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Reflection Functors in the Representation Theory of Quivers

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Reflection Functors in the Representation Theory of Quivers

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Abstract: The paper "Coxeter Functors and Gabriel's Theorem" written by I.N. Bernstein, I.M. Gel'fand, and V.A. Ponomarev explores the concept of reflection functors. A thorough proof of several results used by Bernstein et al in their paper is presented. The focus is on the category of representations and reflection functors, both negative and positive. The quadratic form is the bridge between the results on quivers and the techniques of Lie algebras. The Dynkin diagrams mentioned in Gabriel's Theorem are discussed.

Executive Summary:

The purpose of this paper is to thoroughly prove some of the important results that are used in the paper "Coxeter Functors and Gabriel's Theorem" by Bernstein et al [1]. The focus is mostly on the category of representations and the reflection functors to better understand how they can be used to prove Gabriel's Theorem. Gabriel's Theorem was initially not proved through Lie algebra or representation theory but it gave results about the Dynkin Diagrams which were previously only related to those two fields. Bernstein et al wrote another proof of Gabriel's Theorem using tools from representation theory, namely the reflection functors. This offers a relation between these fields of mathematics.

Consider a graph which is a set of a finite number of vertices and edges, namely Γ. Then we place an orientation on it which makes the edges arrows so that they have an orientation, namely Λ. The category $\mathscr{L}(\Gamma, \Lambda)$ has objects and morphisms. Objects are collections of vector spaces and linear mappings which go between the vector spaces. Morphisms are a logical way to compare objects.

We showed that $\mathscr{L}(\Gamma, \Lambda)$ satisfies the following conditions and therefore is a category:

1. The composition of morphisms is a morphism and the composition is associative
2. For all morphisms $\phi : (U, f) \to (V, g)$, $1_{(V,g)}\phi = \phi 1_{(U,g)} = \phi$

Reflection functors change representations. For example look at an orientation Λ where there is a vertex β such that all of the arrows that are connected to β are going into the vertex (referred to as a sink), then F_β^+ (referred to as a positive reflection functor) changes $\mathscr{L}(\Gamma, \Lambda)$ to $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$ where $\sigma_\beta\Lambda$ looks exactly like Λ except that instead of all of the arrows going into β all of the arrows are coming out of β (referred to as a source). The vertices are vector spaces and the arrows are linear mappings, therefore since the vertices don't change between Λ and $\sigma_\beta\Lambda$, but the arrows do then the vector spaces don't change and the linear mappings do. Therefore we must check that how we defined the reflection functors, both positive and negative for a sink and a source respectively, work properly.

We show that $F_\beta^+ : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\beta\Lambda)$ satisfies the following conditions and therefore is a functor:

1. $F_\beta^+(1_{(U,f)}) = 1_{(X,r)}$
2. $F_\beta^+(\psi\phi) = (F_\beta^+(\psi))(F_\beta^+(\phi))$

Similarly we can show that F_α^- is a functor.

After proving that F_β^+ and F_α^- are both functors, we can now use Theorem 1, and Lemma 1. We use statements and mappings that we used earlier to prove the Theorem 1 and Lemma 1. From the Theorem and Lemma we can immediatly prove Corollary 1. These proofs give us more insight in how the functors can be used, and what properites that they have in a more abstract way.

We discuss the quadratic form in order to bridge the relationship between the results on quivers and the techniques of Lie algebras. This brings us closer to our goal of abstractly showing how these different fields of mathematics are related.

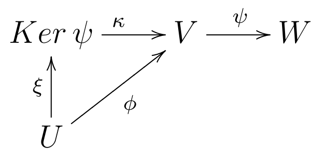
Now to show the main idea of this paper we will show how the reflection functors F_β^+ and F_α^- were used to prove part 2 of the famous Gabriel's Theorem. This is not the first way that Gabriel's Theorem was proven, therefore the two fields of mathematics which the two different proofs came from are connected in this way.

Contents

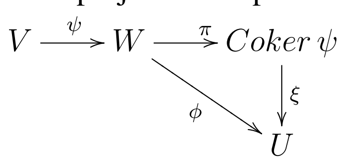
1 Introduction

This project is about representations of quivers which is an area of mathematics that uses methods of linear algebra, combinatorics and category theory.
Recall some necessary definitions from linear algebra.

Let V and W be vector spaces over a fixed field K. A function $\psi : V \to W$ is a **linear mapping** if $\psi(u + v) = \psi(u) + \psi(v)$ and $\psi(cu) = c\psi(u)$ for all $u, v \in V$ and $c \in K$. If $\phi : U \to V$ is another linear mapping, then the composition $\psi \circ \phi : U \to W$ is defined by $[\psi \circ \phi](u) = \psi(\phi(u))$. Sometimes we write $\psi\phi$ instead of $\psi\circ\phi$. The following two definitions are from the text Homology by Saunders Mac Lane. The **kernel** of a morphism $h : V \to W$, Ker ψ, consists of all $v \in V$ such that $\psi(v) = 0$. The following is a universal property: for each $\phi : U \to V$ satisfying $\psi\phi = 0$, there exists a unique $\xi : U \to$ Ker ψ with $\phi = \kappa\xi$, κ the inclusion map.



The **cokernel** of a morphism $\widetilde{h} : V \to W$, Coker $\widetilde{h}$, is equal to the quotient module $W/\mathrm{Im}\,\widetilde{h}$. The following is a universal property: for each $\phi : W \to U$ satisfying $\phi\psi = 0$, there exists a unique ξ : Coker $\psi \to U$ with $\phi = \xi\pi$, π the natural projection map.



The **identity mapping** $1_U : U \to U$ is given by $1_U(u) = u$ for all $u \in U$. We use the fact that the composition of linear mappings is associative, i.e. if ϕ and ψ are as above and $\xi : W \to Y$ is a linear mapping, then $(\xi \circ \psi) \circ \phi = \xi \circ (\psi \circ \phi)$. We also use the fact that $1_V \circ \phi = \phi \circ 1_U = \phi$ for all ϕ as above. Recall that the vector space V is finite dimensional if it has a finite spanning set.

A linear map $\psi : V \to W$ is an isomporhpism if there exists a linear map $\zeta : W \to V$ satisfying $\psi \circ \zeta = 1_W$ and $\zeta \circ \psi = 1_V$. It is a standard fact that a linear map is an isomorphism if and only if it is both injective and surjective. Vector spaces V and W are isomorphic if there exists an isomporphism $V \to W$.

If V and W are vector spaces, the direct sum $V \oplus W$ is the set of all pairs (v, w) such that $v \in V$ and $w \in W$ with component-wise addition and scalar multiplication. If $\mu : V \to V'$ and $\nu : W \to W'$ are linear maps, then the direct sum $\mu \oplus \nu : V \oplus W \to V' \oplus W'$ is defined by $(\mu \oplus \nu)(v, w) = (\nu(v), \mu(w))$. If $\phi : V' \to V''$, $\psi : W' \to W''$ are linear maps, then $(\phi \oplus \psi)(\mu \oplus \nu) = \phi\mu \oplus \psi\nu$. A categorical definition of a direct sum is that a vector space X is isomorphic to $V \oplus W$ if and only if there exist four linear maps $V \underset{\pi_V}{\overset{\iota_V}{\rightleftarrows}} X \underset{\pi_W}{\overset{\iota_W}{\leftrightarrows}} W$ satisfying $\pi_V\iota_V = 1_V$, $\pi_W\iota_W = 1_W$, and $\iota_V\pi_V + \iota_W\pi_W = 1_X$. In the special case when $X = V \oplus W$ as above then the maps are defined as follows: $\iota_V : V \to X$, $\iota_W : W \to X$, $\pi_V : X \to V$, and $\pi_W : X \to W$ such that $\iota(v) = (v, 0)$, $\iota(w) = (0, w)$, $\pi_V(v, w) = v$, and $\pi_W(v, w) = w$ where $v \in V$, $w \in W$, and $(v, w) \in X$.

We present the following facts from the "Coxeter Functors and Gabriel's Theorem" paper written by I.N. Bernstein, I.M. Gel'fand, and V.A. Ponomarev.

Define Γ as a finite connected graph with the set of vertices Γ_0 and the set of edges Γ_1. Fix an orientation Λ of the graph Γ which assigns to each edge $\ell \in \Gamma_1$ a starting point $\alpha(\ell) \in \Gamma_0$ and an end-point $\beta(\ell) \in \Gamma_0$. We obtain a directed (oriented) graph which we call a quiver and denote by (Γ, Λ).

