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Middle Grades Mathematics Teachers’ Learning through Designing Structured Exercises and Learner Generated Examples

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ABSTRACT

Exemplification is significant for the teaching of mathematics and revealing mathematical structure to learners. Structured exercises and learner generated examples (LGEs) are pedagogical tools intended to provide opportunities for learners to discern and generalize mathematical structure. The aim of this study was to understand how middle school mathematics teachers develop their knowledge about designing and implementing sets of structured exercises and LGEs and the factors that influenced their use and implementation of structured exercises and LGEs. Four middle grades mathematics teachers participated in a series of four learning study cycles focused on the design and implementation of tasks that incorporated structured exercises and/or LGEs. The data was analyzed through a lens of variation theory and theory of example spaces to understand the changes in teachers' use and views of examples over time and the purpose, design, and enactment of examples. The Principle of Explicit Contrast emerged as a principle of both design and enactment with a number of associated design and enactment strategies. The Principle of Attending to Generation and Response emerged for the design of LGEs. The teachers thought about and developed their knowledge of task design and enactment through deliberate practice that included careful consideration of their own thinking about a class of examples and the aspects of that class of examples they attended to, the collaborative exchange of ideas with colleagues, the collective and individual design of sets of examples using patterns of variation, including structured exercises, and LGEs, and the revision or potential revision of such tasks. Factors that influenced and shaped teachers' conceptualization and operationalization of structured exercises and LGEs included teachers' perceptions of control of the examples, or lack thereof, teachers' notions of student success, and teachers' prior opportunities and experience with task design.

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by

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Persistence and resilience only come from having been given the chance to work through difficult problems.

Gever Tulley

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CHAPTER 1 - INTRODUCTION

Examples and exemplification have had a significant role in the teaching and learning of mathematics throughout history. Since the turn of the century, there has been a renewed interest in examples and exemplification, and a growing body of literature concerning how powerful student learning can be achieved through examples and exemplification. Much of this literature has fallen under the theme of design. Researchers have asked questions about characteristics and features of chosen examples and questions about sequencing, in order to understand the optimal variation of such sets of examples to best support students' learning of a particular mathematical concept or procedure. Not surprisingly, much of this literature takes the perspective of examples and exercises as authority-provided, meaning that they are presented to students via the teacher, the textbook, or some other means. A more recent vein of research in the exemplification literature has begun to examine examples and exercises as student-provided. Some researchers have suggested that when students generate examples, they are engaging in a cognitive act and through the use of learner generated examples (LGEs) can come to develop a deeper conceptual understanding of mathematics. A second theme has emerged within this vein of the exemplification literature – the enactment of examples and exercises within lessons. Taken together, this research has helped us to understand the complexities of both designing and implementing sets of examples and exercises that result in powerful student learning. In my study, I investigated the development of middle school mathematics teachers' knowledge about teaching and learning through structured exercises and LGEs. I examined changes in teacher knowledge related to the design of tasks that incorporated structured exercises and LGEs and the pedagogical implementation of such tasks. In the following pages, I will discuss the aims of this research, the rationale for my study, and a framework and methodology for the investigation.

The Aim of this Study

Over a number of years, I became more and more intrigued by the possibility of using examples and exercises in ways that support students in becoming aware of mathematical structure in order to use this awareness for the benefit of a deeply connected understanding of mathematics. In the course of attempting to design and implement such examples and exercises in my own teaching, I quickly realized what a complex endeavor this was! As a middle school mathematics teacher, I felt as though I needed additional support and opportunities to design these kinds of tasks with my colleagues and find design and implementation strategies that were helpful and supported students' learning. The aim of this study was to understand how middle school mathematics teachers develop their knowledge about designing and implementing sets of examples and exercises for the purpose of revealing mathematical structure. I investigated the ways in which middle school mathematics teachers' use and views of examples and exercises changed, and factors that influenced this change, after participating in a series of learning studies focused on designing and implementing structured exercises and LGEs. I sought to understand how teachers' knowledge about designing and enacting tasks that incorporated structured exercises and LGEs developed and sought to characterize this development. The following research questions guided my study:

1. How do teachers conceptualize and develop their knowledge about task design that structures students' experiences of learning algebraic constructs?
2. How do teachers develop their knowledge about enacting tasks that incorporate structured exercises or learner generated examples (LGEs) in ways that support students in developing an awareness of algebraic structure?

3. What factors influence and shape teachers' conceptualization and implementation of structured exercises and learner generated examples (LGEs)?

Pursuing these questions allowed me to better understand the ways in which teachers' pedagogical knowledge about the design and implementation of sets of examples and exercises developed, and factors that influenced that development, through a deliberate focus on these issues through a series of four learning study cycles.

Rationale for this Study

Examples are a means by which mathematics has always been taught. One might also argue that it is through examples that mathematics has always been learned. It would probably be a rare occurrence for a theorem to be proven or a generalization to be made without first considering particular instantiations of the concept under study. Examples of an object are the most basic unit of some larger class of mathematical objects that one might wish to consider or understand. Because of this, examples are central to the teaching and learning of mathematics.

Learning algebra is central to the learning of mathematics. Algebra emphasizes the relationships among quantities and includes ways of representing those relationships and analyzing change. The National Council of Teachers of Mathematics (NCTM) (2000) describes the mismatch between what algebra is perceived as and what algebra actually is:

> Many adults equate school algebra with symbol manipulation – solving complicated equations and simplifying algebraic expressions. Indeed, the algebraic symbols and the procedures for working with them are a towering, historic mathematical accomplishment and are critical in mathematical work. But algebra is more than moving symbols around. Students need to understand the concepts of algebra, the *structures* [emphasis mine] and

> principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations. (p. 37)

Understanding structure is one of the basic tenets to understanding algebra. More recently, the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) reaffirmed the importance of structural awareness in learning mathematics. The Standards for Mathematical Practice describe forms of mathematical expertise that teachers should seek to help students develop at every stage of their K-12 mathematics education. One of these standards calls for students to "Look for and make use of structure" (p. 8). Another one of these standards calls for students to "Look for and express regularity in repeated reasoning" (p. 8). These two standards combined support the mathematical proficiencies of being able to discern structure and exploit that structure to conjecture, generalize, and abstract.

The NCTM (2000) recognizes that all students should learn algebra. "Algebraic competence is important in adult life, both on the job and as preparation for postsecondary education" (p. 37). Despite this, too many American students, particularly students of color, are being failed by our educational system, right at the juncture where students first encounter algebra. While U.S. eighth graders have consistently scored above the average number of score points on the Trends in International Mathematics and Science Study (TIMSS) since 2003 (Provasnik et al., 2012), Kieran (2007) points out that the performance of U.S. students is still disturbingly low, citing the poor results on a particular test item in 2003 about a patterning relationship and another about the meaning of a variable in a given situation. Findings from the Programme for International Student Assessment (PISA) 2015 administration paint a picture of U.S. mathematics performance that is even more discouraging. Among the 44 Organisation for

Economic Cooperation and Development (OECD) countries included in the assessment, the U.S. ranked 35th. Socio-economic background was found to have a significant impact on the performance of U.S. students, with 11% of variation in student performance explained by this factor (OECD, 2016). As mathematics teachers, "the most important theory we want learners to construct is that they do actually possess the requisite powers to do mathematics and to think mathematically. Then they can make an informed choice as to whether to develop and make use of those powers within mathematics in the future" (Mason, 2006, p. 66).

For all students to learn algebra, we need to look closely at the teaching of algebra, and ways in which mathematics teaching can be improved, in general. The Third International Mathematics and Science Study (TIMSS) noted the marked differences in teaching methods between teachers in the U.S., Germany, and Japan (Stigler & Hiebert, 2009). This study made apparent that the U.S. has few mechanisms in place for the continuous improvement of teaching. Improving teachers' abilities to teach increases students' opportunities to learn (Darling-Hammond, 2010). Teachers can improve their teaching when they are given opportunities to collaborate and learn from each other (Lewis, 2002).

This study will bring together multiple bodies of mathematics education literature in order to shed light on the complex endeavor of teaching algebra for structural awareness and understanding. This literature encompasses the various roles that examples and exercises have taken in mathematics education, including an illustrative role, a practice-oriented role, and a cognitive role. Examples can generally be thought of as authority-provided or learner-provided. There is, by far, more abundant literature on examples as authority-provided. The research encompasses aspects of both design and implementation. Research about the design of examples includes studies about the variation in sets of examples and exercises and the use of nonexamples

(Charles, 1980; Petty & Jansson, 1987; Wilson, 1986), the sequencing and arrangement of examples and exercises (Petty & Jansson, 1987; Sun, 2011; Watson & Mason, 2006; Wilson, 1986), and the frequency of features of examples (Wilson, 1986). Some research found that teachers' choice of examples could support or impede students' learning and their perception of generality (Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008). Students' difficulties with perceiving generality through a particular example put an onus on teachers to draw attention to relevant and irrelevant aspects of examples (Mason, 2006, 2011; Mason & Pimm, 1984; Sinclair, Watson, Zazkis, & Mason, 2011; Zaslavsky & Zodik, 2007), implicating the importance of the implementation of examples, in addition to the design of examples.

Research on examples as learner-provided, or LGEs, is more recent and has focused on ways that students generate examples and what those processes reveal about student understanding (Aydin, 2014; Antonini, 2011; Hazzan & Zazkis, 1999; Sinclair et al., 2011), the power of LGEs to reveal aspects of what students' consider, in that moment, to qualify as an example of a particular type (Goldenberg & Mason, 2008; Sinclair et al., 2011; Zaslavsky & Zodik, 2014), stimulate learning events (Dahlberg & Housman, 1997; Zaslavsky & Zodik, 2014), and support students in developing a deep conceptual understanding of the particular content under consideration (Dahlberg & Housman, 1997; Watson & Mason, 2002; Zaslavsky & Peled, 1996). Research about the pedagogical implications for teaching through structured exercises and LGEs found that teacher action is significant in directing students' attention (Arzarello, Ascari, & Sabena, 2011), examples should be imbued with a purpose that matches how learners will engage with them (Watson & Chick, 2011), and in-the-moment decision making is paramount to being able to teach in responsive ways to the examples generated by students (Zaslavksy & Zodik, 2007, 2014; Zodik & Zaslavsky, 2008). I will discuss this literature, along with literature related to teacher learning through teacher noticing and participation in learning studies, in Chapter 2.

Theoretical Framework

The purpose of this study was to understand how teachers' knowledge about designing and enacting tasks that incorporate structured exercises and LGEs develops, and possible factors that influence and shape teachers' conceptualization and implementation of structured exercises and LGEs. Because of the variation in the teachers' knowledge at the outset of the study, and variation in their thinking and learning throughout the study, variation theory provided a useful framework for focusing on the design and implementation of structured exercises and LGEs as an object of learning. In variation theory, the object of learning refers to the content that is intended to be learned. From a variation theory perspective, learning is the development of a capability, where a capability is described as seeing, experiencing, or understanding something in a particular way that is relative to the object of learning and the learners themselves. Variation theory posits that learning is individual, but that differences in learning arise not so much from differences in what learners do, but rather differences in how learners see. How one sees or perceives of a problem, and the awarenesses that are brought to the fore of one's attention affect how one subsequently acts (Marton, Runesson, & Tsui, 2004). Variation theory, as a theoretical framework, provided a means for me to examine the opportunities for teacher learning that were either opened or narrowed during the course of enactment (in this case, the intervention of learning study), both individually and collectively.

