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N-bakry Emery Ricci Curvature & N-quasi Einstein Metrics

Alice Wu Lim
Syracuse University, candalice@gmail.com

Abstract

We begin the thesis by giving an intuitive introduction to calculus on manifolds for the non-mathematician. We then give a semi-intuitive description on Ricci curvature for the non-geometer. We give a description of the N-Bakry-Émery Ricci curvature and the N-quasi Einstein metric. The main results in this thesis are related to the N-Bakry-Émery Ricci curvature and the N-quasi Einstein metric.

Our first set of main results are as follows. We generalize topological results known for noncompact manifolds with nonnegative Ricci curvature to spaces with nonnegative N-Bakry Émery Ricci curvature. We study the Splitting Theorem and a property called the geodesic loops to infinity property in relation to spaces with nonnegative N-Bakry Émery Ricci Curvature. In addition, we show that if M^n is a complete, noncompact Riemannian manifold with nonnegative N-Bakry Émery Ricci curvature where $N > n$, then $H_{n-1}(M, \mathbb{Z})$ is 0.

For our second set of main results, we classify the compact locally homogeneous non-gradient N-quasi Einstein 3-manifolds. Along the way, we also prove that given a compact quotient of a Lie group of any dimension that is

N-quasi Einstein, the potential vector field X must be left invariant and Killing. We also classify the nontrivial N-quasi Einstein metrics that are a compact quotient of be the product of two Einstein metrics. We also show that S^1 is the only compact manifold of any dimension which admits a metric which is nontrivially N-quasi Einstein and Einstein.

N-Bakry Emery Ricci curvature & N-quasi Einstein Metrics

by

Alice Wu Lim

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1 Calculus on manifolds

This chapter is meant to be an intuitive explanation of objects like manifolds and Euclidean space. There are few definitions and many figures and images. For mathematical definitions of such objects, the author recommends Do Carmo's *Riemannian Geometry* [8] and Lawson's *Topology: A Geometric Approach* [21].

1.1 What is Euclidean space?

While the name "Euclidean space" sounds like a difficult mathematical term, the space itself is easy to understand as it is modeled after our world.

The world we live in is an example of 3-dimensional Euclidean space. In our world, we can move forward, backward, left, and right. If we get in an airplane, we can also move up and down.

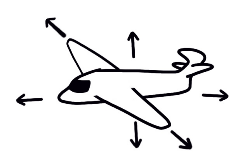


Figure 1.1: If we are in an airplane in our world, we can move upward, downward, forward, backward, left and right.

The forward and backward motion corresponds to one of the dimensions in 3-dimensional Euclidean space. The other two dimensions are the left and right movements and the up and down movements, respectively.

Two-dimensional Euclidean space is similar to the world that lies on a chalk board or a white board. If we are a stick figure on the chalk board with the ability to move around on the chalk board, we can move up and down, as well as left and right. The up and down movement corresponds to one dimension and the left and right movement corresponds to the second dimension. We lose the ability to move forward and backward when we switch from 3-dimensions to 2-dimensions.

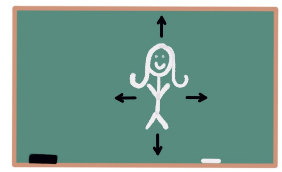


Figure 1.2: We can only move left, right, up, down, and a combination of the four directions in two-dimensional Euclidean space.

For an example of 1-dimensional Euclidean space, imagine a piece of string laid flat on a table with the left end of the string toward the left side of the table and the right end of the string toward the right side of the table. Some segments of the string are colored in with pink marker. In this scenario, we are the pink segments in the string. Imagine that we are able to move along the string. The only movements we can make are left and right. This is the one-dimension in one-dimensional Euclidean space.

Figure 1.3: The string is an example of Euclidean space. The pink segment can only move left and right on the string.

Euclidean space in 0-dimensions is the least exciting of the ones we can picture. This space looks like a point. If we are living on a point, then we must be the entire point, and we are unable to move. This inability to move corresponds to the 0-dimensions.

Figure 1.4: Zero-dimensional Euclidean space

While 0, 1, 2, and 3 dimensional Euclidean spaces are the only spaces we can easily picture, the concepts of n-dimensional Euclidean space for $n > 3$ exist and can be interpreted in a similar way to the lower dimensional Euclidean spaces. Note that in each of our examples of Euclidean space, the spaces themselves are “flat" in some sense. Our chalkboard is a flat surface and our string is laid out flat from left to right. This is a defining characteristic of Euclidean space. But what about objects that aren't flat, like a rubber ball or an unfilled donut? Can we talk about living on objects which aren't flat in a meaningful way?

1.2 *What is a manifold?*

In Section 1.1, we explored the concept of the Euclidean space and how the dimensions of the Euclidean space affect the available movements to an object that lives in such a space. In this section, we give some intuition for an object called a manifold.

A manifold is an object which locally looks like Euclidean space. In other words, if we zoom in closely enough to a manifold of dimension n, the manifold will start to look like Euclidean space of dimension n. We often call manifolds of dimension n M^n.

As a first example, we will look at the sphere. When we say a "sphere", we mean a hollow object: one which resembles a beach ball or a basketball, rather than a baseball.

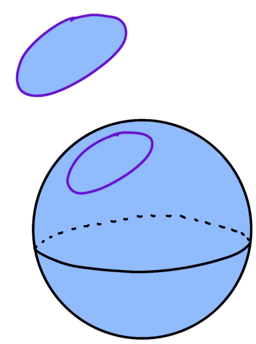


Figure 1.5: The patch outlined in purple on the sphere resembles 2-dimensional Euclidean space.

While we often think of a sphere as a 3-dimensional object since they exist in our world, we classify the sphere as a 2-dimensional manifold. If we zoom in closely to a sphere, we see that the sphere resembles two-dimensional Euclidean space.

It is important to note that when picturing ourselves living on a 2-dimensional manifold such as a sphere, we should not imagine the sphere in the context of a sphere living in 3-dimensional Euclidean space. If we are living on the 2-dimensional sphere, we are not sitting on the sphere the way we are currently sitting on the earth. Rather, we are a 2-dimensional object living on the sphere, in the same way we were in the chalk board example.

Figure 1.6: This beach ball is an example of a 2-manifold. Since we are living on the beach ball, we are also a 2-D figure and we can move left and right as well as up and down.

An object like the interior of a baseball is a 3-manifold since if we zoom in closely enough to the baseball, the baseball resembles three-dimensional Euclidean space. Again, we shouldn't imagine this baseball in our world; rather, the baseball is the entire world. If we were to live in the baseball, we would be a 3-dimensional object within the baseball, rather than sitting on top of the baseball.

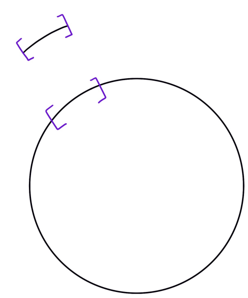


Figure 1.7: The line segment within the purple brackets on this circle resembles 1-dimensional Euclidean space.

An example of a 1-dimensional manifold is a circle. If we were to be living in the circle, we would be segments of the circle.

An example of a 0-dimensional manifold is a set of disjoint points. If we were to be living in the 0-manifold, we would be a subset of the disjoint points.

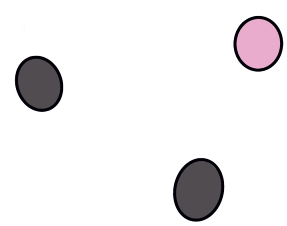


Figure 1.8: The three points in this graphic represent a 0-dimensional manifold. If we are living on this 0-manifold, then we must be one of the points. In this graphic, we are the pink point.

1.3 *Differentiable manifold: an intuitive summary*

Now that we have some intuition for the concept of an n-dimensional manifold, we can talk about the objects that we will work with in this thesis: the differentiable manifold. Although the definition of a differentiable manifold is quite abstract, we'd like to give intuition for the question, "How can we take derivatives on a manifold?"

The theme of single variable calculus is as follows: if we have functions with one independent variable, x, and one dependent variable, $y = f(x)$, what can we say about the slope, or derivative, of such a function? What can we say about the area under the curve, or the integral of such a function? How are the function, derivative, and integral related?

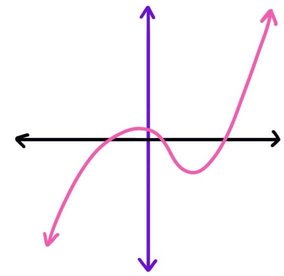


Figure 1.9: In this image, domain is $\mathbb{R}$, with coordinates x, which is depicted in black. The range is also $\mathbb{R}$, depicted in purple and with coordinates y. The function is depicted in pink.

In multivariable (specifically two variable) calculus classes, we have two independendent variables, x and y, and one dependent variable, $z = f(x, y)$. The main theme in two variable calculus is to generalize the concept of the derivative and the integral to the two independent variable and one dependent variable case.

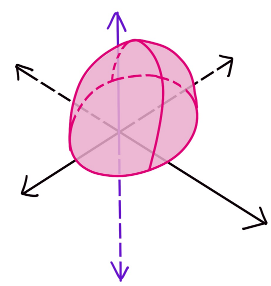


Figure 1.10: In two-variable calculus, the domain is $\mathbb{R}^2$, with coordinates (x, y). This is depicted in black. The range is $\mathbb{R}$ with coordinates z, which is depicted in purple. The function is depicted in pink.

In differential geometry, we want to find derivatives and integrals for functions on differentiable manifolds. To do this, we first need to understand some nuances of the single variable calculus case and the multivariable calculus case. In single variable calculus, when we draw the functions, we draw our functions in 2-dimensions out of convenience. However, the domain actually $\mathbb{R}$, or 1-dimensional Euclidean space, and the range is also $\mathbb{R}$. In two-variable calculus, the domain is $\mathbb{R}^2$, or 2-dimensional Euclidean space, and the range is $\mathbb{R}$. Thus, we draw the functions in $\mathbb{R}^3$, or 3-dimensional Euclidean space for convenience.

In differential geometry, rather than considering functions from n-dimensional Euclidean space to 1-dimensional Euclidean space, we'd like to consider functions from n-dimensional manifolds to 1-dimensional Euclidean space. However, the n-dimensional manifolds in Section 1.2 do not have enough structure to allow us to do calculus. The differentiable manifold essentially puts the correct type of coordinates which allows us to find derivatives of the functions from our manifolds M^n to $\mathbb{R}$.

In single variable calculus, since Euclidean space is flat, when we draw the x-axis and the y-axis, both are straight lines. Manifolds are not necessarily flat, so we have to piece together the coordinates in a way that allows us to still do calculus. Essentially, manifolds look like pieces of Euclidean space glued together. In order to do calculus, we put coordinates (ie the x-axis and y-axis in the two variable case) on each of the pieces of Euclidean space, and we piece them together in a way that the coordinates line up nicely. What could go wrong? If we have coordinates that line up in a way which is not differentiable, such as a vertical tangent or a cusp, this would be a problem if we try to do calculus on such a manifold with coordinates.

Finally, what does a function on a differentiable manifold look like? In the following figure, we consider the manifold $M^2 = S^2$.

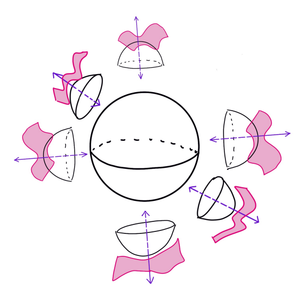


Figure 1.11: This figure a function (in pink) with domain S^2 (in black) and range $\mathbb{R}$ (in purple).

Rather than attempt to draw $\mathbb{R}$ through every point on S^2, we split up the sphere into six pieces which look like 2-dimensional Euclidean space. The coordinates on S^2 are (x, y) and the coordinates on $\mathbb{R}$, depicted in purple, are z. The function is depicted in pink. We can then piece the function together in the same way we'd piece the sphere together. As long as the coordinates on each piece meet in a way so that the function is differentiable, we can take derivatives and integrals of this function.

2 Ricci curvature

2.1 What is Ricci curvature?

We are already familiar with the distinction between curved and flat objects in everyday terminology. For example, we know that a flat basketball is less desirable than a round basketball. We know that a piece of paper is flat, while a computer mouse is not flat. We have a children's ride called the merry-go-round and a useful tool called a straight-edge. In each of our real-world examples, we have words like “flat" and “straight" which oppose words like “round" and “curve". In this chapter, we will discuss how mathematical curvature is defined in such a way that the smaller, or closer to zero, the curvature is, the closer the manifold is to flat, or Euclidean space. In some sense, the larger, or further from zero, the curvature is, the further the manifold is from Euclidean space.

Now we will define the Riemannian curvature and the Ricci curvature, the latter of which is the main curvature we will be studying in this thesis. This definition requires knowledge of the Riemannian metric, which we will denote g, orthonormal bases, and covariant derivatives, which we will denote ∇. See

Do Carmo's *Riemannian Geometry* [8] for detailed descriptions of these objects.

Definition 2.1.1. *Let M^n be a Riemannian manifold. Let X and Y be vectors in $T(M^n)$. Then the Riemannian curvature is defined as follows:*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for any $Z \in T(M^n)$.

Definition 2.1.2. *Let M^n be a Riemannian manifold. Let X be a vector in T_pM, where p is a point on the manifold, and let $\{X_i\}_{i=1}^{n-1}$ be an orthonormal basis such that each X_i is orthogonal to X. Then*

$$\mathrm{Ric}_p(X,Y) = \sum_{i=1}^{n-1} g(R(X,X_i)Y,X_i).$$

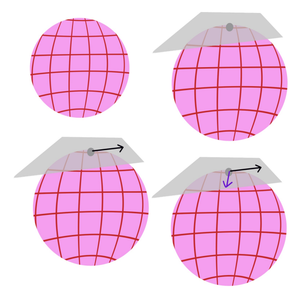


Figure 2.1: The top left image is a given manifold M^2. The top right image shows T_pM, the tangent space at a point p. The bottom left image depicts a vector X in T_pM. The bottom right image shows X_1 orthogonal to X.

To parse this definition, we will consider a Riemannian n-manifold, M^n. In order to get a good sense of the Ricci curvature of M^n at a point p, we will consider $\{X_i\}_{i=1}^{n}$ to be an orthonormal basis in $T_p(M^n)$. If we calculate the following:

$$\begin{matrix} \mathrm{Ric}(X_1,X_1) & \mathrm{Ric}(X_1,X_2) & \cdots & \mathrm{Ric}(X_1,X_n) \\ \mathrm{Ric}(X_2,X_1) & \mathrm{Ric}(X_2,X_2) & \cdots & \mathrm{Ric}(X_2,X_n) \\ \vdots & & \ddots & \vdots \\ \mathrm{Ric}(X_n,X_1) & \mathrm{Ric}(X_3,X_2) & \cdots & \mathrm{Ric}(X_n,X_n) \end{matrix}$$

Table 2.1: If we calculate Ricci of each pair of orthonormal vectors in $T_p(M^3)$, then we have a good sense of what the curvature of the manifold looks like at the point p.

then we have a good sense of what the curvature of the manifold looks like at the point p.

The author was relieved to read the following line in Besse's *Einstein Manifolds*, "Throughout our long life we have found the Ricci curvature quite hard to FEEL." As such, we will illustrate the main idea of Ricci curvature by instead looking at a similar concept, sectional curvature. Consider a complete, simply connected manifold of dimension n with constant sectional curvature. If the manifold has positive sectional curvature, then the space must be a sphere. If the manifold has negative sectional curvature, then this manifold must be hyperbolic space. If the manifold has sectional curvature zero, then the manifold must be Euclidean space.

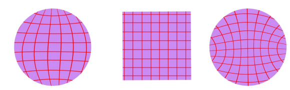


Figure 2.2: If M^n is a manifold with constant sectional curvature, then if sec > 0, then $M^n = S^n$ (leftmost figure), if sec $= 0$, then $M^n = \mathbb{R}$ (center figure), and if sec < 0, then $M^n = H^n$ (rightmost figure).

While Ricci curvature and sectional curvature are not the same, their intuition is similar.

2.2 *How do we calculate Ricci curvature given a specific manifold with a specific metric?*

We will work through an example to illustrate the calculation of the Ricci curvature.

Example 2.2.1. *Consider the rotationally symmetric metric of S^2. In other words, let $M^2 = S^2$ and let $dr^2 + \varphi^2(r)dx^2$ where $\{ \frac{\partial}{\partial r}, \frac{1}{\varphi(r)} \frac{\partial}{\partial x} \}$ is an orthonormal basis. We want to calculate the following:*

$$\begin{array}{cc} \mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) & \mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}\right) \\ \mathrm{Ric}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial r}\right) & \mathrm{Ric}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \end{array}$$

First, we will calculate the covariant derivatives, $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}$, $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$ and $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}$.

To find $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}$, we first calculate the inner product of $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x}$:

$g(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}, \frac{\partial}{\partial r}) = g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{1}{2} \frac{\partial}{\partial x} g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0.$

This tells us that $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}$ *is zero in the* $\frac{\partial}{\partial r}$ *direction. Thus,* $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x} = A \frac{\partial}{\partial x}$ *where* A *is some function with respect to* r *and* x.

Next, we calculate the inner product of $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial x}$:

$g(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{2} \frac{\partial}{\partial r} g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{2} \frac{\partial}{\partial r}(\varphi^2) = \varphi\varphi'$*, so then:*

$\varphi\varphi' = g(A \frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = A\varphi^2$.

We finally get that $A = \dfrac{\varphi'}{\varphi}$.

Thus, $\boxed{\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x} = \frac{\varphi'}{\varphi} \frac{\partial}{\partial x}}$

We do a similar set of calculations to find $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial r}$, $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$, *and* $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}$.

$$g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial r}, \frac{\partial}{\partial x}) = \frac{1}{2} \frac{\partial}{\partial r} g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) - g([\frac{\partial}{\partial r}, \frac{\partial}{\partial x}], \frac{\partial}{\partial x}) = \varphi\varphi'$$

$$\Rightarrow g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{1}{2} \frac{\partial}{\partial x} g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$$

$$\Rightarrow \varphi\varphi' = g(A \frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = A\varphi^2$$

$$\Rightarrow A = \frac{\varphi'}{\varphi}$$

$$\boxed{= \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial r} = \frac{\varphi'}{\varphi} \frac{\partial}{\partial x}}$$

$$g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \frac{\partial}{\partial r}) = -\frac{1}{2} \frac{\partial}{\partial r} g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) - g(\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial r}\right], \frac{\partial}{\partial x}) = -\frac{1}{2} \frac{\partial}{\partial r} \varphi^2 = -\varphi\varphi'$$

$$g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{2} \frac{\partial}{\partial x} g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{2} \frac{\partial}{\partial x} \varphi^2 = 0$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = A \frac{\partial}{\partial r}$$

$$\Rightarrow g(A \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = A$$

$$\Rightarrow A = -\varphi\varphi'$$

$$\boxed{\Rightarrow \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\varphi\varphi' \frac{\partial}{\partial r}}$$

$$g(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{1}{2} \frac{\partial}{\partial r} g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{1}{2} \frac{\partial}{\partial r}(1) = 0$$

$$g(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial x}) = -\frac{1}{2} \frac{\partial}{\partial x} g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) - g([\frac{\partial}{\partial r}, \frac{\partial}{\partial x}], \frac{\partial}{\partial r}) = -\frac{1}{2} \frac{\partial}{\partial x}(1) = 0$$

$$\boxed{\Rightarrow \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0}$$

Now that we've calculated the covariant derivatives of each pair of orthogonal basis vectors, we can calculate the Ricci curvatures of each pair of orthogonal basis vecotrs.

