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Properties of the Toric Rings of a Chordal Bipartite Family of Graphs

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Abstract

This project concerns the classification and study of a group of Koszul algebras coming from the toric ideals of a chordal bipartite infinite family of graphs (alternately, these rings may be interpreted as coming from determinants of certain ladder-like structures). We determine a linear system of parameters for each ring and explicitly determine the Hilbert series for the resulting Artinian reduction. As corollaries, we obtain the multiplicity and regularity of the original rings. This work extends results known for a subfamily coming from a two-sided ladder and includes constructive proofs which may be useful in future study of these rings and others. We also develop explicit elements in the Priddy complex which correspond via known isomorphisms to Tate variables in the acyclic closure of the residue field over the localization of our rings at their homogeneous maximal ideals.

Properties of the Toric Rings of a Chordal Bipartite Family of Graphs

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Contents

1 Introduction

1.1 Road Map

Toric ideals and toric rings are algebraic objects which are studied across various mathematical fields, including algebraic geometry, algebra, and graph theory. The goal of this work is to study the properties of the toric rings of a particular family of graphs which generalizes a family of two-sided ladder determinantal rings. We hope that this non-traditional approach to exploring toric rings will lead to new techniques in studying similar properties for related rings and graphs.

In Chapter 2, we provide the necessary background information to understand the work in this dissertation, including information about rings in general and toric rings in particular, as well as basic graph theory. We include definitions that will be relevant for understanding some of the algebraic properties of these rings, which include dimension, Hilbert series for each ring modulo a linear regular sequence, and regularity. We also include background on Lie algebras and free resolutions, which will be necessary for understanding the work we have done to construct explicit Tate variables for these rings.

Chapter 3 introduces a family $\mathcal{F}$ of chordal bipartite graphs coming from generalized determinantal ideals of a particular family of ladder-like structures. This is a generalization of a family $\mathcal{F}_1 \subset \mathcal{F}$ of two-sided ladder determinantal ideals (for large τ), introduced in Example 3.1.3. While the rings coming from $\mathcal{F}_1$ come from a distributive lattice and have well-known properties, we show that the rings associated to $\mathcal{F}$ do not come from a lattice at all in general, and merit closer study.

Chapter 4 covers some algebraic properties of $\mathcal{F}$, particularly dimension, multiplicity, and regularity. We prove our generators are a Gröbner basis to work with initial ideals, and we develop a system of parameters that allows us to work with Artinian rings in part of our treatment.

In Chapter 5, we explicitly construct the variables in a Tate resolution for a localization of each of the rings in our families, using the associated homotopy Lie algebra for a localization of each ring and its universal enveloping algebra, the Koszul dual algebra. To do this, we first construct the Priddy complex for each ring and then use a Gröbner basis argument to establish basis elements for the Lie algebra embedded in the dual of the Priddy complex. It is our hope that this explicit construction will yield further homological results for these rings and others.

We conclude with a brief chapter of unanswered questions and suggested future work on these and related rings.

1.2 Motivation

This project was sparked by a paper by Jennifer Biermann, Augustine O'Keefe, and Adam Van Tuyl titled *Bounds on the regularity of toric ideals of graphs*. We had already been studying toric ideals, and doing so from the perspective of graph theory was appealing. A project was inspired from this point, not only to understand the regularity of toric rings coming from basic graphs, but also to understand other algebraic properties of these rings, through the study of iterated families of graphs.

In this work, to begin an understanding of toric rings coming from graphs, we begin with the most basic graph with a non-trivial toric ideal, a square, and construct an iterated family with toric ideals that get progressively larger but which are still tractable. We aim to create the "simplest" family of graphs that have interesting toric ideals. There are natural benefits to the construction we use, namely that each graph is a subgraph of the next, and that all of the graphs are chordal bipartite. These properties are relevant to the proofs concerning regularity.

A subfamily constructed turns out to be a well-known family that has been studied from various perspectives different from our approach. Our goal in Chapter 3 is to understand the general family and then to extend our understanding in Chapter 4 to algebraic properties of this family.

1.3 Results

We establish the dimension and multiplicity of the toric rings $R(\tau, e)$ developed in this thesis, associated to a family $\mathcal{F}$ of graphs G_τ^e. We also establish the Hilbert series of $R(\tau, e)/(\overline{X_\tau})$, the quotient of $R(\tau, e)$ by a linear regular sequence. We show in Chapter 5 that a particular subset of $R(\tau, e)^\perp$, the quadratic dual algebra of $R(\tau, e)$, is a basis within the Priddy complex which corresponds to the Tate variables in a minimal resolution of k over a localization of $R(\tau, e)$.

First, we establish that the dimension of the toric rings associated to this family depends only on τ (as is true of the remaining results in Chapter 4).

Theorem 1.3.1 (Theorem 4.2.1)**.** *The Krull dimension of $R(\tau, e)$ is*

$$\dim R(\tau, e) = \frac{\tau}{2} + 3.$$

As a corollary, we obtain the projective dimension of the rings $R(\tau, e)$.

Corollary 1.3.2 (Corollary 4.2.2)**.** *The projective dimension of $R(\tau, e)$ over $Q(\tau)$ is*

$$\operatorname{pd}_{Q(\tau)} R(\tau, e) = \tau/2 + 1.$$

We then develop a linear system of parameters for the rings $R(\tau, e)$.

Proposition 1.3.3 (Proposition 4.2.8). *Let* $R(\tau, e) = S(\tau)/I_{G_\tau^e}$, *let*

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

so that $\mathfrak{X}_\tau$, *the image of* X_τ *in* $S(\tau)/(in_> I_{G_\tau^e})$, *is the system of parameters from Remark 4.2.3. Then the image of* X_τ *in* $R(\tau, e)$ *is a system of parameters for* $R(\tau, e)$.

Since the rings $R(\tau, e)$ are proved to be Cohen-Macaulay, the linear system of parameters above is actually a regular sequence (Corollary 4.2.9). We proceed to show the coefficients of the Hilbert series for $R(\tau, e)$. We note that in the below, $\widehat{R(\tau, e)}$ does not denote the completion, but rather is isomorphic to the quotient of $R(\tau, e)$ by the linear regular sequence X_τ. We explain the choice of notation in Notation 4.2.4.

Theorem 1.3.4 (Theorem 4.3.4). *If* $R(\tau, e) = S(\tau)/I_{G_\tau^e}$ *and* $\widehat{R(\tau, e)} \cong R(\tau, e)/(\overline{X_\tau})$, *we have*

$$\dim_k(\widehat{R(\tau, e)})_n = \begin{cases} 1 & n = 0 \\ \dfrac{2^{-n}}{n!} \displaystyle\prod_{j=1}^{n} (\tau + 2j - 4(n-1)) & 1 \leq n \leq \tau/4 + 1 \\ 0 & \textit{else}. \end{cases}$$

As a corollary, we obtain the regularity of $R(\tau, e)$.

Corollary 1.3.5 (Corollary 4.3.7). *For* $G_\tau^e \in \mathcal{F}$,

$$\textit{reg}\ R(\tau, e) = \lfloor \tau/4 \rfloor + 1.$$

We go on to establish a Fibonacci relationship between the lengths of the Artinian rings above, and obtain the multiplicity of $R(\tau, e)$ as a corollary.

Proposition 1.3.6 (Proposition 4.3.8)**.** *The lengths of the rings $R(\tau, e)$ satisfy the recursive formula (where we drop e for convenience)*

$$\ell(\widehat{R(\tau)}) = \ell(\widehat{R(\tau-2)}) + \ell(\widehat{R(\tau-4)})$$

for $\tau \geq 4$. Consequently, if $F(n)$ is the Fibonacci sequence, with $F(0) = 0$ and $F(1) = 1$, then

$$\ell(\widehat{R(\tau)}) = F\left(\frac{\tau}{2}+3\right) = \frac{(1+\sqrt{5})^{\frac{\tau}{2}+3} - (1-\sqrt{5})^{\frac{\tau}{2}+3}}{2^{\frac{\tau}{2}+3}\sqrt{5}}.$$

Corollary 1.3.7 (Corollary 4.3.10)**.** *For even $\tau \geq 4$, there is an equality of multiplicities*

$$e(R(\tau)) = e(R(\tau-2)) + e(R(\tau-4)).$$

In particular,

$$e(R(\tau)) = F\left(\frac{\tau}{2}+3\right) = \frac{(1+\sqrt{5})^{\frac{\tau}{2}+3} - (1-\sqrt{5})^{\frac{\tau}{2}+3}}{2^{\frac{\tau}{2}+3}\sqrt{5}}.$$

In Chapter 5, we find an explicit Tate resolution of k over $R(\tau, e)_{\mathfrak{m}}$ (i.e., the minimal model for $R(\tau, e)$) in the following way: Since it is a minimal resolution, it must be isomorphic to the localization of the dual Priddy resolution $(R(\tau, e) \otimes R(\tau, e)^{\perp}_{\bullet}, \partial)$, so it suffices to identify images of the Tate variables under this isomorphism. The following results do so.

In the following, $Q^{\perp}_{\tau,e}$ is a particular generating set for the defining ideal of the Koszul dual algebra $R(\tau, e)^{\perp}$.

Theorem 1.3.8 (Theorem 5.5.1)**.** *The elements of $Q^{\perp}_{\tau,e}$ are a Gröbner-Shirshov basis for the ideal they generate.*

As a corollary, we obtain

Corollary 1.3.9 (Corollary 5.5.2)**.** *The images of the $Q^{\perp}_{\tau,e}$-reduced super-Lyndon-Shirshov Lie monomials form a basis for*

$$L(\tau) = \frac{Lie(X^*)}{\overline{\langle Q^{\perp}_{\tau,e} \rangle}},$$

the dual of which may be taken to be the set of Tate variables in a minimal Tate resolution of k over $R(\tau, e)_{\mathfrak{m}}$.

1.4 Main Techniques

One technique that proves to be useful in this dissertation is the computation of special bases, of Gröbner bases in Chapter 4 and a Gröbner-Shirshov basis in Chapter 5. We use initial ideals throughout Chapter 4 to establish properties for our rings. Other techniques include the use of a Fibonacci relationship naturally arising between the lengths of related Artinian rings, graph theoretic properties for an alternate proof of regularity, and homological isomorphisms relating our constructions through the Priddy complex to a minimal resolution of k over $R(\tau, e)$ in Chapter 5.

1.5 Contribution

The theoretical contributions of this thesis are:

- This work gives rare examples of an explicit computation of the Tate variables.
- The proofs are often constructive and may prove useful in proving properties about similar families.
- The graphs introduced are basic enough that they may often show up as subgraphs of larger graphs. It is possible that some properties may be traceable between these graphs and graphs that contain them.

2 Background

This section covers the necessary details for understanding the mathematics in Chapters 3, 4, and 5. We begin with basic algebra and graph theory and conclude with Lie algebras and more advanced homological algebra. We include forays into various subdisciplines of algebra as prove useful to the work herein.

In the following, we assume basic knowledge about the following:

- set theory (including the concept of an ordered set)
- group theory
- matrices and determinants
- vector spaces (including duals)
- rings and modules

2.1 Basic Algebra, Graph Theory, and Lattice Theory

The following definitions outline basic notions in algebra and graph theory that are necessary for understanding the content of this dissertation. We begin with some algebraic definitions. For further treatment, see [Mat87] and [AM69]; for graded items in particular see [Pee11].

2.1.1 Algebra

This work will involve both commutative and noncommutative rings with unity. All rings in Chapters 3 and 4 will be commutative rings with unity; we will encounter noncommutative rings in Chapter 5 when we get into the Priddy complex, and an associated Lie algebra. The field k will be a field of characteristic zero throughout.

Definition 2.1.1. A (commutative) *polynomial ring* $k[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ over a field k is the set of polynomials in variables $x_1, \ldots, x_n$ over a field k with the usual addition and multiplication. A polynomial where all of the terms are linear is called a *linear form*, and a monic polynomial with a single term is called a *monomial*.

This work will deal with both commutative and noncommutative polynomial rings with degree one variables; we introduce the noncommutative version in Definition 2.5.5.

Definition 2.1.2. A *graded ring* R is a ring $R = \oplus_{i=0}^{\infty} R_i$, such that if $r_s \in R_s$ and $r_t \in R_t$, then $r_s r_t \in R_{s+t}$. If $r \in R_s$, then r is a homogeneous element of R; we say that the degree of r is $|r| = s$. A homogeneous or graded ideal I of R is an ideal generated by homogeneous elements or equivalently, an ideal $I = \oplus_{i=0}^{\infty} I_n$ that is a direct sum of its graded pieces. A *connected graded ring* R is a commutative graded ring R such that R_0 is a field. A graded R-module N is $N = \oplus_{i=0}^{\infty} N_i$, where $R_i N_j \in N_{i+j}$.

Definition 2.1.3. A *graded k-algebra* A is a k-algebra $A = \oplus_{i=0}^{\infty} A_i$ such that $k \subseteq A_0$ and such that A is both an algebra and a graded ring. We say that A is standard graded if $k = A_0$ and A is generated by A_1 as a k-algebra.

We note that the polynomial ring $k[x_1, \ldots, x_n]$ is a standard graded algebra over k, as is $k[x_1, \ldots, x_n]/I$ for a graded ideal I.

We now introduce the general notions of dimension, system of parameters, regular sequence, and depth, which allows us to define a Cohen-Macaulay ring. The rings defined in Chapter 3 are shown to be Cohen-Macaulay in Corollary 3.2.2.

Definition 2.1.4. The *(Krull) dimension* of a ring R is equal to the length n of the longest chain of the form $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$, where the P_i are prime ideals in R.

Definition 2.1.5. A *system of parameters* for a local (or graded) ring R is a minimal set of (homogeneous) elements $r_1, \ldots, r_n$ such that the quotient ring $R/(r_1, \ldots, r_n)$ has dimension zero. It is a well-known result that $n = \dim R$. When R is graded, a system of parameters is called *linear* if it consists of linear homogeneous elements.

Definition 2.1.6. A *regular sequence* for a ring R is a sequence of elements $m_1, m_2, \ldots, m_n$ such that

- $R/(m_1, \ldots, m_n) \neq 0$
- m_i is a nonzerodivisor on $R/(m_1, \ldots, m_{i-1}) \neq 0$ for $1 \leq i \leq n$, where by convention, when $i = 1$, the quotient is R.

Definition 2.1.7. The *depth* of a graded ring R is the length of the longest regular sequence of homogeneous elements in R

Definition 2.1.8. A ring R is *Cohen-Macaulay* if depth $R = \dim R$

In a Cohen-Macaulay ring, a system of parameters is a regular sequence.

We now review Hilbert functions, which we use to prove results about multiplicity and regularity in Chapter 4. For a full treatment, see [AM69], [Pee11], or [Mat87].

Definition 2.1.9. For a graded ring R, the Hilbert function $H_R(n)$ is the vector space dimension $\dim_k R_n$. The Hilbert series $\mathrm{Hilb}_R(t)$ is

$$\mathrm{Hilb}_R(t) = \sum_n H_R(n) t^n.$$

Definition 2.1.10. The length of a graded ring R is $\ell(R) := \mathrm{Hilb}_R(1)$.

We include a well-known result which we use in the manner of [Pee11, Th 16.7] to define the multiplicity of a standard graded ring R.

Theorem 2.1.11. *Let R be a quotient of a polynomial ring in n variables of degree one, with $d = \dim R$. Then we have*

$$Hilb_R(t) = \frac{f(t)}{(1-t)^n}$$

for some polynomial $f(t)$. Furthermore, if we cancel all possible factors of $(1-t)$, we get a reduced polynomial $h_R(t)$ such that $(1-t) \nmid h_R(t)$ and

$$Hilb_R(t) = \frac{h_R(t)}{(1-t)^d}.$$

Definition 2.1.12. When R satisfies the hypotheses of Theorem 2.1.11, we define the *multiplicity* of R to be $e(R) = h_R(1)$, and note that when $d = 0$ (when R is Artinian), $h_R(1) = \mathrm{Hilb}_R(1) = \ell(R)$.

Multiplicity is also (more traditionally) defined to be the leading coefficient of the Hilbert-Samuel polynomial of R for $d \geq 0$, or as the leading coefficient of the Hilbert polynomial when $d > 0$ and the length $\ell(R)$ when $d = 0$. The definition given is equivalent.

2.1.2 Graph Theory

We move on to some graph theory, establishing a few basic notions and then defining the toric ideal of a graph. For the graph theory, see [Tuc95]; particularly for toric ideals of graphs, see [HHO18], Section 5.3.

Definition 2.1.13. A *graph* G is a set of vertices V together with a set of edges of the form $\{u, v\}$, where $u, v \in V$. The vertices u and v are called the *endpoints* of the edge $\{u, v\}$. The *degree of a vertex* v in a graph G is the number of edges that have v as an endpoint.

This work will only consider *simple graphs*, that is, graphs which do not contain any multiple edges (an edge $\{u, v\}$ appearing twice in the set of edges) or loops (edges $\{v, v\}$).

Definition 2.1.14. An *induced matching* on a graph G is a set of edges who do not share endpoints and whose set of endpoints S has the following property: no edge in G has both its endpoints in S unless it is part of the induced matching.

Definition 2.1.15. A graph G is said to be *bipartite* if the set of vertices of G can be split into two sets V and W (a bipartition) such that every edge of G has exactly one endpoint in V and one in W.

Definition 2.1.16. If a graph G has edge set E and vertex set V, an *(n-)cycle* in G is a subgraph of G with vertex set $\{v_1, \ldots, v_0 = v_n\} \subset V$ and edge set

$$\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_n\} \mid v_0 = v_n\} \subset E.$$

A cycle is called *odd* if n is odd, and *even* if n is even.

A *closed walk (of length n)* in a graph G may be defined the same way as an (n-)cycle, but allows repeated edges and vertices. A closed walk is also called odd if n is odd and even if n is even.

We note that an n-cycle is a closed loop containing n vertices and n edges. We further note that in a bipartite graph, there are no odd cycles.

Definition 2.1.17. We say that a cycle C with edge set E and vertex set V in a graph G has a chord if there is an additional edge e in the edge set of G with the following properties:

- $e \notin E$, and
- the endpoints of e are in V.

Definition 2.1.18. A bipartite graph is said to be *chordal bipartite* if every n-cycle with $n \geq 6$ has a chord.

Essentially, if a graph is chordal bipartite, all of its cycles are even and can be split up into 4-cycles. All of the graphs considered in this work will be chordal bipartite.

We now introduce the toric ideal of a graph, which ties some of the algebraic and graph theoretic notions together.

Definition 2.1.19. Let V be the set of vertices of a graph G and E the set of edges. Let $k[E]$ be the polynomial ring in the edges over k and $k[V]$ the polynomial ring in the vertices over k. Let $\pi : k[E] \to k[V]$ be the ring map induced by assigning to each edge the product of its endpoints. Then the kernel of π is denoted I_G and is called the *toric ideal of* G. We use notation $k[G]$ for the image of π in $k[V]$. The *edge ring of* G is

$$k[G] = \text{im}\, \pi \cong \frac{k[E]}{I_G}.$$

In this dissertation, we study $\frac{k[E]}{\ker \pi}$ and call it the *toric ring of* G (not uncommon in the literature).

It is known that a set of generators for I_G comes from closed even walks in G in the following way: If a closed even walk in G has edge set

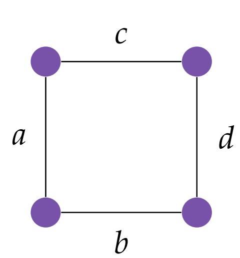
$$\{e_1 = \{v_0, v_1\}, e_2 = \{v_1, v_2\}, \cdots, e_t = \{v_{t-1}, v_t\}\}$$

(where some edges and vertices may repeat), then

$$e_1 e_3 \cdots e_{t-3} e_{t-1} - e_2 e_4 \cdots e_{t-2} e_t \in \ker \pi = I_G;$$

see for example [HHO18], Lemma 5.9.

Example 2.1.20. The graph G below has toric ideal $I_G = (ad - bc)$.



The toric ideal of a graph is a special case of the general notion of toric ideals, particularly the classical notion in algebraic geometry. We focus in this dissertation on properties of the toric rings of a particular family of graphs.

2.1.3 Lattice Theory

We spend a bit of time developing some lattice theory for the development of our family in Chapter 3. One subfamily of ideals we work with comes from a two-sided ladder and hence from a lattice; we show that not every ideal in the general family comes from a lattice in a natural way, so that results from lattice theory do not obviously apply to the general family.

For the treatment below on join-meet ideals, we adopt notation found in [HHO18]. We note that in this source, a lattice of indexed variables is defined from an underlying poset of indices, but we combine these notions to simplify the treatment in this dissertation.

Definition 2.1.21. A *(classical) lattice* L is a partially ordered set of elements $x_1, \ldots, x_n$ with the property that any two elements in L share a common upper bound and a common lower bound. The *join* of two elements x_i and x_j in a lattice L is their least upper bound in L. We denote this by $x_i \vee x_j$. The *meet* of two elements x_i and x_j in a lattice L is their greatest lower bound in L. We denote this by $x_i \wedge x_j$.

Suppose L is a lattice on the variables $x_1, \ldots, x_n$. Then the *join-meet ideal* of L is the ideal of $k[x_1, \ldots, x_n]$ generated by the elements $x_i x_j - (x_i \vee x_j)(x_i \wedge x_j)$. We note that for comparable x_i and x_j, one has $x_i x_j - (x_i \vee x_j)(x_i \wedge x_j) = 0$.

Definition 2.1.22. Given a, b in a lattice L, we say that $\{a, b\}$ is a *comparable pair* if $a < b$ or $b < a$ in L, and we say $\{a, b\}$ is an *incomparable pair* if a and b are not comparable in L.

2.2 Gröbner Bases and Initial Ideals

The following definitions and notes provide the background necessary to understand Gröbner bases, initial ideals, and a criterion used to find Gröbner bases. We use this information in Chapters 3 and 4. For more information on Gröbner bases and initial ideals, see [KR00].

Definition 2.2.1. A *monomial order* on a polynomial ring Q is a total order on the monomials m_i in Q that have a k-coefficient of 1 such that:

- If $m_i \leq m_j$, then $m_\ell m_i \leq m_\ell m_j$ for all ℓ.

We define the degree reverse lexicographic order, which we use throughout Chapters 3 and 4. In Section 2.5.2, we define the noncommutative version of the lexicographic monomial order for use in Chapter 5.

Definition 2.2.2. Let $x_1 > \ldots > x_n$. To a monomial $m = \Pi_{i=1}^{n} x_i^{r_i}$, we associate the n-tuple $\widehat{m} = (r_1, \ldots, r_n)$. The exponent r_i is called the *multiplicity* of the variable x_i. We define the *degree* of m to be $\deg m = \sum_{i=1}^{n} r_i$. We define the *degree reverse lexicographic order* as follows: we have $m > n$ if

- $\deg m > \deg n$ or if
- $\deg m = \deg n$ and the last nonzero entry in $\widehat{m} - \widehat{n}$ is negative.

Definition 2.2.3. The *leading term* of a polynomial is the term whose monomial (it may have a coefficient) is largest in the monomial ordering. We denote the leading term of a polynomial f by $LT(f)$.

Theorem 2.2.4. *(Macaulay's Basis Theorem) Let $Q = k[x_1, \ldots, x_n]$ and let I be an ideal of Q. Let B be the set of (monic) monomials b in Q such that $LT(f) \nmid b$ for any f in I. Then the residue classes of the elements of B are a k-vector space basis for Q/I.*

This is actually a less general version; for the original theorem see [KR00], Theorem 1.5.7. To apply the above result, we need to find a way to characterize the leading terms of an ideal I.

Definition 2.2.5. The *initial ideal* of an ideal I in a polynomial ring Q is denoted $\text{in}_{>}(I)$ and is equal to the ideal generated by the leading terms of all polynomials in I.

By Proposition 9.3.4 and Proposition 9.3.12 of [CLO07], an ideal I and its initial ideal $\text{in}_{>}(I)$ of a polynomial ring Q have the same Hilbert function, and in particular, the quotients by them have the same dimension and multiplicity. By Theorem 4.1.3 of [BH93], the Krull dimension of the quotient of a ring by an ideal may be established from its Hilbert polynomial. Since $Q/\text{in}_{>} I$ and Q/I have the same Hilbert polynomial, we have the following proposition.

Proposition 2.2.6. *For a polynomial ring Q and an ideal I of Q, the quotient ring $Q/(in_{>} I)$ has the same Krull dimension as Q/I.*

We use this information in Chapter 4.

Definition 2.2.7. A *Gröbner basis* of an ideal I is a generating set for I whose leading terms generate the initial ideal $\text{in}_{>}(I)$.

Gröbner bases are very helpful when one wishes to work with the initial ideal. Below is a criterion for finding a Gröbner basis from an existing generating set; we first introduce some helpful definitions.

Definition 2.2.8. Let $G = \{g_1, \ldots, g_m\}$. The S-polynomial of g_i and g_j is

$$S_{i,j} = \frac{LT(g_j)}{gcd(LT(g_i), LT(g_j))}(g_i) - \frac{LT(g_i)}{gcd(LT(g_i), LT(g_j))}(g_j)$$

We note that for $i = j$, $S_{i,j} = 0$.

Definition 2.2.9. Let $f \in k[x_1, \ldots, x_n]$ and $G = (g_1, \ldots, g_m)$ be an ordered m-tuple of polynomials. The *remainder on division of f by G* is denoted $\overline{f}^G$ (or $\overline{f}$ when G is understood) and is defined as follows in the case when f and g_i are monic:

- Let i be the first index such that $LT(g_i) \mid LT(f)$. Then replace f with the reduction $f - \frac{LT(f)}{LT(g_i)} g_i$.
- Repeat until there is no longer any such i.
- The final reduction is called the *remainder on division of f by G*.

See for example [CLO07, Th 2.3.3]. We note that this algorithm terminates, since each successive leading term is less than the previous one and monomial orderings satisfy the descending chain condition. When G is a Gröbner basis, the remainder of $f \in k[x_1, \ldots, x_n]$ is unique regardless of the ordering on G.

Theorem 2.2.10 (Buchberger's Criterion)**.** *Let $G = \{g_1, \ldots, g_m\}$ be a generating set for an ideal $I \in k[x_1, \ldots, x_n]$. If the remainder on division of $S_{i,j}$ by G (where the elements of G are listed in some order) is 0 for all i, j, then G is a Gröbner basis for I.*

For this theorem, see for example [CLO07, Th 2.6.6]. We note that the ordering of the generators of G may differ for each pair $\{i, j\}$. In this dissertation, we denote the remainder on division of $S_{i,j}$ by G as $\overline{S_{i,j}}$ and call it the *reduced form* of $S_{i,j}$.

There are some shortcuts that may be used when applying the Buchberger's criterion. For instance, if $gcd(LT(g_i), LT(g_j)) = 0$, then the reduced S-polynomial of g_i and g_j is zero. Also, the (reduced) S-polynomial of two monomials is zero.

2.3 Homological Algebra

First, we begin with definitions pertaining to resolutions, especially in the graded setting, and the numerical invariant regularity, which will be used in Chapter 4. Then we recall the definition of the Tate resolution (in the local setting), which will be used in Chapter 5. For more information on chain complexes and the Tate resolution, see [Avr98].

Definition 2.3.1. A *chain complex* of modules is a sequence of modules M_i and module homomorphisms f_i of the following form:

$$\cdots \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} \cdots$$

such that $f_{i-1} \circ f_i = 0$ for $i \in \mathbb{Z}$. We say that a chain complex is *exact* if in addition $\ker f_{i-1} = \operatorname{im} f_i$ for all i.

Definition 2.3.2. A *free resolution* of a module M over a ring R is an exact chain complex with free R-modules F_i of the following truncated form:

$$\cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\pi} M \to 0,$$

where π is surjective. The maps f_i may be represented with matrices A_i. A resolution is *minimal* if the entries in A_i are not units. A resolution is *graded* if R and M are graded and the entries in the A_i are homogeneous.

Definition 2.3.3. One can decompose each F_i as $F_i = \oplus_j R(-j)^{\beta_{i,j}}$. The exponent $\beta_{i,j}$ is called the (i, j)-th *graded Betti number* of M.

We will be looking at resolutions of a field k as an $R = k[x_1, \ldots, x_n]/I$-module, where I is a homogeneous ideal of $k[x_1, \ldots, x_n]$.

Definition 2.3.4. The *Castelnuovo-Mumford regularity* of an ideal I in a ring R is

$$\operatorname{reg}_R I = \operatorname{reg} I = \max\{j - i \mid \beta_{i,j}(I) \neq 0\},$$

where $\beta_{i,j}$ is the (i, j)-th graded Betti number in a graded minimal free resolution of I over R. The regularity of R/I is $\operatorname{reg} I - 1$.

The regularity of a standard graded ring $R \cong Q/I$ for a polynomial ring Q and graded ideal I is $\operatorname{reg} R = \operatorname{reg}_Q R$. It is well-known that this is independent of the choice of Q and I.

We use a result in Chapter 4 relating the top nonzero degree of an Artinian quotient of a polynomial ring with its regularity over the polynomial ring; see for example [Pee11], Theorem 18.4.

Theorem 2.3.5. *If Q is a polynomial ring with an ideal B such that Q/B is Artinian, then $\operatorname{reg}_Q Q/B$ is equal to*

$$\max\{n \mid (Q/B)_n \neq 0\},$$

the top nonzero degree of Q/B.

We now move on to the construction of a differential graded algebra resolution.

Definition 2.3.6. A *differential graded k-algebra* is a graded k-algebra

$$A = \bigoplus_{n \geq 0} A_n$$

equipped with a map $d : A \to A$ of degree -1 such that $d^2 = 0$ and the Leibniz rule $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ holds. When k is understood, we will sometimes refer to a differential graded k-algebra as a *dg-algebra*. Notice that the set $(\{A_n\}_{n \geq 0}, d)$ forms a complex, which we will again call A. A resolution that is also a dg-algebra is called a *dg-algebra resolution*.

We recall the classic construction by Tate [Tat57] of a dg-algebra resolution, but focus on the setting of resolutions of the residue field k over a graded k-algebra localized at the homogeneous maximal ideal. We introduce some preliminary definitions and then move to the acyclic closure of k over R.

Definition 2.3.7. Given an odd cycle z in the complex A (i.e., $|z|$ is odd and $\partial(z) = 0$), one adjoins a *divided powers variable* x of even degree $|z| + 1$ to obtain a complex $A\langle x \rangle$ with $\partial(x) = z$ in the following way:

- The graded algebra $A\langle x \rangle$ is generated as a free module over A by the set

  $$\{x^{(i)} : |x^{(i)}| = i|x|\}_{i \geq 0}$$

  with $x^{(0)} = 1$ and $x^{(1)} = x$

- The divided power $x^{(i)}$ commutes with all other variables and we have

  $$x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)}$$

  for $i, j \geq 0$.

- The new (compatible) differential is $\partial(\Sigma a_i x^{(i)} = \Sigma \partial(a_i) x^{(i)} + \Sigma(-1)^{|a_i|} z x^{(i-1)}$ for $a_i \in A$ homogeneous.

Definition 2.3.8. Given an even cycle z in the complex A, one adjoins an *exterior variable* y of odd degree $|z| + 1$ to obtain a complex $A\langle y \rangle$ with $\partial(y) = z$ in the following way:

- The graded algebra $A\langle y \rangle$ is generated as a free module over A by the set $\{1, y\}$.

- The variable y commutes with divided powers variables and anticommutes with exterior variables.

- The new (compatible) differential is $\partial(ay) = \partial(a)y + (-1)^{|a_i|} z$ for $a \in A$ homogeneous.

Definition 2.3.9. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} \cong k$. The *acyclic closure* of k over R involves a very specific iterative construction of a differential graded k-algebra that is a resolution of k over R. This construction was first introduced by John Tate:

- We begin with the natural surjective ring map π from R to k.
- Let $x_1, \ldots, x_n$ be a set of minimal generators of $\mathfrak{m} = \ker \pi$. Adjoin a set of $Y_1 = \{y_1, \ldots, y_n\}$ variables of degree one to obtain $R\langle Y_1 \rangle := R\langle y_1, \ldots, y_n \rangle$ with $\partial(y_i) = x_i$.
- Iteratively, we find the ith homology of $R\langle Y_1, \ldots, Y_i \rangle$ and adjoin a minimal set Y_{i+1} of polynomial or divided powers variables that map under the differential to elements whose images minimally generate the ith homology.

Theorem 2.3.10. *The acyclic closure of k over R is a minimal resolution.*

This was proved independently by Gulliksen [Gul68] and Schoeller [Sch67].

2.4 The Priddy Complex

The following definitions will aid in understanding the construction of the Priddy complex, which is a chain complex constructed by Stewart Priddy that yields a resolution of the residue field of certain k-algebras that is much smaller than the one given by the bar construction: He showed its dual gives an R-free resolution of k in the case when R is a Koszul k-algebra (see Definition 2.4.3); it was in fact Priddy who developed the notion of a Koszul algebra [Pri70]. The dual of the Priddy complex will be the minimal graded free resolution of k over R. For further reading on the Priddy complex, see [PP05].

Definition 2.4.1. Let V be a vector space. The *tensor algebra of V* is $T(V) = \oplus_{i=0}^{\infty} V^{\otimes i}$, where $V^{\otimes 0}$ is taken to be k. It is an algebra with formal addition and concatenary multiplication. One calls $T(V^*)$ the *dual tensor algebra* of V.

Note that the tensor algebra is not commutative when $\dim_k V \geq 2$. If V is the vector space generated by $y_1, \ldots, y_n$, then $T(V)$ is isomorphic to $k\langle y_1, \ldots, y_n \rangle$, the ring of polynomials in noncommutative variables $y_1, \ldots, y_n$.

