}(S)(e_1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \operatorname{div}(S)(e_6) \\ \operatorname{div}(S)(e_1) & \cdots & \operatorname{div}(S)(e_6) & 0 \end{pmatrix}.$$

3. $\mathrm{scal}_D = \mathrm{scal}^H - \operatorname{tr}(S^2)$.

4. $Q_D = \begin{pmatrix} Q_H & \\ & q_{e_7} \end{pmatrix} - P$ *where*

$$\begin{aligned} Q_H = &\mathrm{Ric}_H + \frac{1}{2}[D,D^t] - \frac{1}{4}\operatorname{tr}(D+D^t)(D+D^t) \\ &+ \frac{1}{2}[-(D+D^t)^2 + J(D+D^t)B + BJ(D+D^t) + B^2] \\ &- \frac{1}{3}[\mathrm{scal}_H - (\operatorname{tr} S)^2 - \operatorname{tr}(S^2)] I_{6\times 6}, \end{aligned}$$

and $q_{e_7} = -\frac{1}{3}\mathrm{scal}_H + \frac{1}{3}(\operatorname{tr} S)^2 - \frac{2}{3}\operatorname{tr}(S^2)$.

Proof. Recall there exists an orthonormal basis $(e_i)_{i=1}^7$ such that $\mathfrak{h} = \mathrm{Span}(e_i)_{i=1}^6$ and $\varphi = \omega \wedge e^7 + \rho^+$. Since the structure is determined by D, we write $\tau_{\mathfrak{g}} = \tau_D$. We compute the intrinsic torsion $\tau_{\mathfrak{g}} = - * d_{\mathfrak{g}} * \varphi$ for closed G_2-structure φ using linear algebra properties of $*$ on $\Lambda^k \mathfrak{g}^*$ and $*_{\mathfrak{h}}$ on $\Lambda^k \mathfrak{h}^*$ from [[Lau17a], Lemma 5.11]. Taking the Hodge star of φ we get

$$*\varphi = *(\omega \wedge e^7) + *\rho^+ = (-1)^2 *_{\mathfrak{h}} \omega + *_{\mathfrak{h}} \rho^+ \wedge e^7 = \frac{1}{2}\omega \wedge \omega + \rho^- \wedge e^7,$$

where the first equality follows from [[Lau17a], Lemma 5.11 (i) and (ii)], while the second equality follows from [[Lau17a], Lemma 5.11 (iii) and (iv)]. Taking the differential yields

$$d_{\mathfrak{g}}(*\varphi) = d_{\mathfrak{g}}(\frac{1}{2}\omega\wedge\omega) + d_{\mathfrak{g}}(\rho^- \wedge e^7) = \frac{1}{2}d_{\mathfrak{g}}(\omega\wedge\omega) + d_{\mathfrak{h}}\rho^- \wedge e^7$$

as $\rho^- \in \Lambda^3\mathfrak{h}^*$. The first term in the last expression is

$$\begin{aligned}
\frac{1}{2}d_{\mathfrak{g}}(\omega\wedge\omega) &= \frac{1}{2}[d_{\mathfrak{h}}(\omega\wedge\omega) + (-1)^{4+1}(\theta(D)(\omega\wedge\omega))\wedge e^7] \\
&= \frac{1}{2}[(d_{\mathfrak{h}}\omega\wedge\omega + (-1)^2\omega\wedge d_{\mathfrak{h}}\omega) - (\theta(D)(\omega\wedge\omega))\wedge e^7] \\
&= -\frac{1}{2}(\theta(D)(\omega\wedge\omega))\wedge e^7 \\
&= -\theta(D)(\frac{1}{2}\omega\wedge\omega)\wedge e^7 \\
&= -\theta(D) *_{\mathfrak{h}} \omega\wedge e^7 \\
&= *_{\mathfrak{h}}\theta(D^t)\omega\wedge e^7,
\end{aligned}$$

where we used $d_{\mathfrak{h}}\omega = 0$ in the third equality, [[Lau17a], Lemma 5.11 (iii)] in the fifth, and [[Lau17a], Lemma 5.1.2 (vi)] in the last as $\operatorname{tr} D = 0$. If $d_{\mathfrak{h}}\rho^- \wedge e^7 \neq 0$, we get

$$d*\varphi = -*_{\mathfrak{h}}\theta(D^t)\omega\wedge e^7 + d_{\mathfrak{h}}\rho^- \wedge e^7.$$

Taking the Hodge star again gives

$$\begin{aligned}
d_{\mathfrak{g}}\varphi &= *(-*_{\mathfrak{h}}\theta(D^t)\omega\wedge e^7 + d_{\mathfrak{h}}\rho^- \wedge e^7) \\
&= (-1)^4 *_{\mathfrak{h}} (-*_{\mathfrak{h}}\theta(D^t)\omega) + (-1)^4 *_{\mathfrak{h}} d_{\mathfrak{h}}\rho^- \\
&= -*_{\mathfrak{h}}^2\theta(D^t)\omega + *_{\mathfrak{h}}d_{\mathfrak{h}}\rho^- \\
&= -(-1)^2\theta(D^t)\omega + *_{\mathfrak{h}}d_{\mathfrak{h}}\rho^- \\
&= -\theta(D^t)\omega + *_{\mathfrak{h}}d_{\mathfrak{h}}\rho^-.
\end{aligned}$$

where we used [[Lau17a], Lemma 5.11 (ii)] for the second equality and [[Lau17a], Lemma 5.11 (v)] for the second to last equality. Then

$$\tau_D = - * d_{\mathfrak{g}} * \varphi = \theta(D^t)\omega - *_{\mathfrak{h}} d_{\mathfrak{h}} \rho^-.$$

Note, $\rho^- \in \Lambda^3\mathfrak{h}^*$ implies $d_{\mathfrak{h}}\rho^- \in \Lambda^4\mathfrak{h}^*$. Taking the Hodge star yields $*_{\mathfrak{h}} d_{\mathfrak{h}}\rho^- \in \Lambda^2\mathfrak{h}^*$, i.e., $*_{\mathfrak{h}} d_{\mathfrak{h}}\rho^-$ is a 2-form on $\mathfrak{h}^*$ and thus can be written as a matrix with respect to 2-forms $(e^{ij})_{i,j}$; $i, j = 1, \ldots, 6$. Set $B := *_{\mathfrak{h}} d_{\mathfrak{h}}\rho^-$. Then the matrix representation of the torsion 2-form is

$$\tau_D = \begin{pmatrix} -J(D+D^t) & \\ & 0 \end{pmatrix} - \begin{pmatrix} B & \\ & 0 \end{pmatrix}.$$

Taking the square of τ_D yields the matrix representation of τ_D^2.

We use Lemma 5.1.2 to obtain Ric_D. Let g_H denote the metric on H corresponding to $\mathfrak{h}$. By Lemma 5.1.2 (1) with $\xi = e_7$, we have

$$\mathrm{Ric}_D(e_7, e_7) = -\operatorname{tr}(S^2).$$

Also, Lemma 5.1.2 (2) and symmetry of Ric_D yields

$$\mathrm{Ric}_D(e_7, e_i) = \mathrm{Ric}_D(e_i, e_7) = -\operatorname{div}(S)(e_i) \quad \forall\, i = 1, \ldots, 6.$$

Note $-[S,A] = [A,S] = \frac{1}{2}[D,D^t]$. Then for $i, j = 1, \ldots, 6$, Lemma 5.1.2 (3) gives

$$\begin{aligned} \mathrm{Ric}_D(e_i, e_j) = &\mathrm{Ric}^H(e_i, e_j) \\ &- \frac{1}{4}\operatorname{tr}(D+D^t) g_H((D+D^t)(e_i), e_j) + g_H(\frac{1}{2}[D,D^t](e_i), e_j). \end{aligned}$$

Putting these observations together and using the fact that $\operatorname{tr} D = \operatorname{tr} D^t = 0$ yields the matrix representation of Ric_D.

The expression for scal_D follows from taking the trace of Ric_D and the fact that $\mathrm{tr}[D, D^t] = 0$. The matrix formula for Q_D follows from equation 3.2 and the preceding results. □

Remark 5.2.2. A one-dimensional extension is a product metric if and only if the derivation D is anti-symmetric. By the discussion preceding Proposition 5.2.1, a product metric $N^6 \times \mathbb{R}$ has a closed G_2-structure if N^6 admits an anti-symmetric derivation and a symplectic half-flat $\mathrm{SU}(3)$-structure. We do not know of any examples of symplectic half-flat $\mathrm{SU}(3)$ structures that admit an anti-symmetric derivation. Moreover, in order for such a metric to be a closed gradient Laplacian soliton, by Proposition 5.2.1, we see that $\mathrm{Ric}^N - \frac{1}{2}B^2$ must be a constant multiple of the metric g_N.

5.3 Structure 2(b) with $\nabla r = e_7$

We first prove a proposition regarding closed gradient Laplacian solitons with potential function of the form $f = ar + b$ on Lie groups with Lie algebra $\mathfrak{h} \oplus_D \mathbb{R}e_7$, where $\mathfrak{h}$ is a general Lie subalgebra and $\nabla r = e_7$.

Proposition 5.3.1. *Suppose G_D has Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ and admits closed G_2-structure φ. If*

$$(\varphi = \omega \wedge e^7 + \rho^+, \nabla f, \lambda)$$

is a closed gradient Laplacian soliton where $f(r) = ar + b$ and $\nabla r = e_7$, then

1. $\mathrm{div}(S)(X) = 0 \;\; \forall X \in \mathfrak{h}$.

2. $\mathrm{div}(S)(\nabla r) = \mathrm{div}(S)(e_7) = \mathrm{tr}(S^2)$.

3. $\lambda = \mathrm{scal}^H + 2\,\mathrm{tr}(S^2)$.

4. $\Delta f = -2\,\mathrm{tr}(JSB) - \frac{1}{2}\,\mathrm{tr}\,B^2 - 4\,\mathrm{tr}(S^2)$.

Proof. By Proposition 5.2.1 (2) we have

$$\mathrm{Ric}_D = \begin{pmatrix} \mathrm{Ric}^H & \\ & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}[D, D^t] & \\ & -\frac{1}{4}\operatorname{tr}((D+D^t)^2) \end{pmatrix} - P,$$

where P is the matrix with 0 entries for all (i,j) except for the $(i,7),(7,i)$-entries, where it is $\operatorname{div}(S)(e_i)$ for $i = 1,\ldots,6$. Since $\operatorname{Hess} f(\nabla r) = a\nabla_{\nabla r}\nabla r = 0$, the gradient Laplacian soliton equation applied to $\nabla r = e_7$ becomes

$$0 = -\mathrm{Ric}_D(e_7) - \frac{1}{2}\tau_D^2(e_7) + \frac{1}{3}(\mathrm{scal}_D - \lambda)I(e_7) \tag{5.1}$$

By Lemma 5.1.2 (1) and equation (5.1), we get

$$\begin{aligned} -\operatorname{div}(S)(e_i) &= \mathrm{Ric}_D(e_7, e_i) \\ &= -\frac{1}{2}g(\underbrace{\tau_D^2(e_7)}_{=0}, e_i) + \frac{1}{3}(\mathrm{scal}_D - \lambda)\underbrace{g(e_7, e_i)}_{=0} = 0 \end{aligned}$$

for $i = 1,\ldots,6$. Thus

$$\operatorname{div}(S)(X) = 0 \quad \forall\, X \in \mathfrak{h}.$$

Recall that by [[HPW15], Proposition 2.7], the shape operator $T(X) = \nabla_X^{G_D} e_7$ is related to symmetrization S by $T = -S$. So with $e_7 = \nabla r$, we have

$$S(X) = -T(X) = -\nabla_X \nabla r.$$

Then

$$\operatorname{div}(S)(\nabla r) = -\mathrm{Ric}_D(\nabla r, \nabla r) - D_{\nabla r}(\Delta r) = -\mathrm{Ric}_D(\nabla r, \nabla r) = -\mathrm{Ric}_D(e_7, e_7) = \operatorname{tr}(S^2),$$

where the first equality follows from a Bochner formula (see Appendix B.4); the second equality follows from Δr being constant on one-dimensional extensions; and the last equality follows from Lemma 5.1.2 (1).

The expression for λ is obtained from solving for λ in equation (5.1) and using the expression for scal_D from Proposition 5.2.1 (3). Finally, Δf is obtained from taking the trace of the soliton equation $\mathrm{Hess}\, f = -Q_D - (1/3)\lambda I$ and substituting the expression in (3) for λ. □

Remark 5.3.2. Equation (5.1) in the proof of Proposition 5.3.1 requires that the gradient soliton has potential function of the form $f = ar + b$ and that $\nabla r = e_7$. In particular, the hypothesis $\nabla r = e_7$ is needed to obtain the explicit expressions for λ and Δf in terms of S.

We obtain the following corollary.

Corollary 5.3.3. *If $\mathfrak{h}$ is an abelian ideal in addition to the hypotheses of Proposition 5.3.1, then*

$$\lambda = -2\,\mathrm{scal}_D\,.$$

That is, such a gradient soliton must be expanding and is steady if and only if φ is torsion-free. Moreover,

(i) $\Delta f = 4\,\mathrm{scal}_D$*;*

(ii) $-\frac{1}{2}\operatorname{div}\tau_D^2(e_i) = \begin{cases} 0 & i = 1,\ldots,6 \\ -\mathrm{tr}(S^2) = \mathrm{scal}_D & i = 7; \quad (e_7 = \nabla r) \end{cases}.$

Proof. In the setting of almost abelian solvmanifolds admitting closed G_2-structure φ, the terms $B, \mathrm{tr}\, D, \mathrm{tr}\, D^t, \mathrm{tr}\, S, \mathrm{Ric}^H, \mathrm{scal}^H$ are all 0. The formulas for λ and Δf immediately follow from these observations and Proposition 5.3.1. Since $\mathrm{scal}^H = 0$, $\mathrm{scal}_D = -\mathrm{tr}(S^2)$. By the Key Lemma and Lemma 5.1.2 (1) and Lemma 5.1.2 (2), we have

$$-\frac{1}{2}\operatorname{div}\tau_D^2(e_i) = \mathrm{Ric}_D(\nabla r, e_i) = -\operatorname{div}(S)(e_i),$$

which by Proposition 5.3.1 is 0 for $i = 1,\ldots,6$ and $-\mathrm{tr}(S^2)$ for $i = 7$. □

Theorem 5.3.4. *Let G_D be a Lie group with Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ where $\mathfrak{h}$ is a codimension-one abelian ideal. Suppose G_D admits closed gradient Laplacian soliton*

$$(\varphi = \omega \wedge e^7 + \rho^+, \nabla f, \lambda).$$

Suppose the potential function is either of the form $f = ar + b$ or $f(x,y) = ar(x) + v(y)$ with $\nabla r = e_7$. Then G_D is flat, φ is torsion-free, and the soliton is steady.