$$\text{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = g\big(R(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \frac{\partial}{\partial r}, \frac{\partial}{\partial r})\big) + g\big(R(\frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x}) \frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x})\big)$$

$$= g\big(\nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} \frac{\partial}{\partial r} - \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x} \big) - g\big(\nabla_{[\frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x}]} (\frac{\partial}{\partial r}), \frac{1}{\varphi} \frac{\partial}{\partial x} \big)$$

$$= g\big(\nabla_{\frac{\partial}{\partial r}} (\frac{\varphi'}{\varphi^2} \frac{\partial}{\partial x}), \frac{1}{\varphi} \frac{\partial}{\partial x} \big) - \frac{1}{\varphi} g\big(- \nabla_{\frac{\varphi'}{\varphi^2} \frac{\partial}{\partial x}} (\frac{\partial}{\partial r}), \frac{\partial}{\partial x} \big)$$

$$= g\big(\frac{\varphi''}{\varphi^2} \frac{\partial}{\partial x} - \frac{2(\varphi')^2}{\varphi^3} \frac{\partial}{\partial x} + \frac{\varphi'}{\varphi} \frac{\varphi'}{\varphi} \frac{\partial}{\partial x}, \frac{1}{\varphi} \frac{\partial}{\partial x} \big) + \frac{1}{\varphi} g\big(- [\frac{\partial}{\partial r}, \frac{\varphi'}{\varphi^2} \frac{\partial}{\partial x}] + \nabla_{\frac{\partial}{\partial r}} (\frac{\varphi'}{\varphi^2} \frac{\partial}{\partial x}), \frac{\partial}{\partial x} \big)$$

$$= \frac{\varphi''}{\varphi} - \frac{2(\varphi')^2}{\varphi^2} + \frac{(\varphi')^2}{\varphi} + \frac{1}{\varphi} g\big(\frac{(\varphi')^2}{\varphi^3} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \big)$$

$$\boxed{= \frac{\varphi''}{\varphi} - \frac{(\varphi')^2}{\varphi^2} + \frac{(\varphi')^2}{\varphi}}$$

$$\text{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}) = g\big(R(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \frac{\partial}{\partial x}, \frac{\partial}{\partial r})\big) + g\big(R(\frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x}) \frac{\partial}{\partial x}, \frac{1}{\varphi} \frac{\partial}{\partial x})\big)$$

$$= g\big(\nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} \frac{\partial}{\partial x} - \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x}, \frac{1}{\varphi} \frac{\partial}{\partial x} \big) - \frac{1}{\varphi} g\big(\nabla_{[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}]} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \big)$$

$$= \frac{1}{\varphi} g\big(\nabla_{\frac{\partial}{\partial r}} (-\varphi' \frac{\partial}{\partial r}) - \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} (\frac{\varphi'}{\varphi} \frac{\partial}{\partial x}), \frac{\partial}{\partial x} \big) - \frac{1}{\varphi} g\big(\nabla_{\frac{\partial}{\partial r}(\frac{1}{\varphi}) \frac{\partial}{\partial x} + \frac{1}{\varphi}[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}]} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \big)$$

$$= \frac{1}{\varphi} g\big((- \frac{\partial}{\partial r}(\varphi') \frac{\partial}{\partial r} - \varphi' \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}) - \Big((\frac{1}{\varphi} \frac{\partial}{\partial x})(\frac{\varphi'}{\varphi}) \frac{\partial}{\partial x} + \frac{\varphi'}{\varphi} \nabla_{\frac{1}{\varphi} \frac{\partial}{\partial x}} (\frac{\partial}{\partial x}) \Big), \frac{\partial}{\partial x} \big) -$$

$$\frac{1}{\varphi} g\big(\nabla_{[\frac{\partial}{\partial r}, \frac{1}{\varphi} \frac{\partial}{\partial x}]} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \big)$$

$$= \frac{1}{\varphi} g\big(- \varphi'' \frac{\partial}{\partial r} - \Big(\frac{\varphi'}{\varphi} (-\varphi' \frac{\partial}{\partial r}) \Big), \frac{\partial}{\partial x} \big) - \frac{1}{\varphi} g\big(\frac{(\varphi')^2}{\varphi} \frac{\partial}{\partial r}, \frac{\partial}{\partial x} \big)$$

$$= 0$$

$$\mathrm{Ric}(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = g\big(R(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial r})\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial r})\big) + g\big(R(\tfrac{\partial}{\partial x}, \tfrac{1}{\varphi}\tfrac{\partial}{\partial x})\tfrac{\partial}{\partial x}, \tfrac{1}{\varphi}\tfrac{\partial}{\partial x})\big)$$

$$= g\big(\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial r}}\tfrac{\partial}{\partial x} - \nabla_{\frac{\partial}{\partial r}}\nabla_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial r}\big) + g\big(\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{1}{\varphi}\frac{\partial}{\partial x}}\tfrac{\partial}{\partial x} - \nabla_{\frac{1}{\varphi}\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial x} - \nabla_{[\frac{\partial}{\partial x}, \frac{1}{\varphi}\frac{\partial}{\partial x}]}\tfrac{\partial}{\partial x}, \tfrac{1}{\varphi}\tfrac{\partial}{\partial x}\big)$$

$$= g\big(\nabla_{\frac{\partial}{\partial x}}(\tfrac{\varphi'}{\varphi}\tfrac{\partial}{\partial x}) - \nabla_{\frac{\partial}{\partial r}}(-\varphi\varphi'\tfrac{\partial}{\partial r}), \tfrac{\partial}{\partial r}\big) + g\big(\nabla_{\frac{\partial}{\partial x}}(-\varphi'\tfrac{\partial}{\partial r}) - \nabla_{\frac{1}{\varphi}\frac{\partial}{\partial x}}(-\varphi\varphi'\tfrac{\partial}{\partial r}), \tfrac{1}{\varphi}\tfrac{\partial}{\partial x}\big)$$

$$= g\big(\tfrac{\varphi'}{\varphi}\nabla_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial r}(\varphi\varphi')\tfrac{\partial}{\partial r} + \varphi\varphi'\nabla_{\frac{\partial}{\partial r}}\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}\big) + g\big(-\varphi'\nabla_{\frac{\partial}{\partial x}}\tfrac{\partial}{\partial r} + \varphi\varphi'\nabla_{\frac{1}{\varphi}\frac{\partial}{\partial x}}\tfrac{\partial}{\partial r}, \tfrac{1}{\varphi}\tfrac{\partial}{\partial x}\big)$$

$$= g\big(\tfrac{\varphi'}{\varphi}(-\varphi\varphi'\tfrac{\partial}{\partial r}) + (\varphi'\varphi' + \varphi\varphi'')\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}\big) + \tfrac{1}{\varphi}g\big(-\varphi'\tfrac{\varphi'}{\varphi}\tfrac{\partial}{\partial x} + \tfrac{\varphi'\varphi'\varphi}{\varphi^2}\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\big)$$

$$= \varphi\varphi''$$

2.3 *Why do we study manifolds with nonnegative Ricci curvature?*

In general, it's interesting in math to apply bounds on functions to see how that restricts the types of objects we can get. In our case, we apply a lower bound on the Ricci tensor to see if this restricts the topology of a given manifold.[1]

[1] Although we say we are studying spaces which satisfy Ric ≥ 0, it is more accurate to say we are studying spaces with $\mathrm{Ric}(X, X) \geq 0 \cdot g(X, X)$ for all $X \in TM$ where M is the manifold.

There are many examples of topological results with the nonnegative Ricci curvature restriction on the manifold. The following examples and theorems can be found in Petersen's *Riemannian Geometry*. [33, page 288]. Many of these theorems follow from the Cheeger-Gromoll Splitting Theorem which we will go over in detail in Chapter 3. The first theorem we review is named Myers Theorem. We will reference this theorem throughout the thesis.

Theorem 2.3.1. *[33, Theorem 25] Suppose (M, g) is a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k > 0$. *Then* $diam(M, g) \leq \frac{\pi}{\sqrt{k}}$. *Furthermore,* (M, g) *has finite fundamental group.*

Theorem 2.3.2. *[33, Corollary 26]* $S^p \times S^1$ *does not admit any metrics such that* $\mathrm{Ric} = 0$ *everywhere.*

Theorem 2.3.3. *[33, Corollary 27] Suppose* (M, g) *is a complete, compact Riemannian manifold with* $\mathrm{Ric} \geq 0$. *If the universal cover is contractible, then* (M, g) *is a flat manifold.*

Theorem 2.3.4. *[33, Corollary 28] If* (M, g) *is compact with* $\mathrm{Ric} \geq 0$ *and has* $\mathrm{Ric} > 0$ *on some tangent space* T_pM, *then* $\pi_1(M)$ *is finite.*

2.4 *What is an Einstein manifold?*

In this section, we will discuss the notion of Einstein manifolds. For a great reference, see Besse's *Einstein Manifolds* [3].

Einstein manifolds satisfy the following condition on the Ricci curvature:

$$\mathrm{Ric} = \lambda g$$

where λ is a constant and g is the metric. Manifolds which satisfy the Einstein equation are constant in the sense that the Ricci quadratic form is constantly λ if and only if $\mathrm{Ric} = \lambda g$.

The main question posed in the beginning of Besse's *Einstein Manifolds* is as follows, "Are there any best (or nicest, or distinguished) Riemannian structures on M?" For surfaces, or manifolds of dimension two, the best Riemannian structures on a compact manifold are those with constant Gauss curvature. It makes sense to think that a good generalization of the best Riemannian structures on manifolds of dimension $n \geq 3$ would be a manifold of constant Ricci or sectional curvature.

There are only three complete, simply connected n-manifolds of constant sectional curvature up to diffeomorphism: namely, the sphere when $\sec > 0$, Euclidean space when $\sec = 0$, and hyperbolic space when $\sec < 0$, each with the standard metric. On the other hand, any compact Riemannian manifold of any dimension admits a metric of constant scalar curvature. Therefore, it makes sense that we would like to study manifolds with constant Ricci curvature.

For some interesting examples of Einstein manifolds, see Besse's [3, Chapter 0 Section D].

3 *Topology and Geometry of Riemannian manifolds*

This chapter contains the main topological and geometric definitions needed in chapters 5, 6, 7, 8, and 9.

3.1 *What are some topological definitions that we'll use?*

In this section, we'll go over the main topological definitions that we'll go over in the remaining chapters in this thesis.

First, we introduce the term homotopy in order to define the notion of a loop being homotopic to another loop along a ray.

Definition 3.1.1. *Consider two continuous functions $f : X \to Y$ and $g : X \to Y$. A homotopy between f and g is defined as a continuous function H such that $H : X \times [0,1] \to Y$, $H(x,0) = f(x)$, and $H(x,1) = g(x)$ for all $x \in X$. If such an H exists, we say that f is homotopic to g.*

Next, we introduce the notion of a loop being homotopic to another loop along a ray. We use this topological definition in order to define the notion of the geodesic loops to infinity property which we will also define.

Definition 3.1.2. *Given a ray γ and a loop $C : [0, L] \to M$ based at $\gamma(0)$, we say that a loop $\widetilde{C} : [0, L] \to M$ is homotopic to C along γ if there exists $r > 0$ with $\widetilde{C}(0) = \widetilde{C}(L) = \gamma(r)$ and the loop, constructed by joining γ from 0 to r with C from 0 to L and then with γ from r to 0 is homotopic to C, in $\pi_1(M, \gamma(0))$.*

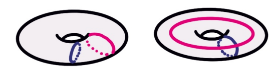


Figure 3.1: In the figure on the left, the two black loops are homotopic to each other. In the figure on the right, the black loop is not homotopic to the gray loop.

Next, we introduce the geodesic loops to infinity property, which we will use to prove our main results in Chapter 6.

Definition 3.1.3. *An element $h \in \pi_1(M, \gamma(0))$ has the geodesic loops to infinity property along γ if for any $A \subset M$ compact, there exists a loop $\widetilde{C} \subset M \setminus A$ which is homotopic to a representative loop, C of h along γ.*

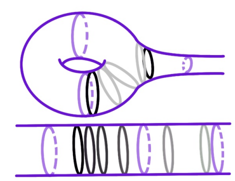


Figure 3.2: In the figure above, we have a punctured torus, which does not have the geodesic loops to infinity property. The geodesic loop in black on the left gets "stuck" and cannot reach the geodesic loop in black on the right. In the figure below, we have a cylinder, which does have the geodesic loops to infinity property. We see that the black loop is able to homotope to any of the gray loops.

Next, we define locally homogeneous and locally homogeneous manifolds, which we will reference in chapters 6 and 7.

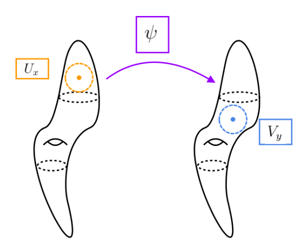


Figure 3.3: This shows a locally homogeneous manifold as in Definition 3.1.4.

Definition 3.1.4. *Let (M, g) be a Riemannian manifold. Then (M, g) is locally homogeneous if for every pair of points $x, y \in M$, there exists neighborhoods U_x of x and V_y of y such that there is an isometry ψ mapping $(U_x, g|_{U_x})$ to $(V_y, g|_{V_y})$, with $\psi(x) = y$.*

Definition 3.1.5. *Let (M, g) be a Riemannian manifold. Then (M, g) is homogeneous if for every pair of points $x, y \in M$, there exists an isometry $\psi : M \to M$, $\psi(x) = y$.*

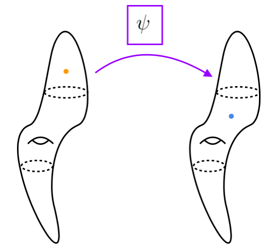


Figure 3.4: This shows a homogeneous manifold as in Definition 3.1.5.

The next definition is that of a one-ended manifold. We use this definition in chapter 6.

Definition 3.1.6. *A manifold is k-ended with $k \leq K$ if given any compact set $A \subset M$, $M \setminus A$ has at most K unbounded components*

Figure 3.5: This shows a two-ended manifold on the left, and a one-ended manifold on the right, as in Definition 3.1.6.

3.2 What are some geometric definitions that we'll need?

In this section, we give the main geometric definitions which we will use throughout the rest of this thesis. We start with the definition of a line, which we will reference in chapter 6.

Definition 3.2.1. *A line is a geodesic $\gamma : (-\infty, \infty) \to M$, which is minimizing between any two points.*

Figure 3.6: The figure on the left depicts a line on a cylinder as in Definition 3.2.1. The figure on the right does not depict a line on a cylinder because the geodesic is not minimizing.

Next, we give definitions for the following related terms, a product splitting, a warped product splitting and a ray lying in the split direction.

Definition 3.2.2. *(M, g) has a product splitting if M is isometric to $\mathbb{R} \times L$ where L is an $(n-1)$-dimensional manifold and $g = dr^2 + g_L$.*

Definition 3.2.3. *(M, g) has a warped product splitting if M is diffeomorphic to $\mathbb{R} \times L$ where L is an $(n-1)$-dimensional manifold and there exists $u : \mathbb{R} \to \mathbb{R}^+$ such* *that* $g = dr^2 + u^2(r) g_0$ *for a fixed metric* g_0*. We call* g *a warped product over* $\mathbb{R}$ *and we call* $u(r)$ *the warping function.*

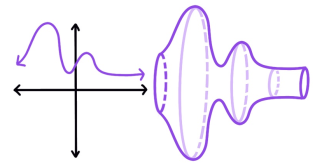


Figure 3.7: The figure on the left depicts a function, $u(r)$ as in Definition 3.2.1. The figure on the right depicts a manifold which has a warped product splitting with warping function $u(r)$.

Definition 3.2.4. *In a warped product splitting,* $M^n = N \times \mathbb{R}$ *with* $g = e^{\frac{\phi(r)}{n-1}} g_N + dr^2$*, we say that* γ*, a ray, lies in the split direction if* $\gamma(r) = (x_0, r)$*, where* $x_0 \in N$*.*

Figure 3.8: The figure on the left depicts two rays in the split direction on a cylinder. The figure on the right depicts two rays not in the split direction.

4 Bakry Émery Ricci Curvature

In this chapter, we will give an overview of the N-Bakry-Émery Ricci curvature. The results given in chapters 5, 6, 7, 8, and 9 involve various forms of the N-Bakry-Émery Ricci curvature.

4.1 What is N-Bakry Émery Ricci curvature?

We define the N-Bakry-Émery Ricci tensors as follows:

Definition 4.1.1. *Let X be a vector field on (M^n, g), a Riemannian manifold. The N-Bakry-Émery tensor is*

$$\mathrm{Ric}_X^N := \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^*$$

where $\mathcal{L}_X g$ is the Lie derivative of g with respect to X, defined as follows:

$$\mathcal{L}_X g : T_pM \times T_pM \to \mathbb{R} \tag{4.1}$$

$$(Y, Z) \mapsto \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle \tag{4.2}$$

and

$$X^* : T_pM \to \mathbb{R}$$

$$Y \mapsto g(X, Y).$$

If $X = \nabla\phi$ where $\phi : M \to \mathbb{R}$ is a smooth function, the N-Bakry-Émery Ricci tensor is

$$\mathrm{Ric}_\phi^N := \mathrm{Ric} + \mathrm{Hess}\,\phi - \frac{1}{N-n} d\phi \otimes d\phi.$$

If $X = \nabla\phi$ and $N = \infty$, then we denote $\mathrm{Ric}_\phi := \mathrm{Ric}_\phi^\infty = \mathrm{Ric} + \mathrm{Hess}\,\phi$.

Remark 4.1.2. *Note that Ric_X^N is a generalization of Ric_ϕ^N because if $X = \nabla\phi$, then $\mathrm{Ric}_X^N = \mathrm{Ric}_\phi^N$. Similarly, we call Ric_ϕ^N a generalization of Ric because if ϕ is constant, then $\mathrm{Ric}_\phi^N = \mathrm{Ric}$.* [1]

[1]
Ric_X^N
$\downarrow X = \nabla\phi$
Ric_ϕ^N
$\downarrow \phi$=constant
Ric

The constant N is also called the synthetic dimension.

4.2 Why do we study manifolds with nonnegative N-Bakry Émery Ricci curvature?

Riemannian manifolds with smooth positive density function $e^{-\phi}$ were first studied by Lichnerowicz in 1971 [22]. Bakry and Émery studied this further in order to study diffusion processes [2]. More recently, Bakry-Émery Ricci tensors have been studied in optimal transport, Ricci flow, and general relativity. Qian proved in [36] that Myers' Theorem[2] holds for gradient N-Bakry Émery Ricci curvature. In [26], Lott gives topological consequences to nonnegative and positive Bakry-Émery Ricci curvature, as well as relations between

[2] See Theorem 2.3.1

the Bakry-Émery Ricci curvature bounded below and measured Gromov-Hausdorff limits. In [40], Wei-Wylie proved Bakry-Émery Ricci curvature versions of the comparison theorems and the volume comparison theorem. Fang-Li-Zhang in [9], Khuri-Woolgar-Wylie in [16], Munteanu-Wang in [30], and Wylie in [43] also prove different versions of the Splitting Theorem for nonnegative Bakry-Émery Ricci curvature.

According to Lott in [26], if M is compact and satisfies $\mathrm{Ric}_\phi^N > 0$, then the warped product metric[3] on $M \times S^{n-N}$ satisfies $\mathrm{Ric} > 0$. When $N = \infty$, the ∞-quasi Einstein equation is the Ricci soliton equation. When $N < n$, Wylie-Woolgar study Ric_ϕ^N in the context of Lorentzian scalar-tensor gravitational theories in cosomology in [41]. Milman [27], Ohta [12], and Wylie [43] also give descriptions of the condition Ric_X^N bounded above with $N < n$.

[3] See Definition 3.2.3

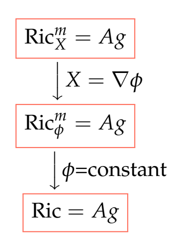
4.3 What is an N-quasi Einstein manifold?

We are ready to define the N-quasi Einstein equation.

Definition 4.3.1. *A manifold (M, g) satisfies the N-quasi Einstein equation if $\mathrm{Ric}_X^N = Ag$ for some constants A.*

Remark 4.3.2. *Many authors only consider the gradient case and/or the manifolds with boundary case of the N-quasi Einstein equation. We will assume neither condition in this thesis.*

[4]



[4]If (M, g) is N-quasi Einstein and if $X = \nabla\phi$, then we call the space gradient N-quasi Einstein. If $X = 0$, then we call the space trivial.

4.4 Why do we study N-quasi Einstein manifolds?

Non-gradient N-quasi Einstein manifolds are of particular interest in the study of near-horizon geometries (See [15], [17], and [19]). in this thesis, we study non-gradient N-quasi Einstein manifolds as a generalization of Einstein manifolds, gradient N-quasi Einstein manifolds, and Ricci solitons.

The $m = \infty$ case of the N-quasi Einstein equation corresponds to the Ricci soliton equation, $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = Ag$. Ivey showed in [14] that compact Ricci solitons must be shrinking, i.e. A must be positive. Perelman showed in [32] that compact shrinking Ricci solitons must be gradient. Then Petersen-Wylie showed in [34] that any compact locally homogeneous[5] gradient Ricci soliton is Einstein. Therefore, by Ivey, Perelman, and Petersen-Wylie, here are no non-Einstein non-trivial locally homogeneous compact Ricci solitons.