In the case of teachers' learning about task design and implementation that incorporates structured exercises and LGEs, I would want teachers to "see" that designing and enacting such tasks should be grounded in a theory about what makes learning possible. Variation theory posits that learning is associated with discerning difference (Marton, 2015). It is through variation that learners discern, and as such, the variation enacted in a lesson opens or narrows opportunities for student learning. Mason, Stephens, and Watson (2009) explain:

> This idea [variation theory] is particularly powerful in mathematics, because variables and their variation are our stock in trade. Furthermore, in mathematics we like to vary whatever *can* vary, so rather than considering given dimensions of variation we also think about what dimensions can possibly be varied, and in what ways they can vary." (p. 12)

Thinking about the aspects of an object of learning that can and cannot be varied, how those aspects can be varied, and the relationships between those aspects, is in line with what mathematicians regularly seek to do to make sense of a class of mathematical objects. Initiating students, then, into this way of perceiving of mathematics, and teachers into this way of perceiving of the teaching of mathematics, can awaken teachers and students, alike, to the sensitivities and awarenesses related to developing mathematical expertise.

When considering what aspects of an example that can change and the possible range of that change, one creates an *example space*, comprised of particular examples and construction methods to generate additional examples. The theory of example spaces grew out of Watson and Mason's (2005) experiences of individual examples being interconnected, and hence, structured in some way. This theory is both influenced by and closely related to variation theory. Watson and Mason perceived of an example space as a theoretical "space" that could be explored, as in a topological landscape, the boundaries of which could be expanded, and its interior more deeply connected. In order to expand upon or further develop an example space, the dimensions of possible variation could be identified, and therefore changed, and the range of permissible change associated with that particular dimension could be explored. In this sense, example spaces are dynamic. The further development of an example space is also closely associated with Vygotsky and a zone of proximal development, as Watson and Mason indicate that “extending example spaces involves indicating what might be slightly out of reach but attainable using what is at hand” (p. 81). In this way, one’s current knowledge can be used to construct new knowledge, or alternatively, the collective example space formed in a classroom can trigger new understandings and insight.

The collective example space can be conceptualized as a component of the shared space of learning. According to Marton et al. (2004), the space of learning is, in a sense, shared in that the classroom environment is a shared and social place. The collective example space is the collection of examples made available temporally to a group of learners, such as the learners in a classroom at a particular time. A variation theory perspective says that current experience needs to be juxtaposed with previous experience in order to discern difference, and it is against this backdrop that current experience is made sense of. The current experience may consist of examples that a student had never considered before being put forth by classmates. This current experience is juxtaposed with the student’s past experience in which other examples, but not this one or ones like it, were made available for consideration. When the examples are aligned with the purpose of a particular object of learning, the collective example space can create opportunities for a learner to discern difference in dimensions of variation among examples, begin to make sense of that difference, and expand on his or her understanding of the object of learning.

The interplay between current and past experiences and their influences on each other is important for discernment and sense-making.

> This is an ever-changing and interactive process in which the features that we discern in a situation and what we discern as critical keep being revised. One crucial means in which this is done is by talking over events with people and by reflecting on the events after they have occurred. (Marton et al., 2004, p. 34-35)

The community around us mediates what we discern, what we view as critical, and provides opportunities for talk and reflection to allow for both individuals and communities of learners to construct new knowledge. Variation theory does not locate learning either within an individual (as in constructivism) or a community (as in socio-cultural theory), but rather only makes claims about how learning comes about through discernment of variation, and in particular, difference. There are multiple ways for learners to experience variation, either within oneself or in a collective space, but it is variation related to the particular object of learning that is required for learning about that particular object of learning to take place. We know that teachers have experience with using examples in their teaching, and that many teachers have experiences designing their own sets of examples. We do not know how teachers develop their knowledge about designing and implementing examples on the basis of variation. This study examined the process whereby teachers developed their knowledge about designing and enacting tasks that incorporated structured exercises and LGEs to begin to understand possible trajectories of teachers' learning about using variation in design and implementation of sets of examples. The learning study intervention (Lo, 2012) provided opportunities for teachers to juxtapose their past experience of designing and implementing sets of examples with designing and implementing tasks that incorporated structured exercises and LGEs. Learning study also provided opportunities for talk and reflection within the community of teacher learners, allowing for the variation in their own thinking to be made available for consideration by the community.

In Chapter 2, I review the literature on exemplification and its roles in school mathematics. As mentioned previously, much of the literature is focused on examples as authority-provided, either by the teacher or a textbook. More recent research has begun to look at the opportunities provided to learners when they generate their own examples. Collectively, this research demonstrates the potential for learning through structured exercises and LGEs, but also reveals the complexities that are involved in teaching through these kinds of tasks. Literature related to teacher learning through noticing and teacher learning through lesson and learning study may help in understanding how teachers develop their pedagogical knowledge about designing and implementing tasks that reveal and respond to student thinking.

In Chapter 3, I discuss the methodology for this study. In brief, this is a case study about a team of middle grades mathematics teachers learning to design and implement sets of examples using specific patterns of variation through the process of learning study. I recruited a team of four middle grades mathematics teachers who were interested in exploring ideas related to examples and exercises through the process of learning study. The team of teachers participated in four learning study cycles in which they designed and implemented tasks involving either structured exercises or LGEs. I interviewed and observed each teacher prior to the learning study intervention and after the learning study intervention. Each interview was transcribed, and I took field notes for each of the observations. The planning meetings for each of the four research lessons were audio-recorded, and episodes pertaining to the design of examples were transcribed. The research lessons were video-recorded, and episodes within the videos that pertained to the implementation of examples were transcribed and merged with my field notes from the lesson. The evaluation meetings with the teachers, immediately after each research lesson, were audio-recorded, and episodes pertaining to the design or implementation of examples were transcribed.

The data was coded and analyzed through a lens of variation theory with the theory of example spaces serving as an analytical framework, in addition to my own analytical framework of purpose, design, and enactment. In Chapter 4, I present the results of my analysis, and in Chapter 5, I discuss the results in light of the research literature and the implications that my study raises for the field.

Definition of Terms

A number of terms in this study are taken to have specific meanings. The following definitions will be used:

Example – a particular instantiation of a larger class of mathematical objects, an illustration of a principle or concept, or questions that are worked through to demonstrate a procedure

Exercise – a type of example which is a specific question that is meant to be worked through by the learner

Structured exercise – a collection of procedural questions or tasks combined in a way that allows for individual disturbance, connections between various *elements* (individual questions in the set), mathematical sense-making, and potentially, generalization and abstraction around the concept to be learned (Rowland, 2008; Watson & Mason, 2006; Watson & Shipman, 2008)

Learner generated examples (LGEs) – instances of a particular class of examples generated by a learner. Throughout this study, I will refer to the design and enactment of an LGE. This refers to the design of the prompt given to learners to provoke the generation of examples.

Exemplification – any situation in which a particular instance is offered to represent a more general class

Mathematical structure – the "identification of general properties which are instantiated in particular situations as relationships between elements" (Watson and Mason, 2006, p. 10). Watson and Mason contend that discerning mathematical structure requires an awareness of what features can change and which must remain invariant to maintain examplehood.

Task – any mathematical activity that a learner is asked to do.

Learning – the development of capabilities, where a capability is described as seeing, experiencing, or understanding something in a certain way (Marton, Runesson, and Tsui, 2004).

Object of learning – what is to be learned, comprised of two components: the direct object of learning and the indirect object of learning

Direct – refers to the content of learning (e.g., multiplication of multi-digit numbers, systems of linear equations in two unknowns, etc.)

Indirect – what the learner is expected to become able to do with that content (Marton, 2015, p. 37)

Critical aspects – the aspects of an object of learning that the learner has to notice, but is not yet able to, relative to both the object of learning and the learners.

Critical features – refers to the particular values of a critical aspect that need to be discerned. Often some, but not all, of the critical features need to be discerned in conjunction with the critical aspect. For instance, if the critical aspect is number, students may need to discern the critical features whole numbers and fractions, but not yet integers, rational numbers, irrational numbers, or complex numbers.

Example space – "an experience of having come to mind one or more classes of mathematical objects together with construction methods and associations" (Goldenberg & Mason, 2008, p. 189).

CHAPTER 2 – RELATED LITERATURE

Researchers have discussed the various roles that examples and exercises have in mathematics education, including an illustrative role, a practice-oriented role, and a cognitive role. Examples are at the heart of mathematics, used to communicate past knowledge to new learners and to provide opportunities to explore new knowledge. While examples and exercises in mathematics are often presented to the learner, either through a teacher or a textbook, example generation has been suggested by textbook authors, included in assessments, and more recently, been used to assist learners in constructing new knowledge (Watson & Mason, 2005). In this chapter, I focus on the cognitive role of examples and exercises as a means to promote mathematical sense-making and an appreciation of mathematical structure, which has been heavily informed by variation theory. I will discuss the variation theory and describe how it has impacted the development of the theory of example spaces and the use of structured exercises and learner generated examples (LGEs) in instruction. Teaching and learning through structured exercises and LGEs embraces the cognitive role of examples and exercises. Teaching through structured exercises and LGEs allows for learning to take place on two levels: that of the students and that of the teachers. I will first discuss student learning through structured exercises and LGEs, and then teacher learning. I will then discuss teaching through structured exercises and LGEs, including implications for the design and implementation of tasks that incorporate structured exercises and LGEs and relate this to the literature about teacher learning through professional noticing and participation in lesson and learning study.

Roles of Examples and Exercises

In recent years, exemplification has attracted renewed attention in mathematics education (Bills et al., 2006), particularly in the use and role of examples as a pedagogical tool.

Exemplification describes any situation in which a particular instance is offered to represent a more general class. I use example in a broad sense, as a particular instantiation of a larger class of mathematical objects, an illustration of a principle or concept, questions that are worked through to demonstrate a procedure, or specific questions that are meant to be worked through by the learner, often called exercises (Watson & Mason, 2005). In the following sections, I will discuss three roles of examples and exercises as described in the literature: illustrative, practice-oriented, and cognitive. These three roles are not necessarily distinct. The role of a particular example or exercise often rests with the individual. For instance, a teacher may provide an example to illustrate a general procedure, but a student may perceive of it as simply another exercise to complete and miss the important features of the example that the teacher perceived as generalizable and had intended for students to also perceive. Lastly, whether an example is presented or generated has a bearing on its role for the teacher and the learner. A teacher's presentation of an example might be for the purposes of illustration, but a student's generation of the same example might be for the purposes of mathematical sense-making and, hence, takes on a cognitive role.

Illustrative Role

The importance of examples for the learning of mathematics can be seen in the earliest mathematical records. The inclusion of examples was often meant to illustrate some procedure or algorithm, but more so in the general sense, as indicated by statements such as, "do it thus," and "thus it is done" (Watson & Mason, 2005). Examples within a text are presented to the learner, with the clear intent of illustrating either a mathematical object or a procedure. Within the illustrative role of examples, there are further categories that serve to separate what examples illustrate, whether it be a new concept, a basic or standard instance, a general case, or refutations of a statement. Later research about teachers' perspectives and use of examples in instruction further expanded on these categories, but often remained situated in the illustrative role of examples. I will begin by discussing some early research about the pedagogy of examples and continue with more recent work about the use and role of examples from teachers' perspectives.