Proof. In the case of almost abelian solvmanifolds, by Proposition 5.2.1 we have

$$\operatorname{Ric}_D = \begin{pmatrix} \frac{1}{2}[D,D^t] & \\ & -\frac{1}{4}\operatorname{tr}(D+D^t)^2 \end{pmatrix} \quad \text{and} \quad \tau_D^2 = \begin{pmatrix} -(D+D^t)^2 & \\ & 0 \end{pmatrix}.$$

[These matrix expressions also follow from results of Lauret (see [Arr13, Lau11], and [Lau17a]).] Suppose $(\varphi, \nabla f, \lambda)$ is a gradient Laplacian soliton where the potential function is of the form $f = ar + b$ with $\nabla r = e_7$. Since $\mathfrak{h}$ is abelian, Corollary 5.3.3 says that $\lambda = -2\operatorname{scal}_D$. Substituting this expression for λ in the soliton equation gives

$$\begin{aligned} \operatorname{Hess} f = a \operatorname{Hess} r &= -\operatorname{Ric}_D - \frac{1}{2}\tau_D^2 + \operatorname{scal}_D I \\ &= -\begin{pmatrix} \frac{1}{2}[D,D^t] & \\ & \operatorname{scal}_D \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -(D+D^t)^2 & \\ & 0 \end{pmatrix} + \operatorname{scal}_D I_{7\times 7} \end{aligned}$$

Taking the trace yields

$$\Delta f = -\operatorname{scal}_D + 2\operatorname{tr}(S^2) + 7\operatorname{scal}_D = 4\operatorname{scal}_D$$

where we used that $\operatorname{tr}[D,D^t] = 0$. By Proposition 5.2.1 (3), $\operatorname{scal}_D = -\operatorname{tr}(S^2)$, and so

$$\Delta f = -4\operatorname{tr}(S^2).$$

Recall the shape operator $T = \nabla_{\cdot} e_7 = -S$ and so

$$-S(X) = T(X) = \nabla_X e_7 = \nabla_X \nabla r = \frac{1}{a} \nabla_X \nabla f = \frac{1}{a} \operatorname{Hess} f(X).$$

Hence $\operatorname{Hess} f = -aS$ and taking the trace yields $\Delta f = -a \operatorname{tr}(S) = 0$, where the last equality follows from $\operatorname{tr}(D) = \operatorname{tr}(D^t) = 0$. Putting this together with the expression obtained for Δf above gives $\operatorname{tr}(S^2) = 0$. Thus $S = 0$ and by Proposition 5.2.1 (3) we get $\operatorname{scal}_D = 0$.

Note $S = 0$ if and only if $D = -D^t$, i.e., D is antisymmetric. This together with $\operatorname{tr}(S^2) = 0$ gives $\operatorname{Ric}_D = 0$, i.e., the space is Ricci-flat. By a result of Alekseevskiĭ-Kimel′fel′d in [AK75], Ricci-flat homogeneous spaces are flat, and so (G_D, g_φ) is flat. Moreover, $\lambda = -2 \operatorname{scal}_D = 0$, i.e., the soliton is steady. Furthermore, since $\operatorname{scal}_D = 0$, φ is torsion-free.

In the case where the potential function is of the form $f(x,y) = ar(x) + v(y)$, recall that the function v on the Euclidean factor is in $\{\operatorname{Hess} v = 0\}$. Then $\operatorname{Hess} f = a \operatorname{Hess} r$ and so we can run through the same arguments above to get that the space is flat, the soliton is steady, and φ is torsion-free. $\square$

Remark 5.3.5. If we start with a flat space, we can choose potential function $f = ar + b$ such that the gradient Laplacian soliton equation is satisfied by taking r to be the coordinate of one of the unit basis vectors, $\nabla r = e_i$, and $\lambda = 0$. One can also construct a Gaussian on flat space $\mathbb{R}^7$ (see, e.g., the case $\mathfrak{n}_1$ in Chapter 4).

5.4 Structure 2(b) when $\nabla r \neq \pm e_7$

We now show that if an almost abelian solvmanifold has Lie algebra decompositions $\mathfrak{h} \oplus_D \mathbb{R} e_7 = \mathfrak{h} \oplus_{D'} \mathbb{R} \nabla r$, with potential function $f = ar + b$ where $\nabla r \neq \pm e_7$, then the space is flat, φ is torsion-free, and the soliton is steady. The idea for the proof is as follows. By using observations from the two decompositions, properties of the Hessian, and the gradient soliton equation, we show that the symmetrization of D is zero, i.e., we show that $S = 0$. The rest of the proof follows similar arguments as in the proof of Theorem 5.3.4.

Theorem 5.4.1. *Let D and D' be derivations of 6-dimensional subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of a 7-dimensional Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ defined by $D(X)=[e_7,X]$ and $D'(Y)=[\nabla r,Y]$, respectively. Suppose $\mathfrak{h}$ is codimension-one abelian ideal. Let M be the Lie group corresponding to Lie algebra $\mathfrak{g}$ with decompositions*

$$\mathfrak{h}\oplus_D\mathbb{R}e_7=\mathfrak{h}'\oplus_{D'}\mathbb{R}\nabla r,$$

where $\nabla r\neq\pm e_7$. Suppose (M,g_φ) admits a closed gradient Laplacian soliton $(\varphi,\nabla f,\lambda)$ with potential function either of the form $f=ar+b$ or $f(x,y)=ar(x)+v(y)$. Then (M,g_φ) must be flat, φ is torsion-free, and the soliton is steady.

Proof. To prove this, we make several observations leading to $\operatorname{tr}(S^2)=0$. For any two vectors $X,Y\in\mathfrak{g}$, we can write $X=h_1+ae_7$ and $Y=h_2+be_7$. In the case of decomposition $\mathfrak{h}\oplus_D\mathbb{R}e_7$, we have

$$\begin{aligned}[X,Y]&=[h_1,h_2]+a[h_1,e_7]+b[e_7,h_2]+ab[e_7,e_7]\\&=[h_1,h_2]-aD(h_1)+bD(h_2)\in\mathfrak{h}.\end{aligned}$$

By similar arguments in the case of decomposition $\mathfrak{h}'\oplus_{D'}\mathbb{R}\nabla r$, we also get $[X,Y]\in\mathfrak{h}'$. Thus $[X,Y]\in\mathfrak{h}\cap\mathfrak{h}'\ \forall\, X,Y\in\mathfrak{g}$. This shows that $D:\mathfrak{h}\to\mathfrak{h}\cap\mathfrak{h}'\subset\mathfrak{h}$ and $D':\mathfrak{h}'\to\mathfrak{h}\cap\mathfrak{h}'$. In particular, $[e_7,\nabla r],[\nabla r,e_7]\in\mathfrak{h}\cap\mathfrak{h}'$.

Note that

$$\begin{aligned}\operatorname{Hess}f(e_7,e_7)&=g(\nabla_{e_7}\nabla f,e_7)=ag(\nabla_{e_7}\nabla r,e_7)\\&=ag([e_7,\nabla r]+\nabla_{\nabla r}e_7,e_7)=^{*}ag(\nabla_{\nabla r}e_7,e_7)=\frac{a}{2}D_{\nabla r}\|e_7\|^2=0,\end{aligned}$$

where equality $*$ follows from the fact that $[e_7,\nabla r]\perp e_7$ as $[e_7,\nabla r]\in\mathfrak{h}\cap\mathfrak{h}'\subset\mathfrak{h}$. So $\operatorname{Hess}f(e_7,e_7)=$

0. Then from the soliton equation,

$$0 = \operatorname{Hess} f(e_7, e_7) = -\operatorname{Ric}_D(e_7, e_7) - \frac{1}{2}\tau_D^2(e_7, e_7) + \frac{1}{3}(\operatorname{scal}_D - \lambda).$$

Since $\tau_D^2(e_7) = 0$ by Proposition 5.2.1 (1), the preceding soliton equation holds if and only if

$$\frac{1}{3}(\operatorname{scal}_D - \lambda) = \operatorname{Ric}_D(e_7, e_7) = -\operatorname{tr}(S^2). \tag{5.2}$$

Moreover, since $-\operatorname{tr}(S^2) = -\frac{1}{4}\operatorname{tr}(D + D^t)^2 = \operatorname{scal}_D$, we get

$$\lambda = -2\operatorname{scal}_D.$$

We claim $e_7 \in \ker \operatorname{Hess} r$. For $i \neq 7$, we also have from the soliton equation that

$$\begin{aligned} a \operatorname{Hess} r(e_7, e_i) = \operatorname{Hess} f(e_7, e_i) &= -g(\operatorname{Ric}_D(e_7), e_i) - \frac{1}{2} g(\tau_D^2(e_7), e_i) + g(\frac{1}{3}(\operatorname{scal}_D - \lambda)e_7, e_i) \\ &= g(-(-\operatorname{tr}(S^2)e_7) - \operatorname{tr}(S^2)e_7, e_i) = 0. \end{aligned}$$

Thus $e_7 \in \ker \operatorname{Hess} r$ and we have $\operatorname{Span}\{e_7, \nabla r\} \subset \ker \operatorname{Hess} r$.

Recall $\mathfrak{h} = \operatorname{Span}\{e_1, \ldots, e_6\}$. Consider $\mathfrak{h} \cap \operatorname{Span}\{\nabla r, e_7\}$, which is at least a one-dimensional subspace containing ∇r, and suppose η is in this intersection. Then we can write $\eta = \alpha \nabla r + \beta e_7$ for some $\alpha, \beta \in \mathbb{R}$. If $\eta = 0$, since ∇r and e_7 are both of unit length, we would get $\nabla r = \pm e_7$, contradicting our assumption. So $\eta \neq 0$.

We claim that both $D(\eta), D^t(\eta)$ are 0. To show this, we first show $D^t(\eta) = 0$. We then show $S(\eta) = 0$, from which we get $D(\eta) = 0$. Since $\eta \in \mathfrak{h}$ and $D : \mathfrak{h} \to \mathfrak{h} \cap \mathfrak{h}'$, it follows that $D(\eta) \in \mathfrak{h} \cap \mathfrak{h}'$. By assumption, $e_7 \perp \mathfrak{h}$ and so $e_7 \perp \mathfrak{h} \cap \mathfrak{h}'$. Similarly, $\nabla r \perp \mathfrak{h} \cap \mathfrak{h}'$. Hence $D(\eta) \perp \eta$. More generally, $D(v) \perp \eta$ for any $v \in \mathfrak{g}$. This means $0 = g(D(v), \eta) = g(v, D^t(\eta))$ for any $v \in \mathfrak{g}$. Thus $D^t(\eta) = 0$.

To show $S(\eta) = 0$, we need to following.

1. $\nabla_{e_7} e_7 = 0$ since by Koszul formula $g(\nabla_{e_7} e_7, e_j) = g([e_j, e_7], e_7) = 0$ for all $j = 1, \ldots, 6$ as $[e_j, e_7] \in \mathfrak{h}$ for all $j = 1, \ldots, 6$; clearly $g(\nabla_{e_7} e_7, e_7) = 0$.

2. We show $\nabla_{\nabla r} e_7 = 0$. Let X be any invariant vector field. Then

$$\begin{aligned} g(\nabla_{\nabla r} e_7, X) &= \nabla_{\nabla r}(\underbrace{g(e_7, X)}_{\text{constant}}) - g(e_7, \nabla_{\nabla r} X) \\ &= -g(e_7, \nabla_{\nabla r} X) \\ &= -[g(e_7, \underbrace{[\nabla r, X]}_{\in \mathfrak{h} \cap \mathfrak{h}'}) + g(e_7, \nabla_X \nabla r)] \\ &= -g(\nabla_X \nabla r, e_7) = -g(\nabla_{e_7} \nabla r, X) = 0, \end{aligned}$$

   where the last equality follows from $e_7 \in \ker \operatorname{Hess} r$.

Now recall that the shape operator corresponding to decomposition $\mathfrak{h} \oplus_D \mathbb{R} e_7$ is

$$T(X) = \nabla_X e_7 = -S(X) = -2^{-1}(D + D^t)(X).$$

Then

$$-S(\eta) = T(\eta) = \nabla_\eta e_7 = \nabla_{\alpha \nabla r + \beta e_7} e_7 = \alpha \nabla_{\nabla r} e_7 + \beta \nabla_{e_7} e_7 = 0.$$

Thus $S(\eta) = 0$. This together with $D^t(\eta) = 0$ yields $D(\eta) = 0$.

