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Ambient Obstruction Solitons And Homogeneous Gradient Bach Solitons

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Abstract

Differential geometry is a diverse field which applies principles from calculus to a more general set of objects. Endowing a smooth manifold with a Riemannian metric allows us to measure length and angle in a way such that length is positive. This enables us to examine measures of curvature on a manifold. The study of manifolds with such metrics is called Riemannian geometry. Using geometric flows associated with tensors, we are able to analyze the relationship between metrics and curvature. Examining solitons, specifically gradient solitons, is one way we investigate this relationship.

This thesis focuses on the geometric flows associated with the Bach tensor and the ambient obstruction tensor. The Bach tensor is realized as the gradient of the Weyl energy functional. Consequently, the minimizers of the Weyl energy are the metrics where the Bach tensor vanishes. There are a number of metrics that are widely considered interesting that are known to be Bach flat. Studying the Bach flow and broadening our understanding of Bach flat metrics could produce other such metrics. At the crux of our investigation is the fact that the Bach tensor is divergence-free (in dimension 4) and trace-free. To generalize this to higher dimensions and maintain these properties, we consider the ambient obstruction tensor, $\mathcal{O}$. For $n = 4$ the ambient obstruction tensor is the Bach tensor.

In this thesis we begin a new program of studying ambient obstruction solitons and homogeneous gradient Bach solitons. Examining higher dimensions, we establish a number of results for solitons to the geometric flow for a general tensor q and apply these result to the ambient obstruction flow. This method enables us to prove that any compact ambient obstruction soliton with constant scalar curvature is trivial. For $n = 4$, we show that any homogeneous gradient Bach soliton that is steady must be Bach flat, and that the only non-Bach-flat, shrinking gradient solitons are product metrics on $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^2 \times H^2$. Moreover, we construct a non-Bach-flat expanding homogeneous gradient Bach soliton.

Ambient Obstruction Solitons and Homogeneous Gradient Bach Solitons

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Introduction

Research in Riemannian geometry aims to answer the question "what is the best metric?" with the hope that the answer to this question will provide valuable insights into the universe we inhabit. The word "best" takes on different meanings in different contexts, leading to a variety of approaches.

Motivating Riemannian Geometry

Before discussing this notion of a "best" metric, we examine the intuition behind some of the basic objects and tools of Riemannian geometry.

A Riemannian manifold, (M, g), is a smooth (C^∞) manifold M paired with a Riemannian metric, g. Like the dot product we learn in calculus, Riemannian metrics are positive definite inner products defined on the tangent space, T_pM: the space spanned by all tangent vectors at a point p on M. This metric provides a way to measure length and angle on the manifold.

The idea of congruence inspires our investigation of the origins of curvature. For a more thorough discussion, refer to [Lee18]. We know that for polygons, combinations of angles and lengths are sufficient means to guarantee congruence. However, when examining curves we quickly see that we need a systematic way to consider and quantify the "curviness" of a curve. The notion of concavity that we learned in calculus seems to do this, so we will use it as a foundation to build up a notion of curvature.

Broadly speaking, we use the concavity of a curve to identify a best fit circle. The best fit circle of radius R at a point p is called the osculating circle. The curvature at that point is the same as the curvature of the osculating circle:

$$\kappa = \frac{1}{R}.$$

This definition is actually quite intuitive. Briefly consider the case where our curve is a circle of radius R. Since the congruence of two circles depends only on their radii, their curvatures should depend only on their radii. Furthermore, our intuition says that the smaller a circle is the "curvier" it is (and vice versa), so it makes sense that the curvature should have an inverse relationship with the radius.

The last thing we need to do is distinguish between curving up and down. To do so we need to choose a normal direction and assign a sign based on whether our curve is curving towards or away from that direction. This is signed curvature. For example, consider a curve with normal direction defined to be in the direction of the positive y-axis. Then our curve is curving towards the normal direction and has, say, positive curvature when it is concave up. It is curving away from the normal direction and has, correspondingly, negative curvature when it is concave down.

To examine the curvature of a surface, M, at a point, p, we look at the signed curvature of the curve formed by intersecting a plane, Π, with M at p. Rotating Π produces infinitely many curves and consequently infinitely many signed curvatures. The largest and smallest signed curvatures are the principle curvatures κ_1 and κ_2, respectively. Using the principal curvatures we can calculate the Gaussian curvature:

$$K = \kappa_1 \kappa_2.$$

The Gaussian curvature, though seemingly simple, plays a huge role in our understanding of Riemannian 2-manifolds.

Theorem (Theorema Egregium, Gauss)**.** *The Gaussian curvature is intrinsic to a surface. That is, K is preserved by isometries.*

This theorem is hugely influential in mathematics and in our daily lives, informing things even as mundane as how we eat pizza.

Metrics and curvature provide a clear distinction between geometry and topology. Where topology is focused on examining a manifold regardless of its shape, geometry is focused on determining that shape. The Gauss-Bonnet Theorem shows that though these two subjects are different, they are necessarily linked.

Theorem (Gauss-Bonnet Theorem)**.** *Consider a compact Riemannian 2-manifold, (M, g). Then*

$$\int_M K \, dA = 2\pi\chi(M),$$

where K is the Gaussian curvature of g and $\chi(M)$ is its Euler characteristic.

A manifold's Euler characteristic depends only on its topology (genus). So we have some sort of constraint on Gaussian curvature given by the manifold's topology and some constraint on topology from the Gaussian curvature, ultimately allowing us to classify compact manifolds.

In Section 1.1 we will discuss the tools we use to analyze curvature for higher dimensional manifolds.

Basics of Geometric Flow

Returning to the question of finding the best metric, limiting our scope to Riemannian metrics allows us to use curvature as a tool to help define what "best" might mean. As we point out above, there are many ways to measure a manifold's curvature. We will use curvature to mean these measures in general. The study of geometric flow evolved as a way to use curvature to identify best metrics.

A geometric flow is a differential equation in which the metric is considered as a function of time, $g(t)$, and is changed over time in accordance with the curvature of the manifold. Specifically, we define a geometric flow for a general tensor q (or for a general measure of curvature) as a one parameter family of smooth metrics such that

$$\begin{cases} \partial_t g = q \\ g(0) = h. \end{cases}$$

We call this the q-flow. The geometric flow associated with a tensor enables us to use tools from differential equations to analyze the relationship between metrics and curvature. This shift in perspective allows us to examine the behavior of the flow itself, to better understand how the curvature behaves, and, consequently, to refine the idea of what "best" might mean for a specific measure of curvature.

One of the major ideas from differential equations is locating and classifying fixed points. In the study of geometric flows, this manifests as the examination and classification of solitons. Solitons are solutions to the flow that, over time, change only by diffeomorphism and/or rescaling. A (normalized) soliton of the q-flow (where q is a general-tensor) is a metric that satisfies the equation:

$$\frac{1}{2} \mathcal{L}_X \, g = cg + \frac{1}{2} q,$$

where X is a vector field and $\mathcal{L}_X \, g$ is the Lie derivative of the metric g in the direction of the vector field X. Note, that we have normalized the equation by scaling q by $\frac{1}{2}$. This scaling enables us to show that solitons are in fact solutions to the geometric flow in Theorem 2.1.13.

Studying the solitons of a geometric flow provides insight into the nature of the flow while narrowing down the number of metrics that one is considering. We classify these solitons as expanding, steady, and shrinking when $c < 0$, $c = 0$, and $c > 0$, respectively.

Letting $X = \nabla f$, the resulting solitons are called gradient solitons. Here f is a function called the potential function. Thus, for a general tensor q, we can say a (normalized) gradient q-soliton satisfies:

$$\operatorname{Hess} f = cg + \frac{1}{2}q.$$

Note that we've used the fact that $\mathcal{L}_{\nabla f}\, g = 2 \operatorname{Hess} f$, where Hess is the Hessian (the matrix of second derivatives). This choice of vector field serves to improve our understanding of what we mean by "best" and to get us closer to finding a best metric.

A Quick Note on Homogeneous Manifolds

When beginning the examination of solitons, it is useful to first consider only homogeneous manifolds. As such, we focus on examining gradient solitons on such manifolds.

A Riemannian manifold (M, g) is homogeneous if for each p and q on M there exists an isometry, f, such that $f(p) = q$. Broadly, this means that each point of a manifold "looks like" all of the other points on the manifold. More concretely, they share specific attributes such as curvature. From this we see that all homogeneous manifolds have constant scalar curvature. Classic examples include $\mathbb{R}^n$, S^n, and H^n.

An Overview of our Tensors

We will delve into the following topics more in Chapter 1, but wanted to give the reader a more condensed overview and to show how this work contributes to the overall goals of Riemannian geometry.

One way that something can be "best" in mathematics is that it minimizes a functional. Indeed, we see even the shape of many objects in nature is explained by minimizing functionals. The shape of soap bubbles, for example, minimizes surface area. We know from calculus that to find the minimum of a function we need to examine its derivative.

To understand why the Bach tensor would be a helpful in our search for the best metric, we must begin by considering the Weyl tensor. The Weyl tensor is the conformally invariant component of the Riemannian curvature tensor. One can think of the Weyl tensor as the obstruction to a manifold being locally conformally flat. For $n = 4$, the Bach tensor is the gradient of the Weyl energy functional. The minimizers, then, are where the Bach tensor vanishes. It is known that the Bach tensor vanishes for Einstein metrics and (anti)self-dual metrics. These three types of metric have historically been considered as candidates for a best metric, so studying the Bach flow and broadening our understanding of Bach flat metrics could produce other such metrics. To do so, we investigate homogeneous gradient Bach solitons. That is, we will look at gradient solitons on homogeneous manifolds using the Bach tensor, B, as my measure of curvature.

The Bach tensor itself has properties that are useful in expanding the field of geometric flow. There is an explicit representation of the Bach tensor in arbitrary dimension. Like the Weyl tensor, the Bach tensor is trace-free for arbitrary n. Moreover, in dimension $n = 4$, the Bach tensor is conformally invariant (of weight -2) and divergence-free. Since these properties only hold in this dimension, we limit our examination of the Bach flow to dimension $n = 4$.

Given the utility of the Bach flow, it would be helpful to be able to examine manifolds where $n \neq 4$. However, because the Weyl energy is no longer conformally invariant for $n \neq 4$, the Bach tensor loses many of its properties. We look to changing the functional to get a better higher dimensional generalization. For even dimensions $n \geqslant 4$ this functional is the Q-energy: a similar functional to the Weyl energy that uses the Q-curvature instead of the Weyl tensor. The gradient of the Q-energy is the ambient obstruction tensor, $\mathcal{O}$. Like the Bach tensor, the ambient obstruction tensor is trace-free, divergence-free, and conformally invariant (of weight $2 - n$). In fact, for $n = 4$ the ambient obstruction tensor is the Bach tensor.

Summary of Results

My work begins a new program of studying homogeneous ambient obstruction solitons and homogeneous gradient Bach solitons. In the subsequent sections I will explain the specific aspects of these flows that make them ideal tools in our search for finding the best metric.

Focusing first on dimension 4, I was able to show that any homogeneous gradient Bach soliton that is steady must be Bach flat, and that the only non-Bach-flat, shrinking gradient solitons are product metrics on $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^2 \times H^2$. Moreover, I constructed a non-Bach-flat expanding homogeneous gradient Bach soliton. To extend my work to higher dimensions, I established a number of results for solitons to the geometric flow for a general tensor q. Applying these result to the ambient obstruction flow resulted in proving that any compact ambient obstruction soliton with constant scalar curvature is trivial.

Overview

The dissertation is organized as follows. We begin Chapter 1 with a discussion of the major curvature tensors. Proceeding, we introduce the Weyl, Bach, and ambient obstruction tensors. We conclude with a section detailing the geometric flows we will examine and discussing some of the results from Ricci flow that inspired our search. Next, in Chapter 2 we begin by establishing a number of results for a general tensor q and applying them to the ambient obstruction tensor. Then we move our focus onto the Bach tensor, beginning to classify the gradient Bach tensors of homogeneous 4-manifolds. The results of this partial classification are summarized in Table 2.1.

Chapter 1

Background

1.1 Riemannian Geometry

We begin the background section with a brief review of Riemannian geometry to get the reader acquainted with the conventions used.

Recall from the introduction that a Riemannian manifold, (M, g), is a smooth (C^∞) manifold M paired with a Riemannian metric, g, defined on the tangent space of the manifold, T_pM, at a point, p. For the duration of this thesis, manifolds can be assumed to be Riemannian.

1.1.1 Einstein Notation

For the reader unfamiliar with Einstein notation, we provide a brief explanation of the notation. Einstein notation is a notational shorthand in which we replace a sum with repeated indices, where one is a superscript and the other is a subscript. For example, if we are working over an n-dimensional manifold:

$$a^i q_i = \sum_{i=1}^{n} a^i q_i = a^1 q_1 + a^2 q_2 + \cdots + a^n q_n.$$

This notation serves as a useful shorthand when working with equations expressed in terms of local coordinates. Moreover, raised indices within a tensor can be lowered using elements of our metric:

$$T_i{}^k{}_j = g^{km} T_{imj}.$$

This type of change is particularly useful in Appendix B.

In addition to using this shorthand for summation, mathematicians will also use subscripts of semi-colons or commas to represent derivatives such as:

$$T_{ij;k} = \nabla_k T_{ij} \qquad \text{and} \qquad T_{ij,k} = \nabla_k T_{ij}.$$

It should be notated that this notation is not always convenient to write as proper superscript and subscript pairs. For the sake of this thesis, any repeated indices can be understood as being summed over.

1.1.2 Basics of Curvature

Continuing to develop an intuition behind curvature of curves and surfaces from the introduction, this section will focus on the tools that we use to measure curvature in a more general sense. One of the biggest differences in this discussion will be the use of properties of tensors, such as type changes. For a review of tensors, we refer the reader to [Lee18, Appendix B].

We know intuitively that measuring curvature means we want to see how "non-flat" a manifold is. In order to use this intuition we will rely on the flatness criterion as discussed in [Lee18, Chapter 7], which says a Riemannian manifold is flat if the connection, ∇ satisfies the following condition:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

A manifold is flat and, in particular, satisfies this equation if it is locally isometric to Euclidean space. Consequently, the first measure of curvature we will examine is the Riemannain curvature tensor as a (3,1)-tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Based on the flatness criterion above, it is clear that this tensor measures how much a manifold differs from being flat. Putting this equation in terms of local coordinates:

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}, \tag{1.1}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

is the Christoffel symbol. The Riemannian curvature tensor can also be presented as a $(4,0)$ tensor:

$$Rm(X,Y,Z,W) = g\left(R(X,Y)Z, W \right),$$

with corresponding equation in local coordinates given by:

$$R_{ijkl} = g_{lm} \left(\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp} \right).$$

Taking the trace of the Riemannian curvature tensor we get the symmetric 2-tensor, the Ricci curvature:

$$R_{ij} = R_{kij}{}^{k} = g^{km} R_{kijm}. \tag{1.2}$$

Taking the trace of the Ricci curvature yields the scalar curvature:

$$S = g^{ij} R_{ij}. \tag{1.3}$$

While the Ricci curvature and scalar curvature have a geometric interpretation, the tools we've presented are not sufficient to thoroughly explain this interpretation.

These two measures of curvature have played a major role in modern mathematics. For example, we define what it means for a metric to be an Einstein metric by:

$$\mathrm{Ric} = \lambda g \qquad \lambda \in \mathbb{R}.$$

These metrics are considered interesting for a number of reasons including their connections to physics and their potential uses in higher dimensions. The topic of Einstein metrics for $n = 4$ is something that is often researched. We discuss the impacts of the Ricci flow and Einstein metrics further in Section 1.5. Further, Hilbert showed that Einstein metrics are critical points of the total scalar curvature functional:

$$\mathcal{S} = \int_M S \, dV_g.$$

This notion of examining metrics that are the critical points of functionals can be seen in the motivation behind examining the Bach and ambient obstruction tensors.

1.2 Weyl Tensor

The Weyl tensor has been an object of interest for mathematicians and physicists for decades. Though the work in this paper focuses on the Weyl energy, we will spend our time here discussing the origins of the Weyl tensor, its properties, and its self-duality in dimension $n = 4$. We provide reader with additional background and demonstrate the nature of calculations using the Weyl tensor in Appendix A. We will only consider dimensions $n \geqslant 4$ in our calculation, since the Weyl tensor is identically zero for $n = 2, 3$.

In the broadest sense, the Weyl tensor measures how close a manifold is to being conformally flat. More explicitly, a manifold is conformally flat if and only if its Weyl tensor vanishes [Bes08].

The Weyl tensor is typically considered as a (3,1) tensor, but can be given as a (4,0) tensor:

$$W_{abcd} = R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac},$$

where P is the Schouten tensor given in terms of the Ricci and scalar curvature by:

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right).$$

For $n \geqslant 4$, the Weyl tensor is also conformally invariant, so if $\tilde{g} = e^{-\omega}g$ then $\tilde{W} = W$. This property has proved imperative in both the study conformal geometry and the study of geometric flows.

1.2.1 The Cotton Tensor

Briefly moving away from the Weyl tensor, we take a moment to exam some identities of the Cotton tensor.

We noted above that the Weyl tensor is identically 0 for $n = 3$. However, for $n = 3$, $C_{ijk} = 0$ if and only if the manifold is locally conformally flat and thus plays the role of the Weyl tensor in this dimension. The Cotton tensor is given locally by:

$$C_{ijk} = \nabla_i P_{jk} - \nabla_j P_{ik}.$$

For $n \geqslant 4$, the Cotton tensor can also be realized as the divergence of the Weyl tensor, up to a constant [CC13]. This can be seen in the following definition of the Cotton tensor:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}.$$

Like the Weyl tensor in $n \geqslant 4$, for dimension $n = 3$ the Cotton tensor is conformally invariant. In fact, for 3-dimensional Riemannian manifolds, any conformally invariant irreducible natural tensors are equivalent with a multiple of the Cotton tensor, modulo a conformally invariant natural tensor of degree at least 2 in curvature.[GH08, Theorem 1.2].

Lemma 1.2.1 (Properties of the Cotton Tensor)**.**

a. $C_{ijk} = -C_{jik}$

b. $g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0.$

Proof. a.

$$\begin{aligned} C_{ijk} &= 2\nabla_l W_{ijkl} = \nabla_i P_{jk} - \nabla_j P_{ik} \\ &= -2\nabla_l W_{jikl} = \nabla_j P_{ik} - \nabla_i P_{jk} \\ &= -C_{jik} \end{aligned}$$

b.

$$g^{ij}C_{ijk} = g^{ij}\nabla_l W_{ijkl} = \nabla_l g^{ij} W_{ijkl} = \nabla_l 0 = 0$$

$$g^{ik}C_{ijk} = g^{ik}\nabla_l W_{ijkl} = \nabla_l g^{ik} W_{ijkl} = \nabla_l 0 = 0$$

□

1.2.2 The Duality of the Weyl Tensor

The Weyl tensor decomposes into self-dual and anti-self-dual components only in dimension $n = 4$. As such, we limit the scope of the following section to $n = 4$.

In general, an object is self-dual if it equals its dual. Likewise an object is anti-self-dual if it equals the opposite its dual.

To understand precisely what it means for a tensor to be (anti)self-dual, we consider the Hodge $*$ operator as presented by [Jos17, Section 1.8]. In dimension n, the Hodge $*$ operator maps from k forms to $n - k$ forms. Examining $n = 4$, is governed by the following equivalences for an orthonormal frame $e_1, \dots e_4$:

$$*(e_1 \wedge e_2) = e_3 \wedge e_4 \qquad *(e_1 \wedge e_3) = e_4 \wedge e_2 \qquad *(e_1 \wedge e_4) = e_2 \wedge e_3$$
$$*(e_2 \wedge e_3) = e_1 \wedge e_4 \qquad *(e_2 \wedge e_4) = e_3 \wedge e_1 \qquad *(e_3 \wedge e_4) = e_1 \wedge e_2$$

Since $** = 1$, we see that $*$ has eigenvalues ± 1 for corresponding eigenspaces $\Lambda^{2,\pm}$. Thus, the Hodge $*$ operator induces the following decomposition of exterior 2-forms:

$$\Lambda^2 = \Lambda^{2,+} \oplus \Lambda^{2,-}$$

into self-dual and anti-self-dual components, respectively. Similarly, we are able to decompose the Weyl tensor into self-dual and anti-self dual components:

$$W = W^+ \oplus W^-.$$

Appendix A details how the matrix representation of the Riemannian curvature operator formalizes this decomposition. Further, we discuss the resulting eigenbasis and use it to prove facts about the Weyl tensor and its components.

1.3 Bach Tensor

The Bach tensor was defined by Rudolph Bach in [Bac21] in 1920 to study conformal relativity [CC13]. While the Bach tensor takes on more significance in dimension 4, we will begin by looking at this tensor for general $n \geqslant 4$.

The Bach tensor is given in local coordinates in terms of the Weyl tensor by, [CC13]:

$$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R^{kl} W_{ikjl}. \tag{1.4}$$

The Bach tensor can be given in terms of the Schouten tensor, P, [Hel20]:

$$B_{ij} = g^{lq}P_{ij;lq} - g^{lq}P_{il;jq} + P^{kl}W_{kijl}, \quad \text{where} \quad P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right). \tag{1.5}$$

The Bach tensor can also be given in terms of the Cotton tensor and Weyl tensor. This will prove helpful in proving the following fact. Consider the Cotton tensor for arbitrary dimensions, given by

$$C_{ijk} = -\frac{n-2}{n-3}\nabla^l W_{ijkl}.$$

Then the Bach tensor can be given as follows.

$$\begin{aligned} B_{ij} &= \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl} \\ &= \frac{1}{n-3}\nabla^k\left(-\frac{(n-3)}{n-2}C_{ikj}\right) + \frac{1}{n-2}R^{kl}W_{ikjl} \\ &= \frac{1}{n-2}\nabla^k C_{kij} + \frac{1}{n-2}R^{kl}W_{ikjl} \\ &= \frac{1}{n-2}\left(\nabla^k C_{kij} + R^{kl}W_{ikjl}\right). \end{aligned}$$

Proceeding, I will prove some well established properties of the Bach tensor which hold for all dimensions $n \geqslant 4$.

Fact 1.3.1. *The Bach tensor is a symmetric tensor, that is* $B_{ij} = B_{ji}$.

Proof. We know from the Bianchi identity that $W_{ikjl} = W_{jkil} + W_{ijkl}$. (The proof of this identity is in Appendix A.) Thus:

$$B_{ij} - B_{ji} = \frac{1}{n-3}\nabla^k\nabla^l W_{ijkl} + \frac{1}{n-2}R^{kl}W_{ijkl}.$$

Examining each term we see that:

$$\nabla^k\nabla^l W_{ijkl} = \sum_{k,l=1}^{n} \nabla_k\nabla_l W_{ijkl}.$$

For $k = l$ $W_{ijkl} = 0$ (Appendix A). Moreover, we can choose a basis such that $\nabla_{E_l} E_k = 0$. Splitting the sum and reindexing, we see that the two sums in fact cancel out:

$$\begin{aligned}
\sum_{k,l=1}^{n} \nabla_k \nabla_l W_{ijkl} &= \sum_{k<l} \nabla_k \nabla_l W_{ijkl} + \sum_{k>l} \nabla_k \nabla_l W_{ijkl} \\
&= \sum_{k<l} \nabla_k \nabla_l W_{ijkl} - \sum_{k>l} \nabla_k \nabla_l W_{ijlk} \\
&= \sum_{k<l} \nabla_k \nabla_l W_{ijkl} - \sum_{k'<l'} \nabla_{l'} \nabla_{k'} W_{ijk'l'} \\
&= \sum_{k<l} \nabla_k \nabla_l W_{ijkl} - \sum_{k'<l'} \nabla_{k'} \nabla_{l'} W_{ijkl} \\
&= 0.
\end{aligned}$$

Similarly, we see the same thing can be done to the second term using the symmetry of the Ricci tensor:

$$\begin{aligned}
\sum_{k,l=1}^{n} R_{kl} W_{ijkl} &= \sum_{k<l} R_{kl} W_{ijkl} + \sum_{k>l} R_{kl} W_{ijkl} \\
&= \sum_{k<l} R_{kl} W_{ijkl} - \sum_{k>l} R_{lk} W_{ijlk} \\
&= \sum_{k<l} \nabla_k \nabla_l W_{ijkl} - \sum_{k'<l'} R_{k'l'} W_{ijk'l'} \\
&= 0.
\end{aligned}$$

Thus $B_{ij} - B_{ji} = 0$. Therefore the Bach tensor is symmetric. □

Fact 1.3.2. *The Bach tensor is trace-free in arbitrary dimension $n \geqslant 4$. That is,* $\mathrm{tr}(B) = 0$.

Proof. We know from above that:

$$B_{ij} = \frac{1}{n-2} \left(\nabla_k C_{kij} + R^{kl} W_{ikjl} \right).$$

Since both the Cotton and Weyl tensors are trace-free ([CC13], Appendix A), it is clear that $\mathrm{tr}(B) = 0$. □

Fact 1.3.3. *Einstein metrics are Bach flat.*

Proof. An Einstein metric is one in which $R_{ij} = \lambda g_{ij}$. Taking the trace of both sides, we see that $S = \lambda n$. Note that this forces $R_{ij} = \frac{S}{n} g_{ij}$.

Examining the Schouten tensor, we see:

$$P_{ij} = \left(\frac{S}{n} - \frac{S}{6} \right) g_{ij} = \left(1 - \frac{n}{6} \right) \lambda g_{ij}.$$

Using the Schouten tensor definition of the Cotton tensor, we see that:

$$C_{ijk} = \nabla_i P_{jk} - \nabla_j P_{ik} = \left(\lambda - \frac{\lambda n}{6} \right) (\nabla_i g_{jk} - \nabla_j g_{ik}) = 0.$$

Examining the second term in the Bach tensor:

$$R^{kl} W_{ikjl} = \frac{R}{n} g^{kl} W_{ikjl} = \frac{S}{n} \cdot 0 = 0.$$

because the Weyl tensor is trace-free. Thus,

$$B_{ij} = \frac{1}{n-2} \left(\nabla_k C_{kij} + R^{kl} W_{ikjl} \right) = 0.$$

□

Fact 1.3.4. *If (M^n, g) is locally conformally flat, that is, if $W_{ijkl} = 0$, then $B_{ij} = 0$.*

Proof. This follows from the fact that

$$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}.$$

□

1.3.1 Dimension 4

Switching gears, we will focus only on dimension $n = 4$. This is, in fact, a very natural setting for the study of the Bach tensor. As mentioned in the introduction, in dimension $n = 4$ the Bach tensor is realized as the negative gradient of the conformally invariant functional given by:

$$\mathcal{W}(g) = \int_M |W_g|^2 \mathrm{d}V_g.$$

where W_g is the Weyl tensor and $|W_g|^2 = g^{ip}g^{jq}g^{kr}g^{ls}W_{ijkl}W_{pqrs}$. This functional has been studied for decades in the context of physics. This functional, known as the Weyl energy, has been used historically to study relativity [Bes08]. One commonly referenced fact is that the Weyl energy is only conformally invariant in dimension 4 [Der83]. Though this fact is well established in the literature, we prove it here for completeness.

Fact 1.3.5. *The Weyl energy is only conformally invariant in dimension 4.*

Proof. Consider the Weyl energy:

$$\mathcal{W}(g) = \int_M |W_g|^2 \mathrm{d}V_g = \int_M g^{ap}g^{bq}g^{cr}g^{ds}W_{abcd}W_{pqrs}\,\mathrm{d}V_g.$$

Consider the conformal mapping such that $\tilde{g} = e^{\omega}g$ for some $\omega \in C^{\infty}(M)$. We want to show that in dimension 4, $\mathcal{W}(\tilde{g}) = \mathcal{W}(g)$. First note that:

$$\mathrm{d}V_{\tilde{g}} = e^{n\omega}\mathrm{d}V_g \qquad \tilde{g}^{ij} = e^{-\omega}g^{ij}.$$

Examining the Weyl energy, we see that:

$$\begin{aligned}
\mathcal{W}(\tilde{g}) &= \int_M |W_{\tilde{g}}|^2 \, \mathrm{d}V_{\tilde{g}} \\
&= \int_M \tilde{g}^{ap}\tilde{g}^{bq}\tilde{g}^{cr}\tilde{g}^{ds}\tilde{W}_{abcd}\tilde{W}_{pqrs} \, \mathrm{d}V_{\tilde{g}}. \\
&= \int_M \left(e^{-\omega}g^{ap}\right)\left(e^{-\omega}g^{bq}\right)\left(e^{-\omega}g^{cr}\right)\left(e^{-\omega}g^{ds}\right) W_{abcd}W_{pqrs}\, e^{n\omega}\mathrm{d}V_g \\
&= \int_M e^{-4\omega}g^{ap}g^{bq}g^{cr}g^{ds}W_{abcd}W_{pqrs}\, e^{n\omega}\mathrm{d}V_g \\
&= \int_M e^{(n-4)\omega}|W_g|^2 \, \mathrm{d}V_g.
\end{aligned}$$

Thus $\mathcal{W}(\tilde{g}) = \mathcal{W}(g)$ for all g if and only if $n = 4$. □

Continuing our investigation of dimension $n = 4$, we note that the Weyl tensor is self-dual in dimension 4. These considerations make $n = 4$ a natural setting in which to consider the Bach tensor. Moreover, the four dimensional Bach tensor arises naturally when examining Huygen's principle in physics [Sze68].

Examining (1.4) in this context, we see that for $n = 4$ the Bach tensor is given by:

$$B_{ij} = \nabla^k\nabla^l W_{ikjl} + \frac{1}{2}R^{kl}W_{ikjl}. \tag{1.6}$$

In addition to being trace-free, for $n = 4$ the Bach tensor is symmetric, divergence-free, and conformally invariant of weight -2. Note, we say that a function is conformally invariant of weight -2 if for a positive, smooth function ρ, $\tilde{g} = \rho^2 g$ then $\tilde{B} = \frac{1}{\rho^2}B$. Fefferman-Graham detail why the Bach tensor is only conformally invariant for $n = 4$ in [FG12, Chapter 6]. We show below that it is divergence-free.

Fact 1.3.6. *The Bach tensor is only necessarily divergence-free for $n = 4$.*

Proof. From [CC13] we know the divergence of the Bach tensor is given by:

$$\mathrm{div}B = \nabla_j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R_{jk}.$$

Clearly for $n = 4$ $\mathrm{div}B = 0$.

For $n \neq 4$, there are conditions that we can place on the manifold (such as being Ricci-flat) that would force the Bach tensor to be divergence-free. However, the Bach tensor is not necessarily divergence-free as in dimension 4. □

Examining $n = 4$ allows us to consider self-dual and anti-self-dual metrics, as described in Section 1.2.

Proposition 1.3.1. *(Anti)self-dual metrics are Bach flat.*

Proof. Using [Der83, Lemma 6]. We are able to rewrite the equation for the Weyl energy using the self-dual and anti-self-dual components:

$$\int_M |W(g)|^2 = \int_M |W^+(g)|^2 + \int_M |W^-(g)|^2.$$

We also consider the signature formula given by:

$$\tau(M) = \frac{1}{12\pi^2} \left(\int_M |W^+(g)|^2 - \int_M |W^-(g)|^2 \right).$$

Note that, like the Euler characteristic of a manifold, the signature of a manifold is a topological invariant [Der83]. (In fact, the relationship between the characteristic and signature of a manifold is given by the Thorpe inequality.)

Manipulating these equations we see that:

$$\mathcal{W}(g) = \int |W(g)|^2 = 12\pi^2 \tau(M) + 2 \int |W^-(g)|^2.$$

Since the signature of a manifold is a topological invariant, $\tau(M)$ and, consequently, $12\pi^2\tau(M)$ are fixed. Therefore $\mathcal{W}$ is minimized when $W^-(g) = 0$. That is, when out metric, g, is self-dual.

By [Der83, Lemma 1], we know that a metric on a compact oriented four-manifold M is a critical point of $g \to \mathcal{W}$ if and only if its Bach tenser vanishes identically. Since self-dual metrics are minimizers of $\mathcal{W}$, they are critical points and are therefore Bach Flat.

On the other hand, consider an anti-self-dual metric g, so $W^+(g) = 0$. By definition, $\tau(M) \leqslant 0$. Since

$$\mathcal{W}(g) = 2\int |W^+(g)|^2 - \int |W(g)|^2 = 12\pi^2\tau(M)$$

is minimized when $W^+(g) = 0$, we know by the same argument as above that g is Bach flat. □

1.3.2 Bach Tensor on Product Manifolds

For a manifold $M = N_{(1)} \times N_{(2)}$ with product metric $g = g_{(1)} + g_{(2)}$ the Bach tensor acts differently on the components depending on their dimensions. For simplicity, we will refer to a manifold where $\dim\left(N^{(1)}\right) = a$ and $\dim\left(N^{(2)}\right) = b$ as an $a \times b$ product manifold. Following the conventions set by Helliwell [Hel20], we use Greek indices for $N^{(1)}$ and lower case roman indices for $N^{(2)}$. Moreover, it should be noted that indices begin at 0.

As Helliwell points out, for a general product manifold

$$R_{\alpha\beta} = R^{(1)}_{\alpha\beta}, \qquad R_{ij} = R^{(2)}_{ij}, \qquad R_{\alpha j} = 0, \qquad S = S^{(1)} + S^{(2)}.$$

Specifically for a 1×3 product manifold $R_{00} = 0$ and $S = S^{(2)}$.