Early research in exemplification was situated in cognitive theories of learning (Charles, 1980; Michener, 1978). The conception of mathematical knowledge was seen as comprising interrelated knowledge about concepts, results, and examples (Michener, 1978). Michener (1978) called the cognitive organization of these aspects concepts-space, results-space, and examples-space, respectively, and the inter-space relations dual relations. Within the examples-space, Michener distinguished the various roles that examples could play in the acquisition of mathematical knowledge, identifying start-up, reference, model, and counter examples. According to Michener, start-up examples are used for the introduction of a new subject and motivate basic definitions and results. Reference examples are examples that are continually referred back to for a check of one's understanding. Model examples are general examples that are meant to capture the essence of the situation. Counter examples are used to show that a statement is not true. Distinguishing between different types of examples highlighted the pedagogical implications and the choices that teachers could make about examples in support of students' learning. Despite Michener's contention that understanding mathematics was an active process, teachers' presentation of examples, rather than students' use of examples, permeated the research literature. The selection and arrangement of examples within a presentation are meant to illustrate the teacher's conception of a mathematical object or concept to her students. Thus, a focus on teachers' presentations of examples highlighted the illustrative role of examples in the literature.

The use of examples and nonexamples in instruction, and in particular, the usefulness of nonexamples in facilitating concept acquisition was frequently studied in the early mathematics education literature (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986). Charles (1980) found that nonexamples could be instructive for some concepts, such as bilateral symmetry, but were not found to have a significant contribution for rotational symmetry. Charles conjectured that examples may be more instructive for "easy" concepts, while nonexamples might be more instructive for "difficult" concepts (p. 19). This suggests that the choice of examples (and nonexamples) is context dependent.

Later research explored the interaction of examples and nonexamples with sequence and frequency of irrelevant features. Petty and Jansson (1987) explored the effects of the sequence of examples and nonexamples on students' acquisition of the concept of quadrilateral. They found that a rational sequence of examples and nonexamples enhanced concept attainment, perhaps allowing students to more easily discriminate between relevant and irrelevant attributes. In a rational sequence of examples and nonexamples, each example is matched with a nonexample so that their irrelevant attributes are as similar as possible. Example/nonexample matched pairs are chosen so that they are divergent from other example/nonexample pairs in the set, meaning that the irrelevant attributes are as different as possible. Finally, the pairs are arranged in ascending order of difficulty. Meanwhile, Wilson (1986) explored the interaction between sequences of examples versus sequences of mixed instances of examples and nonexamples of the altitude of a triangle with the frequency of irrelevant features. She found that a mixture of examples and nonexamples was significant for helping students to discern the relevant features of the altitude of a triangle (right angle and vertex included) when only one feature of each of the irrelevant dimensions (above/below, vertical/not vertical, inside/outside) appeared at a high frequency

(90% of the time). Both Petty and Jansson and Wilson cautioned about overgeneralizing the results to other geometrical concepts, enforcing the notion that the optimal example choice and arrangement may be context dependent. These two studies also revealed that variation in relevant and irrelevant attributes is important for concept acquisition, but too much variation may impede concept acquisition. Hence, matched pairs in Petty and Jansson's rational sequences included irrelevant attributes that were as similar as possible, while Wilson found that holding one irrelevant feature relatively invariant supported students in discerning relevant features.

While research showed that concept attainment could be supported by careful attention to examples and nonexamples, their sequence, and the frequency of irrelevant attributes, research on students' perceptions of examples revealed difficulties related to concept acquisition, including the development of figural concepts and a lack of a perception of generality (Fischbein, 1993; Mason & Pimm, 1984). A figural concept can develop when the figure associated with a concept is fused with the students' conception of the concept itself, to the point where the concept definition is neglected or rejected (Fischbein, 1993). For instance, if a student has a figural concept of altitude of a triangle as presented in Figure 2.1a, the student may have difficulty drawing the altitude from vertex B in Figure 2.1b, and instead draw BD, despite knowing the definition of the altitude of a triangle. The student may have even more difficulty drawing an altitude from vertex C, since AB is not parallel to the edge of the paper. The notion of figural concepts can be linked to the notion of prototypes. A prototype is an example that best represents all other examples. Exposure to a high frequency of examples that have an irrelevant attribute of a concept may result in that attribute being incorporated into the students' concept prototype (Kellogg, 1980). The difficulties attributed to figural concepts and concept prototypes arise when students fail to distinguish between relevant and irrelevant attributes or fail to perceive the generality within a given example.

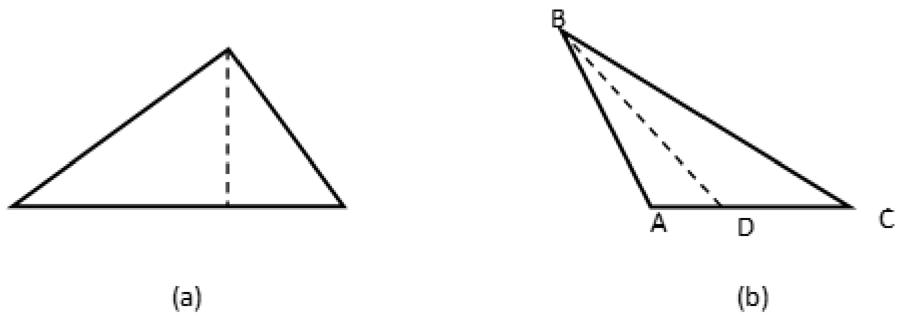


Figure 2.1. A figural concept of altitude, presented in (a), may cause difficulties for a student drawing an altitude from vertex B in (b) (Fischbein, 1993).

The failure to perceive the generality within a given example was highlighted in studies that explored the difference between expert and novice views of examples (Mason & Pimm, 1984). A teacher might regard an example as representative of an entire class of examples, while students see only a single, particular example. For instance, when giving $f(x) = |x|$ as an example of a non-differentiable function, the teacher may recognize that this represents the whole class of examples: $f(x) = k|x + a| + c$, while students might only recognize the particular example given. Mason and Pimm argued for the use of generic examples, but cautioned about the important role of the teacher in helping to bring about students' perceptions of generality:

> A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general. This is done by means of stressing and ignoring various key features, of attempting to structure one's perception of it. Different ways of seeing lead to different ways of knowing. (p. 287)

As described by Mason and Pimm, the presentation of a generic example is meant to illustrate the generality of particular key features and the irrelevance of others. This implies that it is not enough to discern relevant from irrelevant attributes; the range of possible instances of relevant attributes also has to be recognized in order to perceive the generality of a given example.

Both the development of a figural concept and the failure to perceive the generality inherent in a given example have a basis in a failure to perceive enough variability in the relevant and irrelevant attributes. In the case of a figural concept, not enough instances of a particular concept have been illustrated that do not include the irrelevant attribute. As shown in Figure 2.1, for the case of the altitude of a triangle, the variation presented did not allow for the student to discern that whether the altitude is inside or outside the triangle is irrelevant. In the case of presenting $f(x) = |x|$ as a generic example of a non-differentiable function, students' attention may not be drawn to the family of functions that this single function represents through the infinite number of transformations that can be performed on this function, all resulting in an example of a non-differentiable function.

Early research pointed to the pedagogical choices that teachers have to make when it comes to selecting and sequencing examples. In particular, variation is necessary in sets of examples to allow for students to discern the range of relevant and irrelevant attributes, but at the same time, too much variation may not allow for the relevant attributes to be distinguished from the irrelevant attributes in the first place. This suggests a dilemma of design, as teachers need to determine the optimal amount of variation in both the relevant and irrelevant attributes in order to support students' concept acquisition. Within the early research, the role of examples remained mostly illustrative. The teachers would be the ones to select, arrange, and present examples to learners. More recent research focused on teachers' choice of examples, including their characteristics and how those examples may support or impede learning (Rowland, 2008;

Zaskis & Chernoff, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008) continues to be situated within the illustrative role of examples in instruction.

Rowland (2008) examined the choices of mathematics examples by preservice elementary school teachers. He identified four categories of exemplification that teachers could attend to: (1) variables, (2) sequencing, (3) representations, and (4) learning objectives. Zodik and Zaslavsky (2008) reaffirmed Rowland’s assertion of the pedagogic importance of examples along with potential pitfalls: “The specific choice of examples may facilitate or impede students’ learning, thus it presents the teacher with a challenge, entailing many considerations that should be weighted” (p. 166). Zodik and Zaslavsky considered secondary in-service mathematics teachers’ choices, characterizing them on the basis of either planned or spontaneous examples, as well as teachers’ considerations and intentions in choosing examples. The principles that Zodik and Zaslavsky identified as guiding teachers’ choice or generation of examples, loosely map onto the four factors identified by Rowland that teachers should attend to in choosing or generating examples. The alignment of Rowland’s factors with Zodik and Zaslavsky’s principles is shown in Table 2.1.

Table 2.1

Alignment of Rowland’s Factors and Zodik and Zaslavsky’s Principles

Rowland’s Factors	Zodik and Zaslavsky’s Principles
Variables	• Draw attention to relevant features • Convey generality by random choice • Include uncommon cases
Sequencing	• Start with a simple or familiar case
Representations	• Keep unnecessary work to minimum
Learning Objectives	• Attend to students’ errors

Particular example characteristics may be discouraged, as well as sought after. Rowland (2008) and Zaslavsky and Zodik (2007) identified common pitfalls in teachers' choice of examples. Rowland's analysis of preservice elementary teachers' poor choices of examples was based on variation theory and on Watson and Mason's (2005) notions of dimension of possible variation and associated range of permissible change. Poor choices of examples included instances where the preservice teacher:

- obscured the role of the variable by using the same value for two (or more) dimensions of possible variation (as in choosing the example 4 – 2 = 2, in which the subtrahend and the difference have the same value).
- randomly generated examples by using dice to determine values, for instance, when control of the examples would have been better suited to bring about awareness of a particular aspect of the content.
- chose 'non-sensible' numbers for a particular procedure when a different strategy would make more sense, as when choosing 19×4 to demonstrate the standard algorithm for multiplication when a mental strategy of rounding up and compensating (e. g. $(20 \times 4) - 4)$ would be more suitable.

Rowland acknowledged that there are appropriate times to indicate generality by the random generation of examples. When the teacher's intention is to bring about a particular awareness regarding the structure of a mathematical concept, a controlled sequence is more suitable. This echoes the earlier work by Petty and Jansson (1987) and Wilson (1986), who also advocated for controlled sequences of examples and nonexamples.

Students' difficulties with perceiving generality, as intended by the teacher, continued to be a source of pedagogical difficulty (Mason & Pimm, 1984). Zaslavsky and Zodik (2007) observed 54 lessons at the middle school level (seventh through ninth grades) taught by five experienced mathematics teachers in order to characterize teachers' choices of examples. Zaslavsky and Zodik found that enacting the potential of a sequence of examples was not trivial for the teacher and that the students may not perceive the same generality as intended by the teacher. Differences in expert versus novice views of examples, in particular with regard to generality and conceiving of a particular example as representative of an entire class, is a common theme in the exemplification literature (Mason, 2006; Mason & Pimm, 1984; Sinclair et al., 2011).

Some research related to learners' perceptions of examples regarding generality, or the lack thereof, suggests tactics that can push the boundaries of learners' personal example spaces. Mason (2006) and Zazkis and Chernoff (2008) both considered the exemplariness of an example (or a counterexample), indicating that exemplariness resides in how the example is perceived by the learner. Both studies rely on what Zazkis and Chernoff call Mason's "methodology of noticing," in which phenomena of interest are explored within the authors' own experiences, where the goal is to "highlight or even awaken sensitivities and awarenesses" (Mason, 2006, p. 43). Both studies suggest that this methodology can inform one's future actions, impacting future practice, although the authors do not claim that their own experiences are generalizable.