Now we develop the notion of a quadratic algebra and its quadratic dual algebra.

Definition 2.4.2. Suppose Q is a linearly independent set of degree 2 elements in the tensor algebra of V which is a basis for the subspace (Q). The *quadratic algebra* defined by Q is $T(V)/\langle Q \rangle$, where $\langle Q \rangle$ is the two-sided ideal generated by Q in $T(V)$.

Definition 2.4.3. A quadratic algebra R is said to be *Koszul* if the minimal graded resolution of k over R is linear, that is, if the entries of the matrices describing the maps in the resolution are zero or homogeneous of degree one.

Definition 2.4.4. We define $Q^{\perp}$ to be a basis of the *perpendicular subspace* $(Q)^{\perp}$ defined by $(Q)^{\perp} = \{f \in V^{*\otimes 2} | f(v) = 0 \text{ for all } v \in Q\}$, where one identifies $(V^*)^{\otimes 2} \cong (V^{\otimes 2})^*$.

Definition 2.4.5. Using the above notation, one calls $T(V^*)/\langle Q^{\perp} \rangle$ the *quadratic dual algebra* to the quadratic algebra $T(V)/\langle Q \rangle$. When $R = T(V)/\langle Q \rangle$, the quadratic dual algebra $T(V^*)/\langle Q^{\perp} \rangle$ is denoted by $R^{\perp}$ or by $R^{!}$.

In the below, we write the quadratic dual algebras R and $R^{\perp} = R^{!}$ as quotients of the noncommutative polynomial rings $k\langle z_1, \ldots, z_n \rangle$ and $k\langle z_1^*, \ldots, z_n^* \rangle$, instead of using the tensor algebras $T(V)$ and $T(V^*)$, where $z_1, \ldots, z_n$ and $z_1^*, \ldots, z_n^*$ are bases for the vector space V and its dual V^*, respectively. We may do this because the tensor algebra in a set of variables X over the field k is isomorphic to the noncommutative polynomial ring in the variables X over the field k. We use the same notation for the two-sided ideal generated by an ideal in the noncommutative ring as we do in the tensor algebra. We have

$$R^{\perp} \cong \frac{k\langle z_1^*, \ldots, z_n^* \rangle}{\langle Q^{\perp} \rangle},$$

when

$$R \cong \frac{k\langle z_1, \dots, z_n\rangle}{\langle Q\rangle}.$$

Both R and $R^\perp$ are quadratic algebras.

We now define the Priddy complex, the dual of which gives us a resolution of k over R when R is Koszul.

Definition 2.4.6. Suppose that V is a vector space with basis $x_1, \dots, x_n$ and dual basis $x_1^*, \dots, x_n^* \in V^*$. Then

$$R = T(V)/\langle Q\rangle$$

and

$$R^\perp = T(V^*)/\langle Q^\perp\rangle$$

are quadratic k-algebras, and the Priddy complex of R (which Priddy called the Koszul complex of R) is

$$P(R) = R \to R \otimes_k V^* \to R \otimes_k R_2^\perp \to \cdots \to R \otimes_k R_d^\perp \to \cdots$$

where the maps are multiplication on the right by

$$t = \sum_{i=1}^{n} x_i \otimes x_i^*;$$

see [Eis89].

Definition 2.4.7. The *dual complex* of a chain complex of free R-modules

$$\dots \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} F_0 \xrightarrow{f_0} \cdots,$$

where each f_i is represented by a matrix M_i, is the chain complex

$$\cdots \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_n^*} F_n^* \xrightarrow{f_{n+1}^*} \cdots,$$

where f_i^* is represented by the matrix M_i^T. We use the notation $F_i^* = \mathrm{Hom}_R(F_i, R)$ to denote the set of all R-module maps from F_i to R. This is again isomorphic to F_i.

Theorem 2.4.8. *If R is a Koszul graded k-algebra, then the dual of the Priddy complex gives a (minimal graded) resolution of k over R.*

2.5 Graded Lie Algebras and Gröbner-Shirshov Bases

Here we define Lie algebras, their universal enveloping algebras, and Gröbner-Shirshov bases, which will be used in Chapter 5. For the following, we use [Avr98] and [BKLM99].

2.5.1 Lie Algebras and Their Universal Enveloping Algebras

Definition 2.5.1. A *graded Lie algebra* over k is a k-module $L = \{L_n\}_{n\in\mathbb{Z}}$ with a k-bilinear pairing, called the *Lie bracket*

$$[-,-] : L_i \times L_j \to L_{i+j} \text{ for } i, j \in \mathbb{Z}, (\alpha, \beta) \mapsto [\alpha, \beta], \tag{2.1}$$

such that for all $\alpha, \beta, \gamma \in L$

- $[\alpha, \beta] = -(-1)^{|\alpha||\beta|}[\beta, \alpha]$
- $[[\alpha, \beta], \gamma] = [\alpha, [\beta, \gamma]] - (-1)^{|\alpha||\beta|}[\beta, [\alpha, \gamma]]$.

Definition 2.5.2. For every graded associative algebra A, we get a graded Lie algebra $\mathrm{Lie}(A)$ by forgetting some of the associative structure and using the *graded commutator*

$$[x, y] = xy - (-1)^{|x||y|}yx \tag{2.2}$$

for $x, y \in A$.

Definition 2.5.3. The ring $k\langle X\rangle$ becomes a graded Lie (super)algebra $\mathrm{Lie}(k\langle X\rangle)$ by introducing the (super)bracket

$$[a, b] = ab - (-1)^{|a||b|}ba$$

for $a, b \in k\langle X\rangle$. The *free Lie (super)algebra generated by* X, $\mathrm{Lie}(X)$, is the (graded/super) Lie subalgebra of $\mathrm{Lie}(k\langle X\rangle)$ generated by X.

Definition 2.5.4. For a Lie algebra L, there is a unique *universal enveloping algebra* $U(L)$, given by allowing associative multiplication between all elements of L and identifying $[a, b]$ with $ab - (-1)^{|a||b|}ba$ in $U(L)$.

The universal enveloping algebra $U(L)$ has a universal property and is unique up to isomorphism of associative algebras, and L is embedded in $U(L)$ by the Poincaré-Birkhoff-Witt Theorem. In some cases, we may begin with an associative algebra A and obtain a lie algebra L such that $A = U(L)$; see [MM65].

2.5.2 Gröbner-Shirshov bases

The following definitions and notes provide the background necessary to understand Gröbner-Shirshov bases. We provide further background in Chapter 5. For more information on Gröbner-Shirshov bases, see [BKLM99].

Definition 2.5.5. We use the notation $k\langle X\rangle$ to denote the noncommutative polynomial ring in a set of variables X. It has the obvious noncommutative multiplication. We note that $k\langle X\rangle$ is the free associative algebra generated by X.

Definition 2.5.6. A *monomial* (or *word*) in $k\langle X\rangle$ is a monic noncommutative polynomial with a single term, and a (noncommutative) polynomial is a linear combination of monomials in $k\langle X\rangle$ with coefficients in k. If $u = x_{n_1}x_{n_2}\cdots x_{n_t}$ is a monomial in $k\langle X\rangle$ with

$x_{n_1}, x_{n_2}, \ldots, x_{n_t} \in X$, we say u has *length* (or *degree*) t and write $|u| = t$. The monomial $w = 1$ is called the *empty word*, and its degree is 0.

The lexicographic monomial order defined below is a “dictionary" ordering, where larger variables come earlier in the alphabet. For a commutative ring, it is merely the multiplicity of each x_i that counts, while in a noncommutative ring, it is also the order that counts. When we are in the noncommutative setting in Chapter 5, we will use this monomial order.

Definition 2.5.7. Let $X = \{x_1, \ldots, x_n\}$ with $x_1 > \ldots > x_n$. The *lexicographic ordering* is defined as follows: For any nonempty words v and w in $k\langle X \rangle$,

- $1 > v$
- $v > w$ if there are (possibly empty) words u, v', and w' in $k\langle X \rangle$ and variables $x_{n_1}, x_{n_2} \in X$ such that $v = ux_{n_1}v'$ and $w = ux_{n_2}w'$ with $x_{n_1} > x_{n_2}$.

Definition 2.5.8. The *leading term* of a polynomial in $k\langle X \rangle$ is the term whose monomial (it may have a nonzero coefficient) is largest in the monomial ordering. We denote the leading term of a polynomial f by $LT(f)$.

2.6 Notation

We provide for the reader's reference a list of notations which will be defined in later chapters.

- M_τ^e is a ladder-like structure depending on even τ and $e \in \mathbb{F}_2^{\tau/2+1}$.
- The graph G_τ^e is defined from the ladder-like structure M_τ^e and has edge set $E = \{x_0, x_2, x_3, \ldots, x_{\tau+4}\}$.
- The family of graphs G_τ^e for even τ and $e \in \mathbb{F}_2^{\tau/2+1}$ is denoted $\mathcal{F}$.
    * The subfamily where all entries of e are one is denoted $\mathcal{F}_1$. In this family, M_τ^e is a two-sided ladder for even $\tau \geq 10$.
    * The subfamily where all entries of e are zero is denoted $\mathcal{F}_2$.
- $S(\tau) := k[x_0, x_2, x_3, \ldots, x_{\tau+4}]$, where τ is even, will denote the polynomial ring over the field k on variables $x_0, x_2, x_3, \ldots, x_{\tau+4}$.
- $I_{G_\tau^e}$ is the toric ideal associated with the graph G_τ^e.
- $R(\tau, e) := S(\tau)/I_{G_\tau^e}$ is the toric ring of G_τ^e, isomorphic to the edge ring $k[G_\tau^e]$.
- We use X_τ to denote a particular linear set in $S(\tau)$
    - The image $\mathfrak{X}$ of X_τ in $S(\tau)/\mathrm{in}_> I_{G_\tau^e}$ is proven to be a linear system of parameters for the quotient.

  The image $\overline{X_\tau}$ of X_τ in $R(\tau, e)$ is proven to be a linear system of parameters for $R(\tau, e)$.
- We use notation $z_n := x_n^*$ in Chapter 2.3.6, and use $k\langle z_0, z_2, z_3, \ldots, z_{\tau+4}\rangle$ to denote the noncommutative polynomial ring in the variables $z_0, z_2, z_3, \ldots, z_{\tau+4}$ over the field k.
- $R(\tau, e)^\perp = k\langle X^* \rangle / \langle Q_{\tau,e}^\perp \rangle$ is the quadratic dual algebra of $R(\tau, e) = k\langle X \rangle / \langle Q_{\tau,e} \rangle$.

3 The Family of Toric Rings

In the following, we define a family of toric rings coming from an iterative chordal bipartite family of graphs, $\mathcal{F}$. We show that although one subfamily of these rings comes from join-meet ideals of a (distributive) lattice and has many known results, this is not true in general. Throughout, k is a field with characteristic zero, and τ and i are even indices.

3.1 The Family $\mathcal{F}$ of Graphs

Below, we define the family $\mathcal{F}$ of graphs iteratively from a family of ladder-like structures M_τ^e. We note that the quantities involved in the following definition follow patterns as follows:

τ	$\tau/2+1$	$\lfloor \tau/4 \rfloor + 2$	$\lceil \tau/4 \rceil + 2$
0	1	2	2
2	2	2	3
4	3	3	3
6	4	3	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Definition 3.1.1. For each even $\tau \geq 0$ and for each $e \in \mathbb{F}_2^{\tau/2+1}$, we construct a ladder-like structure M_τ^e with $(\lfloor \tau/4 \rfloor + 2)$ rows and $(\lceil \tau/4 \rceil + 2)$ columns and nonzero entries in the set $X = \{x_0, x_2, x_3, \ldots, x_{\tau+4}\}$. To do so, we use the notation $\widehat{e} \in \mathbb{F}_2^{\tau/2}$ for the first $\tau/2$ entries of e, that is, all except the last entry. The construction is as follows, where throughout, indices of entries in M_τ^e are strictly increasing from left to right in each row and from top to bottom in each column. We note that M_τ^e does not depend on e for $\tau \leq 2$, but does for $\tau \geq 4$.

- For $\tau = 0$, the ladder-like structure $M_0^0 = M_0^1$ is

$$\begin{matrix} x_0 & x_2 \\ x_3 & x_4 \end{matrix}$$

- For $\tau = 2$, to create M_2^e (regardless of what e is in $\mathbb{F}_2^2$), we add another column with the entries x_5 and x_6 to the right of M_0 to obtain

$$\begin{matrix} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \end{matrix}$$

- For $4 \leq \tau \equiv 0 \mod 4$, to create M_τ^e, we add another row with the entries $x_{\tau+3}, x_{\tau+4}$ below $M_{\tau-2}^{\hat{e}}$ in the following way:
    - The entry $x_{\tau+4}$ is in the new row (row $\lfloor \tau/4 \rfloor + 2$) and the rightmost column (column $\lceil \tau/4 \rceil + 2$).
    - The entry $x_{\tau+3}$ is in the new row (row $\lfloor \tau/4 \rfloor + 2$) in a position directly below another nonzero entry in M_τ^e.
        - If the last entry of e is 0, $x_{\tau+3}$ is directly beneath the first nonzero entry in the previous row.
        - If the last entry of e is 1, $x_{\tau+3}$ is directly beneath the second nonzero entry in the previous row.
- For $2 \leq \tau \equiv 2 \mod 4$, to create M_τ^e, we add another column with the entries $x_{\tau+3}, x_{\tau+4}$ to the right of $M_{\tau-2}^{\hat{e}}$ in the following way:
    - The entry $x_{\tau+4}$ is in the new column (column $(\lceil \tau/4 \rceil + 2)$) and the bottom row (row $\lfloor \tau/4 \rfloor + 2$).

- ◦ The entry $x_{\tau+3}$ is in the new column (column $(\lceil \tau/4 \rceil + 2)$) in a position directly to the right of another nonzero entry of M_τ^e.
  - ∗ If the last entry of e is 0, $x_{\tau+3}$ is directly to the right of the first nonzero entry in the previous column.
  - ∗ If the last entry of e is 1, $x_{\tau+3}$ is directly to the right of the second nonzero entry in the previous column.

In this way, the entries in e determine the choice at each stage for the placement of $x_{\tau+3}$.

Remark 3.1.2. We note a few things about this construction:

- We note that $x_{\tau+4}$ is in row $\lfloor \tau/4 \rfloor + 2$ and column $\lceil \tau/4 \rceil + 2$ for $\tau \geq 0$, and in particular, that $x_{\tau+4}$ is directly beneath $x_{\tau+2}$ for $\tau \equiv 0 \mod 4$ and that $x_{\tau+4}$ is directly to the right of $x_{\tau+2}$ for $\tau \equiv 2 \mod 4$.

- We note that for $\tau \equiv 0 \mod 4$ ($\equiv 2 \mod 4$) the only entries in row $\lfloor \tau/4 \rfloor + 2$ (column $\lceil \tau/4 \rceil + 2$) are $x_{\tau-1}$, x_τ, and $x_{\tau+2}$, so that the choices listed for placement of $x_{\tau+3}$ are the only cases. In particular, $e_{\tau/2+1} = 0$ if and only if $x_{\tau+3}$ is directly beneath (to the right of) $x_{\tau-1}$, and $e_{\tau/2+1} = 1$ if and only if $x_{\tau+3}$ is directly beneath (to the right of) x_τ.

Example 3.1.3. For example, we have

$$M_4^{(1,1,1)} = \begin{matrix} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \\ & x_7 & x_8 \end{matrix}$$

$$M_4^{(0,0,0)} = \begin{matrix} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \\ x_7 & & x_8 \end{matrix}$$

In either of the above cases, we could go on to construct M_6^e and M_8^e in the following way: For $\tau = 6$, add x_{10} to the right of x_8 and add x_9 to the right of either x_5 or x_6, depending whether the last entry of e is 0 or 1, respectively. Then for $\tau = 8$, add x_{12} below x_{10} and add x_{11} below either x_7 or x_8, depending whether the last entry of e is 0 or 1, respectively.

In fact, when the entries of e are all ones, we see that $M_\tau^{(1,1,\ldots,1)}$ has a ladder shape (is a two-sided ladder for $\tau \geq 6$), shown below in the case when $4 \leq \tau \equiv 0 \mod 4$:

$$\begin{array}{cccccccccc}
x_0 & x_2 & x_5 & & & & & & & \\
x_3 & x_4 & x_6 & x_9 & & & & & & \\
 & x_7 & x_8 & x_{10} & x_{13} & & & & & \\
 & & x_{11} & x_{12} & x_{14} & x_{17} & & & & \\
 & & & x_{15} & x_{16} & x_{18} & x_{21} & & & \\
 & & & & x_{19} & x_{20} & x_{22} & x_{25} & & \\
 & & & & & x_{23} & x_{24} & x_{26} & \ddots & \\
 & & & & & & x_{27} & x_{28} & \ddots & x_{\tau+1} \\
 & & & & & & & \ddots & \ddots & x_{\tau+2} \\
 & & & & & & & & x_{\tau+3} & x_{\tau+4}.
\end{array}$$

We denote the subfamily of graphs coming from $e = (1, 1, \ldots, 1)$ by $\mathcal{F}_1 \subset \mathcal{F}$.

When the entries of e are all zeros, $M_\tau^{(0,0,\ldots,0)}$ has the following structure, shown below in the case when $4 \le \tau \equiv 0 \mod 4$:

$$
\begin{array}{cccccccccc}
x_0 & x_2 & x_5 & x_9 & x_{13} & x_{17} & x_{21} & x_{25} & \cdots & x_{\tau+1} \\
x_3 & x_4 & x_6 & & & & & & & \\
x_7 & & x_8 & x_{10} & & & & & & \\
x_{11} & & & x_{12} & x_{14} & & & & & \\
x_{15} & & & & x_{16} & x_{18} & & & & \\
x_{19} & & & & & x_{20} & x_{22} & & & \\
x_{23} & & & & & & x_{24} & x_{26} & & \\
x_{27} & & & & & & & x_{28} & \ddots & \\
\vdots & & & & & & & & \ddots & x_{\tau+2} \\
x_{\tau+3} & & & & & & & & & x_{\tau+4}.
\end{array}
$$

We denote the subfamily of graphs coming from $e = (0, 0, \ldots, 0)$ by $\mathcal{F}_2 \subset \mathcal{F}$.

For a more varied example, we have $M_{32}^{(1,0,1,0,1,1,0,0,1,1,1,0,0,0,1,0,0)}$ equal to

$$
\begin{array}{cccccccccc}
x_0 & x_2 & x_5 & x_9 & & & & & & \\
x_3 & x_4 & x_6 & & & & & & & \\
 & x_7 & x_8 & x_{10} & x_{13} & x_{17} & & & & \\
 & & x_{11} & x_{12} & x_{14} & & & & & \\
 & & x_{15} & & x_{16} & x_{18} & x_{21} & x_{25} & x_{29} & x_{33} \\
 & & & & x_{19} & x_{20} & x_{22} & & & \\
 & & & & & x_{23} & x_{24} & x_{26} & & \\
 & & & & & x_{27} & & x_{28} & x_{30} & \\
 & & & & & & & x_{31} & x_{32} & x_{34} \\
 & & & & & & & x_{35} & & x_{36}.
\end{array}
$$

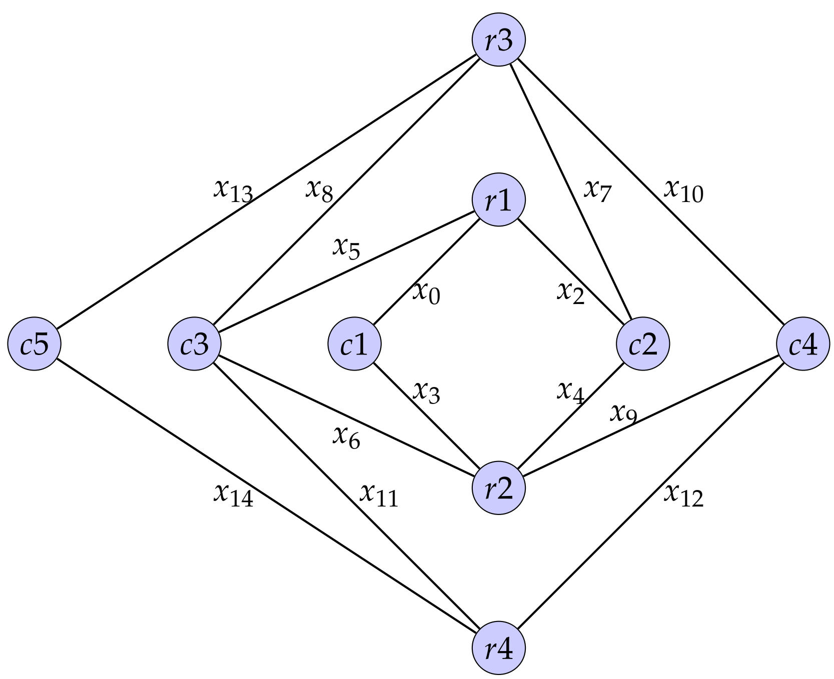
Remark 3.1.4. If we associate a vertex to each row and each column and an edge to each nonzero entry of M_τ^e, we have a finite simple connected bipartite graph G_τ^e. The set V_1 of vertices corresponding to rows and the set V_2 of vertices corresponding to columns form a bipartition of the vertices of G_τ^e, of cardinalities

$$|V_1| = \left\lfloor \frac{\tau}{4} \right\rfloor + 2$$
$$|V_2| = \left\lceil \frac{\tau}{4} \right\rceil + 2.$$

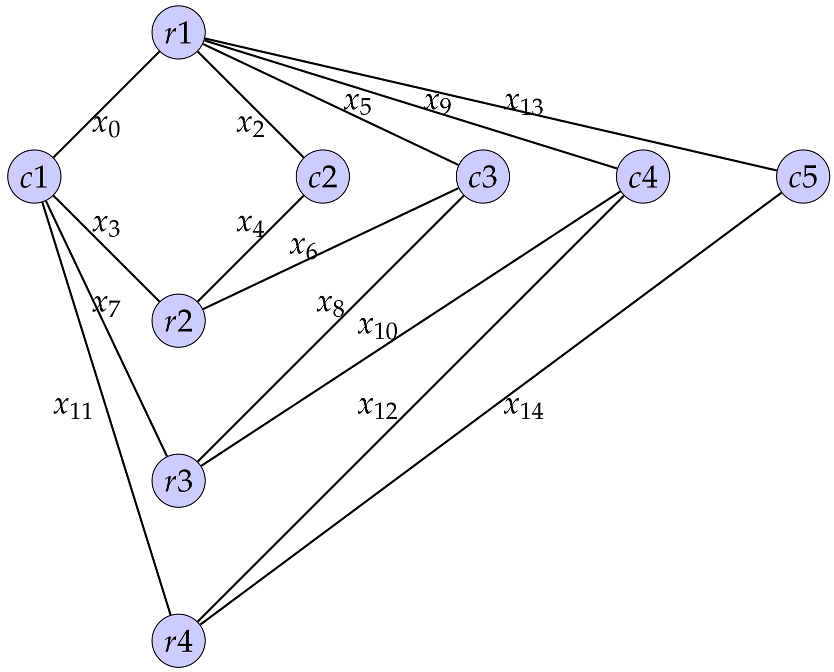
We note that by construction G_τ^e has no vertices of degree one, since each row and each column of M_τ^e has more than one nonzero entry. We also note that the vertex that is an endpoint of both $x_{\tau+3}$ and $x_{\tau+4}$ has degree two, since the row or column containing these has no other nonzero entries.

Definition 3.1.5. We say a graph G is in $\mathcal{F}$ if $G = G_\tau^e$ for some even $\tau \geq 0$ and some $e \in \mathbb{F}_2^{\tau/2+1}$.

Example 3.1.6. When $\tau = 10$, $G_{10}^{(1,1,\ldots,1)} \in \mathcal{F}_1$ is



When $\tau = 10$, $G_{10}^{(0,0,\ldots,0)} \in \mathcal{F}_2$ is



We develop properties of M_τ^e which allow us to show in Section 3.2 that certain minors of M_τ^e are generators for the toric ring of G_τ^e.

Definition 3.1.7. For this dissertation, a *distinguished minor* of M_τ^e is a 2-minor involving only (nonzero) entries of the ladder-like structure M_τ^e, coming from a $2x2$ subarray of M_τ^e.

Proposition 3.1.8. *For each even $i \geq 2$ and each $f \in \mathbb{F}_2^{i/2+1}$, the entry x_{i+3} and the entry x_{i+4} each appear in exactly two distinguished minors in M_i^f. For $i \equiv 0 \mod 4 (\equiv 2 \mod 4)$, these minors are of the form*

$$t_i := x_{i+1}x_{i+3} - x_{j_i}x_{i+4}$$

coming from the subarray

$$\begin{bmatrix} x_{j_i} & x_{i+1} \\ x_{i+3} & x_{i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_i} & x_{i+3} \\ x_{i+1} & x_{i+4} \end{bmatrix} \right)$$

for some $j_i \in \{0, 2, 3, \ldots, i-2\}$ *and the other of the form*

$$t_{i+1} := x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}$$

coming from the subarray

$$\begin{bmatrix} x_{j_{i+1}} & x_{i+2} \\ x_{i+3} & x_{i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{i+1}} & x_{i+3} \\ x_{i+2} & x_{i+4} \end{bmatrix} \right)$$

for some $j_{i+1} \in \{i-1, i\}$, *and the only distinguished minor of* M^e_τ *with indices all less than* 5 *is* $t_1 := x_2x_3 - x_0x_4$.

Proof. The last statement is clear; we prove the remaining statements by induction on even i. For $i = 2$, we have the distinguished minors $t_2 = x_3x_5 - x_0x_6$ and $t_3 = x_4x_5 - x_2x_6$ coming from the subarrays

$$\begin{bmatrix} x_0 & x_5 \\ x_3 & x_6 \end{bmatrix}$$

and

$$\begin{bmatrix} x_2 & x_5 \\ x_4 & x_6 \end{bmatrix}$$

where $j_2 = 0 \in \{0\}$ and $j_3 = 2 \in \{1, 2\}$, so we have our base case. Now suppose the statement is true for even i with $2 \leq i < \tau$, and let $\tau \equiv 0 \mod 4 (\equiv 2 \mod 4)$ and $e \in \mathbb{F}_2^{\tau/2+1}$.

Case 1: If $e_{\tau/2+1} = 0$, then by Remark 3.1.2, $x_{\tau+3}$ is in the same column (row) as $x_{\tau-1}$. By induction, we have the distinguished minor $t_{\tau-2} = x_{\tau-1}x_{\tau+1} - x_{j_{\tau-2}}x_{\tau+2}$ coming from the subarray

$$\begin{bmatrix} x_{j_{\tau-2}} & x_{\tau+1} \\ x_{\tau-1} & x_{\tau+2} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{\tau-2}} & x_{\tau-1} \\ x_{\tau+1} & x_{\tau+2} \end{bmatrix} \right).$$

Then in fact we have a subarray of the form

$$\begin{bmatrix} x_{j_{\tau-2}} & x_{\tau+1} \\ x_{\tau-1} & x_{\tau+2} \\ x_{\tau+3} & x_{\tau+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{\tau-2}} & x_{\tau-1} & x_{\tau+3} \\ x_{\tau+1} & x_{\tau+2} & x_{\tau+4} \end{bmatrix} \right),$$

so that we have the distinguished minors

$$\begin{aligned} t_\tau &= x_{\tau+1}x_{\tau+3} - x_{j_{\tau-2}}x_{\tau+4} \\ t_{\tau+1} &= x_{\tau+2}x_{\tau+3} - x_{\tau-1}x_{\tau+4} \end{aligned}$$

with

$$j_\tau = j_{\tau-2} \in \{0, 2, 3, \ldots, \tau - 4\} \subset \{0, 2, 3, \ldots, \tau - 2\}$$

by induction and with

$$j_{\tau+1} = \tau - 1 \in \{\tau - 1, \tau\}.$$

Since the only entries in row $\lfloor \tau/4 \rfloor + 2$ (column $\lceil \tau/4 \rceil + 2$) of M_τ^e are $x_{\tau+3}$ and $x_{\tau+4}$ and since the only entries in column $\lceil \tau/4 \rceil + 2$ (row $\lfloor \tau/4 \rfloor + 2$) of M_τ^e are $x_{\tau+1}$, $x_{\tau+2}$, and $x_{\tau+4}$ by Remark 3.1.2, these are the only distinguished minors of M_τ^e containing either $x_{\tau+3}$ or $x_{\tau+4}$.

Case 2: If $e_{\tau/2+1} = 1$, then by Remark 3.1.2, $x_{\tau+3}$ is in the same column (row) as x_τ. By induction, we have the distinguished minor $t_{\tau-1} = x_\tau x_{\tau+1} - x_{j_{\tau-1}} x_{\tau+2}$ coming from the subarray

$$\begin{bmatrix} x_{j_{\tau-1}} & x_{\tau+1} \\ x_\tau & x_{\tau+2} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{\tau-1}} & x_\tau \\ x_{\tau+1} & x_{\tau+2} \end{bmatrix} \right).$$

Then in fact we have a subarray of the form

$$\begin{bmatrix} x_{j_{\tau-1}} & x_{\tau+1} \\ x_\tau & x_{\tau+2} \\ x_{\tau+3} & x_{\tau+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_{\tau-1}} & x_\tau & x_{\tau+3} \\ x_{\tau+1} & x_{\tau+2} & x_{\tau+4} \end{bmatrix} \right),$$

so that we have the distinguished minors

$$\begin{aligned} t_\tau &= x_{\tau+1}x_{\tau+3} - x_{j_{\tau-1}}x_{\tau+4} \\ t_{\tau+1} &= x_{\tau+2}x_{\tau+3} - x_\tau x_{\tau+4} \end{aligned}$$

with

$$j_\tau = j_{\tau-1} \in \{\tau-3, \tau-2\} \subset \{0, 2, 3, \ldots, \tau-2\}$$

by induction and with

$$j_{\tau+1} = \tau \in \{\tau-1, \tau\}.$$

Since the only entries in row $\lfloor \tau/4 \rfloor + 2$ (column $\lceil \tau/4 \rceil + 2$) of M^e_τ are $x_{\tau+3}$ and $x_{\tau+4}$ and since the only entries in column $\lceil \tau/4 \rceil + 2$ (row $\lfloor \tau/4 \rfloor + 2$) of M^e_τ are $x_{\tau+1}$, $x_{\tau+2}$, and $x_{\tau+4}$ by Remark 3.1.2, these are the only distinguished minors of M^e_τ containing either $x_{\tau+3}$ or $x_{\tau+4}$, as desired. $\square$

Definition 3.1.9. For even i, define the integers j_i, j_{i+1} for $j_1, \ldots, j_{\tau+1}$ as in the statement of Lemma 3.1.8. We note in the remark below some properties of the j_n.

Remark 3.1.10. From the proof of Lemma 3.1.8, we note that $j_2 = 0$, $j_3 = 2$, and that for even $i \geq 4$, we have the following:

$$e_{i/2+1} = 0 \iff j_i = j_{i-2} \iff j_{i+1} = i-1$$

$$e_{i/2+1} = 1 \iff j_i = j_{i-1} \iff j_{i+1} = i.$$

For the sake of later proofs, we extend the notion of j_n naturally to $t_1 = x_2x_3 - x_0x_4$ and say that $j_1 = 0$, and note the following properties of the j_n for $1 \leq n \leq \tau + 1$:

- For even i, $j_i \in \{j_{i-2}, j_{i-1}\}$ and $j_i \leq i - 2$. For $i = 2$, $j_2 = j_1 = 0$, and for $i \geq 4$, this is clear from Proposition 3.1.8 and the above statement, since $e_{i/2+1} \in \{0, 1\}$.

- For even i, we have $j_{i+1} \in \{i - 1, i\}$. For $i = 0$, $j_1 = 0 \in \{-1, 0\}$, and for $i \geq 2$, this follows from Proposition 3.1.8.

- For even i, $j_i < j_{i+1}$. Indeed, $j_i \leq i - 2 < j_{i+1}$.

- The j_{i+1} form an increasing sequence for even i. This is clear by the fact that $j_{i+1} \in \{i - 1, i\}$ for $i \geq 0$.

- The j_i form a non-decreasing sequence for even i. Indeed, for $i \geq 4$, either $j_i = j_{i-2}$ or $j_i = j_{i-1} \geq i - 3 > i - 4 \geq j_{i-2}$.

Remark 3.1.11. We also note from the above proof that the following is a subarray of M_τ^e for all even $i \equiv 0 \mod 4 (\equiv 2 \mod 4)$ such that $2 \leq i \leq \tau$:

$$\begin{bmatrix} x_{j_i} & x_{i+1} \\ x_{j_{i+1}} & x_{i+2} \\ x_{i+3} & x_{i+4} \end{bmatrix} \qquad \left(\begin{bmatrix} x_{j_i} & x_{j_{i+1}} & x_{i+3} \\ x_{i+1} & x_{i+2} & x_{i+4} \end{bmatrix} \right)$$

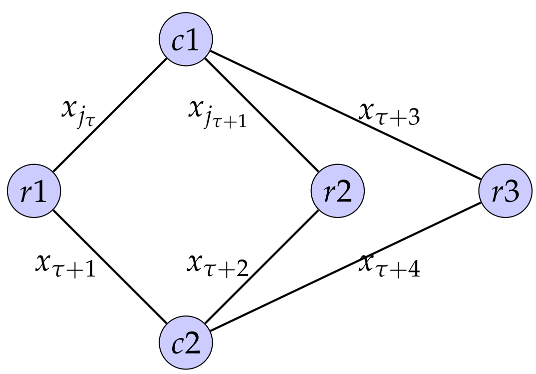
Proposition 3.1.12. *For even $\tau \geq 0$, each graph $G_\tau^e \in \mathcal{F}$ is chordal bipartite with vertex bipartition $V_1 \cup V_2$ of cardinalities*

$$|V_1| = \left\lfloor \frac{\tau}{4} \right\rfloor + 2$$
$$|V_2| = \left\lceil \frac{\tau}{4} \right\rceil + 2.$$

Proof. We prove this by induction on the even subscript. We already know by Remark 3.1.4 that every graph G_τ^e is bipartite for $\tau \geq 0$, with the above bipartition. Let $f \in \mathbb{F}_2^{i/2+1}$.