[5] See Definition 3.1.4

5 The splitting theorem for spaces with nonnegative Bakry Émery Ricci curvature

5.1 What question are we trying to answer?

First, we will review an important result in Riemannian geometry by Cheeger and Gromoll, called the Cheeger-Gromoll Splitting Theorem:

Theorem 5.1.1. *[6] Let M be a complete manifold of nonnegative Ricci curvature. Then M is the isometric product $N \times \mathbb{R}^k$ where N contains no lines and R^k has its standard flat metric.*

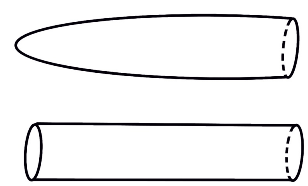


Figure 5.1: The figure above contains lines and is an example of the $k = 0$ case of the Cheeger-Gromoll Splitting Theorem. The figure below does split off $\mathbb{R}$, so the manifold could possibly satisfy Ric nonnegative everywhere.

This leads us to the main question in this chapter:

Question 5.1.2. *What assumptions do we need to generalize the Cheeger-Gromoll Splitting Theorem to the nonnegative N-Bakry-Émery Ricci curvature case?*

5.2 Why is this question interesting?

The $\text{Ric}_X^N \geq 0$ assumption becomes a weaker hypothesis as N increases. $N > n$ is our strongest premise and the Splitting Theorem holds with no further assumptions, as shown by Khuri-Woolgar-Wylie in [16, Theorem 2] and Fang-Li-Zhang in [9, Theorem 1.3]. $N < 1$ or $N = \infty$ is a weaker premise and the Splitting Theorem does not hold in general; however, Wylie showed that including the additional assumptions $X = \nabla\phi$ and $\phi < K$ for K constant gives a splitting [43, Corollary 1.3]. Munteanu-Wang also showed in [30, Theorem 1.6] that if $N = \infty$ and $X = \nabla\phi$ where ϕ has linear growth with a weighted entropy condition, then the Splitting Theorem holds or M is connected at infinity. If $N = 1$ the Splitting Theorem does not hold, even when ϕ is bounded. However, if $X = \nabla\phi$ with $\phi < K$, then Wylie showed in [43, Theorem 1.2] that there is a more general warped product splitting[1].

[1] See Definition 3.2.3

5.3 What are the main results in this section?

We had two main results in this chapter. First, we contributed to the splitting theorem results for various N-Bakry-Émery Ricci curvature constraints by proving the following proposition:

Proposition 5.3.1. *[24] If* $\text{Ric}_\phi \geq 0$ *and* $\nabla\phi \to 0$ *at* ∞*, then the Splitting Theorem holds.*

Table 5.1 summarizes the known versions of the Splitting Theorem for $\text{Ric}_X^N \geq 0$.

If $\mathrm{Ric}_X^N \geq 0$, then:		
$N > n$	$\Rightarrow$	Splitting Theorem, [16], [9]
$N < 1$, $X = \nabla\phi$, $\phi < K$	$\Rightarrow$	Splitting Theorem, [43]
$N = \infty$, $X = \nabla\phi$, $\phi < K$	$\Rightarrow$	Splitting Theorem, [9]
$N = 1$, $X = \nabla\phi$, $\phi < K$	$\Rightarrow$	Warped Product Splitting, [43]
$N = \infty$, $X = \nabla\phi$, $\nabla\phi \to 0$ at ∞	$\Rightarrow$	Splitting Theorem, [9]

Table 5.1: Splitting Theorem results for $\mathrm{Ric}_X^N \geq 0$

Our second main result in this chapter was the construction of an example where $\mathrm{Ric}_\phi > 0$, $\nabla\phi$ is bounded, $\lim_{\rho\to\infty} \frac{1}{\rho^2}\phi(\gamma(s))ds$ is nonzero and finite, and the Splitting Theorem doesn't hold.

5.4 *How do we prove the main results?*

Our first main results is as follows:

Proposition 5.3.1 [24] If $\mathrm{Ric}_\phi \geq 0$ and $\nabla\phi \to 0$ at ∞, then the Splitting Theorem holds.

Proof of Proposition 5.3.1. Suppose $\nabla\phi \to 0$ at ∞. Then, for each $\varepsilon > 0$, there exists $R > 0$ such that for all $x \in M \setminus \overline{B(\gamma(0), R)}$, $|\nabla\phi|(x) < \varepsilon$.

Let $\gamma(t)$ be a unit speed ray. Then,

$$(\phi \circ \gamma)'(t)) = \langle \nabla\phi, \dot{\gamma} \rangle \leq |\nabla\phi| \leq \varepsilon.$$

After integrating, for the same $\varepsilon > 0$ and $R > 0$, we get $\phi(\gamma(t)) < \varepsilon t + C$, where C is a constant. Then,

$$\lim_{\rho\to\infty}\frac{1}{\rho^2}\int_0^\rho \phi(\gamma(s))ds = \lim_{\rho\to\infty}\frac{1}{\rho^2}\left(\int_0^R \phi(\gamma(s))ds + \int_R^\rho \phi(\gamma(s))ds\right)$$
$$< \lim_{\rho\to\infty}\left(\frac{1}{\rho^2}\left(\int_0^R \phi(\gamma(s))ds\right) + \frac{\varepsilon}{2} + \frac{C}{\rho} - \frac{\varepsilon R^2}{2\rho^2} - \frac{CR}{\rho^2}\right)$$
$$= \frac{\varepsilon}{2}.$$

Letting $\varepsilon \to 0$, we get

$$\lim_{\rho\to\infty}\frac{1}{\rho^2}\int_0^\rho \phi(\gamma(s))ds \leq 0.$$

Thus, by [9, Remark 3.1], the Splitting Theorem holds. □

Now, we will give an example where $\mathrm{Ric}_\phi > 0$, $\nabla\phi$ is bounded, $\lim_{\rho\to\infty}\frac{1}{\rho^2}\phi(\gamma(s))ds$ is nonzero and finite, and the Splitting Theorem doesn't hold. This shows that this version of the splitting theorem is optimal. We will use this example again in chapter 6 to show that our main homology theorem is optimal.

Example 5.4.1. *[24] Let $M = \mathbb{R} \times S^{n-1}$, where our metric is $g = dr^2 + \rho^2(r)g_N$. We wish to construct $\rho(r)$ and $\phi(r)$ such that $\mathrm{Ric}_\phi > 0$ everywhere and $\phi(r)$ and $\rho(r)$ are smooth.*

Let ρ be a function such that

$$\begin{cases} \rho > 0 & \text{everywhere} \\ |\dot\rho| < 1 & \text{everywhere} \\ -C < \ddot\rho < 0 & |r| > A \end{cases}$$

where A and C are constants. Figure 5.2 is an example of what ρ might look like:

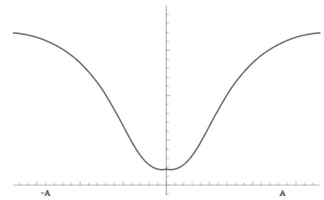


Figure 5.2: $\rho(r)$

Later in the example, we will consider $\varepsilon\rho$ *where* $\varepsilon > 0$*, so the space will look like a cylinder with a small dip around* 0*. We proceed with our calculations:*

Let V *be a vector* TS^{n-1}*. Given our metric,* $\mathrm{Ric}_\phi(\frac{\partial}{\partial \mathrm{r}}, \frac{\partial}{\partial \mathrm{r}}) = -(n-1)\frac{\ddot{\rho}}{\rho} + \ddot{\phi}$ *and* $\mathrm{Ric}_\phi(V,V) = (n-2)(1-\dot{\rho}^2) - \rho\ddot{\rho} + \dot{\phi}\dot{\rho}$ *(See [33], page 69).*

On $|r| < A$*, there exists a smooth function,* $\alpha(r)$*, larger than* $(n-1)\dfrac{\ddot{\rho}}{\rho}$ *on* $|r| < A$ *such that* $\alpha(A) = \alpha(-A) = 0$*, because* $-(n-1)\dfrac{\ddot{\rho}}{\rho}(\pm A) < 0$*.*

Let ϕ *be a function such that* $\phi''(r) = \alpha(r)$*.Then,* $\mathrm{Ric}_\phi(\frac{\partial}{\partial \mathrm{r}}, \frac{\partial}{\partial \mathrm{r}}) > 0$ *everywhere.*

Now, consider $\varepsilon\rho$ *in place of* ρ *where* $\varepsilon > 0$*.*

Then we still get $\mathrm{Ric}_\phi(\frac{\partial}{\partial \mathrm{r}}, \frac{\partial}{\partial \mathrm{r}}) = -(n-1)\frac{\ddot{\rho}}{\rho} > 0$*.*

$\mathrm{Ric}_\phi(V,V) = (n-2)(1-\varepsilon^2\dot{\rho}^2) - \varepsilon^3\rho\ddot{\rho} + \dot{\phi}\varepsilon^2\dot{\rho}\rho$*. Letting* $\varepsilon \to 0$*,* $\mathrm{Ric}_\phi(V,V) \to n-2 > 0$ *since* $\dot{\phi}$ *is bounded.*

Finally, we have $\mathrm{Ric}_\phi > 0$ *everywhere.*

On $|r| > A$*,* $\phi''(r) = 0$*, which means* $\phi(r) = Br + E$ *on* $|r| > A$*.*

Thus, $\displaystyle\lim_{t\to\infty} \frac{1}{t^2}\int_0^t \phi \circ \gamma dr = \lim_{t\to\infty} \frac{1}{t^2}\int_0^A \phi\circ\gamma(r)dr + \frac{1}{t^2}\int_A^t (Br+E)dr = \frac{B}{2}$*.*

Note that in the previous example, there is a splitting of the manifold, but there is not an isometric splitting of the metric as in the Cheeger-Gromoll Splitting Theorem. Rather, there is a warped product splitting.

The next example, which Wei-Wylie constructed in [40, Example 2.2], satisfies $\mathrm{Ric}_\phi^\infty > 0$ and $\displaystyle\lim_{\rho\to\infty}\frac{1}{\rho^2}\int_0^\rho \phi(\gamma(s))ds = \infty$, yet the Splitting Theorem does not hold.

Example 5.4.2. *Consider* $M^n = \mathbb{H}^n$*. Fix* $p \in M$*. Let* $\phi(x) = (n-1)d(x,p)^2$*, where* $d(x,p)$ *is the distance to* p*. Then,* $\mathrm{Ric}_\phi \geq 0$ *and the Splitting Theorem does not hold, as in [40, Example 2.2]. Also, if* $\gamma(0) = p$*, then*

$$\begin{aligned}
&\lim_{\rho\to\infty} \frac{1}{\rho^2}\int_0^\rho \phi(\gamma(s))ds \\
&= \lim_{\rho\to\infty} \frac{1}{\rho^2}\int_0^\rho (n-1)d(\gamma(s),\gamma(0))^2 ds \\
&= \lim_{\rho\to\infty} (n-1)\frac{\rho}{3} = \infty.
\end{aligned}$$

6 Homology of manifolds with nonnegative N-Bakry-Émery Ricci curvature

6.1 What question are we trying to answer?

In this chapter, we will use various topological techniques to answer the following question:

Question 6.1.1. *Given a manifold with nonnegative N-Bakry-Émery Ricci curvature, can we say something about the homology of such a manifold?*

6.2 Why is this question interesting?

One of the themes of Riemannian geometry is analyzing the topological implications of a manifold admitting a metric with a curvature constraint. In 1976, Yau proved that if M^n is a complete, noncompact manifold with $\mathrm{Ric} > 0$, then $H_{n-1}(M, \mathbb{R}) = 0$ [44]. In 2000, Shen-Sormani generalized this to show that such a space has $H_{n-1}(M, \mathbb{Z}) = 0$ by studying topological properties like the loops to infinity property[1] [37]. An important result by Shen-Sormani is as follows:

[1] See Definition 3.1.3

Theorem 6.2.1. *[37, Theorem 1.2] Let M^n be a complete noncompact unorientable manifold with nonnegative Ricci curvature and $G = \mathbb{Z}_2$ or $\mathbb{Z}$. Then one of the following holds:*

1. $H_{n-1}(M, G) = 0$
2. $H_{n-1}(M, G) = G$

In this thesis, we will generalize these results to that of Riemannian manifolds with non-negative and positive Bakry-Émery Ricci curvature. Our results for positive curvature are optimal in the sense that none of the assumptions can be removed (See Examples 5.4.1, 6.4.3 and 5.4.2).

6.3 What are the main results in this chapter?

Our next theorem answers the question, "What can we say about the $(n-1)$st integral homology of manifolds with stictly positive N-Bakry-Émery Ricci curvature?"

Theorem 6.3.1. *[24] Let M^n be complete and noncompact.*

1. *If $\mathrm{Ric}_X^N > 0$ for $N > n$, then $H_{n-1}(M, \mathbb{Z}) = 0$.*
2. *If $\mathrm{Ric}_\phi^N > 0$ with $\phi < K$ for some $K \in \mathbb{R}$ and $N \leq 1$, then $H_{n-1}(M, \mathbb{Z}) = 0$.*
3. *If $\mathrm{Ric}_\phi^\infty > 0$ with $\nabla\phi \to 0$ at ∞, then $H_{n-1}(M, \mathbb{Z}) = 0$.*

We also revisit Example 5.4.1 to see that Theorem 6.3.1 (3) is optimal.

The next theorem, Theorem 6.4.2, gives a more general statement than Theorem 6.3.1.

Theorem 6.4.2 [24]: Let M^n be complete and noncompact.

1. If $\mathrm{Ric}_X^N \geq 0$ for $N > n$, then $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$.

2. If $\mathrm{Ric}_\phi^N \geq 0$ with $\phi < K$ for some $K \in \mathbb{R}$ and $N < 1$, then $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$.

3. If $\mathrm{Ric}_\phi^N \geq 0$ with $|\phi| < K$ for some $K \in \mathbb{R}$ and $N = 1$, then $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$.

4. If $\mathrm{Ric}_\phi^\infty \geq 0$ with $\nabla\phi \to 0$ at ∞, then $H_{n-1}(M, \mathbb{Z}) = 0$ or $\mathbb{Z}$. [2]

[2] The assumption here matches the assumption in Proposition 5.3.1.

The main lemma in this chapter, Lemma 6.4.12, is as follows:

Lemma 6.4.12 [24]: Let (M, g, ϕ) be a Riemannian manifold with $\mathrm{Ric}_\phi^1 \geq 0$ and $|\phi| \leq K$ for $K > 0$. Suppose there exists $h \in \pi_1(M)$ which does not satisfy the geodesic loops to infinity property[3] along a given ray γ. Then the lift $\widetilde{\gamma}$ of γ is in the split direction,

[3] See Definition 3.1.3

$$\widetilde{\gamma}(t) = (x(0), y(t))$$

and

$$h_*(\widetilde{\gamma}'(t)) = -\widetilde{\gamma}'(t).$$

6.4 What results and proofs do we need to answer our main question?

First, we prove our homology result for positive N-Bakry-Émery Ricci curvature which we restate below.

Theorem 6.3.1 [24] Let M^n be complete and noncompact.

1. If $\mathrm{Ric}_X^N > 0$ for $N > n$, then $H_{n-1}(M, \mathbb{Z}) = 0$.
2. If $\mathrm{Ric}_\phi^N > 0$ with $\phi < K$ for some $K \in \mathbb{R}$ and $N \leq 1$, then $H_{n-1}(M, \mathbb{Z}) = 0$.
3. If $\mathrm{Ric}_\phi^\infty > 0$ with $\nabla\phi \to 0$ at ∞, then $H_{n-1}(M, \mathbb{Z}) = 0$.

Since Theorem 6.3.1(3) follows directly from the version of the splitting theorem which we proved in Chapter 5, we revisit **Example 5.4.1**. In Example 5.4.1, we let $M = \mathbb{R} \times S^{n-1}$, and we constructed a metric and potential function which had $\mathrm{Ric}_\phi > 0$, $\nabla\phi$ is bounded, $\lim_{\rho\to\infty} \frac{1}{\rho^2}\phi(\gamma(s))ds$ nonzero and finite, and the Splitting Theorem didn't hold. Computing the $(n-1)$st homology, we get that $H_{n-1}(M, \mathbb{Z})$ is $\mathbb{Z}$, rather than 0. Therefore, Theorem 6.3.1 is optimal.

Next, we review a proposition by Carron and Pedon, which we will use to prove one of our main theorems in this section (Theorem 6.4.2) in the orientable case.

Proposition 6.4.1. *[4, Proposition 5.2] If M^n is an orientable open manifold having one end, and if every twofold normal covering of M also has one end*[4]*, then* $H_{n-1}(M, \mathbb{Z}) = 0$.

[4] See Definition 3.1.6

We are now ready to state and prove one of our main theorems in this section.

Theorem 6.4.2. *[24] Let M^n be complete and noncompact.*

1. *If* $\mathrm{Ric}_X^N \geq 0$ *for* $N > n$*, then* $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$*.*

2. *If* $\mathrm{Ric}_\phi^N \geq 0$ *with* $\phi < K$ *for some* $K \in \mathbb{R}$ *and* $N < 1$ *, then* $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$*.*

3. *If* $\mathrm{Ric}_\phi^N \geq 0$ *with* $|\phi| < K$ *for some* $K \in \mathbb{R}$ *and* $N = 1$ *, then* $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$*.*

4. *If* $\mathrm{Ric}_\phi^\infty \geq 0$ *with* $\nabla\phi \to 0$ *at* ∞*, then* $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$*.* [5]

[5] The assumption here matches the assumption in Proposition 5.3.1.

We will now prove Theorem 6.4.2 for M^n orientable. To do this, we will follow [4, Proposition 5.3] in using Proposition 6.4.1. Later, we will prove Theorem 6.4.2 in the M^n non-orientable case by using the loops to infinity property[6].

[6] See Definition 3.1.3

Proof of Theorem 6.4.2 (orientable case). Suppose M is one-ended[7] and suppose every double cover of M is one-ended. By Proposition 6.4.1, $H_{n-1}(M, \mathbb{Z}) = 0$.

[7] See Definition 3.1.6

Suppose M is one-ended and there exists a double cover, $\widetilde{M}$, which is two-ended. Then $\widetilde{M}$ splits isometrically as $\widetilde{L} \times \mathbb{R}$, where $\widetilde{L}$ is compact by the Splitting Theorem. Let h be the nontrivial deck transformation acting on $\widetilde{M}$. Then $H_{n-1}(M, \mathbb{Z}) = H_{n-1}(M, \mathbb{Z}) = H_{n-1}\left(\frac{\widetilde{L} \times \mathbb{R}}{\langle h \rangle}, \mathbb{Z}\right)$. Since M is one-ended, M is orientable if and only if $\widetilde{L}$ is non-orientable. Thus, $H_{n-1}(M, \mathbb{Z}) = H_{n-1}\left(\frac{\widetilde{L} \times \mathbb{R}}{\langle h \rangle}, \mathbb{Z}\right) = 0$.

Suppose M is two ended. By the Splitting Theorem, M is isometric to $L \times \mathbb{R}$ where L is compact and has the same orientability as M. Then, since M is orientable, $H_{n-1}(M, \mathbb{Z}) = H_{n-1}(N, \mathbb{Z}) = \mathbb{Z}$. □

In the following example, we will give a space and metric where $\mathrm{Ric}_\phi^N > 0$ for $N = \infty$ and $N \leq 1$, ϕ is unbounded, and $H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}$.

Example 6.4.3. *[24] Consider $M = S^{n-1} \times \mathbb{R}$.*

Let $\phi : M^n \to \mathbb{R}$, $\phi(r) = r^2$.

In the S^{n-1} direction, $\mathrm{Ric} > 0$ and $\mathrm{Hess}\,\phi = 0$. In the $\mathbb{R}$ direction, $\mathrm{Ric} = 0$ and $\mathrm{Hess} > 0$.

If $N = \infty$ or $N \leq 1$, then $-\dfrac{1}{N-n}\nabla\phi^* \otimes \nabla\phi^* \geq 0$.

Therefore, $\mathrm{Ric}_\phi^N > 0$ for $N = \infty$ and $N \leq 1$. However, $H_{n-1}(S^{n-1} \times \mathbb{R}, \mathbb{Z}) = H_{n-1}(S^{n-1}, \mathbb{Z}) = \mathbb{Z}$. Notice that ϕ is unbounded.

Observe that in Example 6.4.3, the Splitting Theorem does hold.

We will present the proof of Sormani's Line Theorem, which, along with the Splitting Theorem for $\mathrm{Ric}_X^N \geq 0$, allows us to prove our main lemma, Lemma 6.4.12. We then prove Theorem 6.4.2 in the non-orientable case. We are ready to present Sormani's Line Theorem.