Recognizing that not all examples have the same convincing power for learners, Zazkis and Chernoff introduced notions of pivotal examples and bridging examples. Pivotal examples should push the boundaries of a learner's personal example space, creating a turning point in a learner's awareness, perhaps by introducing a conflict. A bridging example assists in conflict resolution, and "serves as a bridge from learner's initial (naïve, incorrect or incomplete conceptions) towards appropriate mathematical conceptions" (Zazkis & Chernoff, 2008, p. 197).

Zazkis and Chernoff specify that pivotal and bridging examples are situational and learner-dependent. While these kinds of examples and their full effects can only be recognized after their implementation, they can be anticipated. The ability to anticipate potential examples that can serve as pivotal in the development of students' awareness and understanding is significant for teachers' choice of examples and exercises in instruction.

Summary of illustrative role. Research related to the presentation of examples in the mathematics classroom has shown that teachers' choices and design of examples for illustrative purposes is complex, comprising a multitude of factors that have the potential to either support or impede student understanding (Rowland, 2008; Zaslavsky & Zodik, 2007). Researchers suggest a number of design dilemmas. First, variation, but not too much variation, in the relevant and irrelevant features of examples is required for the discernment of important aspects of examples illustrating a particular mathematical concept or procedure (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986), as well as the perception of generality that an example affords (Fischbein, 1993; Kellogg, 1980; Mason & Pimm, 1984). Second, in addition to the variation in the relevant and irrelevant features, researchers suggest that teachers need to attend to the sequencing, representations, and student errors within the design (Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008). Third, teachers need to anticipate examples that will push the boundaries of learners' example spaces and bridging examples that support students in developing appropriate mathematical conceptions from their initial, naïve, or incomplete conceptions (Zazkis & Chernoff, 2008). Examples must be relative to the learners and are dependent on the implementation. Hence, teachers cannot know if their anticipated examples will be successful until they are implemented. Lastly, the content matters. The optimal sequencing and frequency of relevant and irrelevant features depends on the content to be learned. The use of nonexamples, for instance, may be useful for illustrating the relevant and irrelevant features of a mathematical construct with some content, but not others (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986). The design of sets of examples should be intentional and relative to the learners and the content.

Practice-Oriented Role

It is clear from stepping into any mathematics classroom that a common use of examples in mathematics instruction is practice. Examples with a practice-oriented role are more typically called exercises, but as Rowland (2008) says, “Such exercises *are* examples nevertheless, selected from a class of possible such examples” (p. 150). While the research literature is rich with studies that examine teacher choice of examples and the use of examples for illustrative or expository use, as described in the previous section, there is very little recent research literature about the choice of examples or “exercises” chosen for practice.

The ubiquitous role of examples used in a practice-oriented fashion can be seen in the extensive research on textbooks. Özer and Sezer (2014) did a comparative analysis of questions presented in American, Singaporean, and Turkish mathematics textbooks. In the selected American textbook, 81% of the problems were identified as procedural practice. Likewise, 83% of the problems in the Singaporean textbook were identified as procedural practice, while the Turkish textbook included 67% procedural practice problems. While this research supports the notion that examples within a practice-oriented role are common place across cultures, Özer and Sezer do not suggest rationales for teachers’ choice of a selection of exercises from such texts, nor rationales from textbook authors regarding choice and arrangement of such exercise examples.

Rowland (2008) outlined some possible rationales for guiding the choice of exercises: (1) to assist retention of the procedure by repetition, (2) to develop fluency, and (3) as an instrument for assessment. Rowland's study, however, examined teachers' choices of examples within the course of a lesson, which tend to fall within the illustrative role, and not examples or exercises chosen specifically for practice. In their examination of a Chinese textbook series and the transition from concrete to abstract representations of the distributive property, Ding and Li (2014) suggested a fourth possible rationale for choosing exercises: to deepen or extend what was taught in the worked example from the lesson. Ding and Li observed comparison among carefully structured practice exercises as a technique to support students in deepening their understanding of the distributive property. Table 2.2 displays the worked example, as well as the practice examples and their associated forms. Ding and Li described the practice problems as contrasting cases, as they differ in two aspects: (1) the direction in which the distributive property is used and (2) the position of the common factor. The first practice exercise has the same form as the worked example. The second practice problem reverses the direction in which the distributive property is applied. From the second practice exercise to the third, the direction remains the same, but the position of the common factor is changed, followed by a reversal of the direction again. Ding and Li suggest that, "the use of contrasting cases with progressive variations may deepen students' understanding of the distributive property" (p. 114). In this sense, examples as exercises take on a dual-role of illustrative and practice-oriented.

Like the practice examples described by Ding and Li (2014), past research has acknowledged the role that practice examples or exercises can play in developing students' awareness of problem structure. Sweller and Cooper (1985) found that students applied more efficient problem-solving strategies and appeared to attend to more structural aspects of the problems when practice exercises were matched to worked examples. In a review of the worked examples literature by Atkinson, Derry, Renkl, and Wortham (2000), a number of inter-example lesson features that pertained to the role of examples as practice were identified:

- examples in proximity to matched problems,
- multiple examples per problem type, and
- surface features that encourage search for deep structure, which includes multiple solutions to the same problem from different points of view. (p. 206)

These features potentially lend some guidance to teachers' choices of examples as exercises, although the question remains: How can surface features of sets of examples or exercises be designed to promote a search for deep structure?

Table 2.2

Comparison Between the Worked Example and Practice Exercises in a Chinese Textbook

	Form
Example: $(65+45)\times 5 = 65\times 5 + 45\times 5$	$(a+b)c = ac+bc$
Practice 1: $(42+35)\times 2 = 42\times \square + 35\times \square$	$(a+b)c = ac+bc$
Practice 2: $27\times 12 + 43\times 12 = (27+\square)\times \square$	$ac+bc = (a+b)c$
Practice 3: $15\times 26 + 15\times 24 = \square\bigcirc(\square\bigcirc\square)$	$ab+ac = a(b+c)$
Practice 4: $72\times(30+6) = \square\bigcirc\square\bigcirc\square\bigcirc\square$	$a(b+c) = ab+ac$

Hewitt (1996) took a different approach to practice, suggesting that the target skill to be practiced should have a subordinate role to the task at hand. Hewitt noted the problem of students remembering for now versus learning for life, and claimed that those things that are learned through being subordinated to another task are retained longer than the task that was actually attended to. He attributed this to the effects of simultaneity, where attention to the task at hand occurred through a somewhat unconscious attention to the skill or action required to bring about changes in the object of attention. Hewitt pointed out that fluency is recognized as an ability to perform a given skill with little attention being paid to it, and suggested that learning to be fluent in arithmetic and algebraic operations arises from practicing a skill through this same inattentiveness.

Hewitt (1996) called his model of subordination practice through progress, which required "practicing something whilst progressing with something else" (p. 30). He provided the following task as an illustration:

> Enter the following number in a calculator:
>
> 52846173
>
> The task is to *zap* the digit "1" (turn it into a zero), whilst keeping all the other digits as they are. The only operation allowed is a single subtraction. Next, the digit "2" has to be *zapped*, then the digit "3", etc., until all the digits have disappeared. (p. 30)

The main task has to do with zapping each of the digits in turn. Hewitt said that this task is understandable, even if one does not possess a good understanding of place value. In fact, practicing place value is subordinated to the task of zapping digits, but through a coordinated focus on zapping digits, one simultaneously practices place value. The focus is on the results of practicing, rather than on practicing itself. Hewitt further notes that students can see whether they are correct or not through the consequences of the subtraction they make, and whether they are right or wrong, the consequence provides some information to the learner about place value.

Hewitt's (1996) task on zapping digits illustrated three principles of practice that he suggested for greater retention and increased fluency: (1) practicing takes place while progressing forward with something else, (2) attention is on the purpose, or the result, of the task, not on practicing, and (3) students are able to self-evaluate. Unlike Atkinson et al. (2000), who suggested practice exercises that are in close proximity to the examples illustrated during the lesson, Hewitt suggested that practice exercises or a task be chosen such that the intended practice is not the focus, but rather the medium through which something else can be understood. In this way, the necessity for the required skill is created and students are granted an opportunity to understand the consequences of the results of the skill on another entity.

In yet another approach, Corlu, Capraro, and Corlu (2011) found that science-contextualized drill exercises produced higher gains in computational fluency for middle and high school students than more traditional, decontextualized drill exercises. This study situated the use of practice-oriented examples within a rationale of fluency. The authors surmised that, particularly for middle school students, science-contextualized drill exercises increased students' attentive behavior when compared to traditional drill exercises. This could possibly be linked to Hewitt's model of subordination, as the purpose of the drill exercises, for the students, may not have been fluency with the mathematics, but rather, an understanding of the science concepts. Hence, contextualizing drill exercises within science curricula motivated the need for computational fluency in mathematics.

Summary of practice-oriented role. A practice-oriented role of examples and exercises generally focuses on students' retention through repetition, fluency, and an assessment of students' understanding (Rowland, 2008). Chinese variation problems further suggest an opportunity to deepen and extend students' understanding through practice (Ding & Li, 2014). There is no doubt that fluency is a crucial component to mathematical proficiency, as supported by the CCSSM (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and, more often than not, research has assumed that fluency results from practice. There are divergent views as to what kind of practice best supports students in retention and the development of fluency. While Atkinson et al. (2000) recommend that practice exercises be matched to presented examples, Hewitt (1996) suggested that practice take a subordinate role to the learning of something else, or the task of something else. While the research on examples chosen for illustration is robust, there is a lack of clear guidance in the research literature about how teachers can select and design appropriate practice examples and exercises, which is perhaps related to both the divergent views of practice expressed in the research literature, and the lack of research focused on this issue in the first place.

Cognitive Role

In her seminal work on understanding what it means to understand mathematics, Michener (1978) indicated the active process required of the learner: “To discover what makes an individual item or a whole theory really work, one must do quite a bit other than passively waiting for understanding to happen” (p. 373). Unlike a pure illustrative role, which generally can be perceived of as a demonstration directed at a passive learner, a cognitive role highlights the active cognition of the learner in making sense of mathematical concepts and procedures by using examples and exercises in some way.

At the heart of exemplification is the need for learners to understand the mathematical structure of a construct by distinguishing between essential and non-essential aspects of examples. This requires cognitive action on the part of the learner. Mason et al. (2009) defined mathematical structure as the “identification of general properties which are instantiated in particular situations as relationships between elements” (p. 10). Watson and Mason (2006) contend that discerning mathematical structure requires an awareness of what features can change and which must remain invariant to maintain examplehood. They drew on variation theory, developed by Marton and colleagues (1997, 2004), which will be discussed in the next section, to describe how learners can “see” or develop an awareness of mathematical structure. Watson and Mason extended Marton’s notion of dimensions of variation to dimensions of possible variation to indicate that some dimensions are not noticed by some learners at some particular times. Dimensions of possible variation refer to discerning that which can change. Each dimension of possible variation has an associated range of permissible change. Discerning what dimensions of an example are relevant and irrelevant, the ways in which relevant dimensions of variation can change, and relationships between relevant features are necessary for perceiving the generality that an example affords and the basis for understanding mathematical structure.