With the reference to a general definition of a category in Homology by Saunders Mac Lane we define a **category** $\mathscr{L}(\Gamma, \Lambda)$ as follows. A category consists of objects and morphisms which may sometimes be composed. An object of $\mathscr{L}(\Gamma, \Lambda)$ is any collection (V, f) of finite dimensional vector spaces V_α $(\alpha \in \Gamma_0)$ and linear mappings $f_\ell(\ell \in \Gamma_1)$. There is a particular representation where all the vector spaces are zero and all the maps are the zero maps, called 0. A **morphism** $\phi : (V, f) \to (W, g)$ is a collection of linear mappings $\phi_\alpha : V_\alpha \to W_\alpha(\alpha \in \Gamma_0)$ such that for each edge $\ell \in \Gamma_1$ the following diagram

$$\begin{array}{ccc} V_{\alpha(\ell)} & \xrightarrow{f_\ell} & V_{\beta(\ell)} \\ \big\downarrow{\scriptstyle \varphi_{\alpha(\ell)}} & & \big\downarrow{\scriptstyle \varphi_{\beta(\ell)}} \\ W_{\alpha(\ell)} & \xrightarrow[g_\ell]{} & W_{\beta(\ell)} \end{array}$$

is commutative, that is, $\phi_{\beta(\ell)} f_\ell = g_\ell \phi_{\alpha(\ell)}$. The objects of $\mathscr{L}(\Gamma, \Lambda)$ are called representations of the quiver (Γ, Λ) and the category $\mathscr{L}(\Gamma, \Lambda)$ is called the category of representations of (Γ, Λ).

We define the law of composition for morphisms as follows. Let $\phi : (U, f) \to (V, g)$ and $\psi : (V, g) \to (W, h)$ be morphisms where $\phi = (\phi_\alpha)_{\alpha \in \Gamma_0}$ and $\psi = (\psi_\alpha)_{\alpha \in \Gamma_0}$. Then $\psi \circ \phi : (U, f) \to (W, h)$ is given by $(\psi \circ \phi)_\alpha = \psi_\alpha \circ \phi_\alpha$.

Define the **identity morphism** $1_{(V,f)}$ for an object (V, f) by $1_{(V,f)} = (1_{V_\alpha})_{\alpha \in \Gamma_0}$. We prove that $\mathscr{L}(\Gamma, \Lambda)$ is a category in the next section.

2 The Category of Representations

We show that $\mathscr{L}(\Gamma, \Lambda)$ satisfies the following conditions and therefore is a category:

1. The composition of morphisms is a morphism and the composition is associative
2. For all morphisms $\phi : (U, f) \to (V, g)$, $1_{(V,g)}\phi = \phi 1_{(U,g)} = \phi$

For any objects (U, f), (V, g), and (W, h) in $\mathscr{L}(\Gamma, \Lambda)$, let $\phi : (U, f) \to (V, g)$ and $\psi : (V, g) \to (W, h)$ be morphisms. Then we have a commutative diagram,

$$\begin{array}{ccc} U_{\alpha(\ell)} & \xrightarrow{f_\ell} & U_{\beta(\ell)} \\ {\scriptstyle \phi_{\alpha(\ell)}}\downarrow & & \downarrow{\scriptstyle \phi_{\beta(\ell)}} \\ V_{\alpha(\ell)} & \xrightarrow{g_\ell} & V_{\beta(\ell)} \\ {\scriptstyle \psi_{\alpha(\ell)}}\downarrow & & \downarrow{\scriptstyle \psi_{\beta(\ell)}} \\ W_{\alpha(\ell)} & \xrightarrow{h_\ell} & W_{\beta(\ell)} \end{array}$$

that is

$$\phi_{\beta(\ell)} f_\ell = g_\ell \phi_{\alpha(\ell)} \tag{1}$$

and $\psi_{\beta(\ell)} g_\ell = h_\ell \psi_{\alpha(\ell)}$. Then

$$\psi_{\beta(\ell)}\phi_{\beta(\ell)} f_\ell = \psi_{\beta(\ell)}(\phi_{\beta(\ell)} f_\ell) = \psi_{\beta(\ell)}(g_\ell \phi_{\alpha(\ell)}) = \\ (\psi_{\beta(\ell)} g_\ell)\phi_{\alpha(\ell)} = (h_\ell \psi_{\alpha(\ell)})\phi_{\alpha(\ell)} = h_\ell \psi_{\alpha(\ell)}\phi_{\alpha(\ell)}$$

which shows that $\psi \circ \phi : (U, f) \to (W, h)$ is a morphism, that is the diagram

$$\begin{array}{ccc} U_{\alpha(\ell)} & \xrightarrow{f_\ell} & U_{\beta(\ell)} \\ {\scriptstyle [\psi\circ\phi]_{\alpha(\ell)}}\downarrow & & \downarrow{\scriptstyle [\psi\circ\phi]_{\beta(\ell)}} \\ W_{\alpha(\ell)} & \xrightarrow{h_\ell} & W_{\beta(\ell)} \end{array}$$

commutes.

We have shown that the composition of morphisms is well-defined.

Suppose that ϕ and ψ are as above, and $\xi : (W, h) \to (Y, j)$ is a morphism in $\mathscr{L}(\Gamma, \Lambda)$ where $\xi = (\xi_\alpha)$, $\alpha \in \Gamma_0$ Then, using the associativity of composition of linear mappings we get

$$[(\xi \circ \psi) \circ \phi]_\alpha = (\xi \circ \psi)_\alpha \circ \phi_\alpha = (\xi_\alpha \circ \psi_\alpha) \circ \phi_\alpha = \xi_\alpha \circ (\psi_\alpha \circ \phi_\alpha) = \xi_\alpha \circ (\psi \circ \phi)_\alpha = [\xi \circ (\psi \circ \phi)]_\alpha.$$

Therefore, $(\xi \circ \psi) \circ \phi = \xi \circ (\psi \circ \phi)$. We have shown the composition of morphisms is associative. Thus $\mathscr{L}(\Gamma, \Lambda)$ satisfies the first property.

For a morphism $\phi : (U, f) \to (V, g)$ as above, we have

$$[1_{(V,g)} \circ \phi]_\alpha = (1_{(V,g)})_\alpha \circ \phi_\alpha = \phi_\alpha \quad \text{and} \quad [\phi \circ 1_{(U,f)}]_\alpha = \phi_\alpha \circ (1_{(U,f)})_\alpha = \phi_\alpha.$$

Therefore $1_{(V,g)} \circ \phi = \phi \circ 1_{(U,f)} = \phi$. We have shown that $\mathscr{L}(\Gamma, \Lambda)$ satisfies the second property. We have shown that all of the axioms of a category defined in Homology by Saunders Mac Lane meaning that we have shown $\mathscr{L}(\Gamma, \Lambda)$ is a category.

A morphism $\psi : (V, g) \to (W, h)$ is an isomorphism if there exists a morphism $\zeta : (W, h) \to (V, g)$ satisfying $\psi \circ \zeta = 1_{(W,h)}$ and $\zeta \circ \psi = 1_{(V,g)}$. Representations of quivers (V, g) and (W, h) of the quiver (Γ, Λ) are isomorphic if there exists an isomorhpism $(V, g) \to (W, h)$. If (V, g), (W, h) are representations then the set of morphisms $(V, g) \to (W, h)$ is a finite dimensional vector space over the field K.

$$\phi = (\phi_\alpha)_{\alpha \in \Gamma_0}\,,\ \psi = (\psi_\alpha)_{\alpha \in \Gamma_0}$$

We define $\phi + \psi$ by

$$(\phi + \psi)_\alpha = \phi_\alpha + \psi_\alpha$$

and, for $c \in K$ we define $c\phi$ by

$$(c\phi)_\alpha = c\phi_\alpha.$$

Referencing Equation (1) we have $\phi_{\beta(\ell)} f_\ell = g_\ell \phi_{\alpha(\ell)}$ and $\psi_{\beta(\ell)} f_\ell = g_\ell \psi_{\alpha(\ell)}$. Adding the left hand sides and right hand sides gives us $(\phi_{\beta(\ell)} + \psi_{\beta(\ell)}) f_\ell = g_\ell(\phi_{\alpha(\ell)} + \psi_{\alpha(\ell)})$ which shows $\phi + \psi$ is a morphism.

The verification that $c\phi$ is a morphism is similar.

In view of our definition of the sums of the morphisms, and the scalar multiplication, the above verification also shows that $\mathrm{Hom}_{\mathscr{L}(\Gamma,\Lambda)}((U, f), (V, g)) \subset \underset{\oplus}{\alpha \in} \Gamma_0\ \mathrm{Hom}_K(U_\alpha, V_\alpha)$ is a subspace. Therefore since we know that $\underset{\oplus}{\alpha \in} \Gamma_0\ \mathrm{Hom}_K(U_\alpha, V_\alpha)$ is finite dimensional, then
$\mathrm{Hom}_{\mathscr{L}(\Gamma,\Lambda)}((U, f), (V, g))$ is finite dimensional.

A verification similar to above shows that $\phi(\psi + \xi) = \phi\psi + \phi\xi$ and $(\phi + \xi)\psi = \phi\psi + \xi\psi$ is true for $\mathscr{L}(\Gamma, \Lambda)$, therefore we know that $\mathscr{L}(\Gamma, \Lambda)$ is a preadditive. It is easy to verify that $c(\phi\psi) = (c\phi)\psi = \phi(c\psi)$ so $\mathscr{L}(\Gamma, \Lambda)$ is a k-category.