For a 1×3 product manifold, we see in [DK12] and [Hel20] that the equations for the component of the Bach tensor are:

$$
\begin{aligned}
B_{00} =& \left(-\frac{1}{12}(\Delta^{(2)} S^{(2)}) - \frac{1}{4}\left[(|\operatorname{Ric}|^{(2)})^2 - \frac{1}{3}(S^{(2)})^2\right]\right) g_{00},\\
B_{jk} =& \frac{1}{2}\Delta^{(2)} R^{(2)}_{jk} - \frac{1}{12}\Delta^{(2)} S^{(2)} g_{jk} - \frac{1}{6} S^{(2)}_{;jk} - 2\operatorname{tr}^{(2)}(\operatorname{Ric}^{(2)} \otimes \operatorname{Ric}^{(2)})_{jk} \\
&+ \frac{7}{6} S^{(2)} R^{(2)}_{jk} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{jk} - \frac{5}{12}(S^{(2)})^2 g_{jk}.
\end{aligned} \tag{1.7}
$$

Here $\operatorname{tr}(\operatorname{Ric} \otimes \operatorname{Ric})_{jk} = g^{il} R_{ij} R_{lk}$

From [DK12], [Ho18], and [Hel20], we see that the 2×2 product manifold breaks down as follows:

$$
\begin{aligned}
B_{\mu\nu} &= -\frac{1}{6}\nabla_\mu \nabla_\nu S^{(1)} + \frac{1}{6} g^{(1)}_{\mu\nu}\left[\nabla^\alpha \nabla_\alpha S^{(1)} - \frac{1}{2}\nabla^k \nabla_k S^{(2)} + \frac{1}{4}\left(\left(S^{(2)}\right)^2 - \left(S^{(1)}\right)^2\right)\right],\\
B_{ij} &= -\frac{1}{6}\nabla_i \nabla_j S^{(2)} + \frac{1}{6} g^{(2)}_{ij}\left[\nabla^k \nabla_k S^{(2)} - \frac{1}{2}\nabla^\alpha \nabla_\alpha S^{(1)} + \frac{1}{4}\left(\left(S^{(2)}\right)^2 - \left(S^{(1)}\right)^2\right)\right],\\
B_{\alpha j} &= 0.
\end{aligned} \tag{1.8}
$$

Note that we've used the equations as stated in [Hel20].

We use these equations to find an explicit representation for the Bach tensor in terms of the metric. The cases investigated in this theses use structure constants to complete this calculation. This is discussed in more detail in Appendix B, where we go through an example of computing the Bach tensor of a manifold.

1.4 Ambient Obstruction Tensor

For our study of Bach solitons, it is particularly important that the Bach tensor is divergence-free and conformally invariant of weight -2. But these properties are only guaranteed for $n = 4$. Consequently, in order to find a higher dimensional equivalent we examine the first variation of the functional for even n:

$$
\mathcal{F}^n_Q(g) = \int_M Q(g)\, \mathrm{d}V_g,
$$

where $Q(g)$ is Branson's Q-curvature described in [Bra93].

The use of this functional is interesting. The Q curvature is itself a scalar quantity defined on even-dimensional manifolds. We see that Q lacks some of the conformal properties of $|W|^2$, specifically Q is not pointwise conformally covariant. However, the functionals $\mathcal{F}_Q^n$ are conformally invariant for arbitrary even n. Moreover Branson uses the Chern-Gauss-Bonnet theorem to show that, in dimension $n = 4$, $\mathcal{F}_Q^n$ is related to $\mathcal{W}$ by the equation:

$$\mathcal{F}_Q^4 = 8\pi^2\chi(M) - \frac{1}{4}\mathcal{W},$$

where $\chi(M)$ is the Euler characteristic of M. Since $\chi(M)$ is a topological invariant and a constant, the functionals have the same critical metrics.

In [FG12], Fefferman and Graham examine the gradient of $\mathcal{F}_Q^n$ and introduce the resulting symmetric 2-tensor, the ambient obstruction tensor, $\mathcal{O}$, for even $n \geqslant 4$. This tensor can be also characterized as the obstruction to an n-manifold having a formal power series of asymptotically hyperbolic Einstein metric (or Poincaré metric) in dimension $n+1$ [BH11], [GH08]. In fact, this characterization provides the relationship between the Q-curvature and obstruction tensor, as established in [GZ03], [FG02].

Like the Bach tensor in dimension 4, the ambient obstruction tensor is symmetric, trace-free, divergence-free, and conformally invariant of weight $2-n$. (A tensor, q, is of weight w if $\hat{g} = \rho^2 g$, $\hat{q} = \rho^w q$, for $0 < \rho \in C^\infty(M)$.) The ambient obstruction tensor can be viewed as a family of even dimensional tensors, where the dimension 4 ambient obstruction tensor is the Bach tensor.

Explicitly, the ambient obstruction tensor is given by the equation found in [BH11]:

$$\begin{aligned} \mathcal{O}_n = \frac{1}{(-2)^{\frac{n}{2}-2}\left(\frac{n}{2}-2\right)!}\left(\Delta^{\frac{n}{2}-1}P - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 S\right) + T_{n-1} \\ P = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{1}{2(n-1)}Sg\right), \end{aligned} \tag{1.9}$$

where P is the Schouten tensor and T_{n-1} is a polynomial natural tensor of order $n-1$. It should be noted that the ambient obstruction tensor is given slightly differently in [FG12], [GH08], and [Lop18]:

$$\begin{aligned} \mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}\left(P_{ij,k}{}^{k} - P_{k}{}^{k}{}_{,ij}\right) + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm) \\ T_k^m(A) &= \sum_{i_1+\cdots+i_k=m} \nabla^{i_1} A * \cdots * \nabla^{i_k} A. \end{aligned} \tag{1.10}$$

The reader should note that (1.10) uses Einstein notation to represent the same operations on the Schouten tensors. This is detailed in Proposition 1.4.1 below. We include (1.10) because it provides a representation in local coordinate and it illuminates the nature of the lower order terms. Furthermore, using the definition of the Weyl tensor as seen in Section 1.2, (1.10) quickly yields the following:

$$\mathcal{O}_{ij} = \frac{1}{3-n}\Delta^{\frac{n}{2}-2}\nabla^l\nabla^k W_{kijl} + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm).$$

It is worthwhile to show that (1.9) is, in fact, the same (up to a constant) as (1.10). For readers unfamiliar with Einstein notation, this will also serve to illuminate some of the notation used in (1.10).

Proposition 1.4.1. *Equation (1.9) is equivalent to (1.10) up to a constant.*

Proof. Following the steps shown by Lopez in [Lop18, Proposition 2.3], we first note that:

$$\begin{aligned} P_k{}^k &= P_{kj}g^{jk} \\ &= \frac{1}{n-2}\left[g^{jk}R_{kj} - \frac{1}{2(n-1)}Sg^{jk}g_{kj}\right] \\ &= \frac{1}{n-2}\left[S - \frac{n}{2(n-1)}S\right] \\ &= \frac{1}{2(n-1)}S. \end{aligned}$$

Using this in our equation and expanding some of the Einstein notation used by [FG12], [GH08], we see that:

$$
\begin{aligned}
\mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}\left(P_{ij,k}{}^{k} - P_{k}{}^{k}{}_{,ij}\right) + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm) \\
&= \Delta^{\frac{n}{2}-2}\left(\Delta P_{ij} - \nabla_j\nabla_i P_k^{\ k}\right) + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm) \\
&= \Delta^{\frac{n}{2}-1} P_{ij} - \Delta^{\frac{n}{2}-2}\nabla_j\nabla_i P_k^{\ k} + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm) \\
&= \Delta^{\frac{n}{2}-1} P_{ij} - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla_j\nabla_i S + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm).
\end{aligned}
$$

□

Note that equation 1.9 is scaled by the constant:

$$
c_n = \frac{1}{(-2)^{\frac{n}{2}-2}\left(\frac{n}{2}-2\right)!}
$$

This constant will ultimately change how Bahuaud-Helliwell determine $\hat{\mathcal{O}}$ and consequently will change the way they define the ambient obstruction flow in [BH11], [BH15].

In any formulation of the equation, the lower order terms present an obstacle for working with the ambient obstruction tensor. However, in dimension $n = 4$ we know that $\mathcal{O}_{ij} = B_{ij}$ and for $n = 6$:

$$
\begin{aligned}
\mathcal{O}_{ij} = B_{ij,k}{}^{k} - 2W_{kijl}B^{kl} - 4P_k^{\ k}B_{ij} + 8P^{kl}C_{(ij)k,l} - 4C^{k}{}_{i}{}^{l}C_{ljk} \\
+2C_i^{\ kl}C_{jkl} + 4P^{k}{}_{k,l}C^{\ \ l}_{(ij)} - 4W_{kijl}P^{k}{}_{m}P^{ml}.
\end{aligned}
$$

1.5 Geometric Flows

Below we will restate important definitions in the study of geometric flows, establish the ambient obstruction and Bach flows, and discuss results from the study of Ricci flow that we will generalize in Section 2.1. Please refer to the introduction for a more detailed explanation of the origin and motivation of geometric flows and solitons.

As stated in the introduction, a geometric flow is a differential equation in which the metric, $g(t)$, is changed over time in accordance with the chosen tensor. For a general tensor, q, the q-flow is the one parameter family of smooth metrics such that:

$$\begin{cases} \partial_t g = q \\ g(0) = h. \end{cases} \tag{1.11}$$

Solitons are self-similar solutions to this flow, meaning they are metrics that the flow changes by diffeomorphism and/or rescaling. More specifically, a (normalized) q-soliton is a metric that satisfies the equation:

$$\frac{1}{2}\mathcal{L}_X\, g = cg + \frac{1}{2}q, \tag{1.12}$$

for vector field X and Lie derivative $\mathcal{L}_X\, g$. As in the introduction, we normalize the equation to prove Theorem 2.1.13. We classify these solitons as expanding, steady, and shrinking when $c < 0$, $c = 0$, and $c > 0$, respectively.

Letting $X = \nabla f$, where f is the potential function, a (normalized) gradient q-soliton satisfies:

$$\operatorname{Hess} f = cg + \frac{1}{2}q. \tag{1.13}$$

1.5.1 Ambient Obstruction Flow

As we saw in Section 1.4, the Bach tensor is the four dimensional ambient obstruction tensor. While we will discuss the Bach flow specifically, one should remember that the definitions and results for the ambient obstruction flow apply to the Bach flow as well.

In the last decade Bahuaud-Helliwell, Helliwell, and Lopez have studied flowing a metric by the ambient obstruction tensor. Bahuaud and Helliwell, in [BH11, Theorem C], consider the flow given by:

$$\begin{cases} \partial_t g = \mathcal{O}_n + c_n(-1)^{\frac{n}{2}} \left(\Delta^{\frac{n}{2}-1} S\right) g \\ g(0) = h, \end{cases} \tag{1.14}$$

where h is a smooth metric on a compact manifold of even dimension $n \geqslant 4$ and

$$c_n = \frac{1}{2^{\frac{n}{2}-2}\left(\frac{n}{2}-2\right)!(n-2)(n-1)}.$$

For $n = 4$ we will call this flow the Bach flow, which is given by:

$$\begin{cases} \partial_t g = B + \frac{1}{12}\Delta S g \\ g(0) = h. \end{cases} \tag{1.15}$$

In [BH11, BH15] Bahuaud and Helliwell show short time existence and uniqueness on compact manifolds for this flow. As Lopez explains in [Lop18], the scalar curvature term "counteracts the invariance of $\mathcal{O}$ under the action of the conformal group on the space of metrics on M." Moreover, the addition of this terms serves as a way to make the geometric flow strongly parabolic, allowing the use of the first part of the DeTurk trick. In [Lop18], Lopez finds pointwise smoothing estimates and uses them to find an obstruction to long-time existence and to prove a compactness theorem for the flow (1.14).

Since homogeneous manifolds have constant scalar curvature, the equations for the ambient obstruction flow and Bach flow on homogeneous manifolds are given by:

$$\begin{cases} \partial_t g = \mathcal{O}_n \\ g(0) = h \end{cases} \quad \text{and} \quad \begin{cases} \partial_t g = B \\ g(0) = h, \end{cases} \tag{1.16}$$

respectively. Helliwell uses the latter equation in [Hel20] to study the Bach flow on homogeneous compact product manifolds of the form $S^1 \times K^3$.

The solitons of these flows are defined as follows.

Definition 1.5.1. *An ambient obstruction soliton is a solution, (M, g), to the equation:*

$$\frac{1}{2}\mathcal{L}_X\, g = cg + \frac{1}{2}\left(\mathcal{O}_n + c_n(-1)^{\frac{n}{2}}\left(\Delta^{\frac{n}{2}-1}S\right)g\right),$$

where c_n is defined as above. In dimension $n = 4$, the ambient obstruction soliton is the Bach soliton, given by:

$$\frac{1}{2}\mathcal{L}_X\, g = cg + \frac{1}{2}\left(B + \frac{1}{12}\Delta S g\right).$$

These are called gradient if $X = \nabla f$, and the corresponding equations are

$$\begin{aligned} \operatorname{Hess} f &= cg + \frac{1}{2}\left(\mathcal{O}_n + c_n(-1)^{\frac{n}{2}}\left(\Delta^{\frac{n}{2}-1}S\right)g\right) \\ \operatorname{Hess} f &= cg + \frac{1}{2}\left(B + \frac{1}{12}\Delta S g\right). \end{aligned} \tag{1.17}$$

This change comes from the following identity:

$$\mathcal{L}_{\nabla f}\, g(Y,Z) = g(\nabla_Y \nabla f, Z) + g(Y, \nabla_Z \nabla f) = \operatorname{Hess} f(Y,Z) + \operatorname{Hess} f(Z,Y) = 2\operatorname{Hess} f(Y,Z).$$

Note that, like our general soliton equation, this equation has also been normalized. As such it is slightly different than the definition in [Ho18]. Moreover, Ho only considers a flow that aligns with the definition in the case of constant scalar curvature. Taking this into account and letting $c = -\frac{1}{2}\lambda$ we see that the two equations are equivalent.

$$\begin{aligned} B &= \lambda g + \mathcal{L}_X\, g(f) \\ \mathcal{L}_X\, g(f) &= -\lambda g + B \\ \frac{1}{2}\,\mathcal{L}_X\, g(f) &= -\frac{1}{2}\lambda g + \frac{1}{2}B \\ \frac{1}{2}\,\mathcal{L}_X\, g(f) - \frac{1}{2}B &= cg. \end{aligned}$$

Using these definitions, we begin by examining general solitons then focus more on examining specific Bach solitons.

1.5.2 Ricci Flow Results

Historically, analyzing gradient solitons has provided a lot of insight into the Ricci flow. The work of Hamilton, Ivey, and Perelman combine to classify 3-dimensional shrinking gradient Ricci solitons [PW10]. Further, in [Per02], Perelman proves that any compact Ricci soliton is a gradient Ricci soliton. As we will briefly discuss in Appendix B, Perelman also used Ricci flow to prove Thurston's geometrization theorem. Most notably, the study of Ricci solitons was imperative in Perelman's proof of the Poincaré Conjecture.

The popularity of the Ricci flow has lead to a great deal of results about Ricci solitons that the author has used as a basis for generalizations in this paper. On such well known result is as follows

Theorem 1.5.2. *[PW09, Theorem 3.1] A compact Ricci soliton with constant scalar curvature is Einstein.*

By the (twice-contracted) second Bianchi identity, we know that:

$$\operatorname{div} \mathrm{Ric} = \frac{1}{2}\nabla S.$$

Thus, the Ricci tensor is divergence-free if and only if the scalar curvature is constant.

Moreover, for the Ricci tensor requiring constant scalar curvature is similar to the trace-free condition. Looking to this for inspiration, we get the following result for general q.

Theorem 1.5.3. *For a divergence-free, trace-free tensor q, any compact q-soliton is q-flat.*

We also establish a generalization of the following theorem and apply said generalization to the ambient obstruction tensor.

Theorem 1.5.4 (Theorem 1.1, [PW09])**.** *A shrinking compact gradient soliton is rigid with trivial f if*

$$\int_M \mathrm{Ric}(\nabla f, \nabla f) \leqslant 0.$$

Using this as inspiration, we investigate the implications of this for gradient ambient obstruction solitons in Theorem 2.1.8 which is as follows.

Theorem. *For any compact gradient ambient obstruction soliton*

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\ dvol_g \geqslant 0,$$

where the integral is zero if and only if f is constant.

The study of Ricci solitons has continued to prove a bountiful source of information and is still a very large area of research. It is reasonable to hope that the study of gradient solitons for other flows (specifically the Bach flow and ambient obstruction flow) would prove similarly fruitful in the understanding of the behavior of the flows and consequently the behaviors of the tensors themselves.

For further background on Ricci flow, refer to [CLN06, MT07, Top06]. For further reading about Ricci solitons refer to [Cao06, CK04, CLN06, Der12].

Chapter 2

Results

2.1 Results for General Tensor

In this section, we prove a number of statements for a general trace-free and/or divergence-free tensor q. Applications of the theorem to the ambient obstruction tensor will follow in subsequent corollaries. For the sake of simplicity, full proofs of these corollaries have been omitted, but appropriate connections will be made.

Recall from Section 1.4 that the ambient obstruction tensor, $\mathcal{O}_n$ (n even), is trace-free and divergence-free. However, the reader should note that the tensor affiliated with the general flow (1.14) does not possess all of these properties. That said, we will often focus on the homogeneous case in order to define the flow as in (1.16) and to use these properties of the ambient obstruction tensor.

The following proposition is useful in examining gradient solitons and will be used to prove later results.

Proposition 2.1.1. *Let q be a symmetric two tensor and (M, g, f) a gradient q-soliton*

(1.13). The potential function, f, has the property that

$$\mathrm{Ric}(\nabla f) = \mathrm{div}Q - \frac{1}{2}\nabla(\mathrm{tr}\, Q),$$

where Q is the dual (1,1)-tensor of q with respect to g.

Proof. Consider a gradient soliton of the q-flow, given by

$$\mathrm{Hess}\, f = cg_{ij} + \frac{1}{2}q_{ij}.$$

Type changing into (1,1) tensor

$$\nabla\nabla f = cI + \frac{1}{2}Q.$$

If we simply take the trace of each of the terms, we see that then $\Delta f = cn + \frac{1}{2}\,\mathrm{tr}\, Q$.

Taking the divergence of each term in our soliton equation we see that:

$$\begin{aligned}
\mathrm{div}Q &= \mathrm{div}(\nabla\nabla f) \\
&= \mathrm{Ric}(\nabla f) + \nabla(\Delta f) \\
&= \mathrm{Ric}(\nabla f) + \nabla(cn + \frac{1}{2}\,\mathrm{tr}\, Q) \\
&= \mathrm{Ric}(\nabla f) + \frac{1}{2}\nabla(\mathrm{tr}\, Q).
\end{aligned}$$

Thus:

$$\mathrm{Ric}(\nabla f) = \mathrm{div}Q - \frac{1}{2}\nabla(\mathrm{tr}\, Q).$$

□

Using this theorem, we are able to quickly generalize [Ho18, Theorem 3.4] as follows.

Corollary 2.1.2. *For any constant trace, divergence-free tensor q, the gradient solitons of its flow has that property that* $\mathrm{Ric}(\nabla f) = 0$.

For the ambient obstruction flow on a non-homogeneous manifold, we see that a gradient soliton is given by:

$$\operatorname{Hess} f = cg + \frac{1}{2}\left(\mathcal{O}_n + a_n\left(\Delta^{\frac{n}{2}-1}S\right)g\right),$$

where

$$a_n = \frac{(-1)^{\frac{n}{2}}}{2^{\frac{n}{2}-2}\left(\frac{n}{2}-2\right)!(n-2)(n-1)}.$$

Note that a_n simply combines constant terms in our original definition to help with notation. Examining this soliton, we get the following corollary.

Corollary 2.1.3. *A gradient ambient obstruction soliton with potential function f satisfies* $\operatorname{Ric}(\nabla f) = a_n(1-n)\,\nabla\left(\Delta^{\frac{n}{2}-1}S\right)$.

Proof. Consider a gradient ambient obstruction soliton with potential function f. Then $q = \mathcal{O}_n + a_n\left(\Delta^{\frac{n}{2}-1}S\right)g$ and consequently

$$\operatorname{div} q = a_n d\left(\Delta^{\frac{n}{2}-1}S\right),$$

$$\operatorname{tr} q = n a_n\left(\Delta^{\frac{n}{2}-1}S\right),$$

$$\nabla \operatorname{tr} q = n a_n \nabla\left(\Delta^{\frac{n}{2}-1}S\right).$$

Using Proposition 2.1.1:

$$\operatorname{Ric}(\nabla f) = a_n(1-n)\nabla\left(\Delta^{\frac{n}{2}-1}S\right).$$

□

Remark 2.1.4. *For a gradient ambient obstruction soliton with constant scalar curvature (specifically for homogeneous manifolds) we see that* $\Delta^{\frac{n}{2}-1}S = 0$*, so* $\operatorname{Ric}(\nabla f) = 0$.

The following lemma appears to be well known, but we include the proof for completeness.

Lemma 2.1.5. *For any symmetric (0,2)-tensor field ψ and vector field ξ:*

$$\langle \mathcal{L}_\xi \, g, \psi \rangle = 2\text{div}(i_\xi \psi) - 2(\text{div}\psi)\xi,$$

where $i_\xi \psi$ is a 1-form such that $i_\xi \psi(\cdot) = \psi(\xi, \cdot)$

Proof. Consider a symmetric (0,2)-tensor field ψ and a vector field ξ. For a (0,2)-tensor A, we know that $A(x, y) = g(A(x), y)$, so:

$$\langle A, B \rangle = \sum_i g(A(e_i), B(e_i)) = \sum_i A(e_i, B(e_i)),$$

where B is a (1,1)-tensor.

Consider the Lie derivative as our (0,2)-tensor, and ψ a (1,1)-tensor. First, examining the type change, consider ψ as a (0,2)-tensor:

$$\psi(X, Y) = g(\psi(X), Y) \Longrightarrow \psi(X, E_j) = g(\psi(X), E_j) \Longrightarrow \psi(X) = \sum_j g(\psi(X), E_j)E_j.$$

Next, we know that:

$$\text{div}(\iota_\xi \psi) = \sum_i (\nabla_{E_i} \iota_\xi \psi)(E_i) = \sum_i \nabla_{E_i} \psi \, (\xi, E_i) = \sum_i \nabla_{E_i} g \, (\psi(E_i), \xi) \,,$$

$$(\text{div}\psi)(\xi) = \sum_i g(\xi, \nabla_{E_i}(\psi(E_i))).$$

Then

$$\begin{aligned}
\langle \mathcal{L}_\xi \, g, \psi \rangle &= \sum_i \mathcal{L}_\xi \, g(E_i, \psi(E_i)) \\
&= \sum_i g\left(\nabla_{E_i}\xi, \psi(E_i)\right) + \sum_i g\left(E_i, \nabla_{\psi(E_i)}\xi\right) \\
&= \sum_i g\left(\nabla_{E_i}\xi, g(\psi(E_i), E_j)E_j\right) + \sum_i g\left(E_i, \nabla_{g(\psi(E_i),E_j)E_j}\xi\right) \\
&= \sum_i g(\psi(E_i), E_j) g(\nabla_{E_i}\xi, E_j) + \sum_i g(\psi(E_i), E_j) g(E_i, \nabla_{E_j}\xi) \\
&= 2\sum_i g(\psi(E_i), E_j) g(\nabla_{E_i}\xi, E_j) \\
&= 2\sum_i (g(\nabla_{E_i}\xi, \psi(E_i)) \\
&= 2\sum_i \left[\nabla_{E_i} g(\xi, \psi(E_i)) - g(X, \nabla_{E_i}(\psi(E_i)))\right] \\
&= 2\mathrm{div}\iota_\xi\psi - 2(\mathrm{div}\psi)(\xi).
\end{aligned}$$

Thus, the identity holds. □

We use this fact to prove the following lemma for compact solitons of a general q-flow. Note that these solitons are not necessarily gradient solitons.

Lemma 2.1.6. *Let (M, g, X) be an n-dimensional compact soliton to the q-flow, (1.12). Then:*

a. $\displaystyle\int_M ||\, \mathcal{L}_X \, g||^2 \; dvol_g = -2\int_M \mathrm{div}(q)(X) \; dvol_g.$

b. If q is divergence-free, then X is Killing.

c. If q is divergence-free and trace-free, then (M, g_{ij}) must be q-flat.

Proof. a. Consider the q-soliton, $\frac{1}{2}\mathcal{L}_X \, g = cg + \frac{1}{2}q$. We know that for any vector field ξ on M

$$\langle \mathcal{L}_\xi \, g, \psi \rangle = 2\mathrm{div}(i_\xi\psi) - 2(\mathrm{div}\psi)(\xi)$$

where $i_\xi\psi(\cdot) = \psi(\xi, \cdot)$.

Note that the soliton can be written as $q = \mathcal{L}_X - 2cg$. Examining the divergence of this equation:

$$\mathrm{div} q_{ij} = \mathrm{div}(\mathcal{L}_X g) - 2c\mathrm{div}(g_{ij}) = \mathrm{div}(\mathcal{L}_X g).$$

Using Lemma 2.1.5, we see that letting $\psi = \mathcal{L}_X g$ and $\xi = X$:

$$\langle \mathcal{L}_X g, \mathcal{L}_X g \rangle = ||\mathcal{L}_X g||^2 = 2\mathrm{div}(i_X \mathcal{L}_x g) - 2\mathrm{div}(\mathcal{L}_X g)(X) = 2\mathrm{div}(i_X \mathcal{L}_x g) - 2\mathrm{div}(q)(X).$$

Integrating over M we see that since M is compact and has no boundary:

$$\begin{aligned}\int_M ||\mathcal{L}_X g||^2 \, dvol_g &= 2\int_M \mathrm{div}(i_X \mathcal{L}_x g) \, dvol_g - 2\int_M \mathrm{div}(q)(X) \, dvol_g \\ &= -2\int_M \mathrm{div}(q)(X)) \, dvol_g.\end{aligned}$$

b. If q is divergence-free part (a) shows that $\int_M ||\mathcal{L}_X g||^2 \, dvol_g = 0$. Thus, $\mathcal{L}_X g = 0$ and consequently X is Killing.

c. Suppose that q is divergence-free and trace-free. From (b), this means that $q_{ij} = -2cg_{ij}$. Taking the trace of both sides we see that $0 = -2nc$ and thus $c = 0$. Thus $q_{ij} = 0$ and subsequently (M, g_{ij}) is q-flat.

□

Corollary 2.1.7. *Let (M, g, X) be an n-dimensional compact ambient obstruction soliton with constant scalar curvature. Then $\int_M ||\mathcal{L}_X g||^2 \, dvol_g = 0$, X is Killing, and M is $\mathcal{O}$-flat.*

Proof. Since M has constant scalar curvature we know that the flow is given by (1.16). Thus, we consider $q = \mathcal{O}_n$. Since $\mathcal{O}$ is divergence-free and trace-free, the conclusion follows directly from Lemma 2.1.6 □

In particular, Corollary 2.1.7 shows that any homogeneous compact ambient obstruction soliton is $\mathcal{O}$-flat.

Proceeding to examine the non-homogeneous, gradient case we have the following inequality. This inequality was inspired by [PW09, Theorem 1.1] as mentioned in Section 1.5.

Theorem 2.1.8. *For any compact gradient ambient obstruction soliton* (M, g, f)

$$\int_M \operatorname{Ric}(\nabla f, \nabla f)\ dvol_g \geqslant 0,$$

where the integral is zero if and only if f is constant.

Proof. Consider an n-dimensional compact gradient ambient obstruction soliton, (M, g, f). Applying Lemma 2.1.6, let $q = \mathcal{O}$ and let $X = \nabla f$. From Corollary 2.1.3:

$$\operatorname{div} Q = a_n \nabla \left(\Delta^{\frac{n}{2}-1} S\right) = \frac{a_n}{1-n}(1-n)\nabla \left(\Delta^{\frac{n}{2}-1} S\right) = \frac{1}{1-n} \operatorname{Ric}(\nabla f).$$

By Lemma 2.1.6:

$$0 \leqslant \int_M ||\, \mathcal{L}_{\nabla f}\, g||^2 \ \mathrm{d}vol_g = -2 \int_M \operatorname{div}(q)(\nabla f))\ \mathrm{d}vol_g = \frac{2}{n-1} \int_M \operatorname{Ric}\left(\nabla f, \nabla f\right)\ \mathrm{d}vol_g.$$

Thus $\int_M \operatorname{Ric}(\nabla f, \nabla f)\ \mathrm{d}vol_g \geqslant 0$.

Suppose $\int_M \operatorname{Ric}(\nabla f, \nabla f)\ \mathrm{d}vol_g = 0$.

Since

$$\int_M ||\, \mathcal{L}_{\nabla f}\, g||^2 \ \mathrm{d}vol_g = \frac{2}{n-1} \int_M \operatorname{Ric}\left(\nabla f, \nabla f\right)\ \mathrm{d}vol_g,$$

if the right hand side is zero then $\mathcal{L}_{\nabla f}(g) = 0$ and consequently $\operatorname{Hess} f = 0$. Since M is compact this implies that f is constant. If f is constant $\nabla f = 0$ then clearly $\operatorname{Ric}(\nabla f) = 0$. Therefore, the integral is zero if and only if f is constant □

Remark 2.1.9. *A soliton is defined to be stationary if f is constant. Thus Theorem 2.1.8 implies that a compact gradient ambient obstruction soliton with non-positive Ricci curvature must be stationary.*

We note that in general, stationary gradient ambient obstruction solitons are characterized by the following proposition.

Proposition 2.1.10. *If (M, g, f) is a stationary gradient ambient obstruction soliton, then (M, g) is $\mathcal{O}$-flat. If (M, g) is also compact then S is constant.*

Proof. Consider a stationary gradient ambient obstruction soliton, (M, g, f). Since the soliton is stationary, f is constant. Consequently $\operatorname{Hess} f = 0$ and thus $q = -2cg$. Since $q = \mathcal{O}_n + a_n \left(\Delta^{\frac{n}{2}-1} S\right) g$,

$$\mathcal{O}_n = \left(-a_n \left(\Delta^{\frac{n}{2}-1} S\right) - 2c\right) g.$$

Taking the trace of both sides:

$$0 = n \left(-a_n \left(\Delta^{\frac{n}{2}-1} S\right) - 2c\right).$$

Thus

$$0 = -a_n \left(\Delta^{\frac{n}{2}-1} S\right) - 2c.$$

This forces $\mathcal{O}_n = 0$, so that soliton is $\mathcal{O}$-flat. Furthermore:

$$\Delta^{\frac{n}{2}-1} S = \frac{2c}{a_n}$$

is constant. If M is compact, this implies that S is constant. □

Remark 2.1.11. *The converse of Proposition 2.1.10 is true in the compact case. That is, a compact gradient ambient obstruction soliton that is $\mathcal{O}$-flat and has constant scalar curvature is stationary. Constant scalar curvature and $\mathcal{O}$-flat imply that* $\operatorname{Hess} f = cg$. *Compactness* *forces the manifold to have a maximum and minimum so* $\operatorname{Hess} f = 0$. *Appealing once more to compactness, this forces* f *to be constant and our soliton to be stationary.*

Though the following lemma is not necessary when studying when ambient obstruction solitons are stationary (this was taken care of in Corollary 2.1.7), it does give another criteria for when a q-soliton is stationary.

Proposition 2.1.12. *For a trace-free tensor* q*, any compact gradient soliton to the* q*-flow must be* q*-flat.*

Proof. Generalizing from [Ho18], consider a gradient q-soliton (1.13). By assumption $\operatorname{tr}(q) = 0$, so taking the trace of both sides yields $\Delta f = cn$. Integrating over M:

$$0 = \int_M cn - \Delta f \; dvol_g = cn \, Vol(M, g).$$

Thus $c = 0$. Further, $\Delta f = 0n + 0$ so $\Delta f = 0$, that is, f is harmonic. Since M is compact, f must be constant.

Therefore $q_{ij} = 2 \operatorname{Hess} f - 2cg_{ij} = 0$, so any compact gradient soliton is q-flat. □

Changing directions slightly, we will show that for a general tensor q with certain scaling properties that a gradient q-soliton is a self similar solution to the q-flow. This observation appears to be made first by Lauret [Lau16]. To do so we will follow the proof from [CLN06, Chapter 4] which shows that gradient Ricci solitons are self-similar solutions to the Ricci flow. Following our proof, we will apply the theorem to the ambient obstruction flow in both the homogeneous and non-homogeneous cases. In [Lau19] and [Lau16], Lauret shows that the following theorem is true for general, non-gradient solitons and can be made into an if and only if statement. We have chosen to focus on the case of gradient solitons. Our goal in including the following proof is to motivate our choice to modify the equation for a soliton by including a factor of $\frac{1}{2}$ and to show a more explicit proof of this theorem.

Theorem 2.1.13. *Consider any tensor q with the property that when the metric is scaled by a constant $\lambda \in \mathbb{R}$:*

$$\tilde{g} = \lambda g \quad \Longrightarrow \quad \tilde{q} = \lambda^{\frac{w}{2}} q.$$

Consider a complete gradient q soliton (M^n, h, f_0, c), that is:

$$\operatorname{Hess}_h f_0 = ch + \frac{1}{2} q(h).$$

There exists an $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$ there is a solution g_t of the q flow with $g_0 = h$, diffeomorphisms φ_t with $\varphi_0 = \mathbb{1}_{M^n}$, and functions $f(t) = f_t$ with $f(0) = f_0$, such that:

1. *τ is scales the metric according to the function:*

$$\tau_t := \begin{cases} e^{1-2ct} & w = 2 \\ \left(1 - 2c\left(1 - \frac{w}{2}\right) t\right)^{\frac{1}{1-\frac{w}{2}}} & w \neq 2, \end{cases}$$

2. *The vector field $X_t := \tau_t^{\frac{w}{2}-1} \nabla_h f_0$ exists,*
3. *$\varphi_t : M^n \to M^n$ is the 1-parameter family of diffeomorphisms generated by X_t. So:*

$$\frac{\partial}{\partial t} \varphi_t(x) = \tau_t^{\frac{w}{2}-1} \left(\nabla_h f_0\right) (\varphi_t(x)),$$

4. *g_t is the pull back by φ_t of h up to the scale factor τ_t:*

$$g_t = \tau_t \varphi_t^* h,$$

5. f_t is the pull back by φ_t of f_0:

$$f_t = f_0 \circ \varphi_t = \varphi_t^*(f_0).$$

Moreover

$$\operatorname{Hess}_{g_t} f_t = \frac{c}{\tau_t} g_t + \frac{1}{2}(q(g_t))$$

or equivalently

$$q(g_t) = -\frac{2c}{\tau_t} g_t + 2 \operatorname{Hess}_{g_t} f_t$$

and

$$\frac{\partial f}{\partial t}(t) = \tau^{\frac{w}{2}} \left| \nabla_{g_t} f_t \right|^2_{g_t}.$$

Proof. Construct a 1-parameter family of diffeomorphisms $\varphi_t : M^n \to M^n$ generated by vector field $X_t = \tau^{\frac{w}{2}-1} \nabla_h f_0$ defined for all t such that $t \in (-\varepsilon, \varepsilon)$. Define $f_t = f_0 \circ \varphi_t$ and $g_t = \tau_t \varphi_t^* h$.