Drawing on their own experiences as learners, Watson and Mason (2002) began to stress the cognitive role that examples could play and, in particular, the cognitive actions that result when examples are generated by students, rather than presented to students. Prior to the late 1990s, there was very little research on student-generated examples, and even less on the use of student-generated examples as a cognitive act (Watson & Mason, 2002). The research that did exist generally used student-generated examples for the purposes of assessment, rather than learning. The earliest work with LGEs involved constructive tasks of the form, "Give an example of…" (Hazzan & Zazkis, 1999). Much of the literature followed a constructivist epistemology and, in general, was qualitative in nature (Dahlberg & Housman, 1997; Hazzan & Zazkis, 1999). Situated within the constructivist epistemology, authors drew on theories related to schema construction and concept image to describe how students were constructing their knowledge from such examples (Dahlberg & Housman, 1997; Hazzan & Zazkis, 1999; Zaslavsky & Peled, 1996).

Watson and Mason (2002) furthered work related to student-generated examples by situating the act of exemplification as an act of cognition. This work considered what students could learn from engaging in the act of exemplification and identified five techniques for teachers and researchers to use to encourage students to generate examples. These techniques were described in terms of the students' experience: experiencing structure; experiencing and extending the range of variation; experiencing generality; experiencing constraints and the meanings of conventions; extending example-spaces and exploring boundaries.

Soon afterwards, Watson and Mason (2005) began to draw on variation theory, as a means to further their work with LGEs and investigate ways in which the generality and mathematical structure intended by the use of examples could be conveyed to learners. The techniques put forth by Watson and Mason (2002) to encourage student generation of examples reflect a variation theory perspective, which contends that discernment arises out of experience (Marton, 2015). Through the experience of examples, rather than simply viewing examples, as could be interpreted using an illustrative role of examples, learners have an opportunity to discern the relevant versus the irrelevant features of the examples, and the relationships between the features, that are significant to a particular concept or procedure. Experience is the crux of a cognitive role of examples; learners must engage in the experience of considering the example and its relevant and irrelevant features in order to "see" the underlying mathematical structure, rather than only see the example as presented.

To address the problem of students' perceptions of generality and structure (Mason & Pimm, 1984), Mason and Watson (2006), among others, advocated for approaches that encouraged an appreciation of mathematical structure. While some of their suggestions focused on the particular arrangement or sequencing of examples or exercises, as in structured exercises, much of their work suggested the use of LGEs (Bills et al., 2004; Mason, 2006; Mason et al., 2009; Sinclair et al., 2011; Watson & Mason, 2005). Both structured exercises and LGEs require action and mathematical sense-making on the part of the learner, indicating their cognitive role.

Through the experience of a set of carefully structured examples or exercises, or the experience of generating examples, learners have the opportunity to engage in discernment of key features of the examples, and either explicit or implicit conjecturing, verification, and generalization.

Summary of cognitive role. Some researchers have recently situated examples within a cognitive role that highlights the active engagement and cognition of the learner in discerning key features and attributes of examples, as well as the relationships between features, through the act of experiencing examples, rather than simply viewing examples (Watson & Mason, 2002, 2005). Learners need opportunities to engage with and think about examples beyond a presentation of such examples. Structured exercises and LGEs offer opportunities for students to engage in discerning the important features of examples, conjecture about relationships between features, and generalize beyond the specific examples and exercises included in the task.

Summary of the Roles of Examples and Exercises

In the preceding sections, I discussed three roles of examples and exercises as described in the literature: illustrative, practice-oriented, and cognitive. The illustrative role is the most common perspective taken by the research literature, and the focus of the research is on the presentation of examples and exercises to students. A practice-oriented role of examples, more often called exercises when employed for the purpose of practice, is less common in the research literature, despite the ubiquitous use of practice associated with learning mathematics. Lastly, a cognitive role is focused on students' use of and engagement with examples, and in particular, the mathematical sense-making and active cognition that occurs when students are given opportunities to experience examples.

As mentioned above, the three roles of examples, illustrative, practice-oriented, and cognitive, are not distinct. Even a student who sits in her seat listening to a lecture may be cognitively engaged in and experiencing an example being presented by the teacher if she is able to attend to and discern important features of the example and consider the relationships between the features of the example. As such, the role of a particular example or exercise often rests with the individual. Likewise, even when the use of examples is situated within a particular perspective, such as the use of LGEs within a cognitive perspective, this may overlap with other roles in either explicit or implicit ways. For instance, the teacher's intention in having students generate graphs of functions that have an average rate of change of a particular value over their domains may be to illustrate that the line segments adjoining the endpoints of such graphs over an interval are parallel, provided that the scale is consistent across graphs. In the process of generating such functions, however, students will invariably engage in practice in calculating the average rate of change over an interval. Students will likely also engage in mathematical sense-making as they either consciously or unconsciously conjecture about the relevant and irrelevant attributes of such functions with that particular condition. So once again, the roles of examples are not distinct.

Research about exemplification in mathematics does suggest some possible design considerations for sets of examples and exercises employed in instruction. Example choice is a complex act, as it encompasses a range of attributes that can be taken into account. Taking account of – or a lack thereof – can result in examples that either support or impede learners. The research literature suggests the delicate balance in variation across sets of examples and exercises. Too much variation may result in the inability of students to discern the relevant features of the concept, while a lack of variation in some aspects could result in the construction of figural concepts or erroneous or incomplete prototypes. Further complicating the issue of design further, the optimal amount of variation and arrangement of examples and exercises may be content and learner dependent. This suggests some possible challenges that teachers might face when designing such sets of exercises and examples. It seems likely that teachers' initial attempts at task design will not strike the right balance between variance and invariance, inevitably requiring revision to the design to address issues that arise in implementation. Further, particular patterns of variance and invariance that teachers' may have found success with in the past, may or may not translate to success with other content. This could lead to a perpetual state of uncertainty for teachers about task design, and could potentially prevent them from coming to any sort of generalizable takeaways about task design and example choice, if in teachers' minds, "it all depends." Teachers may also be likely to encounter challenges about example and exercise choice that fits their intended purpose, and potential mismatches between how examples and exercises were intended by the teacher and how they were perceived by the students. Attention to purpose is also significant when choosing examples and exercises, and choosing tasks that require practice subordinated to the purpose may be an effective approach. Lastly, teachers need to have an awareness of how examples and exercises are used by students, or engaged with by students, as understanding mathematics requires active cognition on the part of the learner.

A cognitive perspective on examples has been recently taken up by a number of researchers, as evidenced by a number of studies about the teaching and learning of mathematics through structured exercises and LGEs. The remainder of this literature review will situate examples and exercises within their cognitive role by discussing what research has found about learning through structured exercises and LGEs and teaching through structured exercises and LGEs. I will first discuss variation theory and the influence of variation theory on the theory of example spaces. These perspectives provide the theoretical basis for research on and the design of structured exercises and LGEs.

Variation Theory

Variation theory is a theory of learning, arising out of phenomenography, “the empirical study of the limited number of qualitatively different ways in which various phenomena in, and aspects of, the world around us are experienced, conceptualized, understood, perceived, and apprehended” (Marton, 1994, p. 4424). While constructivism focuses on the individual, and socio-cultural theory focuses on a community of learners and their social interactions, variation theory focuses on what is called the object of learning, or what is to be learned.

Marton, Runesson, and Tsui (2004) define learning as the development of capabilities, where a capability is described as seeing, experiencing, or understanding something in a certain way. In order to develop a particular capability (way of seeing, experiencing, or understanding), one must simultaneously focus on the critical aspects of the particular object of learning. The object of learning is comprised of three components: (1) the intended, (2) the enacted, and (3) the lived (Marton et al., 2004). The intended object of learning is what the teacher intends the students learn at the outset or in the planning of a lesson (i.e., what should be learned?). The enacted object of learning is what was actually made possible for students to learn in the implementation of a lesson (i.e., what can possibly be learned?). The lived object of learning is what the students did learn at the completion of the lesson, and beyond (i.e., what is actually learned?).

According to Marton et al. (2004), aspects of a particular object of learning represent dimensions of variation, and there are values along that dimension of variation that an aspect of an object of learning could take. For instance, an aspect of a dog could be coat color, and possible values along the dimension of variation coat color might be black, brown, white, grey, and so on. In mathematics, visual variations often relate to a variation of the meaning of an underlying object. For instance, an aspect of function is the input. Possible values along the dimension of variation input might be a numeric value, a quantity, a variable that represents a quantity, an expression, an expression that represents a quantity, or another function. Recognizing expressions and functions as possible values for the input requires seeing those constructs as objects themselves than can be acted upon (A. Watson, personal communication, August 13, 2015). It is important to note that not all aspects of an object of learning are critical aspects, but it is the critical aspects that need to be focused on to bring about an intended understanding. A variation theory perspective claims that we can only focus on that which we discern; we can only discern what we experience to vary; we can only experience variation if we have experienced different instances previously and can juxtapose our previous experiences with our current experience simultaneously.

Marton (2015) contends that people discern through difference, thus it is necessary to be concerned with what varies and what remains invariant in a learning situation. Marton describes what varies and what remains invariant as a pattern of variation and identified three of these: (1) contrast, (2) generalization, and (3) fusion. The first two patterns of variation, contrast and generalization, serve to separate the critical aspects from other aspects, or in other words, relevant features from irrelevant features. For instance, the first pattern of variation, contrast, refers to difference against a background of sameness, in order to allow the critical aspect to be discerned via difference. This separates out the critical aspect that is intended for students to notice. In general, contrast can be thought of as the comparison between what something is and what it is not. Generalization serves to separate the critical aspects from the optional aspects. Once a particular critical aspect has been discerned via difference, becoming aware of the sameness between different instantiations of the same phenomenon or concept allows the learner to determine what features can change while the object or concept itself remains invariant. Following contrast and generalization, the critical aspects need to be fused so that students see them working together as a single object. Fusion, the last pattern of variation, is the experiencing of variation in multiple critical aspects simultaneously and the discernment of relationships between these critical aspects. Through fusion, learners develop the ability to make generalizations that link the critical aspects of an object of learning (Holmqvist, 2011).

Students' understanding of why they are engaging in a particular task, and for what purposes, influences what they attend to and what they are able to discern. Marton (2015) refers to this purpose as the relevance structure. In reading a text, for example, if the relevance structure, for the student, is to remember the text, they approach it in a significantly different way than those students who approach the same text with a relevance structure of finding a solution to a problem. Holmqvist, Gustavsson, and Wernberg (2008) claim that:

> To teach someone to experience in a new way requires building a structure of relevance and the architecture of variation. A structure of relevance means an awareness of a purpose, its demands, and information about where they will lead. To sum up, in a learning situation, there must be a structure of relevance and a variation to make it possible to discern critical aspects. (p. 114)

In other words, in order to learn, students need to be presented with a meaningful situation in which they can become aware of the variation inherent in the critical aspects of an object of learning.

With its focus on variation, particularly patterns of variation and the need for a relevance structure, variation theory explains earlier research in exemplification that found that controlled variation within examples was needed for the discernment of relevant and irrelevant aspects of a concept (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986), attention to purpose was significant when choosing examples and exercises (Hewitt, 1996; Rowland, 2008) and, perhaps most importantly, content matters (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986). According to Marton et al. (2004):

> It is highly unlikely that there is any one particular way of arranging for learning that is conducive to *all* kinds of learning. In order to find effective ways of arranging for learning, researchers need to first address *what* it is that should be learned in each case, and find the different conditions that are conducive to different kinds of learning. (p. 3)

With a clear focus on what is to be learned, variation theory can offer guidance to teachers and researchers as they explore the optimal conditions of variation for students' learning.