If (U, f) and (V, g) are representations of (Γ, Λ) the direct sum of $(U, f) \oplus (V, g)$ is the representation (X, s) where $X_\alpha = U_\alpha \oplus V_\alpha$, $\alpha \in \Gamma_0$ and $s_\ell : X_{\alpha(\ell)} \to X_{\beta(\ell)}$ is the linear map $s_\ell = f_\ell \oplus g_\ell : U_{\alpha(\ell)} \oplus V_{\alpha(\ell)} \to U_{\beta(\ell)} \oplus V_{\beta_\ell}$ where $\ell \in \Gamma_1$. Since the direct sums exist $\mathscr{L}(\Gamma, \Lambda)$ is an additive k-category. An object is **indecomposable** if it is not isomorphic to the direct sum of two nonzero representations.

3 Reflection Functors

We present the following facts from the "Coxeter Functors and Gabriel's Theorem" paper written by Bernstein, Gel'fand, and Ponomarev.

For each vertex $\alpha \in \Gamma_0$ we denote by Γ^α the set of edges containing α. If Λ is some orientation of the graph Γ, we denote by $\sigma_\alpha \Lambda$ the orientation obtained from Λ by changing the directions of all edges $\ell \in \Gamma^\alpha$.

We say that a vertex α is a source of (Γ, Λ) if $\beta(\ell) \neq \alpha$ for all $\ell \in \Gamma_1$ (this means that all the edges containing α start there and that there are no loops in Γ with vertex at α). Similarly we say that a vertex β is a sink of (Γ, Λ) if $\alpha(\ell) \neq \beta$, for all $\ell \in \Gamma_1$.

To study indecomposable objects in the category $\mathscr{L}(\Gamma, \Lambda)$ we consider **refection functors** $F_\beta^+ : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\beta \Lambda)$ and $F_\alpha^- : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\alpha \Lambda)$. These functors send an indecomposible representation to either an indecomposible representation or to zero. We construct such a functor for each vertex α at which all the edges have the same direction.

We will prove that F_β^+ is a functor in section 3.1, and that F_α^- is a functor in section 3.2.

3.1 A Positive Reflection Functor

Suppose that the vertex β of the graph Γ is a sink with respect to the orientation Λ. From an object (U, f) in $\mathscr{L}(\Gamma, \Lambda)$ we construct a new object $F_\beta^+(U, f) = (X, r)$ in $\mathscr{L}(\Gamma, \sigma_\beta \Lambda)$.

Namely, we put $X_\gamma = U_\gamma$ for $\gamma \neq \beta$. To construct X_β we consider all the edges $\ell_1, \ell_2, \ldots, \ell_k$ that end at β (that is, all edges of Γ^β). We denote by X_β the subspace in the direct sum $\bigoplus_{i=1}^{k} U_{\alpha(\ell_i)}$ consisting of the vectors $u = (u_1, \ldots, u_k)$ (here $u_i \in U_{\alpha(\ell_i)}$) for which $f_{\ell_i}(u_1) + \ldots + f_{\ell_k}(u_k) = 0$. In other words, if we denote by h_U the mapping $h_U : \bigoplus_{i=1}^{k} U_{\alpha(\ell_i)} \to U_\beta$ defined by the formula $h_U(u_1, u_2, \ldots, u_k) = f_{\ell_1}(u_1) + \ldots + f_{\ell_k}(u_k)$, then $X_\beta = \operatorname{Ker} h$.

We now define the mappings r_{ℓ_j}. For $\ell_j \notin \Gamma^\beta$ we put $r_{\ell_j} = f_{\ell_j}$. If $\ell = \ell_j \in \Gamma^\beta$, then r_{ℓ_j} is defined as the composition of the natural embedding $\kappa_U : X_\beta \to \oplus U_{\alpha(\ell_i)}$ of X_β in $\oplus U_{\alpha(\ell_i)}$ and the projection $\pi_{U,\alpha(\ell_j)} : \oplus U_{\alpha(\ell_i)} \to U_{\alpha(\ell_j)}$ of the sum $\oplus U_{\alpha(\ell_i)}$ onto the term $U_{\alpha(\ell_j)} = X_{\alpha(\ell_j)}$. In other words, $r_{\ell_j} = \pi_{U,\alpha(\ell_j)} \kappa_U$. We note that on all edges $\ell_j \in \Gamma^\beta$ the orientation has been changed, that is, the resulting object (X, r) belongs to $\mathscr{L}(\Gamma, \sigma_\beta \Lambda)$. Let $\phi = (\phi_\alpha) : (U, f) \to (V, g)$ be a morphism in $\mathscr{L}(\Gamma, \Lambda)$, let $(X, r) = F_\beta^+(U, f)$ and $(Y, s) = F_\beta^+(V, g)$. We construct $F_\beta^+(\phi) = \xi = (\xi_\alpha)_{\alpha \in \Gamma_0} : (X, r) \to (Y, s)$. If $\alpha \neq \beta$, then $X_\alpha = U_\alpha$, $Y_\alpha = V_\alpha$, and we set $\xi_\alpha = \phi_\alpha : U_\alpha \to V_\alpha$. To construct $\xi_\beta : X_\beta \to Y_\beta$, we consider the following diagram of vector spaces and linear maps

$$\begin{array}{ccccc} X_\beta & \xrightarrow{\kappa_U} & \oplus_{i=1}^k U_{\alpha(\ell_i)} & \xrightarrow{h_U} & U_\beta \\ {\scriptstyle \xi_\beta}\downarrow & & \downarrow{\scriptstyle \oplus \phi_{\alpha(\ell_i)}} & & \downarrow{\scriptstyle \phi_\beta} \\ Y_\beta & \xrightarrow{\kappa_V} & \oplus_{i=1}^k V_{\alpha(\ell_i)} & \xrightarrow{h_V} & V_\beta \end{array} \tag{2}$$

where $X_\beta = \operatorname{Ker} h_U$, $Y_\beta = \operatorname{Ker} h_V$, and κ_U and κ_V are the inclusion maps. It is easy to verify that the right square of the diagram commutes.

$$\phi_\beta h_U = h_V(\oplus_{i=1}^k \phi_{\alpha(\ell_i)})$$

Since $h_V(\overset{k}{\underset{i=1}{\oplus}}\phi_{\alpha(\ell_i)})\kappa_U = \phi_\beta h_U \kappa_U = \phi_\beta 0 = 0$, the universal property of the kernel (see Introduction) says that there exists a unique k-linear map $\xi_\beta : X_\beta \to Y_\beta$ satisfying $\kappa_V \xi_\beta = (\overset{k}{\underset{i=1}{\oplus}}\phi_{\alpha(\ell_i)})\kappa_U$. This finishes the construction of $\xi = F^+_\beta(\phi)$. We now verify that it is a morphism in $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$. For each edge $\ell = \ell_j : \beta \to \alpha_{(\ell_j)}$ in Γ^β (in the orientation $\sigma_\beta\Lambda$), we have

$$\xi_{\alpha(\ell_j)}\pi_{U,\alpha(\ell_j)}(u_1, \ldots, u_k) = \xi_{\alpha(\ell_j)}(u_j) = \phi_{\alpha(\ell_j)}(u_j) \text{ and}$$
$$\pi_{V_{\alpha(\ell_j)}}[\oplus\phi_{\alpha(\ell_i)}](u_1, \ldots, u_k) = \pi_{V_{\alpha(\ell_j)}}(\phi_{\alpha(\ell_1)}(u_1), \ldots, \phi_{\alpha(\ell_k)}(u_k)) = \phi_{\alpha(\ell_j)}(u_j). \text{ Hence}$$
$$\xi_{\alpha(\ell_j)}\pi_{U,\alpha(\ell_j)} = \pi_{V,\alpha(\ell_j)}[\oplus\phi_{\alpha(\ell_i)}] \text{ and we have}$$
$$\xi_{\alpha(\ell_j)}r_{\ell_j} = \xi_{\alpha(\ell_j)}\pi_{U,\alpha(\ell_j)}\kappa_U = \pi_{V,\alpha(\ell_j)}[\oplus\phi_{\alpha(\ell_i)}]\kappa_U = \pi_{V,\alpha(\ell_j)}\kappa_V\xi_\beta = s_{\ell_j}\xi_\beta.$$

For each edge $\ell \in \Gamma_1$ not incident to β, we have $\alpha(\ell) \neq \beta$, $\beta(\ell) \neq \beta$, so

$$\begin{array}{ccc} U_{\alpha(\ell)} & \xrightarrow{f_\ell} & U_{\beta(\ell)} \\ \phi_{\alpha(\ell)}\downarrow & & \downarrow\phi_{\alpha(\ell)} \\ V_{\alpha(\ell)} & \xrightarrow{g_\ell} & V_{\beta(\ell)} \end{array}$$

is a commutative diagram because $\phi : (U, f) \to (V, g)$ is a morphism. Hence the above construction yields the commutative diagram

$$\begin{array}{ccc} X_{\alpha(\ell)} & \xrightarrow{r_\ell} & X_{\beta(\ell)} \\ \xi_{\alpha(\ell)}\downarrow & & \downarrow\xi_{\beta(\ell)} \\ Y_{\beta(\ell)} & \xrightarrow{s_\ell} & Y_{\beta(\ell)} \end{array}$$

as required.