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} g_t = \left.\frac{\partial}{\partial t}\right|_{t=t_0} (\tau_t \varphi_t^* h) = \left(\frac{\partial}{\partial t} \tau_t\right) \varphi_{t_0}^* h + \tau_{t_0} \frac{\partial}{\partial t}\Big|_{t=t_0} \varphi_t^* h.$$

Using Remark 1.24 from [CLN06] we are able to assess the derivative of the pullback:

$$\tau_{t_0} \left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi_t^* h = \tau_{t_0} \mathcal{L}_{Y(t)} \left(\varphi_{t_0}^* h\right) = \mathcal{L}_{Y(t)} \left(\tau_{t_0} \varphi_{t_0}^* h\right),$$

where

$$Y(t) := \left.\frac{\partial}{\partial t}\right|_{t=t_0} \left(\varphi_{t_0}^{-1} \circ \varphi_t\right) = (\varphi_{t_0}^{-1})_* \left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi_t.$$

Note that for $\tilde{g} = \lambda g$:

$$g(\nabla_g f, X) = df(X) = \tilde{g}(\nabla_{\tilde{g}} f, X) = \lambda g(\nabla_{\tilde{g}} f, X).$$

So $\frac{1}{\lambda}\nabla_g f = \nabla_{\tilde{g}} f$. Therefore:

$$\nabla_{g_{t_0}} f_{t_0} = \nabla_{\tau_{t_0}\varphi_{t_0}^* h} f_{t_0} = \frac{1}{\tau_{t_0}}\nabla_{\varphi_{t_0}^* h} f_{t_0} = \frac{1}{\tau_{t_0}}\nabla_{\varphi_{t_0}^* h}\varphi_{t_0}^* f_0 = \frac{1}{\tau_{t_0}}\varphi_{t_0}^*(\nabla_h f_0) = \varphi_{t_0}^*\left(\frac{1}{\tau_{t_0}}\nabla_h f_0\right).$$

Thus

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi_t = \tau_{t_0}^{\frac{w}{2}-1}\nabla_h f_0 = \tau_{t_0}^{\frac{w}{2}}\left(\frac{1}{\tau_{t_0}}\nabla_h f_0\right) = \tau_{t_0}^{\frac{w}{2}}\left((\varphi_{t_0})_*\left(\nabla_{g_{t_0}} f_{t_0}\right)\right).$$

Using this, we are able to evaluate the desired derivative and find one term of our initial sum:

$$\tau_{t_0}\left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi_t^* h = \tau_{t_0}\mathcal{L}_{Y(t)}\left(\varphi_{t_0}^* h\right) = \mathcal{L}_{\tau_{t_0}^{\frac{w}{2}}\nabla_{g_{t_0}} f_{t_0}}\left(\tau_{t_0}\varphi_{t_0}^* h\right) = \tau_{t_0}^{\frac{w}{2}}\mathcal{L}_{\nabla_{g_{t_0}} f_{t_0}} g_{t_0}.$$

To evaluate the derivative of τ we must consider each case.

Case 1. *For $w = 2$ define $\tau_t = e^{1-2ct}$. Then:*

$$\begin{aligned}\left(\frac{\partial}{\partial t}\tau_t\right)\varphi_{t_0}^* h &= -2c\tau\varphi_{t_0}^* h \\ &= -2cg(t_0).\end{aligned}$$

Case 2. *For $w \neq 2$ define $\tau_t = \left(1 - 2c\left(1 - \frac{w}{2}\right)t\right)^{\frac{1}{1-\frac{w}{2}}}$. We can compute the following:*

$$\begin{aligned}\left(\frac{\partial}{\partial t}\tau_t\right)\varphi_{t_0}^* h &= \frac{1}{1-\frac{w}{2}}\left(1 - 2c\left(1-\frac{w}{2}\right)t_0\right)^{\frac{1}{1-\frac{w}{2}}-1}\left(-2c\left(1-\frac{w}{2}\right)\right)\left(\varphi_{t_0}^* h\right) \\ &= -2c\left(1 - 2c\left(1-\frac{w}{2}\right)t_0\right)^{\frac{w/2}{1-\frac{w}{2}}}\left(\varphi_{t_0}^* h\right) \\ &= -2c\tau_{t_0}^{\frac{w}{2}}\left(\frac{\tau_{t_0}\varphi_{t_0}^* h}{\tau_{t_0}}\right) \\ &= -2c\tau_{t_0}^{\frac{w}{2}-1} g(t_0).\end{aligned}$$

Thus we see that for any w,

$$\left(\frac{\partial}{\partial t}\tau_t\right)\varphi_{t_0}^* h = -2c\tau_{t_0}^{\frac{w}{2}-1} g(t_0).$$

Returning to our original derivative, we see that for general t:

$$\begin{aligned}\frac{\partial}{\partial t} g_t =& -2c\tau_{t_0}^{\frac{w}{2}-1} g_t + \tau_{t_0}^{\frac{w}{2}} \mathcal{L}_{\nabla_{g_t} f_t}\, g(t) \\ =& \tau_{t_0}^{\frac{w}{2}} \left(\frac{-2c}{\tau_t} g(t) + 2\nabla^{g_t}\nabla^{g_t} f_t \right).\end{aligned}$$

Applying [CLN06] Exercise 1.23 to q we see:

$$\begin{aligned}q(g_t) &= q(\tau_t \varphi_t^* h) \\ &= \tau_t^{\frac{w}{2}} \varphi_t^*(q(h)) \\ &= \tau_t^{\frac{w}{2}} \varphi_t^* \left(-2ch + 2\operatorname{Hess}_h f_0\right) \\ &= \tau_t^{\frac{w}{2}} \varphi_t^* \left(-2ch + \mathcal{L}_{\nabla_h f_0}\, h\right) \\ &= \tau_t^{\frac{w}{2}} \left(\frac{-2c}{\tau_t} g_t + \mathcal{L}_{\nabla_{g_t} f_t}\, g(t) \right) \\ &= \tau_t^{\frac{w}{2}} \left(\frac{-2c}{\tau_t} g_t + 2\operatorname{Hess}_{g_t} f_t \right) \\ &= \frac{\partial}{\partial t} g_t.\end{aligned}$$

Hence, there exists a solution g_t to the flow with the desired properties.

Looking at the derivative of the potential function we see that:

$$\begin{aligned}\frac{\partial f_t(x)}{\partial t} &= \frac{\partial}{\partial t} f_0(\varphi_t(x)) \\ &= \lim_{\eta \to 0} \frac{f_0(\varphi_{t+\eta}(x)) - f_0(\varphi_t(x))}{\eta} \\ &= h\left(\nabla_h f_0,\ \frac{\partial}{\partial t}\varphi_t \right) \\ &= h\left(\nabla_h f_0,\ \tau^{\frac{w}{2}-1} \nabla_h f_0(\varphi_t(x)) \right) \\ &= \tau^{\frac{w}{2}-1} h\left(\nabla_h f_t, \nabla_h f_t(x) \right) \\ &= \tau^{\frac{w}{2}-1} \frac{1}{\tau} g_t\left(\tau \nabla_{g_t} f_t,\ \tau \nabla_{g_t} f_t(x) \right) \\ &= \tau^{\frac{w}{2}} \left| \nabla_{g_t} f_t \right|_{g_t}^2 .\end{aligned}$$

$\square$

Remark 2.1.14. *If the vector field* $X_t = \tau_t^{\frac{w}{2}-1}\nabla_h f_0$ *is complete then the flow exists for all* t *such that* $\tau_t > 0$*.*

Remark 2.1.15. *One such tensor* q *with the necessary weighting property is a conformally invariant tensor of weight* w*. That is, a tensor* T *such that for* $\tilde{g} = \rho^2 g$*, then* $\tilde{T} = \rho^w T$ *for a smooth positive function* ρ*.*

Corollary 2.1.16. *The gradient solitons of the ambient obstruction flow are self similar solutions to the ambient obstruction flow.*

Proof. Consider the tensor provided by the ambient obstruction flow:

$$\mathcal{O}_n + c_n(-1)^{\frac{n}{2}}\left(\Delta^{\frac{n}{2}-1}S\right)g.$$

We know that the ambient obstruction tensor is of conformal weight $2-n$, and is consequently a tensor q described by Theorem 2.1.13. In the homogeneous case, or more generally the constant scalar curvature case, we are able to directly apply the theorem.

To examine the non-homogeneous case we must also investigate the scaling properties of the scalar curvature term. A simple calculation (shown in Appendix C) shows that for $\tilde{g} = \lambda^2 g$:

$$\tilde{\Delta}\tilde{S}\tilde{g} = \frac{1}{\lambda^2}\Delta S g.$$

Using induction one can show that this generalizes to:

$$\tilde{\Delta}^k\tilde{S}\tilde{g} = \frac{1}{\lambda^{2k}}\Delta^k S g.$$

Thus for $k = \frac{n}{2} - 1$

$$\tilde{\Delta}^{\frac{n}{2}-1}\tilde{S}\tilde{g} = \frac{1}{\lambda^{n-2}}\Delta^k S g = \lambda^{2-n}\Delta^k S g.$$

That is, the scalar curvature term is scaled by a factor of $2-n$ and consequently has the same scaling properties as the ambient obstruction tensor.

Applying Theorem 2.1.13 with $w = 2-n$, we see that this implies that with the appropriate choice of τ and φ a gradient ambient obstruction soliton is a self-similar solution to the ambient obstruction flow. □

As Lauret shows, Corollary 2.1.16 is also true for non-gradient solitons. Turning our attention to noncompact, homogeneous solitons we consider recent theorem of Petersen and Wylie [PW20]. This theorem is a key part of understanding homogeneous gradient Bach solitons as we see in Section 2.2.

Theorem 2.1.17 (Petersen-Wylie)**.** *Let (M, g) be a homogeneous manifold and $\tilde{q}$ an isometry invariant symmetric two-tensor which is divergence-free. If there is a non-constant function such that* $\mathrm{Hess} f = \tilde{q}$ *then (M, g) is a product metric $N \times \mathbb{R}^k$ and f is a function on the Euclidean factor.*

For a divergence-free tensor q, we apply this theorem to homogeneous gradient q solitons by simply letting $\tilde{q} = cg + \frac{1}{2}q$. Then $\tilde{q}$ is the sum of isometry invariant symmetric two-tensors that are divergence-free and is itself such a tensor. Applying this theorem to homogeneous manifolds, we are able limit the ambient obstruction flow to the flow given by (1.16). Since $\mathcal{O}$ is a divergence-free, isometry invariant, symmetric two-tensor, we can let $q = \mathcal{O}_n$ resulting in the following corollary.

Corollary 2.1.18. *If (M, g) is a homogeneous gradient ambient obstruction soliton, then either M is stationary or it splits as a product $\mathbb{R}^k \times N$ and f is a function on the Euclidean factor.*

This theorem informs our approach to classifying homogeneous gradient Bach solitons in the next section.

2.2 Gradient Bach Solitons

In order to examine and classify the gradient solitons of the Bach flow on homogeneous 4-manifolds, we consider the four configurations of homogeneous 4-manifolds that are found by "pulling off copies of $\mathbb{R}$". More explicitly, by Theorem 2.1.17, the solitons will be of the form $\mathbb{R}^4$, $\mathbb{R}^3 \times N^1$, $\mathbb{R}^2 \times N^2$, $\mathbb{R} \times N^3$, or N^4 (where N^k is necessarily homogeneous). The first and last case we will call non-split manifolds, the others may be called the 3×1, 2×2, and 1×3 cases respectively. For each of these cases (and for the remainder of the paper) it will be assumed that the product manifolds are equipped with the appropriate product metric $g = g_0 \times g_N$. Table 1 summarizes our findings regarding each type and thus proves the following general theorem.

Theorem 2.2.1. *Any homogeneous gradient Bach soliton that is steady must be Bach flat and the only non-Bach-flat shrinking solitons are product metrics on $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^2 \times H^2$.*

Remark 2.2.2. *There are non-trivial homogeneous 4-dimensional Bach flat metrics. For example, Einstein metrics and (anti)self-dual metrics are Bach flat. Moreover there is a classification of simply connected homogeneous Bach-flat 4-manifolds. (See [AGS13] and [CnLGMGR$^+$19].)*

Remark 2.2.3. *There are non-Bach-flat expanding homogeneous gradient Bach solitons. We find one such soliton on $\mathbb{R} \times S^3$ with metric $g = g_0 \times g_{SU(2)}$. We show this is the only expanding soliton on a manifold of the form $\mathbb{R} \times N^3$ where N^3 is a unimodular Lie group.*

Setting up the conventions used throughout this section, recall from (1.13) we know that for homogeneous manifolds the equation for a gradient Bach soliton is given by:

$$\operatorname{Hess} f = cg + \frac{1}{2}B,$$

and can be represented in coordinates as:

$$\nabla_i \nabla_j f = c g_{ij} + \frac{1}{2} B_{ij}.$$

In order to make the following proofs more clear, we will consider how the above equation can be given by matrices. In order to do this we will establish conventions that will hold for the remainder of the section unless otherwise noted. We will always choose a basis so both the metric and the Bach tensor are diagonal. (This is always possible, per the spectral theorem.) Since the metric and the Bach tensor are diagonal, Hess f must also be diagonal so $\nabla_i \nabla_j f = 0$ for $i \neq j$. One very important statement in Theorem 2.1.17 is that the potential function depends on only the Euclidean factor of the product manifold. Let $\nabla_i \nabla_i f = f_{ii}$. Thus, in general we see that the gradient Bach solitons can be represented by the following equality:

$$\begin{bmatrix} f_{00} & 0 & 0 & 0 \\ 0 & f_{11} & 0 & 0 \\ 0 & 0 & f_{22} & 0 \\ 0 & 0 & 0 & f_{33} \end{bmatrix} = c \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} B_{00} & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 \\ 0 & 0 & B_{22} & 0 \\ 0 & 0 & 0 & B_{33} \end{bmatrix}.$$

To prove Theorem 2.2.1, we will simply examine each type of manifold and assess the solitons. The following table will summarize this investigation with one notable exception: in the $\mathbb{R} \times N^3$ case we are able to prove that non-Bach-flat gradient solitons must be expanding. It should also be noted that we have not completed the classification of manifolds of type $\mathbb{R} \times N^3$, but this is not necessary to prove Theorem 2.2.1. We expand on our choice of 3-manifolds in Appendix B.

Split	Manifold	Type of Soliton	Permissible Metrics	Potential Function
N^4	$\mathbb{R}^4$	Gaussian	Bach flat (any)	$f(x,y,z,w) = c(x^2+y^2+z^2+w^2)+ax+by+dz+hw+k$
	N^4	Stationary	Bach flat	$f(x,y,z,w) = k$
$\mathbb{R}^3 \times N^1$		Steady	Bach flat (any)	$f(x,y,z) = ax+by+dz+k$
$\mathbb{R}^2 \times N^2$	$\mathbb{R}^2 \times \mathbb{R}^2$	Steady	Bach flat (any)	$f(x,y) = ax+by+d$
	$\mathbb{R}^2 \times S^2$	Shrinking	See [Ho18]	$f(x,y) = c(x^2+y^2)+ax+by+d$
	$\mathbb{R}^2 \times H^2$	Shrinking	See [Ho18]	$f(x,y) = c(x^2+y^2)+ax+by+d$
$\mathbb{R} \times N^3$	$\mathbb{R} \times \mathbb{R}^3$	Steady	Bach flat (any)	$f(x) = ax+b$
	$\mathbb{R} \times Nil$	—	None	—
	$\mathbb{R} \times Solv$	—	None	—
	$\mathbb{R} \times \widehat{SL}(2,\mathbb{R})$	—	None	—
	$\mathbb{R} \times (\mathbb{R} \times H^2)$	—	None	—
	$\mathbb{R} \times (\mathbb{R} \times S^2)$	—	None	—
	$\mathbb{R} \times E(2)$	Steady	Bach flat ($g_{11} = g_{22}$)	$f(x) = ax+b$
	$\mathbb{R} \times H^3$	Steady	Bach flat	$f(x) = ax+b$
	$\mathbb{R} \times S^3$	Steady	Bach flat ($g_{11} = g_{22} = g_{33}$)	$f(x) = ax+b$
		Expanding	$g_{11} = g_{22} = 4g_{33}$	$f(x) = 2cx^2+ax+b$

Table 2.1: Summary of Results

2.2.1 Non-split Manifolds

Theorem 2.2.4. $(\mathbb{R}^4, g_0)$ *is a Gaussian soliton.*

Proof. We know from the equation for the Bach tensor that $(\mathbb{R}^4, g_0)$ is Bach flat, that is, $B_{ij} = 0$ for all $i, j = 0, 1, 2, 3$, so $\operatorname{Hess} f = cg$. By Theorem 2.1.17, f is a function on $\mathbb{R}^4$. Thus for any orthonormal basis, $\mathbb{R}^4$ is a gradient Bach soliton with potential function

$$f(x, y, z, w) = \frac{1}{2}c(x^2 + y^2 + z^2 + w^2) + ax + by + dz + hw + k$$

for $a, b, d, h, k \in \mathbb{R}$.

Since there are no restrictions on c, we see that this is the Gaussian soliton. $\square$

Proposition 2.2.5. *Consider a non-split, homogeneous 4-manifold $N^4 \neq \mathbb{R}^4$ with metric g_N. Then N^4 is a gradient Bach soliton if and only if it is Bach flat.*

Proof. Consider a non-split, homogeneous 4-manifold N^4 with metric g_N. By the converse of Theorem 2.1.17, since N^4 is not a product manifold, it must have constant potential function and is therefore stationary. Since the potential function is constant, $\operatorname{Hess} f = 0$. Consequently, any soliton has the form $-\frac{1}{2}B = cg$. Taking the trace of each side we see that

$$0 = -\frac{1}{2} \operatorname{tr} B = \operatorname{tr} cg = 4c,$$

and so it is necessarily true that $c = 0$ and the soliton is steady.

Since $c = 0$ always, $B = 0$ always and thus the manifold must be Bach flat. $\square$

2.2.2 Manifolds of the form $\mathbb{R}^3 \times N^1$

Remark 2.2.6. *For a homogeneous manifold of the form $\mathbb{R}^3 \times N^1$ with metric $g = g_0 \times g_N$, we know that $N^1 = \mathbb{R}^1$ or S^1. Thus any manifold of this form is flat and consequently Bach flat.*

Proposition 2.2.7. *Homogeneous manifolds of the form $\mathbb{R}^3 \times N^1$ with metric $g = g_0 \times g_N$ are steady gradient Bach solitons with linear potential functions.*

Proof. Consider a homogeneous manifold of the form $\mathbb{R}^3 \times N^1$ with metric $g = g_0 \times g_N$. We know from Remark 2.2.6 that any manifold of this form is Bach flat. So for any gradient Bach soliton $\operatorname{Hess} f = cg$. By Theorem 2.1.17 we know that $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$. So $\nabla_3 \nabla_3 f = 0 = cg_{33}$. Since the metric is positive definite, $c = 0$. Therefore, the gradient Bach solitons are steady.

Consequently $\operatorname{Hess} f = 0$, so $f_{xx} = f_{yy} = f_{zz} = 0$. Thus $f(x, y, z) = ax + by + cz + d$. $\square$

2.2.3 Manifolds of the form $\mathbb{R}^2 \times N^2$

In his 2018 paper, [Ho18], Ho finds homogeneous gradient solitons of the form $\mathbb{R}^2 \times N^2$. Ho proves that both $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^2 \times H^2$ is a nontrivial soliton of the form:

$$\operatorname{Hess} f = B + \frac{1}{12} g$$

for any function f of the form $f(x, y) = \frac{1}{12}(x^2 + y^2) + k$. Note the difference between Ho's definition of a gradient Bach soliton and that of this paper. Ho has chosen to place the metric term on the right hand side of the equation switching the conventions of shrinking/ expanding. We will prove that Ho's examples are the only examples of this type.

Theorem 2.2.8. *If a homogeneous manifold of the form $\mathbb{R}^2 \times N^2$ equipped with product metric $g_0 \times g_N$ is a non-Bach-flat gradient Bach soliton, then it is a shrinking soliton. Fur-*

thermore, the soliton is steady if and only if it is Bach flat.

Proof. Consider a homogeneous manifold of $\mathbb{R}^2 \times N^2$. Using (1.8) where $M^{(1)} = \mathbb{R}^2$, $M^{(2)} = N^2$, $S^{(1)} = 0$ and $S^{(2)} = S_N$ are the respective scalar curvatures, and g_0 and g_N are their respective metrics. Recall that homogeneous 2-manifolds have constant scalar curvature, thus we see that:

$$B_{00} = \frac{1}{12}(S_N)^2 g_{00} \quad B_{11} = \frac{1}{12}(S_N)^2 g_{11} \quad B_{22} = -\frac{1}{12}(S_N)^2 g_{22} \quad B_{33} = -\frac{1}{12}(S_N)^2 g_{33}.$$

Since $\mathbb{R}^2 \times N^2$ is a gradient Bach soliton, the following system must hold.

$$\begin{bmatrix} f_{xx} g_{00} & 0 & 0 & 0 \\ 0 & f_{yy} g_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \left(\frac{1}{24}(S_N)^2 + c\right) g_{00} & 0 & 0 & 0 \\ 0 & \left(\frac{1}{24}(S_N)^2 + c\right) g_{11} & 0 & 0 \\ 0 & 0 & \left(\frac{-1}{24}(S_N)^2 + c\right) g_{22} & 0 \\ 0 & 0 & 0 & \left(\frac{-1}{24}(S_N)^2 + c\right) g_{33} \end{bmatrix}.$$

Thus $0 = \left(\frac{-1}{24}(S_N)^2 + c\right) g_{ii}$ for $i = 2, 3$. Since the metric is positive definite, we know that $c = \frac{1}{24}(S_N)^2$. Thus $c \geqslant 0$ and the soliton must be steady or shrinking.

The soliton is steady if and only if $S_N = 0$ which happens if and only if the manifold is Bach flat.

If the manifold is non-Bach-flat, then $c > 0$ and soliton must be shrinking. □

Scaling S^2 and H^2 so that $S_{S^2} = 1 = -S_{H^2}$, we see that $c = \frac{1}{24}$ and the potential function is of the form $f(x,y) = \frac{1}{24}(x+y)^2 + ax + by + d$. Again, this differs slightly from Ho because of our initial definition of a gradient Bach soliton. This confirms that the gradient solitons found by Ho are in fact the only gradient solitons on $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^2 \times H^2$ up to scaling.

Corollary 2.2.9. *The potential function of a steady gradient Bach soliton of the form* $\mathbb{R}^2 \times N^2$ *equipped with product metric* $g_0 \times g_N$ *must be linear.*

Proof. Since $\mathbb{R}^2 \times N^2$ must be steady, we know that $f_{xx} = f_{yy} = 0$. Recall from the beginning of this section that the Hessian must be diagonal, and consequently $f_{xy} = f_{yx} = 0$. It follows that $f(x,y) = ax + by + k$. □

Corollary 2.2.10. *The manifold* $\mathbb{R}^2 \times \mathbb{R}^2$ *with metric* $g = g_0 \times g_N$*, where* g_N *is a flat metric, is a steady gradient Bach soliton with linear potential function.*

Proof. Consider a homogeneous manifold of $\mathbb{R}^2 \times \mathbb{R}^2$. Using (1.8), we know that $\mathbb{R}^2 \times \mathbb{R}^2$ is Bach flat. By Theorem 2.2.8 we know that the soliton is steady. By Corollary 2.2.9 the potential function must be linear. □

2.2.4 Manifolds of the form $\mathbb{R} \times N^3$

We begin by stating and proving statements that apply to all homogeneous manifolds of the form $\mathbb{R} \times N^3$, then we will examine specific manifolds of this form.

A few notes before stating the theorem. We will look at a potential function $f : \mathbb{R} \to \mathbb{R}$. Since we use x in later computations to mean something else, we have chosen to make f a function of $r \in \mathbb{R}$. Furthermore, note that in this potential function $c \in \mathbb{R}$ is the same c such that $\text{Hess} f = cg + \frac{1}{2}B$. Thus, we have a steady soliton, the potential function necessarily lacks that term.

Lemma 2.2.11. *A homogeneous gradient Bach soliton of the form* $\mathbb{R} \times N^3$ *with metric* $g = g_0 \times g_N$ *has potential function of the form* $f(r) = 2cr^2 + ar + b$ *for* $a, b \in \mathbb{R}$*.*

Proof. Since the manifold is a soliton, we know that $\operatorname{Hess} f = cg + \frac{1}{2}B$. By Theorem 2.1.17 that f is a function on $r \in \mathbb{R}$ and consequently $\operatorname{tr} \operatorname{Hess} f = f''(r)$. Since the Bach tensor is trace-free:

$$\operatorname{tr} \operatorname{Hess} f = \operatorname{tr}(cg) + \operatorname{tr} B \Longrightarrow f''(r) = 4c.$$

Using calculus we see that this implies that $f(r) = 2cr^2 + ar + b$ for $a, b \in \mathbb{R}$. ☐

In order to examine specific manifolds, we will need the following theorem. This theorem enables us to use algebra to determine which metrics will produce solitons.

Theorem 2.2.12. *Consider a homogeneous manifold of the form* $\mathbb{R} \times N^3$ *equipped with metric* $g = g_0 \times g_N$*. The manifold is a gradient Bach soliton if and only if*

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -2c \qquad \textit{for } c \in \mathbb{R}. \tag{2.1}$$

Proof. Consider a manifold of the form $\mathbb{R} \times N^3$ equipped with metric $g = g_0 \times g_N$. Suppose that this manifold is a gradient Bach soliton. Then:

$$\operatorname{Hess} f = cg + \frac{1}{2}B,$$

where $f : \mathbb{R} \to \mathbb{R}$. Examining the components of the flow:

$$\begin{bmatrix} f''g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = c \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} B_{00} & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 \\ 0 & 0 & B_{22} & 0 \\ 0 & 0 & 0 & B_{33} \end{bmatrix}.$$

This system yields the following equalities.

$$f''g_{00} - \frac{1}{2}B_{00} = cg_{00} \qquad -\frac{1}{2}B_{11} = cg_{11} \qquad -\frac{1}{2}B_{22} = cg_{22} \qquad -\frac{1}{2}B_{33} = cg_{33}.$$

It follows that:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -2c \qquad \text{for } c \in \mathbb{R}.$$

Thus the desired equality holds. Furthermore, since $B_{00} = -2cg_{00} + 2f''(r)g_{00}$, by Lemma 2.2.11, $B_{00} = 6cg_{00}$ and consequently $\frac{B_{00}}{g_{00}} = 6c$.

On the other hand, suppose that

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -2c \qquad \text{for } c \in \mathbb{R}.$$

Then $-\frac{1}{2}B_{11} = cg_{11}$, $-\frac{1}{2}B_{22} = cg_{22}$, and $-\frac{1}{2}B_{33} = cg_{33}$. Taking the trace of the Bach tensor:

$$\begin{aligned}
\operatorname{tr} B &= g^{ij}B_{ij} \\
&= g^{00}B_{00} + g^{11}B_{11} + g^{22}B_{22} + g^{33}B_{33} \\
&= g^{00}B_{00} - 2g^{11}cg_{11} - 2g^{22}cg_{22} - 2g^{33}cg_{33} \\
&= g^{00}B_{00} - 6c.
\end{aligned}$$

Since B is trace-free, we see that $B_{00} = 6cg_{00}$. By Lemma 2.2.11 $f''(r) = 4c$, so:

$$f''g_{00} - \frac{1}{2}B_{00} = 4cg_{00} - \frac{1}{2}(6cg_{00}) = cg_{00}.$$

Thus, $\nabla_i\nabla_j f - \frac{1}{2}B_{ij} = cg_{ij}$ for all $i, j = 0, 1, 2, 3$, so $\operatorname{Hess} f = cg + \frac{1}{2}B$. Therefore, $\mathbb{R} \times N^3$ is a gradient Bach soliton. □

This theorem enables a classification of the resulting solitons of the form $\mathbb{R} \times N^3$. We see that we can apply it broadly to the following case to classify Bach-flat homogeneous gradient Bach solitons.

Corollary 2.2.13. *If a homogeneous manifold of the form* $\mathbb{R} \times N^3$ *equipped with metric* $g = g_0 \times g_N$ *is a non-Bach-flat gradient Bach soliton, then it is an expanding soliton. The soliton is steady if and only if it is Bach flat.*

Proof. Consider a manifold of the form $\mathbb{R} \times N^3$ equipped with metric $g = g_0 \times g_N$.

From Theorem 2.2.12 we know that:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -2c.$$

Since the Bach tensor is trace-free we know that:

$$\begin{aligned} -B_{00} &= \frac{B_{11}}{g_{11}} g_{11} + \frac{B_{22}}{g_{22}} g_{22} + \frac{B_{33}}{g_{33}} g_{33} \\ &= -2c(g_{11} + g_{22} + g_{33}) \\ B_{00} &= 2c(g_{11} + g_{22} + g_{33}). \end{aligned}$$

Using (1.7), since S is constant:

$$B_{00} = -\frac{1}{4}\left[(|\operatorname{Ric}|^{(2)})^2 - \frac{1}{3}(S^{(2)})^2 \right] g_{00}.$$

By Cauchy-Schwartz, we know

$$|\operatorname{Ric}^{(2)}|^2 \geqslant \frac{\operatorname{tr}\left(\operatorname{Ric}^{(2)}\right)}{3} = \frac{1}{3}(S^{(2)})^2,$$

and thus $B_{00} \leqslant 0$. Since the metric is positive definite, this implies $c \leqslant 0$, where $c = 0$ if and only if $B_{00} = 0$. By definition a soliton is expanding if $c < 0$.

If $c = 0$, $B_{00} = 0$ then:

$$\begin{bmatrix} f_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 \\ 0 & 0 & B_{22} & 0 \\ 0 & 0 & 0 & B_{33} \end{bmatrix}.$$

Clearly, this implies that $B_{ii} = 0$ for $i = 1, 2, 3$. Thus, if the soliton is steady, the manifold is Bach flat.

If the soliton is Bach flat then $\operatorname{Hess} f = cg$, so $0 = cg_{ii}$ for $i = 1, 2, 3$ so $c = 0$ and the soliton is steady. □

Remark 2.2.14. *In [Hel20, Proposition 2.2], Helliwell notes that g_{00} is static if and only if the manifold is Bach flat. Moreover, if this is not the case then g_{00} is strictly decreasing. This seems to contradict the condition that a soliton be expanding.*

Recall that rescaling is a diffeomorphism of $\mathbb{R}$. That is, contracting is the same as stretching after diffeomorphism. Thus, we see that though $\frac{\partial}{\partial t} g_{00} < 0$ under the Bach flow, our soliton $\mathbb{R} \times N^3$ can still be classified as expanding.

In order to use this theorem to find metrics that produce solitons, we will need explicit representations of the Bach tensor. These can be found using (1.7) with $M^{(1)} = \mathbb{R}$ and $M^{(2)} = N^3$. The Bach tensor for solitons of the form $\mathbb{R} \times N^3$ where N^3 3-dimensional unimodular Lie group is given in [Hel20]. In Appendix B we provide more background on the choice of manifolds and their affiliated structure constants, present the equations for calculating the Bach tensor in terms of those structure constants, and demonstrate how one would calculate the components of the Bach tensor.

We begin investigating manifolds of the form $\mathbb{R} \times N^3$ by examining the covering spaces for the nine manifolds with compact quotient. The qualitative behavior of the compact quotients is examined in [Hel20]. The gradient solitons of the compact quotients themselves are easily classified by Corollary 2.1.7. We, however, are interested in the solitons on the covering spaces themselves. As such, we will examine the 9 manifolds in [Hel20] to see if there is a metric that produces a gradient Bach solitons. The Lie groups with compact quotient are given by the unimodular, solvable Bianchi classes. That is, Bianchi classes I, II, VI_0, VII_0, VIII, and IX. There are three additional cases which are not Lie groups, but have compact quotient.

By Theorem 2.2.12 we need only show that a metric satisfies (2.1). If there are no metrics that satisfy the string of equalities, then the manifold produces no solitons. The general methodology is to use the explicit representation for the Bach tensor in the above equality, then see what conditions must be placed on the metric to produce a soliton. We show how to find an explicit representation for the Bach tensor in Appendix B and work through the example of $\mathbb{R} \times Nil$.

For ease of notation in calculations, we will let:

$$x = g_{11}, \quad y = g_{22}, \quad z = g_{33}, \quad \beta = \frac{1}{6(\det g)^2}.$$

To clarify the consequences of each example, the metric notations will be used. These proofs heavily rely on the fact that Riemannain metrics are positive definite. That is, $g_{ii} > 0$ is a strict inequality. This allows us the use the quotients in (2.1) and to rule out potential solitons. A summary of our results is as follows. The proofs will be in subsequent sections.