It is clear for $i = 0$ and $i = 2$ that G_i^f is chordal bipartite, since these graphs have fewer than six vertices. Now suppose G_i^f is chordal bipartite for even i with $2 \leq i < \tau \equiv 0 \mod 4 (\equiv 2 \mod 4)$, and consider G_τ^e for $e \in \mathbb{F}_2^{\tau/2+1}$. We know that the following array (or its transpose) is a subarray of M_τ^e by Remark 3.1.11, and we include for reference the corresponding subgraph of G_τ^e with vertices labeled as in the argument below.

$$\begin{bmatrix} x_{j_\tau} & x_{\tau+1} \\ x_{j_{\tau+1}} & x_{\tau+2} \\ x_{\tau+3} & x_{\tau+4} \end{bmatrix}$$



We know the only difference between G_τ^e and $G_{\tau-2}^{\widehat{e}}$ is one vertex r_3 corresponding to row $\lfloor \tau/4 \rfloor + 2$ (column $\lceil \tau/4 \rceil + 2$) and two edges $\{r_3, c_2\} = x_{\tau+4}$ and $\{r_3, c_1\} = x_{\tau+3}$, where c_2 corresponds to column $\lceil \tau/4 \rceil + 2$ (row $\lfloor \tau/4 \rfloor + 2$) and c_1 corresponds to the column containing $x_{\tau+3}$. Any even cycle containing r_3 must also contain $x_{\tau+4}$ and $x_{\tau+3}$, since $\deg r_3 = 2$ by Remark 3.1.4. By Remark 3.1.2, the only other edges with endpoint c_2 are $x_{\tau+1}$ and $x_{\tau+2}$, the entries added to make $M_{\tau-2}$, so we know that any even cycle containing $x_{\tau+4}$ and $x_{\tau+3}$ must contain either $x_{\tau+1}$ or $x_{\tau+2}$. We see that any even cycle containing r_3 and $x_{\tau+1}$ is either a 4-cycle or has x_{j_τ} as a chord, and any even cycle containing r_3 and $x_{\tau+2}$ is either a 4-cycle or has $x_{j_{\tau+1}}$ as a chord. Thus every graph G_τ^e is chordal bipartite for $\tau \geq 0$, with the above bipartition. $\square$

3.2 Toric Rings for $\mathcal{F}$

In this section, we develop the toric ring $R(\tau, e)$ for each of the graphs G_τ^e in the family $\mathcal{F}$. We first show that the toric ideal $I_{G_\tau^e}$ of the graph G_τ^e is the same as the ideal $I(\tau, e)$ generated by the distinguished minors of M_τ^e. We then demonstrate that for some τ and e, these ideals do not arise from the join-meet ideals of lattices in a natural way, so that results in lattice theory do not apply to the general family $\mathcal{F}$ in an obvious way.

3.2.1 From Toric Ideals of Graphs

Let $S(\tau) = k[x_0, x_2, x_3, \ldots, x_{\tau+4}]$. The edge ring for $G_\tau^e \in \mathcal{F}$ is denoted by $k[G_\tau^e]$ and is isomorphic to the following toric ring, which we call $R(\tau, e)$.

$$R(\tau, e) = \frac{S(\tau)}{I_{G_\tau^e}},$$

where $I_{G_\tau^e}$ is the toric ideal of G_τ^e. Our goal is to show that

$$I_{G_\tau^e} = I(\tau, e) = (\{\text{distinguished minors of } M_\tau^e\}).$$

Proposition 3.2.1. *Let $S(\tau) = k[x_0, x_2, x_3, \ldots, x_{\tau+4}]$. For $G_\tau^e \in \mathcal{F}$, we have*

$$R(\tau, e) = \frac{S(\tau)}{I(\tau, e)},$$

where

$$I(\tau, e) = (\{\textit{distinguished minors of } M_\tau^e\}).$$

Proof. To prove this, we need only show that $I(\tau, e)$ is the toric ideal $I_{G_\tau^e}$ of the graph G_τ^e. It is clear that the distinguished minors of M_τ^e are in $I_{G_\tau^e}$, corresponding to the 4-cycles of G_τ^e. Since G is chordal bipartite, these are the only generators of $I_{G_\tau^e}$ ([HHO18], Corollary 5.15). □

Corollary 3.2.2. *The rings $R(\tau, e)$ are Cohen Macaulay.*

Proof. By Remark 3.1.4 and Proposition 3.2.1, the ring $R(\tau, e)$ is the toric ring of a finite simple connected bipartite graph, and hence by Corollary 5.26 in [HHO18], $R(\tau, e)$ is Cohen Macaulay for each τ and e. $\square$

Because we know the distinguished minors of M_τ^e, we are now able to characterize the generators for the toric ideal $R(\tau, e)$ of G_τ^e.

Remark 3.2.3. By Proposition 3.1.8 and Remark 3.1.10 that the generators $t_1, \ldots, t_{\tau+1}$ for $I_{G_\tau^e}$ may be summarized as follows. For even integers i such that $2 \leq i \leq \tau$, set

$$\begin{aligned} t_1 &= x_2x_3 - x_{j_1}x_4 \\ t_i &= x_{i+1}x_{i+3} - x_{j_i}x_{i+4} \\ t_{i+1} &= x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}, \end{aligned}$$

where the nonnegative integers j_n are as in Remark 3.1.10, that is, $j_1 = j_2 = 0$, $j_3 = 2$, and for even $i \geq 4$, we have

$$e_{i/2+1} = 0 \iff j_i = j_{i-2} \iff j_{i+1} = i - 1$$

$$e_{i/2+1} = 1 \iff j_i = j_{i-1} \iff j_{i+1} = i.$$

We note that the number of generators depends on τ and that the j_n depend on e, but we may ignore dependence on e when working with general j_n. We sometimes call $t_1, \ldots, t_{\tau+1}$ the *standard generators* of $I_{G_\tau^e}$, and show in Section 3.2.2 that for certain τ and e, they are not equal to the usual generators for the join-meet ideal of any lattice L.

Example 3.2.4. We first look at the toric ideal of a graph in $\mathcal{F}_1$. For $\tau = 10$ and $e = (1, 1, \ldots, 1)$, by Remark 3.1.10 we have $j_1 = j_2 = 0$, $j_3 = 2$, $j_i = j_{i-1}$ and $j_{i+1} = i$ for even

$i \geq 4$, so that

$$R(\tau, e) = \frac{k[x_0, x_2, \ldots, x_{14}]}{I_{G_\tau^e}},$$

where $I_{G_\tau^e}$ is generated by

$$\begin{aligned}
t_1 &= x_2x_3 - x_0x_4 & t_2 &= x_3x_5 - x_0x_6 \\
t_3 &= x_4x_5 - x_2x_6 & t_4 &= x_5x_7 - x_2x_8 \\
t_5 &= x_6x_7 - x_4x_8 & t_6 &= x_7x_9 - x_4x_{10} \\
t_7 &= x_8x_9 - x_6x_{10} & t_8 &= x_9x_{11} - x_6x_{12} \\
t_9 &= x_{10}x_{11} - x_8x_{12} & t_{10} &= x_{11}x_{13} - x_8x_{14} \\
t_{11} &= x_{12}x_{13} - x_{10}x_{14}.
\end{aligned}$$

We now consider a graph in $\mathcal{F}_2$. For $\tau = 10$ and $e = (0, 0, \ldots, 0)$, by Remark 3.1.10 we have $j_1 = j_2 = 0$, $j_3 = 2$, $j_i = j_{i-2}$ and $j_{i+1} = i - 1$ for even $i \geq 4$, so that

$$R(\tau, e) = \frac{k[x_0, x_2, \ldots, x_{14}]}{I_{G_\tau^e}},$$

where $I_{G_\tau^e}$ is generated by

$$\begin{aligned}
t_1 &= x_2x_3 - x_0x_4 & t_2 &= x_3x_5 - x_0x_6 \\
t_3 &= x_4x_5 - x_2x_6 & t_4 &= x_5x_7 - x_0x_8 \\
t_5 &= x_6x_7 - x_3x_8 & t_6 &= x_7x_9 - x_0x_{10} \\
t_7 &= x_8x_9 - x_5x_{10} & t_8 &= x_9x_{11} - x_0x_{12} \\
t_9 &= x_{10}x_{11} - x_7x_{12} & t_{10} &= x_{11}x_{13} - x_0x_{14} \\
t_{11} &= x_{12}x_{13} - x_9x_{14}.
\end{aligned}$$

3.2.2 Distinction From Join-Meet Ideals of Lattices

We saw in Example 3.1.3 and Proposition 3.2.1 that if $G^e_\tau \in \mathcal{F}_1 \subset \mathcal{F}$, then $I_{G^e_\tau}$ is a ladder determinantal ideal for $\tau \geq 4$. It is known that a ladder determinantal ideal is equal to the join-meet ideal of a (distributive) lattice (indeed, with a natural partial ordering which decreases along rows and columns of M^e_τ we obtain such a lattice), so much is known about $R(\tau, e)$ via distributive lattice theory and the theory of ladder determinantal ideals. We spend some time in this section establishing that not all rings $R(\tau, e) \in \mathcal{F}$ arise from a lattice in a natural way (see Remark 3.2.6), and so there does not seem to be any obvious way to obtain our results in Chapter 4 from the literature on join-meet ideals of distributive lattices or on ladder determinantal ideals. The results in Chapter 4 may be viewed as an extension of what is already known for the family $\mathcal{F}_1$ from the existing literature.

The following five lemmas serve to provide machinery to show that there is at least one ring in the family $\mathcal{F}$, namely $k[G_{14}^{(1,1,1,1,1,0)}]$, whose toric ideal does not come from a lattice on the set $\{x_0, \ldots, x_{14}\}$ in any obvious way. That is, we show that the standard generators of $I_{G^e_{14}}$, the t_n from Remark 3.2.3, are not equal to the standard generators (see Definition 3.2.5) for any lattice L on $\{x_0, \ldots, x_{14}\}$.

Before we begin, we introduce some definitions and notation that we will use extensively throughout:

Definition 3.2.5. In this dissertation, a *standard generator* of the join-meet ideal of a lattice L is a nonzero element of one of the following four forms:

$$\begin{aligned} x_a x_b - (x_a \vee x_b)(x_a \wedge x_b) &= x_a x_b - (x_a \wedge x_b)(x_a \vee x_b) \\ (x_a \vee x_b)(x_a \wedge x_b) - x_a x_b &= (x_a \wedge x_b)(x_a \vee x_b) - x_a x_b \end{aligned}$$

for $x_a, x_b \in L$. We will sometimes refer to such an element as a *standard generator of L*. We note that for a standard generator, the pair $\{x_a, x_b\}$ is an incomparable pair, and the pair

$\{(x_a \vee x_b), (x_a \wedge x_b)\}$ is a comparable pair.

Though we are in a commutative ring, we provide all possible orderings for factors within the terms of a standard generator to emphasize that either factor of the monomial $(x_a \vee x_b)(x_a \wedge x_b) = (x_a \wedge x_b)(x_a \vee x_b)$ may be the join or the meet of x_a and x_b.

Remark 3.2.6. We give an explanation of why it makes sense to focus only on the standard generators of a join-meet ideal. We recall that the standard generators t_n for $I_{G^e_\tau}$ from Remark 3.2.3 come from distinct 2×2 arrays within the ladder-like structure M^e_τ and recognize that either monomial of t_n determines its 2×2 array. Then an element of the form $ab - cd$ in $I_{G^e_{14}}$ with $a, b, c, d \in \{x_0, x_2, x_3, \ldots, x_{14}\}$ must be equal to $\pm t_s$ for some s, since a nontrivial sum of t_n with coefficients in $\{-1, 1\}$ either has more than two terms or is equal to t_s for some s, and other coefficients would be extraneous. Then any generating set for $I_{G^e_{14}}$ where each element has the form $ab - cd$ in $I_{G^e_{14}}$ with $a, b, c, d \in \{x_0, x_2, x_3, \ldots, x_{14}\}$ must consist of all the t_n (up to sign). We conclude that it is natural to check whether the t_n are standard generators of a lattice L, instead of non-standard generators.

For the next definition, we note that for a standard generator coming from $x_a, x_b \in L$, $\{x_a, x_b\}$ is an incomparable pair and $\{(x_a \vee x_b), (x_a \wedge x_b)\}$ is a comparable pair.

Definition 3.2.7. Given a standard generator $s = uz - wv$ of a lattice L, let $F_s \in \mathbb{F}_2$ be defined as follows:

> If $F_s = 0$, the elements in the first monomial of s are not comparable in L (so the elements in the second monomial of s are comparable in L).
>
> If $F_s = 1$, the elements in the second monomial of s are not comparable in L (so the elements in the first monomial of s are comparable in L).

For a given list $s_1, s_2, \ldots, s_n$ of standard generators of a lattice L, we will use

$$F = (F_1, \ldots, F_n) \in \mathbb{F}_2^n,$$

where $F_j = F_{s_j}$, to encode the comparability of the variables in these generators.

We note that exactly one of $F_j = 0$ or $F_j = 1$ happens for each j; we are merely encoding which monomial in each relation corresponds to $x_a x_b$, and which to $(x_a \vee x_b)(x_a \wedge x_b) = (x_a \wedge x_b)(x_a \vee x_b)$.

Notation 3.2.8. We will use the notation $u > \{w, v\}$ if $u > w$ and $u > v$ in a lattice L, and $\{w, v\} > z$ if $w > z$ and $v > z$ in L.

In the first lemma, we begin by showing what restrictions we must have on a lattice whose join-meet ideal contains the 2-minors of the following array as standard generators:

$$\begin{array}{ccc} a & b & e \\ c & d & f \end{array}$$

Lemma 3.2.9. *Suppose*

$$\begin{aligned} s_1 &= bc - ad \\ s_2 &= ce - af \\ s_3 &= de - bf \end{aligned}$$

are standard generators of a lattice L. Let $F \in \mathbb{F}_2^3$ be defined for these three elements as in 3.2.7. Then up to relabeling of variables,

$$F \in \{\{0,0,0\}, \{0,0,1\}, \{0,1,1\}\}.$$

Proof. We first note that some of the cases we consider are equivalent. If we relabel variables according to the permutation $(ac)(bd)(ef)$, we see that

$$F = \{i, j, k\} \equiv \{1-i, 1-j, 1-k\}.$$

This limits the cases we need to consider to

$$F \in \{\{0,0,0\},\{0,0,1\},\{0,1,0\},\{0,1,1\}\}.$$

That is, we only need to show that the case $F = \{0,1,0\}$ is impossible.

Let $F_1 = 0$. Then without loss of generality, up to reversing the order in the lattice (which does not affect the join-meet ideal), we have $a > \{b,c\} > d$. If $F_2 = 1$, we have $e > \{a,f\} > c$, so $e > \{b,f\} > d$ and hence $F_3 = 1$. We conclude that the case $F = \{0,1,0\}$ is impossible. ☐

In the second lemma, we show what restrictions we must have on a lattice whose join-meet ideal contains the 2-minors of the following ladder as standard generators, and which meets certain comparability conditions.

$$\begin{array}{ccc} a & b & e \\ c & d & f \\ & g & h \end{array}$$

Lemma 3.2.10. *Suppose*

$$\begin{aligned} s_1 &= bc - ad \\ s_2 &= ce - af \\ s_3 &= de - bf \\ s_4 &= eg - bh \\ s_5 &= fg - dh \end{aligned}$$

are standard generators of a lattice L, and that $\{a,g\}$,$\{a,h\}$,$\{c,g\}$, and $\{c,h\}$ are comparable pairs in L. Let $F \in \mathbb{F}_2^5$ be defined for these five elements as in 3.2.7. Then up to relabeling of variables, $F = \{0,0,0,0,0\}$.

Proof. We first note that with natural relabeling, both $\{s_1, s_2, s_3\}$ and $\{s_3, s_4, s_5\}$ satisfy the hypotheses of Lemma 3.2.9, so if we let F be defined as in 3.2.7, this limits the cases we need to consider to 5-tuples whose first three elements and whose last three elements satisfy the conclusion of Lemma 3.2.9. We note that some of the cases we consider are equivalent. If we relabel variables according to the permutation $(ac)(bd)(ef)$, we see that $F = \{i, j, k, l, m\} \equiv \{1-i, 1-j, 1-k, m, l\}$, and if we relabel the variables according to the permutation $(be)(df)(gh)$, we have $F = \{i, j, k, l, m\} \equiv \{j, i, 1-k, 1-l, 1-m\}$. The permutation $(ah)(cg)(bf)$ yields $F = \{i, j, k, l, m\} \equiv \{m, l, k, j, i\}$. Then by Lemma 3.2.9 we have the eighteen cases

$$\{0,0,0,0,0\} \equiv \{1,1,1,0,0\} \equiv \{1,1,0,1,1\} \equiv \{0,0,1,1,1\}$$

$$\begin{aligned}\{0,0,0,0,1\} \equiv \{1,1,1,1,0\} \equiv \{1,1,0,0,1\} \equiv \{0,0,1,1,0\} &\equiv \{0,1,1,0,0\} \\ &\equiv \{1,0,0,0,0\} \\ &\equiv \{0,1,1,1,1\} \\ &\equiv \{1,0,0,1,1\}\end{aligned}$$

$$\{0,0,0,1,1\} \equiv \{1,1,1,1,1\} \equiv \{1,1,0,0,0\} \equiv \{0,0,1,0,0\}$$

$$\{0,1,1,1,0\} \equiv \{1,0,0,0,1\}$$

We need only show that the cases $\{0,0,0,0,1\}$, $\{0,1,1,1,0\}$, and $\{0,0,0,1,1\}$ are impossible.

Case 1: $F = \{0,0,0,0,1\}$. Since $F_1 = 0$, without loss of generality (reversing the order on the entire lattice if needed) we have $a > \{b, c\} > d$. Then $F_2 = F_3 = F_4 = 0$ and $F_5 = 1$, with the ordering chosen, yield

$$a > \{c, e\} > f$$

$$b > \{d, e\} > f$$

$$b > \{e, g\} > h$$

$$g > \{d, h\} > f.$$

If $c > g$, then $a > \{b, c\} > g > d$, but then $bc - ad$ is not a standard generator of L, and this is a contradiction. If $c < g$, then $c < g < b$ so that both $\{b, c\}$ and $\{a, d\}$ from s_1 are comparable pairs, but this is a contradiction. We conclude that the case $\{0, 0, 0, 0, 1\}$ is impossible.

Case 2: $F = \{0, 1, 1, 1, 0\}$. Since $F_1 = 0$, without loss of generality we have $a > \{b, c\} > d$. Then $F_2 = F_3 = F_4 = 1$ and $F_5 = 0$, with the ordering chosen, yield

$$e > \{a, f\} > c$$

$$e > \{b, f\} > d$$

$$e > \{b, h\} > g$$

$$h > \{f, g\} > d.$$

If $c > g$, then $f > c > g$ so that both $\{f, g\}$ and $\{d, h\}$ from s_5 are comparable pairs, but this is a contradiction. If $c < g$, then $c < g < b$ so that both $\{b, c\}$ and $\{a, d\}$ from s_1 are comparable pairs, but this is a contradiction. We conclude that the case $\{0, 1, 1, 1, 0\}$ is impossible.

Case 3: $F = \{0,0,0,1,1\}$. Since $F_1 = 0$, without loss of generality we have $a > \{b,c\} > d$. Then $F_2 = F_3 = 0$ and $F_4 = F_5 = 1$, with the ordering chosen, yield

$$a > \{c,e\} > f$$
$$b > \{d,e\} > f$$
$$g > \{b,h\} > e$$
$$g > \{d,h\} > f.$$

If $c > g$, then $c > g > b$ so that both $\{b,c\}$ and $\{a,d\}$ from s_1 are comparable pairs, but this is a contradiction. Then $c < g$ with $d < \{b,c\} < a < g$, since $bc - ad$ is a standard generator of L.

> Case 3a: If $a < h$, then $b < a < h$ so that both $\{b,h\}$ and $\{e,g\}$ from s_4 are comparable pairs, but this is a contradiction.
>
> Case 3b: If $a > h$, then $g > a > \{d,h\} > f$, so that $fg - dh$ is not a standard generator of L, but this is a contradiction. We conclude that the case $\{0,0,0,1,1\}$ is impossible.

We note that these cases are compatible with all three possible relabelings. Then up to relabeling, $F = \{0,0,0,0,0\}$. □

In the third lemma, we show what restrictions we must have on a lattice whose join-meet ideal contains the 2-minors of the following ladder as standard generators and which meets certain comparability conditions.

$$\begin{array}{cccc} a & b & e & \\ c & d & f & i \\ & g & h & j \end{array}$$

Lemma 3.2.11. *Suppose*

$$\begin{aligned} s_1 &= bc - ad \\ s_2 &= ce - af \\ s_3 &= de - bf \\ s_4 &= eg - bh \\ s_5 &= fg - dh \\ s_6 &= gi - dj \\ s_7 &= hi - fj \end{aligned}$$

are standard generators of a lattice L, and that $\{a, g\}$, $\{a, h\}$, $\{c, g\}$, $\{c, h\}$, $\{b, i\}$, $\{b, j\}$, $\{e, i\}$, *and* $\{e, j\}$ *are comparable pairs in L. Let* $F \in \mathbb{F}_2^7$ *be defined for these seven elements as in 3.2.7. Then up to relabeling of variables,* $F = \{0, 0, 0, 0, 0, 0, 0\}$.

Proof. We first note that with natural relabeling, both $\{s_1, s_2, s_3, s_4, s_5\}$ and $\{s_3, s_4, s_5, s_6, s_7\}$ satisfy the hypotheses of Lemma 3.2.10, so if we let F be defined as in 3.2.7, this limits the cases we need to consider to 7-tuples whose first five elements and whose last five elements satisfy the conclusion of Lemma 3.2.10, so the only possible cases are $\{0, 0, 0, 0, 0, 0, 0\}$ and $\{0, 0, 1, 1, 1, 0, 0\}$. If we relabel variables according to the permutation $(be)(df)(gh)$, we see that $F = \{i, j, k, l, m, n, o\} \equiv \{j, i, 1-k, 1-l, 1-m, o, n\}$, so that these two cases are equivalent. Then up to relabeling of variables, $F = \{0, 0, 0, 0, 0, 0, 0\}$. $\square$

In the fourth lemma, we show what restrictions we must have on a lattice whose join-meet ideal contains the 2-minors of the following ladder as standard generators, and which meets certain comparability conditions.

$$\begin{matrix} a & b & e & \\ c & d & f & i \\ & g & h & j \\ & & k & l \end{matrix}$$

Lemma 3.2.12. *Suppose*

$$\begin{aligned} s_1 &= bc - ad \\ s_2 &= ce - af \\ s_3 &= de - bf \\ s_4 &= eg - bh \\ s_5 &= fg - dh \\ s_6 &= gi - dj \\ s_7 &= hi - fj \\ s_8 &= ik - fl \\ s_9 &= jk - hl \end{aligned}$$

are standard generators of a lattice L, and that $\{a, g\}$, $\{a, h\}$, $\{c, g\}$, $\{c, h\}$, $\{b, i\}$, $\{b, j\}$, $\{e, i\}$, $\{e, j\}$, $\{d, k\}$, $\{d, l\}$, $\{g, k\}$, *and* $\{g, l\}$ *are comparable pairs in L. Let* $F \in \mathbb{F}_2^9$ *be defined for these nine elements as in 3.2.7. Then* $F = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$.

Proof. We first note that with natural relabeling, both $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ and $\{s_3, s_4, s_5, s_6, s_7, s_8, s_9\}$ satisfy the hypotheses of Lemma 3.2.11, so if we let F be defined as in 3.2.7, this limits the cases we need to consider to 9-tuples whose first seven entries and whose last seven entries satisfy the conclusion of Lemma 3.2.11. We see by Lemma 3.2.11 that $F = \{0,0,0,0,0,0,0,0,0\}$. □

We now have the machinery necessary to show that for $e = (1,1,1,1,1,0)$, $I_{G_{14}^e}$ does not come from a lattice. In our proof, we use the previous four lemmas and the fact that $I_{G_{14}^e}$ is generated by the 2-minors of the following ladder-like structure:

$$\begin{array}{ccccc} x_0 & x_2 & x_5 & & \\ x_3 & x_4 & x_6 & x_9 & x_{13} \\ & x_7 & x_8 & x_{10} & \\ & & x_{11} & x_{12} & x_{14} \end{array}$$

Proposition 3.2.13. *Let $\tau = 10$ and $e = (1,1,1,1,1,0)$. Then the set of standard generators for $I_{G_{14}^e}$, as given in 3.2.3, is not equal to the complete set of standard generators (up to sign) of any (classical) lattice.*

Proof. By choice of $\tau = 10$ and $e = (1,1,1,1,1,0)$, the generators of $I_{G_{14}^e}$ are

$$\begin{aligned}
t_1 &= x_2x_3 - x_0x_4 & t_2 &= x_3x_5 - x_0x_6 \\
t_3 &= x_4x_5 - x_2x_6 & t_4 &= x_5x_7 - x_2x_8 \\
t_5 &= x_6x_7 - x_4x_8 & t_6 &= x_7x_9 - x_4x_{10} \\
t_7 &= x_8x_9 - x_6x_{10} & t_8 &= x_9x_{11} - x_6x_{12} \\
t_9 &= x_{10}x_{11} - x_8x_{12} & t_{10} &= x_{11}x_{13} - x_6x_{14} \\
t_{11} &= x_{12}x_{13} - x_9x_{14} & &
\end{aligned}$$

Suppose a lattice L exists whose complete set of standard generators (up to sign) equals $\{t_1, \ldots, t_{11}\}$. We note that if the monomial x_ix_j does not appear in any of the t_n, then $\{x_i, x_j\}$ is a comparable pair, since otherwise $\pm(x_i x_j - (x_i \vee x_j)(x_i \wedge x_j))$ would be in the set of standard generators of L. Thus the pairs $\{x_0, x_7\}$, $\{x_0, x_8\}$, $\{x_3, x_7\}$, $\{x_3, x_8\}$, $\{x_2, x_9\}$, $\{x_2, x_{10}\}$, $\{x_5, x_9\}$, $\{x_5, x_{10}\}$, $\{x_4, x_{11}\}$, $\{x_4, x_{12}\}$, $\{x_7, x_{11}\}$, and $\{x_7, x_{12}\}$ are comparable pairs in L. Let $F \in \mathbb{F}_2^{11}$ be defined as in 3.2.7. Then with natural relabeling of the first nine relations, this lattice satisfies the hypotheses of Lemma 3.2.12, so the only cases we need to consider are $F = \{0,0,0,0,0,0,0,0,0,a,b\}$.

Since $F_1 = 0$, without loss of generality, we have $x_0 > \{x_2, x_3\} > x_4$. Then $F_3 = F_5 = F_7 = F_9 = 0$, with the ordering chosen, yields

$$x_2 > \{x_4, x_5\} > x_6$$

$$x_4 > \{x_6, x_7\} > x_8$$

$$x_6 > \{x_8, x_9\} > x_{10}$$

$$x_8 > \{x_{10}, x_{11}\} > x_{12}$$

Case 1: Suppose $b = F_{11} = 0$. With the ordering chosen, this yields $x_9 > \{x_{12}, x_{13}\} > x_{14}$. If $\{x_{10}, x_{13}\}$ is an incomparable pair, we would have $\pm(x_{10}x_{13} - (x_{10} \vee x_{13})(x_{10} \wedge x_{13}))$ in our set of standard generators, so $\{x_{10}, x_{13}\}$ must be a comparable pair.

Case 1a: If $x_{10} > x_{13}$, then $x_9 > x_{10} > \{x_{12}, x_{13}\} > x_{14}$, so that $x_{12}x_{13} - x_9x_{14}$ is not a standard generator of L, but this is a contradiction.

Case 1b: If $x_{10} < x_{13}$, then $x_{12} < x_{10} < x_{13}$ so that both $\{x_{12}, x_{13}\}$ and $\{x_9, x_{14}\}$ from t_{11} are comparable pairs, but this is a contradiction.

Case 2: Suppose $b = F_{11} = 1$. With the ordering chosen, this yields $x_{13} > \{x_9, x_{14}\} > x_{12}$, since $x_9 > x_{10} > x_{12}$. If $\{x_{10}, x_{14}\}$ is an incomparable pair, we would have $\pm(x_{10}x_{14} - (x_{10} \vee x_{14})(x_{10} \wedge x_{14}))$ in our set of standard generators, so $\{x_{10}, x_{14}\}$ must be a comparable pair.

Case 2a: If $x_{10} < x_{14}$, then $x_{13} > \{x_9, x_{14}\} > x_{10} > x_{12}$, so that $x_{12}x_{13} - x_9x_{14}$ is not a standard generator of L, but this is a contradiction.

Case 2b: If $x_{10} > x_{14}$, then $x_{14} < x_{10} < x_9$ so that both $\{x_9, x_{14}\}$ and $\{x_{12}, x_{13}\}$ from t_{11} are comparable pairs, but this is a contradiction.

We conclude that there is no lattice whose complete set of standard generators (up to sign) equals the set of standard generators of $I_{G_{14}^e}$. $\square$

4 Properties of the Family of Toric Rings

In Chapter 3, we defined a family of toric rings, the toric rings $R(\tau, e)$ coming from the family $\mathcal{F}$; we demonstrated some context for these rings in the area of graph theory. Now we investigate some of the algebraic properties of $R(\tau, e)$. We develop proofs for properties such as dimension and regularity and build a short exact sequence associated with the family $\mathcal{F}$.

4.1 Gröbner Basis and Initial Ideal

We use the the degree reverse lexicographic monomial ordering with $x_0 > x_2 > x_3 > \dots$ throughout this chapter, and denote it by $>$. We show that the standard generators t_i given in Remark 3.2.3 are a Gröbner basis for $I_{G_\tau^e}$ with respect to $>$.

Lemma 4.1.1. *If $t_1, \dots, t_{\tau+1}$ are as in Remark 3.2.3, then $G = \{t_1, \dots, t_{\tau+1}\}$ is a Gröbner basis for $I_{G_\tau^e}$ with respect to $>$.*

Remark 4.1.2. We recall parts of Remark 3.2.3 verbatim here for use in the proof below. The generators $t_1, \dots, t_{\tau+1}$ for $I_{G_\tau^e}$ may be summarized as follows. For even integers i such that $2 \le i \le \tau$, set

$$\begin{aligned} t_1 &= x_2x_3 - x_{j_1}x_4 \\ t_i &= x_{i+1}x_{i+3} - x_{j_i}x_{i+4} \\ t_{i+1} &= x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}, \end{aligned}$$

where the nonnegative integers j_n are as in Remark 3.1.10, that is, $j_1 = j_2 = 0$, $j_3 = 2$, and for even $i \ge 4$, we have

$$e_{i/2+1} = 0 \iff j_i = j_{i-2} \iff j_{i+1} = i - 1$$

$$e_{i/2+1} = 1 \iff j_i = j_{i-1} \iff j_{i+1} = i.$$

Proof. We use Buchberger's criterion, as described in Theorem 2.2.10 and show that in each case, the reduced form of the S-polynomial $S_{i,j}$ is zero. We look at the various cases for

$$S_{i,j} = \frac{LT(t_j)}{gcd(LT(t_i, t_j)}(t_i) - \frac{LT(t_i)}{gcd(LT(t_i, t_j)}(t_j)$$

and denote the reduced form of $S_{i,j}$ by $\overline{S_{i,j}}$. We note that $S_{i,j} = S_{j,i}$ up to sign, that $S_{i,i} = 0$, and that if t_i and t_j have no shared variables in their leading terms, then $\overline{S_{i,j}} = 0$.

As the even- and odd-indexed generators follow a different pattern, we consider just even i and $\widehat{i}$ with $0 \leq \{i, \widehat{i}\} \leq \tau$, obtaining odd indices as $i+1$ and $\widehat{i}+1$.

Case 1: First we consider $S_{i,\widehat{i}}$ for even i with $0 \leq \{i, \widehat{i}\} \leq \tau$. We have $t_i = x_{i+1}x_{i+3} - x_{j_i}x_{i+4}$ and $t_{\widehat{i}} = x_{\widehat{i}+1}x_{\widehat{i}+3} - x_{j_{\widehat{i}}}x_{\widehat{i}+4}$. The only case where t_i and $t_{\widehat{i}}$ have shared variables in their leading terms, without loss of generality, is when $\widehat{i} = i+2$, for $2 \leq i \leq \tau - 2$. We have

$$\begin{aligned} S_{i,\widehat{i}} = S_{i,i+2} &= x_{i+5}(x_{i+1}x_{i+3} - x_{j_i}x_{i+4}) - x_{i+1}(x_{i+3}x_{i+5} - x_{j_{i+2}}x_{i+6}) \\ &= x_{j_{i+2}}x_{i+1}x_{i+6} - x_{j_i}x_{i+4}x_{i+5}. \end{aligned}$$

Case 1.1: If $e_{(i+2)/2+1} = 0$, then $j_{i+2} = j_i$ and $j_{i+3} = i+1$, so

$$S_{i,i+2} = x_{j_i}x_{i+1}x_{i+6} - x_{j_i}x_{i+4}x_{i+5}$$

and $t_{i+3} = x_{i+4}x_{i+5} - x_{i+1}x_{i+6}$. Adding $x_{j_i}t_{i+3}$ yields $\overline{S_{i,i+2}} = 0$.