We show that $F^+_\beta : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\beta\Lambda)$ satisfies the following conditions and therefore is a functor:

1. $F^+_\beta(1_{(U,f)}) = 1_{(X,r)}$
2. $F^+_\beta(\psi\phi) = (F^+_\beta(\psi))(F^+_\beta(\phi))$

As previously defined, $1_{(U,f)} : (U, f) \to (U, f)$, and $F^+_\beta(1_{(U,f)}) = \xi = (\xi_\alpha)_{\alpha\in\Gamma_0} : (X, r) \to (X, r)$. To show: $\xi_\alpha = 1_{X_\alpha}$, $\alpha \in \Gamma_0$.

If $\alpha \neq \beta$, then $\xi_\alpha = \phi_\alpha$, but $\phi_\alpha = 1_{U_\alpha} = 1_{X_\alpha}$ since $\alpha \neq \beta$.

To show $\xi_\beta = 1_{X_\beta}$, we specialize the diagram (2) to the case where $\phi = 1_{(U,f)} : (U, f) \to (U, f)$. We obtain the following commutative diagram

$$\begin{array}{ccccc} X_\beta & \xrightarrow{\kappa_U} & \oplus U_{\alpha(\ell_i)} & \xrightarrow{h_U} & U_\beta \\ \xi_\beta\downarrow & & \oplus 1_{U_{\alpha(\ell_i)}}\downarrow & & 1_{U_\beta}\downarrow \\ X_\beta & \xrightarrow{\kappa_U} & \oplus U_{\alpha(\ell_i)} & \xrightarrow{h_U} & U_\beta \end{array}$$

It is clear that replacing ξ_β with 1_{X_β} preserves the commutativity of the left square of the diagram: $\kappa_U 1_{X_\beta} = (\oplus 1_{U_{\alpha(\ell_i)}})\kappa_U = (1_{\oplus U_{\alpha(\ell_i)}})\kappa_U = \kappa_U$. By the uniqueness of ξ_β we must have $\xi_\beta = 1_{X_\beta}$. Hence, $F^+_\beta(1_{(U,f)}) = 1_{(X,r)}$.

Now we check if $F_\beta^+(\psi\phi) = (F_\beta^+(\psi))(F_\beta^+(\phi))$.
For any objects (U, f), (V, g), and (W, h) in $\mathscr{L}(\Gamma, \Lambda)$, let $\phi : (U, f) \to (V, g)$ and $\psi : (V, g) \to (W, h)$ be morphisms.
Set

$$\begin{gathered} F_\beta^+(\phi) = \xi = (\xi_\alpha)_{\alpha\in\Gamma_0} \\ F_\beta^+(\psi) = \zeta = (\zeta_\alpha)_{\alpha\in\Gamma_0} \\ F_\beta^+(\psi\phi) = \theta = (\theta_\alpha)_{\alpha\in\Gamma_0} \end{gathered}$$

We want to show that $\theta_\alpha = \zeta_\alpha\xi_\alpha$, $\alpha \in \Gamma_0$.
a) For $\alpha \neq \beta$

$$\theta_\alpha = [F_\beta^+(\psi\phi)]_\alpha = (\psi\phi)_\alpha = \psi_\alpha\phi_\alpha = [F_\beta^+(\psi)]_\alpha[F_\beta^+(\phi)]_\alpha = \zeta_\alpha\xi_\alpha$$

b) For $\alpha = \beta$ we set $X_\beta = \operatorname{Ker} h_U$, $Y_\beta = \operatorname{Ker} h_V$, and $Z_\beta = \operatorname{Ker} h_W$

$$\begin{CD} X_\beta @>{\kappa_U}>> \oplus_{i=1}^k U_{\alpha(\ell_i)} @>{h_U}>> U_\beta \\ @V{\xi_\beta}VV @V{\oplus_{i=1}^k \phi_{\alpha(\ell_i)}}VV @VV{\phi_\beta}V \\ Y_\beta @>{\kappa_V}>> \oplus_{i=1}^k V_{\alpha(\ell_i)} @>{h_V}>> V_\beta \\ @V{\zeta_\beta}VV @V{\oplus_{i=1}^k \psi_{\alpha(\ell_i)}}VV @VV{\psi_\beta}V \\ Z_\beta @>{\kappa_W}>> \oplus_{i=1}^k W_{\alpha(\ell_i)} @>{h_W}>> W_\beta \end{CD}$$

By (2) the above diagram commutes so

$$[\oplus(\psi\phi)_{\alpha(\ell_i)}]\kappa_U = (\bigoplus_{i=1}^k \psi_{\alpha(\ell_i)}\phi_{\alpha(\ell_i)})\kappa_U = (\bigoplus_{i=1}^k \psi_{\alpha(\ell_i)})(\bigoplus_{i=1}^k \phi_{\alpha(\ell_i)})\kappa_U = (\bigoplus_{i=1}^k \psi_{\alpha(\ell_i)})\kappa_V\xi_\beta = \kappa_W\zeta_\beta\xi_\beta$$

By (2), the diagram below commutes.

$$\begin{CD} X_\beta @>{\kappa_U}>> \bigoplus_{i=1}^k U_{\alpha(\ell_i)} @>{h_U}>> U_\beta \\ @V{\theta_\beta}VV @V{\bigoplus_{i=1}^k (\psi\phi)_{\alpha(\ell_i)}}VV @VV{(\psi\phi)_\beta}V \\ Z_\beta @>{\kappa_W}>> \bigoplus_{i=1}^k W_{\alpha(\ell_i)} @>{h_W}>> W_\beta \end{CD}$$

We have

$$[\oplus(\psi\phi)_{\alpha(\ell_i)}]\kappa_U = \kappa_W\theta_\beta$$

So both $\zeta_\beta\xi_\beta$ and θ_β make the left square of the above diagram commute. By the uniqueness of θ_β, we must have $\theta_\beta = \zeta_\beta\xi_\beta$. Therefore $F_\beta^+(\psi\phi) = (F_\beta^+(\psi))(F_\beta^+(\phi))$ and $F_\beta^+(1_{(U,f)}) = 1_{(X,r)}$. Thus F_β^+ is a functor.

It is easy to see that $F_\beta^+(\phi + \psi) = F_\beta^+(\phi) + F_\beta^+(\psi)$ and $F_\beta^+(c\phi) = cF_\beta^+(\phi)$. Therefore F_β^+ is a k-linear functor.

3.2 A Negative Reflection Functor

Suppose that the vertex α of the graph Γ is a source with respect to the orientation Λ. From an object (U, f) in $\mathscr{L}(\Gamma, \Lambda)$ we construct a new object $F_\alpha^-(U, f) = (X, r)$ in $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$.

Namely, we put $X_\gamma = U_\gamma$ for $\gamma \neq \alpha$.

Next we consider all the edges $\ell_1, \ell_2, \ldots, \ell_k$ that start at α (that is, all edges of Γ^α). We denote by $\widetilde{h}_U$ the mapping $\widetilde{h}_U : U_\alpha \to \bigoplus_{i=1}^{k} U_{\beta(\ell_i)}$ defined by the formula $\widetilde{h}_U(u) = (f_{\ell_1}(u), \ldots, f_{\ell_k}(u))$, and set $X_\alpha =$Coker $\widetilde{h}_U = \bigoplus_{i=1}^{k} U_{\beta(\ell_i)}/\mathrm{Im}\,\widetilde{h}_U$. Denote by $\pi_U : \oplus U_{\beta(\ell_i)} \to X_\alpha$ the canonical map.

We now define the mappings r_ℓ. For $\ell \notin \Gamma^\alpha$ we put $r_\ell = f_\ell$. If $\ell = \ell_j \in \Gamma^\alpha$, then r_{ℓ_j} is defined as the composition of the natural embedding $\kappa_{U,\ell_j} : U_{\beta(\ell_j)} \to \bigoplus_{i=1}^{k} U_{\beta(\ell_i)}$ and the canonical map $\pi_U : \oplus U_{\beta(\ell_i)} \to X_\alpha$. In other words, $r_{\ell_j} = \pi_U \kappa_{U,\beta(\ell_j)}$. We note that on all edges $\ell \in \Gamma^\alpha$ the orientation has been changed, that is, the resulting object (X, r) belongs to $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$. Let $\phi = (\phi_\beta) : (U, f) \to (V, g)$ be a morphism in $\mathscr{L}(\Gamma, \Lambda)$, let $(X, r) = F_\alpha^-(U, f)$ and $(Y, s) = F_\alpha^-(V, g)$. We construct $F_\alpha^-(\phi) = \xi = (\xi_\beta)_{\beta \in \Gamma_0} : (X, r) \to (Y, s)$. If $\beta \neq \alpha$, then $X_\beta = U_\beta$, $Y_\beta = V_\beta$, and we set $\xi_\beta = \phi_\beta : U_\beta \to V_\beta$. To construct $\xi_\alpha : X_\alpha \to Y_\alpha$, we consider the following diagram of vector spaces and linear maps

$$\begin{array}{ccccc} U_\alpha & \xrightarrow{\widetilde{h_U}} & \oplus_{i=1}^k U_{\beta(\ell_i)} & \xrightarrow{\pi_U} & X_\alpha \\ {\scriptstyle \phi_\alpha}\downarrow & & \downarrow{\scriptstyle \oplus\phi_{\beta(\ell_i)}} & & \downarrow{\scriptstyle \xi_\alpha} \\ V_\alpha & \xrightarrow{\widetilde{h_V}} & \oplus_{i=1}^k V_{\beta(\ell_i)} & \xrightarrow{\pi_V} & Y_\alpha \end{array} \qquad (3)$$

where $X_\alpha =$ Coker $\widetilde{h_U}$, $Y_\alpha =$ Coker $\widetilde{h_V}$, and π_U and π_U are the canonical maps. It is easy to verify that the left square of the diagram commutes.