Theorem 2.2.15. *For a homogeneous manifold of type $M = \mathbb{R}^1 \times N^3$ equipped with the metric $g = g_0 \times g_N$ the following hold:*

a. If $N^3 = \mathbb{R}^3$, then a metric $g = g_0 \times g_N$, where g_N is a flat metric, produces a gradient Bach soliton with linear potential function.

b. If $N^3 = Nil,\ Solv,\ \widetilde{SL}(2,\mathbb{R}),\ \mathbb{R} \times S^2,\ \mathbb{R} \times H^2$ then g is not a gradient Bach soliton

c. *If* $N^3 = E(2), H^3$*, then* g *produces a Bach soliton if and only if it is Bach flat.*

d. *If* $N^3 = S^3$*, then a gradient Bach soliton is produced if and only if the metric is of the from* $g_{11} = g_{22} = g_{33}$ *or if it is isometric to* $g_{11} = g_{22} = 4g_{33}$*. These solitons are categorized in Theorems 2.2.26 and 2.2.28 respectively.*

$\mathbb{R} \times \mathbb{R}^3$

Proposition 2.2.16. *The manifold* $\mathbb{R} \times \mathbb{R}^3$ *with metric* $g = g_0 \times g_N$*, where* g_N *is a flat metric, is a gradient Bach soliton with potential function* $f(r) = ar + b$ *or some* $a \in \mathbb{R}$*.*

Proof. We know from (1.7) that $B_{ii} = 0$ for $i = 0, 1, 2, 3$. By Corollary 2.2.13 we know that the soliton is steady, so $c = 0$. So by Lemma 2.2.11 $f(r) = ar + b$ for $a, b \in \mathbb{R}$. □

$\mathbb{R} \times Nil$

We know from [Hel20]

$$B_{00} = -\beta(g_{00})^3(g_{11})^4 \qquad B_{11} = -5\beta(g_{00})^2(g_{11})^5$$
$$B_{22} = 3\beta(g_{00})^2(g_{11})^4 g_{22} \qquad B_{33} = 3\beta(g_{00})^2(g_{11})^4 g_{33}.$$

Proposition 2.2.17. *The manifold* $\mathbb{R} \times Nil$ *with metric* $g = g_0 \times g_{Nil}$ *is not a gradient Bach soliton.*

Proof. Proceeding by contradiction, suppose $\mathbb{R} \times Nil$ with metric $g = g_0 \times g_{Nil}$ is a gradient Bach soliton. Then using (2.1) we see that:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} \quad \Longrightarrow \quad -5\beta(g_{00})^2(g_{11})^4 = 3\beta(g_{00})^2(g_{11})^4.$$

However, this implies that $-5 = 3$. Thus $\mathbb{R} \times Nil$ is not a gradient Bach soliton. □

$\mathbb{R} \times Solv$

We know from [Hel20]

$$B_{00} = -\beta p(g_{11}, g_{22})(g_{00})^3 \qquad B_{11} = -\beta q(g_{11}, g_{22})(g_{00})^2 g_{11}$$
$$B_{22} = -\beta q(g_{22}, g_{11})(g_{00})^2 g_{22} \qquad B_{33} = 3\beta p(g_{11}, g_{22})(g_{00})^2 g_{33}$$

where

$$p(x,y) = x^4 + x^3y + xy^3 + y^4 \qquad q(x,y) = 5x^4 + 3x^3y - xy^3 - 3y^4.$$

Proposition 2.2.18. *The manifold $\mathbb{R} \times Solv$ with metric $g = g_0 \times g_{Solv}$ is not a gradient Bach soliton.*

Proof. Proceeding by contradiction, suppose $\mathbb{R} \times Solv$ with metric $g = g_0 \times g_{Solv}$ is a gradient Bach soliton. Using (2.1) we see that:

$$\frac{B_{11}}{g_{11}} = \frac{B_{33}}{g_{33}} \quad \Longrightarrow \quad -\beta\, q(g_{11}, g_{22})(g_{00})^2 = 3\beta\, p(g_{11}, g_{22})(g_{00})^2.$$

Letting $x = g_{11}$ and $y = g_{22}$:

$$\begin{aligned} -q(x,y) &= 3p(x,y) \\ -(5x^4 + 3x^3y - xy^3 - 3y^4) &= 3(x^4 + x^3y + xy^3 + y^4) \\ -5x^4 - 3x^3y + xy^3 + 3y^4 &= 3x^4 + 3x^3y + 3xy^3 + 3y^4 \\ -8x^4 - 6x^3y - 2xy^3 &= 0 \\ -2x(4x^3 + 6x^2y + y^3) &= 0 \\ x = 0 \quad \text{or} \quad 4x^3 + 6x^2y + y^3 &= 0. \end{aligned}$$

Then either $x = 0$ or $4x^3 + 6x^2y + y^3 = 0$. The first statement is not possible because the metric is positive definite. The latter statement holds if and only if $x = y = 0$ forcing either

$g_{11} = 0$ or $g_{11} = g_{22} = 0$, contradicting positive definiteness. Thus $\mathbb{R} \times Solv$ is not a gradient Bach soliton. $\square$

$\mathbb{R} \times \widehat{SL}(2, \mathbb{R})$

We know from [Hel20]

$$B_{00} = -\beta p(-g_{11}, g_{22}, g_{33})(g_{00})^3 \qquad B_{11} = -\beta q(-g_{11}, g_{22}, g_{33})(g_{00})^2 g_{11}$$
$$B_{22} = -\beta q(g_{22}, -g_{11}, g_{33})(g_{00})^2 g_{22} \qquad B_{33} = -\beta q(g_{33}, -g_{11}, g_{22})(g_{00})^2 g_{33}$$

where

$$p(x, y, z) = x^4 - x^3(y + z) + x^2yz + x(-y^3 + y^2z + yz^2 - z^3) + y^4 - y^3z - yz^3 + z^4$$
$$q(x, y, z) = 5x^4 - 3x^3(y + z) + x^2yz + x(y^3 - y^2z - yz^2 + z^3) - 3y^4 + 3y^3z + 3yz^3 - 3z^4.$$

Proposition 2.2.19. *The manifold $\mathbb{R} \times \widehat{SL}(2, \mathbb{R})$ with metric $g = g_0 \times g_{\widehat{SL}(2,\mathbb{R})}$ cannot be a gradient Bach soliton.*

Proof. Proceeding by contradiction, suppose $\mathbb{R} \times \widehat{SL}(2, \mathbb{R})$ with metric $g = g_0 \times g_{\widehat{SL}(2,\mathbb{R})}$ is a gradient Bach soliton. Using (2.1) we see that:

$$\frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}}$$
$$-\beta\, q(g_{22}, -g_{11}, g_{33})(g_{00})^2 = -\beta\, q(g_{33}, -g_{11}, g_{22})(g_{00})^2$$
$$q(y, -x, z) = q(z, -x, y)$$
$$\begin{array}{c} 5y^4 - 3y^3(-x + z) - y^2xz + y(-x^3 - x^2z \\ +xz^2 + z^3) - 3x^4 - 3x^3z - 3xz^3 - 3z^4 \end{array} = \begin{array}{c} 5z^4 - 3z^3(-x + z) - z^2xy + z(-x^3 - x^2y \\ +xy^2 + y^3) - 3x^4 - 3x^3y - 3xy^3 - 3y^4 \end{array}$$

$$\begin{aligned}
5y^4 + 3xy^3 - 3y^3z - xy^2z - x^3y - x^2yz \\ +xyz^2 + yz^3 - 3x^4 - 3x^3z - 3xz^3 - 3z^4 &= \begin{aligned} 5z^4 + 3xz^3 - 3yz^3 - xyz^2 - x^3z - x^2yz \\ +xzy^2 + y^3z - 3x^4 - 3x^3y - 3xy^3 - 3y^4 \end{aligned} \\
8y^4 + 6xy^3 - 4y^3z - 2xy^2z + 2x^3y \\ +2xyz^2 + 4yz^3 - 2x^3z - 6xz^3 - 8z^4 &= 0 \\
2(y-z)(x^3 + 3xy^2 + 2xyz + 3xz^2 \\ +4y^3 + 2y^2z + 2yz^2 + 4z^3) &= 0.
\end{aligned}$$

The only potential real solution is that $y = z$. As above, because the metric is positive definite, the last term in the product is nonzero. Examining the consequences of this using the other equations in (2.1) we see that the following must hold.

$$\begin{aligned}
\frac{B_{11}}{g_{11}} &= \frac{B_{22}}{g_{22}} \\
-\beta\, q(-g_{11}, g_{22}, g_{33})(g_{00})^2 &= -\beta\, q(g_{22}, -g_{11}, g_{33})(g_{00})^2 \\
q(-x, y, z) &= q(y, -x, z) \\
5x^4 + 3x^3(y+z) + x^2yz - x(y^3 - y^2z \\ -yz^2 + z^3) - 3y^4 + 3y^3z + 3yz^3 - 3z^4 &= \begin{aligned} 5y^4 - 3y^3(-x+z) - y^2xz + y(-x^3 - x^2z \\ +xz^2 + z^3) - 3x^4 - 3x^3z - 3xz^3 - 3z^4 \end{aligned} \\
5x^4 + 3x^3y + 3x^3z + x^2yz - xy^3 + xy^2z \\ +xyz^2 - xz^3 - 3y^4 + 3y^3z + 3yz^3 - 3z^4 &= \begin{aligned} 5y^4 + 3xy^3 - 3y^3z - xy^2z - x^3y - x^2yz \\ +xyz^2 + yz^3 - 3x^4 - 3x^3z - 3xz^3 - 3z^4 \end{aligned} \\
8x^4 + 4x^3y + 6x^3z + 2x^2yz - 4xy^3 + 2xy^2z \\ +2xz^3 - 8y^4 + 6y^3z + 2yz^3 &= 0.
\end{aligned}$$

However, if $y = z$ then:

$$\begin{aligned}
8x^4 + 4x^3y + 6x^3z + 2x^2yz - 4xy^3 \\ +2xy^2z + 2xz^3 - 8y^4 + 6y^3z + 2yz^3 &= \begin{aligned} 8x^4 + 4x^3y + 6x^3y + 2x^2y^2 - 4xy^3 \\ +2xy^3 + 2xy^3 - 8y^4 + 6y^4 + 2y^4 \end{aligned} \\
&= 8x^4 + 10x^3y + 2x^2y^2 \\
&\neq 0.
\end{aligned}$$

Therefore if $y = z$, then $B_{11} \,/\, g_{11} \neq B_{22} \,/\, g_{22}$. Thus $y \neq z$. Therefore, $\mathbb{R} \times \widehat{SL}(2, \mathbb{R})$ is not a gradient Bach soliton. $\square$

$\mathbb{R} \times (\mathbb{R} \times S^2)$

Proposition 2.2.20. *There are no gradient Bach solitons on* $\mathbb{R} \times (\mathbb{R} \times S^2)$ *with metric* $g = g_0 \times (g_{\mathbb{R}} \times g_{S^2})$.

Proof. Consider the manifold $\mathbb{R} \times (\mathbb{R} \times S^2)$ with metric $g = g_0 \times (g_{\mathbb{R}} \times g_{S^2})$. Rescaling the sphere to have scalar curvature $S_{S^2} = 1$, from Theorem 2.2.8 we know:

$$B_{00} = \frac{1}{12} g_{00} \qquad B_{11} = \frac{1}{12} g_{11} \qquad B_{22} = -\frac{1}{12} g_{22} \qquad B_{33} = -\frac{1}{12} g_{33}.$$

This contradicts Theorem 2.2.12. Therefore, there are no gradient Bach solitons on $\mathbb{R} \times (\mathbb{R} \times S^2)$ with potential function on $\mathbb{R}$.

$\square$

$\mathbb{R} \times (\mathbb{R} \times H^2)$

Proposition 2.2.21. *There are no gradient Bach solitons on* $\mathbb{R} \times (\mathbb{R} \times H^2)$ *with metric* $g = g_0 \times (g_{\mathbb{R}} \times g_{H^2})$.

Proof. Rescaling the H^2 to have scalar curvature $S_{H^2} = -1$, from Theorem 2.2.8 we know:

$$B_{00} = \frac{1}{12} g_{00} \qquad B_{11} = \frac{1}{12} g_{11} \qquad B_{22} = -\frac{1}{12} g_{22} \qquad B_{33} = -\frac{1}{12} g_{33},$$

and thus the proof follows exactly as in the proof for $\mathbb{R} \times \mathbb{R} \times S^2$ above. $\square$

$\mathbb{R} \times E(2)$

We know from [Hel20]

$$B_{00} = -\beta p(-g_{11}, g_{22})(g_{00})^3 \qquad B_{11} = -\beta q(-g_{11}, g_{22})(g_{00})^2 g_{11}$$
$$B_{22} = -\beta q(g_{22}, -g_{11})(g_{00})^2 g_{22} \qquad B_{33} = 3\beta p(-g_{11}, g_{22})(g_{00})^2 g_{33}$$

where $p(x, y)$ and $q(x, y)$ are as above.

Proposition 2.2.22. *The manifold* $\mathbb{R} \times E(2)$ *with metric* $g = g_0 \times g_{E(2)}$ *is a gradient Bach soliton if and only if it is Bach flat.*

Proof. Consider the manifold $\mathbb{R} \times E(2)$ with metric $g = g_0 \times g_{E(2)}$. Using (2.1) we see that:

$$\begin{aligned}
\frac{B_{11}}{g_{11}} &= \frac{B_{22}}{g_{22}} \\
-\beta\, q(-g_{11}, g_{22})(g_{00})^2 &= -\beta\, q(g_{22}, -g_{11})(g_{00})^2 \\
q(-x, y) &= q(y, -x) \\
5x^4 - 3x^3y + xy^3 - 3y^4 &= 5y^4 - 3y^3x + yx^3 - 3x^4 \\
8x^4 - 4x^3y + 4xy^3 - 8y^4 &= 0 \\
2x^4 - x^3y + xy^3 - 2y^4 &= 0 \\
(x - y)(x + y)(2x^2 - xy + 2y^2) &= 0.
\end{aligned}$$

The only two real, nonzero solutions are that $x = y$ or $x = -y$. Since our metric is positive definite $x \neq -y$. Thus $x = y$ is the only candidate. Proceeding, we will see that the equalities from (2.1) are satisfied if and only if $x = y$.

$$\frac{B_{11}}{g_{11}} = \frac{B_{33}}{g_{33}}$$

$$
\begin{aligned}
-\beta\, q(-g_{11}, g_{22})(g_{00})^2 &= 3\beta\, p(-g_{11}, g_{22})(g_{00})^2 \\
-q(-x, y) &= 3p(-x, y) \\
-(5x^4 - 3x^3 y + xy^3 - 3y^4) &= 3(x^4 - x^3 y - xy^3 + y^4) \\
-5x^4 + 3x^3 y - xy^3 + 3y^4) &= 3x^4 - 3x^3 y - 3xy^3 + 3y^4 \\
-8x^4 + 6x^3 y + 2xy^3 &= 0 \\
-2x(4x^3 - 3x^2 y - y^3) &= 0.
\end{aligned}
$$

Since $x \neq 0$, $4x^3 - 3x^2 y - y^3 = 0$. We see that $x = y$ holds.

$$
\begin{aligned}
\frac{B_{22}}{g_{22}} &= \frac{B_{33}}{g_{33}} \\
-\beta\, q(g_{22}, -g_{11})(g_{00})^2 &= 3\beta\, p(-g_{11}, g_{22})(g_{00})^2 \\
-q(y, -x) &= 3p(-x, y) \\
-(5y^4 - 3y^3 x + yx^3 - 3x^4) &= 3(x^4 - x^3 y - xy^3 + y^4) \\
-5y^4 + 3xy^3 - x^3 y + 3x^4 &= 3x^4 - 3x^3 y - 3xy^3 + 3y^4 \\
-8y^4 + 6xy^3 + 2x^3 y &= 0 \\
-2y(4y^3 - 3xy^2 - x^3) &= 0.
\end{aligned}
$$

Since $y \neq 0$, $4y^3 - 3xy^2 - 2x^3 = 0$. Again, we see that $x = y$ holds.

Thus, $g_{11} = g_{22}$. This condition is equivalent to being Bach flat by the following lemma. Therefore, by Theorem 2.2.12 and Lemma 2.2.23, $\mathbb{R} \times E(2)$ is a gradient Bach soliton if and only if it is Bach flat. □

Lemma 2.2.23. *The manifold* $\mathbb{R} \times E(2)$ *with metric* $g = g_0 \times g_{E(2)}$ *is Bach flat if and only if* $g_{11} = g_{22}$.

Proof. Factoring the components of the Bach tensor for $\mathbb{R} \times E(2)$:

$$B_{00} = -\beta\,(g_{11} - g_{22})^2\left((g_{11})^2 + g_{11}g_{22} + (g_{22})^2\right)(g_{00})^3,$$

$$B_{11} = -\beta\,(g_{11} - g_{22})\left(5(g_{11})^3 + 2(g_{11})^2(g_{22}) + 2(g_{11})(g_{22})^2 + 3(g_{22})^3\right)(g_{00})^2 g_{11},$$

$$B_{22} = -\beta\,(g_{22} - g_{11})\left(3(g_{11})^3 + 2(g_{11})^2(g_{22}) + 2(g_{11})(g_{22})^2 + 3(g_{22})^3\right)(g_{00})^2 g_{22},$$

$$B_{33} = 3\beta\,(g_{11} - g_{22})^2\left((g_{11})^2 + g_{11}g_{22} + (g_{22})^2\right)(g_{00})^2 g_{11}.$$

Since our metric is positive definite $B_{ii} = 0$ if and only if $g_{11} - g_{22} = 0$ if and only if $g_{11} = g_{22}$. □

$\mathbb{R} \times H^3$

Proposition 2.2.24. *The manifold $\mathbb{R} \times H^3$ with metric $g = g_0 \times g_{H^3}$ is the trivial gradient Bach soliton. That is, $\mathbb{R} \times H^3$ is a Bach soliton if and only if it is Bach-flat.*

Proof. Following the explanation from [Hel20], we know that H^3 is a one parameter family of homogeneous metrics. Consequently all metrics are Einstein since they are scalar multiples of the standard metric. Thus, as Helliwell concludes, the flat metric remains flat in the Bach flow. Therefore, the Bach flat metric produces a gradient soliton. □

$\mathbb{R} \times S^3$

Before delving into this case, it is important that the reader note that I mean S^3 to be synonymous with $SU(2)$. That is, the manifold does NOT necessarily have the round metric, but rather has any left invariant metric on Lie group $SU(2)$. My choice to call this S^3 was motivated by wanting to maintain consistency between the cases presented by Helliwell in [Hel20] and this paper.

We know from [Hel20]

$$B_{00} = -\beta\ p(g_{11}, g_{22}, g_{33})(g_{00})^3 \qquad B_{11} = -\beta\ q(g_{11}, g_{22}, g_{33})(g_{00})^2 g_{11}$$

$$B_{22} = -\beta\ q(g_{22}, g_{33}, g_{11})(g_{00})^2 g_{22} \qquad B_{33} = -\beta\ q(g_{33}, g_{11}, g_{22})(g_{00})^2 g_{33}$$

where

$$p(x, y, z) = x^4 - x^3(y + z) + x^2yz + x(-y^3 + y^2z + yz^2 - z^3) + y^4 - y^3z - yz^3 + z^4,$$

$$q(x, y, z) = 5x^4 - 3x^3(y + z) + x^2yz + x(y^3 - y^2z - yz^2 + z^3) - 3y^4 + 3y^3z + 3yz^3 - 3z^4.$$

Proposition 2.2.25. *The manifold $\mathbb{R} \times S^3$ with metric $g = g_0 \times g_{SU(2)}$ is a gradient Bach soliton if and only if our metric is $g_{11} = g_{22} = g_{33}$ or if it is isometric to $g_{11} = g_{22} = 4g_{33}$.*

Proof. Proceeding, consider $\mathbb{R} \times S^3$ with metric $g = g_0 \times g_{SU(2)}$. We will show that the (2.1) holds if and only if $x = y = z$, $x = y = 4z$, $x = 4y = z$, or $4x = y = z$.

We will first consider that case where $x = y = z$:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -\beta\ q(g_{11}, g_{11}, g_{11})(g_{00})^2.$$

This clearly satisfies (2.1).

Proceeding to examine the equalities in general we see that:

$$
\begin{aligned}
\frac{B_{11}}{g_{11}} &= \frac{B_{22}}{g_{22}} \\
-\beta\, q(g_{11}, g_{22}, g_{33})(g_{00})^2 &= -\beta\, q(g_{22}, g_{33}, g_{11})(g_{00})^2 \\
q(x,y,z) &= q(y,z,x) \\
\begin{matrix} 5x^4 - 3x^3(y+z) + x^2yz + x(y^3 - y^2z \\ -yz^2 + z^3) - 3y^4 + 3y^3z + 3yz^3 - 3z^4 \end{matrix} &= \begin{matrix} 5y^4 - 3y^3(z+x) + y^2zx + y(z^3 - z^2x \\ -zx^2 + x^3) - 3z^4 + 3z^3x + 3zx^3 - 3x^4 \end{matrix} \\
\begin{matrix} 5x^4 - 3x^3y - 3x^3z + x^2yz + xy^3 - xy^2z \\ -xyz^2 + xz^3 - 3y^4 + 3y^3z + 3yz^3 - 3z^4 \end{matrix} &= \begin{matrix} 5y^4 - 3y^3z - 3xy^3 + xy^2z + yz^3 - xyz^2 \\ -x^2yz + x^3y - 3z^4 + 3xz^3 + 3x^3z - 3x^4 \end{matrix} \\
\begin{matrix} 8x^4 - 4x^3y - 6x^3z + 2x^2yz + 4xy^3 - 2xy^2z \\ -2xz^3 - 8y^4 + 6y^3z + 2yz^3 \end{matrix} &= 0 \\
\begin{matrix} 2(x-y)(4x^3 + 2x^2y - 3x^2z + 2xy^2 \\ -2xyz + 4y^3 - 3y^2z - z^3) \end{matrix} &= 0.
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\frac{B_{11}}{g_{11}} &= \frac{B_{33}}{g_{33}} \\
-\beta\, q(g_{11}, g_{22}, g_{33})(g_{00})^2 &= -\beta\, q(g_{33}, g_{11}, g_{22})(g_{00})^2 \\
q(x,y,z) &= q(y,z,x) \\
\begin{matrix} 5x^4 - 3x^3(y+z) + x^2yz + x(y^3 - y^2z \\ -yz^2 + z^3) - 3y^4 + 3y^3z + 3yz^3 - 3z^4 \end{matrix} &= \begin{matrix} 5z^4 - 3z^3(x+y) + z^2xy + z(x^3 - x^2y \\ -xy^2 + y^3) - 3x^4 + 3x^3y + 3xy^3 - 3y^4 \end{matrix} \\
\begin{matrix} 5x^4 - 3x^3y - 3x^3z + x^2yz + xy^3 - xy^2z \\ -xyz^2 + xz^3 - 3y^4 + 3y^3z + 3yz^3 - 3z^4 \end{matrix} &= \begin{matrix} 5z^4 - 3xz^3 - 3yz^3 + xyz^2 + x^3z - x^2yz \\ -xy^2z + y^3z - 3x^4 + 3x^3y + 3xy^3 - 3y^4 \end{matrix} \\
\begin{matrix} 8x^4 - 6x^3y - 4x^3z + 2x^2yz - 2xy^3 \\ -2xyz^2 + 4xz^3 + 2y^3z + 6yz^3 - 8z^4 \end{matrix} &= 0 \\
\begin{matrix} 2(x-z)(4x^3 - 3x^2y + 2x^2z - 2xyz \\ +2xz^2 - y^3 - 3yz^2 + 4z^3) \end{matrix} &= 0.
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\frac{B_{22}}{g_{22}} &= \frac{B_{33}}{g_{33}} \\
-\beta\, q(g_{22}, g_{33}, g_{11})(g_{00})^2 &= -\beta\, q(g_{33}, g_{11}, g_{22})(g_{00})^2 \\
q(y,z,x) &= q(y,z,x) \\
\begin{matrix} 5y^4 - 3y^3(z+x) + y^2zx + y(z^3 - z^2x \\ -zx^2 + x^3) - 3z^4 + 3z^3x + 3zx^3 - 3x^4 \end{matrix} &= \begin{matrix} 5z^4 - 3z^3(x+y) + z^2xy + z(x^3 - x^2y \\ -xy^2 + y^3) - 3x^4 + 3x^3y + 3xy^3 - 3y^4 \end{matrix} \\
\begin{matrix} 5y^4 - 3y^3z - 3xy^3 + xy^2z + yz^3 - xyz^2 \\ -x^2yz + x^3y - 3z^4 + 3xz^3 + 3x^3z - 3x^4 \end{matrix} &= \begin{matrix} 5z^4 - 3xz^3 - 3yz^3 + xyz^2 + x^3z - x^2yz \\ -xy^2z + y^3z - 3x^4 + 3x^3y + 3xy^3 - 3y^4 \end{matrix} \\
\begin{matrix} 8y^4 - 4y^3z - 6xy^3 + 2xy^2z + 4yz^3 - 2xyz^2 \\ -2x^3y - 8z^4 + 6xz^3 + 2x^3z \end{matrix} &= 0 \\
\begin{matrix} -2(y-z)(x^3 + 3xy^2 + 2xyz + 3xz^2 \\ -4y^3 - 2y^2z - 2yz^2 - 4z^3) \end{matrix} &= 0.
\end{aligned} \tag{2.4}
$$

Case 1. *Suppose that $x = y$. Then (2.2) is satisfied. Moreover this means that in order for (2.3) to be satisfied:*

$$
\begin{aligned}
0 &= 4x^3 - 3x^3 + 2x^2z - 2x^2z + 2xz^2 - x^3 - 3xz^2 + 4z^3 \\
&= 4z^3 - xz^2 \\
&= z^2(4z - x)
\end{aligned}
$$

Consequently $x = 4z$. We see that this equality not only holds in 2.4, but is forced:

$$
\begin{aligned}
0 &= x^3 + 3x^3 + 2x^2z + 3xz^2 - 4x^3 - 2x^2z - 2xz^2 - 4z^3 \\
&= xz^2 - 4z^3 \\
&= z^2(x - 4z).
\end{aligned}
$$

Thus $x = y = 4z$ maintains all three equalities.

Case 2. *Suppose that* $x = z$*. Then (2.3) is satisfied. Moreover this means that in order for (2.2) to be satisfied:*

$$\begin{aligned}
0 &= 4x^3 + 2x^2y - 3x^3 + 2xy^2 - 2x^2y + 4y^3 - 3y^2x - x^3 \\
&= 4y^3 - y^2x \\
&= y^2(4y - x).
\end{aligned}$$

Consequently $x = 4y$*. We see that this equality not only holds in (2.4), but is forced:*

$$\begin{aligned}
0 &= x^3 + 3xy^2 + 2x^2y + 3x^3 - 4y^3 - 2xy^2 - 2x^2y - 4x^3 \\
&= xy^2 - 4y^3 \\
&= y^2(x - 4y).
\end{aligned}$$

Thus $x = 4y = z$ *maintains all three equalities.*

Case 3. *Suppose that* $y = z$*. Then (2.4) is satisfied. Moreover this means that in order for (2.2) to be satisfied:*

$$\begin{aligned}
0 &= 4x^3 + 2x^2y - 3x^2y + 2xy^2 - 2xy^2 + 4y^3 - 3y^3 - y^3 \\
&= 4x^3 - y^2x \\
&= x^2(4x - y).
\end{aligned}$$

Consequently $4x = y$*. We see that this equality not only holds in (2.4), but is forced:*

$$\begin{aligned}
0 &= 4x^3 - 3x^2y + 2x^2y - 2xy^2 + 2xy^2 - y^3 - 3y^3 + 4y^3 \\
&= 4x^3 - x^2y \\
&= x^2(4x - y).
\end{aligned}$$

Thus $4x = y = z$ maintains all three equalities.

Case 4. *Suppose that $x \neq y$, $x \neq z$, $y \neq z$. Then only other permissible metric would need to satisfy the system of equations:*

$$\begin{cases} 4x^3 + 2x^2y - 3x^2z + 2xy^2 - 2xyz + 4y^3 - 3y^2z - z^3 = 0, \\ 4x^3 - 3x^2y + 2x^2z - 2xyz + 2xz^2 - y^3 - 3yz^2 + 4z^3 = 0, \\ x^3 + 3xy^2 + 2xyz + 3xz^2 - 4y^3 - 2y^2z - 2yz^2 - 4z^3 = 0. \end{cases}$$

Subtracting the first equation from the second yields:

$$5x^2y - 5x^2z + 2xy^2 - 2xz^2 + 5y^3 - 3y^2z + 3yz^2 - 5z^3 = 0$$
$$(y - z)(5x^2 + 2xy + 2xz + 5y^2 + 2yz + 5z^2) = 0.$$

Thus $y = z$ contradicting the original assertion. Moreover, the metric is positive definite. Thus, this case yields no potential metrics.

Therefore, by Theorem 2.2.12, $\mathbb{R} \times S^3$ is a Bach soliton if and only if $g_{11} = g_{22} = g_{33}$, $g_{11} = g_{22} = 4g_{33}$, $g_{11} = 4g_{22} = g_{33}$, or $4g_{11} = g_{22} = g_{33}$. □

Theorem 2.2.26. *If $g_{11} = g_{22} = g_{33}$ then the soliton produced by $\mathbb{R} \times S^3$ is Bach flat and steady.*

Proof. Suppose $g_{11} = g_{22} = g_{33}$. We know by Theorem 2.2.25 that this is the metric of a soliton on $\mathbb{R} \times S^3$. Then:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -\beta\, q(g_{11}, g_{11}, g_{11})(g_{00})^2 = -\beta(0)(g_{00})^2 = 0.$$

Thus $c = 0$, so the soliton is steady.

Moreover, since

$$p(x,x,x) = x^4 - x^3(2x) + x^4 + x(-x^3 + x^3 + x^3 - x^3) + x^4 - x^4 - x^4 + x^4 = 0$$

$$q(x,x,x) = 5x^4 - 3x^3(2x) + x^4 + x(x^3 - x^3 - x^3 + x^3) - 3x^4 + 3x^4 + 3x^4 - 3x^4 = 0.$$

We know that $B_{ii} = 0$ for all $i = 0, 1, 2, 3$. Therefore the metric is Bach flat. □

Remark 2.2.27. *Note that in the previous proof, one could have referenced Corollary 2.2.13 instead of calculating the Bach tensor. The calculation was included to demonstrate an alternate method using known components of the Bach tensor.*

Theorem 2.2.28. *If $g_{11} = g_{22} = 4g_{33}$ then the soliton produced by $\mathbb{R} \times S^3$ is expanding and immortal.*

Proof. Without loss of generality, suppose $g_{11} \leqslant g_{22} \leqslant g_{33}$. Consider $g_{11} = g_{22} = 4g_{33}$. We know by Theorem 2.2.25 that this is the metric of a soliton on $\mathbb{R} \times S^3$. Then:

$$\frac{B_{11}}{g_{11}} = \frac{B_{22}}{g_{22}} = \frac{B_{33}}{g_{33}} = -\beta\, q(g_{11}, g_{11}, 4g_{11})(g_{00})^2 = -2c$$

Observe that:

$$\begin{aligned} q\left(x, x, \frac{1}{4}x\right) &= 5x^4 - 3x^3\left(\frac{5}{4}x\right) + \frac{1}{4}x^4 + x\left(x^3 - \frac{1}{4}x^3 - \frac{1}{16}x^3 + \frac{1}{64}x^3\right) \\ &\qquad - 3x^4 + \frac{3}{4}x^4 + \frac{3}{64}x^4 - \frac{3}{256}x^4 \\ &= x^4\left(5 - \frac{15}{4} + \frac{1}{4} + 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - 3 + \frac{3}{4} + \frac{3}{64} - \frac{3}{256}\right) \\ &= -\frac{3}{256}x^4. \end{aligned}$$

Thus $\beta\frac{3}{256}(g_{11})^4(g_{00})^2 > 0$. Since

$$-2c = \beta\frac{3}{256}(g_{11})^4(g_{00})^2$$

we see that $c < 0$. Recall the soliton is of the form Hess $f - \frac{1}{2}B = cg$. Thus, the soliton with the given metric is expanding.

Using Theorem 2.1.13. The Bach tensor is conformally invariant of weight $w = -2$, so $\tau_t = \sqrt{1 - 4ct}$. Since $c < 0$, we see that τ_t is defined for $t \in \left(\frac{1}{4c}, \infty\right)$. Thus the soliton is immortal. □

Remark 2.2.29. *This result aligns with the analysis of the Bach flow of $\mathbb{R} \times S^3$ in [Hel20].*

Chapter 3

Future Directions

This thesis begins the program of studying ambient obstruction solitons and there is still much to be learned. The following research objectives outline a few of the different directions and applications of this program of study.

Open Question 3.1. *Complete classification of homogeneous gradient Bach solitons and extend methodology to dimension* $n = 6$*.*

In order to complete the classification of homogeneous gradient Bach solitons, we need to classify expanding solitons. These solitons will be of the form $\mathbb{R} \times N^3$ where N^3 is a non-unimodular Lie group, and thus can be classified using the same methodology as in my previous work, [Gri20]. Thus, once one calculates the explicit representations of the components of the Bach tensor using computing software (following methods used in [Hel20]), the method used in Section 2.2 will determine the existence and nature of solitons on those manifolds. The cases of non-gradient solitons and co-homogeneity one manifolds are also completely open.

Extending these results to higher dimensions, we can examine homogeneous gradient ambient obstruction solitons for $n = 6$. Since Theorem 2.1.17 applies to every dimension of the ambient obstruction tensor, we can use methods similar to those used in Section 2.2

to investigate the homogeneous 6-dimensional ambient obstruction solitons. This project will provide additional insight into the nature of ambient obstruction solitons to inform the understanding of the conformal invariance of solitons for $n = 6$.