Case 1.2: If $e_{(i+2)/2+1} = 1$, then $j_{i+2} = j_{i+1}$ and $j_{i+3} = i+2$, so

$$S_{i,i+2} = x_{j_{i+1}} x_{i+1} x_{i+6} - x_{j_i} x_{i+4} x_{i+5}$$

and $t_{i+3} = x_{i+4}x_{i+5} - x_{i+2}x_{i+6}$.

Case 1.2a: If $i \geq 4$ and $e_{i/2+1} = 0$, then $j_i = j_{i-2}$ and $j_{i+1} = i-1$, so

$$S_{i,i+2} = x_{i-1} x_{i+1} x_{i+6} - x_{j_{i-2}} x_{i+4} x_{i+5}$$

and $t_{i-2} = x_{i-1}x_{i+1} - x_{j_{i-2}} x_{i+2}$. Adding $x_{j_{i-2}} t_{i+3}$ yields

$$x_{i-1} x_{i+1} x_{i+6} - x_{j_{i-2}} x_{i+2} x_{i+6},$$

and adding $-x_{i+6} t_{i-2}$ yields $\overline{S_{i,i+2}} = 0$.

Case 1.2b: If $i = 2$, or $i \geq 4$ and $e_{i/2+1} = 1$, then $j_i = j_{i-1}$ and $j_{i+1} = i$, so

$$S_{i,i+2} = x_i x_{i+1} x_{i+6} - x_{j_{i-1}} x_{i+4} x_{i+5}$$

and $t_{i-1} = x_i x_{i+1} - x_{j_{i-1}} x_{i+2}$. Adding $x_{j_{i-1}} t_{i+3}$ yields

$$x_i x_{i+1} x_{i+6} - x_{j_{i-1}} x_{i+2} x_{i+6},$$

and adding $-x_{i+6} t_{i-1}$ yields $\overline{S_{i,i+2}} = 0$.

Case 2: Now we consider $S_{i+1,\widehat{i}}$. We have $t_{i+1} = x_{i+2}x_{i+3} - x_{j_{i+1}} x_{i+4}$ and $t_{\widehat{i}} = x_{\widehat{i}+1} x_{\widehat{i}+3} - x_{j_{\widehat{i}}} x_{\widehat{i}+4}$. The only cases where t_{i+1} and $t_{\widehat{i}}$ have shared variables in their leading terms are when $\widehat{i} = i$ and when $\widehat{i} = i+2$.

Case 2.1: Suppose $\widehat{i} = i$ for even i such that $2 \le i \le \tau$. We have

$$\begin{aligned} S_{i+1,\widehat{i}} = S_{i+1,i} &= x_{i+2}(x_{i+1}x_{i+3} - x_{j_i}x_{i+4}) - x_{i+1}(x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}) \\ &= x_{j_{i+1}}x_{i+1}x_{i+4} - x_{j_i}x_{i+2}x_{i+4}. \end{aligned}$$

Case 2.1.1: If $i \ge 4$ and $e_{i/2+1} = 0$, then $j_i = j_{i-2}$ and $j_{i+1} = i-1$, so

$$S_{i+1,i} = x_{i-1}x_{i+1}x_{i+4} - x_{j_{i-2}}x_{i+2}x_{i+4}$$

and $t_{i-2} = x_{i-1}x_{i+1} - x_{j_{i-2}}x_{i+2}$. Adding $-x_{i+4}t_{i-2}$ yields $\overline{S_{i+1,i}} = 0$.

Case 2.1.2: If $i = 2$, or $i \ge 4$ and $e_{i/2+1} = 1$, then $j_i = j_{i-1}$ and $j_{i+1} = i$, so

$$S_{i+1,i} = x_i x_{i+1}x_{i+4} - x_{j_{i-1}}x_{i+2}x_{i+4}$$

and $t_{i-1} = x_i x_{i+1} - x_{j_{i-1}}x_{i+2}$. Adding $-x_{i+4}t_{i-1}$ yields $\overline{S_{i+1,i}} = 0$.

Case 2.2: Suppose $\widehat{i} = i+2$ for even i such that $0 \le i \le \tau - 2$. We have

$$\begin{aligned} S_{i+1,\widehat{i}} = S_{i+1,i+2} &= x_{i+5}(x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}) - x_{i+2}(x_{i+3}x_{i+5} - x_{j_{i+2}}x_{i+6}) \\ &= x_{j_{i+2}}x_{i+2}x_{i+6} - x_{j_{i+1}}x_{i+4}x_{i+5}. \end{aligned}$$

Case 2.2.1: If $i \ge 2$ and $e_{(i+2)/2+1} = 0$, then $j_{i+2} = j_i$ and $j_{i+3} = i+1$, so

$$S_{i+1,i+2} = x_{j_i}x_{i+2}x_{i+6} - x_{j_{i+1}}x_{i+4}x_{i+5}$$

and $t_{i+3} = x_{i+4}x_{i+5} - x_{i+1}x_{i+6}$.

Case 2.2.1a: If $i \geq 4$ and $e_{i/2+1} = 0$, then $j_i = j_{i-2}$ and $j_{i+1} = i - 1$, so

$$S_{i+1,i+2} = x_{j_{i-2}} x_{i+2} x_{i+6} - x_{i-1} x_{i+4} x_{i+5}$$

and $t_{i-2} = x_{i-1}x_{i+1} - x_{j_{i-2}} x_{i+2}$. Adding $x_{i-1}t_{i+3}$ yields

$$x_{j_{i-2}} x_{i+2} x_{i+6} - x_{i-1} x_{i+1} x_{i+6},$$

and adding $x_{i+6}t_{i-2}$ yields $\overline{S_{i+1,i+2}} = 0$.

Case 2.2.1b: If $i = 2$, or $i \geq 4$ and $e_{i/2+1} = 1$, then $j_i = j_{i-1}$ and $j_{i+1} = i$, so

$$S_{i+1,i+2} = x_{j_{i-1}} x_{i+2} x_{i+6} - x_i x_{i+4} x_{i+5}$$

and $t_{i-1} = x_i x_{i+1} - x_{j_{i-1}} x_{i+2}$. Adding $x_i t_{i+3}$ yields

$$x_{j_{i-1}} x_{i+2} x_{i+6} - x_i x_{i+1} x_{i+6},$$

and adding $x_{i+6}t_{i-1}$ yields $\overline{S_{i+1,i+2}} = 0$.

Case 2.2.2: If $i = 0$, or $i \geq 2$ and $e_{(i+2)/2+1} = 1$, then $j_{i+2} = j_{i+1}$ and $j_{i+3} = i + 2$, so

$$S_{i+1,i+2} = x_{j_{i+1}} x_{i+2} x_{i+6} - x_{j_{i+1}} x_{i+4} x_{i+5}$$

and $t_{i+3} = x_{i+4}x_{i+5} - x_{i+2}x_{i+6}$. Adding $x_{j_{i+1}} t_{i+3}$ yields $\overline{S_{i+1,i+2}} = 0$.

Case 3: There is nothing to consider for $S_{i+1,\hat{i}+1}$, since the leading terms of $t_{i+1} = x_{i+2}x_{i+3} - x_{j_{i+1}} x_{i+4}$ and $t_{\hat{i}+1} = x_{\hat{i}+2} x_{\hat{i}+3} - x_{j_{\hat{i}+1}} x_{\hat{i}+4}$ have no shared variables for i

distinct from $\widehat{i}$.

Thus, G is a Gröbner basis for $I_{G_\tau^e}$ with respect to $>$. $\square$

Corollary 4.1.3. *The initial ideal for $I_{G_\tau^e}$ with respect to the degree reverse lexicographic monomial order $>$ is*

$$in_> I_{G_\tau^e} = (x_2x_3, \{x_{i+1}x_{i+3}, x_{i+2}x_{i+3} \mid i \text{ even}, 2 \le i \le \tau\}).$$

We also obtain the following corollary, establishing that $R(\tau, e)$ is Koszul, which becomes useful in Chapter 5.

Corollary 4.1.4. *Since $I_{G_\tau^e}$ has a quadratic Gröbner basis, the ring $R(\tau, e)$ is Koszul for all τ due to [HHO18, Th 2.28]. This becomes relevant in Chapter 5.*

Proof. This is clear; these are the leading terms of the standard generators for $I_{G_\tau^e}$ from Remark 4.1.2, established as a Gröbner basis in Lemma 4.1.1. $\square$

Remark 4.1.5. We note that $\text{in}_> I_{G_\tau^e}$ does not depend on e, which will be important for the following sections.

4.2 Dimension and System of Parameters

We use the initial ideal $\text{in}_> I_{G_\tau^e}$ found in Corollary 4.1.3 and direct computation to show that the Krull dimension $\dim R(\tau, e) = \tau/2 + 3$. As a corollary, we obtain the projective dimension of $R(\tau, e)$. We note that the Krull dimension, like the initial ideal, does not depend on e. We refer the reader to Proposition 3.2.1 and Remark 4.1.2 for a reminder of how to think of the toric ring

$$R(\tau, e) = \frac{S(\tau)}{I_{G_\tau^e}}$$

in the context of this dissertation.

Theorem 4.2.1. *The Krull dimension of $R(\tau, e)$ is*

$$\dim R(\tau, e) = \tau/2 + 3.$$

Proof. Let $>$ be the degree reverse lexicographic monomial ordering with

$$x_0 > x_2 > x_3 > \cdots > x_{\tau+4}.$$

By Corollary 4.1.3, the initial ideal of $I_{G_\tau^e}$ with respect to $>$ is

$$\text{in}_> I_{G_\tau^e} = (x_2x_3, \{x_{i+1}x_{i+3}, x_{i+2}x_{i+3} \mid i \text{ even}, 2 \leq i \leq \tau\}).$$

Since $S(\tau)/(\text{in}_> I_{G_\tau^e})$ and $S(\tau)/I_{G_\tau^e}$ have the same Krull dimension by Proposition 2.2.6, it suffices to prove that

$$\dim S(\tau)/(\text{in}_> I_{G_\tau^e}) = \tau/2 + 3.$$

To see that the dimension is at least $\tau/2 + 3$, we construct a chain of prime ideals in $S(\tau)$ containing $\text{in}_> I_{G_\tau^e}$. Since every monomial generator of $\text{in}_> I_{G_\tau^e}$ contains a variable of odd index, we begin with $P_\tau = (\{x_k \mid k \text{ odd}, 2 < k < \tau + 4\})$, a prime ideal containing $\text{in}_> I_{G_\tau^e}$. Then we have the chain of prime ideals $P_\tau \subsetneq P_\tau + (x_0) \subsetneq P_\tau + (x_0, x_2) \subsetneq P_\tau + (x_0, x_2, x_4) \subsetneq \cdots \subsetneq P_\tau + (\{x_i \mid i \text{ even}, 0 \leq i \leq \tau + 4\})$. Since the list

$$x_0, x_{2(1)}, x_{2(2)}, \ldots, x_{2(\tau/2+2)}$$

contains $\tau/2 + 3$ variables of even index, this chain has length $\tau/2 + 3$, so that

$$\dim S(\tau)/(\text{in}_> I_{G_\tau^e}) \geq \tau/2 + 3.$$

To see that the dimension is at most $\tau/2+3$, we find a sequence $\mathfrak{X}_\tau$ of $\tau/2+3$ elements in $S(\tau)/(\mathrm{in}_> I_{G^e_\tau})$ such that $\frac{S(\tau)/(\mathrm{in}_> I_{G^e_\tau})}{(\mathfrak{X}_\tau)}$ has dimension zero. Let

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

in $S(\tau)$, and take the quotient of $S(\tau)/(\mathrm{in}_> I_{G^e_\tau})$ by $(\mathfrak{X}_\tau)$, where $\mathfrak{X}_\tau$ is the image of X_τ in $S(\tau)/(\mathrm{in}_> I_{G^e_\tau})$, to obtain the following. In the last step, we rewrite the quotient of $S(\tau)$ and $(\mathrm{in}_> I_{G^e_\tau}) + (X_\tau)$ by (X_τ) by setting x_0 and $x_{\tau+4}$ equal to 0 and replacing x_i by x_{i+1} for even i with $2 \leq i \leq \tau+2$:

$$\begin{aligned}
\frac{S(\tau)/(\mathrm{in}_> I_{G^e_\tau})}{(\mathfrak{X}_\tau)} &\cong \frac{S(\tau)}{((\mathrm{in}_> I_{G^e_\tau}) + (X_\tau))} \\
&\cong \frac{S(\tau)/(X_\tau)}{((\mathrm{in}_> I_{G^e_\tau}) + (X_\tau))/(X_\tau)} \\
&\cong \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{(x_3^2, \{x_{i+1}x_{i+3}, x_{i+3}^2 \mid i \text{ even}, 2 \leq i \leq \tau\})}.
\end{aligned}$$

Now let

$$\mathfrak{I} = (x_3^2, (x_3^2, \{x_{i+1}x_{i+3}, x_{i+3}^2 \mid i \text{ even}, 2 \leq i \leq \tau\})).$$

We show that

$$\frac{S(\tau)/(\mathrm{in}_> I_{G^e_\tau})}{(\mathfrak{X}_\tau)} \cong \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{\mathfrak{I}}$$

has dimension zero by showing that $k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]/\sqrt{\mathfrak{I}}$ has dimension zero, since $\dim R/I = \dim R/\sqrt{I}$ for an ideal I of a ring R. Since $x_{i+1}^2 \in \mathfrak{I}$ for even i with $2 \leq i \leq \tau+2$, we have

$$(x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}) \subseteq \sqrt{\mathfrak{I}} \subseteq (x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}),$$

so that $\sqrt{\mathfrak{I}} = (x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3})$. Then

$$\begin{aligned}
\dim \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{\mathfrak{I}} &= \dim \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{\sqrt{\mathfrak{I}}} \\
&= \dim \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{(x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3})} \\
&= 0.
\end{aligned}$$

Thus,

$$\dim S(\tau)/(\mathrm{in}_> I_{G_\tau^e}) \leq \tau/2 + 3.$$

We conclude that $\dim R(\tau, e) = \dim S(\tau)/(\mathrm{in}_> I_{G_\tau^e}) = \tau/2 + 3$. □

Corollary 4.2.2. *The projective dimension of $R(\tau, e)$ over $Q(\tau)$ is*

$$\mathrm{pd}_{\,Q(\tau)} R(\tau, e) = \tau/2 + 1.$$

Proof. We know the Krull dimension of the polynomial ring $Q(\tau)$ is $\tau + 4$. The result follows from the fact that $R(\tau, e)$ is Cohen-Macaulay and from the graded version of the Auslander-Buchsbaum formula. □

Remark 4.2.3. The proof of the previous theorem shows that the image of

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

in $S(\tau)/(\mathrm{in}_> I_{G_\tau^e})$ (which we call $\mathfrak{X}_\tau$) is a system of parameters for $S(\tau)/(\mathrm{in}_> I_{G_\tau^e})$. We prove in the next theorem that the image of X_τ in $R(\tau, e)$ (which we call $\overline{X_\tau}$) is also a system of parameters for $R(\tau, e)$. Before doing so, we introduce some notation and a definition which will allow us to better grapple with the quotient ring $R(\tau, e)/(\overline{X_\tau})$.

Notation 4.2.4. In this dissertation, we have already seen some different quotient rings, and we are about to extensively work with a quotient of a quotient ring. We use mathfrak notation $\mathfrak{X}$ and $\mathfrak{I}$ when we take the quotient by $\mathrm{in}_> I_{G^e_\tau}$ and we use overline notation $\overline{X_\tau}$ for the image of

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

from Remark 4.2.3 in the quotient $R(\tau, e) = S(\tau)/I_{G^e_\tau}$. When we take the quotient of $R(\tau, e)$ by $(\overline{X_\tau})$, we introduce widehat notation for a ring $\widehat{R(\tau, e)}$ described below that is isomorphic to $R(\tau, e)/\overline{X_\tau}$. We find this notation natural since it is often used for the removal of variables, and the quotient by $\overline{X_\tau}$ may be viewed as identifying and removing variables. Since this dissertation has no completions in it, there should be no conflict of notation.

Definition 4.2.5. Here we define a ring $\widehat{R(\tau, e)}$ which is isomorphic to the ring

$$\frac{R(\tau, e)}{(\overline{X_\tau})},$$

where $\overline{X_\tau}$ is the image of X_τ from Remark 4.2.9 in $R(\tau, e)$. We use this ring extensively, for the following theorem as well as throughout Section 4.3. To motivate the definition of $R(\tau, e)$, note the following isomorphism:

$$R(\tau, e)/(\overline{X_\tau}) \cong S(\tau)/(I_{G^e_\tau} + (X_\tau)) \cong \frac{S(\tau)/(X_\tau)}{(I_{G^e_\tau} + (X_\tau))/(X_\tau)}.$$

We will define a ring $\widehat{S(\tau)} \cong S(\tau)/(X_\tau)$ with an ideal $\widehat{I_{G^e_\tau}} \cong (I_{G^e_\tau} + (X_\tau))/(X_\tau)$ and set

$$\widehat{R(\tau, e)} := \widehat{S(\tau)}/\widehat{I_{G^e_\tau}}.$$

To define $\widehat{S(\tau)}$ and $\widehat{I_{G^e_\tau}}$, we view taking the quotient by (X_τ) as setting x_0 and x_τ equal to 0 and replacing x_i with x_{i+1} for even i. Then we obtain

$$\widehat{S(\tau)} = k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}] \cong S(\tau)/(X_\tau).$$

To define $\widehat{I_{G^e_\tau}}$, we recall the standard generators of $I_{G^e_\tau}$ and introduce notation to describe the generators of $\widehat{I_{G^e_\tau}} \cong (I_{G^e_\tau} + (X_\tau))/(X_\tau)$. By Remark 4.1.2, the standard generators of $I_{G^e_\tau}$ are

$$\begin{aligned} t_1 &= x_2 x_3 - x_{j_1} x_4 \\ t_i &= x_{i+1} x_{i+3} - x_{j_i} x_{i+4} \\ t_{i+1} &= x_{i+2} x_{i+3} - x_{j_{i+1}} x_{i+4}, \end{aligned}$$

for $2 \le i \le \tau$, where the nonnegative integers j_n are as in Remark 3.1.10.

Let $\widehat{i}$ be the largest even index such that $j_{\widehat{i}} = 0$. By Remark 3.1.10, we see that for even i, the j_i are defined recursively and form a non-decreasing sequence. Then

$$j_2 = j_4 = j_6 = \cdots = j_{\widehat{i}} = 0,$$

and since we view taking the quotient as setting $x_0 (= x_{j_1} = x_{j_2})$ and $x_\tau (= x_{j_\tau} = x_{j_{\tau+1}})$ equal to 0 in our system of parameters, we define $\widehat{I_{G^e_\tau}}$ by replacing x_{j_n} with x_{J_n}, where

$$x_{J_n} = \begin{cases} 0 & \text{if } n \text{ is even and } n \le \widehat{i}, \text{ or if } n \in \{1, \tau, \tau+1\} \\ x_{j_n+1} & \text{if } \widehat{i} < j_n < \tau \text{ and } j_n \text{ is even} \\ x_{j_n} & \text{if } j_n \text{ is odd} \end{cases}$$

To define the generators for $\widehat{I_{G_\tau^e}}$, we set x_0 and x_τ equal to 0 and replace x_i with x_{i+1} for even i to obtain

$$\begin{aligned}
\widehat{t_1} &= x_3^2 - x_{J_1}x_5 \\
\widehat{t_i} &= x_{i+1}x_{i+3} - x_{J_i}x_{i+5} \\
\widehat{t_{i+1}} &= x_{i+3}^2 - x_{J_{i+1}}x_{i+5}
\end{aligned}$$

for $2 \leq i \leq \tau$.

By properties of the original j_n from Remark 3.1.10, we have the following:

- $x_{J_1} = x_{J_2} = x_{J_\tau} = x_{J_{\tau+1}} = 0$, $J_3 = 3$, and
- for even $i \geq 4$,
  * $e_{i/2+1} = 0 \iff x_{J_i} = x_{J_{i-2}} \iff J_{i+1} = i - 1$
  * $e_{i/2+1} = 1 \iff x_{J_i} = x_{J_{i-1}} \iff J_{i+1} = i + 1.$

Example 4.2.6. For the graph $G_4^{(0,0,0)} \in \mathcal{F}_2 \subset \mathcal{F}$, we have the toric ring

$$R(4,e) = \frac{k[x_0, x_2, x_3, \ldots, x_8]}{(x_2x_3 - x_0x_4, x_3x_5 - x_0x_6, x_4x_5 - x_2x_6, x_5x_7 - x_0x_8, x_6x_7 - x_3x_8)}$$

coming from the ladder-like structure

$$M_4^{(0,0,0)} = \begin{matrix} x_0 & x_2 & x_5 \\ x_3 & x_4 & x_6 \\ x_7 & & x_8 \end{matrix}$$

from Example 3.1.3. We know

$$X_4 = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_4 - x_5, x_6 - x_7, x_8,$$

so that $\frac{R(4,e)}{(\overline{X_4})}$ is isomorphic to

$$\frac{k[x_0, x_2, x_3, x_4, x_5, x_6, x_7, x_8]}{(x_2x_3 - x_0x_4, x_3x_5 - x_0x_6, x_4x_5 - x_2x_6, x_5x_7 - x_0x_8, x_6x_7 - x_3x_8, x_0, x_2 - x_3, \ldots, x_8)}$$

$$\cong \frac{k[x_3, x_5, x_7]}{(x_3^2, x_3x_5, x_5^2 - x_3x_7, x_5x_7, x_7^2)} = \widehat{R(4,e)}.$$

Remark 4.2.7. We may summarize the information in Definition 4.2.5 by saying that

$$R(\tau, e)/(\overline{X_\tau}) \cong \widehat{R(\tau, e)} = \widehat{S(\tau)}/\widehat{I_{G^e_\tau}},$$

where

$$\widehat{S(\tau)} = k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}],$$

and where $\widehat{I_{G^e_\tau}}$ is generated by

$$\begin{aligned} \widehat{t_1} &= x_3^2 - x_{J_1}x_5 \\ \widehat{t_i} &= x_{i+1}x_{i+3} - x_{J_i}x_{i+5} \\ \widehat{t_{i+1}} &= x_{i+3}^2 - x_{J_{i+1}}x_{i+5} \end{aligned}$$

for even i with $2 \leq i \leq \tau$, such that

- $x_{J_1} = x_{J_2} = x_{J_\tau} = x_{J_{\tau+1}} = 0$, $J_3 = 3$, and
- for even $i \geq 4$,
  - * $e_{i/2+1} = 0 \iff x_{J_i} = x_{J_{i-2}} \iff J_{i+1} = i - 1$
  - * $e_{i/2+1} = 1 \iff x_{J_i} = x_{J_{i-1}} \iff J_{i+1} = i + 1.$

In particular, we note that for $\tau = 0$ we obtain

$$R(0, e)/(\overline{X_0}) \cong \widehat{R(0, e)} = \widehat{Q(0)}/\widehat{I_{G^e_0}} = \frac{k[x_3]}{(x_3^2)}$$

and that for $\tau = 2$ we obtain

$$R(2,e)/(\overline{X_2}) \cong \widehat{R(2,e)} = \widehat{Q(2)}/\widehat{I_{G_2^e}} = \frac{k[x_3, x_5]}{(x_3^2, x_3x_5, x_5^2)}.$$

Proposition 4.2.8. *Let $R(\tau, e) = S(\tau)/I_{G_\tau^e}$, let*

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \ldots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

so that $\mathfrak{X}_\tau$, the image of X_τ in $S(\tau)/(in_> I_{G_\tau^e})$, is the system of parameters from Remark 4.2.3. Then the image of X_τ in $R(\tau, e)$ is a system of parameters for $R(\tau, e)$.

Proof. Let X_τ be defined as above. Then by Theorem 4.2.1 and Definition 4.2.5 we need only show that $\dim \widehat{R(\tau,e)} = 0$. By Remark 4.2.7, we have for $\tau = 0$

$$\widehat{R(0,e)} = \frac{k[x_3]}{(x_3^2)},$$

for $\tau = 2$

$$\widehat{R(2,e)} = \frac{k[x_3, x_5]}{(x_3^2, x_3x_5, x_5^2)},$$

and for even $\tau > 2$

$$\widehat{R(\tau,e)} = \frac{\widehat{S(\tau)}}{\widehat{I_{G_\tau^e}}} = \frac{k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]}{(\{\widehat{t_1}, \widehat{t_i}, \widehat{t_{i+1}} \mid i \text{ even}, 2 \leq i \leq \tau\})},$$

where

$$\begin{aligned} \widehat{t_1} &= x_3^2 - x_{J_1}x_5 \\ \widehat{t_i} &= x_{i+1}x_{i+3} - x_{J_i}x_{i+5} \\ \widehat{t_{i+1}} &= x_{i+3}^2 - x_{J_{i+1}}x_{i+5}. \end{aligned}$$

We know $\dim \widehat{R(\tau,e)}) = \dim \widehat{S(\tau)}/\widehat{I_{G_\tau^e}} = \dim \widehat{S(\tau)}/\sqrt{\widehat{I_{G_\tau^e}}}$. We claim that

$$\sqrt{\widehat{I_{G_\tau^e}}} = (x_3, x_5, \dots, x_{\tau+1}, x_{\tau+3}).$$

This is clear for $\tau \in \{0,2\}$. For $\tau > 2$, we will prove this by induction. Since $\widehat{t_1} = x_3^2$ and $\widehat{t_{\tau+1}} = x_{\tau+3}^2$ are in $\widehat{I_{G_\tau^e}}$, we have $x_3 \in \sqrt{\widehat{I_{G_\tau^e}}}$ and $x_{\tau+3} \in \sqrt{\widehat{I_{G_\tau^e}}}$. Since

$$\widehat{t_3} = x_5^2 - x_3x_7 \in \widehat{I_{G_\tau^e}} \subseteq \sqrt{\widehat{I_{G_\tau^e}}}$$

and $x_3 \in \sqrt{\widehat{I_{G_\tau^e}}}$, we get $x_5^2 \in \sqrt{\widehat{I_{G_\tau^e}}}$, so that $x_5 \in \sqrt{\widehat{I_{G_\tau^e}}}$. Now suppose $x_{i-1}, x_{i+1} \in \sqrt{\widehat{I_{G_\tau^e}}}$ for some even i, $4 \le i \le \tau - 2$. Then

$$\widehat{t_{i+1}} = x_{i+3}^2 - x_{J_{i+1}}x_{i+5} \in \widehat{I_{G_\tau^e}} \subseteq \sqrt{\widehat{I_{G_\tau^e}}}.$$

But $x_{J_{i+1}} \in \{x_{i-1}, x_{i+1}\}$ by Remark 4.2.7 and $\{x_{i-1}, x_{i+1}\} \subseteq \sqrt{\widehat{I_{G_\tau^e}}}$ by induction, so that $x_{i+3}^2 \in \sqrt{\widehat{I_{G_\tau^e}}}$, and hence we get $x_{i+3} \in \sqrt{\widehat{I_{G_\tau^e}}}$. We conclude that

$$(x_3, x_5, \dots, x_{\tau+1}, x_{\tau+3}) \subseteq \sqrt{\widehat{I_{G_\tau^e}}} \subseteq (x_3, x_5, \dots, x_{\tau+1}, x_{\tau+3}),$$

so $\sqrt{\widehat{I_{G_\tau^e}}} = (x_3, x_5, \dots, x_{\tau+1}, x_{\tau+3})$. Since $\widehat{R(\tau,e)} = \widehat{S(\tau)}/\sqrt{\widehat{I_{G_\tau^e}}} \cong k$ has dimension zero, so does $R(\tau,e)/(\overline{X_\tau})$. Thus, $\overline{X_\tau}$ is a system of parameters for $R(\tau,e)$. □

Corollary 4.2.9. *With the same hypotheses as in Proposition 4.2.8, the image of*

$$X_\tau = x_0, x_2 - x_3, x_4 - x_5, \dots, x_\tau - x_{\tau+1}, x_{\tau+2} - x_{\tau+3}, x_{\tau+4}$$

in $R(\tau,e)$ is a regular sequence on $R(\tau,e)$.

Proof. We know by Proposition 4.2.8 that the image of X_τ in $R(\tau,e)$ is a linear system of parameters. Since the rings $R(\tau,e)$ are Cohen-Macaulay (Corollary 3.2.2), we are done.

$\square$

4.3 Length, Multiplicity, and Regularity

In this section, we determine the multiplicity and Castelnuovo-Mumford regularity of the toric rings $R(\tau, e)$ coming from graphs $G_\tau^e \in \mathcal{F}$ by computing the length of the rings $\widehat{R(\tau, e)} \cong \widehat{R(\tau, e)}/\overline{X_\tau}$. We know by Proposition 4.2.8 that these rings are Artinian and by Corollary 4.2.9 that $\overline{X_\tau}$ is a linear regular sequence, which allows us to compute the multiplicity of the rings $R(\tau, e)$. As a corollary of Theorem 4.3.4, which establishes the Hilbert function for $\widehat{R(\tau, e)}$, we obtain the multiplicity and regularity of $R(\tau, e)$. We also develop an alternate graph-theoretic proof for the regularity of $R(\tau, e)$, which is included at the end of this section. We refer the reader to the background for definitions of multiplicity and regularity.

We begin with a lemma establishing a vector space basis for the ring

$$\widehat{R(\tau, e)} \cong R(\tau, e)/(\overline{X_\tau})$$

from Remark 4.2.7, which we use extensively for our results.

Lemma 4.3.1. *The image of all squarefree monomials with only odd indices whose indices are at least four apart, together with the image of 1_k, forms a vector space basis of $\widehat{R(\tau, e)}$ over k.*

Proof. We remind the reader of the definition of $\widehat{R(\tau, e)}$ and then find the initial ideal of $\widehat{I_{G_\tau^e}}$ and use Macaulay's Basis Theorem to show that the desired representatives form a basis of $\widehat{R(\tau, e)}$ as a vector space over k.

We recall for the reader that from Remark 4.2.7, we have

$$R(\tau, e)/(\overline{X_\tau}) \cong \widehat{R(\tau, e)} = \widehat{S(\tau)}/\widehat{I_{G_\tau^e}},$$

where

$$\widehat{S(\tau)} = k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}],$$

and where $\widehat{I_{G_\tau^e}}$ is generated by

$$\begin{aligned}
\widehat{t_1} &= x_3^2 - x_{J_1}x_5 \\
\widehat{t_i} &= x_{i+1}x_{i+3} - x_{J_i}x_{i+5} \\
\widehat{t_{i+1}} &= x_{i+3}^2 - x_{J_{i+1}}x_{i+5}
\end{aligned}$$

for i even with $2 \leq i \leq \tau$, such that

- $x_{J_1} = x_{J_2} = x_{J_\tau} = x_{J_{\tau+1}} = 0$, $J_3 = 3$, and
- for even $i \geq 4$,
  - * $e_{i/2+1} = 0 \iff x_{J_i} = x_{J_{i-2}} \iff J_{i+1} = i - 1$
  - * $e_{i/2+1} = 1 \iff x_{J_i} = x_{J_{i-1}} \iff J_{i+1} = i + 1.$

We first find a set of monomials with the desired property whose image is a basis in the quotient $\frac{\widehat{S(\tau)}}{\mathrm{in}_> \widehat{I_{G_\tau^e}}}$ by the initial ideal. By Macaulay's Basis Theorem, which is Theorem 1.5.7 in [KR00], the image of these monomials in $\widehat{R(\tau, e)}$ is also a basis.

To find the initial ideal of $\widehat{I_{G_\tau^e}}$, we show that the given generators $\widehat{t_n}$ are a Gröbner basis for $\widehat{I_{G_\tau^e}}$ with respect to the degree reverse lexicographic order $>$. We do this by applying Buchberger's algorithm, and note cases where $x_{J_n} = 0$ affects the application of the algorithm.

We use the notation $LT(\widehat{t_n})$ to denote the leading term of a generator $\widehat{t_n}$. We show that in each case, the reduced form $\overline{S_{m,n}}$ of $S_{m,n}$ is zero, where

$$S_{m,n} = \frac{LT(\widehat{t_n})}{gcd(LT(\widehat{t_m}), LT(\widehat{t_n}))}(\widehat{t_m}) - \frac{LT(\widehat{t_m})}{gcd(LT(\widehat{t_m}), LT(\widehat{t_n}))}(\widehat{t_n}).$$

We need not consider cases where $gcd(LT(\widehat{t_m}), LT(\widehat{t_n})) = 1$, as $\overline{S_{m,n}} = 0$ in these cases. Then the only interesting cases involve $S_{n,n+1}$ and $S_{2n,2n+2}$.

Case $(i, i+1)$: When $2 \leq i \leq \tau$ and i is even, we have

$$\begin{aligned} S_{i,i+1} &= x_{i+3}(x_{i+1}x_{i+3} - x_{J_i}x_{i+5}) - x_{i+1}(x_{i+3}^2 - x_{J_{i+1}}x_{i+5}) \\ &= x_{J_{i+1}}x_{i+1}x_{i+5} - x_{J_i}x_{i+3}x_{i+5}. \end{aligned}$$

Case $(i, i+1)$.1: If $i = 2$, then $x_{J_i} = 0$ and $x_{J_{i+1}} = 3$, so we have

$$S_{2,3} = x_3^2 x_7.$$

Since $\widehat{t_1} = x_3^2$, adding $-x_7\widehat{t_1}$ yields $\overline{S_{2,3}} = 0$.

Case $(i, i+1)$.2: If $i = \tau$, then $x_{J_i} = x_{J_{i+1}} = 0$ and we are done.