$$\widetilde{h_V}\phi_\alpha = (\oplus_{i=1}^k \phi_{\beta(\ell_i)})\widetilde{h_U}$$

Since $\pi_V(\oplus\phi_{\beta(\ell_i)})\widetilde{h}_U = \pi_V \widetilde{h}_V \phi_\alpha = 0$, the universal property of the cokernel (see Introduction) says that there exists a unique k-linear map $\xi_\alpha : X_\alpha \to Y_\alpha$ satisfying $\pi_V(\oplus\phi_{\beta(\ell_i)}) = \xi_\alpha \pi_U$. This finishes the construction of $\xi = F_\alpha^-(\phi)$. We now verify that it is a morphism in $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$. For each edge $\ell = \ell_j : \beta_{(\ell_j)} \to \alpha$ in Γ^α (in the orientation $\sigma_\alpha\Lambda$), we claim that

$$[\oplus\phi_{\beta(\ell_i)}]\kappa_{U,\beta(\ell_j)} = \kappa_{V,\beta(\ell_j)}\xi_{\beta(\ell_j)}.$$

Indeed

$$[\oplus\phi_{\beta(\ell_i)}]\kappa_{U,\beta(\ell_i)}(u_j) = [\oplus\phi_{\beta(\ell_i)}](0, \ldots, u_j, \ldots, 0) = (0, \ldots, \phi_{\beta(\ell_j)}(u_j), \ldots, 0)$$

and

$$\kappa_{V,\beta(\ell_j)}\xi_{\beta(\ell_i)}(u_j) = \kappa_{V,\beta(\ell_j)}(\phi_{\beta(\ell_i)}(u_j)) = (0, \ldots, \phi_{\beta(\ell_j)}(u_j), \ldots, 0).$$

Therefore

$$\xi_\alpha r_{\ell_j} = \pi_V[\oplus\phi_{\beta(\ell_i)}]\kappa_{U,\beta(\ell_j)} = \pi_V \kappa_{V,\beta(\ell_j)}\xi_{\beta(\ell_j)} = s_{\ell_j}\xi_{\beta(\ell_j)}.$$

For each edge $\ell \in \Gamma_1$ not incident to α, we have $\beta(\ell) \neq \alpha$, $\alpha(\ell) \neq \alpha$, so

$$\begin{CD} U_{\alpha(\ell)} @>{f_\ell}>> U_{\beta(\ell)} \\ @V{\phi_{\alpha(\ell)}}VV @VV{\phi_{\alpha(\ell)}}V \\ V_{\alpha(\ell)} @>{g_\ell}>> V_{\beta(\ell)} \end{CD}$$

is a commutative diagram because $\phi : (U, f) \to (V, g)$ is a morphism. Hence the above construction yield the commutative diagram

$$\begin{CD} X_{\alpha(\ell)} @>{r_\ell}>> X_{\beta(\ell)} \\ @V{\xi_{\alpha(\ell)}}VV @VV{\xi_{\beta(\ell)}}V \\ Y_{\beta(\ell)} @>{s_\ell}>> Y_{\beta(\ell)} \end{CD}$$

as required.

We show that $F_\alpha^- : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\alpha \Lambda)$ satisfies the following conditions and therefore is a functor:

1. $F_\alpha^-(1_{(U,f)}) = 1_{(X,r)}$
2. $F_\alpha^-(\psi\phi) = (F_\alpha^-(\psi))(F_\alpha^-(\phi))$

As previously defined, $1_{(U,f)} : (U, f) \to (U, f)$, and $F_\alpha^-(1_{(U,f)}) = \xi = (\xi_\beta)_{\beta \in \Gamma_0} : (X, r) \to (X, r)$. To show: $\xi_\beta = 1_{X_\beta}$, $\beta \in \Gamma_0$.

If $\beta \neq \alpha$, then $\xi_\beta = \phi_\beta$, but $\phi_\beta = 1_{U_\beta} = 1_{X_\beta}$ since $\beta \neq \alpha$.

To show $\xi_\alpha = 1_{X_\alpha}$, we specialize the diagram (3) to the case where $\phi = 1_{(U,f)} : (U, f) \to (U, f)$. We obtain the following commutative diagram

$$\begin{CD} U_\alpha @>{\widetilde{h}_U}>> \oplus U_{\beta(\ell_i)} @>{\pi_U}>> X_\alpha \\ @V{1_{U_\alpha}}VV @V{\oplus 1_{U_{\beta(\ell_i)}}}VV @V{\xi_\alpha}VV \\ U_\alpha @>{\widetilde{h}_U}>> \oplus U_{\beta(\ell_i)} @>{\pi_U}>> X_\alpha \end{CD}$$

It is clear that replacing ξ_α with 1_{X_α} preserves the commutativity of the right square of the diagram: $\pi_U = 1_{X_\alpha}\pi_U = \pi_U(1_{\oplus U_{\beta(\ell_i)}}) = \pi_U(\oplus 1_{U_{\beta(\ell_i)}})$. By the uniqueness of ξ_α we must have $\xi_\alpha = 1_{X_\alpha}$. Hence, $F_\alpha^-(1_{(U,f)}) = 1_{(X,r)}$.

Now we check if $F_\alpha^-(\psi\phi) = (F_\alpha^-(\psi))(F_\alpha^-(\phi))$.
For any objects (U, f), (V, g), and (W, h) in $\mathscr{L}(\Gamma, \Lambda)$, let $\phi : (U, f) \to (V, g)$ and $\psi : (V, g) \to (W, h)$ be morphisms.
Set

$$F_\alpha^-(\phi) = \xi = (\xi_\beta)_{\beta \in \Gamma_0}$$
$$F_\alpha^-(\psi) = \zeta = (\zeta_\beta)_{\beta \in \Gamma_0}$$
$$F_\alpha^-(\psi\phi) = \theta = (\theta_\beta)_{\beta \in \Gamma_0}$$

We want to show that $\theta_\beta = \zeta_\beta \xi_\beta$, $\beta \in \Gamma_0$.
a) For $\beta \neq \alpha$

$$\theta_\beta = [F_\alpha^-(\psi\phi)]_\beta = (\psi\phi)_\beta = \psi_\beta\phi_\beta = [F_\alpha^-(\psi)]_\beta[F_\alpha^-(\phi)]_\beta = \zeta_\beta\xi_\beta$$

b) For $\beta = \alpha$ we set $X_\alpha = \text{Coker}\,\widetilde{h}_U$, $Y_\alpha = \text{Coker}\,\widetilde{h}_V$, and $Z_\alpha = \text{Coker}\,\widetilde{h}_W$

$$\begin{array}{ccccc}
U_\alpha & \xrightarrow{\widetilde{h}_U} & \bigoplus_{i=1}^{k} U_{\beta(\ell_i)} & \xrightarrow{\pi_U} & X_\alpha \\
\downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \bigoplus_{i=1}^{k}\phi_{\beta(\ell_i)}} & & \downarrow{\scriptstyle \xi_\alpha} \\
V_\alpha & \xrightarrow{\widetilde{h}_V} & \bigoplus_{i=1}^{k} V_{\beta(\ell_i)} & \xrightarrow{\pi_V} & Y_\alpha \\
\downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \bigoplus_{i=1}^{k}\psi_{\beta(\ell_i)}} & & \downarrow{\scriptstyle \zeta_\alpha} \\
W_\alpha & \xrightarrow{\widetilde{h}_W} & \bigoplus_{i=1}^{k} W_{\beta(\ell_i)} & \xrightarrow{\pi_W} & Z_\alpha
\end{array}$$

By (3) the above diagram commutes so

$$\pi_W[\oplus(\psi\phi)_{\beta(\ell_i)}] = \pi_W(\bigoplus_{i=1}^{k}\psi_{\beta(\ell_i)}\phi_{\beta(\ell_i)}) = \pi_W(\bigoplus_{i=1}^{k}\psi_{\beta(\ell_i)})(\bigoplus_{i=1}^{k}\phi_{\beta(\ell_i)}) = \zeta_\alpha\pi_V(\bigoplus_{i=1}^{k}\phi_{\beta(\ell_i)}) = \zeta_\alpha\xi_\alpha\pi_U$$

By (3), the diagram below commutes.

$$\begin{array}{ccccc}
U_\alpha & \xrightarrow{\widetilde{h}_U} & \bigoplus_{i=1}^{k} U_{\beta(\ell_i)} & \xrightarrow{\pi_U} & X_\alpha \\
\downarrow{\scriptstyle (\psi\phi)_\alpha} & & \downarrow{\scriptstyle \bigoplus_{i=1}^{k}(\psi\phi)_{\beta(\ell_i)}} & & \downarrow{\scriptstyle \theta_\alpha} \\
W_\alpha & \xrightarrow{\widetilde{h}_W} & \bigoplus_{i=1}^{k} W_{\beta(\ell_i)} & \xrightarrow{\pi_W} & Z_\alpha
\end{array}$$

We have

$$\pi_W[\oplus(\psi\phi)_{\beta(\ell_i)}] = \theta_\alpha\pi_U$$

So both $\zeta_\alpha\xi_\alpha$ and θ_α make the left square of the above diagram commute. By the uniqueness of θ_α, we must have $\theta_\alpha = \zeta_\alpha\xi_\alpha$. Therefore $F_\alpha^-(\psi\phi) = (F_\alpha^-(\psi))(F_\alpha^-(\phi))$ and $F_\alpha^-(1_{(U,f)}) = 1_{(X,r)}$. Thus F_α^- is a functor.