Open Question 3.2. *Continue studying the solitons of the q-flow via properties of the general tensor q, specifically considering q with a well defined conformal transformation law.*

Using similar techniques to Section 2.1, we can continue examining solitons of the q-flow where q is a general tensor with selected properties. Specifically, we are interested in examining the case where q has a well defined transformation law under conformal change. In the case when q is conformally invariant, this question reduces to examining modified solitons of the form $\frac{1}{2}\mathcal{L}_X g = \lambda g + \frac{1}{2}q$ where λ is function. This change allows us to see how conformal class is preserved by the flow and look towards finding conformal classes of solitons. This work bridges the fields of conformal geometry and geometric flow, and has proven incredibly fruitful.

Since the conformal invariance (of weight $2-n$) of the ambient obstruction tensor is well established, we will use this tensor as a guiding example. To improve generalizations, we look to [BH11] to guide how we account for other transformation laws when constructing the geometric flows.

Open Question 3.3. *Use the known connection between the divergence of the Ricci tensor and scalar curvature to continue to generalize theorems from Ricci flow.*

As we point out in Section 1.5, $\operatorname{div}\operatorname{Ric} = \frac{1}{2}\nabla S$. This implies that on a manifold with curvature the Ricci tensor is divergence-free. We note that in Section 2.1 many of our generalizations rely on one or both of these properties. Researching this equivalence could allow us to loosen conditions, to generalize a greater number of theorems, to consider the homogeneous non-gradient case, etcetera. One such theorem that could be generalized is as follows.

Theorem 3.3.1 (Theorem 1.2, [PW09])**.** *A gradient Ricci soliton is rigid if and only if it* *has constant scalar curvature and is radially flat.*

(A gradient soliton is rigid if it is of type $N \times_\Gamma \mathbb{R}^k$ where N is Einstein with Einstein constant λ, $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$, and where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$.)

This theorem comments on complete non-homogeneous manifolds, something that is not discussed in this thesis. A first step in generalizing this result would be to investigate the condition that the soliton has constant scalar curvature. Is it a necessary attribute of the manifold? Or, is it necessary to force the Ricci tensor to be divergence-free? Answering this question would allow us to give a similar result for a general tensor q.

Open Question 3.4. *Examine extrinsic analogies of the Bach flow and ambient obstruction flow.*

The Ambient obstruction tensor and, consequently, the Bach tensor are intrinsic tensors, meaning that they do not depend on the ambient space a manifold is immersed in. Another way to expand our work is by flowing manifolds by extrinsic tensors with similar conformal properties. Studying the Bach flow built an understanding of how conformal properties impact intrinsic flows and has led to a number of generalizations. Seeking to do this for extrinsic flows, we will be led by the example of the first variation of the Willmore energy: the Willmore invariant. Just as mean curvature flow is an extrinsic analog of Ricci flow, flowing a surface by the Willmore invariant is the extrinsic analog of the Bach flow. This can be extended to higher dimensions by considering the first variation of the conformally invariant generalization of the Willmore energy, as established by Graham-Reichert in [GR17]. This can be thought of as the extrinsic analog of the ambient obstruction tensor. Though the tools needed for this investigation seem to be fundamentally different from our previous work, the connections between the Willmore invariant and Bach tensor (and that of their higher dimensional equivalents) will allow us to draw conclusions about extrinsic geometric flows by q with conformal transformation laws, and determine the necessary considerations for extrinsic q and thus make further generalizations.

Appendix A

Weyl Tensor

This appendix is intended to serve as a continuation of Section 1.2. We begin by proving a number of identities and properties of the Weyl tensor in arbitrary dimension $n \geqslant 4$. Then, focusing on $n = 4$, we expand upon the notion of self-duality to see that we can use eigenvalues and eigenvectors to prove identities of the self-dual Weyl tensor.

Recall the following definitions from Section 1.2:

$$W_{abcd} = R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac} \tag{A.1}$$

for

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right)$$

Equivalently, we can give the Weyl tensor directly in the terms of its Riemannian, Ricci and scalar curvature tensors. This fact is useful in calculations and the derivation is shown below.

$$W_{abcd} = R_{abcd} + \frac{1}{n-2}\left(R_{bd}g_{ac} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{ac}g_{bd}\right) - \frac{S}{(n-1)(n-2)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right). \tag{A.2}$$

Proposition A.0.1. *(A.1) and (A.2) are equivalent.*

Proof.

$$\begin{aligned}
W_{abcd} &= R_{abcd} + \frac{1}{n-2}\left(g_{ac}\left(R_{bd} - \frac{S}{2(n-1)}g_{bd}\right) - g_{ad}\left(R_{bc} - \frac{S}{2(n-1)}g_{bc}\right)\right. \\
&\qquad \left. -g_{bc}\left(R_{ad} - \frac{S}{2(n-1)}g_{ad}\right) + g_{bd}\left(R_{ac} - \frac{S}{2(n-1)}g_{ac}\right)\right) \\
&= R_{abcd} + \frac{1}{n-2}\left(R_{bd}g_{ac} - \frac{S}{2(n-1)}g_{ac}g_{bd} - R_{bc}g_{ad} + \frac{S}{2(n-1)}g_{ad}g_{bc}\right. \\
&\qquad \left. -R_{ad}g_{bc} + \frac{S}{2(n-1)}g_{bc}g_{ad} + R_{ac}g_{bd} - \frac{S}{2(n-1)}g_{bd}g_{ac}\right) \\
&= R_{abcd} + \frac{1}{n-2}\Bigg(R_{bd}g_{ac} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{ac}g_{bd} \\
&\qquad - \frac{S}{2(n-1)}\left(g_{ac}g_{bd} - g_{ad}g_{bc} - g_{bc}g_{ad} + g_{bd}g_{ac}\right)\Bigg) \\
&= R_{abcd} + \frac{1}{n-2}\left(R_{bd}g_{ac} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{ac}g_{bd}\right) - \frac{S}{(n-1)(n-2)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)
\end{aligned}$$

□

A.0.1 Identities of the Weyl Tensor

In this section we will prove a number of results about the Weyl tensor. These results have been written to include additional steps that one wouldn't otherwise show in order to shed light on the nature of these calculations.

Proposition A.0.2 (Symmetries of the Weyl tensor.)**.**

$$W_{abcd} = -W_{bacd} = -W_{abdc} = W_{cdab}.$$

Proof.

$$W_{abcd} = R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac}$$

$$\begin{aligned}
W_{bacd} &= R_{bacd} + g_{bc}P_{ad} - g_{bd}P_{ac} - g_{ac}P_{bd} + g_{ad}P_{bc} \\
&= -R_{abcd} + -g_{ac}P_{bd} + g_{ad}P_{bc} + g_{bc}P_{ad} - g_{bd}P_{ac} \\
&= -W_{abcd} \\
W_{abdc} &= R_{abdc} + g_{ad}P_{bc} - g_{ac}P_{bd} - g_{bd}P_{ac} + g_{bc}P_{ad} \\
&= -R_{abcd} + -g_{ac}P_{bd} + g_{ad}P_{bc} + g_{bc}P_{ad} - g_{bd}P_{ac} \\
&= -W_{abcd} \\
W_{cdab} &= R_{cdab} + g_{ca}P_{db} - g_{cb}P_{da} - g_{da}P_{cb} + g_{db}P_{ca} \\
&= R_{abcd} + g_{ac}P_{bd} - g_{bc}P_{ad} - g_{ad}P_{bc} + g_{bd}P_{ac} \\
&= R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac} \\
&= W_{abcd}
\end{aligned}$$

□

Note the following symmetries:

$$g_{ij} = g_{ji},$$

$$R_{ij} = g^{ab}R_{aijb} = g^{ab}R_{bjia} = g^{ba}R_{bjia} = R_{ji},$$

$$P_{ij} = \frac{1}{(n-2)}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right) = \frac{1}{(n-2)}\left(R_{ji} - \frac{S}{2(n-1)}g_{ji}\right) = P_{ji}.$$

Lemma A.0.3 (Bianchi identity)**.**

$$W_{abcd} + W_{cabd} + W_{bcad} = 0.$$

Proof.

$$W_{abcd} = R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac}$$

$$
\begin{aligned}
W_{cabd} &= R_{cabd} + g_{cb}P_{ad} - g_{cd}P_{ab} - g_{ab}P_{cd} + g_{ad}P_{cb} \\
W_{bcad} &= R_{abcd} + g_{ba}P_{cd} - g_{bd}P_{ca} - g_{ca}P_{bd} + g_{cd}P_{ba} \\
W_{abcd} + W_{cabd} + W_{bcad} &= R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac} \\
&\quad + R_{cabd} + g_{cb}P_{ad} - g_{cd}P_{ab} - g_{ab}P_{cd} + g_{ad}P_{cb} \\
&\quad + R_{abcd} + g_{ba}P_{cd} - g_{bd}P_{ca} - g_{ca}P_{bd} + g_{cd}P_{ba} \\
&= R_{abcd} + R_{cabd} + R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} \\
&\qquad - g_{bc}P_{ad} + g_{bd}P_{ac} + g_{cb}P_{ad} - g_{cd}P_{ab} - g_{ab}P_{cd} \\
&\qquad + g_{ad}P_{cb} + g_{ba}P_{cd} - g_{bd}P_{ca} - g_{ca}P_{bd} + g_{cd}P_{ba} \\
&= 0 + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{ab}P_{cd} - g_{bc}P_{ad} + g_{bd}P_{ac} - g_{cd}P_{ab} \\
&\qquad - g_{ac}P_{bd} + g_{ad}P_{bc} + g_{ab}P_{cd} + g_{bc}P_{ad} - g_{bd}P_{ac} + g_{cd}P_{ab} \\
&= 0
\end{aligned}
$$

□

Lemma A.0.4. *The Weyl tensor is trace-free.*

Proof. We approach this proof by tracing over the different pairs of indices. First, we examine the results of tracing over the first and fourth indices.

$$
\begin{aligned}
g^{ab}W_{acdb} = g^{ab}\Bigg(& R_{acdb} + \frac{1}{(n-2)}\left(g_{ad}R_{cb} - g_{ab}R_{cd} - g_{cd}R_{ab} + g_{cb}R_{ad}\right) \\
& - \frac{S}{(n-1)(n-2)}\left(g_{ad}g_{cb} - g_{ab}g_{cd}\right)\Bigg) \\
= g^{ab}R_{acdb} &+ \frac{1}{(n-2)}\left(g^{ab}g_{ad}R_{cb} - g^{ab}g_{ab}R_{cd} - g^{ab}g_{cd}R_{ab} + g^{ab}g_{cb}R_{ad}\right) \\
& - \frac{S}{(n-1)(n-2)}\left(g^{ab}g_{ad}g_{cb} - g^{ab}g_{ab}g_{cd}\right) \\
= R_{cd} &+ \frac{1}{(n-2)}\left(\delta^b_d R_{cb} - nR_{cd} - g_{cd}S + \delta^a_c R_{ad}\right) - \frac{S}{(n-1)(n-2)}\left(\delta^b_d g_{cb} - n g_{cd}\right)
\end{aligned}
$$

$$= R_{cd} + \frac{1}{(n-2)}\left(R_{cd} - nR_{cd} - g_{cd}S + R_{cd}\right) - \frac{S}{(n-1)(n-2)}\left(g_{cd} - ng_{cd}\right)$$
$$= R_{cd} + \frac{1}{(n-2)}\left((2-n)R_{cd} - g_{cd}S\right) - \frac{S}{(n-1)(n-2)}\left((1-n)g_{cd}\right)$$
$$= R_{cd} - R_{cd} - \frac{1}{(n-2)}g_{cd}S + \frac{S}{(n-2)}g_{cd}$$
$$= 0.$$

Due to the symmetry of the Ricci tensor, we see that we can switch the inner two indices without consequence to our final calculation when we are not tracing over those two indices.

$$\begin{aligned}
g^{ab}W_{adcb} &= g^{ab}\Bigg(R_{adcb} + \frac{1}{(n-2)}\left(g_{ac}R_{db} - g_{ab}R_{dc} - g_{dc}R_{ab} + g_{db}R_{ac}\right) \\
&\qquad - \frac{S}{(n-1)(n-2)}\left(g_{ac}g_{db} - g_{ab}g_{dc}\right)\Bigg) \\
&= g^{ab}R_{adcb} + \frac{1}{(n-2)}\left(g^{ab}g_{ac}R_{db} - g^{ab}g_{ab}R_{cd} - g^{ab}g_{cd}R_{ab} + g^{ab}g_{db}R_{ac}\right) \\
&\qquad - \frac{S}{(n-1)(n-2)}\left(g^{ab}g_{ac}g_{db} - g^{ab}g_{ab}g_{cd}\right) \\
&= R_{cd} + \frac{1}{(n-2)}\left(\delta^b_c R_{db} - nR_{cd} - g_{cd}S + \delta^a_d R_{ac}\right) - \frac{S}{(n-1)(n-2)}\left(\delta^b_c g_{db} - ng_{cd}\right) \\
&= R_{cd} + \frac{1}{(n-2)}\left((2-n)R_{cd} - g_{cd}S\right) - \frac{S}{(n-1)(n-2)}\left((1-n)g_{cd}\right) \\
&= R_{cd} - R_{cd} - \frac{1}{(n-2)}g_{cd}S + \frac{S}{(n-2)}g_{cd} \\
&= 0.
\end{aligned}$$

Since $W_{bcda} = W_{cbad} = W_{adcb}$, we see that $g^{ab}W_{bcda} = 0$ and likewise $g^{ab}W_{bdca} = 0$.

Applying these calculations to different indices, it is easy to see that:

$$g^{ac}W_{abdc} = g^{ac}W_{adbc} = g^{ac}W_{cbda} = g^{ac}W_{cdba} = 0,$$

$$g^{ad}W_{abcd} = g^{ad}W_{acbd} = g^{ad}W_{dbca} = g^{ad}W_{dcba} = 0,$$

$$g^{bc}W_{badc} = g^{bc}W_{bdac} = g^{bc}W_{cadb} = g^{bc}W_{cdab} = 0,$$

$$g^{bd}W_{bacd} = g^{bd}W_{bcad} = g^{bd}W_{dacb} = g^{bd}W_{dcab} = 0,$$

$$g^{cd}W_{cabd} = g^{cd}W_{cbad} = g^{cd}W_{dabc} = g^{cd}W_{dbac} = 0.$$

Thus, the trace over the first and fourth indices of the Weyl tensor is always 0.

Our method moving forward will be to use identities and symmetries to get the indices we are tracing over in the first and fourth position, then use our previous findings. Proceeding to take the trace over the first and second indices, we use the first Bianchi identity and the symmetries of the Weyl tensor:

$$W_{abcd} = -W_{cabd} - W_{bcad} = -W_{acdb} + W_{bcda}.$$

Taking the trace we see that:

$$g^{ab}W_{abcd} = -g^{ab}W_{acdb} + g^{ab}W_{bcda} = 0.$$

Using symmetries to exhaust other formations:

$$g^{ab}W_{abdc} = -g^{ab}W_{abcd} = 0 \qquad g^{ab}W_{bacd} = -g^{ab}W_{abcd} = 0 \qquad g^{ab}W_{badc} = g^{ab}W_{abcd} = 0.$$

As is the case above, this generalizes, and thus the trace over the first and second indices of the Weyl tensor is always 0.

Proceeding to take the trace over the first and third indices:

$$g^{ac}W_{abcd} = -g^{ac}W_{abdc} = 0 \qquad g^{ac}W_{adcb} = -g^{ac}W_{adbc} = 0$$

$$g^{ac}W_{cbad} = -g^{ac}W_{cbda} = 0 \qquad g^{ac}W_{cdab} = -g^{ac}W_{cdba} = 0$$

As is the case above, this generalizes and thus the trace over the first and third indices of the Weyl tensor is always 0.

Proceeding to take the trace over the second and third indices:

$$g^{bc}W_{abcd} = g^{bc}W_{badc} = 0 \qquad g^{bc}W_{dbca} = g^{bc}W_{bdac} = 0$$
$$g^{bc}W_{acbd} = g^{bc}W_{cadb} = 0 \qquad g^{bc}W_{dcba} = g^{bc}W_{cdab} = 0$$

As is the case above, this generalizes and thus the trace over the second and third indices of the Weyl tensor is always 0.

Proceeding to take the trace over the second and fourth indices:

$$g^{bd}W_{abcd} = -g^{bd}W_{bacd} = 0 \qquad g^{bd}W_{cbad} = -g^{bd}W_{bcad} = 0$$
$$g^{bd}W_{adcb} = -g^{bd}W_{dacb} = 0 \qquad g^{bd}W_{cdab} = -g^{bd}W_{dcab} = 0$$

As is the case above, this generalizes and thus the trace over the second and fourth indices of the Weyl tensor is always 0.

Proceeding to take the trace over the third and fourth indices using the first Bianchi identity:

$$g^{cd}W_{abcd} = g^{cd}\left(-W_{cabd} - W_{bcad}\right) = -g^{cd}W_{cabd} + g^{cd}W_{cbad} = 0.$$

Using symmetries to exhaust other formations:

$$g^{cd}W_{bacd} = -g^{cd}W_{abcd} = 0 \qquad g^{cd}W_{abdc} = -g^{cd}W_{abcd} = 0 \qquad g^{cd}W_{badc} = g^{cd}W_{abcd} = 0.$$

As is the case above, this generalizes and thus the trace over the second and fourth indices of the Weyl tensor is always 0. □

Expansion on Duality

Let $n = 4$. As discussed in Section 1.2, the Weyl tensor decomposes into two parts, W^+ and W^-. In this section we detail this decomposition and examine the linear algebra to prove useful identities for each of the components.

We begin our investigation of the Weyl tensor by noting that the Riemannian curvature tensor defines a self-adjoint transformation $\mathscr{R} : \Lambda^2 \to \Lambda^2$, where Λ^2 is the set of exterior two forms. This transformation is given by:

$$\mathscr{R}(e_i \wedge e_j) = \frac{1}{2}\sum_{k,l} R_{ijkl} e_k \wedge e_l$$

As in [AHS78], we are able to rewrite $\mathscr{R}$ as a block matrix relative to our decomposition of exterior 2-forms:

$$\mathscr{R} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

where B is a homomorphism from $\Lambda^{2,-}$ to $\Lambda^{2,+}$, A is a self-adjoint endomorphism of $\Lambda^{2,+}$, and C is a self-adjoint endomorphism of $\Lambda^{2,-}$.

A classic fact, pointed out in [Bes08], is that there is a natural decomposition of the curvature tensor into components involving the scalar curvature, the trace-free part of its Ricci tensor, and the Weyl tensor. Furthermore since the Weyl component can be decomposed into self-dual and anti-self-dual, per [ST69], this decomposition is given in terms of the matrix by:

$$\mathscr{R} \to (\operatorname{tr} A, B, A - \frac{1}{3}\operatorname{tr} A, C - \frac{1}{3}\operatorname{tr} C),$$

where $\operatorname{tr} A = \operatorname{tr} C = \frac{1}{4}S$, B is the trace-less Ricci tensor, and

$$W^+ = A - \frac{1}{3}\operatorname{tr} A \qquad W^- = C - \frac{1}{3}\operatorname{tr} C \qquad W = W^+ + W^-.$$

Recall that A and C were endomorphisms of $\Lambda^{2,+}, \Lambda^{2,-}$, respectively. Thus, we see how the Weyl tensor decomposes into self-dual and anti-self-dual parts.

Proceeding to examine the Weyl component as a matrix itself, we know from [CGY03] and [Der83] that we can fix a point and diagonalize $W^{\pm}$. Doing so, we get the oriented orthogonal bases $(\omega^+, \eta^+, \theta^+)$ and $(\omega^-, \eta^-, \theta^-)$ of Λ^+ and Λ^- respectively. The eigenvectors of W are such that

$$|\omega^+| = |\eta^+| = |\theta^+| = \sqrt{2} \quad \text{and} \quad |\omega^-| = |\eta^-| = |\theta^-| = \sqrt{2}.$$

Let the three corresponding eigenvalues of W^+ and W^- be given by λ^+, μ^+, ν^+ and λ^-, μ^- and ν^-, respectively. Then we can represent the Weyl tensor as:

$$\begin{aligned} W = & \frac{1}{2}\left(\lambda^+(\omega^+ \otimes \omega^+) + \mu^+(\eta^+ \otimes \eta^+) + \nu^+(\theta^+ \otimes \theta^+)\right) \\ & + \frac{1}{2}\left(\lambda^-(\omega^- \otimes \omega^-) + \mu^-(\eta^- \otimes \eta^-) + \nu^-(\theta^- \otimes \theta^-)\right). \end{aligned}$$

Additionally, the self-dual and anti-self-dual components are given by:

$$\begin{aligned} W^+ &= \frac{1}{2}\left(\lambda^+(\omega^+ \otimes \omega^+) + \mu^+(\eta^+ \otimes \eta^+) + \nu^+(\theta^+ \otimes \theta^+)\right), \\ W^- &= \frac{1}{2}\left(\lambda^-(\omega^- \otimes \omega^-) + \mu^-(\eta^- \otimes \eta^-) + \nu^-(\theta^- \otimes \theta^-)\right). \end{aligned} \tag{A.3}$$

Lemma A.0.5. $|W^+|^2 = (\lambda^+)^2 + (\mu^+)^2 + (\nu^+)^2$.

Proof. In order to avoid issues with normality, we look at each eigenvector as a unit vector in the same direction scaled by $\sqrt{2}$. To do so, let:

$$a = \frac{\omega^+}{\sqrt{2}} \qquad b = \frac{\eta^+}{\sqrt{2}} \qquad c = \frac{\theta^+}{\sqrt{2}}$$

$$\begin{aligned}
|W^+|^2 &= \left|\frac{1}{2}\left(\lambda^+(\omega^+ \otimes \omega^+) + \mu^+(\eta^+ \otimes \eta^+) + \nu^+(\theta^+ \otimes \theta^+)\right)\right|^2 \\
&= \left|\frac{1}{2}\left(\lambda^+(\sqrt{2}a \otimes \sqrt{2}a) + \mu^+(\sqrt{2}b \otimes \sqrt{2}b) + \nu^+(\sqrt{2}c \otimes \sqrt{2}c)\right)\right|^2 \\
&= \left|\frac{1}{2}\left(2\lambda^+(a \otimes a) + 2\mu^+(b \otimes b) + 2\nu^+(c \otimes c)\right)\right|^2 \\
&= \left|\lambda^+(a \otimes a) + \mu^+(b \otimes b) + \nu^+(c \otimes c)\right|^2 \\
&= (\lambda^+)^2 + (\mu^+)^2 + (\nu^+)^2.
\end{aligned}$$

□

Since W is trace-free (and, consequently, so are $W^\pm$), this implies that $\lambda^+ + \mu^+ + \nu^+ = 0$ and $\lambda^- + \mu^- + \nu^- = 0$.

In order to use this basis of eigenvectors, we need to rewrite W^+_{ijkl} as $W^+ : \Lambda^2 \to \Lambda^2$. That is we want type change from a (4,0) tensor to a (2,2) tensor. We know for an arbitrary tensor $T : \Lambda^2 \to \Lambda^2$:

$$T_{ijkl} = T(e_i, e_j, e_k, e_l) = g(T(e_i \wedge e_j), e_k \wedge e_l).$$

This fact will prove invaluable in the following proofs.

Moreover, one should note that results one can quickly find mirroring results for W^-

using the same methods we use below.

Rewriting $e_i \wedge e_j$ in terms of our eigenbasis:

$$\omega^+ = -e_1 \wedge e_2 - e_3 \wedge e_4 \qquad \eta^+ = -e_1 \wedge e_3 - e_4 \wedge e_2 \qquad \theta^+ = -e_1 \wedge e_4 - e_2 \wedge e_3$$
$$\omega^- = e_1 \wedge e_2 - e_3 \wedge e_4 \qquad \eta^- = e_1 \wedge e_3 - e_4 \wedge e_2 \qquad \theta^- = e_1 \wedge e_4 - e_2 \wedge e_3$$

Thus, we see that:

$$\frac{1}{2}\left(-\omega^+ + \omega^-\right) = \frac{1}{2}\left(-(-e_1 \wedge e_2 - e_3 \wedge e_4) + (e_1 \wedge e_2 - e_3 \wedge e_4)\right) = \frac{1}{2}\left(2\, e_1 \wedge e_2\right) = e_1 \wedge e_2,$$
$$\frac{1}{2}\left(-\omega^+ - \omega^-\right) = \frac{1}{2}\left(-(-e_1 \wedge e_2 - e_3 \wedge e_4) - (e_1 \wedge e_2 - e_3 \wedge e_4)\right) = \frac{1}{2}\left(2\, e_3 \wedge e_4\right) = e_3 \wedge e_4.$$

Moreover we use the fact that ω^+ and ω^- are eigenvectors of W^+, W^- to see that:

$$W^+(\omega^+) = \lambda\omega^+, \qquad W^+(\omega^-) = 0.$$

Using this process to rewrite all pairs of $e_i \wedge e_j$ in terms of the corresponding eigenvector and finding the corresponding values of W^+, W^-, and W (when taken as (2,2)-tensors).

$e_i \wedge e_j$	Eigenvector	$W^+(e_i \wedge e_j)$	$W^-(e_i \wedge e_j)$	$W(e_i \wedge e_j)$
$e_1 \wedge e_2$	$\frac{1}{2}(-\omega^+ + \omega^-)$	$-\frac{1}{2}\lambda^+\omega^+$	$\frac{1}{2}\lambda^-\omega^-$	$\frac{1}{2}(-\lambda^+\omega^+ + \lambda^-\omega^-)$
$e_3 \wedge e_4$	$\frac{1}{2}(-\omega^+ - \omega^-)$	$-\frac{1}{2}\lambda^+\omega^+$	$-\frac{1}{2}\lambda^-\omega^-$	$\frac{1}{2}(-\lambda^+\omega^+ - \lambda^-\omega^-)$
$e_1 \wedge e_3$	$\frac{1}{2}(-\eta^+ + \eta^-)$	$-\frac{1}{2}\mu^+\eta^+$	$\frac{1}{2}\mu^-\eta^-$	$\frac{1}{2}(-\mu^+\eta^+ + \mu^-\eta^-)$
$e_4 \wedge e_2$	$\frac{1}{2}(-\eta^+ - \eta^-)$	$-\frac{1}{2}\mu^+\eta^+$	$-\frac{1}{2}\mu^-\eta^-$	$\frac{1}{2}(-\mu^+\eta^+ - \mu^-\eta^-)$
$e_1 \wedge e_4$	$\frac{1}{2}(-\theta^+ + \theta^-)$	$-\frac{1}{2}\nu^+\theta^+$	$\frac{1}{2}\nu^-\theta^-$	$\frac{1}{2}(-\nu^+\theta^+ + \nu^-\theta^-)$
$e_2 \wedge e_3$	$\frac{1}{2}(-\theta^+ - \theta^-)$	$-\frac{1}{2}\nu^+\theta^+$	$-\frac{1}{2}\nu^-\theta^-$	$\frac{1}{2}(-\nu^+\theta^+ - \nu^-\theta^-)$

Table A.1: Weyl tensors given in terms of eigenvalues and eigenvectors

Proceeding, we will use a few lemmas to eliminate cases that we need to consider. We will do this for both the self-dual component of the Weyl tensor and the whole Weyl tensor. It should be noted that there are parallel identities for the anti-self-dual component of the Weyl tensor.

We focus first on the self-dual component.

Lemma A.0.6. *For any i, j, k, l (not necessarily distinct):*

$$W^+_{iikl} = W^+_{ijkk} = 0.$$

Proof. $g(W^+(e_i \wedge e_i), e_k \wedge e_l) = 0 = g(W^+(e_i \wedge e_j), e_k \wedge e_k)$. □

We see this holds for the whole Weyl tensor.

Lemma A.0.7. *For any i, j, k, l (not necessarily distinct):*

$$W_{iikl} = W_{ijkk} = 0.$$

Proof. $g(W(e_i \wedge e_i), e_k \wedge e_l) = 0 = g(W(e_i \wedge e_j), e_k \wedge e_k)$. □

Again, focusing on the self-dual component we get the following identity.

Lemma A.0.8. *If any one index is repeated, the Weyl tensor is zero. That is, for distinct i, j, k, l: $W^+_{ijil} = W^+_{ijki} = W^+_{ijjl} = W^+_{ijkj} = 0$. Note, if two indices are repeated, this is not true.*

Proof. In order to use our eigenbasis, we focus on a specific set of indices that demonstrate this desired repetition. Consider:

$$\begin{aligned} W^+_{1213} &= g(W^+(e_1 \wedge e_2), e_1 \wedge e_3) \\ &= -\frac{1}{4}\lambda^+ g(\omega^+, -\eta^+ + \eta^-) \\ &= -\frac{1}{4}\lambda^+ \left(-g(\omega^+, \eta^+) + g(\omega^+, \eta^-)\right) \\ &= 0 \end{aligned}$$

because the eigenbasis is orthogonal. This calculation can be repeated to the same end for all other such combinations of indices. □

Again, this holds for the entire Weyl tensor.

Lemma A.0.9. *If any one index is repeated, the Weyl tensor is zero. That is, for distinct i, j, k, l: $W_{ijil} = W_{ijki} = W_{ijjl} = W_{ijkj} = 0$. Note, if two indices are repeated, this is not true.*

Proof. Again choosing a specific set of indices for ease of calculation, consider:

$$\begin{aligned}
W_{1213} &= g(W(e_1 \wedge e_2), e_1 \wedge e_3) \\
&= \frac{1}{4} g(-\lambda^+\omega^+ + \lambda^-\omega^-, -\eta^+ + \eta^-) \\
&= \frac{1}{4}\left(-\lambda^+ g(\omega^+, -\eta^+ + \eta^-) + \lambda^- g(\omega^-, -\eta^+ + \eta^-)\right) \\
&= \frac{1}{4}\left(-\lambda^+\left(-g(\omega^+, \eta^+) + g(\omega^+, \eta^-)\right) + \lambda^-\left(-g(\omega^-, \eta^+) + g(\omega^-, \eta^-)\right)\right) \\
&= 0.
\end{aligned}$$

As above, this is due to the orthogonality of our eigenvectors. This calculation can be repeated to the same end for all other such combinations of indices. $\square$

From these two lemmas, we can determine that the only nonzero values of W^+_{ijkl} are those with four distinct indices or with two pairs of indices. These are enumerated as follows.

$$\begin{aligned}
\frac{1}{4}\lambda^+|\omega^+|^2 &= g\left(W^+(e_1 \wedge e_2), e_1 \wedge e_2\right) \\
&= W^+_{1212} = W^+_{1234} = W^+_{3434} = W^+_{3412} = W^+_{2121} = W^+_{2143} = W^+_{4343} = W^+_{4321} \\
-\frac{1}{4}\lambda^+|\omega^+|^2 &= g\left(W^+(e_2 \wedge e_1), e_1 \wedge e_2\right) \\
&= W^+_{1221} = W^+_{1243} = W^+_{3443} = W^+_{3421} = W^+_{2112} = W^+_{2134} = W^+_{4334} = W^+_{4312} \\
\frac{1}{4}\mu^+|\eta^+|^2 &= g\left(W^+(e_1 \wedge e_3), e_1 \wedge e_3\right) \\
&= W^+_{1313} = W^+_{1342} = W^+_{4242} = W^+_{4213} = W^+_{3131} = W^+_{3124} = W^+_{2424} = W^+_{2431} \\
-\frac{1}{4}\mu^+|\eta^+|^2 &= g\left(W^+(e_1 \wedge e_3), e_3 \wedge e_1\right) \\
&= W^+_{1331} = W^+_{1324} = W^+_{4224} = W^+_{4231} = W^+_{3113} = W^+_{3142} = W^+_{2442} = W^+_{2413} \\
\frac{1}{4}\nu^+|\theta^+|^2 &= g\left(W^+(e_1 \wedge e_4), e_1 \wedge e_4\right) \\
&= W^+_{1414} = W^+_{1423} = W^+_{2323} = W^+_{2314} = W^+_{4141} = W^+_{4132} = W^+_{3232} = W^+_{3241}
\end{aligned}$$

$$-\frac{1}{4}\nu^+|\theta^+|^2 = g\left(W^+(e_1\wedge e_4), e_4\wedge e_1\right)$$
$$= W^+_{1441} = W^+_{1432} = W^+_{2332} = W^+_{2341} = W^+_{4114} = W^+_{4123} = W^+_{3223} = W^+_{3214}$$

A sample of the calculation involved:

$$\begin{aligned}
W^+_{1212} &= g\left(W^+(e_1\wedge e_2), e_1\wedge e_2\right) \\
&= g\left(-\frac{1}{2}\lambda^+\omega^+, \frac{1}{2}(-\omega^+ + \omega^-\right) \\
&= \frac{1}{4}\lambda^+ g(\omega^+,\omega^+) - \frac{1}{4}\lambda^+ g(\omega^+,\omega^-) \\
&= \frac{1}{4}\lambda^+|\omega^+|^2
\end{aligned}$$

Similarly, we find the that for the whole Weyl tensor:

$$\begin{aligned}
W_{1212} = W_{3434} &= \frac{1}{4}\left(\lambda^+|\omega^+|^2 + \lambda^-|\omega^-|^2\right) \\
W_{1234} &= \frac{1}{4}\left(\lambda^+|\omega^+|^2 - \lambda^-|\omega^-|^2\right) \\
W_{1313} = W_{4242} &= \frac{1}{4}\left(\mu^+|\eta^+|^2 + \mu^-|\eta^-|^2\right) \\
W_{1342} &= \frac{1}{4}\left(\mu^+|\eta^+|^2 - \mu^-|\eta^-|^2\right) \\
W_{1414} = W_{2323} &= \frac{1}{4}\left(\nu^+|\theta^+|^2 + \nu^-|\theta^-|^2\right) \\
W_{1423} &= \frac{1}{4}\left(\nu^+|\theta^+|^2 - \nu^-|\theta^-|^2\right)
\end{aligned}$$

A sample of the calculation involved:

$$\begin{aligned}
W_{1212} &= g\left(W(e_1\wedge e_2), e_1\wedge e_2\right) \\
&= g\left(\frac{1}{2}(-\lambda^+\omega^+ + \lambda^-\omega^-), \frac{1}{2}(-\omega^+ + \omega^-\right) \\
&= \frac{1}{4}\left(\lambda^+ g(\omega^+,\omega^+) - \lambda^+ g(\omega^+,\omega^-) - \lambda^- g(\omega^-,\omega^+) + \lambda^- g(\omega^-,\omega^-)\right)
\end{aligned}$$

$$= \frac{1}{4}\left(\lambda^+|\omega^+|^2 + \lambda^-|\omega^-|^2\right)$$

The following lemma is widely accepted as true, but an explicit proof was not given in the literature. As such, we've chosen to prove it in this section for completeness. The reader should also note that part (a) of the following Lemma has been edited to the equivalent result in [Der83] as the statement in [CGY03] has a few small errors.