The soliton equation applied to η is

$$\operatorname{Hess} f(\eta, \eta) = -\operatorname{Ric}_D(\eta, \eta) - \frac{1}{2} \tau_D^2(\eta, \eta) + \frac{1}{3}(\operatorname{scal}_D - \lambda) g(\eta, \eta).$$

Since $\eta \in \mathfrak{h}$, the operators Ric_D and τ_D^2 applied η is equal to the restriction to their 6×6 diagonal blocks applied to η. These blocks only involve D and D^t and since $D(\eta) = D^t(\eta) = 0$, these operators applied to η are 0. So $\operatorname{Ric}_D(\eta, \eta), \tau_D^2(\eta, \eta)$ are both 0. As $\eta \in \operatorname{Span}\{\nabla r, e_7\} \subset \ker \operatorname{Hess} r$, $\operatorname{Hess} f(\eta, \eta) = 0$. We get

$$0 = \frac{1}{3}(\operatorname{scal}_D - \lambda) g(\eta, \eta).$$

Since $\eta \neq 0$, $g(\eta, \eta) = \|\eta\|^2 > 0$. So for the equality to hold, we must have $\frac{1}{3}(\mathrm{scal}_D - \lambda) = 0$, from which it follows that $\mathrm{tr}(S^2) = 0$ by 5.2. We can now apply the same arguments as in the proof of Theorem 5.3.4 to conclude that the space is flat, φ is torsion-free, and the soliton is steady. $\square$

6 | The modified conformal Hessian

6.1 Manifolds with density and weighted sectional curvature

Riemannian manifolds (M,g) with smooth density e^{-f} were first studied by Lichnerowicz and further developed by Bakry-Émery and others. These manifolds are also referred to as manifolds with smooth measure μ, denoted (M,g,μ), since choosing a smooth measure μ is equivalent to choosing a density function. Wylie in [Wyl15] introduced the notion of *weighted sectional curvature* for Riemannian manfiolds (M,g) with density φ:

$$\overline{\sec}_\varphi(U,V) = \sec(U,V) + \operatorname{Hess}\varphi(U,V) + d\varphi(U)^2 = g(R^{\nabla^\varphi}(V,U)U,V),$$

where $\nabla^\varphi = \nabla^{g,\varphi}$ and R^{∇^φ} are the *weighted Levi-Civita connection* (in metric g with density φ) and *weighted Riemann curvature tensor*, respectively. They are given by

$$\nabla^\varphi_X Y = \nabla_X Y - d\varphi(X)Y - d\varphi(Y)X,$$

where ∇ is the Levi-Civita connection coming from the fixed metric g and

$$R^{\nabla^\varphi}(X,Y)Z = \nabla^\varphi_X \nabla^\varphi_Y Z - \nabla^\varphi_Y \nabla^\varphi_X Z - \nabla^\varphi_{[X,Y]} Z.$$

We note that the definition of weighted sectional curvature comes from its relationship with the 1-Bakry-Émery Ricci tensor $\mathrm{Ric}^1_f = \mathrm{Ric}^{\nabla^\varphi}$. [In [KWY19], the corresponding measure $\mu = e^{-(n+1)\varphi} d\,\mathrm{vol}_g$;

we also write $\nabla^{\varphi} = \nabla^{g,\varphi} = \nabla^{g,\mu}$.]

Comparison theory for Ricci curvature on manifolds with density are studied in [WY16] and [Wyl16]. In particular, Wylie-Yeroshkin study manifolds with density starting with the general torsion-free affine connection ∇^{α} given by $\nabla^{\alpha}_X Y = \nabla_X Y - \alpha(X)Y - \alpha(Y)X$ where α is a 1-form. The motivation for studying ∇^{α} is due to the fact that it is *projectively equivalent* to ∇, i.e., ∇^{α} has the same geodesics as ∇ up to reparametrization. Comparison theory for sectional curvature on manifolds with density is studied by Kennard-Wylie-Yeroshkin in [KWY19]. It turns out many classical comparison results hold with weighted sectional and weighted Ricci curvature bounds. We refer the reader to [KW17, KWY19, WY16, Wyl15], and [Wyl16] for details.

Given a manifold with density (M, g, φ), consider a conformal metric $\tilde{g} = e^{-2\varphi} g$. It is well known that for any smooth function u, $\operatorname{Hess}_{\tilde{g}} u$ and $\operatorname{Hess}_g u$ are related via

$$\operatorname{Hess}_{\tilde{g}} u = \operatorname{Hess}_g u + d\varphi \otimes du + du \otimes d\varphi - g(\nabla\varphi, \nabla u) g. \tag{6.1}$$

A nice property in the unweighted setting is that ∇r is in $\ker \operatorname{Hess}_g r$ whenever r is some distance function in metric g. However, Kennard-Wylie-Yeroshkin observed via (6.1) that this is not true in the weighted setting, i.e., ∇r is not in $\ker \operatorname{Hess}_{\tilde{g}} r$. To remedy this, Kennard-Wylie-Yeroshkin considered the following lower order perturbation of $\operatorname{Hess}_{\tilde{g}} r$,

$$\operatorname{Hess}_{\tilde{g}} r - d\varphi \otimes dr - dr \otimes d\varphi. \tag{6.2}$$

for which ∇r is at least an eigenvector (∇u is a nullvector when u is a modified distance function). Kennard-Wylie-Yeroshkin also observed that equation (6.2) for a general smooth function u has nice convexity properties along geodesics, namely,

$$(\operatorname{Hess}_{\tilde{g}} u - d\varphi \otimes du - du \otimes d\varphi)(\tilde{\sigma}', \tilde{\sigma}') = u'' - 2\varphi' u', \tag{6.3}$$

where $\tilde{\sigma}$ is a $\tilde{g}$-geodesic; the prime notation denotes the derivative with respect to time parameter

t. To see this, note that

$$\begin{aligned}\operatorname{Hess}_{\tilde{g}} u(\tilde{\sigma}', \tilde{\sigma}') = \tilde{g}(\tilde{\nabla}_{\tilde{\sigma}'(t)} \tilde{\nabla} u(\tilde{\sigma}(t)), \tilde{\sigma}'(t)) &= D_t(\tilde{g}(\tilde{\nabla} u(\tilde{\sigma}(t)), \tilde{\sigma}'(t))) \\ &= \frac{d}{dt}\left(\frac{d}{dt}(u \circ \tilde{\sigma}(t))\right) = \frac{d^2(u \circ \tilde{\sigma}(t))}{dt^2} = \frac{d^2 u}{dt^2}\end{aligned}$$

where the second equality follows from product rule and the fact that $\tilde{\sigma}$ is a $\tilde{g}$-geodesic so that $\tilde{\nabla}_{\tilde{\sigma}'} \tilde{\sigma}' = 0$. The last expression is shorthand notation for the second derivative of the composition $u \circ \tilde{\sigma} : [0, \infty) \to \mathbb{R}$ with respect to t. We also have by chain rule that

$$du(\tilde{\sigma}') = g(\nabla u(\sigma(t)), \tilde{\sigma}'(t)) = \frac{d}{dt}(u \circ \tilde{\sigma}(t)) = \frac{du}{dt}.$$

Similarly, $d\varphi(\tilde{\sigma}') = \dfrac{d\varphi}{dt}$.

We now define the modified conformal Hessian and set notation for it.

Definition 6.1.1. Let (M, g, φ) be a manifold with density and let $\tilde{g} = e^{-2\varphi} g$ be a conformal metric. The *modified conformal Hessian* of a smooth function u on M is

$$\operatorname{MCHess} u := \operatorname{Hess}_{\tilde{g}} u - d\varphi \otimes du - du \otimes d\varphi.$$

Remark 6.1.2. Many of the results in [KWY19] assume u is a modified distance function. In fact, the results in subsequent sections assume u is the modified distance function $u = h \circ r_p = \frac{1}{2} r_p^2$ where $r_p : M \to (0, \infty)$ given by $r_p(x) = d_g(x, p) = |xp|_g$ is the distance function to p and $h : [0, \infty) \to [0, \infty)$ is defined by $x \mapsto \dfrac{1}{2} x^2$ so that $h(0) = h'(0) = 0$, and $h'(r) > 0$ for $r > 0$.

We list some key observations from [KWY19] and [Wyl15].

(a) For a distance function r in metric g, the orthogonal complement of ∇r is a well defined conformal class as conformal change preserves angles and only scales ∇r.

(b) The weighted connection ∇^{φ} is incompatible with g. Note that the two ways of expressing the

Riemannian Hessian in terms of the Levi-Civita connection ∇ are equal:

$$\mathrm{Hess}\, u(U,V) = g(\nabla_U \nabla u, V) = (\nabla_U du)(V).$$

Kennard-Wylie-Yeroshkin observed that when we replace the Levi-Civita connection ∇ with the weighted connection ∇^{φ}, the two expressions yield different tensors:

$$\begin{aligned} g(\nabla^{\varphi}_U \nabla u, V) &= g(\nabla_U \nabla u, V) - d\varphi(\nabla u, V) - d\varphi(\nabla u) g(U,V) \\ &= \mathrm{Hess}\, u(U,V) - d\varphi(U) du(V) - d\varphi(\nabla u) g(U,V); \end{aligned}$$

$$\begin{aligned} (\nabla^{\varphi}_U du)(V) &= D_U du(V) - du(\nabla^{\varphi}_U V) \\ &= D_U du(V) - du(\nabla_U V) + d\varphi(U) du(V) + d\varphi(V) du(U) \\ &= \mathrm{Hess}\, u(U,V) + d\varphi(U) du(V) + d\varphi(V) du(U). \end{aligned}$$

(c) Conformal change from (g,φ) to $(\tilde{g},-\varphi)$ is an involution on the space of Riemannian metrics with density that preserves positivity or negativity of $\overline{\sec}_{\varphi}$.

(d) Let u be a distance function. Kennard-Wylie-Yeroshkin observed that weighted curvatures should control the modified Hessian of distance functions from the formulas

$$\mathrm{Hess}_{\tilde{g}}\, u(U,V) = g(\nabla^{\varphi}_U \nabla u, V)$$

for $U,V \perp \nabla u$ and

$$\nabla^{\tilde{g},-\varphi}_{\cdot}(\cdot) = \mathrm{Hess}_{\tilde{g}}\, u - d\varphi \otimes du - du \otimes d\varphi,$$

where $\nabla^{\tilde{g},-\varphi}$ is the weighted connection in the conformal metric $\tilde{g}$ and density φ.

An instance of observation (d) is [[KWY19], Theorem 3.3], which states that for a simply connected manifold with density (M,g,φ) with $\overline{\sec}_{\varphi} \leq 0$, the modified conformal Hessian

MCHess $u_p > 0$ for all $p \in M$, where u_p is a modified distance function to p. We ask the following related question.

Question. Given a Riemannian manifold with density (M, g, φ) with $\overline{\sec}_\varphi \geq 0$, what conditions are necessary such that

$$\text{MCHess}\, u \leq K$$

for some constant K? The hope is that in obtaining such conditions, we would have some control over MCHess u, which may in turn give us a bound on the number of critical points of modified distance functions to a point $p \in M$ within some open ball centered at p in the weighted setting (i.e., a version of Gromov's critical point theorem also known as the "Baby Soul Theorem" in the weighted setting). Proving this may also give us some insight on how to approach Toponogov's comparison theorem in the weighted setting (see [[Pet16], Theorem 12.2.2 and Lemma 12.4.2]).

We now include a few results from [KWY19] which will be used in the next section.

Proposition 6.1.3 ([KWY19], Proposition 3.2). *Let u be a modified distance function. At points where u is smooth,*

$$\text{Hess}_{\tilde{g}}\, u - d\varphi \otimes du - du \otimes d\varphi = \left(h'' - h' \frac{\partial \varphi}{\partial r}\right) dr \otimes dr + h'(\text{Hess}_g\, r - g(\nabla r, \nabla \varphi) g_r),$$

where $\tilde{g} = e^{-2\varphi} g$, $\varphi' = \frac{\partial \varphi}{\partial r}$, and g_r is the metric on level sets of r_p

We recall that the reparametrized distance defined in [KWY19] is

$$s(r) = \int_0^r e^{-2\varphi(t)} dt, \quad \& \quad s(p,q) = \inf\{s \mid \gamma(0) = p, \gamma(1) = q\}.$$

This is needed for the following Hessian comparison in the weighted setting.

Theorem 6.1.4 ([KWY19], Theorem 4.16). *Suppose that (M, g, φ) is a Riemannian manifold with density. Let r_p be the distance function to p in g. Let q be a point such that r_p is smooth at q and let $Y \in T_qM$ be a unit length vector such that $Y \perp \nabla r$.*

1. *If, for all unit vectors Y perpendicular to the minimizing geodesic from p to q, $\overline{\sec}_\varphi(Y,\nabla r) \geq \kappa e^{-4\varphi}$, then*
$$(\mathrm{Hess}_g\, r - d\varphi(\nabla r)g)(Y,Y) \leq e^{-2\varphi(q)}\frac{\mathrm{cs}_\kappa(s(p,q))}{\mathrm{sn}_\kappa(s(p,q))}.$$

2. *If, for all unit vectors Y perpendicular to the minimizing geodesic from p to q, $\overline{\sec}_\varphi(Y,\nabla r) \leq K e^{-4\varphi}$, then*
$$(\mathrm{Hess}_g\, r - d\varphi(\nabla r)g)(Y,Y) \geq e^{-2\varphi(q)}\frac{\mathrm{cs}_K(s(p,q))}{\mathrm{sn}_K(s(p,q))}.$$

6.2 Modified conformal Hessian bounds

We obtain bounds on the modified conformal Hessian when $\overline{\sec}_\varphi \geq 0$.

Proposition 6.2.1 (MCHess $u \leq Kg$)**.** *Let (M,g,φ) be a Riemannian manifold with density φ such that $a \leq \varphi \leq 0$ and $|\nabla\varphi| \leq \dfrac{c}{r}$ for some constant $c > 0$. Suppose $\overline{\sec}_\varphi \geq 0$ and $u = h \circ r_p = \dfrac{1}{2}r_p^2$ as in Remark 6.1.2. Then*
$$\mathrm{MCHess}\, u \leq Kg$$
for some constant K.

Proof. By Proposition 6.1.3,
$$\mathrm{MCHess}\, u = (h'' - h'\varphi')dr \otimes dr + h'(\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r),$$
where $\varphi' = \partial_r\varphi$; the rest of the derivatives are taken with respect to r. We first set some terminology: we refer to $(h''-h'\varphi')dr\otimes dr$ and $h'(\mathrm{Hess}_g\, r - g(\nabla r,\nabla\varphi)g_r)$ as the "radial term" and "tangential term" of MCHess u, respectively.