It is easy to see that $F_\alpha^-(\phi+\psi) = F_\alpha^-(\phi) + F_\alpha^-(\psi)$ and $F_\alpha^-(c\phi) = cF_\alpha^-(\phi)$. Therefore F_α^- is a k-linear functor.

3.3 Properties of Reflection Functors

Let (Γ, Λ) be a quiver. For each $\gamma \in \Gamma_0$ we denote by L_γ a simple representation defined by the condition $(L_\gamma)_\delta = 0$ for $\delta \neq \gamma$, $(L_\gamma)_\gamma = K$, $f_\ell = 0$ for all $\ell \in \Gamma_1$.

Theorem 1 1) *Let (Γ, Λ) be a quiver and let $\beta \in \Gamma_0$ be a sink. Let $V \in \mathscr{L}(\Gamma, \Lambda)$ be an indecomposable representation. Then two cases are possible:*

a) $V \approx L_\beta$ and $F_\beta^+ V = 0$.

b) $F_\beta^+(V)$ is an indecomposable representation, $F_\beta^- F_\beta^+(V) = V$, and the dimensions of the spaces $F_\beta^+(V)_\gamma$ can be calculated by the formula

$$\begin{aligned}\dim F^+_\beta(V)_\gamma &= \dim V_\gamma \text{ for } \gamma \neq \beta,\\ \dim F^+_\beta(V)_\beta &= -\dim V_\beta + \sum_{\ell\in\Gamma^\beta} \dim V_{\alpha(\ell)}.\end{aligned}$$

2) *If the vertex α is a source, and if $V \in \mathscr{L}(\Gamma,\Lambda)$ is an indecomposable representation, then two cases are possible:*

a) $V \approx L_\alpha$ and $F^-_\alpha(V) = 0$.

b) $F^-_\alpha(V)$ is an indecomposable representation, $F^+_\alpha F^-_\alpha(V) = V$, and

$$\begin{aligned}\dim F^-_\alpha(V)_\gamma &= \dim V_\gamma \text{ for } \gamma \neq \alpha,\\ \dim F^-_\alpha(V)_\alpha &= -\dim V_\alpha + \sum_{\ell\in\Gamma^\alpha} \dim V_{\beta(\ell)}.\end{aligned}$$

Proof. If the vertex β is a sink with respect to Λ, then it is a source with respect to $\sigma_\beta\Lambda$, and so the functor $F^-_\beta F^+_\beta\colon \mathscr{L}(\Gamma,\Lambda) \to \mathscr{L}(\Gamma,\Lambda)$ is defined. For each representation $(V,g) \in \mathscr{L}(\Gamma,\Lambda)$ we set $(Y,s) = F^+_\beta(V,g)$ and $(Z,t) = F^-_\beta(Y,s)$ so that $Z_\beta = (F^-_\beta(Y))_\beta = (F^-_\beta(F^+_\beta(V)))_\beta = (F^-_\beta F^+_\beta)(V)_\beta$. We construct a morphism $i^\beta_V\colon F^-_\beta F^+_\beta(V,g) \to (V,g)$. If $\gamma \neq \beta$, then $F^-_\beta F^+_\beta(V)_\gamma = V_\gamma$, and we put $(i^\beta_V)_\gamma = \mathrm{Id}$, the identity mapping. For the definition of $(i^\beta_V)_\beta$, we consider the following diagram of K-vector spaces.

$$\begin{array}{ccccc} Y_\beta & \xrightarrow{\widetilde{h}_Y=\kappa_V} & \bigoplus_{i=1}^{k} V_{\alpha(\ell_i)} & \xrightarrow{\pi_Y} & Z_\beta \\ & & & \searrow^{h_V} & \downarrow{\scriptstyle (i^\beta_V)_\beta} \\ & & & & V_\beta \end{array} \tag{4}$$

Here the notation is the same as that of formulas (2) and (3). In particular, Y_β =Ker h_V and Z_β =Coker $\widetilde{h}_Y = \bigoplus_{i=1}^{k} V_{\alpha(\ell_i)}/\mathrm{Ker} h_V$. By the First Isomorphism Theorem, there exists a unique linear map $(i^\beta_V)_\beta$ satisfying $h_V = (i^\beta_V)_\beta \pi_Y$. Now we check that i^β_V is a morphism. Let $\ell \in \Gamma_1$, we want to show that the diagram

$$\begin{array}{ccc} Z_{\alpha(\ell)} & \xrightarrow{t_\ell} & Z_{\beta(\ell)} \\ {\scriptstyle (i^\beta_V)_{\alpha(\ell)}}\downarrow & & \downarrow{\scriptstyle (i^\beta_V)_{\beta(\ell)}} \\ V_{\alpha(\ell)} & \xrightarrow{g_\ell} & V_{\beta(\ell)} \end{array}$$

commutes. If $\ell \notin \Gamma^\beta$, the verification is trivial, and we leave it to the reader. Let now $\ell = \ell_j \in \Gamma^\beta$. Then $\alpha(\ell_j) \neq \beta$ so that $Z_{\alpha(\ell_j)} = V_{\alpha(\ell_j)}$ and $(i^\beta_V)_{\alpha(\ell_j)} = \mathrm{Id}$. Since $V_{\alpha(\ell_i)} = Y_{\alpha(\ell_i)}$ for all i, the formulas preceding diagram (3) say that $t_{\ell_j} = \pi_Y \kappa_{V,\alpha(\ell_j)}$. Then

$$(i^\beta_V)_\beta t_{\ell_j} = (i^\beta_V)_\beta \pi_Y \kappa_{V,\alpha(\ell_j)} = h_V \kappa_{V,\alpha(\ell_j)}$$

The latter equality holds, for if $v \in V_{\alpha(\ell_j)}$, then $h_V\kappa_{V,\alpha(\ell_j)}(v) = h_V(0,\ldots,v,\ldots,0) = g_{\ell_1}(0) + \ldots + g_{\ell_j}(v) + \ldots + g_{\ell_k}(0) = g_{\ell_j}(v)$. Here we used the formulas preceding the diagram (2). Therefore the diagram below commutes.

$$\begin{array}{ccc} V_{\alpha(\ell)} & \xrightarrow{\pi_Y\kappa_{V,\alpha(\ell)}} & Z_\beta \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle (i^\beta_V)_\beta} \\ V_{\alpha(\ell)} & \xrightarrow{g_\ell} & V_\beta \end{array}$$

Similarly, for each source vertex α we construct a morphism $p_V^\alpha\colon V \to F_\beta^- F_\beta^+(V)$. Now we state the basic properties of the functors F_α^-, F_β^+ and the morphisms p_V^α, i_V^β.

Lemma 1 $1)F_\alpha^\pm(V_1 \oplus V_2) = F_\alpha^\pm(V_1) \oplus F_\alpha^\pm(V_2)$.
$2)p_V^\alpha$ *is an epimorphism and* i_V^β *is a monomorphism.*
3) *If* i_V^β *is an isomorphism, then the dimensions of the spaces* $F_\beta^+(V)_\gamma$ *can be calculated from* (1.1.1). *If* p_V^α *is an isomorphism, then the dimensions of the spaces* $F_\alpha^-(V)_\gamma$ *can be calculated from* (1.1.2).
4) *The object Ker* p_V^α *is concentrated at* α *(that is, (Ker* $p_V^\alpha)_\gamma = 0$ *for* $\gamma \neq \alpha$*). The representation* $V/$*Im* i_V^β *is concentrated at* β.
5) *If the representation* V *has the form* $F_\beta^- W$ *(*$F_\alpha^+ W$ *respectively), then* i_V^β *(*p_V^α*) is an isomorphism.*
6) *The representation* V *is isomorphic to the direct sum of the representations* $F_\beta^- F_\beta^+(V)$ *and* $V/$*Im* i_V^β *(similarly,* $V \approx F_\alpha^+ F_\alpha^-(V) \oplus$*Ker* p_V^α*).*

Say how define direct sum in category, then use fact about categories then use that they are additive functors to prove 1.