Theorem 6.4.4. *[39, Theorem 1.7] If M^n is a complete non-compact manifold which does not satisfy the geodesic loops to infinity property, then there is a line in its universal cover.*

Proof. Since M^n is a complete, non-compact manifold, there exists a ray, $\gamma : [0, \infty) \to M^n$. Let $h \in \pi_1(M, \gamma(0))$ which does satisfy the loops to infinity property, and let C be a representative of h based at $\gamma(0)$. Because h doesn't satisfy the loops to infinity property, there exists a compact set $A \subset M$ such that any loop homotopic to C along γ intersects A. Let $R > 0$ such that $A \subset B_{\gamma(0)}(R)$. Let $\{r_i\}$ be a sequence such that $\lim_{i\to\infty} r_i = \infty$ and $r_i > R$ for all i.

Now, let $\widetilde{M}$ be the universal cover of M, and let $\pi : \widetilde{M} \to M$ be the covering map. Identifying loops in $\pi_1(M, \gamma(0))$ with deck transformations, let $\widetilde{\gamma}$ and $h \circ \widetilde{\gamma}$ be lifts of γ starting at $\widetilde{\gamma}(0)$ and $h \circ \widetilde{\gamma}(0)$ respectively, and let $\widetilde{C}$ be the lift of C, starting at $\widetilde{\gamma}(0)$ and ending at $h \circ \widetilde{\gamma}(0)$. If $\widetilde{C}_i$ are minimal geodesics from $\widetilde{\gamma}(r_i)$ to $h \circ \widetilde{\gamma}(r_i)$, then, $C_i := \pi(\widetilde{C}_i)$ is a loop based at $\gamma(r_i)$ which is homotopic to C along γ.

Let $L_i = L(\widetilde{C}_i) = L(C_i) = d_{\widetilde{M}}(\widetilde{\gamma}(r_i), h \circ \widetilde{\gamma}(r_i))$. For each C_i, there exists some $t_i \in [0, L_i]$ such that $C_i(t_i) \subset A$.

Let $\widetilde{A}$ be the lift of A to the fundamental domain in $\widetilde{M}$. For all $i \in \mathbb{N}$, there exists $h_i \in \pi_1(M, \gamma(0))$, so that $h_i \circ \widetilde{C}_i(t_i) \in \widetilde{A}$.

Through some computational details which we will omit, (See [39, Theorem 1.7] for more details), $h_i \circ \widetilde{C}_i$ are minimal geodesics from $(t_i - (r_i - R))$ to $(t_i + (r_i - R))$ such that $h_i \circ \widetilde{C}_i(t_i) \in \widetilde{A}$. Letting $r_i \to \infty$, a subsequence of $h_{i*} \circ \widetilde{C}_i'(t_i)$ converges to a vector $\gamma_\infty'(0)$, based at $\gamma_\infty(0)$ in the closure of $\widetilde{A}$. Let γ_∞ be the geodesic with these initial conditions. Then γ_∞ runs from $\lim_{i\to\infty} t_i - (r_i - R) = -\infty$ to $\lim_{i\to\infty} t_i + (r_i - R) = \infty$. Thus, we have constructed a line, namely γ_∞, in $\widetilde{M}$.

□

The following corollary follows from [37] and the generalizations of the Splitting Theorem.

Corollary 6.4.5. *[24] Let M^n be a complete, noncompact Riemannian manifold, and suppose one of the following holds:*

1. $\mathrm{Ric}_X^N \geq 0$ *with* $N > n$.

2. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$, ϕ *bounded above.*

3. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N \leq 1$ *and* ϕ *bounded above.*

4. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$, $\nabla\phi \to 0$ *at* ∞.

Then,

(i) If D be a precompact subset of M and ∂D is simply connected, then $\pi_1(D)$ can only contain elements of order 2.

(ii) If D be a precompact subset of M with smooth boundary, where γ is a ray such that $\gamma(0) \in D$ and if S be any connected component of ∂D containing a point $\gamma(a)$, then the image of the inclusion map

$$i_* : \pi_1(S, \gamma(a)) \to \pi_1(Cl(D), \gamma(a))$$

is $N \subset \pi_1(Cl(D), \gamma(a))$ such that $\pi_1(Cl(D), \gamma(a))/N$ contains at most two elements.

Corollary 6.4.6. *[24] Let M^n be a complete, noncompact Riemannian manifold, and suppose one of the following holds:*

1. $\mathrm{Ric}_X^N \geq 0$ *with* $N > n$*, and there exists a point* $p \in M$ *such that* $(\mathrm{Ric}_X^N)_p > 0$*.*

2. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$*,* ϕ *bounded above, and there exists a point* $p \in M$ *such that* $(\mathrm{Ric}_\phi^N)_p > 0$*.*

3. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N \leq 1$ *and* ϕ *bounded above, and there exists a point* $p \in M$ *such that* $(\mathrm{Ric}_X^N)_p > 0$*.*

4. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$*,* $\nabla\phi \to 0$ *at* ∞*, and there exists a point* $p \in M$ *such that* $(\mathrm{Ric}_\phi^N)_p > 0$*.*

Then, M^n *has the geodesic loops to infinity property*[8]*.*

[8] See Definition 3.1.3

Proof. First, we will show that M and its universal cover, $\widetilde{M}$, have no lines. Suppose for the sake of contradiction that M contains a line. We saw earlier in the paper that each of the four premises gives us a version of the Splitting Theorem. Hence, $M = \mathbb{R} \times N$. However, $\mathrm{Ric}_\phi^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$, which is a contradiction, thus proving our claim.

Ergo, by Sormani's Line Theorem, M has the loops to infinity property. □

Before we prove the next proposition, we will first show that there exist examples of Riemannian manifolds with $\mathrm{Ric}_\phi^1 \geq 0$ not satisfying loops to infinity property along a given ray γ and universal cover which has a warped product splitting[9].

[9] See Definition 3.2.3

Example 6.4.7. *[24] Let* $\phi \in C^2$ *be bounded with bounded first and second derivatives. By [43, Corollary 2.4], there exists* λ *large enough so that* $\mathrm{Ric}_\phi^1 \geq 0$ *and*

$g = dt^2 + e^{\frac{2\phi}{n-1}} S^n_\lambda$. Now consider $M = \left(\mathbb{R} \times S^n\right) / G$, where G is the group generated by $h(t, x) = (a - t, -x)$ for any constant $a > 0$. If we also assume $\phi(a - t) = \phi(t)$, then h is an isometry and (M, g, ϕ) satisfies $\mathrm{Ric}^1_\phi \geq 0$. h does not have the loops to infinity property[10] along $(t, 0) = (-t, a)$, so $\left(\mathbb{R} \times S^n\right) / G$ satisfies all of the necessary properties.

[10] See Definition 3.1.3

Theorem 6.4.8. *[43, Proposition 4.2] Consider a warped product metric[11] of the form $g = dr^2 + v^2(r) g_N$ where $v > 0$ is bounded from above. Let $\gamma : (a, b) \to M$ be a unit speed minimizing geodesic in M and write $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, where γ_1 and γ_2 are projections in the factors $\mathbb{R}$ and N. Then:*

[11] See Definition 3.2.3

(1) γ_2 is either constant or its image is a minimizing geodesic in (N, g_N).

(2) If γ_2 is not a constant and γ is a line[12] in M, then the image of γ_2 is a line in N.

[12] See Definition 3.2.1

Next, we state remarks from [43, Lemma 4.4] and [31, page 208, Remark 8] which we will use in the proof of Lemma 6.4.12.

Remark 6.4.9. *[43, Lemma 4.4] In the context of Theorem 6.4.11, when $g = e^{\frac{2f(r)}{n-1}} g_N + dr^2$ and N does not admit a line, it follows from Theorem 6.4.8 that if $h : M \to M$ is an isometry, then $h = h_1 \times h_2$ where $h_1 \in Isom(N)$ and $h_2 \in Isom(\mathbb{R})$.*

Remark 6.4.10. *[31, page 208, Remark 8] Let $\gamma = (x(t), y(t))$ be a geodesic in the warped product[13], $M = N \times \mathbb{R}$, where the metric tensor is $g = e^{\frac{2f(r)}{n-1}} g_N + dr^2$. Then, the function $e^{\frac{4f(y(t))}{n-1}} |x'(t)|^2$ is a constant C.*

[13] See Definition 3.2.3

The following theorem, which can be found in [43], is the Splitting Theorem for Ric^1_ϕ. We will use this to prove our main lemma, Lemma 6.4.12.

Theorem 6.4.11. *[43, Lemma 4.4] Suppose that (M, g, ϕ) satisfies $\mathrm{Ric}^1_\phi \geq 0$ with ϕ bounded (above and below) and contains a line. Then either the Cheeger Gromoll Splitting Theorem holds or M is diffeomorphic to $N \times \mathbb{R}$ and $g = e^{\frac{2f(r)}{n-1}} g_N + dr^2$ where $\phi = f + f_N$ and (N, g_N) does not admit a line.*

We are prepared to state our main result.

Lemma 6.4.12. *[24] Let (M, g, ϕ) be a Riemannian manifold with $\mathrm{Ric}^1_\phi \geq 0$ and $|\phi| \leq K$ for $K > 0$. Suppose there exists $h \in \pi_1(M)$ which does not satisfy the geodesic loops to infinity property[14] along a given ray γ. Then the lift $\widetilde{\gamma}$ of γ is in the split direction,*

[14] See Definition 3.1.3

$$\widetilde{\gamma}(t) = (x(0), y(t))$$

and

$$h_*(\widetilde{\gamma}'(t)) = -\widetilde{\gamma}'(t).$$

See Figure 6.4 for an image representation of Lemma 6.4.12.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal cover of M. By Theorem 6.4.4, there exists a line in $\widetilde{M}$. By Theorem 6.4.11, we have the following cases: either $\widetilde{M} = N \times \mathbb{R}^k$ and $\widetilde{g} = g_N + g_{\mathbb{R}^k}$, or $\widetilde{M} = N \times \mathbb{R}$ with $\widetilde{g} = e^{\frac{2f(r)}{n-1}} g_N + dr^2$, where N contains no lines.

If $\widetilde{g} = g_N + g_{\mathbb{R}^k}$, then we have a product metric, so we can follow the proof of [39, Proposition 1.9] to obtain the desired conclusion.

Suppose $\widetilde{g} = e^{\frac{2f(r)}{n-1}} g_N + dr^2$, where N contains no lines. Recall the setup of Theorem 6.4.4. We know that there are minimal geodesics $\widetilde{C}_i$ running from $\widetilde{\gamma}(r_i)$ to $h \circ \widetilde{\gamma}(r_i)$. [39, Theorem 1.7]

Let $p_N : \tilde{M} \to N$ and $p_{\mathbb{R}} : \tilde{M} \to \mathbb{R}$ be the projections onto the N component and the $\mathbb{R}$ component, respectively. Let $\widetilde{C}_i(t) = (x_i(t), y_i(t))$, where $x_i(t) := p_N(\widetilde{C}_i(t))$ and $y_i(t) := p_{\mathbb{R}}(\widetilde{C}_i(t))$. We have $h_*(\widetilde{C}_i'(t_i)) = h_*(x_i'(t_i), y_i'(t_i)) = (h_{1*} \circ x_i'(t_i), h_{2*} \circ y_i'(t_i))$ for $t_i \in (0, L_i)$ as in Theorem 6.4.4. The last equality follows from Remark 6.4.9.

We have that $h_*(\widetilde{C}_i'(t_i)) \to \gamma_\infty'(0)$, where we can define $\gamma_\infty'(0) = (x_\infty'(0), y_\infty'(0))$ with $x_\infty(t) = p_N(\gamma_\infty(t))$ and $y_\infty(t) = p_{\mathbb{R}}(\gamma_\infty(t))$. Now, $h_*(x_i'(t_i)) \to x_\infty'(0)$ and $h_*(y_i'(t_i)) \to y_\infty'(0)$. Thus, $\lim_{i\to\infty} |x_i'(t_i)| = \lim_{i\to\infty} |h_*(x_i'(t_i))| = |x_\infty'(0)|$.

By Theorem 6.4.8, since $\gamma_\infty(t)$ is a line, $x_\infty(t)$ is either the image of a line or constant. However, N doesn't contain any lines, so $|x_\infty'(0)| = 0$. Now, using Remark 6.4.10, $e^{\frac{4f(y(t_i))}{n-1}} |x_i'(t_i)|^2 = e^{\frac{4f(y(0))}{n-1}} |x_i'(0)|^2$. Since $|f| \leq K$ for some $K > 0$, for all $t \in \mathbb{R}$,

$$|x_i'(t)|^2 = e^{\frac{4f(y(t_i)) - 4f(y(t))}{n-1}} |x_i'(t_i)|^2 \leq B|x_i'(t_i)|^2, \text{ where } B = e^{\frac{8K}{n-1}}.$$

Then,

$$\lim_{i\to\infty} |x_i'(0)|^2 \leq B \lim_{i\to\infty} |x_i'(t_i)|^2 = B \lim_{i\to\infty} |h_*(x_i'(t_i))| = B|x_\infty'(0)| = 0.$$

So, we know that

$$\lim_{i\to\infty} |x_i'(0)|^2 = 0.$$

We want to show that for any t, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $|y_i'(t)|$ is strictly positive. Suppose for the sake of contradiction that there exists some t_1 such that for all $i \geq i_0$, $y_i'(t_1) = 0$. Then, since $\widetilde{C}_i(t)$ is unit speed, we have

$$|y_i'(t_1)|^2 + e^{\frac{2f(t_1)}{n-1}} |x_i'(t_1)|^2 = 1, \text{ so for all } i \geq i_0,$$

$$|x_i'(t_1)|^2 = e^{\frac{-2f(t_1)}{n-1}}(1 - |y_i'(t_1)|^2) = e^{\frac{-2f(t_1)}{n-1}}.$$

As $i \to \infty$, $|x_i'(t_1)| \to 0$, however,

$$0 < e^{\frac{-2K}{n-1}} < e^{\frac{-2f(t_1)}{n-1}} < e^{\frac{2K}{n-1}},$$

which is a contradiction. Thus, for all t, there exists i large enough so that $|y_i'(t)|$ is strictly positive.

In particular, since $|y_i'(t)|$ is never 0 in $\mathbb{R}$, $|y_i(t)|$ never changes direction, and so

$$d_{\mathbb{R}}(y(r_i), h(y(r_i))) = L(y_i(t)) = \int_0^{L_i} |y_i'(t)| dt.$$

Now, $\dfrac{d_{\mathbb{R}}(y(r_i), h(y(r_i)))}{L_i} = \dfrac{L(y_i(t))}{L_i} = \dfrac{\int_0^{L_i} |y_i'(t)|dt}{\int_0^{L_i} \sqrt{e^{\frac{2f(t)}{n-1}}|x_i'(t)|^2 + |y_i'(t)|^2}dt}$.

Since $e^{\frac{2f(t)}{n-1}}|x_i'(t)|^2 \leq e^{\frac{2K}{n-1}}|x_i'(t)|^2 \to 0$, and $|y_i'(t)|^2 = 1 - e^{\frac{2f(t)}{n-1}}|x_i'(t)|^2$, we get that

$$\frac{\int_0^{L_i} |y_i'(t)|dt}{\int_0^{L_i} \sqrt{e^{\frac{2f(t)}{n-1}}|x_i'(t)|^2 + |y_i'(t)|^2}dt} \geq \frac{\int_0^{L_i} (1-\varepsilon_i)dt}{\int_0^{L_i} \sqrt{e^{\frac{2K}{n-1}}|x_i'(t)|^2 + 1}dt} = \frac{(1-\varepsilon_i)L_i}{\int_0^{L_i} \sqrt{e^{\frac{2K}{n-1}}|x_i'(t)|^2 + 1}dt},$$

where $|x_i'(t)|^2 \to 0$ uniformly by the above as $i \to \infty$, and $\varepsilon_i \to 0$.

Thus,

$$\lim_{i\to\infty} \frac{d_{\mathbb{R}}(y(r_i), h(y(r_i)))}{L_i} \geq 1. \tag{6.1}$$

Since $y(t)$ and $h(y(t))$ are in $\mathbb{R}$, we can write $y(t) = \int_0^t y'(s)ds - y(0)$ and $h(y(t)) = \int_0^t h_*(y'(s))ds - h(y(0))$. Also, the only possible isometries in $\mathbb{R}$ are reflections, translations, and a combination of the two. We want to show that h_* cannot be a translation.

Suppose for the sake of contradiction that $h_*(y'(s)) = y'(s)$.

$$\frac{|h(y(r_i)) - y(r_i)|}{L_i} = \frac{|\int_0^{r_i} y'(s) - \int_0^{r_i} y'(s) - h(y(0)) + y(0)|}{L_i} = \frac{|h(y(0)) - y(0)|}{L_i}.$$

Taking the limit of both sides, we get $\lim_{i\to\infty} \dfrac{|h(y(r_i)) - y(r_i)|}{L_i} = 0$, which is a contradiction. Thus, h_* must be a reflection, and

$$h_*(\widetilde{\gamma}'(0)) = -\widetilde{\gamma}'(0). \tag{6.2}$$

In order to show that $\tilde{\gamma}$ is in the split direction, along with showing (6.2), we must also show that $|x'(s)| = 0$ for all s. We proceed by using (6.2) to show that $\lim_{i\to\infty} \frac{2\int_0^{r_i} |y'(s)|ds}{L_i} = 1$.

By (6.2), we have the following equality:

$$\frac{2|\int_0^{r_i} y'(s)ds|}{L_i} = \frac{|\int_0^{r_i} y'(s)ds - \int_0^{r_i} h_*(y'(s))ds|}{L_i}.$$

By the Fundamental Theorem of Calculus and the Triangle Inequality,

$$= \frac{|y(r_i) - y(0) - h(y(r_i)) + h(y(0))|}{L_i} \geq \frac{|h(y(r_i)) - y(r_i)|}{L_i} - \frac{|h(y(0)) - y(0)|}{L_i}.$$

Taking the limit of both sides, and by (6.1),

$$\lim_{i\to\infty} \frac{2|\int_0^{r_i} y'(s)ds|}{L_i} \geq 1.$$

On the other hand, since $|y'(s)| = 1 - e^{\frac{2f(s)}{n-1}}|x'(s)| \leq 1$,

$\lim_{i\to\infty} \frac{2\int_0^{r_i} |y'(s)|ds}{L_i} \leq \lim_{i\to\infty} \frac{2r_i}{L_i} = 1$. This equality comes from [39, Note 2.1].

Hence, $|y'(s)| = 1$, so $|x'(s)| = 0$, $\tilde{\gamma}(t) = (x(0), y(t))$, and $\tilde{\gamma}$ is in the split direction. □

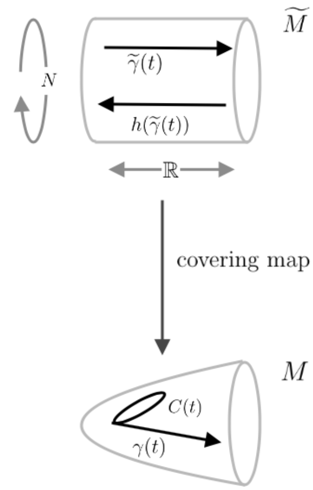


Figure 6.1: $C(t)$ is a representation of h based at $\gamma(0)$. If M satisfies the assumptions in Lemma 6.4.12, then $\widetilde{\gamma}$ is in the split direction and $h(\widetilde{\gamma})$ is also in the split direction but facing the opposite direction of $\widetilde{\gamma}$.

Corollary 6.4.13. *[24] If M^n is a complete noncompact manifold with $\mathrm{Ric}^1_\phi \geq 0$, $|\phi|$ bounded, and there exists an element $h \in \pi_1(M)$ which doesn't satisfy the loops to infinity property*[15] *along a given ray γ, then M^n is a flat normal bundle over a compact totally geodesic soul.*

[15] See Definition 3.1.3

We are now ready to prove Theorem 6.4.2 in the non-orientable case.

Corollary 6.4.14. *[24] Let M^n be a complete non-orientable Riemannian manifold and suppose one of the following holds:*

1. $\mathrm{Ric}^N_X \geq 0$ *with* $N > n$.
2. $\mathrm{Ric}^N_\phi \geq 0$ *with* $N = \infty$, ϕ *bounded above.*

3. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N \leq 1$ *and* ϕ *bounded above.*

4. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$, $\nabla\phi \to 0$ *at* ∞.

Then $H_{n-1}(M, \mathbb{Z}) = 0$ *or* $\mathbb{Z}$.

Proof. We will only prove the $\mathrm{Ric}_\phi^N \geq 0$ with $N \leq 1$ and ϕ bounded above case because the other cases follow similarly to [37].

Suppose M is a two-ended manifold. Then by the Cheeger-Gromoll Splitting Theorem, M splits isometrically as $L \times \mathbb{R}$ where L is compact and has the same orientability as M. Then, since M is non-orientable, $H_{n-1}(M, \mathbb{Z}) = H_{n-1}(N, \mathbb{Z}) = 0$.