Note $|\nabla\varphi| \leq \frac{c}{r}$ if and only if $-\frac{c}{r} \leq \varphi' \leq \frac{c}{r}$. In particular, $-\varphi' \leq \frac{c}{r}$. This together with $u = h(r) = \frac{1}{2}r^2$ yields
$$(h'' - h'\varphi') = 1 - r\varphi' \leq 1 + r\left(\frac{c}{r}\right) = 1 + c$$

for some constant $c > 0$. Thus the radial term

$$(h'' - h'\varphi')dr \otimes dr \leq (1+c)dr \otimes dr.$$

We claim that

$$\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r \leq e^{-2\varphi}\frac{1}{r}g_r, \tag{6.4}$$

from which it follows that the tangential term

$$\begin{aligned} h'(\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r) &\leq r\left(e^{-2\varphi}\frac{1}{r}\right)g_r = e^{-2\varphi}g_r \\ &\leq (1+c)e^{-2\varphi}g_r \leq (1+c)e^{-2a}g_r, \end{aligned}$$

where we used $c > 0$ and $\varphi \geq a$ in the second to last and last inequality, respectively. Since $-2a \geq -2\varphi \geq 0$ if and only if $e^{-2a} \geq e^{-2\varphi} \geq 1$, setting $K := \max\{1+c, e^{-2a}(1+c)\} = e^{-2a}(1+c)$ yields

$$\mathrm{MCHess}\, u \leq K(dr^2 + g_r) = Kg$$

as desired.

We prove inequality (6.4). First, recall the re-parametrized distance defined in [KWY19] is

$$s(r) = \int_0^r e^{-2\varphi(t)}dt, \quad \& \quad s(p,q) = \inf\{s \mid \gamma(0) = p, \gamma(1) = q\}.$$

Since

$$t \to \frac{\mathrm{cs}_\kappa(t)}{\mathrm{sn}_\kappa(t)}$$

is monotonically decreasing in t, we have

$$\varphi \leq 0 \implies s(r) = \int_0^r e^{-2\varphi(t)}dt \geq \int_0^r dt = r_p(q) \implies \frac{\mathrm{cs}_\kappa(s(p,q))}{\mathrm{sn}_\kappa(s(p,q))} \leq \frac{\mathrm{cs}_\kappa(r_p(q))}{\mathrm{sn}_\kappa(r_p(q))}.$$

Since $\overline{\sec}_\varphi \geq 0$, we have $\overline{\sec}_\varphi(Y, \nabla r) \geq 0$ for orthogonal unit length vectors Y and ∇r in T_qM, where q is some point for which r_p is smooth. As $Y \perp \nabla r$, we can view Y as belonging to T_qH where $H = H^{n-1}$ is a hypersurface contained in the level set of r_p at q. Then by Theorem 6.1.4 (1) with $\kappa = 0$ and the preceding inequality, we get

$$\begin{aligned}(\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r)(Y,Y) &= (\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g)(Y,Y) \\ &\le e^{-2\varphi(q)}\frac{\mathrm{cs}_0(s(p,q))}{\mathrm{sn}_0(s(p,q))} \le e^{-2\varphi(q)}\frac{\mathrm{cs}_0(r_p(q))}{\mathrm{sn}_0(r_p(q))} = e^{-2\varphi(q)}\frac{1}{r_p(q)},\end{aligned}$$

where $\mathrm{sn}_0(r) = r$ and $\mathrm{cs}_0(r) = \mathrm{sn}_0'(r) = 1$. So

$$(\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r)(Y,Y) \le e^{-2\varphi(q)}\frac{1}{r_p(q)} \quad \forall\, Y \in T_qH. \tag{6.5}$$

We now show that (6.4) holds for any $Y \in T_qM$. The shape operator S given by $S(Y) = \nabla_Y \nabla r$ and the identity operator I are both self-adjoint $(1,1)$-linear endomorphisms from $T_qM \to T_qM$ with corresponding $(0,2)$-tensors $\mathrm{Hess}_g\, r(\cdot,\cdot)$ and $g(I\cdot,\cdot) = g(\cdot,\cdot)$, respectively. Hence their sum $A := S - g(\nabla r, \nabla\varphi)I$ is also a self-adjoint $(1,1)$-linear endomorphism with corresponding $(0,2)$-tensor $\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g$. Let $H = H^{n-1}$ be the hypersurface contained in the level set of r_p at q. Note that the tangential term $\mathrm{Hess}_g\, r - g(\nabla r, \nabla\varphi)g_r$ does not depend on the radial component of any vector, i.e., it does not depend on the component of a vector that is in the direction of ∇r. To see this, recall any vector field Y along integral curves for $\partial_r = \nabla r$ can be written $Y = Y^\top + Y^\perp = (Y - g(Y,\partial_r)\partial_r) + g(Y,\partial_r)\partial_r$. For each point on the integral curve, we have coordinate basis $\{\partial_r\} \cup \{\partial_i\}_i$ where $\{\partial_i\}_i$ is a coordinate basis for TH and $H \subset r^{-1}(t)$ is a hypersurface. So in local coordinates, we can also write $Y = Y^r\partial_r + Y^i\partial_i$ where $Y^r, Y^i : M \to \mathbb{R}$ smooth. Since $Y^\perp = Y^r\partial_r = g(Y,\partial_r)\partial_r$,

$$\mathrm{Hess}_g(Y^\perp, X) = \mathrm{Hess}_g\, r(Y^r\partial_r, X) = Y^r g(\underbrace{\nabla_{\partial_r}\partial_r}_{=0}, X) = 0$$

for any vector field X along the same curve. With this observation, for general vector fields Y along integral curve for ∂_r, we have

$$\begin{aligned}
&(\mathrm{Hess}_g\, r - g(\nabla r, \nabla \varphi) g_r)(Y,Y)\\
&= \mathrm{Hess}_g\, r(Y^\top + Y^\perp, Y^\top + Y^\perp) - g(\nabla r, \nabla \varphi) g_r(Y^\top + Y^\perp, Y^\top + Y^\perp)\\
&= \mathrm{Hess}_g\, r(Y^\top, Y^\top) - g(\nabla r, \nabla \varphi) g_r(Y^\top, Y^\top)\\
&= (\mathrm{Hess}_g\, r - g(\nabla r, \nabla \varphi) g_r)(Y^\top, Y^\top).
\end{aligned}$$

Thus for any vector $Y \in T_qM$, we need only consider its tangential component $Y^\top \in T_qH$ where $Y^\top \perp \nabla r$. Also note $g|_H = g_r$. The restriction $A\big|_{T_qH} : T_qH \to T_qH$ is a self-adjoint $(1,1)$-linear endomorphism and so by the Spectral Theorem there exists an orthonormal eigenbasis $(E_i)_{i=1}^{n-1}$ for T_qH which diagonalizes $A\big|_{T_qH}$. Since $E_i \in T_qH$, $E_i \perp \nabla r$ and since $(E_i)_i$ is orthonormal, E_i is of unit length in g $\forall\, i = 1, \ldots, n-1$. By (6.5), we have

$$(\mathrm{Hess}_g\, r - g(\nabla r, \nabla \varphi) g_r)(E_i, E_i) \le e^{-2\varphi} \frac{1}{r} = e^{-2\varphi} \frac{1}{r} g(E_i, E_i) = e^{-2\varphi} \frac{1}{r} g_r(E_i, E_i) \tag{6.6}$$

$\forall\, i = 1, \ldots, n-1$. Since $(E_i)_{i=1}^{n-1}$ is an eigenbasis for T_qH, we have $A\big|_{T_qH}(E_i) = \lambda_i E_i$ for $\lambda_i \in \mathbb{R}$ and $i = 1, \ldots, n-1$. Then by 6.6, we get

$$\lambda_i = \lambda_i g(E_i, E_i) = g(\lambda_i E_i, E_i) = g(A\big|_{T_qH}(E_i), E_i) \le e^{-2\varphi} \frac{1}{r}$$

$\forall\, i = 1, \ldots, n-1$. To show (6.4) for any $Y \in T_qM$, it suffices to show

$$(\mathrm{Hess}_g\, r - g(\nabla r, \nabla \varphi) g_r)(Y^\top, Y^\top) \le e^{-2\varphi} \frac{1}{r} g(Y^\top, Y^\top) = e^{-2\varphi} \frac{1}{r} g_r(Y^\top, Y^\top).$$

Since $Y^\top \in T_qH$ and $(E_i)_{i=1}^{n-1}$ is an eigenbasis for T_qH, we write $Y^\top = \sum_i (Y^\top)^i E_i$. Then

$$\begin{aligned}
g(A(Y^\top), Y^\top) &= g(A(\sum_i (Y^\top)^i E_i), \sum_j (Y^\top)^j e_j) \\
&= \sum_i \lambda_i ((Y^\top)^i)^2 g(E_i, E_i) \\
&\le \sum_i e^{-2\varphi} \frac{1}{r} ((Y^\top)^i)^2 g(E_i, E_i) \\
&= e^{-2\varphi} \frac{1}{r} g(\sum_i (Y^\top)^i E_i, \sum_j (Y^\top)^j e_j) = e^{-2\varphi} \frac{1}{r} g(Y^\top, Y^\top) = e^{-2\varphi} \frac{1}{r} g_r(Y^\top, Y^\top).
\end{aligned}$$

Note that the second equality follows from $(E_i)_i$ being an orthonormal eigenbasis. Since the expression on the left-hand side is equal to $(\operatorname{Hess}_g r - g(\nabla r, \nabla\varphi) g_r)(Y^\top, Y^\top)$, we get inequality (6.4) for $Y^\top \in T_qH$, and hence for any $Y \in T_qM$. Equivalently, we have shown that $A \le B$ where $A = S|_{T_qH} - g(\nabla r, \nabla\varphi) I_{n-1}$ and $B = e^{-2\varphi} \frac{1}{r} I_{n-1}$. □

Remark 6.2.2. We can further show that (6.4) holds for any $X, Y \in T_qM$ by similar arguments as in the preceding string of inequalities.

Proposition 6.2.1 gave us $\operatorname{MCHess} u \le Kg$ when the density φ is bounded above by 0 and bounded below by a. It turns out we can show $\operatorname{MCHess} u \le K\tilde{g}$ where the conformal metric $\tilde{g} = e^{-2\varphi} g$ with only the upper bound of 0 on density φ.

Proposition 6.2.3 (MCHess $u \le K\tilde{g}$)**.** *Let (M, g, φ) be a Riemannian manifold with density φ where $\varphi \le 0$ and $|\nabla\varphi| \le \frac{c}{r}$ for some constant $c > 0$. Suppose $\overline{\sec}_\varphi \ge 0$ and $u = h \circ r_p = \frac{1}{2} r_p^2$ as in Remark 6.1.2. Then*

$$\operatorname{MCHess} u \le K\tilde{g}$$

for some constant K.

Proof. As in the proof of Proposition 6.2.1, we have

$$(h'' - h'\varphi') = 1 - r\varphi' \le 1 + r\left(\frac{c}{r}\right) = 1 + c$$

for some constant $c>0$. It follows that the radial term

$$(h''-h'\varphi')dr\otimes dr \leq (1+c)dr\otimes dr = (1+c)e^{2\varphi}e^{-2\varphi}dr\otimes dr \leq (1+c)e^{-2\varphi}dr\otimes dr,$$

where in the last inequality we used that $\varphi \leq 0$ if and only if $e^{2\varphi} \leq e^{\varphi} \leq e^0 = 1$.

By the proof of Proposition 6.2.1, we saw that an application of Theorem 6.1.4 (1) with hypotheses $\overline{\sec}_\varphi \geq 0$ and $\varphi \leq 0$ (for monotonicity) yields

$$(\operatorname{Hess}_g r - g(\nabla r, \nabla\varphi)g_r)(Y,Y) \leq e^{-2\varphi}\frac{1}{r}g_r(Y,Y) \quad \forall\, Y \in T_qM.$$

Then

$$h'(\operatorname{Hess}_g r - g(\nabla r, \nabla\varphi)g_r)(Y,Y) \leq e^{-2\varphi}g_r(Y,Y) \leq (1+c)e^{-2\varphi}g_r(Y,Y),$$

as $h'=r$ and $c>0$. Putting this together with the inequality for the radial term, we get that for any $Y \in T_qM$

$$\begin{aligned}
\operatorname{MCHess} u(Y,Y) &= (h''-h'\varphi')dr\otimes dr + h'(\operatorname{Hess}_g r - g(\nabla r,\nabla\varphi)g_r) \\
&\leq ((1+c)e^{-2\varphi}dr\otimes dr)(Y,Y) + (1+c)e^{-2\varphi}g_r(Y,Y) \\
&= (1+c)e^{-2\varphi}g(Y,Y) \\
&= (1+c)\tilde{g}(Y,Y) \\
&= K\tilde{g}(Y,Y),
\end{aligned}$$

where $K = 1+c$. □

Remark 6.2.4. We need the assumption $\varphi \geq a$ in Proposition 6.2.1 to get the bound e^{-2a} for the factor of $e^{-2\varphi}$ that appears after an application of Theorem 6.1.4 (1). For Proposition 6.2.3, we do not need this assumption as the factor of $e^{-2\varphi}$ is the weight for the conformal metric $\tilde{g}$.

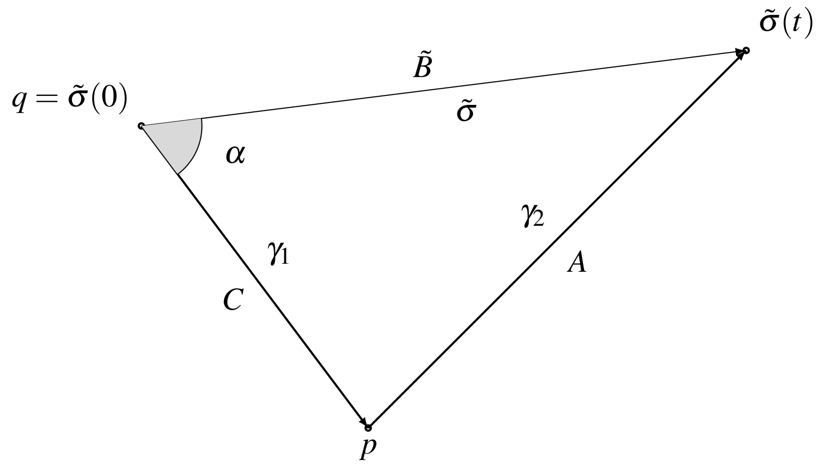

Figure 6.1: Mixed geodesic triangle.

6.3 A warped law of cosines

We now use the bounds obtained from the previous section to obtain inequalities resembling the law of cosines, which we call a "warped law of cosines".