For 2 we have that for it to be an whatever all of it's parts also have to be an whatever.

Proof. 1) We recall the direct sum construction for quiver representations. If $V_1 = (V_1, g_1), V_2 = (V_2, g_2)$, we define $V_1 \oplus V_2 = (V_1 \oplus V_2, h)$ as follows. For all $\gamma \in \Gamma_0$, $(V_1 \oplus V_2)_\gamma = (V_1)_\gamma \oplus (V_2)_\gamma$, and for all $\ell \in \Gamma_1$, $\ell : \alpha(\ell) \to \beta(\ell), h_\ell = (g_1)_\ell \oplus (g_2)_\ell : (V_1)_{\alpha(\ell)} \oplus (V_2)_{\alpha(\ell)} \to (V_1)_{\beta(\ell)} \oplus (V_2)_{\beta(\ell)}$. The maps $\iota_1 : (V_1, g_1) \to (V_1 \oplus V_2, h)$ and $\pi_1 : (V_1 \oplus V_2, h) \to (V_1, g_1)$ are defined as follows. For each $\gamma \in \Gamma_0$, $(i_1)_\gamma : (V_1)_\gamma \to (V_1 \oplus V_2)_\gamma = (V_1)_\gamma \oplus (V_2)_\gamma$ is given by $(i_1)_\gamma(a) = (a, 0)$, and $(\pi_1)_\gamma : (V_1)_\gamma \oplus (V_2)_\gamma \to (V_1)_\gamma$ is given by $(\pi_1)_\gamma(a, b) = a$. Then we define linear maps $i_2 : (V_2, g_2) \to (V_1, g_1) \oplus (V_2, g_2)$ and $\pi_2 : (V_1, g_1) \oplus (V_2, g_2) \to (V_2, g_2)$ analogously. We leave it to the reader to verify that $\iota_j, \pi_j, j = 1, 2$, are morphisms in $\mathscr{L}(\Gamma, \Lambda)$, and that $\pi_j \iota_j = 1_{(V_j, g_j)}, j = 1, 2$, as well as $\iota_1 \pi_1 + \iota_2 \pi_2 = 1_{(V_1 \oplus V_2, h)}$.

Since we know that $\mathscr{L}(\Gamma, \Lambda)$ and $\mathscr{L}(\Gamma, \sigma_\alpha \Lambda)$ are additive categories and that F_α^+ and F_α^- are additive functors then the statement is a consequence of the following general result.

Proposition 1 *Let* $\mathscr{B}$ *and* $\mathscr{C}$ *be preadditive categories, and* $F : \mathscr{B} \to \mathscr{C}$ *an additive functor. If* $A_1 \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} A \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} A_2$ *is a direct sum diagram in* $\mathscr{B}$*, then* $FA_1 \underset{F\pi_1}{\overset{F\iota_1}{\rightleftarrows}} FA \underset{F\pi_2}{\overset{F\iota_2}{\leftrightarrows}} FA_2$ *is a direct sum diagram in* $\mathscr{C}$.

Proof: By assumption, $\pi_j \iota_j = 1_{A_j}$ for $j = 1, 2$, and $\iota_1 \pi_1 + \iota_2 \pi_2 = 1_A$. Applying F, we get

$$F\pi_j F i_j = F(\pi_j i_j) = F(1_{A_j}) = 1_{FA_j}$$
$$F i_1 F\pi_1 + F i_2 F\pi_2 = F(i_1 \pi_1) + F(i_2 \pi_2) = F(i_1 \pi_1 + i_2 \pi_2) = F(1_A) = 1_{FA}$$

2) To show that i_V^β is a monomorphism we need to check that all of its components are monomorphisms. Since $(i_V^\beta)_\gamma = \mathrm{Id}$ for $\gamma \neq \beta$, clearly the identity is a monomorphism. The First Isomorphism Theorem says that the map $(i_V^\beta)_\beta$ in diagram (4) is a monomorphism. Therefore i_V^β is a monomorphism.

Similarly it is easy to verify that p_V^α is an epimorphism.

3) The first of the two formulas in Theorem 1 part 1b) is obvious. Since i_V^β is an isomorphism by assumption, then $(i_V^\beta)_\beta$ is an isomrophism of vector spaces. We know from diagram (4) that $h_V = (i_V^\beta)_\beta \pi_Y$, where π_Y and $(i_V^\beta)_\beta$ are both epimorphisms, so h_V is a composition of epimorphisms making it an epimorhpism. Therefore we obtain an exact sequence of vector spaces.

$$0 \longrightarrow F_\beta^+(V)_\beta \longrightarrow \oplus_{i=1}^k V_{\alpha(\ell_i)} \longrightarrow V_\beta \longrightarrow 0$$

Then $\dim \bigoplus_{i=1}^k V_{\alpha(\ell_i)} = \sum_{i=1}^k \dim V_{\alpha(\ell_i)} = \dim F_\beta^+(V)_\beta + \dim V_\beta$. The Theorem 1 part 1$b$) equations follow.

Likewise, if p_V^α is an isomorphism then the equations from Theorem 1 part 2b) holds.

4) When $\gamma \neq \alpha$ then $(p_V^\alpha)_\gamma = \text{Id}$, therefore $(\text{Ker}\, p_V^\alpha)_\gamma = 0$. For each $\gamma \neq \beta$ we have $(i_V^\beta)_\gamma = \text{Id}\colon V_\gamma \to V_\gamma$, therefore $(\text{Im}\, i_V^\beta)_\gamma = V_\gamma$ so that $(V/\text{Im}\, i_V^\beta)_\gamma = V_\gamma/V_\gamma = 0$.

5) When $\gamma \neq \beta$ then $(i_V^\beta)_\gamma = \text{Id}$ which is an isomorphism. Since V_β is obtained by a negative reflection functor, the map h_V in diagram (4) is an epimorphism. Since $h_V = \pi_Y(i_V^\beta)_\beta$ then $(i_V^\beta)_\beta$ must be a epimorphism. Since we know $(i_V^\beta)_\beta$ is a monomorphism then i_V^β is an isomorphism.

Similarly, the statement regarding p_V^α holds.

6) We have to show that $V \approx F_\alpha^+ F_\alpha^-(V) \oplus \widetilde{V}$, where $\widetilde{V} = V/\text{Im}\, i_V^\beta$. The natural projection $\phi'_\beta\colon V_\beta \to \widetilde{V}_\beta$ has a section $\phi_\beta\colon \widetilde{V}_\beta \to V_\beta$ ($\phi'_\beta \phi_\beta = \text{Id}$). If we put $\phi_\gamma = 0$ for $\gamma \neq \beta$, we obtain a morphism $\phi : \widetilde{V} \to V$. It is clear that the morphisms $\phi : \widetilde{V} \to V$ and $i_V^\beta : F_\beta^- F_\beta^+(V) \to V$ give a decomposition of V into a direct sum. We can prove similarly that $V \approx F_\alpha^+ F_\alpha^-(V) \oplus \text{Ker}\, p_V^\alpha$. We now prove Theorem 1. Let V be an indecomposable representation of the category $\mathscr{L}(\Gamma, \Lambda)$, and β a sink vertex with respect to Λ. Since $V \approx F_\beta^- F_\beta^+(V) \oplus V/\text{Im}\, i_V^\beta$ and V is indecomposable, V coincides with one of the terms.
Case I). $V = V/\text{Im}\, i_V^\beta$. Then $V_\gamma = 0$ for $\gamma \neq \beta$ and, because V is indecomposable, $V \approx L_\beta$.
Case II). $V = F_\beta^- F_\beta^+(V)$, that is, i_V^β is an isomorphism. Then (Theorem 1 part 1) is satisfied by Lemma 1. We show that the representation $W = F_\beta^+(V)$ is indecomposable. For suppose that $W = W_1 \oplus W_2$. Then $V = F_\beta^-(W_1) \oplus F_\beta^-(W_2)$ and so one of the terms (for example, $F_\beta^-(W_2)$) is 0. By (5) of Lemma 1 the morphism $p_V^\beta : W \to F_\beta^+ F_\beta^-(W)$ is an isomorphism, but $p_V^\beta(W_2) \subset F_\beta^+ F_\beta^-(W_2) = 0$, that is, $W_2 = 0$. So we have shown that the representation $F_\beta^+(V)$ is indecomposable. We can similarly prove (2) of Theorem 1.

We say that a sequence of vertices $\alpha_1, \ldots, \alpha_k$ is a sink with respect to Λ if α_1 is a sink with respect to Λ, α_2 is a sink with respect to $\sigma_{\alpha_1}\Lambda$, α_3 is a sink with respect to $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, and so on. We define a source sequence similarly.