Case $(i, i+1)$.3: If $2 < i < \tau$ and $e_{i/2+1} = 0$, then $x_{J_i} = x_{J_{i-2}}$ and $J_{i+1} = i-1$, so we have

$$S_{i,i+1} = x_{i-1}x_{i+1}x_{i+5} - x_{J_{i-2}}x_{i+3}x_{i+5}.$$

Since $\widehat{t_{i-2}} = x_{i-1}x_{i+1} - x_{J_{i-2}}x_{i+3}$, adding $-x_{i+5}\widehat{t_{i-2}}$ yields $\overline{S_{i,i+1}} = 0$.

Case $(i, i+1)$.4: If $2 < i < \tau$ and $e_{i/2+1} = 1$, then $x_{J_i} = x_{J_{i-1}}$ and $J_{i+1} = i+1$, so we have

$$S_{i,i+1} = x_{i+1}^2 x_{i+5} - x_{J_{i-1}}x_{i+3}x_{i+5}.$$

Since $\widehat{t_{i-1}} = x_{i+1}^2 - x_{J_{i-1}}x_{i+3}$, adding $-x_{i+5}\widehat{t_{i-1}}$ yields $\overline{S_{i,i+1}} = 0$.

Case $(i-1,i)$: When $2 \leq i \leq \tau$ and i is even, we have

$$\begin{aligned} S_{i-1,i} &= x_{i+3}(x_{i+1}^2 - x_{J_{i-1}}x_{i+3}) - x_{i+1}(x_{i+1}x_{i+3} - x_{J_i}x_{i+5}) \\ &= x_{J_i}x_{i+1}x_{i+5} - x_{J_{i-1}}x_{i+3}^2. \end{aligned}$$

Case $(i-1,i)$.1: If $i = 2$, then $x_{J_i} = x_{J_{i-1}} = 0$ and we are done.

Case $(i-1,i)$.2: If $i = \tau$, then $x_{J_\tau} = 0$, so we have

$$S_{\tau-1,\tau} = -x_{J_{\tau-1}}x_{\tau+3}^2.$$

Since $\widehat{t_{\tau+1}} = x_{\tau+3}^2$, adding $x_{J_{\tau-1}}\widehat{t_{\tau+1}}$ yields $\overline{S_{\tau-1,\tau}} = 0$.

Case $(i-1,i)$.3: If $2 < i < \tau$ and $e_{i/2+1} = 0$, then $x_{J_i} = x_{J_{i-2}}$ and $J_{i+1} = i-1$, so $\widehat{t_{i+1}} = x_{i+3}^2 - x_{i-1}x_{i+5}$ and we have

$$S_{i-1,i} = x_{J_{i-2}}x_{i+1}x_{i+5} - x_{J_{i-1}}x_{i+3}^2.$$

Adding $x_{J_{i-1}}\widehat{t_{i+1}}$ yields

$$S_{i-1,i} = x_{J_{i-2}}x_{i+1}x_{i+5} - x_{J_{i-1}}x_{i-1}x_{i+5}.$$

Case $(i-1,i)$.3.1: If in addition $4 < i < \tau$ and $e_{(i-2)/2+1} = 0$, then $x_{J_{i-2}} = x_{J_{i-4}}$ and $J_{i-1} = i-3$, so we have

$$S_{i-1,i} = x_{J_{i-4}}x_{i+1}x_{i+5} - x_{i-3}x_{i-1}x_{i+5}.$$

Since $\widehat{t_{i-4}} = x_{i-3}x_{i-1} - x_{J_{i-4}}x_{i+1}$, adding $x_{i+5}\widehat{t_{i-4}}$ yields $\overline{S_{i-1,i}} = 0$.

Case $(i-1,i)$.3.2: If in addition $i=4$, or $4<i<\tau$ and $e_{(i-2)/2+1}=1$, then $x_{J_{i-2}}=x_{J_{i-3}}$ and $J_{i-1}=i-1$, so we have

$$S_{i-1,i}=x_{J_{i-3}}x_{i+1}x_{i+5}-x_{i-1}^2x_{i+5}.$$

Since $\widehat{t_{i-3}}=x_{i-1}^2-x_{J_{i-3}}x_{i+1}$, adding $x_{i+5}\widehat{t_{i-3}}$ yields $\overline{S_{i-1,i}}=0$.

Case $(i-1,i)$.4: If $2<i<\tau$ and $e_{i/2+1}=1$, then $x_{J_i}=x_{J_{i-1}}$ and $J_{i+1}=i+1$, so $\widehat{t_{i+1}}=x_{i+3}^2-x_{i+1}x_{i+5}$ and we have

$$S_{i-1,i}=x_{J_{i-1}}x_{i+1}x_{i+5}-x_{J_{i-1}}x_{i+3}^2.$$

Adding $x_{J_{i-1}}\widehat{t_{i+1}}$ yields $\overline{S_{i-1,i}}=0$.

Case $(i,i+2)$: When $2\le i\le\tau-2$ and i is even, we have

$$\begin{aligned}S_{i,i+2} &= x_{i+5}(x_{i+1}x_{i+3}-x_{J_i}x_{i+5})-x_{i+1}(x_{i+3}x_{i+5}-x_{J_{i+2}}x_{i+7})\\ &= x_{J_{i+2}}x_{i+1}x_{i+7}-x_{J_i}x_{i+5}^2.\end{aligned}$$

Case $(i,i+2)$.1: If $i=2$, then $x_{J_i}=0$, so we have

$$S_{2,4}=x_{J_4}x_3x_9.$$

If $J_4=J_2=0$, we are done. If not, $J_4=J_3=3$, so we have

$$S_{2,4}=x_3^2x_9.$$

Since $\widehat{t_1}=x_3^2$, adding $-x_9\widehat{t_1}$ yields $\overline{S_{2,4}}=0$.

Case $(i, i+2)$.2: If $i = \tau - 2$, then $x_{J_{i+2}} = 0$, so we have

$$S_{\tau-2,\tau} = -x_{J_{\tau-2}} x_{\tau+3}^2.$$

If $J_{\tau-2} = 0$, we are done. If not, since $\widehat{t_{\tau+1}} = x_{\tau+3}^2$, adding $x_{J_{\tau-2}} \widehat{t_{\tau+1}}$ yields $\overline{S_{\tau-2,\tau}} = 0$.

Case $(i, i+2)$.3: If $2 < i < \tau - 2$ and $e_{(i+2)/2+1} = 0$, then $x_{J_{i+2}} = x_{J_i}$ and $J_{i+3} = i+1$, so $\widehat{t_{i+3}} = x_{i+5}^2 - x_{i+1} x_{i+7}$ and we have

$$S_{i,i+2} = x_{J_i} x_{i+1} x_{i+7} - x_{J_i} x_{i+5}^2.$$

If $x_{J_i} = 0$, we are done. If not, adding $x_{J_i} \widehat{t_{i+3}}$ yields $\overline{S_{i,i+2}} = 0$.

Case $(i, i+2)$.4: If $2 < i < \tau - 2$ and $e_{(i+2)/2+1} = 1$, then $x_{J_{i+2}} = x_{J_{i+1}}$ and $J_{i+3} = i+3$, so $\widehat{t_{i+3}} = x_{i+5}^2 - x_{i+3} x_{i+7}$ and we have

$$S_{i,i+2} = x_{J_{i+1}} x_{i+1} x_{i+7} - x_{J_i} x_{i+5}^2.$$

Case $(i, i+2)$.4.1: If in addition $e_{i/2+1} = 0$, then $x_{J_i} = x_{J_{i-2}}$ and $J_{i+1} = i - 1$, so we have

$$S_{i,i+2} = x_{i-1} x_{i+1} x_{i+7} - x_{J_{i-2}} x_{i+5}^2.$$

Since $\widehat{t_{i-2}} = x_{i-1} x_{i+1} - x_{J_{i-2}} x_{i+3}$, adding $-x_{i+7} \widehat{t_{i-2}}$ yields

$$S_{i,i+2} = x_{J_{i-2}} x_{i+3} x_{i+7} - x_{J_{i-2}} x_{i+5}^2.$$

If $J_{i-2} = 0$, we are done. If not, adding $x_{J_{i-2}} \widehat{t_{i+3}}$ yields $\overline{S_{i,i+2}} = 0$.

Case $(i, i+2)$.4.2: If in addition $e_{i/2+1} = 1$, then $x_{J_i} = x_{J_{i-1}}$ and $J_{i+1} = i+1$, so we have

$$S_{i,i+2} = x_{i+1}^2 x_{i+7} - x_{J_{i-1}} x_{i+5}^2.$$

Since $\widehat{t_{i-1}} = x_{i+1}^2 - x_{J_{i-1}} x_{i+3}$, adding $-x_{i+7}\widehat{t_{i-1}}$ yields

$$S_{i,i+2} = x_{J_{i-1}} x_{i+3} x_{i+7} - x_{J_{i-1}} x_{i+5}^2.$$

Adding $x_{J_{i-1}}\widehat{t_{i+3}}$ yields $\overline{S_{i,i+2}} = 0$.

Then the given generators $\widehat{t_n}$ are a Gröbner Basis for $\widehat{I_{G_\tau^e}}$ with respect to the degree reverse lexicographic monomial order $>$, so that

$$\mathrm{in}_> (\widehat{I_{G_\tau^e}}) = (x_3^2, \{x_{i+1}x_{i+3}, x_{i+3}^2 \mid i \text{ even}, 2 \leq i \leq \tau\})$$

in the ring $\widehat{S(\tau)} = k[x_3, x_5, \ldots, x_{\tau+1}, x_{\tau+3}]$. Since $\mathrm{in}_> (\widehat{I_{G_\tau^e}})$ consists precisely of all squares of variables in $\widehat{S(\tau)}$ and all degree two products of variables whose indices differ by exactly two, it follows that the image of the squarefree monomials whose indices are at least four apart, together with the image of 1_k, forms a basis for $\frac{\widehat{S(\tau)}}{\mathrm{in}_> \widehat{I_{G_\tau^e}}}$. Then by Macaulay's Basis Theorem, the image of these monomials in $\widehat{R(\tau, e)} = \frac{\widehat{S(\tau)}}{\widehat{I_{G_\tau^e}}}$ is also a basis. $\square$

We first use the lemma above to establish facts about the vector space dimensions of degree n pieces of $\widehat{R(\tau, e)}$, which are applied further below to establish length and multiplicity.

Notation 4.3.2. Throughout this section, we use $d_{\tau,n} := \dim_k(\widehat{R(\tau, e)})_n$ for the vector space dimension of the degree n piece of $\widehat{R(\tau, e)}$, that is, for the nth coefficient in the Hilbert series of $\widehat{R(\tau, e)}$. By Lemma 4.3.1, these are independent of e.

We establish a recursive relationship on these dimensions by introducing a short exact sequence of vector spaces.

Lemma 4.3.3. *For $\tau \geq 4$ and $n \geq 1$, the vector space dimension $d_{\tau,n} = \dim_k(\widehat{R(\tau,e)})_n$ satisfies the recursive relationship*

$$d_{\tau,n} = d_{\tau-2,n} + d_{\tau-4,n-1}.$$

Proof. We use the vector space bases defined in Lemma 4.3.1. We note that the basis elements described are actually monomial representatives (which do not depend on e) of equivalence classes (which do depend on e), but we suppress this and speak as if they are monomials, not depending on e. We then take the liberty of suppressing e in what follows, for convenience. We recall for the reader that

$$\begin{aligned} \widehat{S(\tau)} &= k[x_3, x_5, \dots, x_{\tau-3}, x_{\tau-1}, x_{\tau+1}, x_{\tau+3}] \\ \widehat{Q(\tau-2)} &= k[x_3, x_5, \dots, x_{\tau-3}, x_{\tau-1}, x_{\tau+1}] \\ \widehat{Q(\tau-4)} &= k[x_3, x_5, \dots, x_{\tau-3}, x_{\tau-1}] \end{aligned}$$

Let $x_{\tau+3} : (\widehat{R(\tau-4)})_{n-1} \to (\widehat{R(\tau)})_n$ be multiplication by $x_{\tau+3}$, and let $\widehat{x_{\tau+3}} : (\widehat{R(\tau)})_n \to (\widehat{R(\tau-2)})_n$ be defined by

$$\widehat{x_{\tau+3}}(b) = \begin{cases} b & \text{if } x_{\tau+3} \nmid b \\ 0 & \text{if } x_{\tau+3} \mid b. \end{cases}$$

for a basis element b. We note that these vector space maps are well-defined, since 1_k or a squarefree monomial with odd indices at least four apart will have an output of 0, 1_k, or a monomial with the same properties. We show that the following sequence is exact.

$$0 \longrightarrow (\widehat{R(\tau-4)})_{n-1} \xrightarrow{x_{\tau+3}} (\widehat{R(\tau)})_n \xrightarrow{\widehat{x_{\tau+3}}} (\widehat{R(\tau-2)})_n \longrightarrow 0$$

Clearly, multiplication by $x_{\tau+3}$ is injective. Any element in $(\widehat{R(\tau-2)})_n$ is not divisible by $x_{\tau+3}$, so is its own preimage under $\widehat{x_{\tau+3}}$. Then the map $\widehat{x_{\tau+3}}$ is surjective. It is easy to see that $\operatorname{im} x_{\tau+3} \subseteq \ker \widehat{x_{\tau+3}}$. Now let $b \in \ker \widehat{x_{\tau+3}}$. Then either $b = 0$ or b is a nonzero monomial such that $x_{\tau+3}|b$. Since b is squarefree, it follows that b is in $\operatorname{im} x_{\tau+3}$. We conclude that the above sequence is exact, and in particular, that

$$d_{\tau,n} = d_{\tau-2,n} + d_{\tau-4,n-1}.$$

$\square$

Applying Lemma 4.3.3 and induction, we achieve the following closed formula for the coefficients of the Hilbert series of $\widehat{R(\tau,e)}$.

Theorem 4.3.4. *If $R(\tau,e) = S(\tau)/I_{G_\tau^e}$ and $\widehat{R(\tau,e)} \cong R(\tau,e)/(\overline{X_\tau})$, we have*

$$\dim_k(\widehat{R(\tau,e)})_n = \begin{cases} 1 & n = 0 \\ \dfrac{2^{-n}}{n!}\displaystyle\prod_{j=1}^{n}(\tau+2j-4(n-1)) & 1 \le n \le \tau/4+1 \\ 0 & \textit{else}. \end{cases}$$

Proof. We show this by induction, using Notation 4.3.2. We begin with the base cases $n \le 2$ and $\tau \in \{0,2\}$, then proceed by induction. It is clear that $d_{\tau,n} = 0$ for $n < 0$. It is also clear that $d_{\tau,0} = 1$, generated by 1_k. By Lemma 4.3.1 and by the fact that $\widehat{R(\tau,e)}$ is a graded quotient, every nonzero element of positive degree n can be represented uniquely as a sum of degree n squarefree monomials with odd indices whose indices are at least four apart. Then $(\widehat{R(\tau,e)})_1$ is generated by the images of all the odd variables

$$x_3, x_{2(1)+3}, \dots, x_{2(\frac{\tau}{2})+3}$$

in $S(\tau)$, so that

$$d_{\tau,1} = \frac{\tau}{2} + 1 = \frac{1}{2}(\tau + 2) = \frac{2^{-1}}{1!} \prod_{j=1}^{1} (\tau + 2j - 4(1-1))$$

matches the given formula.

Now we establish the base cases $\tau = 0$ and $\tau = 2$ and use a preliminary induction to establish the last base case $n = 2$. We obtain $d_{0,n}$ and $d_{2,n}$ by recognizing that the first monomial of degree two with odd indices at least four apart is $x_3 x_7$, which does not exist until $\tau = 4$, so that

$$d_{0,n} = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{else} \end{cases}.$$

and

$$d_{2,n} = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 0 & \text{else} \end{cases}.$$

We recall by Lemma 4.3.3 that we have the recursive relationship

$$d_{\tau,n} = d_{\tau-2,n} + d_{\tau-4,n-1}$$

for $\tau \geq 4$ and $n \geq 1$.

We use a preliminary induction and the above relationship to establish our last base case, when $\tau \geq 4$ and $n = 2$. Suppose that $T \geq 4$ and the desired formula holds for $\tau < T$. Then we have

$$
\begin{aligned}
d_{T,2} &= d_{T-2,2} + d_{T-4,1} \\
&= \frac{2^{-2}}{2!}\prod_{j=1}^{2}((T-2)+2j-4(1)) + \frac{T-4}{2} + 1 \\
&= \frac{1}{8}(T-4)(T-2) + \frac{1}{2}(T-2) \\
&= \frac{1}{8}(T-2)(T) \\
&= \frac{2^{-2}}{2!}\prod_{j=1}^{2}(T+2j-4(1)),
\end{aligned}
$$

as desired.

This gives us the following table of base cases for $d_{\tau,n}$, which match the given formula:

$\tau \setminus n$	0	1	2	3	4	5	6	7	$\cdots$
0	1	1	0	0	0	0	0	0	$\cdots$
2	1	2	0	0	0	0	0	0	$\cdots$
4	1	3	1						
6	1	4	3						
8	1	5	6						
$\vdots$	$\vdots$	$\vdots$	$\vdots$						

Given our base cases and the above recursion, this establishes $d_{\tau,n}$ for all τ and for all n. It remains to show that this inductive relationship yields the formula

$$
d_{\tau,n} = \begin{cases} 1 & n = 0 \\ \dfrac{2^{-n}}{n!}\displaystyle\prod_{j=1}^{n}(\tau+2j-4(n-1)) & 1 \leq n \leq \tau/4+1 \\ 0 & \text{else} \end{cases}
$$

in general. Suppose that $T \geq 4$ and $N \geq 2$, and that we have established the formula for $\tau < T$ and $n \leq N$.

If $2 \leq N < \frac{T}{4} + 1$, then $2 \leq N \leq \frac{T}{4} + \frac{1}{2} = \frac{T-2}{4} + 1$ and $1 \leq N - 1 < \frac{T}{4} = \frac{T-4}{4} + 1$. By induction, we have

$$\begin{aligned}
d_{T,N} &= d_{T-2,N} + d_{T-4,N-1} \\
&= \frac{2^{-N}}{N!} \prod_{j=1}^{N} (T - 2 + 2j - 4(N-1)) + \frac{2^{-(N-1)}}{(N-1)!} \prod_{j=1}^{N-1} (T - 4 + 2j - 4(N-2)) \\
&= \frac{2^{-(N-1)}}{(N-1)!} \left(\frac{2^{-1}}{N} \prod_{j=1}^{N} (T + 2(j-1) - 4(N-1)) + \prod_{j=1}^{N-1} (T + 2j - 4(N-1)) \right) \\
&= \frac{2^{-(N-1)}}{(N-1)!} \left(\frac{1}{2N} \prod_{j=0}^{N-1} (T + 2j - 4(N-1)) + \prod_{j=1}^{N-1} (T + 2j - 4(N-1)) \right) \\
&= \frac{2^{-(N-1)}}{(N-1)!} \left(\prod_{j=1}^{N-1} (T + 2j - 4(N-1)) \right) \left(\frac{1}{2N} (T - 4(N-1)) + 1 \right) \\
&= \frac{2^{-N}}{N!} \left(\prod_{j=1}^{N-1} (T + 2j - 4(N-1)) \right) (T - 4(N-1) + 2N) \\
&= \frac{2^{-N}}{N!} \prod_{j=1}^{N} (T + 2j - 4(N-1)),
\end{aligned}$$

as desired.

For the special case where $N = \frac{T}{4} + 1$, there is only one possible unique representation of an element of degree $\frac{T}{4} + 1$, $\prod_{j=1}^{T/4+1} x_{4j-1}$, so $d_{T,\frac{T}{4}+1} = 1$. We show that this matches the formula

$$\begin{aligned} d_{T,T/4+1} &= \frac{2^{-(T/4+1)}}{(T/4+1)!}\prod_{j=1}^{T/4+1}(T+2j-4((T/4+1)-1)) \\ &= \frac{2^{-(T/4+1)}}{(T/4+1)!}\prod_{j=1}^{T/4+1}2j \\ &= \frac{2^{-(T/4+1)}}{(T/4+1)!}(2^{(T/4+1)})(T/4+1)! \\ &= 1. \end{aligned}$$

The remaining case is $N > \frac{T}{4}+1$. In this case, we have $N-1 > \frac{T}{4} = \frac{T-4}{4}+1$ and $N > \frac{T+2}{4} = \frac{T-2}{4}+1$, so that $d_{T,N} = d_{T-2,N} + d_{T-4,N-1} = 0$ by induction. □

Remark 4.3.5. We note from the proof of the above theorem a few facts for future reference. For $\tau \equiv 0 \mod 4$, we have in the above proof that $d(\tau, \frac{\tau}{4}+1) = 1$, and by our base cases, we have $\ell(\widehat{R(0,e)}) = 1+1 = 2$ and $\ell(\widehat{R(2,e)}) = 1+2 = 3$. Taking the Fibonacci sequence $F(n)$ with $F(0) = 0$ and $F(1) = 1$, we have $F(2) = 1$, $F(3) = 2$, and $F(4) = 3$, so that

$$\begin{aligned} \ell(\widehat{R(0,e)}) &= F(3) \\ \ell(\widehat{R(2,e)}) &= F(4). \end{aligned}$$

These facts become useful in the following corollary as well as in coming results concerning the lengths of the $\widehat{R(\tau,e)}$.

We have now established the dimension of every graded piece of $\widehat{R(\tau,e)}$. We use the following fact in the proof of our next result.

Remark 4.3.6. If S is a polynomial ring with an ideal B such that S/B is Artinian, then reg S/B over S is equal to

$$N = \max\{n \mid (S/B)_n \neq 0\},$$

the top nonzero degree of S/B; see for example [Pee11], Theorem 18.4. We use this below to determine reg $\widehat{R(\tau,e)}$. Furthermore, if an ideal J in a standard graded ring $R = S/I$ is generated by a linear regular sequence, then reg $R =$ reg R/J over S. Indeed, this may be seen from iterated graded short exact sequences coming from regular elements; see for example Corollary 18.7 (2) and (3) and Exercise 18.8 (1) from [Pee11]. Then reg $R(\tau,e) =$ reg $\widehat{R(\tau,e)}$, which we use below.

Corollary 4.3.7. *For $G^e_\tau \in \mathcal{F}$,*

$$\textit{reg}\ R(\tau,e) = \lfloor \tau/4 \rfloor + 1.$$

Proof. We first find reg $\widehat{R(\tau,e)}$ by determining the top nonzero degree of $\widehat{R(\tau,e)}$. By Theorem 4.3.4, we know the top nonzero degree is N for some $N \leq \frac{\tau}{4}+1$, so that $N \leq \lfloor \frac{\tau}{4} \rfloor + 1$. In fact, the top nonzero degree is $\lfloor \frac{\tau}{4} \rfloor + 1$, provided $d_{\tau,\lfloor \tau/4 \rfloor +1} \neq 0$. By Remark 4.3.5, $d(\tau, \frac{\tau}{4}+1) = 1 \neq 0$ when $\tau \equiv 0 \mod 4$, so suppose $\tau \equiv 2 \mod 4$. We have

$$\begin{aligned} d_{\tau,\lfloor \tau/4 \rfloor +1} = d_{\tau,\frac{\tau-2}{4}+1} &= \frac{2^{-(\frac{\tau-2}{4}+1)}}{(\frac{\tau-2}{4}+1)!} \prod_{j=1}^{\frac{\tau-2}{4}+1} \left(\tau + 2j - 4\left(\left(\frac{\tau-2}{4}+1\right)-1\right)\right) \\ &= \frac{2^{-(\frac{\tau-2}{4}+1)}}{(\frac{\tau-2}{4}+1)!} \prod_{j=1}^{\frac{\tau-2}{4}+1} (2j+2) \\ &\neq 0 \end{aligned}$$

for $0 < \tau \equiv 2 \mod 4$. We conclude by Remark 4.3.6 that

$$\text{reg}\ R(\tau,e) = \text{reg}\ \widehat{R(\tau,e)} = \lfloor \tau/4 \rfloor + 1.$$

$\square$

We see from the previous proof that the top nonzero degree of the ring $\widehat{R(\tau,e)}$ is $\lfloor \tau/4 \rfloor + 1$, which was used to establish the regularity of $R(\tau,e)$. For an alternate proof of the regularity of $R(\tau,e)$ which uses different machinery and more graph-theoretic properties, see the end of this section.

In the following, we first compute the lengths of the dimension zero rings $\widehat{R(\tau,e)}$, and then show a closed form for the multiplicity of our original rings $R(\tau,e)$ by using a Fibonacci relationship between the lengths of the rings $\widehat{R(\tau,e)}$ and applying Binet's formula for $F(n)$, the nth number in the Fibonacci sequence:

$$F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

In the theorem and corollaries which follow, we suppress e for convenience, since the statements are independent of e. We warn the reader that e will also be used for multiplicity in the corollaries; we trust that context and placement will minimize confusion.

Proposition 4.3.8. *The lengths of the rings $R(\tau)$ satisfy the recursive formula*

$$\ell(\widehat{R(\tau)}) = \ell(\widehat{R(\tau-2)}) + \ell(\widehat{R(\tau-4)})$$

for $\tau \geq 4$. Consequently, if $F(n)$ is the Fibonacci sequence, with $F(0)=0$ and $F(1)=1$, then

$$\ell(\widehat{R(\tau)}) = F\left(\frac{\tau}{2}+3\right) = \frac{(1+\sqrt{5})^{\frac{\tau}{2}+3} - (1-\sqrt{5})^{\frac{\tau}{2}+3}}{2^{\frac{\tau}{2}+3}\sqrt{5}}.$$

Proof. Again, we use Notation 4.3.2. By our iterative formula, since $d_{\tau,0} = 1$ in general, and since $d_{\tau,n} = 0$ in general for $n > \lfloor \tau/4 \rfloor + 1$ by Theorem 4.3.4, we have

$$\begin{aligned}
\ell(\widehat{R(\tau)}) &= \sum_{n=0}^{\lfloor \tau/4 \rfloor+1} d_{\tau,n} \\
&= d_{\tau,0} + \sum_{n=1}^{\lfloor \tau/4 \rfloor+1} d_{\tau,n} \\
&= d_{\tau-2,0} + \sum_{n=1}^{\lfloor \tau/4 \rfloor+1} (d_{\tau-2,n} + d_{\tau-4,n-1}) \\
&= \sum_{n=0}^{\lfloor \tau/4 \rfloor+1} d_{\tau-2,n} + \sum_{n=0}^{\lfloor \tau/4 \rfloor} d_{\tau-4,n} \\
&= \sum_{n=0}^{\lfloor \tau/4 \rfloor+1} d_{\tau-2,n} + \sum_{n=0}^{\lfloor (\tau-4)/4 \rfloor+1} d_{\tau-4,n} \\
&= \ell(\widehat{R(\tau-2)}) + \ell(\widehat{R(\tau-4)}),
\end{aligned}$$

where the third equality follows from Lemma 4.3.3.

Now we will show the second statement. For our base cases, we see from Remark 4.3.5 that $\ell(\widehat{R(0)}) = F(3) = F(\frac{0}{2} + 3)$ and that $\ell(\widehat{R(2)}) = F(4) = F(\frac{2}{2} + 3)$.

Now suppose that $\ell(\widehat{R(\tau-2)}) = F\left(\frac{\tau-2}{2} + 3\right)$ and $\ell(\widehat{R(\tau-4)}) = F\left(\frac{\tau-4}{2} + 3\right)$. Then we have

$$\begin{aligned}
\ell(\widehat{R(\tau)}) &= \ell(\widehat{R(\tau-2)}) + \ell(\widehat{R(\tau-4)}) \\
&= F\left(\frac{\tau-2}{2} + 3\right) + F\left(\frac{\tau-4}{2} + 3\right) \\
&= F\left(\frac{\tau}{2} + 2\right) + F\left(\frac{\tau}{2} + 1\right) \\
&= F\left(\frac{\tau}{2} + 3\right),
\end{aligned}$$

as desired. The closed form for $\ell(\widehat{R(\tau)})$ follows directly from Binet's formula for the Fibonacci sequence. □

Corollary 4.3.9. *For even $\tau \geq 4$, there is an equality of multiplicities $e(\widehat{R(\tau)}) = e(\widehat{R(\tau-2)}) + e(\widehat{R(\tau-4)})$.*

Proof. We have established the length of the Artinian rings $\widehat{R(\tau)}$, and hence the multiplicity $e(\widehat{R(\tau)})$. $\square$

Corollary 4.3.10. *For even $\tau \geq 4$, there is an equality of multiplicities*

$$e(R(\tau)) = e(R(\tau-2)) + e(R(\tau-4)).$$

In particular,

$$e(R(\tau)) = F\left(\frac{\tau}{2}+3\right) = \frac{(1+\sqrt{5})^{\frac{\tau}{2}+3} - (1-\sqrt{5})^{\frac{\tau}{2}+3}}{2^{\frac{\tau}{2}+3}\sqrt{5}}.$$

Proof. To obtain the multiplicity of $R(\tau)$, we look at $\widehat{R(\tau)} = R(\tau)/(\overline{X_\tau})$, which by Proposition 4.2.8 and Corollary 4.2.9 is the Artinian quotient of $R(\tau)$ by a linear regular sequence. By a standard result, we may calculate length along the obvious short exact sequences coming from multiplication by elements of our regular sequence to obtain the equality

$$\mathrm{Hilb}_{R(\tau)}(t)(1-t)^d = \mathrm{Hilb}_{\widehat{R(\tau)}}(t),$$

where d is the Krull dimension of $R(\tau)$. We refer the reader to Definition 2.1.12 for the definition of multiplicity we use in this dissertation. It follows immediately that

$$e(R(\tau)) = \mathrm{Hilb}_{R(\tau)}(t)(1-t)^d\big|_{t=1} = \mathrm{Hilb}_{\widehat{R(\tau)}}(1) = \ell(\widehat{R(\tau)}).$$

We are done by Proposition 4.3.8. $\square$

We reintroduce e and spend the remainder of this section providing an alternate graph-theoretic proof for the regularity $\operatorname{reg} \widehat{R(\tau, e)}$.

Alternate proof of Corollary 4.3.7. We first show that

$$\operatorname{reg} I_{G^e_\tau} \leq \lfloor \tau/4 \rfloor + 2.$$

By Proposition 3.1.12, we recall that the graph G^e_τ is chordal bipartite with vertex bipartition $V_1 \cup V_2$ of cardinalities

$$\begin{aligned} |V_1| &= \left\lfloor \frac{\tau}{4} \right\rfloor + 2 \\ |V_2| &= \left\lceil \frac{\tau}{4} \right\rceil + 2, \end{aligned}$$

and that G^e_τ does not have any vertices of degree 1. Then by Theorem 4.9 of [BOVT17], we have

$$\operatorname{reg} I_{G^e_\tau} \leq \min\left\{ \left\lfloor \frac{\tau}{4} \right\rfloor + 2, \left\lceil \frac{\tau}{4} \right\rceil + 2 \right\} = \left\lfloor \frac{\tau}{4} \right\rfloor + 2.$$

We now show that $\operatorname{reg} I_{G^e_\tau} \geq \lfloor \tau/4 \rfloor + 2$. Since $I_{G^e_\tau}$ is homogeneous and $\operatorname{in}_> I_{G^e_\tau}$ consists of squarefree monomials, Corollary 2.7 of [CV18] states that $\operatorname{reg} \operatorname{in}_> I_{G^e_\tau} = \operatorname{reg} I_{G^e_\tau}$, so it suffices to prove that $\operatorname{reg} \operatorname{in}_> I_{G^e_\tau} \geq \lfloor \tau/4 \rfloor + 2$. The ideal $\operatorname{in}_> I_{G^e_\tau}$ can be viewed as the edge ideal of a simple graph, a "comb" with $\tau/2 + 1$ tines, with consecutive odd variables corresponding to vertices along the spine, as pictured below:

$$\begin{tikzcd}[arrows=dash]
x_2 \arrow[d] & x_4 \arrow[d] & x_6 \arrow[d] & x_8 \arrow[d] & x_{10} \arrow[d] & x_\tau \arrow[d] & x_{\tau+2} \arrow[d] \\
x_3 \arrow[r] & x_5 \arrow[r] & x_7 \arrow[r] & x_9 \arrow[r] & x_{11} \cdots & x_{\tau+1} \arrow[r] & x_{\tau+3}
\end{tikzcd}$$

We know from Theorem 6.5 of [HVT08] that the regularity of an edge ideal is bounded below by the number of edges in any induced matching plus one, so we choose $\lfloor \tau/4 \rfloor + 1$ edges (tines) corresponding to certain odd variables that create an induced matching. By beginning with the x_3-tine and choosing every other tine corresponding to the variables

$$x_3, x_{3+4(1)}, \ldots, x_{3+4(\lfloor \tau/4 \rfloor)},$$

we obtain $\lfloor \tau/4 \rfloor + 1$ edges that are an induced matching, so we have

$$\operatorname{reg} \operatorname{in}_> I_{G^e_\tau} \geq \lfloor \tau/4 \rfloor + 2,$$

as desired.

We conclude that $\operatorname{reg} I_{G^e_\tau} = \lfloor \tau/4 \rfloor + 2$, and hence that $\operatorname{reg} R(\tau, e) = \lfloor \tau/4 \rfloor + 1$. $\square$

5 DG-Algebra Resolution of k

In the local setting, a minimal Tate resolution, or acyclic closure, of the residue field k is constructed abstractly and is generally difficult to find explicitly, but has advantages when it is explicit, in that it is useful for constructing the cotangent complex, which plays a central role in derived algebraic geometry in characteristic zero. In the graded setting, the dual of the Priddy complex is more concrete and provides a minimal graded resolution of k over a Koszul k-algebra, but does not have the same connections to derived algebraic geometry. This can be localized to provide a minimal resolution of k over the local ring. Since these two complexes are minimal, they are known to be isomorphic, so are both dg-algebra resolutions of k over the local ring. Our goal is to find the Tate resolution by identifying elements inside the dual of the Priddy complex that correspond under such an isomorphism to variables in a minimal Tate resolution, in hopes that their explicit nature might be useful for analysis and future computations, such as the construction of the cotangent complex.