Corollary 6.3.1 (Warped law of cosines)**.** *Let (M, g, φ) be a (complete) Riemannian manifold with density φ such that $a \leq \varphi \leq 0$ and $|\nabla\varphi| \leq \frac{c}{r}$ for some constant $c > 0$. Suppose $\overline{\sec}_\varphi \geq 0$ and $u = h \circ r_p = \frac{1}{2}r_p^2$ as in Propositions 6.2.1 and 6.2.3. Furthermore, suppose the following setup:*

- *let $\tilde{\sigma}$ be a (unit-speed) geodesic in metric $\tilde{g}$ starting at $q = \tilde{\sigma}(0)$ with $\tilde{B} := L_{\tilde{g}}(\tilde{\sigma})$;*
- *let $\gamma_1 : q \to p$ be a (unit-speed) geodesic segment in metric g with $C := |\tilde{\sigma}(0)p|_g$;*
- *and let γ_2 be a (unit-speed) geodesic segment in metric g joining p and $\tilde{\sigma}(t)$ for some $t > 0$ with $A := |\tilde{\sigma}(t)p|_g$.*

Note these three bullets gives a "mixed geodesic triangle" with sides of g-length C, $\tilde{g}$-length $\tilde{B}$, and g-length A (see Figure 6.1). Then

$$A^2 \leq C^2 + k_1\tilde{B}^2 - 2k_2 C\tilde{B}\cos\alpha,$$

where α is the interior angle formed by γ_1 and $\tilde{\sigma}$ at $q = \tilde{\sigma}(0)$ opposite to γ_2 of g-length A; $k_1 = (1+c)e^{-2a}$; $k_2 = e^{-a}$ for $0 \le \alpha < \pi/2$; and $k_2 = e^{-2a}$ for $\pi/2 \le \alpha \le \pi$.

Proof. Note $\tilde{\sigma}(0) = q$, $\tilde{g}(\tilde{\sigma}', \tilde{\sigma}') = |\tilde{\sigma}'|^2_{\tilde{g}} = 1$, and $(\tilde{\sigma}')^\top \perp \nabla r$. Putting (6.3) and Proposition 6.2.3 together gives

$$u'' - 2\varphi' u' = \frac{d^2u}{dt^2} - 2\frac{du}{dt}\frac{d\varphi}{dt} \le 1 + c,$$

where the shorthand notation emphasises that $u = u \circ \tilde{\sigma}$ and $\varphi = \varphi \circ \tilde{\sigma}$ are functions from $[0, \infty) \to \mathbb{R}$. Multiplying both sides of the preceding inequality by $e^{-2\varphi}$ gives

$$e^{-2\varphi}\frac{d^2u}{dt^2} - 2e^{-2\varphi}\frac{du}{dt}\frac{d\varphi}{dt} \le (1+c)e^{-2\varphi} \iff \left(e^{-2\varphi}\frac{du}{dt}\right)' \le (1+c)e^{-2\varphi}.$$

Integrating both sides over $\tilde{\sigma}$ amounts to integrating both sides over $[0, t]$. By the fundamental theorem of calculus, we get

$$\begin{aligned}
\left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(t)) - \left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0)) &= \int_0^t \left(e^{-2\varphi}\frac{du}{dt}\right)' d\nu \\
&\le (1+c)\int_0^t e^{-2\varphi} d\nu \\
&\le (1+c)e^{-2a}\int_0^t d\nu \\
&= (1+c)e^{-2a}t,
\end{aligned}$$

where we used $a \le \varphi$ if and only if $e^{-2a} \ge e^{-2\varphi}$ in the last inequality. Adding $\left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0))$ to both sides and then multiplying through by $e^{2\varphi(\tilde{\sigma}(t))}$ yields

$$\begin{aligned}
\frac{du}{dt}(\tilde{\sigma}(t)) &\le e^{2\varphi(\tilde{\sigma}(t))}\left[(1+c)e^{-2a}t + \left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0))\right] \\
&\le (1+c)e^{-2a}t + \left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0)),
\end{aligned}$$

where we used $\varphi \leq 0$ if and only if $e^{2\varphi} \leq 1$ in the last inequality. Integrating over $[0,t]$ again yields

$$u(\tilde{\sigma}(t)) - u(\tilde{\sigma}(0)) \leq (1+c)e^{-2a}\frac{t^2}{2} + \left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0))t,$$

from which we get

$$u(\tilde{\sigma}(t)) \leq (1+c)e^{-2a}\frac{t^2}{2} + \left(e^{-2\varphi}\frac{du}{dt}\right)(\tilde{\sigma}(0))t + u(\tilde{\sigma}(0)). \tag{6.7}$$

We make some observations:

1.

$$\begin{aligned}
e^{-2\varphi(\tilde{\sigma}(0))}\frac{du}{dt}(\tilde{\sigma}(0)) &\leq e^{-2a}g(\nabla u(\tilde{\sigma}(0)), \tilde{\sigma}'(0)) \\
&= e^{-2a}g(r(\tilde{\sigma}(0))\nabla r(\tilde{\sigma}(0)), \tilde{\sigma}'(0)) \\
&= e^{-2a}r(\tilde{\sigma}(0))g(\nabla r(\tilde{\sigma}(0)), \tilde{\sigma}'(0)) \\
&= e^{-2a}|\tilde{\sigma}(0)p|_g|\nabla r(\tilde{\sigma}(0))|_g|\tilde{\sigma}'(0)|_g \cos\angle(\nabla r(\tilde{\sigma}(0)), \tilde{\sigma}'(0)) \\
&= e^{-2a}Ce^{\varphi}\cos(\pi - \alpha) \\
&= -e^{-2a}e^{\varphi}C|\tilde{\sigma}'(0)|_{\tilde{g}}\cos\alpha \\
&= -e^{\varphi}e^{-2a}C\cos\alpha,
\end{aligned}$$

where we used $C = |\tilde{\sigma}(0)p|_g$, $|\nabla r(\tilde{\sigma}(0))|_g = 1$, and

$$|\tilde{\sigma}'(0)|_g = g(\tilde{\sigma}'(0), \tilde{\sigma}'(0))^{1/2} = e^{\varphi}e^{-\varphi}g(\tilde{\sigma}'(0), \tilde{\sigma}'(0))^{1/2} = e^{\varphi}\tilde{g}(\tilde{\sigma}'(0), \tilde{\sigma}'(0))^{1/2} = e^{\varphi}$$

in the third to last equality. We find possible upper bounds for this expression. Note that since $e^{-2a}C \geq 0$, the upper bounds are dependent on e^{φ} and the sign of $\cos\alpha$ (hence on α). There are two cases to consider.

(a) For $0 \leq \alpha < \pi/2$, we have $\cos\alpha > 0$ if and only if $e^{-2a}C\cos\alpha \geq 0$. Note that since

$a \le \varphi \le 0$ if and only if $-1 \ge -e^{a} \ge -e^{\varphi}$, we get

$$-e^{\varphi}e^{-2a}C\cos\alpha \le -e^{a}e^{-2a}C\cos\alpha = -e^{-a}C\cos\alpha.$$

(b) For $\pi/2 \le \alpha \le \pi$, we have $\cos\alpha \le 0$ if and only if $-e^{-2a}C\cos\alpha \ge 0$. Since $a \le \varphi \le 0$ if and only if $e^{\varphi} \le 1$, we get

$$-e^{\varphi}e^{-2a}C\cos\alpha = e^{\varphi}(-e^{-2a}C\cos\alpha) \le -e^{-2a}C\cos\alpha.$$

2. $t = \int_0^t |\tilde{\sigma}'(\nu)|_{\tilde{g}}d\nu = \tilde{B}$
3. $u(\tilde{\sigma}(t)) = \frac{1}{2}r(\tilde{\sigma}(t))^2 = \frac{1}{2}|\tilde{\sigma}(t)p|_g^2 = \frac{1}{2}A^2$
4. $u(\tilde{\sigma}(0)) = \frac{1}{2}r(\tilde{\sigma}(0))^2 = \frac{1}{2}|\tilde{\sigma}(0)p|_g^2 = \frac{1}{2}C^2.$

Putting observations (1)-(4) and inequality (6.7) together, we get

$$\begin{cases} \frac{A^2}{2} \le (1+c)e^{-2a}\frac{\tilde{B}^2}{2} - e^{-a}C\tilde{B}\cos\alpha + \frac{C^2}{2} & 0 \le \alpha < \frac{\pi}{2} \\ \frac{A^2}{2} \le (1+c)e^{-2a}\frac{\tilde{B}^2}{2} - e^{-2a}C\tilde{B}\cos\alpha + \frac{C^2}{2} & \frac{\pi}{2} \le \alpha \le \pi \end{cases}$$

Setting $k_1 = (1+c)e^{-2a}$ and then multiplying through by 2 gives

$$\begin{cases} A^2 \le C^2 + k_1\tilde{B}^2 - 2e^{-a}C\tilde{B}\cos\alpha & 0 \le \alpha < \frac{\pi}{2} \\ A^2 \le C^2 + k_1\tilde{B}^2 - 2e^{-2a}C\tilde{B}\cos\alpha & \frac{\pi}{2} \le \alpha \le \pi. \end{cases}$$

□

We are currently working on answering the following question.

Question. [Weighted Gromov's critical point estimate] Suppose the hypotheses of Corollary 6.3.1. Is it true that for every $p \in M$ the distance function $r_p(x) = |xp|$ has no critical points outside some ball $B(p,R)$? Does M have the topology of a compact manifold with boundary?

We hope to use Corollary 6.3.1 towards answering this question. So far we have not yet found an application of it. In our attempts, some general questions arise that require further study.

1. What is the relationship between a $\tilde{g}$-geodesic $\tilde{\sigma}$ and a g-geodesic σ? More specifically, what can we say about the angle between a g-geodesic segment σ and a $\tilde{g}$-geodesic segment $\tilde{\sigma}$ where both segments share the same endpoints (e.g., starting at p and ending at y)?

2. Do critical points of a distance function in the metric g remain critical points of a distance function in the conformal metric $\tilde{g} = e^{-2\varphi}g$?

Regarding Question 1, we can at least say something about the g- and $\tilde{g}$-lengths of $\tilde{g}$-geodesic $\tilde{\sigma}$ and g-geodesic σ.

Proposition 6.3.2. *Let (M, g, φ) be a Riemannian manifold with density φ such that $a \le \varphi \le 0$. Let $\tilde{g} = e^{-2\varphi}g$ be the conformal metric. Define*

$$\tilde{B} := L_{\tilde{g}}(\tilde{\sigma}) = \int_a^b \tilde{g}(\tilde{\sigma}'(s), \tilde{\sigma}'(s))^{1/2} ds \;\; \textit{and} \;\; B := L_g(\tilde{\sigma}) = \int_a^b g(\tilde{\sigma}'(s), \tilde{\sigma}'(s))^{1/2} ds.$$

Then $B \le \tilde{B} \le e^{-a}B$.

Proof. We drop parameter s for convenience in notation. Note $a \le \varphi \le 0$ if and only if $e^{-a} \ge e^{-\varphi} \ge 1$. Since

$$L_{\tilde{g}}(\tilde{\sigma}) = \int_a^b \tilde{g}(\tilde{\sigma}', \tilde{\sigma}')^{1/2} ds = \int_a^b e^{-\varphi} g(\tilde{\sigma}', \tilde{\sigma}')^{1/2} ds,$$

we get

$$L_g(\tilde{\sigma}) \le L_{\tilde{g}}(\tilde{\sigma}) \le e^{-a} \int_a^b g(\tilde{\sigma}', \tilde{\sigma}')^{1/2} ds = e^{-a} L_g(\tilde{\sigma}).$$

That is $B \le \tilde{B} \le e^{-a}B$. ☐

Remark 6.3.3. For a g-geodesic σ, setting $A = L_g(\sigma)$ and $\tilde{A} = L_{\tilde{g}}(\sigma)$ also gives similar inequalities: $e^a\tilde{A} \le A \le \tilde{A}$.

A | Tables

This appendix consists of relevant data for Chapter 4. More specifically, we reproduce structure equations for each of the seven nilpotent Lie groups from [Nic18] in Table A.1. Tables A.2 - A.7 are tables of derivatives obtained by the Koszul formula and structure equations of Table A.1.

We reiterate that the structure equations from [Nic18] were obtained from the list of twelve isomorphism classes of nilpotent Lie groups admitting G-invariant closed G_2-structures from [CF11] (see [CF11] for a list of canonical structure equations for $\mathfrak{n}_i$ for $i = 1, \ldots, 12$). Nicolini constructed φ_i for $i = 1, \ldots, 7$ that were either algebraic or semi-algebraic solitons, hence Laplacian solitons by the results of Lauret in [Lau17a]. To obtain the structure equations of Table A.1, one first obtains the exterior derivatives on $(e^i)_i$ from the general structure equations for each of the twelve isomorphism classes of nilpotent Lie groups from [CF11]. Then one uses the closed condition $d\varphi_i = 0$ to determine the appropriate coefficients. Note, the coefficients in the tables below were chosen for convenience; the results of Chapter 4 hold in general by scaling.