Lemma A.0.10 (Chang-Gursky-Yang, Lemma 3.4)**.**

a. $W^+_{ijkl}W^+_{jskl} = -|W^+|^2\delta_{is}$

b. $W^+_{msij}W^+_{ijkl}W^+_{mskl} = 24 \det W^+$

c. $4W_{miks}W^+_{ijkl}W^+_{jmsl} = 48 \det W^+$

a. *Proof.* Using the symmetries of the Weyl tensor one should note that this is equivalent to proving:

$$W^+_{ijkl}W^+_{sjkl} = |W^+|^2\delta_{is}$$

Proceeding, we show by contradiction that $W_{ijkl}W_{sjkl} = 0$ if $i \neq s$. Seeking said contradiction, suppose that there was a nonzero value of $W_{ijkl}W_{sjkl}$. That is, suppose there is a combination of i, s, j, k, l such that both components are nonzero. Without loss of generality, let $i = 1$. From the rules above, we know that $j \neq 1$, and that one of two cases: 1.) One of k or l can equal 1 or 2.) $j \neq k \neq l \neq 1$.

Case 1. *Without loss of generality, suppose $k = 1$. This forces $j = l$ and the first term to be of the form W_{1j1l}. By our original supposition, $s \neq 1$. Consequently we see: W_{sj1l} this means that s, j, and l must be distinct and non-one. However, our supposition forces $j = l$, a contradiction.*

Case 2. *Suppose, on the other hand, that $j \neq k \neq l \neq 1$. Then, when we consider W_{sjkl}, $s = 1$ (else s would equal one of j, k, or l). This contradicts our original supposition that $s \neq i$.*

Therefore, by contradiction, for $i \neq s$ $W_{ijkl}W_{sjkl} = 0$.

Proceeding, suppose $i = s$. Recall from our construction of the eigenbasis, $|\omega^+| = |\eta^+| = |\theta^+| = \sqrt{2}$. Using (A.3) to represent the right hand side in terms of the eigenbasis we appeal to Lemma A.0.5 to see that:

$$|W^+|^2 = \left((\lambda^+)^2 + (\mu^+)^2 + (\nu^+)^2\right)$$

Proceeding to examine the left hand side by enumerating the combinations of indices that produces a nonzero W^+ we see that:

$$\begin{aligned}
W^+_{1jkl}W^+_{1jkl} &= \sum_{jkl} W^+_{1jkl}W^+_{1jkl} \\
&= \sum_{jkl} \left(W^+_{1jkl}\right)^2 \\
&= \Bigg(\underbrace{\frac{1}{4}\lambda^+|\omega^+|^2}_{1212}\Bigg)^2 + \Bigg(\underbrace{\frac{1}{4}\lambda^+|\omega^+|^2}_{1234}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\lambda^+|\omega^+|^2}_{1221}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\lambda^+|\omega^+|^2}_{1243}\Bigg)^2 \\
&\quad + \Bigg(\underbrace{\frac{1}{4}\mu^+|\eta^+|^2}_{1313}\Bigg)^2 + \Bigg(\underbrace{\frac{1}{4}\mu^+|\eta^+|^2}_{1342}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\mu^+|\eta^+|^2}_{1331}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\mu^+|\eta^+|^2}_{1324}\Bigg)^2 \\
&\quad + \Bigg(\underbrace{\frac{1}{4}\nu^+|\theta^+|^2}_{1414}\Bigg)^2 + \Bigg(\underbrace{\frac{1}{4}\nu^+|\theta^+|^2}_{1423}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\nu^+|\theta^+|^2}_{1441}\Bigg)^2 + \Bigg(\underbrace{-\frac{1}{4}\nu^+|\theta^+|^2}_{1432}\Bigg)^2 .
\end{aligned}$$

Since $|\omega^+| = |\eta^+| = |\theta^+| = \sqrt{2}$ we can simplify this:

$$\begin{aligned}
\left(W^+_{1jkl}\right)^2 = &\left(\frac{1}{2}\lambda^+\right)^2 + \left(\frac{1}{2}\lambda^+\right)^2 + \left(-\frac{1}{2}\lambda^+\right)^2 + \left(-\frac{1}{2}\lambda^+\right)^2 \\
&+ \left(\frac{1}{2}\mu^+\right)^2 + \left(\frac{1}{2}\mu^+\right)^2 + \left(-\frac{1}{2}\mu^+\right)^2 + \left(-\frac{1}{2}\mu^+\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\nu^+\right)^2+\left(\frac{1}{2}\nu^+\right)^2+\left(-\frac{1}{2}\nu^+\right)^2+\left(-\frac{1}{2}\nu^+\right)^2\\
=&4\left(\frac{1}{2}\lambda^+\right)^2+4\left(\frac{1}{2}\mu^+\right)^2+4\left(\frac{1}{2}\nu^+\right)^2\\
=&(\lambda^+)^2+(\mu^+)^2+(\nu^+)^2\\
=&|W^+|^2.
\end{aligned}
$$

□

b. *Proof.* In order to show that

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 24\det W^+,$$

note that

$$\det W^+ = \lambda^+\mu^+\nu^+.$$

Moreover, recognize that since $\lambda^+ + \mu^+ + \nu^+ = 0$,

$$
\begin{aligned}
0 &= (\lambda^+ + \mu^+ + \nu^+)^3\\
&= (\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 + 3\lambda^+(\mu^+)^2 + 3\lambda^+(\nu^+)^2 + 3\mu^+(\lambda^+)^2 + 3\mu^+(\nu^+)^2\\
&\qquad + 3\nu^+(\lambda^+)^2 + 3\nu^+(\mu^+)^2 + 6\lambda^+\mu^+\nu^+\\
&= (\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 + 3(\mu^+ + \nu^+)(\lambda^+)^2 + 3(\lambda^+ + \nu^+)(\mu^+)^2\\
&\qquad + 3(\lambda^+ + \mu^+)(\nu^+)^2 + 6\lambda^+\mu^+\nu^+\\
&= (\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 + 3(-\lambda^+)(\lambda^+)^2 + 3(-\mu^+)(\mu^+)^2 + 3(-\nu^+)(\nu^+)^2 + 6\lambda^+\mu^+\nu^+\\
&= (\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 - 3(\lambda^+)^3 - 3(\mu^+)^3 - 3(\nu^+)^3 + 6\lambda^+\mu^+\nu^+\\
&= -2(\lambda^+)^3 - 2(\mu^+)^3 - 2(\nu^+)^3 + 6\lambda^+\mu^+\nu^+\\
&= -2\left((\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3\right) + 6\lambda^+\mu^+\nu^+.
\end{aligned}
$$

Thus $(\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 = 3\lambda^+\mu^+\nu^+$. Therefore, it is equivalent to show that:

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 8\left((\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3\right).$$

The most straightforward way to prove this is to enumerate the possibilities and use some simple counting arguments.

I. Consider the case where $\{m, s\} = \{1, 2\}$, $\{i, j\} = \{1, 2\}$, and $\{k, l\} = \{1, 2\}$,. Then

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = W^+_{1212}W^+_{1212}W^+_{1212} = \frac{1}{64}(\lambda^+)^3(|\omega^+|^2)^3 = \frac{1}{8}\lambda^3.$$

We get this same result when $\{m, s\} = \{3, 4\}$, $\{i, j\} = \{1, 2\}$, and $\{k, l\} = \{1, 2\}$. In fact, if we switch any of the 1's with a 3 (and the corresponding 2 with a 4). So we see that we have the following sets of indices that yield the above result.

- $\{(m, s) = (1, 2),\ (i, j) = (1, 2),\ (k, l) = (1, 2)\}$
- $\{(m, s) = (3, 4),\ (i, j) = (1, 2),\ (k, l) = (1, 2)\}$
- $\{(m, s) = (1, 2),\ (i, j) = (3, 4),\ (k, l) = (1, 2)\}$
- $\{(m, s) = (1, 2),\ (i, j) = (1, 2),\ (k, l) = (3, 4)\}$
- $\{(m, s) = (3, 4),\ (i, j) = (3, 4),\ (k, l) = (1, 2)\}$
- $\{(m, s) = (1, 2),\ (i, j) = (3, 4),\ (k, l) = (3, 4)\}$
- $\{(m, s) = (3, 4),\ (i, j) = (1, 2),\ (k, l) = (3, 4)\}$
- $\{(m, s) = (3, 4),\ (i, j) = (3, 4),\ (k, l) = (3, 4)\}$

Thus, so far we have 8 sets of indices, each of which produce $W^+_{msij}W^+_{ijkl}W^+_{mskl} = \frac{1}{8}\lambda^3$.

Proceeding, we will look at how we can modify these 8 sets by switching pairs of indices.

i. If we switch m and s we generate 8 more sets. Using the symmetries of the Weyl tensor we see that reflecting any single pair of indices results in the same answer:

$$W^+_{smij}W^+_{ijkl}W^+_{smkl} = (-W^+_{msij})W^+_{ijkl}(-W^+_{mskl}).$$

There are 3 ways to reflect one pair of indices. Thus, there are a total of 24 sets of indices with one pair switched, each set producing: $W^+_{msij}W^+_{ijkl}W^+_{mskl} = \frac{1}{8}\lambda^3$.

ii. If we switch m and s and then switch i and j, again we generate 8 more sets. Reflecting two pairs of indices produces the same result:

$$W^+_{smji}W^+_{jikl}W^+_{smkl} = W^+_{msij}(-W^+_{ijkl})(-W^+_{mskl})$$

and there are 3 ways to reflect two pairs of indices. Thus, there are a total of 24 sets of indices with two pairs switched, each set producing: $W^+_{msij}W^+_{ijkl}W^+_{mskl} = \frac{1}{8}\lambda^3$.

iii. Lastly, reflecting all three pairs of indices still maintains the original sign:

$$W^+_{smji}W^+_{jilk}W^+_{smlk} = W^+_{msij}W^+_{ijkl}W^+_{mskl}.$$

There is only 1 way to do this, so there are a total of 8 sets of indices with all three pairs switched, each set producing: $W^+_{msij}W^+_{ijkl}W^+_{mskl} = \frac{1}{8}\lambda^3$.

Therefore, there are a total of $8 + 24 + 24 + 8 = 64$ sets of indices, thus we see that we get

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 8(\lambda^+)^3.$$

II. We can repeat this process with pairs $(1, 3)$ and $(4, 2)$, again producing 64 sets of indices yielding

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 8(\mu^+)^3.$$

III. Lastly, we can repeat this process with pairs $(1,4)$ and $(2,3)$, producing another 64 sets of indices yielding:

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 8(\nu^+)^3.$$

Combining all of these different options we see that indeed:

$$W^+_{msij}W^+_{ijkl}W^+_{mskl} = 8(\lambda^+)^3 + 8(\mu^+)^3 + 8(\nu^+)^3,$$

proving our revised claim. □

c. *Proof.* We will proceed in a similar fashion to show that:

$$4W_{miks}W^+_{ijkl}W^+_{jmsl} = 48 \det W^+.$$

Again, we begin by noting that it is equivalent to prove that

$$W_{miks}W^+_{ijkl}W^+_{jmsl} = 16\lambda^+\mu^+\nu^+.$$

As in part (b), the most straightforward way to do this proof is to enumerate the nonzero options. Recall that the norm of the eigenvectors is $\sqrt{2}$. Moreover, from the symmetries we know that if $m \neq s$, $i \neq k$, and $j \neq l$ then $m \neq i \neq j$ and $s \neq k \neq l$. For example, if $m = 1$ then $i = 2$ and $j = 3$, so:

$$\Rightarrow W_{1212}W^+_{2323}W^+_{3131} = \left(\frac{1}{4}\left(\lambda^+|\omega^+|^2 + \lambda^-|\omega^-|^2\right)\right)\left(\frac{1}{4}\nu^+|\theta^+|^2\right)\left(\frac{1}{4}\mu^+|\eta^+|^2\right)$$

$$= \frac{1}{8}\lambda^+\mu^+\nu^+ + \frac{1}{8}\lambda^-\mu^+\nu^+.$$

The following chart enumerates all of the combinations. Taking the product of the last last three columns of each row and adding the rows together produces $16\lambda^+\mu^+\nu^+$, as desired. □

m	s	i	k	j	l	$imks$	$ijkl$	$mjsl$	W_{imks}	W^+_{ijkl}	W^+_{mjsl}
1	1	2	2	3	3	2121	2323	1313	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\mu^+$
1	1	2	2	4	4	2121	2424	1414	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
1	1	3	3	2	2	3131	3232	1212	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
1	1	3	3	4	4	3131	3434	1414	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
1	1	4	4	2	2	4141	4242	1212	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
1	1	4	4	3	3	4141	4343	1313	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
2	2	1	1	3	3	1212	1313	2323	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
2	2	1	1	4	4	1212	1414	2424	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\mu^+$
2	2	3	3	1	1	3232	3131	2121	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
2	2	3	3	4	4	3232	3434	2424	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
2	2	4	4	1	1	4242	4141	2121	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
2	2	4	4	3	3	4242	4343	2323	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$

3	3	1	1	2	2	1313	1212	3232	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
3	3	1	1	4	4	1313	1414	3434	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
3	3	2	2	1	1	2323	2121	3131	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
3	3	2	2	4	4	2323	2424	3434	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
3	3	4	4	1	1	4343	4141	3131	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\mu^+$
3	3	4	4	2	2	4343	4242	3232	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$

4	4	1	1	2	2	1414	1212	4242	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
4	4	1	1	3	3	1414	1313	4343	$\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
4	4	2	2	1	1	2424	2121	4141	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
4	4	2	2	3	3	2424	2323	4343	$\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
4	4	3	3	1	1	3434	3131	4141	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
4	4	3	3	2	2	3434	3232	4242	$\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$\frac{1}{2}\mu^+$

1	2	2	1	3	4	2112	2314	1324	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
1	2	2	1	4	3	2112	2413	1423	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
1	2	3	4	2	1	3142	3241	1221	$-\frac{1}{2}(\mu^+ - \mu^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
1	2	3	4	4	3	3142	3443	1423	$-\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
1	2	4	3	2	1	4132	4231	1221	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\lambda^+$

1	2	4	3	3	4	4132	4334	1324	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
2	1	1	2	3	4	1221	1324	2314	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
2	1	1	2	4	3	1221	1423	2413	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
2	1	3	4	1	2	3241	3142	2112	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\lambda^+$
2	1	3	4	4	3	3241	3443	2413	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
2	1	4	3	1	2	4231	4132	2112	$-\frac{1}{2}(\mu^+ - \mu^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
2	1	4	3	3	4	4231	4334	2314	$-\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
1	3	2	4	3	1	2143	2341	1331	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
1	3	2	4	4	2	2143	2442	1432	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\nu^+$
1	3	3	1	2	4	3113	3214	1234	$-\frac{1}{2}(\mu^+ + \mu^-)$	$-\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
1	3	3	1	4	2	3113	3412	1432	$-\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
1	3	4	2	2	4	4123	4224	1234	$-\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
1	3	4	2	3	1	4123	4321	1331	$-\frac{1}{2}(\nu^+ - \nu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
3	1	1	3	2	4	1331	1234	3214	$-\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
3	1	1	3	4	2	1331	1432	3412	$-\frac{1}{2}(\mu^+ + \mu^-)$	$-\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
3	1	2	4	1	3	2341	2143	3113	$-\frac{1}{2}(\nu^+ - \nu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
3	1	2	4	4	2	2341	2442	3412	$-\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$

3	1	4	2	2	4	4321	4123	3113	$\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$-\frac{1}{2}\mu^{+}$
3	1	4	2	1	3	4321	4224	3214	$\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$-\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\nu^{+}$
1	4	2	3	3	2	2134	2332	1342	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$\frac{1}{2}\mu^{+}$
1	4	2	3	4	1	2134	2431	1441	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\nu^{+}$
1	4	3	2	2	3	3124	3223	1243	$\frac{1}{2}\left(\mu^{+}-\mu^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$-\frac{1}{2}\lambda^{+}$
1	4	3	2	4	1	3124	3421	1441	$\frac{1}{2}\left(\mu^{+}-\mu^{-}\right)$	$-\frac{1}{2}\lambda^{+}$	$-\frac{1}{2}\nu^{+}$
1	4	4	1	2	3	4114	4213	1243	$-\frac{1}{2}\left(\nu^{+}+\nu^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\lambda^{+}$
1	4	4	1	3	2	4114	4312	1342	$-\frac{1}{2}\left(\nu^{+}+\nu^{-}\right)$	$-\frac{1}{2}\lambda^{+}$	$\frac{1}{2}\mu^{+}$
4	1	1	4	2	3	1441	1243	4213	$-\frac{1}{2}\left(\nu^{+}+\nu^{-}\right)$	$-\frac{1}{2}\lambda^{+}$	$\frac{1}{2}\mu^{+}$
4	1	1	4	3	2	1441	1342	4312	$-\frac{1}{2}\left(\nu^{+}+\nu^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\lambda^{+}$
4	1	2	3	1	4	2431	2134	4114	$\frac{1}{2}\left(\mu^{+}-\mu^{-}\right)$	$-\frac{1}{2}\lambda^{+}$	$-\frac{1}{2}\nu^{+}$
4	1	2	3	3	2	2431	2332	4312	$\frac{1}{2}\left(\mu^{+}-\mu^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$-\frac{1}{2}\lambda^{+}$
4	1	3	2	1	4	3421	3124	4114	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\nu^{+}$
4	1	3	2	2	3	3421	3223	4213	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$\frac{1}{2}\mu^{+}$
2	3	1	4	3	2	1243	1342	2332	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\nu^{+}$
2	3	1	4	4	1	1243	1441	2431	$-\frac{1}{2}\left(\lambda^{+}-\lambda^{-}\right)$	$-\frac{1}{2}\nu^{+}$	$\frac{1}{2}\mu^{+}$
2	3	3	2	1	4	3223	3124	2134	$-\frac{1}{2}\left(\nu^{+}+\nu^{-}\right)$	$\frac{1}{2}\mu^{+}$	$-\frac{1}{2}\lambda^{+}$

2	3	3	2	4	1	3223	3421	2431	$-\frac{1}{2}(\nu^+ + \nu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
2	3	4	1	1	4	4213	4114	2134	$\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
2	3	4	1	3	2	4213	4312	2332	$\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
3	2	1	4	2	3	1342	1243	3223	$\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
3	2	1	4	4	1	1342	1441	3421	$\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
3	2	2	3	1	4	2332	2134	3124	$-\frac{1}{2}(\nu^+ + \nu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\mu^+$
3	2	2	3	4	1	2332	2431	3421	$-\frac{1}{2}(\nu^+ + \nu^-)$	$\frac{1}{2}\mu^+$	$-\frac{1}{2}\lambda^+$
3	2	4	1	1	4	4312	4114	3124	$-\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\nu^+$	$\frac{1}{2}\mu^+$
3	2	4	1	2	3	4312	4213	3223	$-\frac{1}{2}(\lambda^+ - \lambda^-)$	$\frac{1}{2}\mu^+$	$-\frac{1}{2}\nu^+$
2	4	1	3	3	1	1234	1331	2341	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\nu^+$
2	4	1	3	4	2	1234	1432	2442	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
2	4	3	1	1	3	3214	3113	2143	$-\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$
2	4	3	1	4	2	3214	3412	2442	$-\frac{1}{2}(\nu^+ - \nu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
2	4	4	2	1	3	4224	4123	2143	$-\frac{1}{2}(\mu^+ + \mu^-)$	$-\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
2	4	4	2	3	1	4224	4321	2341	$-\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
4	2	1	3	2	4	1432	1234	4224	$-\frac{1}{2}(\nu^+ - \nu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
4	2	1	3	3	1	1432	1331	4321	$-\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\lambda^+$

4	2	2	4	1	3	2442	2143	4123	$-\frac{1}{2}(\mu^+ + \mu^-)$	$\frac{1}{2}\lambda^+$	$-\frac{1}{2}\nu^+$
4	2	2	4	3	1	2442	2341	4321	$-\frac{1}{2}(\mu^+ + \mu^-)$	$-\frac{1}{2}\nu^+$	$\frac{1}{2}\lambda^+$
4	2	3	1	1	3	3412	3113	4123	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\nu^+$
4	2	3	1	2	4	3412	3214	4224	$\frac{1}{2}(\lambda^+ - \lambda^-)$	$-\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
3	4	1	2	2	1	1324	1221	3241	$-\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
3	4	1	2	4	3	1324	1423	3443	$-\frac{1}{2}(\mu^+ - \mu^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
3	4	2	1	1	2	2314	2112	3142	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
3	4	2	1	4	3	2314	2413	3443	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\lambda^+$
3	4	4	3	1	2	4334	4132	3142	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$
3	4	4	3	2	1	4334	4231	3241	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
4	3	1	2	2	1	1423	1221	4231	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\lambda^+$	$-\frac{1}{2}\mu^+$
4	3	1	2	3	4	1423	1324	4334	$\frac{1}{2}(\nu^+ - \nu^-)$	$-\frac{1}{2}\mu^+$	$-\frac{1}{2}\lambda^+$
4	3	2	1	1	2	2413	2112	4132	$-\frac{1}{2}(\mu^+ - \mu^-)$	$-\frac{1}{2}\lambda^+$	$\frac{1}{2}\nu^+$
4	3	2	1	3	4	2413	2314	4334	$-\frac{1}{2}(\mu^+ - \mu^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\lambda^+$
4	3	3	4	1	2	3443	3142	4132	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$-\frac{1}{2}\mu^+$	$\frac{1}{2}\nu^+$
4	3	3	4	2	1	3443	3241	4231	$-\frac{1}{2}(\lambda^+ + \lambda^-)$	$\frac{1}{2}\nu^+$	$-\frac{1}{2}\mu^+$

Table A.3: Computations for proof of Theorem A.0.10 (c)

Appendix B

Structure Constants

In Section 2.2, we use explicit representation of the Bach tensor to determine the existence and nature of solitons. To get those representations we use structure constants. We begin this appendix by looking at the background and meaning behind structure constants, then we will go through an example of such a computation.

Broadly speaking, structure constants provide an $n \times n \times n$ array that describes a Lie algebra structure. Ryan and Shepley, [RS75], call these constants "structure coefficient", which more clearly defines their role in describing a Lie algebra structure. When examining a Lie group with a left invariant metric, the structure constants, C_{ijk} arise when looking at the effects of the Lie bracket on an orthonormal basis $e_1, \ldots, e_n$:

$$[e_i, e_j] = C_{ij}{}^k e_k = \sum_k C_{ijk} e_k.$$

As Milnor notes in [Mil76], this is equivalent to:

$$C_{ijk} = \langle [e_i, e_j], e_k \rangle.$$

The anti-symmetry of the Lie bracket induces an anti-symmetry of the first two indices:

$$C_{ijk} = -C_{jik}.$$

Note, this notation is very similar to the Cotton tensor, but the two are unrelated.

B.1 Geometries

The scope of subsection 2.2.4 is limited to the cases where N^3 is 3-dimensional, unimodular, Lie groups. The choice of manifolds is related to Thurston's geometrization conjecture, which is a 3-dimensional version of the uniformization conjecture in 2-dimensions. Thurston's geometrization conjecture uses the geometries from the following classification with other tools to classifying all 3-manifolds. We refrain from a more thorough discussion of this theorem, but refer the reader to [Lee18, pg.77] for more information. The following classification of a subset of 3-manifolds helps guide our choice of manifolds we investigate in Section 2.2.4. First, define a geometry as a pair (X, G) where X is a set and G is a group acting on X, as in [Sco83]. Then we get the following classification as stated in [Sco83].

Theorem B.1.1. *Any maximal, simply connected, 3-dimensional geometry which admits compact quotients is equivalent to one of the geometries* $(X, \operatorname{Isom} X)$ *where* X *is one of* E^3*,* H^3*,* S^3*,* $\mathbb{R} \times S^2$*,* $\mathbb{R} \times H^2$*,* $\widehat{SL}(2, \mathbb{R})$*, Nil, or Solv*

It's worth noting that this is equivalent to considering to the nine classes discuss by Isenberg-Jackson in [IJ92]. These classes are (nearly) equivalent to Thurston's eight geometries, the notable difference being that the Thurston only considers metrics which have maximal symmetry. Thus both $\mathbb{R}^3$ and the group of isometries of the Euclidean plane, $E(2)$, are considered as one geometry E^3. The connection between the two sets of classifications is summarized in [IJ92, Table 1]. These classifications aid in determining the Bianchi type of many of the manifolds we examine in subsection 2.2.4. The Bianchi type of each of the

Lie algebras is provided in [RS75].

Proceeding, we will see how these Bianchi types enable us to determine the structure constants of our manifolds. In [EM69] we see that for the unimodular Lie groups that are Bianchi types, there is a basis for the Lie algebra such that we can represent the structure constants as:

$$C_{ij}{}^{k} = \varepsilon_{ijs} E^{ks}$$

where ε_{ijs} is the Levi-Civita symbol. The Levi-Civita symbol captures the permutations of the indices and can be defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1),(3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2),(2,1,3) \\ 0 & \text{if } i=j,\ j=k\ ,k=i \end{cases} .$$

The main idea being that each permutation of indices produces a -1. So if we permute an odd number of times we get a -1, and if we permute an even number of times we get $+1$. It's worth noting that this does not permit repeated indices in the structure constants we consider.

Proposition B.1.2. *For a manifold with diagonal E, structure constants with repeated indices are zero*

Proof. By definition of Levi-Civita symbol,

$$C_{ii}{}^{k} = \varepsilon_{iis} E^{ks} = 0 E^{ks} = 0$$

For E diagonal we know that $E^{ij} = 0$ for $i \neq j$. Thus:

$$C_{ij}{}^{i} = \varepsilon_{ijs} E^{is} = 0$$

□

While the first condition is true in general, the latter condition only holds for diagonal E.

This matrix representation of structure constants is also referenced in [RS75, Chapter 6], where they provide a number of charts detailing the structure constants. In [Hel20], Helliwell uses a basis that diagonalizes the initial matrix E. Per [Mil76], this is always possible for a three-dimensional Lie algebra with structure constants of the above form. Using this, we get [Hel20, Figure 1]:

Type	Group	E
I	$\mathbb{R}^3$	0
II	Nil	$\mathrm{diag}(1,0,0)$
VI_0	$Solv$	$\mathrm{diag}(-1,1,0)$
VII_0	$E(2)$	$\mathrm{diag}(-1,-1,0)$
VIII	$\hat{SL}(2,\mathbb{R})$	$\mathrm{diag}(-1,1,1)$
IX	$\mathbb{S}^3$	id

Table B.1: Diagonal representation of structure constants for 3-dimensional unimodular Lie groups.

B.2 Equations

We will now present equations for the Ricci and scalar curvature tensors in terms of structure constants as presented in [Hel20]. In order to use these in determining the Bach tensor, we will also need to use a formula for the Laplacian of a left-invariant (2,0)-tensor. This will be given in terms of a general (2,0)-tensor, T_{ij}.

$$R_{jk} = -\frac{1}{2}\left(C^{l\ \ p}_{\ j} + C^{p\ \ l}_{\ j}\right)C_{lkp} + \frac{1}{4}C^{lp}_{\ \ j}C_{lpk} + \frac{1}{2}C^{lp}_{\ \ l}\left(C_{pjk} + C_{pkj}\right) \tag{B.1}$$

$$S = -\frac{1}{4}C^{lkp}C_{lkp} - \frac{1}{2}C^{pkl}C_{lkp} - C^{lp}_{\ \ l}C^{k}_{\ pk} \tag{B.2}$$

$$\begin{aligned}(\Delta T)_{ij} = \ & \frac{1}{2}T_{pq}\Big(C^{k\ \ p}_{\ i}C_{kj}^{\ \ \ q} + C^{kp}_{\ \ i}C_{k\ j}^{\ q} + C_{i}^{\ pk}C_{j\ k}^{\ q} \\ & \quad - C^{k\ \ p}_{\ i}C_{k\ j}^{\ q} - C^{k\ \ p}_{\ j}C_{k\ i}^{\ q} - C^{k\ \ p}_{\ i}C_{j\ k}^{\ q} \\ & \quad - C^{k\ \ p}_{\ j}C_{i\ k}^{\ q} + C^{kp}_{\ \ i}C_{j\ k}^{\ q} + C^{kp}_{\ \ j}C_{i\ k}^{\ q}\Big) \\ & + \frac{1}{4}T_{qj}\Big((C^{kp}_{\ \ i} - C^{k\ \ p}_{\ i} + C_{i}^{\ pk})(C_{k\ p}^{\ q} - C_{kp}^{\ \ \ q} + C_{p\ k}^{\ q}) \\ & \quad + 2C^{kp}_{\ \ k}(C_{p\ i}^{\ q} - C_{pi}^{\ \ \ q})\Big) \\ & + \frac{1}{4}T_{qi}\Big((C^{kp}_{\ \ j} - C^{k\ \ p}_{\ j} + C_{j}^{\ pk})(C_{k\ p}^{\ q} - C_{kp}^{\ \ \ q} + C_{p\ k}^{\ q}) \\ & \quad + 2C^{kp}_{\ \ k}(C_{p\ j}^{\ q} - C_{pj}^{\ \ \ q})\Big)\end{aligned} \tag{B.3}$$

We will use these equations to get the components of (1.7) from Section 1.3. We restate the equation here for convenience.

$$\begin{aligned}B_{00} &= \left(-\frac{1}{12}(\Delta^{(2)}S^{(2)}) - \frac{1}{4}\left[(|\operatorname{Ric}|^{(2)})^2 - \frac{1}{3}(S^{(2)})^2\right]\right)g_{00} \\ B_{jk} &= \frac{1}{2}\Delta^{(2)}R^{(2)}_{jk} - \frac{1}{12}\Delta^{(2)}S^{(2)}g_{jk} - \frac{1}{6}S^{(2)}_{;jk} - 2g^{il}R^{(2)}_{ij}R^{(2)}_{lk} \\ &\quad + \frac{7}{6}S^{(2)}R^{(2)}_{jk} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{jk} - \frac{5}{12}(S^{(2)})^2 g_{jk}\end{aligned}$$

B.3 Example

In the following example we will calculate the Bach tensor of $\mathbb{R} \times Nil$. It should be noted that this is the most straightforward example and is the only example that is feasible to do by hand. The remaining manifolds require the use of computing software.

From the table above we see that for Nil, the matrix the will yield structure constants is given by:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again we use the fact that:

$$C_{ij}{}^{k} = \varepsilon_{ijs} E^{ks}$$

where ε_{ijs} is the Levi-Civita symbol. We know that E^{11} is the only nonzero entry, so we need only consider the cases where $k = s = 1$:

$$C_{ij}{}^{1} = \varepsilon_{ij1} E^{11}$$

Using the Levi-Civita symbol, we work through a couple of the computations to demonstrate a way one might approach working with the Levi-Civita symbol.

$$C_{23}{}^{1} = \varepsilon_{231} E^{11} = \varepsilon_{231} = -\varepsilon_{213} = \varepsilon_{123} = 1$$
$$C_{32}{}^{1} = \varepsilon_{321} E^{11} = \varepsilon_{321} = -\varepsilon 231 = +\varepsilon 213 = -\varepsilon_{123} = -1$$

Proceeding, we see that these are in fact the only nonzero structure constants.

$$C_{23}{}^{1} = 1$$
$$C_{32}{}^{1} = -1$$
$$C_{11}{}^{1} = C_{12}{}^{1} = C_{13}{}^{1} = C_{21}{}^{1} = C_{22}{}^{1} = C_{31}{}^{1} = C_{33}{}^{1} = 0$$
$$C_{11}{}^{2} = C_{12}{}^{2} == C_{13}{}^{2} = C_{21}{}^{2} = C_{22}{}^{2} = C_{23}{}^{2} = C_{31}{}^{2} = C_{32}{}^{2} = C_{33}{}^{2} = 0$$
$$C_{11}{}^{3} = C_{12}{}^{3} == C_{13}{}^{3} = C_{21}{}^{3} = C_{22}{}^{3} = C_{23}{}^{3} = C_{31}{}^{3} = C_{32}{}^{3} = C_{33}{}^{3} = 0$$

By the spectral theory, we can pick a basis to diagonalize out metric g, so $g_{jk} = 0$ for $j \neq k$.