The Influence of Variation Theory on the Theory of Example Spaces

In past research related to exemplification, variation theory has provided a lens through which to view relevant and irrelevant features of examples, as well as the generality of examples (Rowland, 2008; Watson & Chick, 2011). The work of Watson and Mason (2006), along with others (Bills, Bills, Watson, & Mason, 2004), was informed by comparisons of difference and sameness among examples in an effort to bring about the capability of discerning mathematical structure. In particular, Watson and Mason's (2005) work with LGEs focused on dimensions of possible variation and the associated ranges of permissible change. This was accomplished by the design of questions and prompts that asked learners to vary a particular dimension of variation in order to draw attention to it, to explore its range of permissible change, or to generate examples that were constrained by specific properties in order to inspire learners to seek out new dimensions of variation that had not been perceived yet. When asking learners to generate examples for themselves, Watson and Mason found four main results:

- Exemplification is individual and situational.
- Perceptions of generality are individual.
- Examples can be perceived or experienced as members of structured spaces.
- Example spaces can be explored and extended by the learner, with or without external prompts. (p. 57)

The notion of *example space* arose to explain where examples generated by students came from. Watson and Mason indicated that the important features of example spaces are their scope and interconnectedness and that developing learners' example spaces can lead to a powerful understanding of mathematical structure.

Goldenberg and Mason (2008) defined an example space as "an experience of having come to mind one or more classes of mathematical objects together with construction methods and associations" (p. 189). An example space includes an internal structure, such as the links and associations between examples, classes of examples, mathematical objects, concepts, theorems, procedures, and so forth. Goldenberg and Mason use a pantry metaphor to describe an example space. Some objects in it may come readily to mind – they are at the fore of the pantry because of recent use or because they are favorite ingredients. To find other objects, it may require digging back to the far reaches of the pantry, and this experience may inspire a reorganization of the space.

What comes to mind at a particular moment, due to particular triggers, is called the accessible example space. The goal of exploring one's example space is to extend the reaches of one's accessible example space to include more of one's potential example space by recognizing associations, constructions, and generalizations that were not previously perceived. In this sense, a learner's example space about a concept can be developed; example spaces are dynamic.

Sinclair et al. (2011) presented central characteristics about how a PES appears to be structured: (1) population, referring to the scarcity or availability of examples, (2) generativity, referring to the possibility of generating additional examples within the space, (3) connectedness, referring to examples being disconnected, loosely connected, or well-connected, and (4) generality, referring to the specificity of the example and its representativeness of a class of examples, noting that these four aspects are not independent. Much of the literature regarding teaching and learning mathematics through structured exercises and LGEs, which will be discussed in the next section, rests on the theoretical foundation of example spaces (Sinclair et al., 2011), as researchers and teachers seek to investigate learners' (accessible) example spaces as indicators of learners' knowledge and develop learners' example spaces for a powerful understanding of mathematical structure.

Learning Through Structured Exercises and LGEs

The use of structured exercises and LGEs in instruction creates opportunities for both student and teacher learning. Both students and teachers have opportunities to expand and refine their example spaces about particular mathematical concepts, resulting in learning about mathematical content. Teachers have the additional opportunity to learn about the pedagogy related to responsive teaching and in-the-moment decision making, facets of teaching that are required when examples for consideration are presented by the students, rather than carefully controlled through teacher presentation. In the following sections, I will first discuss the research literature related to student learning through structured exercises and LGEs, followed by teacher learning through structured exercises and LGEs.

Student Learning Through Structured Exercises and LGEs

Structured exercises and LGEs situate examples and exercises within a cognitive role, as they promote mathematical sense-making and encourage students to seek out and develop an understanding of the underlying structure of mathematical representations, concepts, and procedures. In this section, I will first review literature related to student learning through structured exercises, followed by student learning through LGEs. Both of these strategies appear to result in students developing deep conceptual and procedural understanding through underlying processes of conjecture and verification.

Student learning through structured exercises. Mason et al. (2009) contend that while mastering procedures is an important component of making mathematical sense of a concept, it is of little use to learners if it is simply a procedure. Rowland (2008) described exercises as what follows after a student has learned a procedure:

> The student rehearses it on several such ‘exercise’ examples. This is initially to assist retention of the procedure by repetition, then to develop fluency with it. Such exercises are also, invariably, an instrument for assessment, from the teacher’s perspective. Moreover, such ‘mere’ practice might also lead to different kinds of awareness and comprehension. (p. 150)

Drawing on the assumption that practice with exercises can lead to different kinds of awareness and comprehension, as well as variation theory, Watson and Mason (2006) take an exercise to be “a collection of procedural questions or tasks” (p. 91), and theorize about the usefulness of treating such an exercise as a mathematical object that can be structured in such a way, with regard to variation, as to promote mathematical sense-making. Hence, a structured exercise is a collection of procedural questions or tasks combined in a way that allows for individual disturbance, connections between various elements (individual questions in the set), mathematical sense-making, and potentially, generalization and abstraction around the concept to be learned (Rowland, 2008; Watson & Mason, 2006; Watson & Shipman, 2008). In this way, structured exercises can extend the purpose of practice exercises beyond retention and fluency to include opportunities for developing an awareness of mathematical structure.

As discussed in the previous section, researchers from the 1980s considered the combination of examples, particularly the arrangement of examples and nonexamples and the frequency of relevant and irrelevant features, and their effects on student learning (Charles, 1980; Petty & Jansson, 1987; Wilson, 1986). Literature related to structured exercises, however, is theoretical in nature (Mason et al., 2009), or based on observations from semi-formal action research (Watson & Mason, 2006). The theoretical work by Mason et al. posits the importance of an appreciation of mathematical structure, as an awareness, to bring about both conceptual understanding and procedural competence, suggesting structured exercises as a mode for accomplishing this. Their work suggests some pedagogically effective approaches for encouraging an appreciation of mathematical structure, including inviting learners to say what they see as being the same and being different within a pattern, between two patterns, between two exercises, etc., the use of examples that block familiar routines, and asking students to construct similar examples.

Structured exercises should provide learners with opportunities to experience patterns and generalizations. Watson and Mason (2006) discussed an example of a structured exercise written by Krause (1986) to reveal the structure of a taxi circle in taxicab geometry. Over the course of three years, Watson and Mason used this task with approximately 200 people, comprised of both inservice and preservice teachers, as well as two differently-aged classes of school students. After completing the task, all participants were asked to report on their experiences.

> Almost unanimously, people reported that this exercise evoked their natural propensity to look for similarities and to make conjectures to "teach" them something about taxicab geometry; that they started by just "doing" each separate point but were jolted into thinking mathematically by being offered points that broke their current sense of pattern; and that they had not realized they were aware of pattern until they were offered these points. (Watson & Mason, 2006, p. 96-97)

The purpose of structured exercises is to reveal the structure of a particular concept, property, or technique. As described above, learners began by completing each individual exercise, much as one would with any random collection of exercises, but the particular arrangement allowed for the set of exercises to be seen as a whole, whereby learners had the opportunity to discern pattern, conjecture about relationships, generalize, and abstract.

While little has been written about structured exercises in Western culture, Chinese education practices encourage procedural variation or variation problems, a kind of repetitive learning that appears to result in deep conceptual and procedural understanding (Lai & Murray, 2012; Sun, 2011). Sun (2011) presented a comparison between problem sets in American and Chinese textbooks on the topic of fraction division, noting three strategies for the design of such exercises: (1) one problem multiple solutions (OPMS, varying solutions), (2) one problem multiple changes (OPMC, varying conditions and conclusions), and (3) multiple problems one solution (MPOS, varying presentations). Sun found that four of the seven examples presented in the Chinese textbook provided multiple solutions (OPMS), while no examples in the American textbook did. Presenting multiple solutions to an exercise allows learners to focus on the underlying rationale of the algorithm, lending a justification for the procedure, rather than rote repetition. Problems of this type also vertically connect the curriculum by relating new concepts and procedures to previous concepts and procedures. Problem sets that include OPMC (varying conditions and conclusions) emphasize the idea of "simultaneity," in which learners have to simultaneously deal with variation in multiple aspects of a problem. Lastly, problem sets that include MPOS (varying presentations) allow learners to discern the similarities in the underlying structure of the problem by changing the context of the problem, but holding the solution invariant (Sun, 2011).

Lai and Murray (2012) presented a set of exercises of the type OPMC, or one problem in which multiple changes to the conditions are introduced. This particular set of exercises was meant to support learners in their development of a measurement model of division, as well as address the common misconception that "division makes smaller":

> *Problem 1*: There are 9L of apple juice and every 3L is put in a jar. How many jars are needed?
>
> *Problem 2*: There are 9L of apple juice and every 1L is put in a jar. How many jars are needed?
>
> *Problem 3*: There are 9L of apple juice and every 0.3L is put in a jar. How many jars are needed?
>
> *Problem 4*: There are 9L of apple juice and every 0.1L is put in a jar. How many jars are needed?
>
> *Problem 5*: There are 9L of apple juice and every .05L is put in a jar. How many jars are needed? (p. 10)

In this set of exercises, the total amount of apple juice is kept invariant, while the amount per jar is varied, allowing learners to discern the amount of juice per jar as a critical aspect. Generalization arises from the repetitive process, in which learners discern and practice appropriate generalizable procedures for the measurement model of division, while at the same time, discerning key mathematical ideas, such as "the smaller the divisor, the bigger the quotient" (p. 10). Lai and Murray caution that the enactment of such a set of exercises by a teacher can be considered rote drilling if the focus is purely on obtaining correct answers. The teaching of and enactment of such sets of exercises will be discussed further in the section on teaching through structured exercises and LGEs.

The potential for learning through structured exercises is evident in Watson and Mason's (2006) work and the use of procedural variation problems in the Chinese culture. Structured exercises may provide opportunities to support and inspire powerful learning in the mathematics classroom through practice, rather than exercise solely being seen as a means to support retention and develop fluency. It is not clear from the research literature if elements from Chinese variation problems can successfully be adopted into Western mathematics education practices and, furthermore, if such exercises would promote the same deep conceptual and procedural understanding for American students, as seen in Chinese students.

Student learning through LGEs. Much of the research about student-generated examples or LGEs uses them for the purpose of assessing students' understanding. Some studies have shown, however, that the use of LGEs in instruction can have powerful purposes beyond assessment by contributing to students' developing a greater conceptual understanding related to the consolidation and reorganization of knowledge structures (Dahlberg & Housman, 1997; Watson & Mason, 2002; Zaslavksy & Peled, 1996). Watson and Mason (2002) recognized opportunities for learning through LGEs through their work observing classroom teachers and examining the role that generating examples played in their own learning. They claimed that the full potential of students generating examples was rarely exploited.

Studies that involve assessing understanding or learning through LGEs have been conducted with a variety of learners, including post-secondary students (Antonini, 2011; Aydin, 2014; Dahlberg & Housman, 1997), preservice teachers (Hazzan & Zazkis, 1999; Zaslavsky & Peled, 1996; Zaskis & Leikin, 2008), and inservice teachers (Zaslavsky & Zodik, 2014). In this section, I will focus on literature about student learning through LGEs, specifically. The next section will focus on teacher learning through LGEs.

Early work with "Give an example of…" tasks by Zaslavsky and Peled (1996) suggested that example generation supports learning. Zaslavsky and Peled asked 36 in-service mathematics teachers and 67 preservice mathematics teachers to generate an example of a binary operation that was commutative, but not associative. Due to the significant differences in success between the group of experienced teachers, 33% of whom produced a correct example, and preservice teachers, only 4% of whom produced a correct example, the authors concluded that "teaching experience enhances growth" (p. 76), despite no additional formal training. Zaslavsky and Peled attributed the teachers' assumed growth to the example generation inherent in the act of teaching: "The act of teaching, an act constantly requiring the providing of examples, some generated on the spot, seems to have affected their [the teachers] willingness to take risks and their fluency in producing examples" (p. 77). Example generation, therefore, can serve as a catalyst for learning as evidenced by experienced teachers' improved performance. The same growth, however, had yet to be shown with students engaged in the act of example generation.