Table A.1: Structure of $(\mathfrak{n}_i, \varphi_i)$

$(\mathfrak{n}_1, \varphi_1)$	$[e_i, e_j] = 0 \,\forall\, i, j$
$(\mathfrak{n}_2(1,1), \varphi_2)$	$[e_1, e_2] = -e_5, [e_1, e_3] = -e_6$
$(\mathfrak{n}_3(1, 1-c, c), \varphi_3)$	$[e_1, e_2] = -e_4, [e_1, e_3] = (c-1)e_5, [e_2, e_3] = ce_6; 0 < c < 1/2$
$(\mathfrak{n}_4(\sqrt{2}, 1, \sqrt{2}, 1), \varphi_4)$	$[e_1, e_2] = -\sqrt{2}e_3, [e_1, e_3] = -e_6, [e_2, e_4] = -\sqrt{2}e_6, [e_1, e_5] = -e_7$
$(\mathfrak{n}_5(\sqrt{2}, 1, 1, \sqrt{2}), \varphi_5)$	$[e_1, e_2] = -\sqrt{2}e_3, [e_1, e_3] = -e_6, [e_1, e_4] = -e_7, [e_2, e_5] = -\sqrt{2}e_7$
$(\mathfrak{n}_6(\sqrt{2}, \sqrt{2}, 1, 1), \varphi_6)$	$[e_1, e_2] = -\sqrt{2}e_4, \quad [e_1, e_3] = -\sqrt{2}e_5, \quad [e_1, e_4] = -e_6, \quad [e_1, e_5] = -e_7$
$(\mathfrak{n}_7(-4, 2, 2, \sqrt{6}, \sqrt{6}), \varphi_7)$	$[e_1, e_2] = 4e_4, [e_1, e_7] = -2e_6, [e_2, e_7] = -2e_5, [e_5, e_7] = -\sqrt{6}e_3, [e_6, e_7] = -\sqrt{6}e_4$

Table A.2: Derivatives for $\mathfrak{n}_2(1,1)$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{1}{2}e_5$	$-\frac{1}{2}e_6$	0	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	0
2	$\frac{1}{2}e_5$	0	0	0	$\frac{1}{2}e_1$	0	0
3	$\frac{1}{2}e_6$	0	0	0	0	$-\frac{1}{2}e_1$	0
4	0	0	0	0	0	0	0
5	$\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0	0	0	0	0
6	$\frac{1}{2}e_3$	0	$-\frac{1}{2}e_1$	0	0	0	0
7	0	0	0	0	0	0	0

Table A.3: Derivatives for $\mathfrak{n}_3(1,1-c,c)$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{1}{2}e_4$	$\frac{1}{2}(c-1)e_5$	$\frac{1}{2}e_2$	$-\frac{1}{2}(c-1)e_3$	0	0
2	$\frac{1}{2}e_4$	0	$-\frac{1}{2}ce_6$	$-\frac{1}{2}e_1$	0	$\frac{1}{2}ce_3$	0
3	$-\frac{1}{2}(c-1)e_5$	$\frac{1}{2}ce_6$	0	0	$\frac{1}{2}(c-1)e_1$	$-\frac{1}{2}ce_2$	0
4	$\frac{1}{2}e_2$	$-\frac{1}{2}e_1$	0	0	0	0	0
5	$-\frac{1}{2}(c-1)e_3$	0	$\frac{1}{2}(c-1)e_1$	0	0	0	0
6	0	$\frac{1}{2}ce_3$	$-\frac{1}{2}ce_2$	0	0	0	0
7	0	0	0	0	0	0	0

Table A.4: Derivatives for $\mathfrak{n}_4(\sqrt{2},1,\sqrt{2},1)$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{\sqrt{2}}{2}e_3$	$\frac{\sqrt{2}}{2}e_2-\frac{1}{2}e_6$	0	$-\frac{1}{2}e_7$	$\frac{1}{2}e_3$	$\frac{1}{2}e_5$
2	$\frac{\sqrt{2}}{2}e_3$	0	$-\frac{\sqrt{2}}{2}e_1$	$-\frac{\sqrt{2}}{2}e_6$	0	$\frac{\sqrt{2}}{2}e_4$	0
3	$\frac{\sqrt{2}}{2}e_2+\frac{1}{2}e_6$	$-\frac{\sqrt{2}}{2}e_1$	0	0	0	$-\frac{1}{2}e_1$	0
4	0	$\frac{\sqrt{2}}{2}e_6$	0	0	0	$-\frac{\sqrt{2}}{2}e_2$	0
5	$\frac{1}{2}e_7$	0	0	0	0	0	$-\frac{1}{2}e_1$
6	$\frac{1}{2}e_3$	$\frac{\sqrt{2}}{2}e_4$	$-\frac{1}{2}e_1$	$-\frac{\sqrt{2}}{2}e_2$	0	0	0
7	$\frac{1}{2}e_5$	0	0	0	$-\frac{1}{2}e_1$	0	0

Table A.5: Derivatives for $\mathfrak{n}_5(\sqrt{2},1,1,\sqrt{2})$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{\sqrt{2}}{2}e_3$	$\frac{\sqrt{2}}{2}e_2-\frac{1}{2}e_6$	$-\frac{1}{2}e_7$	0	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$
2	$\frac{\sqrt{2}}{2}e_3$	0	$-\frac{\sqrt{2}}{2}e_1$	0	$-\frac{\sqrt{2}}{2}e_7$	0	$\frac{\sqrt{2}}{2}e_5$
3	$\frac{\sqrt{2}}{2}e_2+\frac{1}{2}e_6$	$-\frac{\sqrt{2}}{2}e_1$	0	0	0	$-\frac{1}{2}e_1$	0
4	$\frac{1}{2}e_7$	0	0	0	0	0	$-\frac{1}{2}e_1$
5	0	$\frac{\sqrt{2}}{2}e_7$	0	0	0	0	$-\frac{\sqrt{2}}{2}e_2$
6	$\frac{1}{2}e_3$	0	$-\frac{1}{2}e_1$	0	0	0	0
7	$\frac{1}{2}e_4$	$\frac{\sqrt{2}}{2}e_5$	0	$-\frac{1}{2}e_1$	$-\frac{\sqrt{2}}{2}e_2$	0	0

Table A.6: Derivatives for $\mathfrak{n}_6(\sqrt{2},\sqrt{2},1,1)$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$-\frac{\sqrt{2}}{2}e_4$	$-\frac{\sqrt{2}}{2}e_5$	$\frac{\sqrt{2}}{2}e_2-\frac{1}{2}e_6$	$\frac{\sqrt{2}}{2}e_3-\frac{1}{2}e_7$	$\frac{1}{2}e_4$	$\frac{1}{2}e_5$
2	$\frac{\sqrt{2}}{2}e_4$	0	0	$-\frac{\sqrt{2}}{2}e_1$	0	0	0
3	$\frac{\sqrt{2}}{2}e_5$	0	0	0	$-\frac{\sqrt{2}}{2}e_1$	0	0
4	$\frac{\sqrt{2}}{2}e_2+\frac{1}{2}e_6$	$-\frac{\sqrt{2}}{2}e_1$	0	0	0	$-\frac{1}{2}e_1$	0
5	$\frac{\sqrt{2}}{2}e_3+\frac{1}{2}e_7$	0	$-\frac{\sqrt{2}}{2}e_1$	0	0	0	$-\frac{1}{2}e_1$
6	$\frac{1}{2}e_4$	0	0	$-\frac{1}{2}e_1$	0	0	0
7	$\frac{1}{2}e_5$	0	0	0	$-\frac{1}{2}e_1$	0	0

Table A.7: Derivatives for $\mathfrak{n}_7(-4,2,2,\sqrt{6},\sqrt{6})$

$\nabla_{e_i}e_j$	1	2	3	4	5	6	7
1	0	$2e_4$	0	$-2e_2$	0	e_7	$-e_6$
2	$-2e_4$	0	0	$2e_1$	e_7	0	$-e_5$
3	0	0	0	0	$\frac{\sqrt{6}}{2}e_7$	0	$-\frac{\sqrt{6}}{2}e_5$
4	$-2e_2$	$2e_1$	0	0	0	$\frac{\sqrt{6}}{2}e_7$	$-\frac{\sqrt{6}}{2}e_6$
5	0	e_7	$\frac{\sqrt{6}}{2}e_7$	0	0	0	$-e_2-\frac{\sqrt{6}}{2}e_3$
6	e_7	0	0	$\frac{\sqrt{6}}{2}e_7$	0	0	$-e_1-\frac{\sqrt{6}}{2}e_4$
7	e_6	e_5	$-\frac{\sqrt{6}}{2}e_5$	$-\frac{\sqrt{6}}{2}e_6$	$-e_2+\frac{\sqrt{6}}{2}e_3$	$-e_1+\frac{\sqrt{6}}{2}e_4$	0

B | On computations

B.1 Computations for proof of Proposition 4.1.2

We include computations of div $\tau^2_{\varphi_i}$ for $i = 4, 6, 7$ which were left out of the proof of Proposition 4.1.2. As discussed in the proof of Proposition 4.1.2, (3.5b) = 0 for each of the nilpotent cases. We compute (3.5a).

Case $(\mathfrak{n}_4(\sqrt{2}, 1, \sqrt{2}, 1), \varphi_4)$**.**

$$\nabla_{e_1}(\tau^2_{\varphi_4}(e_1)) = \nabla_{e_1}(-2e_1 + \sqrt{2}e_5) = -\frac{\sqrt{2}}{2}e_7$$
$$\nabla_{e_2}(\tau^2_{\varphi_4}(e_2)) = \nabla_{e_2}(0) = 0$$
$$\nabla_{e_3}(\tau^2_{\varphi_4}(e_3)) = \nabla_{e_3}(-3e_3) = 0$$
$$\nabla_{e_4}(\tau^2_{\varphi_4}(e_4)) = \nabla_{e_4}(-2e_4 + \sqrt{2}e_7) = 0$$
$$\nabla_{e_5}(\tau^2_{\varphi_4}(e_5)) = \nabla_{e_5}(\sqrt{2}e_1 - e_5) = \frac{\sqrt{2}}{2}e_7$$
$$\nabla_{e_6}(\tau^2_{\varphi_4}(e_6)) = \nabla_{e_6}(-3e_6) = 0$$
$$\nabla_{e_7}(\tau^2_{\varphi_4}(e_7)) = \nabla_{e_7}(\sqrt{2}e_4 - e_7) = 0.$$

Thus

$$(3.5\text{a}) = \sum_{i=1}^{7} g(\nabla_{e_i}(\tau^2_{\varphi_4}(e_i)), U) = -\frac{\sqrt{2}}{2}g(e_7, U) + \frac{\sqrt{2}}{2}g(e_7, U) = 0,$$

from which it follows that div $\tau^2_{\varphi_4} = 0$.

Case $(\mathfrak{n}_6(\sqrt{2}, \sqrt{2}, 1, 1), \varphi_6)$**.**

Using that $\tau^2_{\varphi_6}$ was obtained with respect to basis $(e_i)_i$, we have

$$\tau^2_{\varphi_6}(e_1) = 0, \;\; \tau^2_{\varphi_6}(e_2) = -2e_2 - \sqrt{2}e_6, \;\; \tau^2_{\varphi_6}(e_3) = -2e_3 - \sqrt{2}e_7, \;\; \tau^2_{\varphi_6}(e_4) = -3e_4$$

$$\tau^2_{\varphi_6}(e_5) = -3e_5, \;\; \tau^2_{\varphi_6}(e_6) = -\sqrt{2}e_2 - e_6, \;\; \tau^2_{\varphi_6}(e_7) = -\sqrt{2}e_3 - e_7.$$

We make the following observations:

1. $\tau^2_{\varphi_6}(e_i)$ is some linear combination of e_k's where $k \in \{2,...,7\}$ when $i \neq 1$ and $\tau^2_{\varphi_6}(e_1) = 0$ when $i = 1$.

2. From the previous item, $\nabla_{e_i}(\tau^2_{\varphi_6}(e_i))$ is either 0 or a multiple of e_1 when $i = 2,...,7$ since the entries in the bottom right 6×6 block of Table A.6 consists of only 0 or a multiple of e_1. Also note $\nabla_{e_1}(\tau^2_{\varphi_6}(e_1)) = 0$.

3. Then $g(\nabla_{e_i}(\tau^2_{\varphi_6}(e_i)), e_j) = 0 \; \forall \; j = 2,...,7$ as the first component is either a multiple of e_1 or it is 0. It is clear from Table A.6 that $g(\nabla_{e_1}(\tau^2_{\varphi_6}(e_1)), e_j) = 0 \; \forall \; j = 1,...,7$.

It follows from these observations that (3.5a) = 0 and thus div $\tau^2_{\varphi_6} = 0$.

Case $(\mathfrak{n}_7(-4,2,2,\sqrt{6},\sqrt{6}), \varphi_7)$**.** We compute each term of sum (3.5a):

$$\nabla_{e_1}(\tau^2_{\varphi_7}(e_1)) = \nabla_{e_1}(-4e_1 + 2\sqrt{6}e_4) = 2\sqrt{6}(-2e_2) = -4\sqrt{6}e_2$$
$$\nabla_{e_2}(\tau^2_{\varphi_7}(e_2)) = \nabla_{e_2}(-4e_2 + 2\sqrt{6}e_3) = 2\sqrt{6}(0) = 0$$
$$\nabla_{e_3}(\tau^2_{\varphi_7}(e_3)) = \nabla_{e_3}(2\sqrt{6}e_2 - 6e_3) = 2\sqrt{6}(0) = 0$$
$$\nabla_{e_4}(\tau^2_{\varphi_7}(e_4)) = \nabla_{e_4}(2\sqrt{6}e_1 - 22e_4) = 2\sqrt{6}(-2e_2) = -4\sqrt{6}e_2$$
$$\nabla_{e_5}(\tau^2_{\varphi_7}(e_5)) = \nabla_{e_5}(-10e_5 + 4\sqrt{6}e_7) = 4\sqrt{6}(-e_2 - \frac{\sqrt{6}}{2}e_3) = -4\sqrt{6}e_2 - 2\sqrt{6}e_3$$
$$\nabla_{e_6}(\tau^2_{\varphi_7}(e_6)) = \nabla_{e_6}(-10e_6) = 0$$
$$\nabla_{e_7}(\tau^2_{\varphi_7}(e_7)) = \nabla_{e_7}(4\sqrt{6}e_5 - 16e_7) = 4\sqrt{6}(-e_2 + \frac{\sqrt{6}}{2}e_3) = -4\sqrt{6}e_2 + 2\sqrt{6}e_3.$$

Thus

$$(3.5\text{a}) = \sum_{i=1}^{7} g(\nabla_{e_i}(\tau^2_{\varphi_7}(e_i)), U) = -16\sqrt{6}g(e_2, U),$$

from which it follows that div $\tau^2_{\varphi_7} \neq 0$.

B.2 On computation of operators in equation 3.2

We include the list of closed G_2-structures φ_i found by Nicolini in [Nic18] on nilpotent Lie algebras $\mathfrak{n}_i$ for $i = 1, \ldots, 7$.