Suppose M is a one-ended manifold. Suppose M^n satisfies loops to infinity property. M has a double cover $\pi : \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. We first claim that $\widetilde{M}$ has only one end.

Assume for the sake of contradiction that $\widetilde{M}$ has two or more ends. By [43, Lemma 4.4], either $\widetilde{M}$ splits isometrically as $\widetilde{M} = N^{n-1} \times \mathbb{R}$ where N is compact, in which case we follow the proof of [37, Propostion 3.2], or $\widetilde{M} = N \times \mathbb{R}$ with $g = e^{\frac{2\phi}{n-1}} g_N + dr^2$ where N contains no lines. Since $\widetilde{M}$ is orientable, so is the totally geodesic submanifold, N^{n-1}.

Noting that $G(\widetilde{M}) = \mathbb{Z}/2\mathbb{Z}$, let h be the nontrivial deck transformation acting on $\widetilde{M}$ (i.e. $h_{\mathbb{R}}(r) \neq r$). By Theorem 6.4.11, $h = h_{\mathbb{R}} \times h_N$, where $h_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $h_N : N \to N$. Since $h_{\mathbb{R}}$ is an isometry, $h_{\mathbb{R}}(r) = \pm r + r_0$. If $h_{\mathbb{R}}(r) = r + r_0$, then $h_{\mathbb{R}}^2(r) = r + 2r_0$. Since $h_{\mathbb{R}}^2(r) = r$, this implies that $r_0 = 0$, so $h_{\mathbb{R}}(r) = r$, which is a contradiction. Hence, $h_{\mathbb{R}}(r) = -r + r_0$.

Now, we can use the topology of M to show that $\widetilde{M}$ is one-ended and has the loops to infinity property. The interested reader can look at [37, Proposition 3.2] for more details. By [37, Proposition 2.1] , $H_{n-1}(\widetilde{M}, G)$ is trivial. Using the Universal Coefficient Theorem [29, Theorem 55.1], $H_{n-1}(M, \mathbb{Z}) = 0$.

Suppose M^n be a one-ended and doesn't have a ray with the loops to infinity property. Since M^n doesn't have a ray with loops to infinity property, by Corollary 6.4.13, M^n is a flat normal bundle over a compact totally geodesic soul. Since M^n is one-ended, N is orientable if and only if M is non-orientable, so $H_{n-1}(M, G) = H_{n-1}(N, G) = \mathbb{Z}$. □

Next, we prove Theorem 6.4.15, which generalizes Theorem 6.4.2 to classify the $n-1$ homologies with coefficients in Abelian groups of spaces with nonnegative N-Bakry Émery Ricci curvature. This is the N-Bakry Émery Ricci curvature analog of Shen-Sormani's [37, Theorem 1.1] and can be proved in the same way as their theorem except with Theorem 6.4.11 instead of the Cheeger-Gromoll Splitting Theorem. We give a sketch of the proof below.

Theorem 6.4.15. *[24]*

Let M^n be a complete noncompact manifold with either of the following:

1. $\mathrm{Ric}_X^N \geq 0$ *with* $N > n$.
2. $\mathrm{Ric}_\phi^N \geq 0$ *with* ϕ *bounded above and* $N \leq 1$ *or* $N = \infty$.
3. $\mathrm{Ric}_\phi^N \geq 0$ *with* $N = \infty$ *and* $\nabla\phi \to 0$ *at* ∞.

Then we have the following cases:

(i) If M^n has two or more ends[16] *and G is an Abelian group, then*

[16] See Definition 3.1.6

$$H_{n-1}(M,G) = \begin{cases} G & \text{if } M \text{ is orientable} \\ \ker(G \overset{\times 2}{\to} G) & \text{if } M \text{ is not orientable.} \end{cases}$$

(ii) If M^n is one-ended with the loops to infinity property[17]*, then*

[17] See Definition 3.1.3

$$H_{n-1}(M,G) = 0.$$

(iii) If M^n is one-ended and doesn't have a ray with the loops to infinity property, and G is an Abelian group, then

$$H_{n-1}(M,G) = \begin{cases} G & \text{if } M \text{ is not orientable} \\ \ker(G \overset{\times 2}{\to} G) & \text{if } M \text{ is orientable.} \end{cases}$$

Proof of Theorem 6.4.15. Consider M with two or more ends. If $\mathrm{Ric}_\phi^N \geq 0$ with $N = 1$ and ϕ bounded above, then M splits as $N \times \mathbb{R}$ as in Theorem 6.4.11. If N is orientable, then $H_{n-1}(M,G)$ is G, and if N^{n-1} is not orientable, then $H_{n-1}(M,G)$ is $\ker(G \overset{\times 2}{\to} G)$. In all other cases when M has two or more ends, we use the Cheeger-Gromoll Splitting Theorem instead of Theorem 6.4.11, as in [37, Proposition 3.1] to get the same conclusion.

If M is one-ended with the loops to infinity property, then using Poincare Duality, Universal Coefficient Theorem, and other topological arguments, we get the desired result. Since this proof only uses topology, the proof is the same as [37, Proposition 2.1].

Suppose M is one-ended and doesn't have a ray with the loops to infinity property. If $\text{Ric}_\phi^N \geq 0$ with ϕ bounded above, then by Corollary 6.4.13, M^n is a flat normal bundle over a compact totally geodesic soul, N^{n-1}. In all other cases where $\text{Ric}_X^N \geq 0$ and N is not 1, we use [39, Theorem 1.2] to get that M^n is a flat normal bundle over a compact totally geodesic soul, N^{n-1}. Then, using that M is one-ended, we get the desired conclusion. This proof is the same as [37, Proposition 3.3], except we use Corollary 6.4.13 in the $N = 1$ case. □

7 N-quasi Einstein metrics on Lie groups

7.1 What question are we trying to answer?

Question 7.1.1. *What can we say about vector fields on Lie groups which are N-quasi Einstein?*

7.2 Why is this question interesting?

Gradient N-quasi Einstein metrics with $N > n$ where first systematically considered by Case-Shu-Wei in [5] and Kim-Kim in [18]. They show that gradient N-quasi Einstein metrics correspond to warped product[1] Einstein metrics.

[1] See Definition 3.2.3

In [7, Theorem 1.1], Chen-Liang-Zhu proved that if M is a compact Lie group with a left-invariant metric g, and if X is a vector field on M such that $\mathrm{Ric}_X^N = Ag$ for $N \neq n$, then X is a left-invariant. Furthermore, X is a Killing vector field [7, Theorem 2.3].

Chen-Liang-Zhu prove [7, Theorem 1.1] by first proving that X is left-invariant, and then proving that X is Killing using properties of the Ricci tensor. We will consider $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = q$ where q is a left-invariant tensor, which is more general than $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = Ag$. Rather than considering G a compact Lie group, we assume G admits a discrete group of isometries, Γ, which acts cocompactly on G. Next, we give the definition for ad_X in order to state a linear algebra fact to prove that X is Killing given that X is a left-invariant vector field which satisfies $\mathrm{Ric}_X^N = Ag$.

Definition 7.2.1. *If G is a Lie group and if $\mathfrak{g}$ is the Lie algebra of G, then we define $ad_X : \mathfrak{g} \to \mathfrak{g}$ by $ad_X(Y) = [X, Y]$, where X, Y are vector fields in $\mathfrak{g}$.*

If G is a Lie group which admits a discrete subgroup Γ with compact quotient, then G must be unimodular. It is a linear algebra fact that if G is a unimodular Lie group, then there exists a basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$, the Lie Algebra of G, such that $g(ad_X(X_i), X_i) = 0$ for all i. We will use these facts about Lie groups to prove our main lemmas, which are generalizations of Chen-Liang-Zhu's [7, Theorem 1.1] and [7, Theorem 2.3].

7.3 What results and proofs do we need along the way to answering our main result?

We begin by stating our main lemma.

Lemma 7.3.1. *[23] Let G be a connected Lie group and let Γ be a discrete group of isometries which acts cocompactly on G. Let X be a vector field which is invariant under Γ. If (G, g, X) satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = q$, where q and g are left invariant, then X is a left-invariant vector field.*

Proof. Because G is a Lie group which admits a discrete subgroup with compact quotient, G must be unimodular. Let $M = {}^{G}/_{\Gamma}$ and let $\pi : G \to M$. By our discussion above, we can choose a basis, $\{X_i\} \in G$, such that

$g(ad_X(X_i), X_i) = 0$ for all i. Then let $X = \sum_{k=1}^{n} f_k X_k$, where $f_k : G \to \mathbb{R}$.

Using the technique from [7, Theorem 1.1], for all i, we get the following:

$$\begin{aligned}
\frac{1}{2}\mathcal{L}_X g(X_i, X_i) - \frac{1}{N-n} X^* \otimes X^*(X_i, X_i) &= X_i f_i + \sum_{k=1}^{n} f_k g(\nabla_{X_i} X_k, X_i) - \frac{1}{N-n} f_i^2 \\
&= X_i f_i + \sum_{k=1}^{n} f_k g([X_i, X_k], X_i) - \frac{1}{N-n} f_i^2 \\
&= X_i f_i + g(-ad_X(X_i), X_i) - \frac{1}{N-n} f_i^2 \\
&= X_i f_i - \frac{1}{N-n} f_i^2.
\end{aligned}$$

Then, since M is compact, there exists a maximum and a minimum of the function f_i on M. Let r be a point in M such that $f_i(r)$ is maximal and let s be a point in M such that $f_i(s)$ is minimal and let $q(\pi(X_i), \pi(X_i)) = \lambda_i$. Then

$$\begin{aligned}
\lambda_i &= X_i f_i(r) - \frac{1}{N-n} f_i^2(r) \\
&= -\frac{1}{N-n} f_i^2(r)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_i &= X_i f_i(s) - \frac{1}{N-n} f_i^2(s) \\
&= -\frac{1}{N-n} f_i^2(s)
\end{aligned}$$

Then, $f_i^2(r) = f_i^2(s) = -(N-n)\lambda_i$. We will now rule out the case $f_i(r) = -f_i(s)$ in order to show that f_i must be constant.

Let $c(t)$ be an integral curve of X_i. Then along $\pi \circ c(t)$, $f_i'(t) - \frac{1}{N-n} f_i^2(t) = \lambda_i$. Solving this equation (see Lemma 8.4.1), we have that $f_i(t) = \sqrt{-\lambda_i(N-n)}$, $-\sqrt{-\lambda_i(N-n)}$, 0, or $-\sqrt{-\lambda_i(N-n)} \tanh(\frac{\sqrt{-\lambda_i(N-n)}}{(N-n)}(t+C))$.

Assume for the sake of contradiction that $f_i(t)$ is not constant, ie $f_i(t) = -\sqrt{-\lambda_i(N-n)} \tanh(\frac{\sqrt{-\lambda_i(N-n)}}{(N-n)}(t+C))$, where C is a constant. Let $\pi \circ c(t_i)$ be a sequence of points such that $t_i \to \infty$. Since M is compact, there exists a subsequence of $\{\pi \circ c(t_i)\}$ which converges to a point on M.

Now consider the set $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$. Since this set is closed, f_i has a maximal point, t_{max} on this set. Because the supremum of the tanh function is 1, we know that the maximum of $f_i(t)$ on $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$ is $\sqrt{-\lambda_i(N-n)}$.

Let $b(t)$ be an integral curve of X_i such that $b(0) = c(t_{max}) = \sqrt{-\lambda_i(N-n)}$. Now consider the set $\{\pi \circ b(t) : t \in \mathbb{R}\}$. Along $b(t)$, $f_i(t)$ is either $\sqrt{-\lambda_i(N-n)}$ or $-\sqrt{-\lambda_i(N-n)} \tanh(\frac{\sqrt{-\lambda_i(N-n)}}{(N-n)}(t+C))$. Since the supremum of $f_i(t)$ on $\{\pi \circ b(t) : t \in \mathbb{R}\}$ is $\sqrt{-\lambda_i(N-n)}$ and tanh never achieves its maximum on its domain, $f_i(t)$ must be constantly $\sqrt{-\lambda_i(N-n)}$ on the set $\{\pi \circ b(t) : t \in \mathbb{R}\}$.

Finally, since $\overline{\{\pi \circ b(t) : t \in \mathbb{R}\}} = \overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$, $f_i(t)$ is constant on $\overline{\{\pi \circ c(t) : t \in \mathbb{R}\}}$. Then, since $f_i(t)$ is constant along every integral curve and since G is connected, $f_i(t)$ is constant.

□

Lemma 7.3.2. *[23] Let G be a unimodular Lie group with left-invariant metric, g. If X is left-invariant,* $\mathrm{tr}(q \circ ad_X) = 0$, *and* $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = q$, *where q is left-invariant, then X is Killing.*

Proof of Lemma 7.3.2. Let $\{X_i\}$ be an orthonormal basis relative to g and let

$X = a_1X_1 + a_2X_2 + ... + a_nX_n$. Then, plugging in (X_i, X_j) into $q = \frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n}X^* \otimes X^*$, we get

$$q(X_i, X_j) = \frac{1}{2}\big(g([X_i, X], X_j) + g([X_j, X], X_i) - \frac{1}{N-n}g(X, X_i)g(X, X_j).$$

We denote the projection of X_i onto X, as $\operatorname{proj}_X X_i$. Since $\operatorname{proj}_X X_i = \frac{g(X, X_i)X}{|X|^2}$ and $ad_X(X_i) = [X, X_i]$, we have the following:

$$q(X_i, X_j) = \frac{1}{2}\big(g(ad_X(X_i), X_j) + g(ad_X(X_j), X_i)\big) - \frac{|X|^2}{(N-n)}g(\operatorname{proj}_X X_i, X_j).$$

Thus, we have the following equation, where we view q, ad_X, and proj_X as matrices:

$$q = \frac{1}{2}\big(ad_X + ad_X^T\big) - \frac{|X|^2}{(N-n)} \operatorname{proj}_X.$$

We denote "$\cdot$" as the matrix multiplication symbol. Multiplying both sides by the matrix, ad_X, we get:

$$\begin{aligned} q \cdot ad_X &= \frac{1}{2}\big(ad_X + ad_X^T\big) \cdot ad_X - \frac{|X|^2}{(N-n)} \operatorname{proj}_X \cdot ad_X \\ &= \frac{1}{2}\big(ad_X^2 + ad_X^T \cdot ad_X\big) - \frac{|X|^2}{(N-n)} \operatorname{proj}_X \cdot ad_X. \end{aligned}$$

Taking the trace of both sides, we get

$$\operatorname{tr}(q \cdot ad_X) = \frac{1}{2} \operatorname{tr}\left(ad_X^2 + ad_X^T \cdot ad_X\right) - \frac{|X|^2}{(N-n)} \operatorname{tr}(\operatorname{proj}_X \cdot ad_X).$$

Then, since $\operatorname{tr}(q \cdot ad_X) = 0$ and using that for any $n \times n$ matrix A, $\operatorname{tr}(A^2) = \operatorname{tr}((A^T)^2)$, we get

$$0 = \frac{1}{4} \operatorname{tr}\left((ad_X + ad_X^T)^2\right) - \frac{|X|^2}{(N-n)} \operatorname{tr}(\operatorname{proj}_X \cdot ad_X).$$

Now, plugging in X_i, one of the orthonormal basis vectors into $ad_X \cdot \operatorname{proj}_X$ and using that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any two matrices A and B, we get:

$$\begin{aligned} ad_X \cdot \operatorname{proj}_X(X_i) &= \frac{a_i}{|X|^2}[X, X] \\ &= 0. \end{aligned}$$

Thus, we have $0 = \frac{1}{4} \operatorname{tr}\left((ad_X + ad_X^T)^2\right)$.

Now, since $ad_X + ad_X^T$ is symmetric, we can diagonalize $ad_X + ad_X^T$, and call the diagonalized matrix D. Then, $\operatorname{tr}((ad_X + ad_X^T)^2) = \operatorname{tr}(D^2)$. Since the eigenvalues in D^2 are nonnegative and $\operatorname{tr}(D^2)$ is the sum of the eigenenvalues of D^2, $\frac{1}{2}(ad_X + ad_X^T) = 0$. Thus, X is Killing. □

Next, we will apply Lemma 7.3.1 to metrics which satisfy $\operatorname{Ric}_X^N = Ag$.

Theorem 7.3.3. *[23] Let G be a Lie group and let Γ be a discrete group of isometries which acts cocompactly on G, where $\pi : G \to {}^{G}/_{\Gamma}$ is a covering map. If $({}^{G}/_{\Gamma}, g, X)$ satisfies $\operatorname{Ric}_X^N = Ag$, then $\widetilde{X} = \pi^*(X)$ is left invariant and Killing.*

Proof. First, we let $\widetilde{g} = \pi^*(g)$, be the pullback metric of g. Since π is a local isometry, $\operatorname{Ric}_{\widetilde{X}}^m = A\widetilde{g}$

Since $A\widetilde{g} - \operatorname{Ric}_{\widetilde{g}}$ is left-invariant, by Lemmas 7.3.1 and 7.3.2, $\widetilde{X}$ is left-invariant and Killing. □

We immediately get the following corollary, which we will use throughout Section 9.5.

Corollary 7.3.4. *[23] If M^n is a unimodular Lie Group and if $\mathrm{Ric}_X^N = Ag$ with X a left-invariant vector field and g a left-invariant metric, then X is a Killing field.*

Lemma 7.3.5. *[23] Suppose (M^n, g) is a Lie group which satisfies $\mathrm{Ric}_X^N = Ag$ where X is nonzero, left-invariant, and Killing. If $\{X_1, X_2, ...X_n\}$ is an eigenbasis of the Ricci tensor of left invariant fields, then X is a multiple of one of the eigenbasis vectors (ie there exists $1 \leq m \leq n$ such that $X = a_m X_m$).*

Proof. Since X is left-invariant and Killing, we have for all $1 \leq i, j \leq n$ where $i \neq j$,

$$\mathrm{Ric}_X^N(X_i, X_j) = -\frac{1}{N-n} a_i a_j.$$

Now $\mathrm{Ric}_X^N(X_i, X_j) = Ag(X_i, X_j) = 0$ for all sets of i, j if and only if at least $n-1$ sets of a_k are 0. Thus, $X = a_m X_m$ for some $1 \leq m \leq n$. □

7.4 What are the main results in this section?

Lemma 7.3.2 and Theorem 7.3.3, and Corollary 7.3.4 give an us important properties of Lie groups which satisfy the N-quasi Einstein equation. We summarize the main results in this section below.

Lemma 7.3.2 [23]: Let G be a connected Lie group and let Γ be a discrete group of isometries which acts cocompactly on G. Let X be a vector field which is invariant under Γ. If (G, g, X) satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = q$, where q and g are left invariant, then X is a left-invariant vector field.

Theorem 7.3.3 [23]: Let G be a Lie group and let Γ be a discrete group of isometries which acts cocompactly on G, where $\pi : G \to {}^{G}/_{\Gamma}$ is a covering map. If $({}^{G}/_{\Gamma}, g, X)$ satisfies $\mathrm{Ric}_X^N = Ag$, then $\widetilde{X} = \pi^*(X)$ is left invariant and Killing.

Corollary 7.3.4 [23]: If M^n is a unimodular Lie Group and if $\mathrm{Ric}_X^N = Ag$ with X a left-invariant vector field and g a left-invariant metric, then X is a Killing field.

8 Geodesics on manifolds which are N-quasi Einstein and Einstein

8.1 What question are we trying to answer?

Question 8.1.1. *Given a manifold which is Einstein and N-quasi Einstein, what can we say about the geodesics on such manifolds? Does this help us give a characterization of manifolds which are Einstein and N-quasi Einstein?*

8.2 Why is this question interesting?

The results in this chapter were originally meant to be a way to prove the results in Chapter 9. However, it has become clear that the local behavior of manifolds which are N-quasi Einstein and Einstein tells us a lot about the global behavior of such manifolds, which leads us to two of our main results in this chapter, Proposition 8.4.10 and Theorem 8.4.12.

8.3 What are the main results in this chapter?

The following proposition gives us a characterization of geodesics on a manifold which is both N-quasi Einstein and Einstein. For the rest of the chapter, let $m = N - n$. The main theorems of this chapter are as follows. They will be proven in the next section.

Proposition 8.4.3 [23] Let (M, g) be a complete Riemannian manifold and let $\gamma : (-\infty, \infty) \to M$ be a unit speed geodesic. Suppose the equation

$$\frac{1}{2}\mathcal{L}_X g(\dot{\gamma}, \dot{\gamma}) - \frac{1}{N-n} g(X, \dot{\gamma}) g(X, \dot{\gamma}) = \lambda g(\dot{\gamma}, \dot{\gamma})$$

is satisfied at every point on γ.