To do this, we compare two minimal resolutions of k over the localization $R(\tau, e)_{\mathfrak{m}}$ of the rings $R(\tau, e)$, an acyclic closure and the localization of the dual of the Priddy complex, which is a resolution since $R(\tau, e)$ is Koszul (Corollary 4.1.4) via the algebra

$$\mathrm{Ext}_{R(\tau,e)}(k,k) \cong \mathrm{Ext}_{R(\tau,e)_{\mathfrak{m}}}(k,k).$$

We use known isomorphisms to identify $R(\tau, e)^{\perp}$, constructed in the formation of the Priddy complex, with the universal enveloping algebra $\mathrm{Ext}_{R(\tau,e)}(k,k)$ of the homotopy Lie algebra $\pi(R(\tau, e)_{\mathfrak{m}})$ of $R(\tau, e)_{\mathfrak{m}}$, which we call $\pi(\tau, e)$. A basis of $\pi(\tau, e)$ is known to correspond to the Tate variables in a minimal resolution of k over $R(\tau, e)_{\mathfrak{m}}$. We explicitly construct such a basis.

In Section 5.1, we explain the connection between the Tate resolution and the Priddy complex by connecting them through isomorphisms to $\mathrm{Ext}_{R(\tau,e)}(k,k) = U(\pi(\tau,e))$. We make the connection explicit by recalling how $\pi(\tau,e)$ corresponds to the vector space generated by the Tate variables within the second set of isomorphisms in Section 5.1. In Section 5.2, we explicitly compute the Koszul dual $R(\tau,e)^{\perp}$ and the Priddy complex for $R(\tau,e)$. In Section 5.3, we define a Lie algebra $L(\tau,e)$ in $R(\tau,e)^{\perp}$ and show in Remark 5.3.1 that it is isomorpic to $\pi(\tau,e)$. In Section 5.4, we provide the necessary background for Section 5.5, where we find an explicit basis for $L(\tau,e)$ which corresponds to the Tate variables in an acyclic closure of k over $R(\tau,e)_{\mathfrak{m}}$, in Corollary 5.5.2. Throughout this section, char $k = 0$, and we order monomials in the dual associative algebra $k\langle X^* \rangle = k\langle z_0, z_2, z_3, \ldots, z_{\tau+4} \rangle$ using the lexicographic ordering with $z_0 > z_2 > z_3 > \cdots > z_{\tau+4}$.

5.1 Isomorphisms

In this section, we recall the isomorphisms between $\mathrm{Ext}_{R(\tau,e)}(k,k)$ and $R(\tau,e)^{\perp}$ as algebras (see for example [Eis89], Exercise 17.22.f), then between $\mathrm{Ext}_{R(\tau,e)}(k,k)$ and the k-vector space generated by all monomials in the Tate variables. The associative algebra $\mathrm{Ext}_{R(\tau,e)}(k,k)$ is known to be the universal enveloping algebra of $\pi(\tau,e)$, the homotopy Lie algebra of $R(\tau,e)_{\mathfrak{m}}$. For ease of notation, we replace $R(\tau,e)$ with R and $\pi(\tau,e)$ with $\pi(R_{\mathfrak{m}})$ throughout, where $\mathfrak{m}$ is the homogeneous maximal ideal of R.

We recall for the reader the notion of a universal enveloping algebra for a Lie algebra. For a Lie algebra L, there is a unique universal enveloping algebra $U(L)$, given by allowing associative multiplication between all elements of L and identifying $[a,b]$ with $ab - (-1)^{|a||b|}ba$ in $U(L)$. Going from $U(L)$ to L is more subtle; we describe it in particular for $U(L) = Ext_R(k,k)$ and $L = \pi(R_{\mathfrak{m}})$ in the third set of isomorphisms below. For further exposition, see [Avr98, Ch 10]. The Lie algebra L is embedded in $U(L)$ by the Poincaré-Birkhoff-Witt Theorem; see, for example, [Bre14, Th 7.1].

We use explicit isomorphisms from [Avr98] to demonstrate the known fact that the homotopy Lie algebra $\pi(R_{\mathfrak{m}})$ is isomorphic as a vector space to the vector space generated over k by the Tate variables in a natural way.

Let $P = R \otimes_k R^{\perp}$ be the Priddy complex of R; for background on Koszul algebras and the Priddy complex, see Section 2.4. Since R is Koszul (4.1.4), the dual of the Priddy complex, P^{*_R}, is a minimal resolution of k over R, the R-basis of which is given by the k-basis of the quadratic dual algebra $R^{\perp}$; see, for example, [Eis89, Ex 17.22]. Then we have the following known string of algebra isomorphisms between the associative algebras $\mathrm{Ext}_R(k,k)$ and $R^{\perp}$, since there is an algebra structure on $P = R \otimes_k R^{\perp}$, or equivalently, since there is a coalgebra structure on P^{*_R}. Then we can understand $\mathrm{Ext}_R(k,k)$ by investigating $R^{\perp}$, and will spend most of this chapter working with $R^{\perp}$. We have

$$\begin{aligned}
\mathrm{Ext}_R^{\bullet}(k,k) &\cong H(\mathrm{Hom}_R(P_{\bullet}^{*_R},k)) \\
&\cong \mathrm{Hom}_R(P_{\bullet}^{*_R},k) \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(P_{\bullet},R),k) \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R \otimes_k R_{\bullet}^{\perp},R),k) \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_k(R_{\bullet}^{\perp},\mathrm{Hom}_R(R,R)),k) \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_k(R_{\bullet}^{\perp},R),k) \\
&\cong \mathrm{Hom}_R(R \otimes_k \mathrm{Hom}_k(R_{\bullet}^{\perp},k),k) \\
&\cong \mathrm{Hom}_R(R \otimes_k (R_{\bullet}^{\perp})^{*_k},k) \\
&\cong \mathrm{Hom}_k((R^{\perp})_{\bullet}^{*_k},\mathrm{Hom}_R(R,k)) \\
&\cong \mathrm{Hom}_k((R^{\perp})_{\bullet}^{*_k},k) \\
&\cong R_{\bullet}^{\perp},
\end{aligned}$$

where the seventh isomorphism comes from the fact that R is flat over k and that $R_{\bullet}^{\perp}$ is finitely generated as a k-module. Since a minimal Tate resolution $R_{\mathfrak{m}}\langle Y\rangle$ of k over $R_{\mathfrak{m}}$ with set of Tate variables Y is also a minimal resolution of k over $R_{\mathfrak{m}}$, we have

$$\begin{aligned}
R_\bullet^\perp &\cong \operatorname{Ext}_R^\bullet(k,k) \\
&\cong H(\operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\langle Y\rangle_\bullet, k)) \\
&\cong \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\langle Y\rangle_\bullet, k) \\
&\cong \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\otimes_k k\langle Y\rangle_\bullet, k) \\
&\cong \operatorname{Hom}_k(k\langle Y\rangle_\bullet, \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}, k)) \\
&\cong \operatorname{Hom}_k(k\langle Y\rangle_\bullet, k) \\
&\cong k\langle Y\rangle_\bullet^{*_k},
\end{aligned}$$

so that $R^\perp$ is isomorphic as a k-vector space to that generated by all monomials in Y, the set of Tate variables. We note that these isomorphisms are not algebra isomorphisms, since there is not an algebra structure on $k\langle Y\rangle^{*_k}$.

Within the above set of isomorphisms, $\pi(R_{\mathfrak{m}})$ is isomorphic as a vector space to that generated by the Tate variables. For a reference, we use the proof of Theorem 10.2.1 parts (1) and (2) in [Avr98]; we have

$$\begin{aligned}
\pi(R_{\mathfrak{m}})_\bullet &= H\operatorname{Der}^\gamma_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\langle Y\rangle_\bullet, R_{\mathfrak{m}}\langle Y\rangle) \\
&\cong H\operatorname{Der}^\gamma_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\langle Y\rangle_\bullet, k) \\
&= H\operatorname{Der}^\gamma_{k}(k\langle Y\rangle_\bullet, k) \\
&= \operatorname{Der}^\gamma_{k}(k\langle Y\rangle_\bullet, k) \\
&\cong \operatorname{Hom}_k(kY_\bullet, k) \\
&= (kY)_\bullet^{*_k},
\end{aligned}$$

where $\mathrm{Der}^{\gamma}_S(S\langle Y\rangle, B)$, the set of S-linear Γ-derivations from the S-algebra $S\langle Y\rangle$ to the $S\langle Y\rangle^{\sharp}$-module B, naturally sits inside $\mathrm{Hom}_S(S\langle Y\rangle, B)$, so that this set of isomorphisms sits naturally inside the previous set. For more information on Γ-derivations, see [Avr98], Remark 6.2.2.

Remark 5.1.1. We are free to make whatever choice we wish for the acyclic closure $R_{\mathfrak{m}}\langle Y\rangle$ in the above sets of isomorphisms. We exploit this in Remark 5.3.1 to show that the Lie algebra $L(\tau, e)$ that we will define is isomorphic to the homotopy Lie algebra $\pi(\tau, e)$ of $R(\tau, e)$.

We see then that the duals of the Tate variables are identified with a vector space basis of $\pi(R_{\mathfrak{m}})$ inside $R^{\perp}$. Dualizing, the Tate variables correspond to the dual of this basis for $\pi(R_{\mathfrak{m}})$, which sits inside of $(R^{\perp})^{*_k}$. In this way, we identify within the Priddy complex elements corresponding to the Tate variables in a minimal Tate resolution.

5.2 The Priddy Complex

In this section, we construct the Priddy complex $P = R(\tau, e) \otimes_k R(\tau, e)^{\perp}$ for $R(\tau, e)$. We first construct $R(\tau, e)^{\perp}$ and then state the minimal resolution of k over $R(\tau, e)$ obtained from it. For background on Koszul algebras and the Priddy complex, see Section 2.4.

5.2.1 Construction of the Quadratic Dual Algebra

We recall for the reader from Remark 3.2.3 that for $2 \leq i \leq \tau$,

$$\begin{aligned} t_1 &= x_2x_3 - x_{j_1}x_4 \\ t_i &= x_{i+1}x_{i+3} - x_{j_i}x_{i+4} \\ t_{i+1} &= x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4} \end{aligned}$$

are the standard generators for the ideal $I_{G_\tau^e}$.

We rewrite the quadratic algebra $R(\tau, e) = \frac{k[X]}{I_{G_\tau^e}}$ as a quotient of the noncommutative polynomial ring $k\langle X\rangle$ in the set of variables $X = \{x_0, x_2, \ldots, x_{\tau+4}\}$, so that we may define $R(\tau, e)^\perp$. Let $Q_{\tau,e} \subset k\langle X\rangle$ be defined by

$$Q_{\tau,e} = \{ab - ba, dg - ef \mid \text{for } a, b, d, e, f, g \in X \text{ as below}\},$$

where $a > b$ and $d > e > f > g$ with $d^*g^* - e^*f^* = t_n$ for some $1 \leq n \leq \tau + 1$. These elements are linearly independent. Indeed, beginning with a basis $\{ab, ba \mid a > b\}$ for $k\langle X\rangle$, we easily obtain a basis $\{ab, ab - ba, dg - ef, dg - gd \mid a > b, ab \neq dg\}$, of which $Q_{\tau,e}$ is a subset. Then we have

$$R(\tau, e) = \frac{k[X]}{I_{G_\tau^e}} = \frac{k\langle X\rangle}{\langle Q_{\tau,e}\rangle}.$$

and

$$|Q_{\tau,e}| = \binom{\tau+4}{2} + (\tau + 1).$$

We are now going to find a basis $Q_{\tau,e}^\perp \subset k\langle X^*\rangle$ for the perpendicular subspace $(Q_\tau)^\perp$ to define the quadratic dual algebra $R(\tau, e)^\perp$. Let Ω_τ be defined as follows:

$$\Omega_\tau = \{ab + ba, c^2, dg + gd + ef + fe \mid a, b, c, d, e, f, g \in X^* \text{ are as below}\}$$

where

- $a > b$ and $d > e > f > g$,
- $ab \neq dg$ and $ab \neq ef$ for any $t_n = d^*g^* - e^*f^*$, and
- $d^*g^* - e^*f^* = t_n$ for some $1 \leq n \leq \tau + 1$.

Then all of the elements of Ω_τ are orthogonal to all of the elements of $Q_{\tau,e}$, and so we have a containment of subspaces $(\Omega_\tau) \subseteq (Q_{\tau,e})^\perp$, and we now argue by cardinality to show they are equal, since both sets consist of linearly independent elements. We have

$$|\Omega_\tau| = \left(\binom{\tau+4}{2} - 2(\tau+1)\right) + (\tau+4) + (\tau+1).$$

$$\dim_k(k\langle X\rangle)_2 = (\tau+4)^2$$

with

$$\begin{aligned}
|Q_{\tau,e}| + |\Omega_\tau| &= \binom{\tau+4}{2} + (\tau+1) + \left(\binom{\tau+4}{2} - 2(\tau+1)\right) + (\tau+4) + (\tau+1), \\
&= 2\binom{\tau+4}{2} + (\tau+4) \\
&= (\tau+4)(\tau+3) + (\tau+4) \\
&= (\tau+4)^2 \\
&= \dim_k(k\langle X\rangle)_2.
\end{aligned}$$

Since $(\Omega_\tau) \subseteq (Q_{\tau,e})^\perp$, we conclude that $(\Omega_\tau) = (Q_{\tau,e})^\perp$, so that Ω_τ may be taken to be $Q^\perp_{\tau,e}$, a basis for the perpendicular subspace $(Q_{\tau,e})^\perp$. Then

$$R(\tau,e)^\perp = \frac{k\langle X^*\rangle}{\langle Q^\perp_{\tau,e}\rangle},$$

where $Q^\perp_{\tau,e} = \Omega_\tau$ above.

Example 5.2.1. We show a basis $Q^\perp_{0,0}$ for the defining ideal of $R(0,(0))^\perp$, the quadratic dual of

$$R(0,(0)) = \frac{k[x_0, x_2, x_3, x_4]}{(x_2x_3 - x_0x_4)}.$$

Then by the above construction, $Q_{0,0}$ contains the elements

$$x_0x_2 - x_2x_0, x_0x_3 - x_3x_0, x_0x_4 - x_4x_0, x_2x_3 - x_3x_2, x_2x_4 - x_4x_2, x_3x_4 - x_4x_3, x_2x_3 - x_0x_4$$

and $Q_{0,0}^{\perp}$ contains the elements

$$z_0z_2 + z_2z_0, z_0z_3 + z_3z_0, z_2z_4 + z_4z_2, z_3z_4 + z_4z_3, z_0^2, z_2^2, z_3^2, z_4^2, z_0z_4 + z_4z_0 + z_2z_3 + z_3z_2.$$

We return to this ring in Example 5.5.4.

5.2.2 Minimal Resolution

We replace $R(\tau, e)$ with R in the display for simplicity, and let $x_n^* = z_n$ for $n = 0$ and $2 \leq n \leq \tau + 4$. Then the Priddy complex $P = R(\tau, e) \otimes_k R(\tau, e)^{\perp}$ is

$$P = R \to R \otimes_k kX^* \to R \otimes_k R_2^{\perp} \to \cdots \to R \otimes_k R_d^{\perp} \to \cdots$$

where the maps are multiplication on the right by

$$t = \sum_n x_n \otimes z_n.$$

Since R is Koszul, the R-dual of this complex is a minimal resolution of k over R.

5.3 Explicit Correspondence with Tate Variables

We now develop an explicit correspondence with the Tate variables by using what is known about Lie algebras and their universal enveloping algebras in this setting.

We first note that there is a natural Lie algebra L embedded in $R(\tau, e)^{\perp} = k\langle X^* \rangle / \langle Q_{\tau,e}^{\perp} \rangle$ since the generators of $Q_{\tau,e}^{\perp}$ from Section 5.2.1 may all be written as elements in $\mathrm{Lie}(X^*)$, where we recall that the Lie bracket $[u, v]$ in $\mathrm{Lie}(X^*)$ embeds as $uv - (-1)^{|u||v|}vu$ (still denoted by $[u, v]$) in the associative algebra $k\langle X^*\rangle$ and that char $k = 0$. Then we have

$$\begin{aligned} Q^{\perp}_{\tau,e} &= \{ab + ba, c^2, dg + gd + ef + fe \mid a, b, c, d, e, f, g \in X^* \text{ are as below}\} \\ &= \{[a, b], \frac{1}{2}[c, c], [d, g] + [e, f] \mid a, b, c, d, e, f, g \in X^* \text{ are as below}\} \end{aligned}$$

where

- $a > b$ and $d > e > f > g$,
- $ab \neq dg$ and $ab \neq ef$ for any $t_n = d^*g^* - e^*f^*$, and
- $d^*g^* - e^*f^* = t_n$ for some $1 \leq n \leq \tau + 1$.

It is easy to check that these bracket generators are written in terms of the basis for $\mathrm{Lie}(X^*)$ of super-Lyndon-Shirshov monomials given in Remark 5.4.7. The bracket version of $Q^{\perp}_{\tau,e}$ then generates an ideal of $\mathrm{Lie}(X^*)$, which we denote by $\overline{\langle Q^{\perp}_{\tau,e}\rangle}$. We see that the Lie algebra

$$L(\tau, e) = \frac{\mathrm{Lie}(X^*)}{\overline{\langle Q^{\perp}_{\tau,e}\rangle}},$$

where $\mathrm{Lie}(X^*)$ is the free Lie algebra generated by the set X^*, is naturally embedded in the associative algebra

$$R(\tau, e)^{\perp} = \frac{k\langle X^*\rangle}{\langle Q^{\perp}_{\tau,e}\rangle}.$$

In fact, it is known that with this setup, $R(\tau, e)^{\perp}$ is the universal enveloping algebra of $L(\tau, e)$; see for example Theorem 2.8 of [BKLM99]. We show in the following remark that a dual basis for $L(\tau, e)$ can be taken to be the Tate variables in an acyclic closure of $R(\tau, e)$, and via the isomorphisms in Section 5.1 and Remark 5.1.1, that $L(\tau, e)$ is the homotopy Lie algebra of $R(\tau, e)_{\mathfrak{m}}$.

Remark 5.3.1. We show that $L(\tau,e)$ is isomorphic to the homotopy Lie algebra $\pi(\tau,e)$ of $R(\tau,e)_{\mathfrak{m}}$. That is, we show that a dual basis for $L(\tau,e)$ may be taken to be the Tate variables in an acyclic closure, and by Remark 5.1.1 and the preceding isomorphisms, this shows that $L(\tau,e)$ is indeed the homotopy Lie algebra $\pi(\tau,e)$.

We know that as the universal enveloping algebra of a Lie algebra,

$$R(\tau,e)^{\perp} = (U(L(\tau,e))$$

is known to be a Hopf algebra, so that its dual $U(L(\tau,e))^{*_k}$ has an algebra structure. Let Y be a dual k-vector space basis for $L(\tau,e)$. By a dual version of the Poincaré-Birkhoff-Witt Theorem [Blo85, Th 4.9, Prop 4.10], we have an isomorphism of algebras

$$U(L(\tau,e))^{*_k} \cong k\langle Y\rangle,$$

where $k\langle Y\rangle$ is the free Γ-algebra on Y. We now show that Y may be taken to be the Tate variables in an acyclic closure of $R(\tau,e)_{\mathfrak{m}}$, where we replace $R(\tau,e)$ with R and $L(\tau,e)$ with L below for simplicity. Let P be the Priddy complex of R. We have

$$\begin{aligned} P^{*_R} &\cong R\otimes(R^{\perp})^{*_k} \\ &\cong R\otimes(U(L))^{*_k} \\ &\cong R\otimes k\langle Y\rangle \\ &\cong R\langle Y\rangle. \end{aligned}$$

By Lemma 13 of [Bri18], the free product on $R\langle Y\rangle$ is compatible with the differential on the dual Priddy complex under these isomorphisms, making this an isomorphism of dg-algebras. Then the localization of $R\langle Y\rangle$ at $\mathfrak{m}$ is in fact an acyclic closure of $R_{\mathfrak{m}}$, since P^{*_R} is a minimal graded resolution.

Noting Remark 5.1.1, we see that with the choice of Tate resolution $R_{\mathfrak{m}}\langle Y\rangle$ as above, we have $L(\tau,e) = \pi(\tau,e)$, the homotopy Lie algebra of $R(\tau,e)_{\mathfrak{m}}$. These isomorphisms also show that the Lie basis (which we determine in Corollary 5.5.2), sits very naturally inside the dual of the Priddy complex given in Section 5.2.2. In this way, when we localize, Corollary 5.5.2 gives an explicit Tate resolution.

5.4 Necessary Background for Lie Basis

We use a Gröbner-Shirshov basis (generating set) for the defining ideal of $L(\tau,e)$ to develop an explicit k-vector space basis for $L(\tau,e)$. The following definitions and notes provide the background necessary to understand Gröbner-Shirshov bases and one method that may be used to verify such a basis, which we use in Section 5.5. We particularly refer the reader to [Oha], which we used extensively when compiling this information, and where much of the following may be found. We explain what a Gröbner-Shirshov basis of a defining ideal of an associative algebra is, but do not go into detail for what a Gröbner-Shirshov basis of a defining ideal of a Lie algebra is, as we do not need it in our treatment. For more elementary background, see Section 2.5, and for further details, see [BKLM99].

Definition 5.4.1. Let G be a generating set for an ideal J in $\text{Lie}(X)$ and c a monomial in $k\langle X\rangle$. Then we say $f \equiv 0 \mod (G,c)$ if $f - \sum_{i=1}^{n} \alpha_i a_i g_i b_i = 0$ and $LT(a_i g_i b_i) < c$ for each i, where $g_i \in G$, where a_i, b_i, c are monomials in $k\langle X\rangle$, and where $\alpha_i \in k$.

There are two kinds of composition which apply to $g_i, g_j \in G$ when certain conditions are met. We note that c is unique for some (i,j)-pairs, but that others may have multiple possibilities for c (for example, $g_i = yx^2$ and $g_j = x^2y$ could have $c = yx^2y$ or $c = yx^3y$ in the first type of composition below).

Definition 5.4.2. If

$$LT(g_i)a = bLT(g_j) = c$$

for some monomials a, b in $k\langle X\rangle$ with $|LT(g_i)| > |b|$, then the *composition of intersection* $(g_i, g_j)_c$ is $g_i a - b g_j$. That is, c characterizes an overlap c' between g_i and g_j such that $bc'a = c$.

If

$$LT(g_i) = aLT(g_j)b = c$$

for some monomials a, b, then the *composition of inclusion* $(g_i, g_j)_c$ is $g_i - a g_j b$.

In other cases, these compositions are not defined.

We note that in both cases, $LT(g_i, g_j)_c < c$

Definition 5.4.3. We say that $G = \{g_1, \ldots, g_m\}$ is closed under associative composition if $(g_i, g_j)_c \equiv 0 \mod (G, c)$ for every i, j, and c.

Definition 5.4.4. A *Gröbner-Shirshov basis* for an ideal I in $k\langle X\rangle$ generated by elements in $\mathrm{Lie}(X)$ is a generating set $G = \{g_1, \ldots, g_m\}$ for I that is closed under associative composition.

Such a generating set will give us particular results about the algebra $\mathrm{Lie}(X)/J$, where J is the Lie ideal generated by G, but before stating them it would be prudent to define a particular k-vector space basis for the free Lie algebra $\mathrm{Lie}(X)$ and the notion of an G-reduced (Lie) monomial.

Definition 5.4.5. We define below a basis for $\mathrm{Lie}(X)$ consisting of Lie monomials coming from *nonassociative monomials of* $k\langle X\rangle$, that is, noncommutative monomials in $k\langle X\rangle$ that are grouped pairwise. For example, $a(bc) \neq (ab)c$ as nonassociative monomials for variables a, b, and c. We say *$u > v$ or $u = v$ as nonassociative monomials* if this is true in the lexicographic ordering in $k\langle X\rangle$ when we forget the nonassociative structure (remove the parentheses).

Definition 5.4.6 involves an iterative construction of such monomials, and Remark 5.4.7 shows how this creates a basis of Lie monomials with a unique bracket structure.

Definition 5.4.6. A nonassociative monomial w of $k\langle X\rangle$ with $|w| \geq 1$ is a *Lyndon-Shirshov monomial* if $w \in X$, or

- if $w = uv$, then u and v are Lyndon-Shirshov monomials with $u > v$,
- if $w = (xy)v$, then $y \leq v$.

A nonempty, nonassociative monomial w is a *super-Lyndon Shirshov monomial* if w is a Lyndon-Shirshov monomial or if $w = vv$ for a Lyndon-Shirshov monomial v of odd length.

Remark 5.4.7. There is a unique bracket arrangement for each super-Lyndon-Shirshov monomial w which yields a correspondence between super-Lyndon-Shirshov monomials and a basis of Lie monomials for $Lie(X)$, which we call *super-Lyndon-Shirshov Lie monomials*. One forms such a basis for $Lie(X)$ via the following inductive process (see [BMPZ92, Lemma 1.8]):

- If $w \in X$, then w corresponds to the Lie monomial w in $\mathrm{Lie}(X)$.
- If $w = uv$ for $u > v$ Lyndon-Shirshov monomials, where u corresponds to the Lie monomial U and v corresponds to the Lie monomial V, then $w = uv$ corresponds to the Lie monomial $[U, V]$.
- If $w = vv$ for a Lyndon-Shirshov monomial v of odd length, and v corresponds to the Lie monomial V, then w corresponds to the Lie monomial $\frac{1}{2}[V, V]$.

The conditions in the definition of a Lyndon-Shirshov monomial ensure that such bracket arrangements form a basis.

Definition 5.4.8. We say a monomial m in $k\langle X\rangle$ is *G-reduced* if $m \neq aLT(s)b$ for any $g \in G$ and for any monomials $a, b \in k\langle X\rangle$. We say the same for a Lie monomial if it corresponds to an G-reduced monomial via the above correspondence.

We combine results from Theorem 2.8 and 2.10 of [BKLM99] to obtain the following:

Theorem 5.4.9. *If G is a monic generating set for an ideal J of $Lie(X)$ and G is a Gröbner-Shirshov basis for the ideal it generates in $k\langle X\rangle$, then the residue classes of Lie monomials which correspond to G-reduced super-Lyndon-Shirshov monomials form a k-basis for $Lie(X)/J$.*

In their treatment, the authors use the notion of a Gröbner-Shirshov basis for an ideal in the Lie algebra as well as one for an ideal in the universal enveloping algebra. We do not define or explain the Lie version since it is not central to this dissertation.

5.5 Explicit Lie basis

We now go about constructing a basis for $L(\tau, e)$ using the theory of Gröbner-Shirshov bases. By Theorem 5.4.9 from [BKLM99], since $Q^{\perp}_{\tau,e}$ has generators in $\mathrm{Lie}(X^*)$, if it is a Gröbner-Shirshov basis for the ideal it generates in the associative algebra $k\langle X^*\rangle$, then the images of $Q^{\perp}_{\tau,e}$-reduced super-Lyndon-Shirshov Lie monomials form a k-vector space basis for

$$L(\tau, e) = \frac{\mathrm{Lie}(X^*)}{\langle Q^{\perp}_{\tau,e}\rangle},$$

which we know corresponds to the Tate variables in the acyclic closure by the isomorphisms in Section 5.1 and by Remark 5.3.1.

Our strategy, then, is to show that the set $Q^{\perp}_{\tau,e}$ is closed under associative composition, since this will show that it is a Gröbner-Shirshov basis for the ideal $\langle Q^{\perp}_{\tau,e}\rangle$ defining the associative algebra $R(\tau, e)^{\perp}$. In our case, every leading term of the associative version of $Q^{\perp}_{\tau,e}$ has degree two and is distinct, so there are no compositions of inclusion to check; it suffices to check that each composition of intersection is equivalent to zero modulo

$(Q^{\perp}_{\tau,e}, c)$ for every monomial c in $k\langle X^*\rangle$, which we will do in the proof of the theorem below, which follows at the end of this section.

Theorem 5.5.1. *The elements of $Q^{\perp}_{\tau,e}$ are a Gröbner-Shirshov basis for the ideal they generate in $k\langle X^*\rangle$.*

As a corollary, we obtain

Corollary 5.5.2. *The images of the $Q^{\perp}_{\tau,e}$-reduced super-Lyndon-Shirshov Lie monomials form a basis for*

$$L(\tau) = \frac{Lie(X^*)}{\overline{\langle Q^{\perp}_{\tau,e}\rangle}},$$

the dual of which may be taken to be the set of Tate variables in a minimal Tate resolution of k over $R(\tau, e)_{\mathfrak{m}}$.

Proof. This follows immediately by direct application of Theorem 5.4.9 from [BKLM99], by the isomorphisms in Section 5.1, and by Remark 5.3.1. □

Applying the noncommutative version of Macaulay's Basis Theorem (see, for example, Proposition 14.1.11 in [BFKR15]), we also obtain a basis for the universal enveloping algebra $R^{\perp}_{\tau}$.

Corollary 5.5.3. *The images of elements of $k\langle X^*\rangle$ that are $Q^{\perp}_{\tau,e}$-reduced super-Lyndon-Shirshov monomials form a k-basis for $R(\tau, e)^{\perp}$.*

Example 5.5.4. We find explicit Tate variables within the dual Priddy complex for the acyclic closure of k over the ring

$$R(2,e) = \frac{k[x_0, x_2, x_3, x_4, x_5, x_6]}{(x_2x_3 - x_0x_4, x_3x_5 - x_0x_6, x_4x_5 - x_2x_6)}.$$

Using bracket notation $[a,b] = ab - (-1)^{|a||b|}ba$, the set $Q^{\perp}_{2,e}$ consists of the following elements:

$$\begin{aligned}
&[z_0,z_0], [z_0,z_2], [z_0,z_3], [z_0,z_5], [z_2,z_2], [z_2,z_4], [z_2,z_5], [z_3,z_3]\\
&[z_3,z_4], [z_3,z_6], [z_4,z_4], [z_4,z_6], [z_5,z_5], [z_5,z_6], [z_6,z_6]\\
&[z_0,z_4] + [z_2,z_3], [z_0,z_6] + [z_3,z_5], [z_2,z_6] + [z_4,z_5].
\end{aligned}$$

We then obtain the $Q^{\perp}_{2,e}$-reduced super-Lyndon-Shirshov Lie monomials

$$\begin{aligned}
&z_0, z_2, z_3, z_4, z_5, z_6\\
&[z_2,z_3], [z_3,z_5], [z_4,z_5]\\
&[z_2,[z_3,z_5]], [[z_3,z_5],z_4]\\
&[z_2[z_3,z_5]],z_3], [z_2,[[z_3,z_5],z_4], [[z_3,z_5],[z_4,z_5]]\\
&\vdots
\end{aligned}$$

which form a basis for $L(2,e)$. The dual basis corresponds to the Tate variables in an acyclic closure by Corollary 5.5.2.

We now note some things that are useful in the proof of Theorem 5.5.1, where we show that the set $Q^{\perp}_{\tau}$ is a Gröbner-Shirshov basis for the ideal it generates in $k\langle X^*\rangle$. We carefully examine the elements of $Q^{\perp}_{\tau}$ by relating them to previously stated information about the t_n and the indices j_n which appear in the t_n.

We recall for easy reference information about $Q^{\perp}_{\tau,e}$ in the following remark.

Remark 5.5.5. We recall that

$$\begin{aligned} Q^{\perp}_{\tau,e} &= \{ab+ba, c^2, dg+gd+ef+fe \mid a,b,c,d,e,f,g \in X^* \text{ are as below}\} \\ &= \{[a,b], \frac{1}{2}[c,c], [d,g]+[e,f] \mid a,b,c,d,e,f,g \in X^* \text{ are as below}\} \end{aligned}$$

where

- $a > b$ and $d > e > f > g$,
- $ab \neq dg$ and $ab \neq ef$ for any $t_n = d^*g^* - e^*f^*$, and
- $d^*g^* - e^*f^* = t_n$ for some $1 \leq n \leq \tau + 1$.

We use $z_n = x_n^*$, so that $k\langle X^* \rangle = k\langle z_0, z_2, z_3, \ldots, z_{\tau+4} \rangle$. Then more particularly, since

$$\begin{aligned} t_1 &= x_2x_3 - x_{j_1}x_4 \\ t_i &= x_{i+1}x_{i+3} - x_{j_i}x_{i+4} \\ t_{i+1} &= x_{i+2}x_{i+3} - x_{j_{i+1}}x_{i+4}, \end{aligned}$$

for even i, $2 \leq i \leq \tau$, we see that

$$\{a,b\} \notin \{\{z_{j_i}, z_{i+4}\}, \{z_{i+1}, z_{i+3}\}, \{z_{j_{i+1}}, z_{i+4}\}, \{z_{i+2}, z_{i+3}\}\}$$

and

$$\{d,e,f,g\} \in \{\{z_{j_i}, z_{i+1}, z_{i+3}, z_{i+4}\}, \{z_{j_{i+1}}, z_{i+2}, z_{i+3}, z_{i+4}\}\}.$$

Remark 5.5.6. We sometimes refer to pairs $\{a,b\}$ above as *anticommuting pairs* or say that a anticommutes with b (or vice versa), since the image of $ab+ba$ is 0 in $R(\tau,e)^{\perp}$. For the purpose of the proof, the reader should understand that for $a > b$, $\{a,b\}$ is an anticommuting pair if and only if $[a,b]$ is an element of $Q^{\perp}_{\tau,e}$.

We recall and expand upon properties the j_n below, for use in the proof of Theorem 5.5.1.