1. φ_1 can be any closed G_2-structure on the trivial nilpotent Lie algebra $\mathfrak{n}_1$;
2. $\varphi_2 = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346} \in \Lambda^3\mathfrak{n}_2^*$;
3. $\varphi_3 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3\mathfrak{n}_3^*$;
4. $\varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236} \in \Lambda^3\mathfrak{n}_4^*$;
5. $\varphi_5 = e^{134} + e^{457} - e^{246} - e^{125} - e^{356} + e^{167} - e^{237} \in \Lambda^3\mathfrak{n}_5^*$;
6. $\varphi_6 = e^{123} + e^{347} + e^{356} + e^{145} - e^{246} + e^{167} + e^{257} \in \Lambda^3\mathfrak{n}_6^*$;
7. $\varphi_7 = e^{127} + e^{135} - e^{146} - e^{236} - e^{245} + e^{347} + e^{567} \in \Lambda^3\mathfrak{n}_7^*$;

Given a G_2-structure φ and structure equations, we discuss the computations required to obtain relevant operators in 3.2 for the interested reader. We do this for the case of $(\mathfrak{n}_5, \varphi_5)$. Let $(e_i)_{i=1}^7$ be an orthonormal basis for $\mathfrak{n}_5 = \mathfrak{n}_5(a,b,c,d)$, the 7-dimensional nilpotent Lie algebra with structure

$$[e_1,e_2] = -ae_3, \quad [e_1,e_3] = -be_6, \quad [e_1,e_4] = -ce_7, \quad [e_2,e_5] = -de_7, \quad a,b,c,d \in \mathbb{R}^*.$$

Consider the G_2-structure $\varphi_5 = e^{134} + e^{457} - e^{246} - e^{125} - e^{356} + e^{167} - e^{237} \in \Lambda^3\mathfrak{n}_5^*$. Using that the exterior derivative (or differential) $d : \Omega^1(M) \to \Lambda^2(M)$ of invariant 1-forms $\omega \in \Lambda^1(M)$ is $d\omega = -\omega([X,Y])$, [e.g., $de^3 = -e^3([e_1,e_2]) = -e^3(-ae_3) = a$ and so $de^3 = ae^{12}$ as $de^i = -c^i_{jk}e^{jk}$] we get

$$de^1 = de^2 = de^4 = de^5 = 0, \quad de^3 = ae^{12}, \quad de^6 = be^{13}, \quad de^7 = ce^{14} + de^{25}.$$

It is common to write this structure as:

$$\mathfrak{n}_5 = (0,0,ae^{12},0,0,be^{13},ce^{14}+de^{25}).$$

By straightforward computations, one observes that $d\varphi_5 = 0$ if and only if $a = d$ and $b = c$. Thus $\mathfrak{n}_5 = \mathfrak{n}_5(a,b,b,a)$. Nicolini observed that if $a^2 = 2b^2$, then $(\mathfrak{n}_5(a,b,b,a),\varphi_5)$ is a semi-algebraic soliton, hence a Laplacian soliton. The choice of $b = 1$, $a = \sqrt{2}$ yields $(\mathfrak{n}_5(\sqrt{2},1,1,\sqrt{2}),\varphi_5)$ and structure

$$[e_1,e_2] = -\sqrt{2}e_3, \;\; [e_1,e_3] = -e_6, \;\; [e_1,e_4] = -e_7, \;\; [e_2,e_5] = -\sqrt{2}e_7.$$

It is from these structure equations that Table A.5 is obtained via Koszul formula. Then τ_{φ_5} and Ric_{φ_5} are obtained from computing

$$\tau_\varphi = -*d*\varphi_5 \quad \text{and} \quad \mathrm{Ric}_{\varphi_5}(e_j) = \sum_{i=1}^{7} R(e_j,e_i)e_i \;\; \forall j = 1,...7,$$

respectively. We get

$$\tau_{\varphi_5} = -e^{46}+e^{37}-\sqrt{2}e^{35}+\sqrt{2}e^{17} \;\text{ and }\; \mathrm{Ric}_{\varphi_5} = \mathrm{Diag}\left(-2,-2,\frac{1}{2},-\frac{1}{2},-1,\frac{1}{2},\frac{3}{2}\right).$$

Since $\tau_{\varphi_5} = g_{\varphi_5}(\tau_{\varphi_5}\cdot,\cdot)$ is a skew-symmetric 2-form, it has matrix representation:

$$\tau_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tau_{\varphi_5}^2 = \begin{pmatrix} -2 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -3 \end{pmatrix}.$$

Furthermore, $\mathrm{scal}_{\varphi_5} = \mathrm{tr}\,\mathrm{Ric}_{\varphi_5} = -3$ and so by equation (3.2), we get

$$Q_{\varphi_5} = \begin{pmatrix} -2 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 1 \end{pmatrix}.$$

B.3 Derivation of equation 2.8

The purpose of this section is to derive the Laplacian soliton equation (2.8). The arguments below are essentially the same arguments as in the proof of [[LW17], Proposition 9.4]. We include them here to illustrate how one reconciles the differences in notation and convention between Lotay-Wei and Lauret.

Proof of equation (2.8). The goal is to show that the Laplacian soliton equation $\Delta_\varphi \varphi = \lambda\varphi + \mathscr{L}_X \varphi$ is equivalent to

$$i_\varphi(-q_\varphi - \frac{1}{3}\lambda g_\varphi - \frac{1}{2}\mathscr{L}_X g_\varphi) = 0,$$

from which equation (2.8) will follow by injectivity of i_φ. Note that from Lauret's notation and convention for the map i_φ, we have

$$i_\varphi(q_\varphi) = -2\theta(Q_\varphi) = -2\Delta_\varphi \varphi \quad \text{and} \quad i_\varphi(\frac{1}{3}\lambda g_\varphi) = 2\lambda\varphi$$

since $i_\varphi(g_\varphi) = 6\varphi$ (see Subsection 2.1.3 for details). Now the Laplacian soliton equation is

$$\begin{aligned}\Delta_\varphi \varphi = \lambda\varphi + \mathscr{L}_X\varphi \iff 2\Delta_\varphi\varphi = 2\lambda\varphi + 2\mathscr{L}_X\varphi &\iff -i_\varphi(q_\varphi) = i_\varphi(\frac{1}{3}\lambda g_\varphi) + 2\mathscr{L}_X\varphi \\ &\iff -i_\varphi(q_\varphi) - i_\varphi(\frac{1}{3}\lambda g_\varphi) - 2\mathscr{L}_X\varphi = 0.\end{aligned}$$

So by linearity of i_φ, it remains to show $i_\varphi(\frac{1}{2}\mathscr{L}_X g_\varphi) = 2\mathscr{L}_X\varphi$. To do this, we see where i_φ was used in Lotay-Wei's arguments and scale these terms by a factor of 2. Since $\mathscr{L}_X\varphi \in \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$, we can write

$$\mathscr{L}_X\varphi = a\varphi + W \lrcorner \psi + i_\varphi(\eta). \tag{B.1}$$

The term $W \lrcorner \psi \in \Omega^3_7$ vanishes by the same arguments in proof of [[LW17], Proposition 9.4]. More precisely, since $a\varphi \in \Omega^3_1$ and the soliton equation $\lambda\varphi + \mathscr{L}_X\varphi = \Delta_\varphi\varphi \in \Omega^3_1 \oplus \Omega^3_{27}$, these together tell us that $\mathscr{L}_X\varphi$ has no Ω^3_7 component and so $W \lrcorner \psi = 0$.

We now follow through Lotay-Wei's arguments in solving for a and η in (B.1). The computations for a are the same as they do not involve i_φ. Hence $a = \frac{3}{7}\operatorname{div}(X)$. To find η, Lotay-Wei found two equivalent expressions for $\mathscr{L}_X\varphi$, set them equal to each other, and then solved for η. The two expressions for $\mathscr{L}_X\varphi$ are

- $4\operatorname{div}(X)g + 4(\nabla_i X_j - \nabla_j X_i)$; and
- $12ag + 8\eta$,

We check again whether or not the computations of these two terms involved i_φ. The computations for $12ag$ do not involve i_φ, but they do for 8η. Going back in Lotay-Wei's computations, we scale the terms involving $i_\varphi(\eta)$ by 2. Thus in Lauret's convention, the expression $12ag + 8\eta$ is instead $12ag + 16\eta$. The computations for $4\operatorname{div}(X)g + 4(\nabla_i X_j - \nabla_j X_i)$ do not involve i_φ at all. We set $4\operatorname{div}(X)g + 4(\nabla_i X_j - \nabla_j X_i)$ equal to $12ag + 16\eta$ and solve for η to get

$$\eta = -\frac{3}{4}ag + \frac{1}{4}\operatorname{div}(X)g + \frac{1}{4}(\nabla_i X_j - \nabla_j X_i).$$

We now substitute the expressions found for a and η back into the decomposition (B.1). Observe that

$$a\varphi = i_\varphi(\frac{1}{6}ag) = i_\varphi(\frac{1}{6}(\frac{3}{7}\operatorname{div}(X))g) = i_\varphi(\frac{1}{14}\operatorname{div}(X)g)$$

and

$$\begin{aligned} i_\varphi(\eta) &= i_\varphi(-\frac{3}{4}(\frac{3}{7}\operatorname{div}(X))g + \frac{1}{4}\operatorname{div}(X)g + \frac{1}{4}\mathscr{L}_X g) \\ &= i_\varphi(-\frac{1}{14}\operatorname{div}(X)g + \frac{1}{4}\mathscr{L}_X g). \end{aligned}$$

Thus

$$\begin{aligned} \mathscr{L}_X\varphi = a\varphi + i_\varphi(\eta) &= i_\varphi(\frac{1}{14}\operatorname{div}(X)g) + i_\varphi(-\frac{1}{14}\operatorname{div}(X)g + \frac{1}{4}\mathscr{L}_X g) \\ &= i_\varphi(\frac{1}{4}\mathscr{L}_X g) \\ &= \frac{1}{2}i_\varphi(\frac{1}{2}\mathscr{L}_X g), \end{aligned}$$

which holds if and only if $i_\varphi(\frac{1}{2}\mathscr{L}_X g) = 2\mathscr{L}_X\varphi$ as desired. □

B.4 A Bochner formula

We include the proof of a Bochner formula with $f = r$ used in part of the proof of Proposition 5.3.1 (2) for the interested reader. By [[HPW15], Proposition 2.7], the shape operator $T(X) = \nabla^{G_D}_X e_7$ is related to symmetrization S by $T = -S$. With $e_7 = \nabla r$, we have $S(X) = -T(X) = -\nabla_X \nabla r$. Then

$$\begin{aligned}
\operatorname{div}(S)(X) &= g((\nabla_{E_i} S)(E_i), X) \\
&= g(\nabla_{E_i}(S(E_i)) - S(\nabla_{E_i} E_i), X) \\
&= -g(\nabla_{E_i}(\nabla_{E_i} \nabla r), X) + g(\nabla_{\nabla_{E_i} E_i} \nabla r, X) \\
&= -\nabla_{E_i}(g(\nabla_{E_i} \nabla r, X)) + g(\nabla_{E_i} \nabla r, \nabla_{E_i} X) + g(\nabla_X \nabla r, \nabla_{E_i} E_i) \\
&= -\nabla_{E_i}(\operatorname{Hess} r(E_i, X)) + \operatorname{Hess} r(E_i, \nabla_{E_i} X) + \operatorname{Hess} r(X, \nabla_{E_i} E_i).
\end{aligned}$$

Substituting $X = \nabla r$, the first and third terms of the last expression are 0 as $\nabla_{\nabla r} \nabla r = 0$. Then by symmetry of T, [[Pet16], Proposition 3.2.11 (2)], and the fact that covariant differentiation commutes with type change, we have:

$$\begin{aligned}
\operatorname{div}(S)(\nabla r) &= \operatorname{Hess} r(E_i, \nabla_{E_i} \nabla r) = g(\nabla_{E_i} \nabla r, \nabla_{E_i} \nabla r) = g(T(E_i), T(E_i)) \\
&= g(T^2(E_i), E_i) = \operatorname{Hess} r^2(E_i, E_i) \\
&= -R(E_i, \nabla r, \nabla r, E_i) - (\nabla_{\nabla r} \operatorname{Hess} r)(E_i, E_i) \\
&= -R(\nabla r, E_i, E_i, \nabla r) - g((\nabla_{\nabla r} T)(E_i), E_i) \\
&= -R(\nabla r, E_i, E_i, \nabla r) - g(\nabla_{\nabla r}(T(E_i)) - T(\nabla_{\nabla r} E_i), E_i) \\
&= -\operatorname{Ric}_D(\nabla r, \nabla r) - g(\nabla_{\nabla r}(\nabla_{E_i} \nabla r) - \nabla_{\nabla_{\nabla r} E_i} \nabla r, E_i) \\
&= -\operatorname{Ric}_D(\nabla r, \nabla r) - g(\nabla^2_{\nabla r, E_i} \nabla r, E_i) \\
&= -\operatorname{Ric}_D(\nabla r, \nabla r) - D_{\nabla r} \operatorname{div}(\nabla r) \\
&= -\operatorname{Ric}_D(\nabla r, \nabla r) - D_{\nabla r}(\Delta r) \\
&= -\operatorname{Ric}_D(\nabla r, \nabla r),
\end{aligned}$$

where the last equality follows from Δr being constant on one-dimensional extensions. Thus $\operatorname{div}(S)(\nabla r) = -\operatorname{Ric}_D(\nabla r, \nabla r) = -\operatorname{Ric}_D(e_7, e_7) = \operatorname{tr}(S^2)$ by Lemma 5.1.2 (1). The second to last expression is the right-hand side of the Bochner formula $\operatorname{div} \nabla \nabla f = \operatorname{Ric}(\nabla f) + \nabla \Delta f$ with $f = r$.

Bibliography

[Arr13] Romina M. Arroyo, *The Ricci flow in a class of solvmanifolds*, Differential Geom. Appl. **31** (2013), no. 4, 472–485, DOI 10.1016/j.difgeo.2013.04.002. MR3066027

[AK75] D. V. Alekseevskiĭ and B. N. Kimelʹfelʹd, *Structure of homogeneous Riemannian spaces with zero Ricci curvature*, Funkcional. Anal. i PriloŽen. **9** (1975), no. 2, 5–11 (Russian). MR0402650

[Bry06] Robert L. Bryant, *Some remarks on G_2-structures*, Proceedings of Gökova Geometry-Topology Conference 2005, Gökova Geometry/Topology Conference (GGT), Gökova, 2006, pp. 75–109. MR2282011

[BX11] Robert L. Bryant and Feng Xu, *Laplacian flow for closed G_2-structures: short time behavior* (2011), available at arxiv.org/abs/1101.2004.