1. If $\lambda = 0$ for $N \neq n$ at every point along γ, then $\varphi(t) = 0$.

2. If $\lambda m > 0$ at every point along γ, then there are no complete solutions to $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = \lambda g$.

3. If $\lambda m < 0$ along a geodesic, then

$$\varphi(t) = \sqrt{-\lambda(N-n)} \tanh\left(\tfrac{\sqrt{-\lambda(N-n)}}{(N-n)}(t + C)\right) \text{ or}$$
$$\varphi(t) = \pm\sqrt{-\lambda(N-n)}.$$

The next three results follow from Proposition 8.4.3 and give global results about manifolds which are both N-quasi Einstein and Einstein.

Proposition 8.4.8 [23] If M is a compact manifold which satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = \lambda g$ with $X \neq 0$ and $\lambda m < 0$ along every geodesic, then $M = S^1$.

Proposition 8.4.10 [23] Consider $S^j \times \mathbb{R}$ with the product metric and $j \geq 2$, S^j with a constant curvature metric of Ricci curvature ρ, and $\mathbb{R}$ with the flat metric. Then there exists a nontrivial N-quasi Einstein metric, $\text{Ric}_X^N = Ag$ if and only if $A = \rho$ and $N > n$.

Theorem 8.4.12 [23] Consider the compact quotient of $M \times N$ with the product metric, where M and N are simply-connected complete Einstein manifolds. Then the only nontrivial solutions to $\text{Ric}_X^m = Ag$ occurs when either M is $\mathbb{R}$ or N is $\mathbb{R}$.

8.4 *What results and proofs do we need along the way to answering our main result?*

Our first lemma gives a characterization for functions which satisfy the differential equation, $f'(t) - \frac{1}{N-n} f^2(t) = \lambda$. Later in the section, we see that manifolds which are both Einstein and N-quasi Einstein satisfy the same equation.

Lemma 8.4.1. *[23] Let $f'(t) - \frac{1}{N-n} f^2(t) = \lambda$, where $f : \mathbb{R} \to \mathbb{R}$ is defined for all t in $\mathbb{R}$ and λ and m are constants. Then:*

1. *If $\lambda = 0$, then $f(t) = 0$.*
2. *If $\lambda m > 0$, then there are no solutions.*
3. *If $\lambda m < 0$, then $f(t) = \pm\sqrt{-\lambda(N-n)}$ or $\sqrt{-\lambda(N-n)} \tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t + C)\right)$.*

Proof. Suppose $\lambda = 0$. Then it is clear that $f(t) = 0$ is a solution. If $f(0)$ is not

0, then

$$f'(t) = \frac{f(t)^2}{(N-n)}$$
$$\Rightarrow f(t) = \frac{1}{C - \frac{t}{(N-n)}}$$

where C is any real number. However, at $t = mC$, t blows up, which is a contradiction since f has to exist for all time.

If $\lambda m > 0$, then

$$f'(t) = \frac{f(t)^2}{(N-n)} + \lambda.$$

Here, we see that $\frac{f(t)^2}{(N-n)} + \lambda$ is never zero since $\lambda m > 0$. Integrating and rearranging, we get

$$\int \frac{f'(t)}{\frac{f^2(t)}{(N-n)} + \lambda} dt = \int 1 dt$$
$$\Rightarrow \frac{(N-n)}{\lambda} \int \frac{f'(t)}{1 + (\frac{f(t)}{\sqrt{\lambda(N-n)}})^2} dt = t + C$$
$$\Rightarrow \sqrt{\frac{(N-n)}{\lambda}} \tan^{-1}\left(\frac{f(t)}{\sqrt{\lambda(N-n)}}\right) = t + C,$$

so then,

$$f(t) = \sqrt{\lambda(N-n)} \tan\left(\sqrt{\frac{\lambda}{(N-n)}}(t+C)\right).$$

Since the tan function does not exist everywhere, $f(t)$ also does not exist everywhere. Thus, if $\lambda m > 0$, there are no solutions.

If $\lambda m < 0$, then clearly $f(t) = \pm\sqrt{-\lambda(N-n)}$ is a solution to the equation. Assume $f(0)$ is not $\pm\sqrt{-\lambda(N-n)}$. Then we integrate and rearrange as follows:

$$\int \frac{f'(t)}{\frac{f^2(t)}{(N-n)} + \lambda} dt = \int 1 dt$$

$$\frac{(N-n)}{2\sqrt{-\lambda(N-n)}} \ln \left| \frac{1 - \frac{f(t)}{\sqrt{-\lambda(N-n)}}}{1 + \frac{f(t)}{\sqrt{-\lambda(N-n)}}} \right| = t + C$$

$$\Rightarrow \left| \frac{1 - \frac{f(t)}{\sqrt{-\lambda(N-n)}}}{1 + \frac{f(t)}{\sqrt{-\lambda(N-n)}}} \right| = e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}.$$

If $\dfrac{1 - \frac{f(t)}{\sqrt{-\lambda(N-n)}}}{1 + \frac{f(t)}{\sqrt{-\lambda(N-n)}}} = e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}$, then

$$f(t) = \sqrt{-\lambda(N-n)}\left(\frac{1 - e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}}{1 + e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}}\right) = \sqrt{-\lambda(N-n)} \tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)\right).$$

If $\dfrac{1 - \frac{f(t)}{\sqrt{-\lambda(N-n)}}}{1 + \frac{f(t)}{\sqrt{-\lambda(N-n)}}} = -e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}$, then $f(t) = \sqrt{-\lambda(N-n)}\left(\dfrac{1 + e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}}{1 - e^{2\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)}}\right)$.
In this case, at $t = -C$, $f(t)$ does not exist, which is a contradiction. □

Our next definition and proposition deal with analyzing the equation $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = Ag$, which we will use to find N-quasi Einstein solutions to $S^2 \times \mathbb{R}$ and H^3. We will also prove theorems for more general spaces using this analysis.

Definition 8.4.2. *Let $\gamma(t)$ be a unit speed geodesic. We define $\varphi_\gamma(t)$ as $g(X_{\gamma(t)}, \dot{\gamma}(t))$. Note that $\varphi_\gamma(t)$ is well defined for all t that $\gamma(t)$ is defined. If it is clear which $\gamma(t)$ we are defining $\varphi_\gamma(t)$ along, then we will call our function $\varphi(t)$ rather than $\varphi_\gamma(t)$.*

Proposition 8.4.3. *[23] Let (M, g) be a complete Riemannian manifold and let $\gamma : (-\infty, \infty) \to M$ be a unit speed geodesic. Suppose the equation*

$$\frac{1}{2}\mathcal{L}_X g(\dot{\gamma}, \dot{\gamma}) - \frac{1}{N-n} g(X, \dot{\gamma}) g(X, \dot{\gamma}) = \lambda g(\dot{\gamma}, \dot{\gamma})$$

is satisfied at every point on γ.

1. *If $\lambda = 0$ for $N \neq n$ at every point along γ, then $\varphi(t) = 0$.*

2. *If $\lambda m > 0$ at every point along γ, then there are no complete solutions to $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = \lambda g$.*

3. *If $\lambda m < 0$ along a geodesic, then*

$$\varphi(t) = \sqrt{-\lambda(N-n)} \tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t + C)\right) \text{ or}$$
$$\varphi(t) = \pm\sqrt{-\lambda(N-n)}.$$

Proof. We have the following set of equations:

$$\begin{aligned}
\frac{d}{dt}(\varphi(t)) &= \frac{1}{2}\mathcal{L}_X g(\dot{\gamma}, \dot{\gamma}) \\
&= \frac{1}{N-n}(X^* \otimes X^*)(\dot{\gamma}, \dot{\gamma})) + \lambda g(\dot{\gamma}, \dot{\gamma}) \\
&= \frac{1}{N-n} g(X, \dot{\gamma})^2 + \lambda \\
&= \frac{1}{N-n}\varphi^2(t) + \lambda.
\end{aligned}$$

The proposition follows from Lemma 8.4.1. □

Remark 8.4.4. *[23] If M^n is a compact manifold, then we can prove Proposition 8.4.3(2) by using the Divergence Theorem. Taking the trace of both sides of* $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n}X^* \otimes X^* = \lambda g$, *we get* $div(X) - \frac{1}{N-n}|X|^2 = \lambda n$. *Integrating both sides over M, we get*

$$\begin{aligned}\int_M |X|^2 &= -\int_M \lambda mn \\ &= -\lambda mn \,\mathrm{vol}(M)\end{aligned}$$

Either $X = 0$ and $\lambda = 0$ or the left hand side is positive which implies λm must be negative.

In the following example, we provide an example of a manifold which satisfies $\mathrm{Ric}_X^N = \lambda g$ with $\lambda m < 0$.

Example 8.4.5. *[23] Let $M = S^1$ with the usual metric with $\{\frac{\partial}{\partial\theta}\}$ the basis vector. Let $X = \sqrt{-\lambda(N-n)}\frac{\partial}{\partial\theta}$ with $\lambda m < 0$. Since X is Killing and S^1 is Ricci flat, we get* $\mathrm{Ric}_X^N = \lambda g$.

Next, we give a global analysis of $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n}X^* \otimes X^* = \lambda g$ when $\lambda m < 0$. In order to do this, we will first state a definition of critical point originally defined by Grove-Shiohama (Also see [33]).

Definition 8.4.6. *[33] Fix $p \in M$. A point q is a critical point of the distance function to p (is critical point to p) if, for every vector $V \in T_q M$, there is a minimal geodesic γ with $\gamma(0) = p$, $\gamma(d(p,q)) = q$ such that $g(\dot\gamma(d(p,q)), V) \le 0$.*

Lemma 8.4.7. *[33, Corollary 43] Suppose that there are no critical points of the* *distance function to p in the annulus $\{q : a \leq d(p,q) \leq b\}$. Then $B(p,a)$ is homeomorphic to $B(p,b)$ and $B(p,b)$ deformation retracts onto $B(p,a)$. Moreover, if there are no critical points of p in M, then M is diffeomorphic to $\mathbb{R}^n$.*

Using similar techniques to those of Wylie in the proof of [42, Proposition 1], we will look for spaces which admit $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n}X^* \otimes X^* = \lambda g$ with $\lambda m < 0$ everywhere. We will find that the only possibility is S^1 if the space is compact.

Proposition 8.4.8. *[23] If M is a compact manifold which satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n}X^* \otimes X^* = \lambda g$ with $X \neq 0$ and $\lambda m < 0$ along every geodesic, then $M = S^1$.*

Proof. Since M is compact, the function $f(p) = |X(p)|^2$ achieves a maximum and a minimum value. At the minimum, $0 = D_X f = D_X g(X,X) = 2\mathcal{L}_X g(X,X)$. Then,

$$\frac{1}{2}\mathcal{L}_X g(X,X) - \frac{1}{N-n}(X^* \otimes X^*)(X,X) = \lambda g(X,X)$$
$$\Rightarrow -\frac{1}{N-n}|X|^4 = \lambda |X|^2.$$

Then, either $|X|^2 = -\lambda m$ or $|X|^2 = 0$ at the minimum point. By a similar argument, $|X|^2 = -\lambda m$ or $|X|^2 = 0$ at the maximum point as well. Thus, either $|X|^2 = -\lambda m$ for every point on M, or there exists a point $p \in M$ where $X(p) = 0$.

If $|X|^2 = -\lambda m$ for every point in M, then taking the trace of $\frac{1}{2}\mathcal{L}_X g -$

$\frac{1}{N-n}X^* \otimes X^* = \lambda g$, we get

$$\text{div}(X) - \frac{|X|^2}{(N-n)} = \lambda n.$$

Plugging in $|X|^2 = -\lambda m$, we get that

$$\text{div}(X) = \lambda(n-1).$$

Taking the integral of both sides over M and using the Divergence Theorem, we get that $\lambda(n-1)\,\text{vol}(M) = 0$. If $\lambda = 0$ then $X = 0$ by Proposition 8.4.3(1), so n must be 1. Since M is compact, this means that $M = S^1$.

In the case when there exists a point $p \in M$ such that $X(p) = 0$, we will prove that there are no critical points to p in M and we will use Lemma 8.4.7 to show that M must be $\mathbb{R}^n$.

By Definition 8.4.6, we want to show that there exists a vector V such that every geodesic γ with $\gamma(0) = p$, $\gamma(d(p,q)) = q$ such that $g(\dot\gamma(d(p,q)), V) > 0$. Consider the case when $N > n$. Let $\gamma(t)$ be a geodesic with $\gamma(0) = p$ and let $V = X$. If $\varphi(t) = g(X_{\gamma(t)}, \dot\gamma(t))$, then since $X(p) = 0$, $\varphi(0)$ must be 0, so $\varphi(t)$ cannot be constantly nonzero.

Then by Proposition 8.4.3,

$$\varphi(t) = \sqrt{-\lambda(N-n)}\tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}t\right).$$

If $\varphi(t) = \sqrt{-\lambda(N-n)}\tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}t\right)$, then $\varphi(t) > 0$ when $t > 0$, so by Lemma 8.4.7, $M = \mathbb{R}^n$. This is a contradiction because M is compact.

If $N > n$, then we again let $\gamma(t)$ be a geodesic with $\gamma(0) = p$. We will let $V = -X$ so that the differential equation we have to solve is $-\frac{d}{dt}\varphi(t) = \frac{1}{N-n}\varphi^2(t) + \lambda$. Then we get that the solutions are

$$\varphi(t) = \sqrt{-\lambda(N-n)}\tanh\left(\frac{-\sqrt{-\lambda(N-n)}}{(N-n)}t\right) \text{ or } \varphi(t) = \pm\sqrt{-\lambda(N-n)}.$$

$\varphi(t)$ cannot be $\pm\sqrt{-\lambda(N-n)}$ as in the $N > n$ case. If $\varphi(t) = \sqrt{-\lambda(N-n)}\tanh\left(\frac{-\sqrt{-\lambda(N-n)}}{(N-n)}t\right)$, then $\varphi(t)$ is positive for $t > 0$, giving us a contradiction by Lemma 8.4.7. □

Next, we give an example of a space (M, g) which is non Euclidean, N-quasi Einstein and Einstein, and X is not trivial.

Example 8.4.9. *[23] Consider H^2 with the metric $g = dr^2 + e^{2r}dx^2$ and let $X = -m\frac{\partial}{\partial r}$. Then we have the following:*

$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$

$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -e^{2r}\frac{\partial}{\partial r}$

$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0.$

Then, we have the following computations for the Ricci curvature:

$\mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}) = 0$

$\mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -1$

$\mathrm{Ric}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = -e^{2r},$

so we see that our metric satisfies $\mathrm{Ric} = -1g$. *We have the following computations for* Ric_X^N*:*

$\mathrm{Ric}_X^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}) = 0$

$\mathrm{Ric}_X^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -1 - \frac{1}{N-n}(-m)^2 = -1 - m$

$\mathrm{Ric}_X^N(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = e^{2r}(-1 - m),$

so we see that $\mathrm{Ric}_X^N = (-1 - m)g.$

We are now ready to solve for the solutions of the N-quasi Einstein equation for $S^j \times \mathbb{R}$ when $j \geq 2$.

Proposition 8.4.10. *[23] Consider $S^j \times \mathbb{R}$ with the product metric and $j \geq 2$, S^j with a constant curvature metric of Ricci curvature ρ, and $\mathbb{R}$ with the flat metric. Then there exists a nontrivial N-quasi Einstein metric, $\mathrm{Ric}_X^N = Ag$ if and only if $A = \rho$ and $N > n$.*

Proof. Let $\{X_1, X_2, \frac{\partial}{\partial r}\}$ be an orthonormal basis where $\{X_1, X_2\}$ is in TS^2 and $\{\frac{\partial}{\partial r}\}$ is in $T\mathbb{R}$.

First, consider the case $A - \rho = 0$. Let γ_{S^2} be a great circle on S^2 since the geodesics on S^2 are the great circles. We apply Proposition 8.4.3 (1). This says that X restricted to S^2 must be 0. Letting $\gamma_{\mathbb{R}}$ be a unit speed geodesic in $\mathbb{R}$, we have

$$\frac{1}{2}\mathcal{L}_X g(\dot{\gamma}_{\mathbb{R}}, \dot{\gamma}_{\mathbb{R}}) - \frac{1}{N-n} X^* \otimes X^*(\dot{\gamma}_{\mathbb{R}}, \dot{\gamma}_{\mathbb{R}}) = A = \rho.$$

If $A - \rho = 0$ and $N > n$, then by Proposition 8.4.3(3), $\varphi_{\gamma_{\mathbb{R}}}(t)$ is either

$$\sqrt{-\rho(N-n)} \text{ or } \sqrt{-\rho(N-n)} \tanh\big(\frac{\sqrt{-\rho(N-n)}}{(N-n)}(t + C)\big)$$

which implies

$$X = \sqrt{-\rho(N-n)}\frac{\partial}{\partial r} \text{ or } \sqrt{-\rho(N-n)}\tanh\left(\frac{\sqrt{-\rho(N-n)}}{(N-n)}(t+C)\right)\frac{\partial}{\partial r}.$$

If $A-\rho = 0$ and $N > n$, then by Proposition 8.4.3(2), there are no solutions.

If $(A-\rho)m > 0$, then applying Proposition 8.4.3(2) to γ_{S^2} in a similar fashion, we get that there are no solutions.

Consider the case $(A-\rho)m < 0$. Since S^2 has dimension greater than 1, we can choose γ_{S^2} perpendicular to X at 0 so that $\varphi_{\gamma_{S^2}}(0) = 0$. and we apply Proposition 8.4.3(3) to $\gamma_{S^2} \in S^2$. Then $\varphi_{S^2}(t)$ is either

$$\pm\sqrt{-(A-\rho)(N-n)} \text{ or } \sqrt{-(A-\rho)(N-n)}\tanh\left(\frac{\sqrt{(A-\rho)(N-n)}}{(N-n)}(t+C)\right).$$

$\varphi_{S^2}(t)$ cannot be $\sqrt{-(A-\rho)(N-n)}\tanh\left(\frac{\sqrt{(A-\rho)(N-n)}}{(N-n)}(t+C)\right)$ since γ_{S^2} must be periodic and $\varphi_{S^2}(t)$ cannot be $\sqrt{-(A-\rho)(N-n)}$ since $\varphi_{\gamma_{S^2}}(0) = 0$. This is a contradiction, so there are no solutions in this case as well. □

Now, we will generalize Proposition 8.4.10 to compact quotients of manifolds of the form $M \times N$, where M and N are Einstein manifolds. We prove this in a different way from Proposition 8.4.10 because we cannot use the argument that $\varphi(t)$ must be periodic on S^j.

Lemma 8.4.11. *[23] Consider a compact quotient of $M \times N$ with the product metric where M is an Einstein manifold. If there is a nontrivial N-quasi Einstein solution on such a space, then either $X|_M = 0$ or M is one-dimensional.*

Proof. Without loss of generality, assume that M and N are simply connected because if either space is not simply connected, we can lift them to the universal cover. Let $\pi : M \times N \to {(M \times N)}/{\Gamma}$ be the universal covering map and let $\mathrm{Ric}_M = \rho_M g_M$. Let $\gamma_M(t)$ be a unit speed geodesic in M. Then we have

$$\frac{1}{2}\mathcal{L}_X g(\dot{\gamma}_M, \dot{\gamma}_M) - \frac{1}{N-n} X^* \otimes X^*(\dot{\gamma}_M, \dot{\gamma}_M) = A - \rho_M.$$

We aim to show that either $A - \rho_M = 0$ or $M = \mathbb{R}$. If M is not $\mathbb{R}$ then M is not one-dimensional, so we can choose γ_M to be perpendicular to X at 0. In this case, $\varphi_{\gamma_M}(0)$ is zero, so $\varphi_{\gamma_M}(t)$ cannot be constantly nonzero. If $(A - \rho_M)m > 0$, then by Proposition 8.4.3(2), there are no complete solutions. If $(A - \rho_M)m < 0$, then by Proposition 8.4.3(3), and since $\varphi_{\gamma_M}(t)$ $\varphi_{\gamma_M}(t)$ is

$$\sqrt{-(A-\rho_M)(N-n)}\tanh\left(\frac{\sqrt{(A-\rho_M)(N-n)}}{(N-n)}(t+C)\right).$$

To show that $\varphi_{\gamma_M}(t)$ cannot be $\sqrt{-(A-\rho_M)(N-n)}\tanh\left(\frac{\sqrt{(A-\rho_M)(N-n)}}{(N-n)}(t+C)\right)$, we will use an argument similar to the proof of Lemma 7.3.1.