Remark 5.5.7 (Remark 3.1.10 recalled). We recall Remark 3.1.10 verbatim here for use in the remarks below. From the proof of Lemma 3.1.8, we note that $j_2 = 0$, $j_3 = 2$, and that for even $i \geq 4$, we have the following:

$$e_{i/2+1} = 0 \iff j_i = j_{i-2} \iff j_{i+1} = i - 1$$

$$e_{i/2+1} = 1 \iff j_i = j_{i-1} \iff j_{i+1} = i.$$

For the sake of later proofs, we extend the notion of j_n naturally to $t_1 = x_2x_3 - x_0x_4$ and say that $j_1 = 0$, and note the following properties of the j_n for $1 \leq n \leq \tau + 1$:

- For even i, $j_i \in \{j_{i-2}, j_{i-1}\}$ and $j_i \leq i - 2$. For $i = 2$, $j_2 = j_1 = 0$, and for $i \geq 4$, this is clear from Proposition 3.1.8 and the above statement, since $e_{i/2+1} \in \{0, 1\}$.

- For even i, we have $j_{i+1} \in \{i - 1, i\}$. For $i = 0$, $j_1 = 0 \in \{-1, 0\}$, and for $i \geq 2$, this follows from Proposition 3.1.8.

- For even i, $j_i < j_{i+1}$. Indeed, $j_i \leq i - 2 < j_{i+1}$.

- The j_{i+1} form an increasing sequence for even i. This is clear since $j_{i+1} \in \{i - 1, i\}$ for $i \geq 0$.

- The j_i form a non-decreasing sequence for even i. Indeed, for $i \geq 4$, either $j_i = j_{i-2}$ or $j_i = j_{i-1} \geq i - 3 > i - 4 \geq j_{i-2}$.

Remark 5.5.8. We note a few anticommutativity properties illuminated by Remark 5.5.5, which will aid in the following proof. We recall for the reader that $\{a, b\}$ is an anticommuting pair (a anticommutes with b) if and only if $[a, b]$ is an element of $Q^{\perp}_{\tau,e}$. See Remark 5.5.6.

- If $m < i+1$ for even $i \geq 2$, then $\{z_m, z_{i+1}\}$ is an anticommuting pair unless we have $m \in \{i-1, i\}$. In particular, $z_{j_{i+1}}$ never anticommutes with z_{i+1}, since $j_{i+1} \in \{i-1, i\}$ by Remark 3.1.10.

- If $m < i$ for even $i \geq 2$, then z_m anticommutes with z_i unless $m \in \{j_{i-4}, j_{i-3}\}$. This is true for $i \geq 4$ and vacuously true for $i = 2$. In particular z_{j_i} never anticommutes with z_{i+2}, since for $i = 2$ we have $z_{j_2} = z_0$ does not anticommute with z_4 and for $i \geq 4$ we have $j_i \in \{j_{i-2}, j_{i-1}\}$ by Remark 3.1.10, neither of which anticommute with z_{i+2}.

- The variable z_{j_i} always anticommutes with $z_{j_{i+1}}$. Indeed, $z_{j_2} = z_0$ anticommutes with $z_{j_3} = z_2$ by the above statement. For $i \geq 4$, if $j_{i+1} = i-1$, then $j_i = j_{i-2} \leq i-4$ by Remark 3.1.10, so that z_{j_i} anticommutes with $z_{i-1} = z_{j_{i+1}}$. On the other hand, if $j_{i+1} = i$, then $j_i = j_{i-1} > j_{i-3} > j_{i-4}$ by the same remark, so that z_{j_i} anticommutes with $z_i = z_{j_{i+1}}$.

- For $[d, g] + [e, f]$ described in Remark 5.5.5, d and g always anticommute with e and f, but neither of the pairs $\{d, g\}$ or $\{e, f\}$ is an anticommuting pair. Indeed, when $d = z_{j_i}$, we know $j_i \leq i-2$, so $j_i \notin \{i-1, i, i+1, i+2\}$. Then $d = z_{j_i}$ anticommutes with $e = z_{i+1}$ and $f = z_{i+3}$. Similarly, when $d = z_{j_{i+1}}$, we know $j_{i+1} \in \{i-1, i\}$ and $j_{i-2}, j_{i-1} < i-1$, so $d = z_{j_{i+1}}$ anticommutes with $e = z_{i+2}$ and $f = z_{i+3}$. Lastly, since $j_i, j_{i+1} < i+1$, $g = z_{i+4}$ anticommutes with $e \in \{z_{i+1}, z_{i+2}\}$ and $f = z_{i+3}$.

Remark 5.5.9. We also note a few further properties of the j_n which will be useful in the following proof.

- If $j_{i+2n+1} \in \{i-1, i\}$, then $n = 0$. This is clear due to Remark 3.1.10.
- If $j_{i+2n} \in \{i-1, i\}$, then $j_{i+1} = j_{i+2}$ and for even subscripts,

$$j_{i+2} = j_{i+4} = \cdots = j_{i+2n}.$$

  Indeed, because $0 = j_1 = j_2$ and for even $m \geq 4$ all j_m are defined recursively in a nondecreasing sequence by Remark 3.1.10, we see that each j_m for even m comes from an earlier (odd-indexed) j_{m-2k+1} for some k. Such a j_{m-2k+1} is unique by the previous statement.

- For $n \geq 2$, if $j_{i+2n} \in \{j_i, j_{i+1}\}$, then

$$j_{i+2} = j_{i+4} = \cdots = j_{i+2n}.$$

  Indeed, if $j_{i+2n} = j_{i+1}$, this follows directly from the previous statement, since we know $j_{i+1} \in \{i-1, i\}$ by Remark 3.1.10. If $j_{i+2n} = j_i$, we have what we need by the nondecreasing nature of the even-indexed j's.

- If $j_{i+2n+1} \in \{j_i, j_{i+1}\}$ for some $n \geq 0$, then $n = 0$. This is clear due to the fact that for $n > 0$, we have $j_i < j_{i+1} < j_{i+2n+1}$ by Remark 3.1.10.
- For $d \in \{z_{j_i}, z_{j_{i+1}}\}$ and $e \in \{z_{i+1}, z_{i+2}\}$ such that $[d, g] + [e, f]$ is one of our generators, we have $d = z_{j_{i+2}}$ if and only if $e = z_{j_{i+3}}$. Indeed, for $i = 0$ we have $d = z_{j_1} = z_{j_2} = 0$ and $e = z_{j_3} = 2$ by Remark 3.1.10. For $i \geq 2$, this follows directly from the same remark, since $z_{j_{i+2}} = z_{j_i}$ if and only if $z_{j_{i+3}} = z_{i+1}$ and since $z_{j_{i+2}} = z_{j_{i+1}}$ if and only if $z_{j_{i+3}} = z_{i+2}$. We have what we need because $d = z_{j_i}$ if and only if $e = z_{i+1}$, and also $d = z_{j_{i+1}}$ if and only if $e = z_{i+2}$.

Proof of Theorem 5.5.1. As stated before, we need only concern ourselves with showing that compositions of intersection between elements of $Q^{\perp}_{\tau,e}$ are equivalent to zero modulo $(Q^{\perp}_{\tau,e}, c)$ for all noncommutative monomials c in $k\langle X^*\rangle = k\langle z_0, z_2, z_3, \ldots, z_{\tau+4}\rangle$. We note that there is no ambiguity for what c is in our case, since our generators overlap nontrivially in only one way, so we choose to use (noncommutative) S-polynomial notation $S_{u,v} := (u,v)_c$ and omit notation for c. We show that for each $S_{u,v}$, we have $\overline{S_{u,v}} := S_{u,v} - \sum a_i q_i b_i = 0$ for $q_i \in Q^{\perp}_{\tau,e}$ and a_i, b_i monomials in $k\langle z_0, z_2, z_3, \ldots, z_{\tau+4}\rangle$. Since we reduce by leading terms of the q_i, in each case the leading term of $a_i q_i b_i$ is less than c, so that we obtain $S_{u,v} \equiv 0 \mod (Q^{\perp}_{\tau,e}, c)$, as desired.

We arrange cases 1-5 by noncommutative S-polynomials arising from pairs of different types of generators for our ideal. We only need to consider pairs whose leading terms l_1 and l_2 have $l_1 u = v l_2$ with $|LT(l_1)| > |v|$. Since the leading terms of our generators all have degree two, we see $|u| = |v| = 1$ in each case. Cases that are split into "a" and "b" are non-symmetric. For the sake of space and readability, we do not do a lead-reduction (though at each step we reduce by leading terms of the q_i) but show nonetheless that each S-polynomial reduces to $\overline{S_{u,v}} = 0$ as stated above, and hence, that the generators given are a Gröbner-Shirshov basis.

Case 1a: We show that $\overline{S_{[a,a],[a,b]}} = 0$, where $a > b$ are variables such that a and b anticommute, so that $[a,a]$ and $[a,b]$ are two of our generators for $Q^{\perp}_{\tau,e}$. We use the generators

$$\begin{aligned} a^2 &= \frac{1}{2}[a,a] \\ ab + ba &= [a,b] \end{aligned}$$

in our reduction. We have

$$S_{[a,a],[a,b]} = ([a,a])\frac{b}{2} - a([a,b]) = -aba.$$

Adding $(ab+ba)a$ yields ba^2, and adding $-b(a^2)$ yields $\overline{S_{[a,a],[a,b]}} = 0$.

Case 1b: We show that $\overline{S_{[a,b],[b,b]}} = 0$, where $a > b$ are variables such that a and b anticommute, so that $[a,b]$ and $[b,b]$ are two of our generators for $Q_{\tau,e}^{\perp}$. We use the generators

$$\begin{aligned} ab + ba &= [a,b] \\ b^2 &= \frac{1}{2}[b,b] \end{aligned}$$

in our reduction. We have

$$S_{[a,b],[b,b]} = ([a,b])b - \frac{a}{2}([b,b]) = bab.$$

Adding $-b(ab+ba)$ yields $-b^2a$, and adding $(b^2)a$ yields $\overline{S_{[a,b],[b,b]}} = 0$.

Case 2a: We show that $\overline{S_{[d,d],[d,g]+[e,f]}} = 0$, where $d > e > f > g$ are variables such that $[d,g]+[e,f]$ is one of our generators for $Q_{\tau,e}^{\perp}$. Since d anticommutes with e and f by Remark 5.5.8, we use the generators

$$\begin{aligned} d^2 &= \frac{1}{2}[d,d] \\ de + ed &= [d,e] \\ df + fd &= [d,f] \\ dg + gd + ef + fe &= [d,g] + [e,f] \end{aligned}$$

in our reduction. We have

$$S_{[d,d],[d,g]+[e,f]} = ([d,d])\frac{g}{2} - d([d,g]+[e,f]) = -def - dgd - dfe.$$

Adding $(de+ed)f$ yields

$$-dfe - dgd + edf,$$

adding $(df+fd)e$ yields

$$-dgd + edf + fde,$$

adding $(dg+gd+ef+fe)d$ yields

$$edf + fde + gd^2 + efd + fed,$$

and finally, adding $-e(df+fd) - f(de+ed) - g(d^2)$ yields $\overline{S_{[d,d],[d,g]+[e,f]}} = 0$.

Case 2b: We show that $\overline{S_{[d,g]+[e,f],[g,g]}} = 0$, where $d > e > f > g$ are variables such that $[d,g]+[e,f]$ is one of our generators for $Q^{\perp}_{\tau,e}$. Since g anticommutes with e and f by Remark 5.5.8, we use the generators

$$\begin{aligned} g^2 &= \frac{1}{2}[g,g] \\ eg+ge &= [e,g] \\ fg+gf &= [f,g] \\ dg+gd+ef+fe &= [d,g]+[e,f] \end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,g]} = ([d,g]+[e,f])g - \frac{d}{2}([g,g]) = gdg + efg + feg.$$

Adding $-e(fg+gf)$ yields

$$-egf+gdg+feg,$$

adding $-f(eg+ge)$ yields

$$-egf+gdg-fge,$$

adding $-g(dg+gd+ef+fe)$ yields

$$-egf-fge-g^2d-gef-gfe,$$

and finally, adding $(eg+ge)f+g^2d+(fg+gf)e$ yields $\overline{S_{[d,g]+[e,f],[g,g]}}=0$.

Case 3: We show that $\overline{S_{[a,b],[b,c]}}=0$, where $a>b>c$ are variables such that a anticommutes with b and b anticommutes with c, so that $[a,b]$ and $[b,c]$ are two of our generators for $Q^{\perp}_{\tau,e}$.

Claim: If a does not anticommute with c, then $a=d$ and $c=g$ for one of our generators $[d,g]+[e,f]$ with $d>e>f>g$ and either

- $b\in\{e,f\}$ for the same generator, or
- $b>e$ and b anticommutes with both e and f.

Proof of claim:

Suppose a does not anticommute with c. Then

$$\{a,c\}\in\{\{z_{j_i},z_{i+4}\},\{z_{i+1},z_{i+3}\},\{z_{j_{i+1}},z_{i+4}\},\{z_{i+2},z_{i+3}\}\}$$

for some even i, so that

$$\{a,b,c\}\in\{\{z_{j_i},z_m,z_{i+4}\},\{z_{j_{i+1}},z_n,z_{i+4}\},$$

since $a > b > c$ and z_{i+2} does not anticommute with z_{i+3}. This proves the first part of the claim, that ($a = d$ and $c = g$).

We first show that either $z_m \in \{z_{i+1}, z_{i+3}\}$ or both $m < i+1$ (so $z_m > z_{i+1}$) and z_m anticommutes with z_{i+1} and z_{i+3}, which proves the claim for $\{a, b, c\} = \{z_{j_i}, z_m, z_{i+4}\}$. Suppose $m \notin \{i+1, i+3\}$. If z_m does not anticommute with one of z_{i+1} or z_{i+3}, then since $m < i+4$, we have $m \in \{i-1, i, i+2\}$ by Remark 5.5.8. By the same remark, we have $m \neq i+2$, since z_{i+2} does not anticommute with z_{j_i} and z_m does.

If $m = i-1$, then $i \geq 4$, and since z_m anticommutes with z_{i+4} but $z_{j_{i+1}}$ does not for $j_{i+1} \in \{i-1, i\}$, we must have $j_{i+1} = i$ and $j_i = j_{i-1}$, which is in $\{i-3, i-2\}$ by Remark 3.1.10. This is a contradiction since $a = z_{j_i}$ anticommutes with $b = z_m$ but neither z_{i-3} nor z_{i-2} anticommutes with z_{i-1}.

If $m = i$, then either $i = 2$ or $i \geq 4$. It is impossible to have $m = i = 2$, since $z_2 = z_{j_3}$ and does not anticommute with z_6. If $i \geq 4$, then since $b = z_m$ anticommutes with $c = z_{i+4}$ and $z_{j_{i+1}}$ does not, we must have $j_{i+1} = i-1$ and $j_i = j_{i-2}$ by Remark 3.1.10, but this is a contradiction since $a = z_{j_i}$ anticommutes with $b = z_m$ but $z_{j_{i-2}}$ does not anticommute with with z_i by Remark 5.5.8.

Since $m < i+4$, we conclude that if $m \notin \{i+1, i+3\}$, then z_m anticommutes with both z_{i+1} and z_{i+3} and $m < i+1$.

Now we show that either $z_n \in \{z_{i+2}, z_{i+3}\}$ or both $n < i+2$ and z_n anticommutes with z_{i+2} and z_{i+3}, which proves the claim for $\{a, b, c\} = \{z_{j_{i+1}}, z_n, z_{i+4}\}$. Suppose $n \notin \{i+2, i+3\}$. If z_n does not anticommute with one of z_{i+2} or z_{i+3}, then since $n < i+4$, we have $n \in \{j_{i-2}, j_{i-1}, z_{i+1}\}$ by Remark 5.5.8. By the same remark, we have $n \neq i+1$, since z_{i+1} does not anticommute with $z_{j_{i+1}}$ and $b = z_n$ does. By Remark 3.1.10, we also have $j_{i-2} \leq i-4 < j_{i+1} < n$ and $j_{i-1} < j_{i+1} < n$, so that $n \notin \{j_{i-2}, j_{i-1}\}$. Since $n < i+4$, We conclude that if $n \notin \{i+2, i+3\}$, then z_n anticommutes with both z_{i+2} and z_{i+3} and $n < i+2$.

Case 3.1: In this subcase, we show that $\overline{S_{[a,b],[b,c]}} = 0$ when a anticommutes with c. We use the generators

$$\begin{aligned} ab + ba &= [a,b] \\ ac + ca &= [a,c] \\ bc + cb &= [b,c] \end{aligned}$$

in our reduction. We have

$$S_{[a,b],[b,c]} = ([a,b])c - a([b,c]) = -acb + bac.$$

Adding $(ac + ca)b$ yields

$$bac + cab,$$

adding $-b(ac + ca)$ yields

$$-bca + cab,$$

adding $(bc + cb)a$ yields

$$cab + cba,$$

and finally, adding $-c(ab + ba)$ yields $\overline{S_{[a,b],[b,c]}} = 0$.

Case 3.2: In this subcase, we show that $\overline{S_{[a,b],[b,c]}} = 0$ when a and c do not anticommute, that is, by the claim, when $a = d$ and $c = g$ for one of our generators $[d,g] + [e,f]$ with $d > e > f > g$ and either $b \in \{e,f\}$ for the same generator or $b > e$ and b anticommutes with both e and f. We use the generators

$$\begin{aligned} ab + ba &= [a,b] \\ bc + cb &= [b,c] \\ ac + ca + ef + fe &= [a,c] + [e,f] \end{aligned}$$

to begin our reduction.

$$S_{[a,b],[b,c]} = ([a,b])c - a([b,c]) = bac - acb.$$

Adding $(ac + ca + ef + fe)b$ yields

$$bac + cab + efb + feb,$$

adding $-b(ac + ca + ef + fe)$ yields

$$cab + efb + feb - bca - bef - bfe,$$

adding $(bc + cb)a$ yields

$$cab + efb + feb - bef - bfe + cba,$$

and adding $-c(ab + ba)$ yields

$$efb + feb - bef - bfe.$$

Case 3.2.1: If $b = e$, we use the generator

$$e^2 = \frac{1}{2}[e,e]$$

to continue our reduction. We have

$$efb + feb - bef - bfe = fe^2 - e^2 f,$$

so that adding $-f(e^2)$ and $(e^2)f$ yields $\overline{S_{[a,b],[b,c]}} = 0$.

Case 3.2.2: If $b = f$, we use the generator

$$f^2 = \frac{1}{2}[f, f]$$

to continue our reduction. We have

$$efb + feb - bef - bfe = ef^2 - f^2 e,$$

so that adding $-e(f^2)$ and $(f^2)e$ yields $\overline{S_{[a,b],[b,c]}} = 0$.

Case 3.2.3: If $b > e$ and b anticommutes with e and f, we use the generators

$$\begin{aligned} be + eb &= [b, e] \\ bf + fb &= [b, f] \end{aligned}$$

to continue our reduction. We have

$$efb + feb - bef - bfe.$$

Adding $(be + eb)f$ yields

$$efb + feb - bfe + ebf,$$

adding $(bf + fb)e$ yields

$$efb + feb + ebf + fbe,$$

and finally, adding $-e(bf + fb)$ and $-f(be + eb)$ yields $\overline{S_{[a,b],[b,c]}} = 0$.

Case 4a: We show that $\overline{S_{[a,d],[d,g]+[e,f]}} = 0$, where $a > d > e > f > g$ are variables such that a anticommutes with d so that $[a, d]$ and $[d, g] + [e, f]$ are two of our generators for $Q^{\perp}_{\tau,e}$.

Claim: If a does not anticommute with at least one of e, f, and g, then for some even i,

$$\{d, e, f, g\} = \{z_{j_{i+1}}, z_{i+2}, z_{i+3}, z_{i+4}\}$$

and $a = z_{j_i}$.

Proof of claim: If we have $\{d, e, f, g\} = \{z_{j_i}, z_{i+1}, z_{i+3}, z_{i+4}\}$ for some even i, we know that $j_i \leq i - 2 < j_{i+1}$ by Remark 3.1.10. Then if $a = z_n$ with $n < j_i$, Remark 5.5.8 tells us that z_n anticommutes with e, f, and g. Then suppose $\{d, e, f, g\} = \{z_{j_{i+1}}, z_{i+2}, z_{i+3}, z_{i+4}\}$. We know $j_{i+1} \leq i$ by Remark 3.1.10, so if $n < j_{i+1}$ such that z_n does not anticommute with at least one of e, f, and g, we must have $n \in \{j_{i-2}, j_{i-1}, j_i\}$ by Remark 5.5.8. If $i = 2$, $n = j_1 = j_2 = 0$ and $a = z_{j_i}$, so suppose $i \geq 4$. We know then that $j_i \in \{j_{i-2}, j_{i-1}\}$. If $d = z_{j_{i+1}} = z_{i-1}$, then $a = z_{j_i} = z_{j_{i-2}}$ by Remark 3.1.10, since $z_{j_{i-1}}$ does not anticommute with $d = z_{i-1}$ by Remark 5.5.8. If $d = z_{j_{i+1}} = z_i$, then $a = z_{j_i} = z_{j_{i-1}}$ by Remark 3.1.10, since $z_{j_{i-2}}$ does not anticommute with $d = z_i$ by Remark 5.5.8.

Case 4a.1: In this subcase, we show that $\overline{S_{[a,d],[d,g]+[e,f]}} = 0$ when a anticommutes with e, f, and g. We use the generators

$$\begin{aligned}
ae + ea &= [a,e] \\
af + fa &= [a,f] \\
ag + ga &= [a,g] \\
dg + gd + ef + fe &= [d,g] + [e,f]
\end{aligned}$$

in our reduction. We have

$$S_{[a,d],[d,g]+[e,f]} = ([a,d])g - a([d,g] + [e,f]) = dag - agd - aef - afe.$$

Adding $(af + fa)e$ yields

$$dag - agd - aef + fae,$$

adding $(ag + ga)d$ yields

$$dag + gad - aef + fae,$$

adding $-d(ag + ga)$ yields

$$-dga + gad - aef + fae,$$

adding $(ae + ea)f$ yields

$$-dga + gad + eaf + fae,$$

adding $(dg+gd+ef+fe)a$ yields

$$gad+eaf+fae+gda+efa+fea,$$

and finally, adding $-g(ad+da)-e(af+fa)-f(ae+ea)$ yields

$$\overline{S_{[a,d],[d,g]+[e,f]}}=0.$$

Case 4a.2: We show that $\overline{S_{[a,d],[d,g]+[e,f]}}=0$ when

$$\{d,e,f,g\}=\{z_{j_{i+1}},z_{i+2},z_{i+3},z_{i+4}\}$$

and $a=z_{j_i}$. Then we have

$$\begin{aligned} a &= z_{j_i} \\ d &= z_{j_{i+1}} \\ e &= z_{i+2} \\ f &= z_{i+3} \\ g &= z_{i+4} \end{aligned}$$

Since $a=j_i\leq i-2$ by Remark 3.1.10, a anticommutes with $z_{i+3}=f$ by Remark 5.5.8. By the same remark, since $j_{i+1}\in\{i-1,i\}$, d anticommutes with f. We know a anticommutes with d by assumption. When $a=z_{j_i}=z_{j_{i-2}}$, $d=z_{j_{i+1}}=i-1$, so that

$$ae+ea+dz_{i+1}+z_{i+1}d$$

is one of our generators. When $a = z_{j_i} = z_{j_{i-1}}$, $d = z_{j_{i+1}} = i$, so that

$$ae + ea + dz_{i+1} + z_{i+1}d$$

is one of our generators. In either case, we have the same generator, so we use the generators

$$\begin{aligned}
af + fa &= [a, f] \\
df + fd &= [d, f] \\
ad + da &= [a, d] \\
ae + ea + dz_{i+1} + z_{i+1}d &= [a, e] + [d, z_{i+1}] \\
ag + ga + z_{i+1}f + fz_{i+1} &= [a, g] + [z_{i+1}, f] \\
dg + gd + ef + fe &= [d, g] + [e, f].
\end{aligned}$$

in our reduction. We have

$$S_{[a,d],[d,g]+[e,f]} = ([a, d])g - a([d, g] + [e, f]) = dag - agd - aef - afe.$$

Adding $(af + fa)e$ yields

$$dag - agd - aef + fae,$$

adding $(ae + ea + dz_{i+1} + z_{i+1}d)f$ yields

$$dag - agd + fae + eaf + dz_{i+1}f + z_{i+1}df,$$

adding $(ag+ga+z_{i+1}f+fz_{i+1})d$ yields

$$dag+fae+eaf+dz_{i+1}f+z_{i+1}df+gad+z_{i+1}fd+fz_{i+1}d,$$

adding $-e(af+fa)$ yields

$$dag+fae-efa+dz_{i+1}f+z_{i+1}df+gad+z_{i+1}fd+fz_{i+1}d,$$

adding $-z_{i+1}(df+fd)$ yields

$$dag+fae-efa+dz_{i+1}f+gad+fz_{i+1}d,$$

adding $-g(ad+da)$ yields

$$dag+fae-efa+dz_{i+1}f-gda+fz_{i+1}d,$$

adding $-d(ag+ga+z_{i+1}f+fz_{i+1})$ yields

$$fae-efa-gda+fz_{i+1}d-dga-dfz_{i+1},$$

adding $-f(ae+ea+dz_{i+1}+z_{i+1}d)$ yields

$$-efa-gda-dga-dfz_{i+1}-fea-fdz_{i+1},$$

and finally, adding $(dg+gd+ef+fe)a+(df+fd)z_{i+1}$ yields

$$\overline{S_{[a,d],[d,g]+[e,f]}}=0.$$

Case 4b: We show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$, where $d > e > f > g > b$ are variables such that g anticommutes with b so that $[g,b]$ and $[d,g]+[e,f]$ are two of our generators for $Q^{\perp}_{\tau,e}$. Let

$$\{d,e,f,g\} \in \{\{z_{j_i}, z_{i+1}, z_{i+3}, z_{i+4}\}, \{z_{j_{i+1}}, z_{i+2}, z_{i+3}, z_{i+4}\}\}$$

for some even i.

Claim: If b does not anticommute with at least one of d, e, and f, then at least one of d, e, and f is in the set $\{z_{j_{i+2n-4}}, z_{j_{i+2n-3}}\}$ for some $n > 2$. Furthermore, then $b = x_{i+2n}$ and $d \neq z_{j_{i+2n-3}}$.

Proof of claim: If $b = z_{i+2n+1}$ for some $n > 1$ such that b does not anticommute with at least one of d, e, and f, then $b = z_{i+5}$ by Remark 5.5.8, but then b does not anticommute with $g = z_{i+4}$, and this is a contradiction. We conclude since $g > b$ that $b = z_{i+2n}$ for some $n > 2$, and hence that at least one of d, e, and f is in the set $\{z_{j_{i+2n-4}}, z_{j_{i+2n-3}}\}$ by Remark 5.5.8. If $d = z_{j_{i+2n-3}}$, then by Remark 5.5.9, since $d \in \{x_{j_i}, x_{j_{i+1}}\}$, we have $n = 2$, but this is a contradiction.

Case 4b.1: We show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$ when b anticommutes with d, e, and f. We use the generators

$$\begin{aligned}
db + bd &= [d,b] \\
eb + be &= [e,b] \\
fb + bf &= [f,b] \\
gb + bg &= [g,b] \\
dg + gd + ef + fe &= [d,g] + [e,f]
\end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d,g]+[e,f])b - d([g,b]),$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $-g(db+bd)$ yields

$$efb + feb - dbg - gbd,$$

adding $-e(fb+bf)$ yields

$$feb - dbg - gbd - ebf,$$

adding $-f(eb+be)$ yields

$$-fbe - dbg - gbd - ebf,$$

adding $(db+bd)g$ yields

$$-fbe + bdg - gbd - ebf,$$

adding $(gb+bg)d$ yields

$$-fbe + bdg + bgd - ebf,$$

adding $(eb+be)f$ yields

$$-fbe + bdg + bgd + bef,$$

adding $(fb+bf)e$ yields

$$bfe+bdg+bgd+bef,$$

and finally, adding $-b(dg+gd+ef+fe)$ yields $\overline{S_{[d,g]+[e,f],[g,b]}}=0$.

For the next four subcases, we assume b does not anticommute with at least one of d, e, and f, which by the claim is in the set $\{x_{j_{i+2n-4}}, x_{j_{i+2n-3}}\}$

Case 4b.2: In this subcase, we show that $\overline{S_{[d,g]+[e,f],[g,b]}}=0$ when $e=z_{j_{i+2n-4}}$. Since

$$e \in \{z_{i+1}, z_{i+2}\},$$

writing $i+2n-4=(i+2)+2(n-3)$ gives

$$e=j_{i+3}=j_{i+4}=j_{i+6}=\cdots=j_{i+2n-4}$$

by Remark 5.5.9. Since $j_{i+4}=j_{i+3}$, $j_{i+5}=i+4$, so that $z_{i+2n}\neq z_{i+8}$, since z_{i+8} does not anticommute with j_{i+5} by Remark 5.5.8, but $b=z_{i+2n}$ does anticommute with $g=z_{i+4}$ by assumption. Then in fact $n\geq 5$, so that $j_{i+2n-4}=j_{i+2n-6}$ and $j_{i+2n-3}=i+2n-5$. We have $g>z_{i+2n-5}>b$ with

$$\begin{aligned}
d &\in \{z_{j_i}, z_{j_{i+1}}\}\\
e &= z_{j_{i+3}}=z_{j_{i+4}}=z_{j_{i+6}}=\cdots=z_{j_{i+2n-4}}\\
f &= z_{i+3}\\
g &= z_{i+4}=z_{j_{i+5}}\\
z_{i+2n-5} &= z_{j_{i+2n-3}}\\
b &= z_{i+2n}
\end{aligned}$$

Then the only two variables with indices smaller than $i + 2n$ that b does not anticommute with are e and z_{i+2n-5} for $n \geq 5$, so b anticommutes with d and f. Furthermore, since $n > 4$, we have that $i + 3 < i + 2n - 5$, which means that $f = z_{i+3}$ anticommutes with both z_{i+2n-3} and z_{i+2n-1}. We then use the generators

$$\begin{aligned}
db + bd &= [d, b] \\
fb + bf &= [f, b] \\
gb + bg &= [g, b] \\
fz_{i+2n-3} + z_{i+2n-3}f &= [f, z_{i+2n-3}] \\
fz_{i+2n-1} + z_{i+2n-1}f &= [f, z_{i+2n-1}] \\
eb + be + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3} &= [e, b] + [z_{i+2n-3}, z_{i+2n-1}] \\
dg + gd + ef + fe &= [d, g] + [e, f]
\end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d, g] + [e, f])b - d([g, b]),$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $-g(db + bd)$ yields

$$efb + feb - dbg - gbd,$$

adding $-e(fb + bf)$ yields

$$feb - dbg - gbd - ebf,$$

adding $-f(eb+be+z_{i+2n-3}z_{i+2n-1}+z_{i+2n-1}z_{i+2n-3})$ yields

$$-dbg-gbd-ebf-fbe-fz_{i+2n-3}z_{i+2n-1}-fz_{i+2n-1}z_{i+2n-3},$$

adding $(db+bd)g$ yields

$$-gbd-ebf-fbe-fz_{i+2n-3}z_{i+2n-1}-fz_{i+2n-1}z_{i+2n-3}+bdg,$$

adding $(gb+bg)d$ yields

$$-ebf-fbe-fz_{i+2n-3}z_{i+2n-1}-fz_{i+2n-1}z_{i+2n-3}+bdg+bgd,$$

adding $(fb+bf)e$ yields

$$-ebf+bfe-fz_{i+2n-3}z_{i+2n-1}-fz_{i+2n-1}z_{i+2n-3}+bdg+bgd,$$

adding $(fz_{i+2n-3}+z_{i+2n-3}f)z_{i+2n-1}$ yields

$$-ebf+bfe+z_{i+2n-3}fz_{i+2n-1}-fz_{i+2n-1}z_{i+2n-3}+bdg+bgd,$$

adding $(fz_{i+2n-1}+z_{i+2n-1}f)z_{i+2n-3}$ yields

$$-ebf+bfe+z_{i+2n-3}fz_{i+2n-1}+z_{i+2n-1}fz_{i+2n-3}+bdg+bgd,$$

adding $-b(dg+gd+ef+fe)$ yields

$$-ebf+z_{i+2n-3}fz_{i+2n-1}+z_{i+2n-1}fz_{i+2n-3}-bef,$$

adding $(eb + be + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3})f$ yields

$$z_{i+2n-3}fz_{i+2n-1} + z_{i+2n-1}fz_{i+2n-3} + z_{i+2n-3}z_{i+2n-1}f + z_{i+2n-1}z_{i+2n-3}f,$$

and finally, adding

$$-z_{i+2n-3}(fz_{i+2n-1} + z_{i+2n-1}f) - z_{i+2n-1}(fz_{i+2n-3} + z_{i+2n-3}f)$$

yields $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$.