Dahlberg and Housman (1997) found that undergraduate mathematics majors used four learning strategies when presented with a new concept definition: example generation, reformulation, decomposition and synthesis, and memorization. The initial sophistication of the evoked concept image was highest among students who used example generation. The presentation of the concept definition was followed by generation and verification tasks. Dahlberg and Housman claimed that the process of generating, verifying, and reflecting on examples provided stimuli for learning events. Several studies have suggested that verification, as was used in Dahlberg and Housman's study, is a central component to learning through LGEs (Arzarello et al., 2011, Aydin, 2014; Zaskis & Leikin, 2008; Zaslavsky & Zodik, 2014). The act of verifying whether a conjectured example is indeed *an example of* a particular concept, with the required properties, can jolt learners into an awareness of relevant and irrelevant features.

Other research has suggested that the benefits of learning through LGEs may arise more from the process used to generate examples than the examples themselves (Antonini, 2011). Through the process of example generation, students have to ascertain what aspects of examples of a particular concept can be changed and the permissible range of that change. Through this process, learners have opportunities to expand the boundaries of their personal example spaces. Mason (2011) echoes Antonini's claim, suggesting that "tinkering" with basic constructions could be more beneficial for learners than the resultant example itself. Through tinkering with examples, students have the opportunity to discern and generalize processes of example generation. In this sense, a whole range of possible examples can be called upon from one's example space when needed, and individual examples do not need to be memorized and stored. This finding is consistent with Sinclair et al. (2011) who found that the example space of a mathematician related to quadratic functions consisted of a dynamic image, along with means of changing the image to produce additional examples:

> Even though LG [the mathematician] can generate a range of differently “shaped” examples, his example space has been somehow pinched into a single object together with ways to tinker with it by changing parameters, and it can be continuously transformed…The specific examples have been coalesced, and the more valued, more productive, “example” has been collapsed into one that is continuously changing. (p. 301)

In this sense, learning through LGEs is not about the specific examples, themselves, that reveal anything about the mathematical structure, but rather the processes used to generate such examples as learners develop an awareness of permissible ways to alter the example that maintain examplehood.

Despite the potential gains in conceptual understanding, example generation with school-age children tends to be only a rare or sporadic occurrence. In 54 observed lessons of five different middle grades teachers, Zodik and Zaslavsky (2008) identified only 35 student-generated examples out of a total of 639 observed examples. Watson and Mason (2005) presented ample evidence of school-age children learning through example generation, but only two studies (to my knowledge) directly address this group of learners (Arzarello et al., 2011; Watson & Shipman, 2008).

Watson and Shipman (2008) used LGEs for the introduction of a new concept with high-achieving 13-14 year old students and low-achieving 16-year old students. They concluded that learning through exemplification requires conjecturing about the relationships that connect different variables within the examples, rather than simply observing numerical patterns in sequences of examples. While noticing patterns can help the learner discern something about the structure, conjecturing about the relationship within and across examples is a non-trivial shift of perception that allows one to see both the dimensions of possible variation and the range of permissible change within those dimensions. Watson and Shipman suggested that students can make this perceptual shift as a result of their own actions, in particular: (1) discerning critical features by comparing similar examples, and (2) conjecturing from characteristics of special cases. Their study provided evidence that students can learn new concepts from LGEs, although the authors indicated the group nature of the task, noting that "the example spaces generated by the whole class were available for reflection" (p. 108), and warned that "the importance of normal classroom expectations and teacher guidance cannot be overestimated" (p. 108). The role of teacher guidance in students' learning though LGEs will be further discussed in the section on teaching through LGEs.

Arzarello et al. (2011) studied the role of examples in fostering ninth grade students' (14-15 years old) development of calculus concepts through a graphical approach. While the use of LGEs played an important role for bringing out opportunities for discussion related to the verification of provided examples, Arzarello et al. pointed out that the development of a deep conceptual understanding and an understanding of mathematical structure on the part of students, is far from an automatic process. It is perhaps because this is "far from an automatic process," that, as Watson and Mason (2002) claim, student-generated examples are rarely exploited to their full potential. Arzarello et al. found that the teachers' intervention into the space of discussion was crucial "in helping the students to modify a wrong example, to generate the right one for the task and to start the long-term process of building up the structure of their own space of examples" (p. 295). This indicates that LGEs can be powerful for student learning, perhaps through the promotion of acts of conjecturing, verification, and the ensuing discussion, and that teachers play an important role in facilitating these actions.

Teacher Learning Through Structured Exercises and LGEs.

Research on teaching through LGEs, which will be discussed in a later section, has revealed that learning can occur for both teachers and students (Zaslavsky & Zodik, 2014; Zazkis & Leikin, 2008). Student learning is about the mathematical construct under investigation, including the discernment of possible dimensions of variation, the associated ranges of permissible change, and construction methods, as discussed in the previous section. Teacher learning can be about either mathematical content, as students generate possible examples or ranges of permissible change that the teacher had not considered previously, or about pedagogical knowledge needed to teach using LGEs.

Studies with preservice and inservice teachers have generally focused on revealing the nature of the preservice and inservice teachers' mathematical and pedagogical knowledge, rather than on the development of their knowledge. Zazkis and Leikin (2007) discussed using LGEs as a research tool to reveal participants' mathematical and pedagogical knowledge of mathematical concepts, and suggested a framework for analyzing participants' example spaces based on (1) accessibility and correctness, (2) richness, and (3) generality. Zazkis and Leikin (2008) examined the subject matter knowledge and pedagogical content knowledge of 40 prospective secondary mathematics teachers. The preservice teachers were first asked to give as many definitions as possible of a square. In the second task, preservice teachers were given 24 "definitions" to judge the validity of, first individually, then in small groups. Whole class discussion addressed items on which the preservice teachers disagreed. The personal and collective example spaces of the preservice teachers, and the class as a whole, were analyzed on the basis of Zazkis and Leikin's

(2007) earlier framework. This study revealed that LGEs, when used with preservice teachers, could serve as a springboard for pedagogical discussions, producing learning events about pedagogy.

Zaslavsky and Zodik (2014) similarly found that disagreements about the validity of suggested examples led to learning events. In their study, learners were in-service secondary mathematics teachers participating in a workshop taught by the first author. The study found that example generation and verification served as both an indicator of learners' understanding about a particular concept and a catalyst for expanding learners' example spaces. Less familiar examples offered by participants, or suggested by the teacher, served to expand learners' example spaces. This study also revealed the potential for example generation activities to indicate, as well as expand and refine, the pedagogical knowledge of the teacher. The authors offer a framework (see Figure 2.2) "for examining the potential of rich tasks that foster example-generation as indicators and catalysts" (p. 529). This framework was developed by a study in which mathematics teachers participated as learners within a workshop (MTLs, or mathematics teachers as learners), and the mathematics teacher educator (MTE) was positioned as the teacher. The MTE chose to use a task that fostered example-generation, which in turn, engaged MTLs in example-generation. MTLs engagement in example-generation indicated their mathematical knowledge, as shown by the bottom right arrow in Figure 2.2. The MTE, however, needed to respond to the examples that had been generated by the teachers and consider how to further expand the MTLs example spaces. This indicated the MTE's mathematical knowledge, in being able to generate another suitable example, but also the MTE's pedagogical knowledge in generating a productive example that would move the discussion and MTLs thinking forward. The indication of the MTE's knowledge is shown in the framework by the top right arrow. The ensuing discussion that verified the suggested examples served as a catalyst for enhancing the MTE's and the MTLs' mathematical and pedagogical knowledge, as shown by the arrows on the left.

Both the MTLs' and the MTE's mathematical knowledge was enhanced by the social exchanges and the consideration of examples that they themselves had not considered. In considering and verifying such examples, each of the participants' personal example spaces were refined and expanded. Zaslavsky and Zodik note that this kind of responsive teaching requires in-the-moment decision making, but that teaching in this manner also serves as a catalyst for learning how to teach in this manner in future instances, thus offering opportunities for teachers' pedagogical learning. Substituting students in for the MTLs and the teacher for the MTE in this framework suggests that teaching through tasks that involve LGEs results in student opportunities to learn about the content and teacher opportunities to learn about both content and pedagogy.

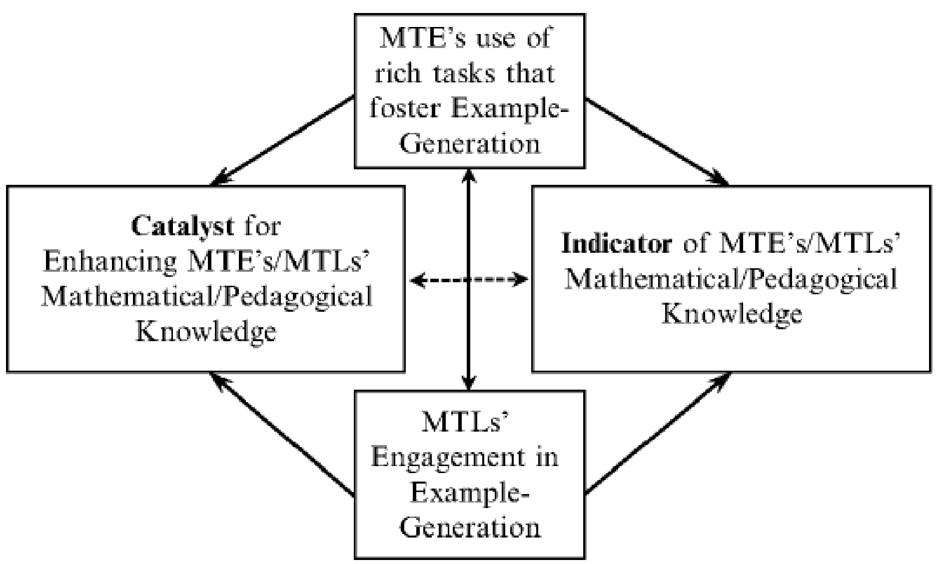


Figure 2.2. A framework presented by Zaslavsky and Zodik (2014) for examining the potential of rich tasks that engage learners in example-generation (p. 529).

Learning about mathematics content through LGEs seems to arise from the dual process of example-generation and example-verification. This process inherently involves conjecture, as well, since changes to an example are conjectured to produce the desired results prior to being tried and verified. Often learning through LGEs takes place in a collective space, namely the classroom, where learners have access to not just their own generated examples, but the collective example space of the class. This shared space of learning opens opportunities for student learning by making unfamiliar examples available for consideration. Research suggests that the process of resolution of disagreements about the validity of generated examples in meeting the constraints of the tasks is a mechanism whereby learning takes place. While research has revealed that teaching through LGEs results in learning opportunities for the teacher related to both mathematics content and pedagogy related to responsive teaching and in-the-moment decision making, teacher learning through structured exercises is absent from the research literature. In the next section, I expand on the research related to teaching through structured exercises and LGEs.

Teaching Through Structured Exercises and LGEs

Teaching through structured exercises and LGEs necessarily requires that the teacher relinquish some control of the lesson, either in terms of the specific examples made available to the group of learners (as through the use of LGEs), or the discernment of relevant features and the relationship between those features (as through the use of both structured exercises and LGEs). Both the use of structured exercises and LGEs ideally result in opportunities for students to conjecture, validate and, potentially, generalize and abstract. Allowing more opportunities for students to do so, places pedagogical demands on the teacher for how to respond. In the following sections, I review the literature related to teaching through structured exercises and

LGEs, specifically, and then summarize implications for the design and enactment of tasks that incorporate these kinds of examples and exercises.