Consider the set $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$. Since this set is closed, $\varphi_{\gamma_M}(t)$ has a maximal point, t_{max} on this set. Because the supremum of the tanh function is 1, we know that the maximum of $\varphi_{\gamma_M}(t)$ on $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$ is $\sqrt{-(A-\rho_M)(N-n)}$.

Let $\beta(t)$ be a geodesic of X such that $\beta(0) = \gamma_M(t_{max}) = \sqrt{-(A-\rho_M)(N-n)}$. Now consider the set $\{\pi \circ \beta(t) : t \in \mathbb{R}\}$. Along $\beta(t)$, $\varphi_\beta(t)$ is either $\sqrt{-(A-\rho_M)(N-n)}$ or $-\sqrt{-(A-\rho_M)(N-n)}\tanh(\frac{\sqrt{-(A-\rho_M)(N-n)}}{(N-n)}(t+C))$. Since the supremum of $\varphi_\beta(t)$ on $\{\beta(t) : t \in \mathbb{R}\}$ is $\sqrt{-(A-\rho_M)(N-n)}$ and the tanh function never achieves its maximum on its domain, $\varphi_\beta(t)$ must be constantly $\sqrt{-(A-\rho_M)(N-n)}$ on the set $\{\pi \circ \beta(t) : t \in \mathbb{R}\}$.

Finally, since $\overline{\{\pi \circ \beta(t) : t \in \mathbb{R}\}} = \overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$, $\varphi_{\gamma_M}(t)$ is constant on $\overline{\{\pi \circ \gamma_M(t) : t \in \mathbb{R}\}}$. Thus, $\varphi_{\gamma_M}(t)$ is constant.

Since $\varphi_{\gamma_M}(0) = 0$, $\varphi_{\gamma_M}(t)$ cannot be $\pm\sqrt{-(A-\rho_M)(N-n)}$, and so we have arrived at a contradiction.

Thus, either $M = \mathbb{R}$ or $A - \rho_M = 0$. If $A - \rho_M = 0$, then by Proposition 8.4.3(1), $\varphi_{\gamma_M} = 0$, which implies $X|_M = 0$. □

Now we can prove the following theorem.

Theorem 8.4.12. *[23] Consider the compact quotient of $M \times N$ with the product metric, where M and N are simply-connected complete Einstein manifolds. Then the only nontrivial solutions to $\mathrm{Ric}_X^m = Ag$ occurs when either M is $\mathbb{R}$ or N is $\mathbb{R}$.*

Proof. Let $\pi : M \times N \to {}^{(M \times N)}/_{\Gamma}$ be the universal covering map and let $\mathrm{Ric}_M = \rho_M g_M$ and $\mathrm{Ric}_N = \rho_N g_N$. Let $\gamma_M(t)$ be a unit speed geodesic in M and let $\gamma_N(t)$ be a unit speed geodesic in N. By Lemma 8.4.11, M is either one-dimensional or $X|_M = 0$ and $A - \rho_M = 0$. By symmetry, either $A - \rho_N = 0$ and $X|_N$ is zero, or $N = \mathbb{R}$.

Suppose without loss of generality that $N = \mathbb{R}$. Then

$$\frac{1}{2}\mathcal{L}_X g(\dot\gamma_N, \dot\gamma_N) - \frac{1}{N-n} X^*(\dot\gamma_N) X^*(\dot\gamma_N) = Ag.$$

By Proposition 8.4.3, $A = 0$, then

$$X = 0,$$

If $Am > 0$, then there are no solutions, and if $Am < 0$, then

$$X = \sqrt{-\lambda(N-n)} \tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)\right)\frac{\partial}{\partial r} \text{ or } X = \pm\sqrt{-\lambda(N-n)}\frac{\partial}{\partial r}.$$

If we consider the set $\overline{\{\pi \circ \gamma_N(t) : t \in \mathbb{R}\}}$ and use the same argument as above, we see that $X = \sqrt{-\lambda(N-n)} \tanh\left(\frac{\sqrt{-\lambda(N-n)}}{(N-n)}(t+C)\right)\frac{\partial}{\partial r}$ is not a solution.

Thus, the only solutions are $X = 0$ when $A = \rho_M = \rho_N \neq 0$, and $X = \pm\sqrt{-A(N-n)}\frac{\partial}{\partial r}$ when either $N = \mathbb{R}$ or $M = \mathbb{R}$.

□

9 Classification of locally homogeneous 3-Manifolds with N-quasi Einstein metrics

9.1 What question are we trying to answer?

Question 9.1.1. *Which locally homogeneous* 3*-manifolds satisfy N-quasi Einstein metrics?*

9.2 Why is this question interesting?

In [5, Theorem 2.1], Case-Shu-Wei prove that a compact gradient N-quasi Einstein with constant curvature must be trivial if $N > n$. Since locally homogeneous manifolds[1] have constant scalar curvature, this shows that compact locally homogeneous manifolds which satisfy $\mathrm{Ric}_\phi^N = Ag$ with $N > n$ must be trivial. The $N > n$ case follows from [35, Theorem 1.9]. In [11, Theorem 1.3], He-Petersen-Wylie prove that if (M^3, g) has no boundary, satisfies $\mathrm{Ric}_\phi^N = Ag$ with $N - n > 1$, and has constant scalar curvature, then M^3 is a quotient of

[1] See Definition 3.1.4

S^3, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $H^2 \times \mathbb{R}$, or H^3 with the standard metric. In [10, Theorem 1.4], He-Petersen-Wylie show that if (M^n, g) is a non-compact Ricci soliton with $N > n$ and $A < 0$, under certain conditions, M admits a non-trivial homogeneous gradient N-quasi Einstein ($\mathrm{Ric}_\phi^N = Ag$) one-dimensional extension. In [20, Theorem 1.1], Lafuente proves a converse to this result.

On the other hand, Chen-Liang-Zhu construct some examples of non-gradient N-quasi Einstein manifolds in [7]. In [19, Corollary 4.1,4.2], Kunduri-Lucietti study the non-gradient N-quasi Einstein metrics with $m = 2$ in the context of vacuum, homogeneous near-horizon geometries, which gives us motivation to study non-gradient N-quasi Einstein metrics.

If M^n is a homogeneous Einstein manifold, where $\mathrm{Ric} = Ag$, then if $A > 0$, then M is compact by Myers' Theorem, if $A = 0$, then M is flat by Alekseevskii-Kimel'fel'd in [1], and if $A < 0$, then M is not compact by Bochner's Theorem, which can be found in Section 9.6. If we compare this to our results in Table 9.1, we see that this structure does not hold for N-quasi Einstein metrics. When $A = 0$, there exist solutions on (compact quotients of) $SU(2)$, which are not flat. Similarly, in the $A < 0$ case, there exist solutions on compact quotients of $SU(2)$.

In [43, Lemma 4.4], we see that if M^n is a compact manifold with infinite fundamental group satisfying $\mathrm{Ric}_\phi^N = Ag$ where $A = 0$, with $m = 1 - n < 0$, then the universal cover has a warped product splitting[2]. By Table 9.1, there exist solutions for the compact quotient of $\widetilde{SL_2(\mathbb{R})}$ if M^n satisfies $\mathrm{Ric}_X^N = Ag$ when $N > n$ and $A = 0$. This is interesting because $\widetilde{SL_2(\mathbb{R})}$ clearly does not

[2] See Definition 3.2.3

split.

9.3 What are the main results in this chapter?

In the following table which can be found in [23], we summarize the solutions of locally homogeneous[3] compact three-manifolds, M^3 which have quasi-Einstein metrics.

[3] See Definition 3.1.4

Manifold	$N > n$ $A > 0$	$N > n$ $A = 0$	$N > n$ $A < 0$	$N < n$ $A > 0$	$N < n$ $A = 0$	$N < n$ $A < 0$
$\mathbb{R}^3$	$\emptyset$	Trivial	$\emptyset$	$\emptyset$	Trivial	$\emptyset$
$SU(2)$	Exists	Exists	Exists	Exists	$\emptyset$	$\emptyset$
$\widetilde{SL_2(\mathbb{R})}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	Exists	$\emptyset$
Nil	$\emptyset$	$\emptyset$	Exists	$\emptyset$	$\emptyset$	$\emptyset$
$E(1,1)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$E(2)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$H^2 \times \mathbb{R}$	$\emptyset$	$\emptyset$	Exists	$\emptyset$	$\emptyset$	$\emptyset$
$S^2 \times \mathbb{R}$	$\emptyset$	$\emptyset$	$\emptyset$	Exists	$\emptyset$	$\emptyset$
H^3	$\emptyset$	$\emptyset$	Trivial	$\emptyset$	$\emptyset$	Trivial

Table 9.1:
Exists: Nontrivial solutions exist to $\mathrm{Ric}_X^N = Ag$

Trivial: The only compact solution to $\mathrm{Ric}_X^N = Ag$ is when $X = 0$

$\emptyset$: No compact solutions to $\mathrm{Ric}_X^N = Ag$ on $M^3 = \widetilde{M^3}/\Gamma$

9.4 What results and proofs do we need along the way to answering our main result?

According to Singer in [38], for every locally homogeneous[4] geometry (M^3, g), the universal cover, $(\widetilde{M}^3, \widetilde{g})$, is homogeneous. If $(\widetilde{M}^3, \widetilde{g})$ is a homogeneous,

[4] See Definition 3.1.4

simply connected manifold that admits a compact quotient, then it is one of the following: $\mathbb{R}^3$, $SU(2)$, $\widetilde{SL_2(\mathbb{R})}$, Nil, $E(1,1)$, $E(2)$, H^3, $S^2 \times \mathbb{R}$, or $H^2 \times \mathbb{R}$ [13, Table 1].

Since $\widetilde{X}$ is a left-invariant solution to $\mathrm{Ric}^m_{\widetilde{X}} = A\widetilde{g}$ if and only if $d\pi(\widetilde{X})$ is a solution to $\mathrm{Ric}^N_X = Ag$, where $\pi : \widetilde{M} \to M$ is the universal covering map, we study these nine geometries in order to classify N-quasi Einstein metrics on locally homogeneous three manifolds. Of the nine geometries, $\mathbb{R}^3$, $SU(2)$, $\widetilde{SL_2(\mathbb{R})}$, Nil, $E(1,1)$, and $E(2)$ are Lie groups. We can also use that H^2 is a Lie group to study $H^2 \times \mathbb{R}$. We will explicitly calculate the metrics on the Lie groups which satisfy $\mathrm{Ric}^N_X = Ag$ using the methods of Section . We will study the equation $\frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^* = \lambda g$ in order to calculate the N-quasi Einstein metrics on $S^2 \times \mathbb{R}$ and H^3.

Throughout this paper, we will use the following computations by Milnor:

Lemma 9.4.1. *[28, pages 305, 307] Let G be a 3-dimensional unimodular Lie group with left invariant metric. If L is self-adjoint, then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ consisting of eigenvectors $LX_i = \lambda_i^* X_i$. We obtain the following:*

$$[X_2, X_3] = \lambda_1^* X_1$$

$$[X_3, X_1] = \lambda_2^* X_2$$

$$[X_1, X_2] = \lambda_3^* X_3.$$

The following chart gives us the signs of λ_i^* for $SU(2)$, $\widetilde{SL_2(\mathbb{R})}$, $E(2)$, $E(1,1)$, *Nil, and* $\mathbb{R}^3$.

Lie Group	λ_1^*	λ_2^*	λ_3^*
Nil	$\lambda_1^* > 0$	$\lambda_2^* = 0$	$\lambda_3^* = 0$
$\widetilde{SL_2(\mathbb{R})}$	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* < 0$
$E(1,1)$	$\lambda_1^* > 0$	$\lambda_2^* < 0$	$\lambda_3^* = 0$
$E(2)$	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* = 0$
$\mathbb{R}^3$	$\lambda_1^* = 0$	$\lambda_2^* = 0$	$\lambda_3^* = 0$
$SU(2)$	$\lambda_1^* > 0$	$\lambda_2^* > 0$	$\lambda_3^* > 0$

Table 9.2: 3-dimensional Lie groups with the signs of their eigenvalues

From now on, let $\lambda_i = |\lambda_i^*|$.

Because we will be using that X is Killing for unimodular Lie groups with $\text{Ric}_X^N = Ag$, it will be useful to calculate $\mathcal{L}_X g$.

Proposition 9.4.2. *[23] Let* $X = a_1X_1 + a_2X_2 + a_3X_3$ *be left-invariant vector field on a 3-dimensional unimodular Lie group with left invariant metric. Then using the same notation as in Lemma 9.4.1, we have the following:*

$$\mathcal{L}_X g(X_i, X_i) = 0 \text{ for all } i$$

$$\mathcal{L}_X g(X_1, X_2) = -a_3\lambda_2^* + a_3\lambda_1^*$$

$$\mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1^* + a_2\lambda_3^*$$

$$\mathcal{L}_X g(X_2, X_3) = -a_1\lambda_3^* + a_1\lambda_2^*$$

Proof. We have the following computation for $\mathcal{L}_X g$:

$\mathcal{L}_X g(X_i, X_j)$

$$= g(\nabla_{X_i}(a_1X_1 + a_2X_2 + a_3X_3), X_j) + g(\nabla_{X_j}(a_1X_1 + a_2X_2 + a_3X_3), X_i)$$

$$= \sum_k a_k g(\nabla_{X_i} X_k, X_j) + a_k g(\nabla_{X_j} X_k, X_i)$$

$$= \sum_k g(\nabla_{X_k} X_i + [X_i, X_k], X_j) + g(\nabla_{X_k} X_j + [X_j, X_k], X_i)$$

$$= \sum_k a_k g([X_i, X_k], X_j) + a_k g([X_j, X_k], X_i) + DX_k g(X_i, X_j)$$

$$= \sum_k a_k g([X_i, X_k], X_j) + a_k g([X_j, X_k], X_i).$$

Then, using Lemma 9.4.1, we get:

$$\mathcal{L}_X g(X_i, X_i) = 0 \text{ for all } i$$

$$\mathcal{L}_X g(X_1, X_2) = -a_3\lambda_2^* + a_3\lambda_1^*$$

$$\mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1^* + a_2\lambda_3^*$$

$$\mathcal{L}_X g(X_2, X_3) = -a_1\lambda_3^* + a_1\lambda_2^*$$

□

Finally, we recall the definition of the Ricci quadratic form, $r(x)$, as introduced by Milnor in [28], and the signatures of the Ricci forms of Nil, $E(1,1)$, $\widetilde{SL_2(\mathbb{R})}$, $E(2)$, $\mathbb{R}^3$, and $SU(2)$ when the metric is left invariant.

Definition 9.4.3. *The Ricci quadratic form, $r(X)$ takes vectors $X \in TM$ to $\mathbb{R}$ and is defined as follows:*

$$g(r(X), Y) = \text{Ric}(X, Y)$$

for all $Y \in TM$.

The collection of signs of $r(e_i)$, namely, $\{sign(r(e_i))\}_{i=1}^n$, is called the signature of the quadratic form r, where $\{e_i\}_{i=1}^n$ is any orthonormal basis for the tangent space.

Lie Group	$r(e_1)$	$r(e_2)$	$r(e_3)$	Reference
Nil	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[28, Corollary 4.6]
$E(1,1)$, $\widetilde{SL_2(\mathbb{R})}$	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	
	$r(e_1) = 0$	$r(e_2) = 0$	$r(e_3) < 0$	[28, Corollary 4.7]
$E(2)$	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[28, Corollary 4.8]
$\mathbb{R}^3$	$r(e_1) = 0$	$r(e_2) = 0$	$r(e_3) < 0$	
$SU(2)$	$r(e_1) > 0$	$r(e_2) > 0$	$r(e_3) > 0$	
	$r(e_1) > 0$	$r(e_2) = 0$	$r(e_3) = 0$	
	$r(e_1) > 0$	$r(e_2) < 0$	$r(e_3) < 0$	[28, Corollary 4.5]

9.5 *N-quasi Einstein Solutions for Nil, $\widetilde{SL_2\mathbb{R}}$, $E(1,1)$, $E(2)$ and $H^2 \times \mathbb{R}$*

In this section, we will compute solutions to the N-quasi Einstein equation for the Lie groups Nil, $\widetilde{SL_2(\mathbb{R})}$, $E(1,1)$, and $E(2)$. We will also compute solutions to $H^2 \times \mathbb{R}$, using the Lie group structure of H^2.

We will use Tables 9.2 and 9.4 as well as the next remark to find examples of X which gives us $\mathrm{Ric}_X^N = Ag$ for $N > n$ and $A < 0$ for the space Nil.

Remark 9.5.1. *By [28, Corollary 4.5], for any left invariant metric on Nil, the principal Ricci curvatures satisfy $|r(e_1)| = |r(e_2)| = |r(e_3)| = |\rho|$.*

Proposition 9.5.2. *[23] Consider Nil with $\mathrm{Ric}_X^N = Ag$. If g is a left-invariant* *metric and if X is a left-invariant vector field, then there exist examples of X such that $\mathrm{Ric}_X^N = Ag$ if and only if $A < 0$ and $N > n$.*

Proof. Let $\{X_1, X_2, X_3\}$ be an orthonormal basis where $\mathrm{Ric}(X_1, X_1) = \rho$, $\mathrm{Ric}(X_2, X_2) = -\rho$, and $\mathrm{Ric}(X_3, X_3) = -\rho$ as in Table 9.4 and Remark 9.5.1. Let $X = a_1X_1 + a_2X_2 + a_3X_3$ where a_1, a_2, and a_3 are all constants. By Corollary 7.3.4, X is a Killing field so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3\lambda_1 = 0$$

$$\mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1 = 0$$

where every other combination of $\mathcal{L}_X g(X_i, X_j)$ is zero by definition of Nil. Thus, $a_2 = a_3 = 0$. We compute Ric_X^N as follows:

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho - \frac{1}{N-n}a_1^2$$

$$\mathrm{Ric}_X^N(X_2, X_2) = -\rho - \frac{1}{N-n}a_2^2 = -\rho$$

$$\mathrm{Ric}_X^N(X_3, X_3) = -\rho - \frac{1}{N-n}a_3^2 = -\rho$$

Thus, $\mathrm{Ric}_X^N = Ag$ if and only if $X = \pm\sqrt{2m\rho}X_1$. In this case, $N > n$ and $A = -\rho < 0$.

□

Now, we will find examples of X which satisfy $\mathrm{Ric}_X^N = Ag$ for the spaces

$E(1,1)$ and $\widetilde{SL_2(\mathbb{R})}$.

Proposition 9.5.3. *[23] Consider $\widetilde{SL_2(\mathbb{R})}$. If g is a left-invariant metric and if X is a left-invariant vector field, then there exist examples of $\mathrm{Ric}_X^N = Ag$ if and only if $N > n$ and $A = 0$.*

Proof. Let g is a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 7.3.4, X must be a Killing field if $\mathrm{Ric}_X^N = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_1 - \lambda_2) = 0$$

$$\mathcal{L}_X g(X_1, X_3) = a_2(-\lambda_1 - \lambda_3) = 0$$

$$\mathcal{L}_X g(X_2, X_3) = a_1(\lambda_2 + \lambda_3) = 0$$

where all other pairs of $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of $\widetilde{SL_2(\mathbb{R})}$. By the above, we must have $a_1 = a_2 = 0$ and either $a_3 = 0$ or $\lambda_1 = \lambda_2$.

By Table 9.4, the signature for the Ricci form is $(+, -, -)$ or $(0, 0, -)$.

If the Ricci form is $(+, -, -)$, let $|\mathrm{Ric}(X_i, X_i)| = \rho_i$. Then, plugging in (X_i, X_j), where $i, j = 1, 2, 3$ into $\mathrm{Ric}_X^N = Ag$, we get the following set of equations:

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n} a_1^2 = \rho_1$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\rho_2 - \frac{1}{N-n} a_2^2 = -\rho_2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n} a_3^2$$

In this case, we cannot have $\mathrm{Ric}_X^N = Ag$ since $\mathrm{Ric}_X^N(X_1, X_1) > 0$ and $\mathrm{Ric}_X^N(X_2, X_2) < 0$.