Corollary 1 *Let* (Γ, Λ) *be an oriented graph and* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *a sink sequence.* 1) *For any* $i (1 \leq i \leq k)$, $F_{\alpha_1}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^-(L_{\alpha_i})$ *is either* 0 *or an indecomposable representation in* $\mathscr{L}(\Gamma, \Lambda)$ *(here* $L_{\alpha_i} \in \mathscr{L}(\Gamma, \sigma_{\alpha_{i-1}} \ldots \sigma_{\alpha_1}\Lambda)$*).*
2) *Let* $V \in \mathscr{L}(\Gamma, \Lambda)$ *be an indecomposable representation, and*

$$F_{\alpha_k}^+ \cdot \ldots \cdot F_{\alpha_1}^+(V) = 0$$

Then for some i

$$V \approx F_{\alpha_1}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^-(L_{\alpha_i}).$$

4 The Quadratic Form

Let Γ be a graph without loops. The following definitions are from Bernstein's paper. We denote by $\mathscr{E}_\Gamma$ the vector space over $\mathbb{Q}$ consisting of sets $x = (x_\alpha)$ of rational numbers $x_\alpha(\alpha \in \Gamma_0)$. We call a vector $x = (x_\alpha)$ positive (written $x > 0$) if $x \neq 0$ and $x_\alpha \geq 0$ for all $\alpha \in \Gamma_0$.

We denote by B the quadratic form on the space $\mathscr{E}_\Gamma$ defined by the formula $B(x) = \sum_{\alpha \in \Gamma_0} x_\alpha^2 - \sum_{\ell \in \Gamma_1} x_{\gamma_1(\ell)} x_{\gamma_2(\ell)}$, where $x = (x_\alpha)$, and $\gamma_1(\ell)$ and $\gamma_2(\ell)$ are the ends of the edge ℓ. We denote by $<, >$ the corresponding symmetric bilinear form.

For each $\beta \in \Gamma_0$ we denote by σ_β the linear transformation in $\mathscr{E}_\Gamma$ defined by the formula $(\sigma_\beta x)_\gamma = x_\gamma$ for $\gamma \neq \beta$, $(\sigma_\beta x)_\beta = -x_\beta + \sum_{\ell \in \Gamma^\beta} x_\gamma(\ell)$, where $\gamma(\ell)$ is the end-point of the edge ℓ other than β.

We denote by W the semigroup of transformations of $\mathscr{E}_\Gamma$ generated by the σ_β $(\beta \in \Gamma_0)$. W is related to the Weyl group and σ_β is often called the reflection.

For each $\beta \in \Gamma_0$ we denote by $\overline{\beta}$ the vector in $\mathscr{E}_\Gamma$ such that $(\overline{\beta})_\alpha = 0$ for $\alpha \neq \beta$ and $(\overline{\beta})_\beta = 1$.

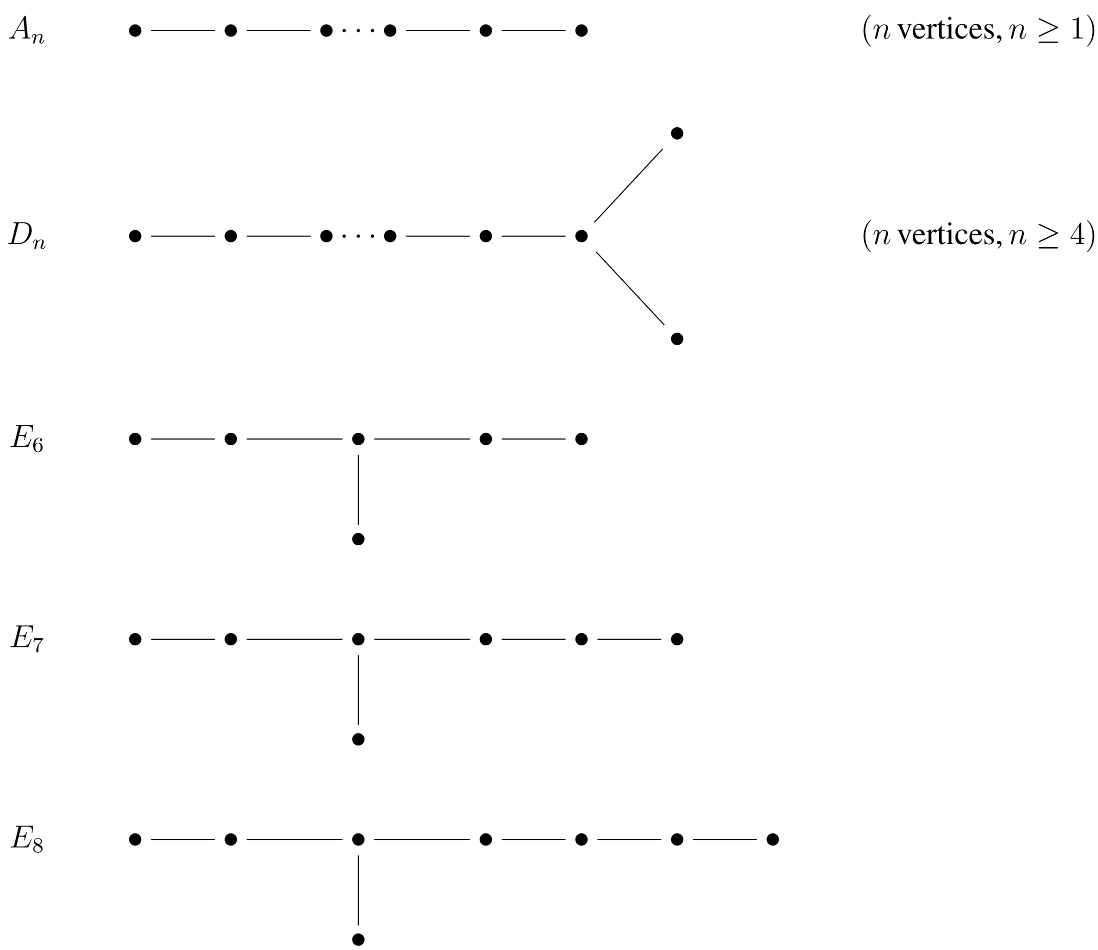
Lemma 2 1) *If* $\alpha, \beta \in \Gamma_0, \alpha \neq \beta$*, then* $< \overline{\alpha}, \overline{\alpha} >= 1$ *and* $2 < \overline{\alpha}, \overline{\beta} >$ *is the negative of the number of edges joining* α *and* β. 2) *Let* $\beta \in \Gamma_0$*. Then* $\sigma_\beta(x) = x - 2 < \overline{\beta}, x > \overline{\beta}, \sigma_\beta^2 = 1$*. In particular,* W *is a group.* 3) *The group* W *preserves the integral lattice in* $\mathscr{E}_\Gamma$ *and preserves the quadratic form* B. 4) *If the form* B *is positive definite (that is,* $B(x) > 0$ *for* $x \neq 0$*), then the group* W *is finite.*

We will skip the proof and move onto more definitions.

Definition 1 *A vector* $x \in \mathscr{E}_\Gamma$ *is called a root if for some* $\beta \in \Gamma_0$*,* $w \in W$ *we have* $x = \omega\overline{\beta}$*. The vectors* $\overline{\beta}(\beta \in \Gamma_0)$ *are called simple roots. A root* x *is called positive if* $x > 0$*.*

5 Applications of Reflection Functors

Let (Γ, Λ) be a finite connected quiver. For each object $V \in \mathscr{L}(\Gamma, \Lambda)$ we regard the set of dimensions $\dim V_\alpha$ as a vector in $\mathscr{E}_\Gamma$ and denote it by $\dim V$. We need the following unoriented graphs to state the main result of the paper, they are known as Dynkin diagrams.



Theorem 2 *(Gabriel [2]).* 1) *If in $\mathscr{L}(\Gamma, \Lambda)$ there are only finitely many non-isomorphic indecomposable objects, then Γ coincides with one of the graphs A_n, D_n, E_6, E_7, E_8.*

2) *Let Γ be a graph of one of the types A_n, D_n, E_6, E_7, E_8, and Λ some orientation of it. Then in $\mathscr{L}(\Gamma, \Lambda)$ there are only finitely many non-isomorphic indecomposable objects. In addition, the mapping $V \mapsto \dim V$ sets up a one-to-one correspondence between classes of isomorphic indecomposable objects and positive roots in $\mathscr{E}_\Gamma$.*

We show how reflection functors F^+_β and F^-_α were used to prove part 2 the following theorem. The following result shows that under the assumptions the quadratic form B is positive definite.

Proposition 2 *The form B is positive definite for the graphs A_n, D_n, E_6, E_7, E_8 and only for them.*

Theorem 1 says that if β is a sink and V is an indecomposable representation of (Γ, Λ), not isomorphic to L_β, then $\dim F^+_\beta V = \sigma_\beta(\dim V)$. Part 2 of Lemma 1 says that σ_β is an invertible linear transformation. Since B is positive definite then σ_β is an orthogonal reflection about a certain hyperplane in $\mathscr{E}_\Gamma$. Due to this fact, F_β^+ got its name as a reflection functor. Repeated use of Corollary 1 implies that there is a bijection between nonisomorphic indecomposable representations of (Γ, Λ) and the positive roots, given by $V \mapsto \dim V$. By part 4 of Lemma 1, the group W is finite. Hence, the set of roots is finite and so is the set of positive roots. Therefore the set of nonisomorphic indecomposable representations is finite.

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