Because we are working on the manifold $\mathbb{R} \times Nil$ we know that:

$$R_{00} = 0 \qquad \text{and} \qquad S^{(1)} = 0$$

Proceeding to examine the components that correspond to Nil we use the equation for the Ricci tensor defined in terms of structure constants.

$$\begin{aligned} R_{jk} = &-\frac{1}{2}\left(g^{ls}C_{sj}{}^{p} + g^{ps}C_{sj}{}^{l}\right) g_{ps}C_{lk}{}^{s} + \frac{1}{4}(g^{ls}g^{pr}g_{jt}C_{sr}{}^{t})(g_{ks}C_{lp}{}^{s}) \\ &+ \frac{1}{2}g^{ls}g^{pr}g_{lt}C_{sr}{}^{t}\left(g_{ks}C_{pj}{}^{s} + g_{js}C_{pk}{}^{s}\right) \end{aligned}$$

Using the fact that the metric is orthogonal we see that:

$$\begin{aligned} R_{jk} = &-\frac{1}{2}\left(g^{ll}C_{lj}{}^{p} + g^{pp}C_{pj}{}^{l}\right) g_{pp}C_{lk}{}^{p} + \frac{1}{4}(g^{ll}g^{pp}g_{jj}C_{lp}{}^{j})(g_{kk}C_{lp}{}^{k}) \\ &+ \frac{1}{2}g^{ll}g^{pp}g_{ll}C_{lp}{}^{l}\left(g_{kk}C_{pj}{}^{k} + g_{jj}C_{pk}{}^{j}\right) \end{aligned}$$

Proposition B.3.1. *A 3-manifold, M, with diagonal E has a diagonal Ricci tensor.*

Proof. Suppose $j \neq k$. From Proposition B.1.2, we know that structure constants with repeated indices are zero. In order for the first and second term to be nonzero, j, k, l, and p need to be distinct:

- $j \neq k$ by assumption
- $l \neq p \neq j$ else we have a repeated index
- $l \neq p \neq k$ for the same reason

Thus $j \neq k \neq l \neq p$. However, because M is a 3-manifold, we only have 3 available indices. Thus, by the pigeonhole principle, one index must be repeated. The last term is zero because of repeated indices.

Thus, the Ricci tensor is diagonal. □

Proceeding, we calculate the remaining pieces of (1.7).

$$\begin{aligned}
R_{11} = &-\frac{1}{2}\left[\left(g^{22}C_{21}{}^{3} + g^{33}C_{31}{}^{2}\right)g_{33}C_{21}{}^{3} + \left(g^{33}C_{31}{}^{2} + g^{22}C_{21}{}^{3}\right)g_{22}C_{31}{}^{2}\right] \\
&+ \frac{1}{4}\left[(g^{22}g^{33}g_{11}C_{23}{}^{1})(g_{11}C_{23}{}^{1}) + (g^{33}g^{22}g_{11}C_{32}{}^{1})(g_{11}C_{32}{}^{1})\right] \\
&+ \frac{1}{2}g^{ll}g^{pp}g_{ll}C_{lp}{}^{l}\left(g_{11}C_{p1}{}^{1} + g_{11}C_{p1}{}^{1}\right) \\
=&\frac{1}{4}\left[(g^{22}g^{33}g_{11}C_{23}{}^{1})(g_{11}C_{23}{}^{1}) + (g^{33}g^{22}g_{11}C_{32}{}^{1})(g_{11}C_{32}{}^{1})\right] \\
=&\frac{1}{4}\left[g^{22}g^{33}(g_{11})^{2} + (g^{33}g^{22}g_{11}(-1))(g_{11}(-1))\right] \\
=&\frac{1}{2}\left[\frac{1}{g_{22}}\frac{1}{g_{33}}(g_{11})^{2}\right] \\
=&\frac{(g_{11})^{2}}{2g_{22}g_{33}}
\end{aligned}$$

$$\begin{aligned}
R_{22} = &-\frac{1}{2}\left(g^{ll}C_{l2}{}^{p} + g^{pp}C_{p2}{}^{l}\right)g_{pp}C_{l2}{}^{p} + \frac{1}{4}(g^{ll}g^{pp}g_{22}C_{lp}{}^{2})(g_{22}C_{lp}{}^{2}) \\
&+ \frac{1}{2}g^{ll}g^{pp}g_{ll}C_{lp}{}^{l}\left(g_{22}C_{p2}{}^{2} + g_{22}C_{p2}{}^{2}\right) \\
= &-\frac{1}{2}\left(g^{11}C_{12}{}^{3} + g^{33}C_{32}{}^{1}\right)g_{33}C_{12}{}^{3} - \frac{1}{2}\left(g^{33}C_{32}{}^{1} + g^{11}C_{12}{}^{3}\right)g_{11}C_{32}{}^{1} \\
= &-\frac{1}{2}\left(\frac{1}{g_{33}}(-1)\right)g_{11}(-1) \\
= &-\frac{g_{11}}{2g_{33}}
\end{aligned}$$

$$\begin{aligned}
R_{33} = &-\frac{1}{2}\left(g^{ll}C_{l3}{}^{p} + g^{pp}C_{p3}{}^{l}\right)g_{pp}C_{l3}{}^{p} + \frac{1}{4}(g^{ll}g^{pp}g_{33}C_{lp}{}^{3})(g_{33}C_{lp}{}^{3}) \\
&+ \frac{1}{2}g^{ll}g^{pp}g_{ll}C_{lp}{}^{l}\left(g_{33}C_{p3}{}^{3} + g_{33}C_{p3}{}^{3}\right) \\
= &-\frac{1}{2}\left(g^{11}C_{13}{}^{2} + g^{22}C_{23}{}^{1}\right)g_{22}C_{13}{}^{2} - \frac{1}{2}\left(g^{22}C_{23}{}^{1} + g^{11}C_{13}{}^{2}\right)g_{11}C_{23}{}^{1} \\
= &-\frac{1}{2}\left(\frac{1}{g_{22}}\right)g_{11}
\end{aligned}$$

$$= -\frac{g_{11}}{2g_{22}}$$

$$\begin{aligned}
(|\operatorname{Ric}|^{(2)})^2 &= \langle \operatorname{Ric}, \operatorname{Ric}\rangle \\
&= g^{il}g^{jk}R_{ij}R_{kl} \\
&= g^{ii}g^{jj}R_{ij}R_{ij} \\
&= \left(g^{ii}R_{ii}\right)^2 \\
&= \left(g^{11}R_{11}\right)^2 + \left(g^{22}R_{22}\right)^2 + \left(g^{33}R_{33}\right)^2 \\
&= \left(g^{11}\frac{(g_{11})^2}{2g_{22}g_{33}}\right)^2 + \left(-g^{22}\frac{g_{11}}{2g_{33}}\right)^2 + \left(-g^{33}\frac{g_{11}}{2g_{22}}\right)^2 \\
&= \left(\frac{g_{11}}{2g_{22}g_{33}}\right)^2 + \left(-\frac{g_{11}}{2g_{22}g_{33}}\right)^2 + \left(-\frac{g_{11}}{2g_{22}g_{33}}\right)^2 \\
&= \frac{(g_{11})^2}{4(g_{22})^2(g_{33})^2} + \frac{(g_{11})^2}{4(g_{22})^2(g_{33})^2} + \frac{(g_{11})^2}{4(g_{22})^2(g_{33})^2} \\
&= \frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2}
\end{aligned}$$

$$\begin{aligned}
S^{(2)} &= -\frac{1}{4}C^{lkp}C_{lkp} - \frac{1}{2}C^{pkl}C_{lkp} - C^{lp}{}_{l}C^{k}{}_{pk} \\
&= -\frac{1}{4}\left[C^{l1p}C_{l1p} + C^{l2p}C_{l2p} + C^{l3p}C_{l3p}\right] \\
&\quad - \frac{1}{2}\left[C^{p1l}C_{l1p} + C^{p2l}C_{l2p} + C^{p3l}C_{l3p}\right] \\
&\quad - \left[C^{l1}{}_{l}C^{k}{}_{1k} + C^{l2}{}_{l}C^{k}{}_{2k} + C^{l3}{}_{l}C^{k}{}_{3k}\right] \\
&= -\frac{1}{4}\left[C^{213}C_{213} + C^{312}C_{312} + C^{123}C_{123} + C^{321}C_{321} + C^{132}C_{132} + C^{231}C_{231}\right] \\
&\quad - \frac{1}{2}\left[C^{312}C_{213} + C^{213}C_{312} + C^{321}C_{123} + C^{123}C_{321} + C^{231}C_{132} + C^{132}C_{231}\right] \\
&\quad - \left[C^{21}{}_{2}C^{3}{}_{13} + C^{31}{}_{3}C^{2}{}_{12} + C^{12}{}_{1}C^{3}{}_{23} + C^{32}{}_{3}C^{1}{}_{21} + C^{13}{}_{1}C^{2}{}_{32} + C^{23}{}_{2}C^{1}{}_{31}\right] \\
&= -\frac{1}{4}\left[g^{2s}g^{1r}C_{sr}{}^{3}g_{3t}C_{21}{}^{t} + g^{3s}g^{1r}C_{sr}{}^{2}g_{2t}C_{31}{}^{t} + g^{1s}g^{2r}C_{sr}{}^{3}g_{3t}C_{12}{}^{t}\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad +g^{3s}g^{2r}C_{sr}{}^{1}g_{1t}C_{32}{}^{t} + g^{1s}g^{3r}C_{sr}{}^{2}g_{2t}C_{13}{}^{t} + g^{2s}g^{3r}C_{sr}{}^{1}g_{1t}C_{23}{}^{t}\big] \\
&-\frac{1}{2}\big[g^{3s}g^{1r}C_{sr}{}^{2}g_{3t}C_{21}{}^{t} + g^{2s}g^{1t}C_{sr}{}^{3}g_{2t}C_{31}{}^{t} + g^{3s}g^{2r}C_{sr}{}^{1}g_{3t}C_{12}{}^{t} \\
&\quad +g^{1s}g^{2r}C_{sr}{}^{3}g_{1t}C_{32}{}^{t} + g^{2s}g^{3r}C_{sr}{}^{1}g_{2t}C_{13}{}^{t} + g^{1s}g^{3r}C_{sr}{}^{2}g_{1t}C_{23}{}^{t}\big] \\
=&-\frac{1}{4}\big[g^{22}g^{11}C_{21}{}^{3}g_{33}C_{21}{}^{3} + g^{33}g^{11}C_{31}{}^{2}g_{22}C_{31}{}^{2} + g^{11}g^{22}C_{12}{}^{3}g_{33}C_{12}{}^{3} \\
&\quad +g^{33}g^{22}C_{32}{}^{1}g_{11}C_{32}{}^{1} + g^{11}g^{33}C_{13}{}^{2}g_{22}C_{13}{}^{2} + g^{22}g^{33}C_{23}{}^{1}g_{11}C_{23}{}^{1}\big] \\
&-\frac{1}{2}\big[g^{33}g^{11}C_{31}{}^{2}g_{33}C_{21}{}^{3} + g^{22}g^{11}C_{21}{}^{3}g_{22}C_{31}{}^{2} + g^{32}g^{22}C_{32}{}^{1}g_{33}C_{12}{}^{3} \\
&\quad +g^{11}g^{22}C_{12}{}^{3}g_{11}C_{32}{}^{1} + g^{22}g^{33}C_{23}{}^{1}g_{22}C_{13}{}^{2} + g^{11}g^{33}C_{13}{}^{2}g_{11}C_{23}{}^{1}\big] \\
=&-\frac{1}{4}\big[g^{33}g^{22}g_{11} + g^{22}g^{33}g_{11}\big] \\
=&-\frac{1}{2}\left[\frac{1}{g_{33}}\frac{1}{g_{22}}g_{11}\right] \\
=&-\frac{g_{11}}{2g_{22}g_{33}} \\
=&-\frac{g_{11}}{2g_{22}g_{33}}\cdot\frac{g_{00}g_{11}}{g_{00}g_{11}} \\
=&-\frac{g_{00}(g_{11})^2}{2\det g}
\end{aligned}
$$

$$
\begin{aligned}
\Delta R_{11} =\ & \frac{1}{2}R_{pq}\Big(C^{k}{}_{1}{}^{p}C_{k1}{}^{q} + C^{kp}{}_{1}C_{k}{}^{q}{}_{1} + C_{1}{}^{pk}C_{1}{}^{q}{}_{k} - C^{k}{}_{1}{}^{p}C_{k}{}^{q}{}_{1} - C^{k}{}_{1}{}^{p}C_{k}{}^{q}{}_{1} \\
&\quad -C^{k}{}_{1}{}^{p}C_{1}{}^{q}{}_{k} - C^{k}{}_{1}{}^{p}C_{1}{}^{q}{}_{k} + C^{kp}{}_{1}C_{1}{}^{q}{}_{k} + C^{kp}{}_{1}C_{1}{}^{q}{}_{k}\Big) \\
&+\frac{1}{2}R_{q1}\Big((C^{kp}{}_{1} - C^{k}{}_{1}{}^{p} + C_{1}{}^{pk})(C_{k}{}^{q}{}_{p} - C_{kp}{}^{q} + C_{p}{}^{q}{}_{k}) + 2C^{kp}{}_{k}(C_{p}{}^{q}{}_{1} - C_{p1}{}^{q})\Big) \\
=\ & \frac{1}{2}R_{pp}\Big(C^{k}{}_{1}{}^{p}C_{k1}{}^{p} + C^{kp}{}_{1}C_{k}{}^{p}{}_{1} + C_{1}{}^{pk}C_{1}{}^{p}{}_{k} - C^{k}{}_{1}{}^{p}C_{k}{}^{p}{}_{1} - C^{k}{}_{1}{}^{p}C_{k}{}^{p}{}_{1} \\
&\quad -C^{k}{}_{1}{}^{p}C_{1}{}^{p}{}_{k} - C^{k}{}_{1}{}^{p}C_{1}{}^{p}{}_{k} + C^{kp}{}_{1}C_{1}{}^{p}{}_{k} + C^{kp}{}_{1}C_{1}{}^{p}{}_{k}\Big) \\
&+\frac{1}{2}R_{11}\Big((C^{kp}{}_{1} - C^{k}{}_{1}{}^{p} + C_{1}{}^{pk})(C_{k}{}^{1}{}_{p} - C_{kp}{}^{1} + C_{p}{}^{1}{}_{k}) + 2C^{kp}{}_{k}(C_{p}{}^{1}{}_{1} - C_{p1}{}^{1})\Big) \\
=\ & \frac{1}{2}R_{11}\big(C^{k}{}_{1}{}^{1}C_{k1}{}^{1} + C^{k1}{}_{1}C_{k}{}^{1}{}_{1} + C_{1}{}^{1k}C_{1}{}^{1}{}_{k} - C^{k}{}_{1}{}^{1}C_{k}{}^{1}{}_{1} - C^{k}{}_{1}{}^{1}C_{k}{}^{1}{}_{1} \\
&\quad -C^{k}{}_{1}{}^{1}C_{1}{}^{1}{}_{k} - C^{k}{}_{1}{}^{1}C_{1}{}^{1}{}_{k} + C^{k1}{}_{1}C_{1}{}^{1}{}_{k} + C^{k1}{}_{1}C_{1}{}^{1}{}_{k}\big)
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{2} R_{22} \left(C^{k}{}_{1}{}^{2} C_{k1}{}^{2} + C^{k2}{}_{1} C_{k}{}^{2}{}_{1} + C_{1}{}^{2k} C_{1}{}^{2}{}_{k} - C^{k}{}_{1}{}^{2} C_{k}{}^{2}{}_{1} - C^{k}{}_{1}{}^{2} C_{k}{}^{2}{}_{1} \right. \\
& \qquad \left. - C^{k}{}_{1}{}^{2} C_{1}{}^{2}{}_{k} - C^{k}{}_{1}{}^{2} C_{1}{}^{2}{}_{k} + C^{k2}{}_{1} C_{1}{}^{2}{}_{k} + C^{k2}{}_{1} C_{1}{}^{2}{}_{k} \right) \\
& + \frac{1}{2} R_{33} \left(C^{k}{}_{1}{}^{3} C_{k1}{}^{3} + C^{k3}{}_{1} C_{k}{}^{3}{}_{1} + C_{1}{}^{3k} C_{1}{}^{3}{}_{k} - C^{k}{}_{1}{}^{3} C_{k}{}^{3}{}_{1} - C^{k}{}_{1}{}^{3} C_{k}{}^{3}{}_{1} \right. \\
& \qquad \left. - C^{k}{}_{1}{}^{3} C_{1}{}^{3}{}_{k} - C^{k}{}_{1}{}^{3} C_{1}{}^{3}{}_{k} + C^{k3}{}_{1} C_{1}{}^{3}{}_{k} + C^{k3}{}_{1} C_{1}{}^{3}{}_{k} \right) \\
& + \frac{1}{2} R_{11} \left((C^{kp}{}_{1} - C^{k}{}_{1}{}^{p} + C_{1}{}^{pk})(C_{k}{}^{1}{}_{p} - C_{kp}{}^{1} + C_{p}{}^{1}{}_{k}) \right) \\
= \; & \frac{1}{2} R_{22} \left(C^{3}{}_{1}{}^{2} C_{31}{}^{2} + C^{32}{}_{1} C_{3}{}^{2}{}_{1} + C_{1}{}^{23} C_{1}{}^{2}{}_{3} - 2C^{3}{}_{1}{}^{2} C_{3}{}^{2}{}_{1} - 2C^{3}{}_{1}{}^{2} C_{1}{}^{2}{}_{3} + 2C^{32}{}_{1} C_{1}{}^{2}{}_{3} \right) \\
& + \frac{1}{2} R_{33} \left(C^{2}{}_{1}{}^{3} C_{21}{}^{3} + C^{23}{}_{1} C_{2}{}^{3}{}_{1} + C_{1}{}^{32} C_{1}{}^{3}{}_{2} - 2C^{2}{}_{1}{}^{3} C_{2}{}^{3}{}_{1} - 2C^{2}{}_{1}{}^{3} C_{1}{}^{3}{}_{2} + 2C^{23}{}_{1} C_{1}{}^{3}{}_{2} \right) \\
& + \frac{1}{2} R_{11} \left((C^{23}{}_{1} - C^{2}{}_{1}{}^{3} + C_{1}{}^{32})(C_{2}{}^{1}{}_{3} - C_{23}{}^{1} + C_{3}{}^{1}{}_{2}) \right. \\
& \qquad \left. + (C^{32}{}_{1} - C^{3}{}_{1}{}^{2} + C_{1}{}^{23})(C_{3}{}^{1}{}_{2} - C_{32}{}^{1} + C_{2}{}^{1}{}_{3}) \right) \\
= \; & \frac{1}{2} R_{22} \left(g^{33} C_{31}{}^{2} C_{31}{}^{2} + g^{33} g^{22} g_{11} C_{32}{}^{1} g^{22} g_{11} C_{32}{}^{1} + g^{22} g_{1} 23 g^{22} g_{33} C_{12}{}^{3} \right. \\
& \qquad \left. - 2g^{33} C_{31}{}^{2} g^{22} g_{11} C_{32}{}^{1} - 2g^{33} C_{31}{}^{2} g^{22} g_{33} C_{12}{}^{3} + 2g^{33} g^{22} g_{11} C_{32}{}^{1} g^{22} g_{33} C_{12}{}^{3} \right) \\
& + \frac{1}{2} R_{33} \left(g^{22} C_{21}{}^{3} C_{21}{}^{3} + g^{22} g^{33} g_{11} C_{23}{}^{1} g^{33} g_{11} C_{23}{}^{1} + g^{33} C_{13}{}^{2} g^{33} g_{22} C_{13}{}^{2} - \right. \\
& \qquad \left. 2g^{22} C_{21}{}^{3} g^{33} g_{11} C_{23}{}^{1} - 2g^{22} C_{21}{}^{3} g^{33} g_{22} C_{13}{}^{2} + 2g^{22} g^{33} g_{11} C_{23}{}^{1} g^{33} g_{22} C_{13}{}^{2} \right) \\
& + \frac{1}{2} R_{11} \left((g^{22} g^{33} g_{11} C_{23}{}^{1} - g^{22} C_{21}{}^{3} + g^{33} C_{13}{}^{2})(g^{11} g_{33} C_{21}{}^{3} - C_{23}{}^{1} + g^{11} g_{22} C_{31}{}^{2}) \right. \\
& \qquad \left. + (g^{33} g^{22} g_{11} C_{32}{}^{1} - g^{33} C_{31}{}^{2} + g^{22} C_{12}{}^{3})(g^{11} g_{22} C_{31}{}^{2} - C_{32}{}^{1} + g^{11} g_{33} C_{21}{}^{3}) \right) \\
= \; & \frac{1}{2} R_{22} \left(g^{33} g^{22} g_{11} C_{32}{}^{1} g^{22} g_{11} C_{32}{}^{1} \right) + \frac{1}{2} R_{33} \left(g^{22} g^{33} g_{11} C_{23}{}^{1} g^{33} g_{11} C_{23}{}^{1} \right) \\
& + \frac{1}{2} R_{11} \left((g^{22} g^{33} g_{11} C_{23}{}^{1})(-C_{23}{}^{1}) + (g^{33} g^{22} g_{11} C_{32}{}^{1})(-C_{32}{}^{1}) \right) \\
= \; & \frac{1}{2} R_{22} \left(g^{33} (g^{22})^{2} (g_{11})^{2} \right) + \frac{1}{2} R_{33} \left(g^{22} (g^{33})^{2} (g_{11})^{2} \right) + \frac{1}{2} R_{11} \left(-2 g^{22} g^{33} g_{11} \right) \\
= \; & \left(-\frac{g_{11}}{4 g_{33}} \right) \left(\frac{(g_{11})^{2}}{g_{33} (g_{22})^{2}} \right) + \left(-\frac{g_{11}}{4 g_{22}} \right) \left(\frac{(g_{11})^{2}}{g_{22} (g_{33})^{2}} \right) + \left(-\frac{(g_{11})^{2}}{2 g_{22} g_{33}} \right) \left(\frac{g_{11}}{g_{22} g_{33}} \right) \\
= \; & - \frac{(g_{11})^{3}}{(g_{33})^{2} (g_{22})^{2}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta R_{22} = \; & \frac{1}{2} R_{pq} \Big(C^{k}{}_{2}{}^{p} C_{k2}{}^{q} + C^{kp}{}_{2} C_{k}{}^{q}{}_{2} + C_{2}{}^{pk} C_{2}{}^{q}{}_{k} - C^{k}{}_{2}{}^{p} C_{k}{}^{q}{}_{2} - C^{k}{}_{2}{}^{p} C_{k}{}^{q}{}_{2} \\
& \qquad - C^{k}{}_{2}{}^{p} C_{2}{}^{q}{}_{k} - C^{k}{}_{2}{}^{p} C_{2}{}^{q}{}_{k} + C^{kp}{}_{2} C_{2}{}^{q}{}_{k} + C^{kp}{}_{2} C_{2}{}^{q}{}_{k} \Big) \\
& + \frac{1}{2} R_{q2} \Big((C^{kp}{}_{2} - C^{k}{}_{2}{}^{p} + C_{2}{}^{pk})(C_{k}{}^{q}{}_{p} - C_{kp}{}^{q} + C_{p}{}^{q}{}_{k}) + 2C^{kp}{}_{k}(C_{p}{}^{q}{}_{2} - C_{p2}{}^{q}) \Big) \\
= \; & \frac{1}{2} R_{pp} \Big(C^{k}{}_{2}{}^{p} C_{k2}{}^{p} + C^{kp}{}_{2} C_{k}{}^{p}{}_{2} + C_{2}{}^{pk} C_{2}{}^{p}{}_{k} - C^{k}{}_{2}{}^{p} C_{k}{}^{p}{}_{2} - C^{k}{}_{2}{}^{p} C_{k}{}^{p}{}_{2} \\
& \qquad - C^{k}{}_{2}{}^{p} C_{2}{}^{p}{}_{k} - C^{k}{}_{2}{}^{p} C_{2}{}^{p}{}_{k} + C^{kp}{}_{2} C_{2}{}^{p}{}_{k} + C^{kp}{}_{2} C_{2}{}^{p}{}_{k} \Big) \\
& + \frac{1}{2} R_{22} \Big((C^{kp}{}_{2} - C^{k}{}_{2}{}^{p} + C_{2}{}^{pk})(C_{k}{}^{2}{}_{p} - C_{kp}{}^{2} + C_{p}{}^{2}{}_{k}) + 2C^{kp}{}_{k}(C_{p}{}^{2}{}_{2} - C_{p2}{}^{2}) \Big) \\
= \; & \frac{1}{2} R_{11} \big(C^{k}{}_{2}{}^{1} C_{k2}{}^{1} + C^{k1}{}_{2} C_{k}{}^{1}{}_{2} + C_{2}{}^{1k} C_{2}{}^{1}{}_{k} - C^{k}{}_{2}{}^{1} C_{k}{}^{1}{}_{2} - C^{k}{}_{2}{}^{1} C_{k}{}^{1}{}_{2} \\
& \qquad - C^{k}{}_{2}{}^{1} C_{2}{}^{1}{}_{k} - C^{k}{}_{2}{}^{1} C_{2}{}^{1}{}_{k} + C^{k1}{}_{2} C_{2}{}^{1}{}_{k} + C^{k1}{}_{2} C_{2}{}^{1}{}_{k} \big) \\
& + \frac{1}{2} R_{33} \big(C^{k}{}_{2}{}^{3} C_{k2}{}^{3} + C^{k3}{}_{2} C_{k}{}^{3}{}_{2} + C_{2}{}^{3k} C_{2}{}^{3}{}_{k} - C^{k}{}_{2}{}^{3} C_{k}{}^{3}{}_{2} - C^{k}{}_{2}{}^{3} C_{k}{}^{3}{}_{2} \\
& \qquad - C^{k}{}_{2}{}^{3} C_{2}{}^{3}{}_{k} - C^{k}{}_{2}{}^{3} C_{2}{}^{3}{}_{k} + C^{k3}{}_{2} C_{2}{}^{3}{}_{k} + C^{k3}{}_{2} C_{2}{}^{3}{}_{k} \big) \\
& + \frac{1}{2} R_{22} \Big((C^{kp}{}_{2} - C^{k}{}_{2}{}^{p} + C_{2}{}^{pk})(C_{k}{}^{2}{}_{p} - C_{kp}{}^{2} + C_{p}{}^{2}{}_{k}) \Big) \\
= \; & \frac{1}{2} R_{11} \big(C^{3}{}_{2}{}^{1} C_{32}{}^{1} + C^{31}{}_{2} C_{3}{}^{1}{}_{2} + C_{2}{}^{13} C_{2}{}^{1}{}_{3} \\
& \qquad - 2C^{3}{}_{2}{}^{1} C_{3}{}^{1}{}_{2} - 2C^{3}{}_{2}{}^{1} C_{2}{}^{1}{}_{3} + 2C^{31}{}_{2} C_{2}{}^{1}{}_{3} \big) \\
& + \frac{1}{2} R_{33} \big(C^{1}{}_{2}{}^{3} C_{12}{}^{3} + C^{13}{}_{2} C_{1}{}^{3}{}_{2} + C_{2}{}^{31} C_{2}{}^{3}{}_{1} \\
& \qquad - 2C^{1}{}_{2}{}^{3} C_{1}{}^{3}{}_{2} - 2C^{1}{}_{2}{}^{3} C_{2}{}^{3}{}_{1} + 2C^{13}{}_{2} C_{2}{}^{3}{}_{1} \big) \\
& + \frac{1}{2} R_{22} \big((C^{13}{}_{2} - C^{1}{}_{2}{}^{3} + C_{2}{}^{31})(C_{1}{}^{2}{}_{3} - C_{13}{}^{2} + C_{3}{}^{2}{}_{1}) \\
& \qquad + (C^{31}{}_{2} - C^{3}{}_{2}{}^{1} + C_{2}{}^{13})(C_{3}{}^{2}{}_{1} - C_{31}{}^{2} + C_{1}{}^{2}{}_{3}) \big) \\
= \; & \frac{1}{2} R_{11} \big(g^{33} C_{32}{}^{1} C_{32}{}^{1} + g^{33} g^{11} g_{22} C_{31}{}^{2} g^{11} g_{22} C_{31}{}^{2} + c^{11} C_{21}{}^{3} g^{11} g_{33} C_{21}{}^{3} \\
& \qquad - 2g^{33} C_{32}{}^{1} g^{11} g_{22} C_{31}{}^{2} - 2g^{33} C_{32}{}^{1} g^{11} g_{33} C_{21}{}^{3} + 2g^{33} g^{11} g_{22} C_{32}{}^{1} g^{11} g_{33} C_{21}{}^{3} \big) \\
& + \frac{1}{2} R_{33} \big(g^{11} C_{12}{}^{3} C_{12}{}^{3} + g^{11} g^{33} g_{22} C_{13}{}^{2} g^{33} g_{22} C_{13}{}^{2} + g^{33} C_{23}{}^{1} g^{33} g_{11} C_{23}{}^{1} \\
& \qquad - 2g^{11} C_{12}{}^{3} g^{33} g_{22} C_{13}{}^{2} - 2g^{11} C_{12}{}^{3} g^{33} g_{11} C_{23}{}^{1} + 2g^{11} g^{33} g_{22} C_{13}{}^{2} g^{33} g_{11} C_{23}{}^{1} \big)
\end{aligned}
$$

$$+\frac{1}{2}R_{22}\left((g^{11}g^{33}g_{22}C_{13}{}^{2}-g^{11}C_{12}{}^{3}+g^{33}C_{23}{}^{1})(g^{22}g_{33}C_{12}{}^{3}-C_{13}{}^{2}+g^{22}g_{11}C_{32}{}^{1})\right.$$

$$\left.+(g^{33}g^{11}g_{22}C_{31}{}^{2}-g^{33}C_{32}{}^{1}+g^{11}C_{21}{}^{3})(g^{22}g_{11}C_{32}{}^{1}-C_{31}{}^{2}+g^{22}g_{33}C_{12}{}^{3})\right)$$

$$=\frac{1}{2}R_{11}g^{33}+\frac{1}{2}R_{33}(g^{33})^2g_{11}+\frac{1}{2}R_{22}\left((g^{33})(-g^{22}g_{11})+(g^{33})(-g^{22}g_{11})\right)$$

$$=\frac{1}{2}\left(\frac{(g_{11})^2}{2g_{22}g_{33}}\right)\left(\frac{1}{g_{33}}\right)+\frac{1}{2}\left(-\frac{g_{11}}{2g_{22}}\right)\left(\frac{g_{11}}{(g_{33})^2}\right)+\left(-\frac{g_{11}}{2g_{33}}\right)\left(\frac{1}{g_{33}}\right)\left(-\frac{g_{11}}{g_{22}}\right)$$

$$=\left(\frac{(g_{11})^2}{4g_{22}(g_{33})^2}\right)-\left(\frac{(g_{11})^2}{4g_{22}(g_{33})^2}\right)+\left(\frac{(g_{11})^2}{2g_{22}(g_{33})^2}\right)$$

$$=\frac{(g_{11})^2}{2g_{22}(g_{33})^2}$$

$$\Delta R_{33}=\frac{1}{2}R_{pq}\left(C^{k}{}_{3}{}^{p}C_{k3}{}^{q}+C^{kp}{}_{3}C_{k}{}^{q}{}_{3}+C_{3}{}^{pk}C_{3}{}^{q}{}_{k}-C^{k}{}_{3}{}^{p}C_{k}{}^{q}{}_{3}-C^{k}{}_{3}{}^{p}C_{k}{}^{q}{}_{3}\right.$$

$$\left.-C^{k}{}_{3}{}^{p}C_{3}{}^{q}{}_{k}-C^{k}{}_{3}{}^{p}C_{3}{}^{q}{}_{k}+C^{kp}{}_{3}C_{3}{}^{q}{}_{k}+C^{kp}{}_{3}C_{3}{}^{q}{}_{k}\right)$$

$$+\frac{1}{2}R_{q3}\left((C^{kp}{}_{3}-C^{k}{}_{3}{}^{p}+C_{3}{}^{pk})(C_{k}{}^{q}{}_{p}-C_{kp}{}^{q}+C_{p}{}^{q}{}_{k})+2C^{kp}{}_{k}(C_{p}{}^{q}{}_{3}-C_{p3}{}^{q})\right)$$

$$=\frac{1}{2}R_{pp}\left(C^{k}{}_{3}{}^{p}C_{k3}{}^{p}+C^{kp}{}_{3}C_{k}{}^{p}{}_{3}+C_{3}{}^{pk}C_{3}{}^{p}{}_{k}-C^{k}{}_{3}{}^{p}C_{k}{}^{p}{}_{3}-C^{k}{}_{3}{}^{p}C_{k}{}^{p}{}_{3}\right.$$

$$\left.-C^{k}{}_{3}{}^{p}C_{3}{}^{p}{}_{k}-C^{k}{}_{3}{}^{p}C_{3}{}^{p}{}_{k}+C^{kp}{}_{3}C_{3}{}^{p}{}_{k}+C^{kp}{}_{3}C_{3}{}^{p}{}_{k}\right)$$

$$+\frac{1}{2}R_{33}\left((C^{kp}{}_{3}-C^{k}{}_{3}{}^{p}+C_{3}{}^{pk})(C_{k}{}^{3}{}_{p}-C_{kp}{}^{3}+C_{p}{}^{3}{}_{k})+2C^{kp}{}_{k}(C_{p}{}^{3}{}_{3}-C_{p3}{}^{3})\right)$$

$$=\frac{1}{2}R_{11}\left(C^{k}{}_{3}{}^{1}C_{k3}{}^{1}+C^{k1}{}_{3}C_{k}{}^{1}{}_{3}+C_{3}{}^{1k}C_{3}{}^{1}{}_{k}\right.$$

$$\left.-2C^{k}{}_{3}{}^{1}C_{k}{}^{1}{}_{3}-2C^{k}{}_{3}{}^{1}C_{3}{}^{1}{}_{k}+2C^{k1}{}_{3}C_{3}{}^{1}{}_{k}\right)$$

$$+\frac{1}{2}R_{22}\left(C^{k}{}_{3}{}^{2}C_{k3}{}^{2}+C^{k2}{}_{3}C_{k}{}^{2}{}_{3}+C_{3}{}^{2k}C_{3}{}^{2}{}_{k}\right.$$

$$\left.-2C^{k}{}_{3}{}^{2}C_{k}{}^{2}{}_{3}-2C^{k}{}_{3}{}^{2}C_{3}{}^{2}{}_{k}+2C^{k2}{}_{3}C_{3}{}^{2}{}_{k}\right)$$