If the Ricci form is $(0, 0, -)$, then we get the following set of equations:

$$\mathrm{Ric}_X^N(X_1, X_1) = -\frac{1}{N-n} a_1^2 = 0$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\frac{1}{N-n} a_2^2 = 0$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n} a_3^2$$

Then, $\mathrm{Ric}_X^N = Ag$ if and only if $a_3 = \sqrt{-m\rho_3}$, $A = 0$, and $N > n$. □

Proposition 9.5.4. *[23] Consider $E(1,1)$. If g is a left-invariant metric and if X is a left-invariant vector field, then there are no solutions to $\mathrm{Ric}_X^N = Ag$.*

Proof. Let g is a left-invariant metric and let X be a left-invariant vector field, where $X = a_1 X_1 + a_2 X_2 + a_3 X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 7.3.4, X must be a Killing field if $\mathrm{Ric}_X^N = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_2 + \lambda_1) = 0$$

$$\mathcal{L}_X g(X_1, X_3) = -a_1\lambda_2 = 0$$

$$\mathcal{L}_X g(X_2, X_3) = -a_2\lambda_1 = 0$$

All other $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of $E(1,1)$. By the three equations above, $a_1 = a_2 = a_3 = 0$. By Table 9.4 , the signature for the Ricci form is $(+,-,-)$ or $(0,0,-)$. If the Ricci form is $(+,-,-)$, let $|\operatorname{Ric}(X_i, X_i)| = \rho_i$. Then, plugging in all iterations of (X_i, X_j), $i, j = 1,2,3$, we get the following:

$$\operatorname{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n}a_1^2 = \rho_1$$
$$\operatorname{Ric}_X^N(X_2, X_2) = -\rho_2 - \frac{1}{N-n}a_2^2 = -\rho_2$$
$$\operatorname{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n}a_3^2 = -\rho_3$$

Ric_X^N cannot equal Ag since $\operatorname{Ric}_X^N(X_1, X_1) > 0$ and $\operatorname{Ric}_X^N(X_2, X_2) < 0$.

If the Ricci form is $(0,0,-)$, then we get the following set of equations:

$$\operatorname{Ric}_X^N(X_1, X_1) = -\frac{1}{N-n}a_1^2 = 0$$
$$\operatorname{Ric}_X^N(X_2, X_2) = -\frac{1}{N-n}a_2^2 = 0$$
$$\operatorname{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n}a_3^2$$

In this case, we cannot have $\mathrm{Ric}_X^N = Ag$ since $\mathrm{Ric}_X^N(X_1, X_1) = \mathrm{Ric}_X^N(X_2, X_2) = 0$ and $\mathrm{Ric}_X^N(X_3, X_3) < 0$. □

Finally, we will find that there are no examples of X on $E(2)$ which give us $\mathrm{Ric}_X^N = Ag$.

Proposition 9.5.5. *[23] Consider $E(2)$. If g is a left-invariant metric and if X is a left-invariant vector field, then there are no solutions to $\mathrm{Ric}_X^N = Ag$.*

Proof. Let g is a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis. By Corollary 7.3.4, X must be a Killing field if $\mathrm{Ric}_X^N = Ag$, so we set $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ as follows:

$$\mathcal{L}_X g(X_1, X_2) = a_3(\lambda_1 - \lambda_2) = 0$$

$$\mathcal{L}_X g(X_1, X_3) = -a_2\lambda_1 = 0$$

$$\mathcal{L}_X g(X_2, X_3) = a_1\lambda_2 = 0$$

All other $\mathcal{L}_X g(X_i, X_j) = 0$ by properties of $E(2)$. By the three equations above, $a_1 = a_2 = 0$ and either $\lambda_1 = \lambda_2$ or $a_3 = 0$. By Table 9.4 , the signature for the Ricci form is $(+, -, -)$. Letting $|\mathrm{Ric}(X_i, X_i)| = \rho_i$, we plug in all iterations of (X_i, X_j), $i, j = 1, 2, 3$ as follows:

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n}a_1^2 = \rho_1$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\rho_2 - \frac{1}{N-n}a_2^2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n}a_3^2$$

Ric_X^N cannot equal Ag since $\mathrm{Ric}_X^N(X_1, X_1) > 0$ and $\mathrm{Ric}_X^N(X_2, X_2) < 0$. □

Proposition 9.5.6. *[23] Consider $\mathbb{R}^3$. If g is a left-invariant metric and if X is a left-invariant vector field, then the only solutions of $\mathrm{Ric}_X^N = Ag$ occur when $N \neq n$, $A = 0$, and $X = 0$.*

Proof. Let g is a left-invariant metric and let X be a left-invariant vector field, where $X = a_1X_1 + a_2X_2 + a_3X_3$ with $\{X_1, X_2, X_3\}$ an orthonormal basis of left-invariant vector fields. By Corollary 7.3.4, X must be a Killing field if $\mathrm{Ric}_X^N = Ag$. By [28, page 307], $\mathcal{L}_X g(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$ and $\mathrm{Ric}(X_i, X_j) = 0$ for all $i, j = 1, 2, 3$, so we have the following sets of equations for $\mathrm{Ric}_X^N(X_i, X_j)$.

$$\mathrm{Ric}_X^N(X_1, X_1) = -\frac{1}{N-n}a_1^2$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\frac{1}{N-n}a_2^2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\frac{1}{N-n}a_3^2$$

Setting $\mathrm{Ric}_X^N = Ag$, the only solutions are when $N \neq n$, $A = 0$, and $X = 0$. □

Remark 9.5.7. *Since $\mathbb{R}^3$ is Ricci flat, Proposition 9.5.6 also follows from Proposition 8.4.8.*

Proposition 9.5.8. *[23] If g is a left-invariant metric on $H^2 \times \mathbb{R}$ and if X is a left-invariant vector field then there exist solutions to $\mathrm{Ric}_X^N = Ag$ if and only if $A < 0$ and $N > n$.*

Proof. Let $\{X_1, X_2, \frac{\partial}{\partial r}\}$ be an orthonormal basis where $\{X_1, X_2\}$ are in TH^2 and $\frac{\partial}{\partial r}$ is in $T\mathbb{R}$. Let $X = a_1 X_1 + a_2 X_2 + a_3 \frac{\partial}{\partial r}$. We compute the Lie derivatives as follows:

$$\mathcal{L}_X g(X_1, X_1) = 2g(\nabla_{X_1} X, X_1) = 2g(-a_2 X_2, X_1) = 0$$

$$\mathcal{L}_X g(X_2, X_2) = 2g(\nabla_{X_2} X, X_2) = 2g(-a_1 X_2 + a_2 X_1, X_2) = -2a_1$$

$$\mathcal{L}_X g(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = 0$$

$$\mathcal{L}_X g(X_1, X_2) = g(\nabla_{X_1} X, X_1) + g(\nabla_{X_1} X, X_1) = g(-a_1 X_2 + a_2 X_1, X_1) = a_2$$

$$\mathcal{L}_X g(X_2, \tfrac{\partial}{\partial r}) = g(\nabla_{X_2} X, \tfrac{\partial}{\partial r}) + g(\nabla_{\frac{\partial}{\partial r}} X, X_2) = 0$$

By Corollary 7.3.4, X must be a Killing field, so we set $\mathcal{L}_X g = 0$ to get that $a_1 = a_2 = 0$. We have that $\mathrm{Ric}(X_1, X_1) = \mathrm{Ric}(X_2, X_2) = -\rho g$ where $\rho > 0$, and $\mathrm{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$, so we can compute Ric_X^N as follows:

$$\text{Ric}_X^N(X_1, X_1) = -\rho$$

$$\text{Ric}_X^N(X_2, X_2) = -\rho$$

$$\text{Ric}_X^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -\frac{1}{N-n}a_3^2$$

Thus, $\text{Ric}_X^N = Ag$ if and only if $X = \pm\sqrt{\rho(N-n)}\frac{\partial}{\partial r}$, where $A = -\rho < 0$ and $N > n$.

□

We will show that we can find examples of X such that $\text{Ric}_X^N = 0$ on $SU(2)$ with left-invariant metric.

Proposition 9.5.9. *[23] Consider $SU(2)$. If g is a left-invariant metric and if X is a left-invariant vector field, then there exist solutions to $\text{Ric}_X^N = Ag$ if and only if either $N > n$ with A any real number or $N > n$ with $A > 0$.*

Proof. Let $X = a_1X_1 + a_2X_2 + a_3X_3$. By Lemma 7.3.5, at least two a_i's must be zero. By Corollary 7.3.4, X is a Killing field, so we compute $\mathcal{L}_X g$ using Proposition 9.4.2 as follows:

$$\begin{aligned}\mathcal{L}_X g(X_1, X_2) &= a_3(\lambda_1 - \lambda_2)\\ \mathcal{L}_X g(X_2, X_3) &= a_1(\lambda_2 - \lambda_3)\\ \mathcal{L}_X g(X_1, X_3) &= a_2(\lambda_3 - \lambda_1).\end{aligned} \tag{9.1}$$

By Table 9.4, the Ricci form is either $(+,+,+)$, $(+,0,0)$, or $(+,-,-)$. Let

$|\,\mathrm{Ric}(X_i, X_i)) = \rho_i$ for $i = 1, 2, 3$. If the Ricci form is $(+,+,+)$, then we have the following computations for Ric_X^N:

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n}a_1^2$$
$$\mathrm{Ric}_X^N(X_2, X_2) = \rho_2 - \frac{1}{N-n}a_2^2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = \rho_3 - \frac{1}{N-n}a_3^2$$

Setting $\mathrm{Ric}_X^N = Ag$, if all three a_i's are zero, then $X = 0$ and $\mathrm{Ric}_X^N = \rho g$ where $\rho = \rho_1 = \rho_2 = \rho_3$.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, and $\rho = \rho_1 = \rho_2$, then

$$X = \pm\sqrt{m(\rho_3 - \rho)}X_3.$$

Similarly, if $a_1 = a_3 = 0$, and $\rho = \rho_1 = \rho_3$, then

$$X = \pm\sqrt{m(\rho_2 - \rho)}X_2.$$

If $a_2 = a_3 = 0$, and $\rho = \rho_2 = \rho_3$, then

$$X = \pm\sqrt{m(\rho_1 - \rho)}X_1.$$

In these cases, $\mathrm{Ric}_X^N = \rho g$, where $\rho > 0$, and m can be positive or negative, depending on the sign of $\rho_3 - \rho$, $\rho_2 - \rho$, and $\rho_1 - \rho$, respectively.

If the Ricci form is $(+, 0, 0)$, then:

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n}a_1^2$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\frac{1}{N-n}a_2^2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\frac{1}{N-n}a_3^2$$

The solutions to the above equations are $X = \pm\sqrt{\rho(N-n)}X_1$ and $\mathrm{Ric}_X^N = 0$. In this case, m must be positive.

If the Ricci form is $(+, -, -)$, then

$$\mathrm{Ric}_X^N(X_1, X_1) = \rho_1 - \frac{1}{N-n}a_1^2$$
$$\mathrm{Ric}_X^N(X_2, X_2) = -\rho_2 - \frac{1}{N-n}a_2^2$$
$$\mathrm{Ric}_X^N(X_3, X_3) = -\rho_3 - \frac{1}{N-n}a_3^2$$

Setting $\mathrm{Ric}_X^N = Ag$, the solutions are $X = \pm\sqrt{m(\rho+\rho_1)}X_1$, where $\rho = \rho_2 = \rho_3$. In this case, $\mathrm{Ric}_X^N = -\rho g$ and m must be positive.

□

9.6 *How does our result relate to the Splitting Theorem, Myers' Theorem and Bochner's Theorem?*

According to Khuri-Woolgar-Wylie, the Splitting Theorem holds for Ric_X^N if $N > n$ [17, Theorem 2]. We also recall that if (M, g) is a noncompact homogenous space, then it contains a line. Using the Ric_X^N version of the Splitting Theorem and the fact about noncompact homogeneous spaces, we will show that of the 9 geometries which are 3-dimensional and homogeneous, the ones which don't split don't have solutions if $N > n$ and $A \geq 0$.

Proposition 9.6.1. *[23] H^3, $\widetilde{SL_2\mathbb{R}}$, Nil,$E(2)$, $H^2 \times \mathbb{R}$, and $E(1,1)$ do not admit metrics such that $\mathrm{Ric}_X^N = Ag$ for $N > n$ and $A \geq 0$.*

Proof. H^3, $\widetilde{SL_2\mathbb{R}}$, Nil,$E(2)$, and $E(1,1)$ all admit lines and don't split as $N \times \mathbb{R}$. Thus, the proposition follows by the Bakry Émery Ricci version of the Splitting Theorem by Khuri-Woolgar-Wylie.

In the case of $H^2 \times \mathbb{R}$, by the Splitting Theorem, $\mathrm{Ric}_X^N \geq 0$ with $N > n$ if and only if $\mathrm{Ric}_X^N \geq 0$ with $N > n$ on H^2. H^2 admits lines and doesn't split as $N \times \mathbb{R}$, so the proposition follows. □

In [36, Theorem 5], Qian proves that Myers' Theorem holds for gradient m-Bakry-Émery Ricci curvature when $N > n$. Limoncu showed in [25, Theorem 1.2] that Myers' Theorem holds for non-gradient m-Bakry-Émery Ricci curvature when $N > n$. In [15] Khuri-Woolgar use Limoncu's version of Myers' Theorem to study Near Horizon Geometries. Using this version of Myers' Theorem, we see that since $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ are both noncompact, $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ do not admit metrics such that $\mathrm{Ric}_X^N = Ag$ for $N > n$ and $A > 0$. In fact, since $SU(2)$ is the only compact simply-connected three-dimensional geometry, it is the only one that can admit a metric such that $\mathrm{Ric}_X^N = Ag$ for $N > n$ and $A > 0$.

Next, we will discuss the $N > n$, $A < 0$ case of the N-quasi Einstein metric. Bochner proved that if (M, g) is compact, oriented and if $\mathrm{Ric} < 0$, then there are no nontrivial Killing fields (See [33, Theorem 36]). This leads us to the next proposition.

Proposition 9.6.2. *[23] If M^n is a compact locally homogeneous[5] Riemannian, and if M^n is a compact quotient of a Lie group, G, then there are no solutions to $\mathrm{Ric}_X^N = Ag$ if $N > n$ and $A < 0$.*

[5] See Definition 3.1.4

Proof. By Lemma 7.3.3, $\widetilde{X}$ is Killing on G. Then, $\mathrm{Ric} = A\widetilde{g} + \frac{1}{N-n}\widetilde{X}^* \otimes \widetilde{X}^*$ which is negative, giving us a contradiction by Bochner's Theorem. □

Corollary 9.6.3. *[23] If M^3 is a compact locally homogeneous[6] Riemannian manifold which satisfies $\mathrm{Ric}_X^N = Ag$ with $N > n$ and $A < 0$, then M^3 cannot be a compact quotient of $\mathbb{R}^3$, $SU(2)$, $\widetilde{SL_2(\mathbb{R})}$, Nil, $E(1,1)$, $H^2 \times \mathbb{R}$, or $E(2)$.*

[6] See Definition 3.1.4

Proposition 9.6.4. *[23] On H^3, $\mathrm{Ric} = -\rho g$ where $\rho > 0$. $\mathrm{Ric}_X^N = Ag$ if and only if $A + \rho = 0$ and $X = 0$.*

Proof. If $(A + \rho)m > 0$, then by Proposition 8.4.3, there are no solutions. If $(A + \rho)m < 0$, then by Proposition 8.4.8, there are no solutions. If $A + \rho = 0$, then by Proposition 8.4.3, $X = 0$. □

Corollary 9.6.5. *[23] There are no solutions to $\mathrm{Ric}_X^N = Ag$ with $A > 0$ on a compact hyperbolic manifold.*

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Alice Wu Lim

Syracuse University Department of Mathematics
Email Address: awlim100@syr.edu
Website: http://awlim100.expressions.syr.edu

Education

Ph.D. in Mathematics, Syracuse University *To be Completed May 2021*
Thesis Advisor: William Wylie
Field of Study: Riemannian Geometry

MS in Mathematics, Syracuse University *Spring 2018*

BS in Mathematics, UCLA *Spring 2015*

Papers

A. Lim, *The Splitting Theorem and Topology of Noncompact Spaces with Nonnegative N-Bakry Émery Ricci Curvature*, preprint: https://arxiv.org/abs/2001.06028v3 (Accepted for publication in *Proceedings of the AMS*, June 2020, DOI: https://doi.org/10.1090/proc/15240).

A. Lim, *Locally Homogeneous Non-gradient Quasi-Einstein 3-Manifolds*, preprint: https://arxiv.org/abs/2009.00720 (Accepted for publication in *Advances in Geometry*, December 2020).

Research Talks

University of Oregon Topology/Geometry Zoom Seminar *October 7, 2020*
Invited Seminar Talk

2020 Virtual Workshop on Ricci and Scalar Curvature *August 21, 2020*
Invited Workshop Talk
https://awlim100.expressions.syr.edu/vwrs-talk/

Dartmouth Geometry Seminar *July 2, 2020*
Invited Seminar Talk

Syracuse University Geometry and Topology Seminar *June 26, 2020*
Seminar Talk

2019 Union College Conference *September 14, 2019*
Conference Talk

2019 Lehigh University Geometry and Topology Conference *June 21, 2019*
Conference Talk

Research Seminar for the Women and Mathematics Program at the Institute for Advanced Studies *May 21, 2019*
Invited Conference Talk

Research Poster Presentations

JMM 2021 AWM Poster Session *To Be Given January 8, 2021*
Invited Poster Presentation at Conference

2020 Mathematics Graduate Organization at Syracuse University *March 28, 2020*
Poster Presentation

Expository Talks

Introduction to Morse Theory *October 25, 2019*
Syracuse University Math Graduate Organization Colloquium

Quasi-Einstein Metrics and How They Relate to Black Holes *November 2018*
Graduate Seminar in Geometry and Topology at Syracuse University

Other Graduate Math Activities

Volunteer Remote Tutor (in Response to COVID-19) *Fall 2020 to present*

- Tutored multiple 4th and 5th grade students in math
- Tutored high school student in statistics with Syracuse University and Salvation Army
- Mentored student in scholarship applications

Committee Member for Career Panel, Association for Women in Mathematics, SU student chapter *Fall 2020*

- Helped the 2020-2021 Association for Women in Mathematics SU student chapter board members plan and execute a career panel consisting of: a postdoctoral researcher in industry, two postdoctoral researchers in academia, a senior program manager in outreach, an assistant professor, and a research mathematician in government.

Research Assistant *Summer 2020, Summer 2019, Summer 2018*

- Research Assistant under William Wylie's NSF grant DMS-1654034

President of Association for Women in Mathematics, SU student chapter *Spring 2019 to Spring 2020*

- Organized events related to women and minorities in math, including:
  - lunches with female seminar speakers
  - career panel with female panelists postdocs, government researcher, outreach, etc (postponed)
  - monthly social events

Vice President of Association for Women in Mathematics, SU student chapter *Spring 2018 to Spring 2019*

- Co-founded the Association for Women in Mathematics, Syracuse University student chapter
- Helped the President plan and execute social events 2-3 times a semester

Directed Reading Program Mentor, Mathematics Department *Fall 2018*

- Read "Elementary Differential Geometry" by Pressley and "The Four Vertex Theorem and its Converse" by DeTurck, Gluck, Pomerleano, and Vick with undergraduate student.
- Worked together with student to find applications to the four vertex theorem in classical differential geometry and some basics in general relativity
- Helped student write an abstract and a 20-minute talk which they gave at the Math Graduate Organization (MGO) Colloquium
- Mentored undergraduate student regarding graduate applications

Women in Science and Engineering Future Professionals Program *Fall 2018 to present*

- Member of Women in Science and Engineering
- Attended seminars related to women and minorities in science and engineering

Future Professoriate Program *Fall 2018 to Spring 2020*

- The Future Professoriate Program is a structured professional development experience for aspiring faculty

Teaching, Syracuse University

Instructor of Record *Fall 2016 to present*

- Taught 2-3 lectures per week and up to one recitation per week
- Assigned homework, classwork, and created and graded 3 to 4 exams per semester

Math 286	Calculus II for Life Sciences	Spring 2020 (Partially online)
Math 285	Calculus I for Life Sciences	Spring 2019, Fall 2019, Fall 2020 (Online, 2 courses)
Math 295	Calculus I	Spring 2017, Spring 2018, Fall 2018
Math 194	Precalculus	Fall 2016, Fall 2017

Teaching Assistant *Fall 2015 to Spring 2016*

- Led four recitations a week
- Wrote quizzes, graded half of exams and quizzes

Math 295	Calculus I	Spring 2016
Math 194	Precalculus	Fall 2015

Math Clinic *Fall 2016 to Spring 2017*

- Helped undergraduate students in courses including Calculus, Linear Algebra, Differential Equations up to two hours a week

Conferences Attended

Virtual Seminar on Geometry with Symmetries *April 2020 to present*
Online

Union College Mathematics Conference 2019 *September 14, 2019*
Union College

2019 Lehigh University Geometry and Topology Conference *June 20, 2019*
Lehigh University

Master Class in Differential Geometry:
The Structure of Limit Spaces *May 27, 2019*
Henri Poincaré Institute

2019 Program for Women and Mathematics:
Topics in Geometric Analysis *May 27, 2019*
IAS and Princeton