[CI07] Richard Cleyton and Stefan Ivanov, *On the geometry of closed G_2-structures*, Comm. Math. Phys. **270** (2007), no. 1, 53–67, DOI 10.1007/s00220-006-0145-7. MR2276440

[CLN06] Bennett Chow, Peng Lu, and Lei Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006. MR2274812

[CF11] Diego Conti and Marisa Fernández, *Nilmanifolds with a calibrated G_2-structure*, Differential Geom. Appl. **29** (2011), no. 4, 493–506, DOI 10.1016/j.difgeo.2011.04.030. MR2811660

[dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty. MR1138207

[Fer87] Marisa Fernández, *An example of a compact calibrated manifold associated with the exceptional Lie group G_2*, J. Differential Geom. **26** (1987), no. 2, 367–370. MR906398

[FFM16] Marisa Fernández, Anna Fino, and Víctor Manero, *Laplacian flow of closed G_2-structures inducing nilsolitons*, J. Geom. Anal. **26** (2016), no. 3, 1808–1837, DOI 10.1007/s12220-015-9609-3. MR3511459

[FG82] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. (4) **132** (1982), 19–45 (1983), DOI 10.1007/BF01760975. MR696037

[FR20] Anna Fino and Alberto Raffero, *Remarks on homogeneous solitons of the* G_2*-Laplacian flow*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 4, 401–406, DOI 10.5802/crmath.39. MR4134249

[Fre13] Marco Freibert, *Calibrated and parallel structures on almost Abelian Lie algebras* (2013), arXiv 1307.2542, primary class math.DG

[Gar22] Augustín Garrone, *New examples of divergence-free G_2-structures* (2022), arXiv 2205.08700, primary class math.DG

[Gri21] Erin Griffin, *Gradient ambient obstruction solitons on homogeneous manifolds*, Ann. Global Anal. Geom. **60** (2021), no. 3, 469–499, DOI 10.1007/s10455-021-09784-3. MR4304859

[Hit00] Nigel Hitchin, *The geometry of three-forms in six dimensions*, J. Differential Geom. **55** (2000), no. 3, 547–576. MR1863733

[HKP22] Mark Haskins, Ilyas Khan, and Alec Payne, *Uniqueness of Asymptotically Conical Gradient Shrinking Solitons in G_2-Laplacian Flow* (2022), arXiv 2210.07954, primary class math.DG

[HN21] Mark Haskins and Johannes Nordström, *Cohomogeneity-one solitons in Laplacian flow: local, smoothly-closing and steady solitons* (2021), arXiv 2112.09095, primary class math.DG

[HPW15] Chenxu He, Peter Petersen, and William Wylie, *Warped product Einstein metrics on homogeneous spaces and homogeneous Ricci solitons*, J. Reine Angew. Math. **707** (2015), 217–245, DOI 10.1515/crelle-2013-0078. MR3403459

[Kar09] Spiro Karigiannis, *Flows of G_2-structures. I*, Q. J. Math. **60** (2009), no. 4, 487–522, DOI 10.1093/qmath/han020. MR2559631

[Kar20] ______, *Introduction to G_2 geometry*, Lectures and surveys on G_2-manifolds and related topics, Fields Inst. Commun., vol. 84, Springer, New York, 2020, pp. 3–50, DOI 10.1007/978-1-0716-0577-6-1. MR4295852

[KW17] Lee Kennard and William Wylie, *Positive weighted sectional curvature*, Indiana Univ. Math. J. **66** (2017), no. 2, 419–462, DOI 10.1512/iumj.2017.66.6013. MR3641482

[KWY19] Lee Kennard, William Wylie, and Dmytro Yeroshkin, *The weighted connection and sectional curvature for manifolds with density*, J. Geom. Anal. **29** (2019), no. 1, 957–1001, DOI 10.1007/s12220-018-0025-3. MR3897040

[Kri21] Anusha M. Krishnan, *Diagonalizing the Ricci tensor*, J. Geom. Anal. **31** (2021), no. 6, 5638–5658, DOI 10.1007/s12220-020-00495-y. MR4267620

[Lau17a] Jorge Lauret, *Laplacian flow of homogeneous G_2-structures and its solitons*, Proc. Lond. Math. Soc. (3) **114** (2017), no. 3, 527–560, DOI 10.1112/plms.12014. MR3653239

[Lau17b] ______, *Laplacian solitons: questions and homogeneous examples*. part B, Differential Geom. Appl. **54** (2017), no. part B, 345–360, DOI 10.1016/j.difgeo.2017.06.002. MR3693936

[Lau11] ______, *Ricci soliton solvmanifolds*, J. Reine Angew. Math. **650** (2011), 1–21, DOI 10.1515/CRELLE.2011.001. MR2770554

[Lee18] John M. Lee, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. Second edition of [MR1468735]. MR3887684

[Lee13] ______, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[LN20] Jorge Lauret and Marina Nicolini, *The classification of ERP G_2-structures on Lie groups*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 6, 2489–2510, DOI 10.1007/s10231-020-00977-4. MR4165690

[Lin13] Christopher Lin, *Laplacian solitons and symmetry in G_2-geometry*, J. Geom. Phys. **64** (2013), 111–119, DOI 10.1016/j.geomphys.2012.11.006. MR3004019

[LW13] Jorge Lauret and Cynthia Will, *On the diagonalization of the Ricci flow on Lie groups*, Proc. Amer. Math. Soc. **141** (2013), no. 10, 3651–3663, DOI 10.1090/S0002-9939-2013-11813-7. MR3080187

[Lee13] John M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[LW17] Jason D. Lotay and Yong Wei, *Laplacian flow for closed* G_2 *structures: Shi-type estimates, uniqueness and compactness*, Geom. Funct. Anal. **27** (2017), no. 1, 165–233, DOI 10.1007/s00039-017-0395-x. MR3613456

[Man20] Víctor Manero, *Compact solvmanifolds with calibrated and cocalibrated* G_2*-structures*, Manuscripta Math. **162** (2020), no. 3-4, 315–339, DOI 10.1007/s00229-019-01133-w. MR4109489

[MT07] John Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR2334563

[Nic18] Marina Nicolini, *Laplacian solitons on nilpotent Lie groups*, Bull. Belg. Math. Soc. Simon Stevin **25** (2018), no. 2, 183–196, DOI 10.36045/bbms/1530065008. MR3819121

[Nic22] ———, *New examples of shrinking Laplacian solitons*, Q. J. Math. **73** (2022), no. 1, 239–259, DOI 10.1093/qmath/haab029. MR4395079

[Pet16] Peter Petersen, *Riemannian geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. MR3469435

[PR19] Fabio Podestà and Alberto Raffero, *On the automorphism group of a closed* G_2*-structure*, Q. J. Math. **70** (2019), no. 1, 195–200, DOI 10.1093/qmath/hay045. MR3927848

[PW09a] Peter Petersen and William Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. **241** (2009), no. 2, 329–345, DOI 10.2140/pjm.2009.241.329. MR2507581

[PW09b] ———, *On gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. **137** (2009), no. 6, 2085–2092, DOI 10.1090/S0002-9939-09-09723-8. MR2480290

[PW10] ———, *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), no. 4, 2277–2300, DOI 10.2140/gt.2010.14.2277. MR2740647

[PW22] ———, *Rigidity of homogeneous gradient soliton metrics and related equations*, Differential Geom. Appl. **84** (2022), Paper No. 101929, 29, DOI 10.1016/j.difgeo.2022.101929. MR4457372

[Wyl15] William Wylie, *Sectional curvature for Riemannian manifolds with density*, Geom. Dedicata **178** (2015), 151–169, DOI 10.1007/s10711-015-0050-3. MR3397488

[Wyl16] ———, *Some curvature pinching results for Riemannian manifolds with density*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 823–836, DOI 10.1090/proc/12853. MR3430857

[WY16] William Wylie and Dmytro Yeroshkin, *On the geometry of Riemannian manifolds with density* (2016), available at arxiv.org/abs/1602.08000.

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M.S., Mathematics, *The City College of New York* — *May 2016*

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Awards & Grants

Kibbey Prize, *Syracuse University* — *2023*
Departmental award for outstanding achievement in the PhD program

AMS Travel Grant – Joint Mathematics Meetings — *2023*
Graduate student travel grant

Outstanding Teaching Assistant Award, *Syracuse University* — *2022*
Annual university-wide award presented to 4% of TAs

Teaching Mentor, *Syracuse University* — *2022-2023*
One of thirty-two selected across the university for excellence in teaching and overall graduate study

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Rich Summer Research Fund, *CCNY* — *2016*
Summer research internship

Teaching Experience

Instructor of Record, *Syracuse University* — *Summer 2018 – Current*

- Prepare and deliver 2-3 Lectures per week; lead recitation once per week
- Construct syllabus, assign homework/in-class assignments, write and grade quizzes/exams

MAT 286 Life Sciences Calculus II	Current
MAT 296 Calculus II	Fall 2022
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MAT 285 Life Sciences Calculus I (2 sections)	Spring 2021
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MAT 221 Elementary Probability and Statistics I	Summer 2020
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Teaching Assistant, *Syracuse University* *Fall 2017 – Current*

- Held 4-5 recitation sections per week
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- Grade homework assignments (current grader for MAT 551)

MAT 551	Fundamental Concepts of Geometry	Current
MAT 221	Elementary Probability and Statistics I (5 sections)	Fall 2021
MAT 221	Elementary Probability and Statistics I (5 sections)	Fall 2019
MAT 221	Elementary Probability and Statistics I (4 sections)	Fall 2018
MAT 221	Elementary Probability and Statistics I (4 sections)	Spring 2018
MAT 221	Elementary Probability and Statistics I (4 sections)	Fall 2017

Math Clinic Instructor, *Syracuse University* *Fall 2018 – Spring 2019*

- Assisted students in undergraduate math courses at the math clinic 1-2 hours per week

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MATH 104	Statistics I	Fall 2016
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Math and Physics Teacher, *C2 Education* *Mar. 2014 – Feb. 2015, Sept. 2016 – Feb. 2017*

- Taught 3-4 students per 2-hour session in various subjects (15-30 hours per week) and taught summer SAT classes

Subjects: SAT/ACT Math & Science, SAT Math & Physics Subject Tests, AP Calculus AB/BC & AP Physics AB/BC, Multivariable Calculus, Pre-calculus, & K-12 Math

Mentoring & Service

Teaching Mentor, *Syracuse University* *May 16, 2022 – Current*
"Core faculty" in Teaching Assistant Orientation Program

- Served as small group leader and mentor to incoming graduate TAs; led microteaching sessions
- Participated in development and implementation of TA program activities throughout academic year
  - TA Orientation Program Session: *"Teaching Online"*
    Session Planning Partner: Ashley Douglass (Department of Psychology)
- Acted as teaching consultant for The Graduate School at SU

Teaching Mentor Selection Committee, *Syracuse University* *Current*

- Examined teaching portfolios of applicants and determined short list of candidates
- Organized and held interviews; recommended Teaching Mentors to the Graduate School

Directed Reading Program Co-organizer, *Syracuse University* *Current*
Organized with Chelsea Sato (Department of Mathematics)

- Pair graduate student mentors with undergraduate student mentees based on mentor expertise and mentee's interest(s)
- Recruit program participants, create flier/poster promoting the program, draft and edit program emails
- Organize program meetings and events including end-of-semester participant presentations

Directed Reading Program Mentor, *Syracuse University*

- Guided undergraduate students through semester-long reading project
- Assigned readings and problems relevant to project
- Assisted mentee in preparation of a 12-minute end-of-semester presentation

Mentee: Lisa Zhang *Fall 2022*
Project: *"Introduction to Lie algebras"*

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Field of Study: Riemannian Geometry

- Read mathematics journal articles, books, and notes on Riemannian geometry
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Research Assistant, *Research Foundation CUNY/CCNY* *Spring 2016, Summer 2016*
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Field of Study: Algebra

Publications

On homogeneous closed gradient Laplacian solitons arxiv.org/abs/2302.11441

(Expository) *Matrix semigroup relations arising from idempotent and nilpotent matrices,* CUNY Research Foundation and Rich Summer Internship – 2016

Talks

"Homogeneous Closed Gradient Laplacian Solitons" Joint Mathematics Meeting, AMS Special Session on New Developments in Differential Geometry and Topology – Boston, Massachusetts, January 5, 2023

"Normed Division Algebras, Vector Cross Products, and G2-Structures" MGO Colloquium – Syracuse University, October 20, 2022

"Homogeneous Closed Gradient Laplacian Solitons" Geometry & Topology Seminar – Syracuse University, October 7, 2022

"Teaching Online" TA Orientation Program – Syracuse University, August 18, 2022

(Expository) "Relations Between Two 2 x 2 Matrices that Determine a Finite Product of Those Two Matrices" Mathematical Association of America, Metropolitan New York Section – Vaughn College, May 1, 2016

Conferences & Workshops Attended

43rd, 44th, 47th, 48th Annual New York State Regional Graduate Mathematics Conference *April 1, 2023*

Syracuse University

Joint Mathematics Meetings *January 4-7, 2023*
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8th Geometry-Topology Summer School *August 15-27, 2022*
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GTA Philadelphia Mathematics Conference *May 20-22, 2022*
Temple University

Rutgers Geometric Analysis Conference *May 16-19, 2022*
Rutgers University, New Brunswick

Mini school on RCD Spaces: Splitting Theorems and Applications *November 9-11, 2021*
National Autonomous University of Mexico

AMS Spring Eastern Virtual Sectional Meeting *March 20-21, 2021*
(formerly at Brown University); Special Session on Recent Developments in Differential Geometry

Binghamton University Graduate Conference in Algebra and Topology *November 7-8, 14-15, 2020*
Binghamton University

AMS Fall Western Virtual Sectional Meeting *October 24-25, 2020*
(formerly at University of Utah); Special Session on Several Complex Variables

MAA Metro New York Section Annual Meeting *May 1, 2016*
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Professional Memberships

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