Case 4b.3 : We show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$ when $e = z_{j_{i+2n-3}}$. Since

$$e \in \{z_{i+1}, z_{i+2}\}$$

for some even i, writing $i + 2n - 3 = i + 2 + 2(n-3) + 1$ gives $n = 3$ and $e = x_{j_{i+3}}$ by Remark 5.5.9, so that $b = z_{i+6}$ by the claim at the beginning of Case 4b. Then $d = z_{j_{i+2}}$ by Remark 5.5.9. We have

$$\begin{aligned} d &= z_{j_{i+2}} \\ e &= z_{j_{i+3}} \\ f &= z_{i+3} \\ g &= z_{i+4} \\ b &= z_{i+6} \end{aligned}$$

We see that b anticommutes with $f = z_{i+3}$ by Remark 5.5.8. We recall by assumption that g anticommutes with b and by Remark 5.5.8 that f anticommutes with g. We use the generators

$$\begin{aligned}
fb + bf &= [f, b] \\
gb + bg &= [g, b] \\
fg + gf &= [f, g] \\
db + bd + fz_{i+5} + z_{i+5}f &= [d, b] + [f, z_{i+5}] \\
eb + be + gz_{i+5} + z_{i+5}g &= [e, b] + [g, z_{i+5}] \\
dg + gd + ef + fe &= [d, g] + [e, f]
\end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d, g] + [e, f])b - d([g, b]),$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $-e(fb + bf)$ yields

$$gdb + feb - dbg - ebf,$$

adding $-g(db + bd + fz_{i+5} + z_{i+5}f)$ yields

$$feb - dbg - ebf - gbd - gfz_{i+5} - gz_{i+5}f,$$

adding $-f(eb + be + gz_{i+5} + z_{i+5}g)$ yields

$$-dbg - ebf - gbd - gfz_{i+5} - gz_{i+5}f - fbe - fgz_{i+5} - fz_{i+5}g,$$

adding $(gb+bg)d$ yields

$$-dbg - ebf + bgd - gfz_{i+5} - gz_{i+5}f - fbe - fgz_{i+5} - fz_{i+5}g,$$

adding $(fb+bf)e$ yields

$$-dbg - ebf + bgd - gfz_{i+5} - gz_{i+5}f + bfe - fgz_{i+5} - fz_{i+5}g,$$

adding $(fg+gf)z_{i+5}$ yields

$$-dbg - ebf + bgd - gz_{i+5}f + bfe - fz_{i+5}g,$$

adding $(db+bd+fz_{i+5}+z_{i+5}f)g$ yields

$$-ebf + bgd - gz_{i+5}f + bfe + bdg + z_{i+5}fg,$$

adding $(eb+be+gz_{i+5}+z_{i+5}g)f$ yields

$$bgd + bfe + bdg + z_{i+5}fg + bef + z_{i+5}gf,$$

and finally, adding $-z_{i+5}(fg+gf) - b(dg+gd+ef+fe)$ yields

$$\overline{S_{[d,g]+[e,f],[g,b]}} = 0.$$

Case 4b.4: We show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$ when $f = z_{i+3} = z_{j_{i+2n-4}}$. Then writing $i+2n-4 = (i+4+2(n-4)$, we get

$$i+3 = j_{i+5} = j_{i+6} = j_{i+8} = \cdots = j_{i+2n-4}$$

for $n \geq 5$ by Remark 5.5.9. We have

$$
\begin{aligned}
d &= z_{j_{i+2}} \\
e &= z_{j_{i+3}} \\
f &= z_{i+3} = z_{j_{i+5}} = z_{j_{i+6}} = z_{j_{i+8}} = \cdots = z_{j_{i+2n-4}} \\
g &= z_{i+4} \\
b &= z_{i+2n}.
\end{aligned}
$$

The only two variables with indices smaller than $i+2n$ that b does not anticommute with are z_{i+3} and $z_{j_{i+2n-3}} \in \{z_{i+2n-5}, z_{i+2n-4}\}$ for $n \geq 5$ by Remark 5.5.8, so that b anticommutes with d and e. We know g anticommutes with b by assumption, and since $n \geq 5$, we have that $i+1, i+2 < i+2n-5$, which means e anticommutes with both z_{i+2n-3} and z_{i+2n-1}. We use the generators

$$
\begin{aligned}
db + bd &= [d, b] \\
eb + be &= [e, b] \\
gb + bg &= [g, b] \\
ez_{i+2n-1} + z_{i+2n-1}e &= [e, z_{i+2n-1}] \\
ez_{i+2n-3} + z_{i+2n-3}e &= [e, z_{i+2n-3}] \\
fb + bf + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3} &= [f, b] + [z_{i+2n-3}, z_{i+2n-1}] \\
dg + gd + ef + fe &= [d, g] + [e, f]
\end{aligned}
$$

in our reduction. We have

$$
S_{[d,g]+[e,f],[g,b]} = ([d, g] + [e, f])b - d([g, b]),
$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $-g(db + bd)$ yields

$$efb + feb - dbg - gbd,$$

adding $-f(eb + be)$ yields

$$efb - dbg - gbd - fbe,$$

adding $-e(fb + bf + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3})$ yields

$$-dbg - gbd - fbe - ebf - ez_{i+2n-3}z_{i+2n-1} - ez_{i+2n-1}z_{i+2n-3},$$

adding $(db + bd)g$ yields

$$bdg - gbd - fbe - ebf - ez_{i+2n-3}z_{i+2n-1} - ez_{i+2n-1}z_{i+2n-3},$$

adding $(gb + bg)d$ yields

$$bdg + bgd - fbe - ebf - ez_{i+2n-3}z_{i+2n-1} - ez_{i+2n-1}z_{i+2n-3},$$

adding $(eb + be)f$ yields

$$bdg + bgd - fbe + bef - ez_{i+2n-3}z_{i+2n-1} - ez_{i+2n-1}z_{i+2n-3},$$

adding $(ez_{i+2n-3} + z_{i+2n-3}e)z_{i+2n-1}$ yields

$$bdg + bgd - fbe + bef + z_{i+2n-3}ez_{i+2n-1} - ez_{i+2n-1}z_{i+2n-3},$$

adding $(ez_{i+2n-1} + z_{i+2n-1}e)z_{i+2n-3}$ yields

$$bdg + bgd - fbe + bef + z_{i+2n-3}ez_{i+2n-1} + z_{i+2n-1}ez_{i+2n-3},$$

adding $-b(dg + gd + ef + fe)$ yields

$$-fbe + z_{i+2n-3}ez_{i+2n-1} + z_{i+2n-1}ez_{i+2n-3} - bfe,$$

adding $(fb + bf + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3})e$ yields

$$z_{i+2n-3}ez_{i+2n-1} + z_{i+2n-1}ez_{i+2n-3} + z_{i+2n-3}z_{i+2n-1}e + z_{i+2n-1}z_{i+2n-3}e,$$

and finally, adding

$$-z_{i+2n-3}(ez_{i+2n-1} + z_{i+2n-1}e) - z_{i+2n-1}(ez_{i+2n-3} + z_{i+2n-3}e)$$

yields $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$.

Case 4b.5: We show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$ when $f = z_{i+3} = z_{j_{i+2n-3}}$. By Remark 5.5.9, writing $i + 2n - 3 = (i + 4) + 2(n - 4) + 1$, one has $n = 4$ and $z_{i+3} = z_{j_{i+5}}$, so by the claim at the beginning of Case 4b, one has $b = z_{i+8}$. Then by Remark 3.1.10 (since $j_{i+5} = i + 3$), $z_{j_{i+4}} = z_{j_{i+2}}$.

Case 4b.5.1: If $z_{j_{i+2}} \neq d$, we have by Remark 5.5.9 that $e \neq z_{j_{i+3}}$, so that

$$\begin{aligned} d &\neq z_{j_{i+4}} = z_{j_{i+2}} \\ e &\neq z_{j_{i+3}} \\ f &= z_{i+3} = z_{j_{i+5}} \\ g &= z_{i+4} \\ b &= z_{i+8}. \end{aligned}$$

Then the only two variables with indices smaller than $i+8$ that b does not anticommute with are f and $z_{j_{i+2}}$ and the only two variables with indices smaller than $i+6$ that z_{i+6} does not anticommute with are $z_{j_{i+3}}$ and $z_{j_{i+4}}$. Since $z_{j_{i+3}}, e \in \{z_{i+1}, z_{i+2}\}$ and $z_{j_{i+2}}, d \in \{z_{j_i}, z_{j_{i+1}}\}$, and $j_i, j_{i+1} \leq i$ by Remark 3.1.10, we see that $e \neq z_{j_{i+2}} = z_{j_{i+4}}$ and $d \neq z_{j_{i+3}}$, so that both d and e anticommute with both b and z_{i+6}. We also know $g = z_{i+4}$ anticommutes with b by assumption, and $e \in \{z_{i+1}, z_{i+2}\}$ and g anticommute with z_{i+7} by Remark 5.5.8. We use the generators

$$\begin{aligned} eb + be &= [e, b] \\ db + bd &= [d, b] \\ gb + bg &= [g, b] \\ ez_{i+6} + z_{i+6}e &= [e, z_{i+6}] \\ ez_{i+7} + z_{i+7}e &= [e, z_{i+7}] \\ fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6} &= [f, b] + [z_{i+6}, z_{i+7}] \\ dg + gd + ef + fe &= [d, g] + [e, f] \end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d,g]+[e,f])b - d([g,b]),$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $-f(eb+be)$ yields

$$gdb + efb - fbe - dbg,$$

adding $-g(db+bd)$ yields

$$-gbd + efb - fbe - dbg,$$

adding $-e(fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6})$ yields

$$-gbd - fbe - dbg - ebf - ez_{i+6}z_{i+7} - ez_{i+7}z_{i+6},$$

adding $(gb+bg)d$ yields

$$bgd - fbe - dbg - ebf - ez_{i+6}z_{i+7} - ez_{i+7}z_{i+6},$$

adding $(db+bd)g$ yields

$$bgd - fbe + bdg - ebf - ez_{i+6}z_{i+7} - ez_{i+7}z_{i+6},$$

adding $(eb+be)f$ yields

$$bgd - fbe + bdg + bef - ez_{i+6}z_{i+7} - ez_{i+7}z_{i+6},$$

adding $(ez_{i+6} + z_{i+6}e)z_{i+7}$ yields

$$bgd - fbe + bdg + bef + z_{i+6}ez_{i+7} - ez_{i+7}z_{i+6},$$

adding $(ez_{i+7} + z_{i+7}e)z_{i+6}$ yields

$$bgd - fbe + bdg + bef + z_{i+6}ez_{i+7} + z_{i+7}ez_{i+6},$$

adding $-b(dg + gd + ef + fe)$ yields

$$-fbe + z_{i+6}ez_{i+7} + z_{i+7}ez_{i+6} - bfe,$$

adding $(fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6})e$ yields

$$z_{i+6}ez_{i+7} + z_{i+7}ez_{i+6} + z_{i+6}z_{i+7}e + z_{i+7}z_{i+6}e,$$

and finally, adding

$$-z_{i+6}(ez_{i+7} + z_{i+7}e) - z_{i+7}(ez_{i+6} + z_{i+6}e)$$

yields $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$.

Case 4b.5.2: If $d = z_{j_{i+2}}$, we have by Remark 5.5.9 that $e = z_{j_{i+3}}$, and by Remark 3.1.10 (since $j_{i+5} = i+3$), $z_{j_{i+4}} = z_{j_{i+2}}$, so that

$$\begin{aligned}
d &= z_{j_{i+4}} = z_{j_{i+2}} \\
e &= z_{j_{i+3}} \\
f &= z_{i+3} = z_{j_{i+5}} \\
g &= z_{i+4} \\
b &= z_{i+8}.
\end{aligned}$$

Then the only two variables with indices smaller than $i+8$ that b does not anticommute with are f and d, so that b anticommutes with e. We know $g = z_{i+4}$ anticommutes with b by assumption, and g and $e \in \{z_{i+1}, z_{i+2}\}$ anticommute with z_{i+7} by Remark 5.5.8. We use the generators

$$\begin{aligned}
eb + be &= [e, b] \\
gb + bg &= [g, b] \\
gz_{i+7} + z_{i+7}g &= [g, z_{i+7}] \\
ez_{i+7} + z_{i+7}e &= [e, z_{i+7}] \\
db + bd + z_{i+5}z_{i+7} + z_{i+7}z_{i+5} &= [d, b] + [z_{i+5}, z_{i+7}] \\
fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6} &= [f, b] + [z_{i+6}, z_{i+7}] \\
ez_{i+6} + z_{i+6}e + gz_{i+5} + z_{i+5}g &= [e, z_{i+6}] + [g, z_{i+5}] \\
dg + gd + ef + fe &= [d, g] + [e, f]
\end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d,g]+[e,f])b - d([g,b]),$$

which expands to

$$gdb + efb + feb - dbg.$$

Adding $(db + bd + z_{i+5}z_{i+7} + z_{i+7}z_{i+5})g$ yields

$$gdb + efb + feb + bdg + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g,$$

adding $-f(eb + be)$ yields

$$gdb + efb - fbe + bdg + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g,$$

adding $-b(dg + gd + ef + fe)$ yields

$$gdb + efb - fbe + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bgd - bef - bfe,$$

adding $(fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6})e$ yields

$$\begin{gathered} gdb + efb + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g \\ -bgd - bef + z_{i+6}z_{i+7}e + z_{i+7}z_{i+6}e, \end{gathered}$$

adding $-g(db + bd + z_{i+5}z_{i+7} + z_{i+7}z_{i+5})$ yields

$$\begin{gathered} efb + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bgd - bef + z_{i+6}z_{i+7}e \\ +z_{i+7}z_{i+6}e - gbd - gz_{i+5}z_{i+7} - gz_{i+7}z_{i+5}, \end{gathered}$$

adding $(gb+bg)d$ yields

$$efb + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bef + z_{i+6}z_{i+7}e$$
$$+z_{i+7}z_{i+6}e - gz_{i+5}z_{i+7} - gz_{i+7}z_{i+5},$$

adding $(gz_{i+7} + z_{i+7}g)z_{i+5}$ yields

$$efb + z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bef + z_{i+6}z_{i+7}e$$
$$+z_{i+7}z_{i+6}e - gz_{i+5}z_{i+7} + z_{i+7}gz_{i+5},$$

adding $-e(fb + bf + z_{i+6}z_{i+7} + z_{i+7}z_{i+6})$ yields

$$z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bef + z_{i+6}z_{i+7}e + z_{i+7}z_{i+6}e$$
$$-gz_{i+5}z_{i+7} + z_{i+7}gz_{i+5} - ebf - ez_{i+6}z_{i+7} - ez_{i+7}z_{i+6},$$

adding $(ez_{i+7} + z_{i+7}e)z_{i+6}$ yields

$$z_{i+5}z_{i+7}g + z_{i+7}z_{i+5}g - bef + z_{i+6}z_{i+7}e + z_{i+7}z_{i+6}e$$
$$-gz_{i+5}z_{i+7} + z_{i+7}gz_{i+5} - ebf - ez_{i+6}z_{i+7} + z_{i+7}ez_{i+6},$$

adding $-z_{i+7}(ez_{i+6} + z_{i+6}e + gz_{i+5} + z_{i+5}g)$ yields

$$z_{i+5}z_{i+7}g - bef + z_{i+6}z_{i+7}e - gz_{i+5}z_{i+7} - ebf - ez_{i+6}z_{i+7},$$

adding $(eb+be)f$ yields

$$z_{i+5}z_{i+7}g + z_{i+6}z_{i+7}e - gz_{i+5}z_{i+7} - ez_{i+6}z_{i+7},$$

adding $(ez_{i+6} + z_{i+6}e + gz_{i+5} + z_{i+5}g)z_{i+7}$ yields

$$z_{i+5}z_{i+7}g + z_{i+6}z_{i+7}e + z_{i+6}ez_{i+7} + z_{i+5}gz_{i+7},$$

and finally, adding $-z_{i+5}(gz_{i+7} + z_{i+7}g) - z_{i+6}(ez_{i+7} + z_{i+7}e)$ yields $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$.

Case 4b.6: In this subcase, we show that $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$ when $d = z_{j_{i+2n-4}}$ for some $n \geq 3$. If $n = 3$, we have $d = z_{j_{i+2}}$ and $e = z_{j_{i+3}}$ by Remark 5.5.9, so we are done by Case 4b.3. If $n = 4$, since $d = z_{j_{i+4}} \in \{z_{j_i}, z_{j_{i+1}}\}$ we have $d = z_{j_{i+2}} = z_{j_{i+4}}$ by Remark 5.5.9, so that $z_{i+3} = z_{j_{i+5}}$ by Remark 3.1.10, and then we are done by Case 4b.5. For $n \geq 5$, since $d = z_{j_{i+4}}$ and $d \in \{x_{j_i}, x_{j_{i+1}}\}$, Remark 5.5.9, we have $d = z_{j_{i+2}} = z_{j_{i+4}} = \cdots = z_{j_{i+2n-4}}$. Thus by the same remark, $e = z_{j_{i+3}}$. Since $n \geq 5$, we also know by Remark 3.1.10 that (since $j_{i+2n-4} = j_{i+2n-6}$), $j_{i+2n-3} = i + 2n - 5$ with $g > z_{i+2n-5} > b$. Then we have

$$\begin{aligned}
d &= z_{j_{i+2}} = z_{j_{i+4}} = \cdots = z_{j_{i+2n-4}} \\
e &= z_{j_{i+3}} \\
f &= z_{i+3} \\
g &= z_{i+4} \\
z_{i+2n-5} &= z_{j_{i+2n-3}} \\
b &= z_{i+2n}.
\end{aligned}$$

by Remark 5.5.9. Then $b = z_{i+2n}$ does not anticommute with d or with z_{i+2n-5}, but does anticommute with e, f, and g by Remark 5.5.8. Since $n \geq 5$, we have $i + 4 < i + 2n - 5$, so that g anticommutes with z_{i+2n-3} and z_{i+2n-1} by the same remark. We use the generators

$$\begin{aligned}
fb+bf &= [f,b] \\
eb+be &= [e,b] \\
gb+bg &= [g,b] \\
gz_{i+2n-3}+z_{i+2n-3}g &= [g,z_{i+2n-3}] \\
gz_{i+2n-1}+z_{i+2n-1}g &= [g,z_{i+2n-1}] \\
db+bd+z_{i+2n-3}z_{i+2n-1}+z_{i+2n-1}z_{i+2n-3} &= [d,b]+[z_{i+2n-3},z_{i+2n-1}] \\
dg+gd+ef+fe &= [d,g]+[e,f]
\end{aligned}$$

in our reduction. We have

$$S_{[d,g]+[e,f],[g,b]} = ([d,g]+[e,f])b - d[g,b],$$

which expands to

$$gdb+efb+feb-dbg.$$

Adding $-e(fb+bf)$ yields

$$gdb-ebf+feb-dbg,$$

adding $-f(eb+be)$ yields

$$gdb-ebf-fbe-dbg,$$

adding $-g(db+bd+z_{i+2n-3}z_{i+2n-1}+z_{i+2n-1}z_{i+2n-3})$ yields

$$-ebf-fbe-dbg-gbd-gz_{i+2n-3}z_{i+2n-1}-gz_{i+2n-1}z_{i+2n-3},$$

adding $(eb+be)f$ yields

$$+bef - fbe - dbg - gbd - gz_{i+2n-3}z_{i+2n-1} - gz_{i+2n-1}z_{i+2n-3},$$

adding $(fb+bf)e$ yields

$$bef + bfe - dbg - gbd - gz_{i+2n-3}z_{i+2n-1} - gz_{i+2n-1}z_{i+2n-3},$$

adding $(gb+bg)d$ yields

$$bef + bfe - dbg + bgd - gz_{i+2n-3}z_{i+2n-1} - gz_{i+2n-1}z_{i+2n-3},$$

adding $(gz_{i+2n-3} + z_{i+2n-3}g)z_{i+2n-1}$ yields

$$bef + bfe - dbg + bgd + z_{i+2n-3}gz_{i+2n-1} - gz_{i+2n-1}z_{i+2n-3},$$

adding $(gz_{i+2n-1} + z_{i+2n-1}g)z_{i+2n-3}$ yields

$$bef + bfe - dbg + bgd + z_{i+2n-3}gz_{i+2n-1} + z_{i+2n-1}gz_{i+2n-3},$$

adding $-z_{i+2n-3}(gz_{i+2n-1} + z_{i+2n-1}g)$ yields

$$bef + bfe - dbg + bgd - z_{i+2n-3}z_{i+2n-1}g + z_{i+2n-1}gz_{i+2n-3},$$

adding $-z_{i+2n-1}(gz_{i+2n-3} + z_{i+2n-3}g)$ yields

$$bef + bfe - dbg + bgd - z_{i+2n-3}z_{i+2n-1}g - z_{i+2n-1}z_{i+2n-3}g,$$

adding $(db + bd + z_{i+2n-3}z_{i+2n-1} + z_{i+2n-1}z_{i+2n-3})g$ yields

$$bef + bfe + bgd + bdg,$$

and finally, adding $-b(dg + gd + ef + fe)$ yields $\overline{S_{[d,g]+[e,f],[g,b]}} = 0$.

Case 5: We show that $\overline{S_{[d,g]+[e,f],[g,m]+[h,l]}} = 0$, where $d > e > f > g > h > l > m$ are variables such that $[g, m] + [h, l]$ and $[d, g] + [e, f]$ are two of our generators for $Q^{\perp}_{\tau,e}$.

Claim: If d does not anticommute with at least one of h, l, and m or if m does not anticommute with at least one of e and f then

$$\{d, e, f, g, h, l, m\} = \{d, e, f, z_{i+4} = z_{j_{i+5}}, z_{i+6}, z_{i+7}, z_{i+8}\},$$

$d = z_{j_{i+2}}$, and $e = z_{j_{i+4}}$.

Proof of Claim: Without loss of generality, by the form of the generators for $Q^{\perp}_{\tau,e}$, we have $g = z_{i+4} \in \{z_{j_{i+2n-4}}, z_{j_{i+2n-3}}\}$ and $m = z_{i+2n}$ for some $n \geq 4$, since $i + 4 \leq i + 2n - 4$ by Remark 3.1.10. If $g = z_{i+4} = z_{j_{i+2n-4}}$, since $j_{i+2n-4} < j_{i+2n-3}$ by Remark 3.1.10, we have d,e, $f \notin \{z_{j_{i+2n-4}}, z_{j_{i+2n-3}}\}$, so that d, e, and f anticommute with m by Remark 5.5.8. Also, $i + 3 \leq i + 2n - 5$ so that d anticommutes with $h = z_{i+2n-3}$ and $l = z_{i+2n-1}$ by the same remark.

On the other hand, if $z_{i+4} = z_{j_{i+2n-3}}$, then $n = 4$ by Remark 5.5.9 and we are in the case above. Since $z_{i+4} = z_{j_{i+5}}$, it follows that $z_{j_{i+4}} = z_{j_{i+3}}$ by Remark 3.1.10. We know $d = z_{j_{i+2}}$ if and only if $e = z_{j_{i+3}} = z_{j_{i+4}}$ by Remark 5.5.9, so suppose $d \neq z_{j_{i+2}}$ and $e \neq z_{j_{i+4}}$. Since $j_i, j_{i+1} \leq i < j_{i+3}$, we see that $d \neq x_{j_{i+3}}$ and that d anticommutes with $h = z_{i+6}$ and $l = z_{i+7}$ by Remark 5.5.8. Furthermore, since $z_{j_{i+4}} \in \{z_{j_{i+2}}, z_{j_{i+3}}\}$ by Remark 3.1.10 and $d \notin \{z_{j_{i+2}}, z_{j_{i+3}}\}$, we see $d \neq z_{j_{i+4}}$. In addition, we have $e \neq z_{j_{i+4}}$ by assumption and

$f = z_{i+3} \neq z_{j_{i+4}}$ since $j_{i+4} \leq i+2 < i+3$ by Remark 3.1.10. Then since $g = z_{j_{i+5}}$, d, e and f all anticommute with m by Remark 5.5.8.

Case 5.1: When d anticommutes with h, l, and m and when e and f anticommute with m, we use the generators

$$\begin{aligned}
dh + hd &= [d,h] \\
dl + ld &= [d,l] \\
dm + md &= [d,m] \\
em + me &= [e,m] \\
fm + mf &= [f,m] \\
dg + gd + ef + fe &= [d,g] + [e,f] \\
gm + mg + hl + lh &= [g,m] + [h,l]
\end{aligned}$$

in our reduction. We have

$$\begin{aligned}
S_{[d,g]+[e,f],[g,m]+[h,l]} &= ([d,g] + [e,f])m - d([g,m] + [h,l]) \\
&= -dhl + gdm + efm + fem - dmg - dlh.
\end{aligned}$$

Adding $(dh + hd)l$ yields

$$-dlh + gdm + efm + fem - dmg + hdl,$$

adding $(dl + ld)h$ yields

$$-dmg + gdm + efm + fem + hdl + ldh,$$

adding $(dm+md)g$ yields

$$efm+gdm+fem+hdl+ldh+mdg,$$

adding $-e(fm+mf)$ yields

$$-emf+gdm+fem+hdl+ldh+mdg,$$

adding $-g(dm+md)$ yields

$$-emf-gmd+fem+hdl+ldh+mdg,$$

adding $-f(em+me)$ yields

$$-emf-gmd-fme+hdl+ldh+mdg,$$

adding $-h(dl+ld)$ yields

$$-emf-gmd-fme-hld+ldh+mdg,$$

adding $-l(dh+hd)$ yields

$$-emf-gmd-fme-hld-lhd+mdg,$$

adding $-m(dg+gd+ef+fe)$ yields

$$-emf-gmd-fme-hld-lhd-mgd-mef-mfe,$$

and finally, adding $(em+me)f+(fm+mf)e+(gm+mg+hl+lh)d$ yields $\overline{S_{[d,g]+[e,f],[g,m]+[h,l]}} = 0$.

Case 5.2: When d does not anticommute with one of h, l, and m, or one of e, f does not anticommute with m, the claim yields that

$$\{d,e,f,g,h,l,m\} = \{d,e,f,z_{i+4} = z_{j_{i+5}}, z_{i+6}, z_{i+7}, z_{i+8}\},$$

$d = z_{j_{i+2}}$, and $e = z_{j_{i+4}}$. By Remark 5.5.9, we have $e = z_{j_{i+3}}$ and so

$$\begin{aligned}
d &= z_{j_{i+2}} \\
e &= z_{j_{i+3}} = z_{j_{i+4}} \\
f &= z_{i+3} \\
g &= z_{i+4} = z_{j_{i+5}} \\
h &= z_{i+6} \\
l &= z_{i+7} \\
m &= z_{i+8}
\end{aligned}$$

Since $d \in \{z_{j_i}, z_{j_{i+1}}\}$ with $j_i, j_{i+1} \leq i$ by Remark 3.1.10, we see that d and f anticommute with l and m by Remark 5.5.8. We use the generators

$$\begin{aligned}
dl + ld &= [d,l] \\
dm + md &= [d,m] \\
fl + lf &= [d,f] \\
fm + mf &= [f,m] \\
dh + hd + fz_{i+5} + z_{i+5}f &= [d,h] + [f,z_{i+5}] \\
em + me + z_{i+5}l + lz_{i+5} &= [e,m] + [z_{i+5},l] \\
dg + gd + ef + fe &= [d,g] + [e,f] \\
gm + mg + hl + lh &= [g,m] + [h,l]
\end{aligned}$$

in our reduction. We have

$$\begin{aligned}
S_{[d,g]+[e,f],[g,m]+[h,l]} &= ([d,g]+[e,f])m - d([g,m]+[h,l]) \\
&= -dhl + gdm + efm + fem - dmg - dlh.
\end{aligned}$$

Adding $-g(dm+md)$ yields

$$-dhl - gmd + efm + fem - dmg - dlh,$$

adding $-e(fm+mf)$ yields

$$-dhl - gmd - emf + fem - dmg - dlh,$$

adding $-f(em+me+z_{i+5}l+lz_{i+5})$ yields

$$-dhl-gmd-emf-dmg-dlh-fme-fz_{i+5}l-flz_{i+5},$$

adding $(dm+md)g$ yields

$$-dhl-gmd-emf+mdg-dlh-fme-fz_{i+5}l-flz_{i+5},$$

adding $(dl+ld)h$ yields

$$-dhl-gmd-emf+mdg+ldh-fme-fz_{i+5}l-flz_{i+5},$$

adding $(fm+mf)e$ yields

$$-dhl-gmd-emf+mdg+ldh+mfe-fz_{i+5}l-flz_{i+5},$$

adding $(fl+lf)z_{i+5}$ yields

$$-dhl-gmd-emf+mdg+ldh+mfe-fz_{i+5}l+lfz_{i+5},$$

adding $(dh+hd+fz_{i+5}+z_{i+5}f)l$ yields

$$-gmd-emf+mdg+ldh+mfe+lfz_{i+5}+hdl+z_{i+5}fl,$$

adding $(gm+mg+hl+lh)d$ yields

$$-emf+mdg+ldh+mfe+lfz_{i+5}+hdl+z_{i+5}fl+mgd+hld+lhd,$$

adding $-m(dg + gd + ef + fe)$ yields

$$-emf + ldh + lfz_{i+5} + hdl + z_{i+5}fl + hld + lhd - mef,$$

adding $(em + me + z_{i+5}l + lz_{i+5})f$ yields

$$ldh + lfz_{i+5} + hdl + z_{i+5}fl + hld + lhd + z_{i+5}lf + lz_{i+5}f,$$

and finally, adding $-h(dl + ld) - z_{i+5}(fl + lf) - l(dh + hd + fz_{i+5} + z_{i+5}f)$ yields $\overline{S_{[d,g]+[e,f],[g,m]+[h,l]}} = 0$.

This concludes our cases, so we see that our set is a stable set for all τ. $\square$

6 Future Work

The work in this dissertation naturally leads to two sets of questions:

1) Could the family of toric rings from Chapter 3 be extended further, or could other families be considered, to achieve results similar to those in Chapter 4?

   - Could this family be extended to a larger family of chordal bipartite graphs, perhaps coming from similar graphs joined by an edge or a single vertex, or more generally, graphs that contain these graphs as subgraphs?
   - Can we characterize the structures within a matrix (such as the ladder-like structures M_τ^e introduced in Definition 3.1.1) that correspond to any chordal bipartite graph in a way that extends our results?
   - Does this family of chordal bipartite graphs generalize to a broader category of graphs that are not necessarily chordal bipartite, but which have similar properties?
   - Are there other families of graphs that generalize in this way, to allow for a broader categorization of some algebraic results?

2) What are the implications of the work done in Chapter 5 to obtain a correspondence to the Tate variables?

   - Is there a meaningful way to work with the Tate variables from Corollary 5.5.2 in the construction of the cotangent complex?
   - Is there a way to generalize the work done in Chapter 5 to apply it to other families of rings or graphs?

Bibliography

[AM69] Michael Francis Atiyah and I. G. MacDonald, *Introduction to commutative algebra.*, Addison-Wesley-Longman, 1969.

[Avr98] Luchezar L. Avramov, *Infinite free resolutions*, Six Lectures on Commutative Algebra (J. Elias, J. M. Giral, R. M. Miró-Roig, and S. Zarzuela, eds.), Birkhäuser Basel, Basel, 1998, pp. 1–118.

[BFKR15] Gilbert Baumslag, Benjamin Fine, Martin Kreuzer, and Gerhard Rosenberger, *A course in mathematical cryptography*, De Gruyter, Berlin, 2015. MR 3558843

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956

[BKLM99] Leonid A. Bokut, Seok-Jin Kang, Kyu-Hwan Lee, and Peter Malcolmson, *Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras*, J. Algebra **217** (1999), no. 2, 461–495. MR 1700511

[Blo85] Richard E. Block, *Commutative Hopf algebras, Lie coalgebras, and divided powers*, J. Algebra **96** (1985), no. 1, 275–306. MR 808852

[BMPZ92] Yuri A. Bahturin, Alexander A. Mikhalev, Viktor M. Petrogradsky, and Mikhail V. Zaicev, *Infinite-dimensional Lie superalgebras*, De Gruyter Expositions in Mathematics, vol. 7, Walter de Gruyter & Co., Berlin, 1992. MR 1192546

[BOVT17] Jennifer Biermann, Augustine O'Keefe, and Adam Van Tuyl, *Bounds on the regularity of toric ideals of graphs*, Advances in Applied Mathematics **85** (2017), 84 – 102.

[Bre14] Murray R. Bremner, *Free associative algebras, noncommutative Gröbner bases, and universal associative envelopes for nonassociative structures*, Comment. Math. Univ. Carolin. **55** (2014), no. 3, 341–379. MR 3225614

[Bri18] Benjamin Briggs, *Local commutative algebra and hochschild cohomology through the lens of koszul duality*, Ph.D. thesis, University of Toronto, 2018.

[CLO07] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2007, An introduction to computational algebraic geometry and commutative algebra. MR 2290010

[CV18] Aldo Conca and Matteo Varbaro, *Square-free groebner degenerations*, 2018.

[Eis89] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Department of Mathematics, Brandeis University, 1989.

[Gul68] Tor Holtedahl Gulliksen, *A proof of the existence of minimal R-algebra resolutions*, Acta Math. **120** (1968), 53–58. MR 224607

[HHO18] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi, *Binomial ideals*, Graduate Texts in Mathematics, vol. 279, Springer, Cham, 2018. MR 3838370

[HVT08] Huy Tài Hà and Adam Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded betti numbers*, Journal of Algebraic Combinatorics **27** (2008), no. 2, 215–245.

[KR00] Martin Kreuzer and Lorenzo Robbiano, *Computational commutative algebra 1*, Springer, Berlin, Heidelberg, 2000.

[Mat87] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1987.

[MM65] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR 174052

[Oha] Mkrtich Ohanyan, *Hochschild cohomology of short gorenstein rings*, Ph.D. thesis, Syracuse University, In preparation.

[Pee11] Irena Peeva, *Syzygies of toric ideals*, pp. 255–287, Springer London, London, 2011.

[PP05] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005. MR 2177131

[Pri70] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60. MR 265437

[Sch67] Colette Schoeller, *Homologie des anneaux locaux noethériens*, C. R. Acad. Sci. Paris Sér. A-B **265** (1967), A768–A771. MR 224682

[Tat57] John Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27. MR 0086072

[Tuc95] Alan Tucker, *Applied combinatorics*, third ed., John Wiley & Sons, Inc., New York, 1995. MR 1287611

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