$$+\frac{1}{2}R_{33}\left((C^{kp}{}_{3}-C^{k}{}_{3}{}^{p}+C_{3}{}^{pk})(C_{k}{}^{3}{}_{p}-C_{kp}{}^{3}+C_{p}{}^{3}{}_{k})\right)$$

$$=\frac{1}{2}R_{11}\left(C^{2}{}_{3}{}^{1}C_{23}{}^{1}+C^{21}{}_{3}C_{2}{}^{1}{}_{3}+C_{3}{}^{12}C_{3}{}^{1}{}_{2}\right.$$

$$\left.-2C^{2}{}_{3}{}^{1}C_{2}{}^{1}{}_{3}-2C^{2}{}_{3}{}^{1}C_{3}{}^{1}{}_{2}+2C^{21}{}_{3}C_{3}{}^{1}{}_{2}\right)$$

$$+\frac{1}{2}R_{22}\left(C^{1}{}_{3}{}^{2}C_{13}{}^{2}+C^{12}{}_{3}C_{1}{}^{2}{}_{3}+C_{3}{}^{21}C_{3}{}^{2}{}_{1}\right.$$

$$\left.-2C^{1}{}_{3}{}^{2}C_{1}{}^{2}{}_{3}-2C^{1}{}_{3}{}^{2}C_{3}{}^{2}{}_{1}+2C^{12}{}_{3}C_{3}{}^{2}{}_{1}\right)$$

$$
\begin{aligned}
&+\frac{1}{2}R_{33}\left((C^{12}{}_3 - C^1{}_3{}^2 + C_3{}^{21})(C_1{}^3{}_2 - C_{12}{}^3 + C_2{}^3{}_1)\right.\\
&\qquad \left. +(C^{21}{}_3 - C^2{}_3{}^1 + C_3{}^{12})(C_2{}^3{}_1 - C_{21}{}^3 + C_1{}^3{}_2)\right)\\
=\ &\frac{1}{2}R_{11}\left(g^{22}C_{23}{}^1C_{23}{}^1 + g^{22}g^{11}g_{33}C_{21}{}^3g^{11}g_{33}C_{21}{}^3 + g^{11}C_{31}{}^2g^{11}g_{22}C_{31}{}^2\right.\\
&\qquad \left. -2g^{22}C_{23}{}^1g^{11}g_{33}C_{21}{}^3 - 2g^{22}C_{23}{}^1g^{11}g_{22}C_{31}{}^2 + 2g^{22}g^{11}g_{33}C_{21}{}^3g^{11}g_{22}C_{31}{}^2\right)\\
&+\frac{1}{2}R_{22}\left(g^{11}C_{13}{}^2C_{13}{}^2 + g^{11}g^{22}g_{33}C_{12}{}^3g^{22}g_{33}C_{12}{}^3 + g^{22}C_{32}{}^1g^{22}g_{11}C_{32}{}^1\right.\\
&\qquad \left. -2g^{11}C_{13}{}^2g^{22}g_{33}C_{12}{}^3 - 2g^{11}C_{13}{}^2g^{22}g_{11}C_{32}{}^1 + 2g^{11}g^{22}g_{33}C_{12}{}^3g^{22}g_{11}C_{32}{}^1\right)\\
&+\frac{1}{2}R_{33}\left((g^{11}g^{22}g_{33}C_{12}{}^3 - g^{11}C_{13}{}^2 + g^{22}C_{32}{}^1)(g^{33}g_{22}C_{13}{}^2 - C_{12}{}^3 + g^{33}g_{11}C_{23}{}^1)\right.\\
&\qquad \left. +(g^{22}g^{11}g_{33}C_{21}{}^3 - g^{22}C_{23}{}^1 + g^{11}C_{31}{}^2)(g^{33}g_{11}C_{23}{}^1 - C_{21}{}^3 + g^{33}g_{22}C_{13}{}^2)\right)\\
=\ &\frac{1}{2}R_{11}\left(g^{22}\right) + \frac{1}{2}R_{22}\left((g^{22})^2g_{11}\right) + \frac{1}{2}R_{33}\left((-g^{22})(g^{33}g_{11}) + (-g^{22})(g^{33}g_{11})\right)\\
=\ &\frac{1}{2}\left(\frac{(g_{11})^2}{2g_{22}g_{33}}\right)\left(\frac{1}{g_{22}}\right) + \frac{1}{2}\left(-\frac{g_{11}}{2g_{33}}\right)\left(\frac{g_{11}}{(g_{22})^2}\right) + \left(-\frac{g_{11}}{2g_{22}}\right)\left(-\frac{1}{g_{22}}\right)\left(\frac{g_{11}}{g_{33}}\right)\\
=\ &\left(\frac{(g_{11})^2}{4(g_{22})^2g_{33}}\right) - \left(\frac{(g_{11})^2}{4(g_{22})^2g_{33}}\right) + \left(\frac{(g_{11})^2}{2(g_{22})^2g_{33}}\right)\\
=\ &\frac{(g_{11})^2}{2(g_{22})^2g_{33}}
\end{aligned}
$$

$$
\Delta S^{(2)} = 0 \quad \text{and} \quad S_{;jk} = \nabla_j\nabla_k S = 0 \quad \text{because } Nil \text{ is homogeneous.}
$$

To keep with the conventions in [Hel20], let

$$
\beta = \frac{1}{6(\det g)^2}.
$$

The component of the Bach tensor corresponding to $\mathbb{R}$ is as follows.

$$
\begin{aligned}
B_{00} &= \left(-\frac{1}{12}(\Delta^{(2)}S^{(2)}) - \frac{1}{4}\left[(|\operatorname{Ric}|^{(2)})^2 - \frac{1}{3}(S^{(2)})^2\right]\right)g_{00}\\
&= \left(-\frac{1}{12}(0) - \frac{1}{4}\left[\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2} - \frac{1}{3}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)^2\right]\right)g_{00}\\
&= \left(-\frac{1}{4}\left[\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2} - \frac{1}{3}\left(-\frac{g_{11}}{2g_{22}g_{33}}\right)^2\right]\right)g_{00}
\end{aligned}
$$

$$= \left(-\frac{1}{4}\left[\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2} - \frac{(g_{11})^2}{12(g_{22})^2(g_{33})^2}\right]\right)$$
$$= -\frac{g_{00}(g_{11})^2}{6(g_{22})^2(g_{33})^2}$$
$$= -\frac{(g_{00})^3(g_{11})^4}{6(\det g)^2}$$
$$= -\beta(g_{00})^3(g_{11})^4$$

For the components corresponding to Nil we use the equation:

$$B_{jk} = \frac{1}{2}\Delta^{(2)}R^{(2)}_{jk} - \frac{1}{12}\Delta^{(2)}S^{(2)}g_{jk} - \frac{1}{6}S^{(2)}_{;jk} - 2g^{il}R^{(2)}_{ij}R^{(2)}_{lk} + \frac{7}{6}S^{(2)}R^{(2)}_{jk} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{jk} - \frac{5}{12}(S^{(2)})^2 g_{jk}$$

The reader should note that we have dropped the (2) notation in our discussion of the components of the Ricci tensor and their corresponding Laplacians. From context indices $i = 1, 2, 3$ correspond to Nil.

$$\begin{aligned}
B_{11} =& \frac{1}{2}\Delta R_{11} - \frac{1}{12}\Delta S^{(2)}g_{11} - \frac{1}{6}S_{;11} - 2g^{il}R_{i1}R_{l1} + \frac{7}{6}S^{(2)}R_{11} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{11} - \frac{5}{12}(S^{(2)})^2 g_{11} \\
=& \frac{1}{2}\Delta R_{11} - 2g^{11}(R_{11})^2 + \frac{7}{6}S^{(2)}R_{11} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{11} - \frac{5}{12}(S^{(2)})^2 g_{11} \\
=& \frac{1}{2}\left(-\frac{(g_{11})^3}{(g_{33})^2(g_{22})^2}\right) - 2g^{11}\left(\frac{(g_{11})^2}{2g_{22}g_{33}}\right)^2 + \frac{7}{6}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)\left(\frac{(g_{11})^2}{2g_{22}g_{33}}\right) \\
&+ \frac{3}{4}\left(\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2}\right)g_{11} - \frac{5}{12}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)^2 g_{11} \\
=& -\frac{1}{2}\frac{(g_{11})^3}{(g_{33})^2(g_{22})^2} - \frac{1}{2}\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} - \frac{7}{24}\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} + \frac{9}{16}\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} - \frac{5}{48}\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} \\
=& \left(-\frac{1}{2} - \frac{1}{2} - \frac{7}{24} + \frac{9}{16} - \frac{5}{48}\right)\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} \\
=& -\frac{5}{6}\frac{(g_{11})^3}{(g_{22})^2(g_{33})^2} \\
=& -\frac{5}{6}\frac{(g_{00})^2(g_{11})^5}{(\det g)^2}
\end{aligned}$$

$$= -5\beta(g_{00})^2(g_{11})^5$$

$$\begin{aligned}
B_{22} =& \frac{1}{2}\Delta R_{22} - \frac{1}{12}\Delta S^{(2)} g_{22} - \frac{1}{6}S_{;22} - 2g^{il}R_{i2}R_{l2} + \frac{7}{6}S^{(2)}R_{22} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{22} - \frac{5}{12}(S^{(2)})^2 g_{22} \\
=& \frac{1}{2}\Delta R_{22} - 2g^{22}(R_{22})^2 + \frac{7}{6}S^{(2)}R_{22} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{22} - \frac{5}{12}(S^{(2)})^2 g_{22} \\
=& \frac{1}{2}\left(\frac{(g_{11})^2}{2g_{22}(g_{33})^2}\right) - 2g^{22}\left(-\frac{g_{11}}{2g_{33}}\right)^2 + \frac{7}{6}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)\left(-\frac{g_{11}}{2g_{33}}\right) \\
&+ \frac{3}{4}\left(\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2}\right) g_{22} - \frac{5}{12}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)^2 g_{22} \\
=& \frac{1}{4}\left(\frac{(g_{11})^2}{g_{22}(g_{33})^2}\right) - \frac{1}{2}\left(\frac{(g_{11})^2}{g_{22}(g_{33})^2}\right) + \frac{7}{24}\left(\frac{(g_{11})^2}{g_{22}(g_{33})^2}\right) + \frac{9}{16}\left(\frac{(g_{11})^2}{g_{22}(g_{33})^2}\right) - \frac{5}{48}\left(\frac{(g_{11})^2}{g_{22}(g_{33})^2}\right) \\
=& \left(\frac{1}{4} - \frac{1}{2} + \frac{7}{24} + \frac{9}{16} - \frac{5}{48}\right)\frac{(g_{11})^2}{g_{22}(g_{33})^2} \\
=& \frac{1}{2}\frac{(g_{11})^2}{g_{22}(g_{33})^2} \\
=& \frac{1}{2}\frac{(g_{00})^2(g_{11})^4 g_{22}}{(\det g)^2} \\
=& 3\beta(g_{00})^2(g_{11})^4 g_{22}
\end{aligned}$$

$$\begin{aligned}
B_{33} =& \frac{1}{2}\Delta R_{33} - \frac{1}{12}\Delta S^{(2)} g_{33} - \frac{1}{6}S_{;33} - 2g^{il}R_{i3}R_{l3} + \frac{7}{6}S^{(2)}R_{33} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{33} - \frac{5}{12}(S^{(2)})^2 g_{33} \\
=& \frac{1}{2}\Delta R_{33} - 2g^{33}(R_{33})^2 + \frac{7}{6}S^{(2)}R_{33} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{33} - \frac{5}{12}(S^{(2)})^2 g_{33} \\
=& \frac{1}{2}\left(\frac{(g_{11})^2}{2(g_{22})^2 g_{33}}\right) - 2g^{33}\left(-\frac{g_{11}}{2g_{22}}\right)^2 + \frac{7}{6}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)\left(-\frac{g_{11}}{2g_{22}}\right) \\
&+ \frac{3}{4}\left(\frac{3(g_{11})^2}{4(g_{22})^2(g_{33})^2}\right) g_{33} - \frac{5}{12}\left(-\frac{g_{00}(g_{11})^2}{2\det g}\right)^2 g_{33} \\
=& \frac{1}{4}\left(\frac{(g_{11})^2}{(g_{22})^2 g_{33}}\right) - \frac{1}{2}\left(\frac{(g_{11})^2}{(g_{22})^2 g_{33}}\right) + \frac{7}{24}\left(\frac{(g_{11})^2}{(g_{22})^2 g_{33}}\right) + \frac{9}{16}\left(\frac{(g_{11})^2}{(g_{22})^2 g_{33}}\right) - \frac{5}{48}\left(\frac{(g_{11})^2}{(g_{22})^2 g_{33}}\right) \\
=& \left(\frac{1}{4} - \frac{1}{2} + \frac{7}{24} + \frac{9}{16} - \frac{5}{48}\right)\frac{(g_{11})^2}{(g_{22})^2 g_{33}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{(g_{11})^2}{(g_{22})^2 g_{33}} \\
&= \frac{1}{2}\frac{(g_{00})^2 (g_{11})^4 g_{33}}{(\det g)^2} \\
&= 3\beta (g_{00})^2 (g_{11})^4 g_{33}
\end{aligned}
$$

Appendix C

Tensors Referenced

This appendix is here to serve as a list of the tensors discussed in this thesis.

- Riemannian Curvature Tensor [$(3, 1)$ version]

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{id} \Gamma^d_{jk} - \Gamma^l_{jd} \Gamma^d_{ik}.$$

- Riemannian Curvature Tensor [$(4, 0)$ version]

$$Rm(X, Y, Z, W) = g\left(R(X, Y)Z, W\right)$$

$$R_{ijkl} = g_{lm}\left(\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp}\right)$$

- Christoffel Symbol

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

- Ricci Curvature Tensor

$$R_{ij} = R_{kij}{}^{k} = g^{km} R_{kijm}$$

- Scalar Curvature Tensor

$$S = g^{ij} R_{ij}$$

- Weyl Tensor ($n \geqslant 4$)

$$W_{abcd} = R_{abcd} + g_{ac}P_{bd} - g_{ad}P_{bc} - g_{bc}P_{ad} + g_{bd}P_{ac}$$

$$\begin{aligned} W_{abcd} = R_{abcd} &+ \frac{1}{n-2}\left(R_{bd}g_{ac} - R_{bc}g_{ad} - R_{ad}g_{bc} + R_{ac}g_{bd}\right) \\ &- \frac{S}{(n-1)(n-2)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \end{aligned}$$

- Schouten

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right)$$

- Cotton Tensor

$$C_{ijk} = -\frac{n-2}{n-3}\nabla^l W_{ijkl}.$$

$$C_{ijk} = \nabla_i P_{jk} - \nabla_j P_{ik}$$

- Bach Tensor

$$B_{ij} = \frac{1}{n-3}\nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl}$$

$$B_{ij} = g^{lq}P_{ij;lq} - g^{lq}P_{il;jq} + P^{kl}W_{kijl}$$

$$B_{ij} = \frac{1}{n-2}\left(\nabla^k C_{kij} + R^{kl}W_{ikjl}\right)$$

- Divergence of Bach Tensor

$$\mathrm{div}B = \nabla_j B_{ij} = \frac{n-4}{(n-2)^2}C_{ijk}R_{jk}$$

- Bach Tensor on Product Manifold

1×3:

$$
\begin{aligned}
B_{00} &= \left(-\frac{1}{12}(\Delta^{(2)}S^{(2)}) - \frac{1}{4}\left[(|\operatorname{Ric}|^{(2)})^2 - \frac{1}{3}(S^{(2)})^2\right]\right) g_{00} \\
B_{jk} &= \frac{1}{2}\Delta^{(2)}R^{(2)}_{jk} - \frac{1}{12}\Delta^{(2)}S^{(2)}g_{jk} - \frac{1}{6}S^{(2)}_{;jk} - 2\operatorname{tr}^{(2)}(\operatorname{Ric}^{(2)} \otimes \operatorname{Ric}^{(2)})_{jk} \\
&\quad + \frac{7}{6}S^{(2)}R^{(2)}_{jk} + \frac{3}{4}(|\operatorname{Ric}|^{(2)})^2 g_{jk} - \frac{5}{12}(S^{(2)})^2 g_{jk}
\end{aligned}
$$

where $\operatorname{tr}(\operatorname{Ric} \otimes \operatorname{Ric})_{jk} = g^{il}R_{ij}R_{lk}$

2×2:

$$
\begin{aligned}
B_{\mu\nu} &= -\frac{1}{6}\nabla_\mu\nabla_\nu S^{(1)} + \frac{1}{6}g^{(1)}_{\mu\nu}\left[\nabla^\alpha\nabla_\alpha S^{(1)} - \frac{1}{2}\nabla^k\nabla_k S^{(2)} + \frac{1}{4}\left(\left(S^{(2)}\right)^2 - \left(S^{(1)}\right)^2\right)\right] \\
B_{ij} &= -\frac{1}{6}\nabla_i\nabla_j S^{(2)} + \frac{1}{6}g^{(2)}_{ij}\left[\nabla^k\nabla_k S^{(2)} - \frac{1}{2}\nabla^\alpha\nabla_\alpha S^{(1)} + \frac{1}{4}\left(\left(S^{(2)}\right)^2 - \left(S^{(1)}\right)^2\right)\right] \\
B_{\alpha j} &= 0
\end{aligned}
$$

- AOT

$$
\mathcal{O}_n = \frac{1}{(-2)^{\frac{n}{2}-2}\left(\frac{n}{2}-2\right)!}\left(\Delta^{\frac{n}{2}-1}P - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R\right) + T_{n-1}
$$

$$
\begin{aligned}
\mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}\left(P_{ij,k}{}^{k} - P_{k}{}^{k}{}_{,ij}\right) + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm) \\
T_k^m(A) &= \sum_{i_1+\cdots+i_k=m} \nabla^{i_1}A * \cdots * \nabla^{i_k}A
\end{aligned}
$$

$$
\mathcal{O}_{ij} = \frac{1}{3-n}\Delta^{\frac{n}{2}-2}\nabla^l\nabla^k W_{kijl} + \sum_{k=2}^{n/2} T_k^{n-2k}(Rm)
$$

$n = 6$:

$$
\begin{aligned}
\mathcal{O}_{ij} &= B_{ij,k}{}^{k} - 2W_{kijl}B^{kl} - 4P_{k}{}^{k}B_{ij} + 8P^{kl}C_{(ij)k,l} - 4C^{k}{}_{i}{}^{l}C_{ljk} \\
&\quad + 2C_i{}^{kl}C_{jkl} + 4P^{k}{}_{k,l}C_{(ij)}{}^{l} - 4W_{kijl}P^{k}{}_{m}P^{ml}
\end{aligned}
$$

C.1 Rescaling Tensors

In this section we will show the effect of rescaling on a tensor. These calculations specifically aid in the proof of Corollary 2.1.16.

Consider the rescaling given by $\tilde{g} = \lambda g$.

$$g(\nabla_g f, X) = df(X) = \tilde{g}(\nabla_{\hat{g}} f, X) = \lambda g(\nabla_{\hat{g}} f, X) \quad \Longrightarrow \quad \frac{1}{\lambda}\nabla_g f = \nabla_{\hat{g}} f$$

$$\begin{aligned}
\tilde{R} &= \sum \tilde{g}\left(\tilde{\mathrm{Ric}}(\tilde{e_j}), \tilde{e_j}\right) \\
&= \sum \lambda g\left(\frac{1}{\lambda^{3/2}}\,\mathrm{Ric}(e_j), \frac{1}{\sqrt{\lambda}} e_j\right) \\
&= \frac{1}{\lambda}\sum g\left(\mathrm{Ric}(e_j), e_j\right) \\
&= \frac{1}{\lambda} R
\end{aligned}$$

$$g(\nabla_g f, X) = df(X) = \tilde{g}(\nabla_{\hat{g}} f, X) = \lambda g(\nabla_{\hat{g}} f, X) \quad \Longrightarrow \quad \frac{1}{\lambda}\nabla_g f = \nabla_{\hat{g}} f$$

$$\tilde{\nabla}_{\tilde{g}} \tilde{R} = \frac{1}{\lambda}\nabla_g \tilde{R} = \frac{1}{\lambda^2}\nabla_g R$$

$$\begin{aligned}
\tilde{\Delta}\tilde{R} &= \sum \tilde{g}\left(\tilde{\nabla}_{\tilde{e_i}} \tilde{\nabla}_{\tilde{g}} \tilde{R}, \tilde{e_i}\right) \\
&= \sum \tilde{g}\left(\nabla_{\frac{1}{\sqrt{\lambda}} e_i} \frac{1}{\lambda}\nabla_g \left(\frac{1}{\lambda} R\right), \frac{1}{\sqrt{\lambda}} e_i\right) \\
&= \sum \lambda g\left(\frac{1}{\lambda^3}\nabla_{e_i}\nabla_g R, e_i\right) \\
&= \frac{1}{\lambda^2}\Delta R
\end{aligned}$$

$$
\begin{aligned}
\tilde{\Delta}^2\tilde{R} &= \tilde{\Delta}\tilde{\Delta}\tilde{R} \\
&= \sum \tilde{g}\left(\tilde{\nabla}_{\tilde{e_i}}\tilde{\nabla}_{\tilde{g}}\left(\tilde{\Delta}\tilde{R}\right), \tilde{e_i}\right) \\
&= \sum \tilde{g}\left(\nabla_{\frac{1}{\sqrt{\lambda}}e_i}\frac{1}{\lambda}\nabla_g\left(\tilde{\Delta}\tilde{R}\right), \frac{1}{\sqrt{\lambda}}e_i\right) \\
&= \sum \tilde{g}\left(\frac{1}{\lambda^{3/2}}\nabla_{e_i}\nabla_g\left(\frac{1}{\lambda^2}\Delta R\right), \frac{1}{\sqrt{\lambda}}e_i\right) \\
&= \sum \lambda g\left(\frac{1}{\lambda^4}\nabla_{e_i}\nabla_g(\Delta R), e_i\right) \\
&= \frac{1}{\lambda^3}\Delta\Delta R \\
&= \frac{1}{\lambda^3}\Delta^2 R
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Delta}^3\tilde{R} &= \tilde{\Delta}\tilde{\Delta}^2\tilde{R} \\
&= \sum \tilde{g}\left(\tilde{\nabla}_{\tilde{e_i}}\tilde{\nabla}_{\tilde{g}}\left(\tilde{\Delta}^2\tilde{R}\right), \tilde{e_i}\right) \\
&= \sum \tilde{g}\left(\nabla_{\frac{1}{\sqrt{\lambda}}e_i}\frac{1}{\lambda}\nabla_g\left(\tilde{\Delta}^2\tilde{R}\right), \frac{1}{\sqrt{\lambda}}e_i\right) \\
&= \sum \tilde{g}\left(\frac{1}{\lambda^{3/2}}\nabla_{e_i}\nabla_g\left(\frac{1}{\lambda^3}\Delta^2 R\right), \frac{1}{\sqrt{\lambda}}e_i\right) \\
&= \sum \lambda g\left(\frac{1}{\lambda^5}\nabla_{e_i}\nabla_g(\Delta^2 R), e_i\right) \\
&= \frac{1}{\lambda^4}\Delta^3 R
\end{aligned}
$$

Suppose that for arbitrary n: $\tilde{\Delta}^n\tilde{R} = \frac{1}{\lambda^{n+1}}\Delta^n R$

$$
\begin{aligned}
\tilde{\Delta}^{n+1}\tilde{R} &= \tilde{\Delta}\tilde{\Delta}^n\tilde{R} \\
&= \sum \tilde{g}\left(\tilde{\nabla}_{\tilde{e_i}}\tilde{\nabla}_{\tilde{g}}\left(\tilde{\Delta}^n\tilde{R}\right), \tilde{e_i}\right) \\
&= \sum \tilde{g}\left(\nabla_{\frac{1}{\sqrt{\lambda}}e_i}\frac{1}{\lambda}\nabla_g\left(\tilde{\Delta}^n\tilde{R}\right), \frac{1}{\sqrt{\lambda}}e_i\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum \tilde{g}\left(\frac{1}{\lambda^{3/2}}\nabla_{e_i}\nabla_g\left(\frac{1}{\lambda^{n+1}}\Delta^n R\right), \frac{1}{\sqrt{\lambda}}e_i\right) \\
&= \sum \lambda g\left(\frac{1}{\lambda^{n+3}}\nabla_{e_i}\nabla_g(\Delta R), e_i\right) \\
&= \frac{1}{\lambda^{n+2}}\Delta^3 R
\end{aligned}
$$

By induction $\tilde{\Delta}^n \tilde{R} = \frac{1}{\lambda^{n+1}}\Delta^n R$.

$$
\tilde{\Delta}^n \tilde{R}\tilde{g} = \left(\frac{1}{\lambda^{n+1}}\Delta^n R\right)(\lambda g) = \frac{1}{\lambda^n}\Delta^n R g.
$$

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Vita

Erin Griffin

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Education

Syracuse University — Expected Graduation: May 2021
PhD Candidate in Mathematics

Syracuse University — August 2016 - December 2018
Masters in Mathematics

California Polytechnic State University, San Luis Obispo — September 2012 - June 2016
Bachelor of Science in Mathematics, Philosophy Minor

Future Employment

Seattle Pacific University — Start Date: September 2021
Assistant Professor of Mathematics
Tenure Track

Awards

Outstanding TA Award (2020) *Syracuse University*, University wide award presented to 4% of TAs annually.

AWM Poster Session Winner (2021) *Joint Mathematics Meeting*, Award presented to two graduate students for the AWM national graduate student poster session.

Teaching Mentor (2020) *Syracuse University*, One of less than ten new mentors selected across the university after competitive application process.

NSF Grant Research Assistanship (2018-2020) *Syracuse University*, One of two students to receive summer funding under Prof. Will Wylie's NSF research grant.

Research

Interests

Riemannian Geometry and Geometric Analysis. Particularly geometric flows, their solitons, and the relation of both to conformal geometry. My recent work has focused on the ambient obstruction flow and Bach flow.

Papers

"Gradient Ambient Obstruction Solitons on Homogeneous Manifolds." (submitted) arXiv:2008.09722

Talks

Research

Online Topology Geometry Seminar: "Homogeneous Gradient Ambient Obstruction Solitons."	March 31, 2021
AMS Special Session, Recent developments in Differential Geometry: "Gradient Ambient Obstruction Solitons on Homogeneous Manifolds."	March 21, 2021
Joint Mathematics Meeting, AWM Workshop: Poster Presentations by Women Graduate Students	January 8, 2021
Syracuse Geometry and Topology Seminar, "Homogeneous Gradient Solitons of the Ambient Obstruction Flow"	July 3, 2020
45th ANYSRGMC, "Gradient Bach Solitons on 4-dimensional Homogeneous Manifolds"	March 28, 2020
Math For All Conference in New Orleans, "Gradient Bach Solitons on 4-dimensional Homogeneous Manifolds"	March 7, 2020

Expository

Mathematics Continued Conference at University of Connecticut, "Eating Pizza Like a Geometer"	November 9, 2019
Worcester Polytechnic Institution Graduate Seminar, "The Volume Comparison Theorem"	November 8, 2019
Syracuse University Math Graduate Organization Seminar, "The Volume Comparison Theorem"	November 1, 2019
44th ANYSRGMC, "A brief look at the Volume Comparison Theorem"	March 23, 2018
Syracuse Geometry and Topology Graduate Seminar, "Volume Comparison Theorem"	November 7, 2018
Syracuse University Math Graduate Organization Colloquium, "Eating Pizza Like a Geometer"	October 19, 2018
42nd ANYSRGMC, "Isometric Immersion of Euclidean Plane in Hyperbolic Space"	April 8, 2017

Employment

Research Assistant *Syracuse University* — Summer 2018- Summer 2020

Three Summer Semesters. Supported by Prof. Will Wylie's NSF research grant NSF-#1654034.

Teaching Mentor *Syracuse University* — May 2020 - May 2021

Created hour-long interactive seminar on developing strategies to motivate students; managed small group of incoming teaching assistants (TAs) during University TA Orientation; facilitated discussions to increase understanding of seminars and find practical ways to implement that knowledge; provided constructive feedback on students' submitted miniature lectures to improve their teaching and equip them with skills to be successful.

Academic Excellence Workshop Curriculum Developer *Syracuse University*

Trained and supervised approximately 50 undergraduate workshop facilitators, ensuring they understood the material well, could teach concepts effectively, and were able manage a classroom; developed and maintained worksheets for the workshop participants; coordinated with course instructors to ensure the worksheets are on track with the course. Transitioned program online for Fall 2020 semester, holding curriculum development meetings virtually and training the facilitators to effectively use technology in their workshops.

- Math 193/194 Precalculus — Fall 2019 - Spring 2021
- Math 295 Calculus I — Fall 2019 - Spring 2021
- Math 296 Calculus II — Fall 2019 - Spring 2021
- Math 331 First Course in Linear Algebra — Spring 2021
- Math 397 Calculus III — Fall 2019 - Spring 2021
- Math 485 Differential Equations and Matrix Algebra for Engineers — Spring 2020 - Spring 2021

Instructor of Record *Syracuse University*

Gave original lectures; created midterms, quizzes, and worksheets; assigned final grades.

- Math 285 Calculus for the Life Sciences I (Online) — Fall 2020
- Math 285 Calculus for the Life Sciences I — Fall 2019
- Math 286 Calculus for the Life Sciences II — Spring 2019
- Math 285 Calculus for the Life Sciences I — Fall 2018
- Math 285 Calculus for the Life Sciences I — Spring 2018
- Math 121 Statistics for the Liberal Arts — Summer 2017

Teaching Assistant

Administered and graded weekly quizzes, graded exams.

- Math 295 Calculus I — Fall 2017
- Math 284 Business Calculus — Spring 2017
- Math 121 Statistics for the Liberal Arts — Fall 2016

Service

Conference Organizer

Organizer, 45th Annual New York State Regional Graduate Mathematics Conference (ANYSRGMC) 2019 - 2020

Organized the longest running graduate student mathematics conference in the United States. Managed NSF conference grant and additional funding from AMS and the Graduate Student Organization. Delegated travel funding to participants. Orchestrated our first ever undergraduate poster session during the conference. (This conference was moved online due to COVID-19.)

Successfully transitioned the conference online within three weeks of the conference date, making it the nation's first entirely remote graduate student conference. The conference included keynote addresses, parallel sessions, and a poster session. Facilitated participant travel reimbursement in spite of cancelled travel.

Co-Organizer, 43rd ANYSRGMC and 44th ANYSRGMC 2017 - 2019

Help with various tasks relating to running a conference including: coordinating catering, helping with registration, and preparing conference materials.

Math Graduate Organization

President 2019 - 2020

Responsible for managing the budget and applying for funding. Organized events for graduate students, including the departmental picnic. Served as graduate student representative to department focused on advocating for graduate student needs.

Vice President 2020 - 2021

Mentored current president through navigating the planning the 46th ANYSRGMC. Responsible for managing the budget; applying for funding; organizing the 45th ANYSRGMC;

Treasurer 2017 - 2019

Apply for funding for various events throughout the year; help maintain budget; co-organized the ANYSRGMC.

Association for Women in Mathematics, Syracuse University Chapter

Vice President 2019 - 2020

Helped plan and apply for funding for various events to promote community and uplift minorities in mathematics; advertised events throughout the department.

Secretary 2018 - 2019

Founding board member of Syracuse University AWM Chapter; helped plan events for academic year.

Contributed Talk Session Moderator

Math for All Conference, Tulane University 2020

43rd ANYSRGMC, Syracuse University 2018

Prospective Student Coordinator 2017 - 2021

Act as the graduate representative for prospective students; answer questions via emails; plan events with other graduate students during propspective student visits; help plan and lead prospective student weekend.

First Year Help Session Leader 2017 - 2018

Lead first year students through old preliminary exam problems to prepare the students for their exams. Secured funding and helped organize sessions.

Funding and Certificates

Travel Funding

2021 **Joint Mathematics Meeting**, *Association for Women in Mathematics*, Selected as presenter and received grant to attend conference.

2020 **Math for All Travel Funding**, *Tulane University*, Partial funding to speak at conference

GSO Travel Grant, *Syracuse University*, Graduate Student Organization grant to attend conference

2019 **Mathematics Continued Conference**, *University of Connecticut*, Funding to travel to conference

Northeast Analysis Network, *University of Connecticut*, Funding to travel to conference

Women and Mathematics Conference, *Institute for Advance Study*, Selected as participant and received grant to attend WAM Conference.

RTG Conference on Geometric Analysis and Diversity in Mathematics, *Princeton University*, Grant to attend conference.

Certificates

2020 **Women in Science and Engineering Future Professionals Program Certificate**

(Pending) **Certificate in Undergraduate Teaching**

Conferences Attended

Joint Mathematics Meeting	January 6-9, 2021
45th ANYSRGMC, Syracuse University	March 28, 2020
Math for All, Tulane University	March 6-7, 2020
Joint Mathematics Meeting	January 15-18, 2020
Mathematics Continued Conference, University of Connecticut	November 9, 2019
Northeast Analysis Network, University of Connecticut	September 21- 22, 2019
RTG Conference on Geometric Analysis and Diversity in Mathematics, Princeton University	June 19-22, 2019
Women and Mathematics Conference, Institute for Advanced Study	May 18- 24, 2019
44th ANYSRGMC, Syracuse University	March 23, 2019
Geometric Analysis Conference at Rutgers University, Rutgers University	November 14-16, 2018
43rd ANYSRGMC, Syracuse University	March 24, 2018
Mini School on Mean Curvature and Ricci Flow, Fields Institute	November 4-5, 2017
Temple University Graduate Student Conference in Algebra, Geometry, and Topology, Temple University	June 3-4, 2017
42nd ANYSRGMC, Syracuse University	April 8, 2017

Professional Memberships

- Association for Women in Mathematics 2015-Present
- American Mathematical Society 2017- Present
- Mathematical Association of America 2017- Present