Teaching Through Structured Exercises

The potential for learning through structured exercises is evident in Watson and Mason's (2006) work and the use of procedural variation problems in the Chinese culture. How sets of structured exercises are designed, and their subsequent enactment by the teacher, are less clear. In the previous section on student learning through structured exercises and LGEs, I discussed a set of exercises presented by Lai and Murray (2012) of the type OPMC (one problem with multiple changes to the conditions). In this set of five exercises the total amount of apple juice (9L) was kept invariant, while the amount of apple juice per jar ranged from 3L to 1L to 0.3L, to 0.1L to .05L. Lai and Murray cautioned that the enactment of such a set of exercises by a teacher can be considered rote drilling if the focus is purely on obtaining correct answers. They suggested that "an experienced mathematics teacher will organise this series of tasks hierarchically and provide scaffolding to illustrate and generalize" (p. 10). Students' attention must to drawn to the variance and invariance inherent in the set of exercises in order to discern the critical aspects of the division involving decimal numbers, while also attending to the generalizability of the procedure and a measurement model of division. The scaffolding that the teacher provides in the enactment of such a set of exercises is significant for students' discernment of the critical aspects and relationships among the critical aspects. Students may independently discern that the amount of juice put into each jar (the divisor) can change and also recognize that the range for this change includes both whole numbers less than 9L and rational values greater than 0L. Students may also likely discern that the number of jars needed (the quotient) varies, and may even discern the relationship between the divisor and the quotient. But without teacher scaffolding, will students independently discern that the amount of juice available, or the dividend, (9L, in this case) can also vary, despite its invariance in this problem set, and the possible range of permissible change for this dimension of variation? Do the students recognize that the context of putting juice in jars can vary? Directing students' attention to the aspects of a problem situation that may vary or must remain invariant, either by design or enactment, is necessary for a deep understanding of mathematical structure.

Watson and Mason (2006) consider the development of structured exercises a design project, in which past responses of participants provide a good prediction of how future participants will respond. Design principles for structured exercises are largely absent from the research literature, although Chinese variation problems offer three possible strategies for designing such tasks: (1) one problem with varying solutions (OPMS), (2) one problem with multiple changes to the conditions and conclusions (OPMC), and (3) multiple problems with a single solution (MPOS) (Sun, 2011). Mason (2006) suggests constraining the variation in a set of exercises to one or two dimensions of possible variation to increase the likelihood that students will notice the critical aspects, and the structure, that the teacher had intended. This strategy is reflected in Chinese variation problems that also vary only one or two dimensions of possible variation at a time. For instance, problem sets of the type OPMS keep the problem itself invariant, while only varying the solution strategies. Problem sets of the type OPMC, vary a single dimension of variation within the exercise and the resulting conclusion, but other aspects remain the same, as was seen in the set of exercises that asked students to distribute 9L of juice among jars by a specific amount per jar. MPOS problem sets vary only the context, while the solution itself remains invariant. Hence, constraining the variation to allow for the discernment of the critical aspects that the teacher intended is a potential design strategy for structured exercises.

At the same time, lack of variation should not give way to tedium. Watson and Mason (2006) suggest drawing on the framework of variation theory to design structured exercises, and comment that:

> Artistry and precision in helping a learner learn does [sic] not come instantly. Constructing tasks that use variation and change optimally is a design project in which reflection about learner responses leads to further refinement and further precision of example choice and sequence. (p. 100)

The optimal amount of change is at the crux of the design of structured exercises. This reflects earlier research on examples, particularly the use of examples and nonexamples and the frequency of relevant and irrelevant attributes, that found that invariance in some irrelevant attributes helps learners to discern the relevant attributes of interest, but that not enough variation may result in figural concepts or flawed prototypes (Petty & Jansson, 1987; Wilson, 1986). In other words, variation, but not too much variation, and at the same time, invariance, but not too much invariance, is important for students' discernment, and hence learning of a particular concept. The optimal amount of variation can only be achieved through a design process that includes revision.

Embedded within Watson and Mason's (2006) claim that, "Artistry and precision in helping a learner learn does [sic] not come instantly" (p. 100), is the enactment of structured exercises. Lai and Murray (2012) indicated that the enactment of structured exercises needs to go beyond getting correct answers. Watson and Mason make a similar point, but claim "that if the teacher offers data that systematically expose mathematical structure" (p. 94) then learners cannot resist seeking generalization. Watson and Mason admit, however, that even when learners seek generalizations, they still may not recognize or express them. Thus, there is a gap in the research literature regarding how teachers can enact structured exercises in ways that support learners in recognizing and expressing generalizations about mathematical structure. Bills et al. (2004) state that classroom discussion should focus on sameness and difference, but the particularities of teaching through structured exercises are absent.

There is some research that suggests approaches for encouraging an appreciation of mathematical structure (Mason et al., 2009; Watson & Mason, 2005, 2006). While some of these suggestions are about the arrangement or sequencing of examples or exercises, much of this research supports the use of LGEs or tasks that incorporate both structured exercises and LGEs (Bills et al., 2004; Mason, 2006; Mason et al., 2009; Sinclair et al., 2011; Watson & Mason, 2005). In the next section, I discuss the literature related to teaching through LGEs.

Teaching Through LGEs

LGEs have been implicitly, or even explicitly, encouraged in various versions of mathematics textbooks throughout history (Watson & Mason, 2005). Their deliberate use as a pedagogical tool, however, is a recent addition to the mathematics education literature. Studies have determined that teaching through LGEs can reveal students' accessible example spaces and support students in developing a deep conceptual understanding of the content (Dahlberg & Housman, 1997; Goldenberg & Mason, 2008; Mason, 2011; Watson & Mason, 2002). Studies focused on teaching through LGEs have identified the significant role of the teacher in facilitating and guiding students' awareness of particular features of examples (Arzarello et al., 2011; Watson & Chick, 2011; Zaslavsky & Zodik, 2014).

Some studies have investigated what constructing examples reveals about learners' understanding of a concept (Hazzan & Zazkis, 1999; Antonini, 2011). This is potentially important to teaching through LGEs, as LGEs can provide insight to teachers about students' current knowledge. In particular, the strategy that students employ to generate examples may indicate the connectedness of their knowledge associated with a particular mathematical concept. Hazzan and Zazkis (1999) identified three main strategies used for example construction and related these strategies to particular types of links among concepts. An occasional link was associated with a trial and error strategy, where a student randomly selects an example and then checks to see if it has the required property. A procedural link was associated with creating an algorithm or a procedure to follow for generating examples. A conceptual link was associated with a mathematical object and the ability to apply operations directly on the object. In other words, the object itself was seen by the student as an entity that could be operated on and transformed. Antonini (2011) studied example generation processes by 14 undergraduate and seven graduate students and also identified three similar processes: trial and error, analysis, and transformation, noting that multiple processes may be used. Similar to Hazzan and Zazkis, Sinclair et al. (2011) related students' abilities to construct examples to the connectedness of a students' understanding, as described through the theory of example spaces.

A number of studies later drew on the theory of example spaces to describe the connectedness of students' knowledge as revealed through LGEs (Goldenberg & Mason, 2008; Sinclair et al., 2011; Zaslavsky & Zodik, 2014). Sinclair et al. (2011) described the connectedness of a learner's example space as either not connected, well-connected, or loosely connected. A trial and error strategy was associated with example spaces that were not connected, while in contrast to Hazzan and Zazkis (1999), using a general structure or algorithm for example generation was associated with a well-connected example space. Students who recognized that some structure did exist and attended to some components of the structure, but not others, to generate additional examples, were seen as having a loosely-connected example space. This study assumed that the learner would use the most sophisticated method that the connectedness of their current example space would allow. In other words, if the learner "saw" the structure of the composition of the examples, she would use that structure, and perhaps an associated algorithm, to construct more examples, rather than rely on a trial and error strategy.

Multiple studies have confirmed the use of LGEs as a pedagogical tool that serves to not only reveal learners' current understandings, but also to stimulate learning events (Dahlberg & Housman, 1997; Zaslavsky & Zodik, 2014). From a teaching perspective, this is valuable for assessing students' prior knowledge and being able to respond in ways that extend students' learning. Zaslavsky and Zodik (2014) note that learning occurs when students go beyond the familiar and accessible examples to push the boundaries of their example spaces. This can occur through individual insight or by insight spurred by the collective example space generated in the classroom. In order for students to go beyond their accessible example spaces and access their potential example space, the teacher should persist and push students to generate more and more examples that are different from the previous ones. While generating such numerous examples can be the catalyst for learning, the teacher plays an important role in helping learners to make sense of the suggested examples and discern the underlying structure of such examples:

> It is also important that the teacher encourages genuine discussions and debates with minimal interference and at the same time offers useful prompts when learners face an impasse or when there is an opportunity to draw learners' attention to a mathematical subtlety that may otherwise be overlooked. When learners seem to have run out of

> examples, the teacher may offer ideas for generating additional examples that could serve to push learners beyond their existing/current concept images and example spaces. (p. 542)

Hence, the stimulation of learning events requires close observation of students' thinking on the part of the teacher, and in-the-moment decision making, as she considers where the boundaries of students' current example spaces are and how those example spaces can be expanded.

Watson and Shipman (2008) suggested that learning through LGEs requires students to make a perceptual shift that allows them to see both the dimensions of possible variation and range of permissible change within those dimensions. While the authors concluded that students could make that perceptual shift as a result of their own actions, making that perceptual shift was not automatic. Classroom norms and teacher guidance were acknowledged as significant for creating opportunities for discernment of dimensions of possible variation and the associated ranges of permissible change. Something about the way that the teacher had taught, not just for this particular lesson, but over the course of the year, situated students in way that allowed them to make the needed perceptual shift, perhaps more independently, than in other classrooms with other teachers.

Watson and Shipman (2008) situated their research in the wider context of the teacher's usual teaching. His teaching included starting and ending lessons with "thinking" tasks, which required more adaptive reasoning than recall or fluency, and a breakdown of about 50% whole group discussion, 30% small group work, and 20% independent work. Homework was created by the students themselves, as each pair of students would be asked to pose a 'hard' question on the current topic for the rest of class. This resulted in 15 to 16 questions, and the students would be involved in marking and discussing these homework exercises. Because of these practices, students were often engaged in mathematical reasoning and sense-making, and may have had many opportunities throughout the course of the school year to either generate exercises or find ones that would fit the criterion as a member of the current topic.

The teacher's beliefs about procedural and conceptual understanding influenced his usual teaching strategies:

> For Steve [the teacher], the purpose of competence with techniques is to think about concepts. For this reason, during episodes of technical work, he sometimes gives directed help to learners so that they can all take part in whole class discussions of mathematical ideas. He also employs a 'gossip' approach to classroom knowledge in which ideas are allowed to spread around the classroom during lessons. (Watson & Shipman, 2008, p. 100)

Help with procedures and algorithms was provided to students to make way for later work where procedural competence would support conceptual reasoning. In this classroom, developing a deep conceptual understanding did not come at the expense of procedural fluency, but rather the two supported each other in tandem.

Lastly, Watson and Shipman (2008) described that the teacher often talked about mathematical strategies, such as "using inverse" and "thinking about special cases" during lessons. Thinking about special cases is perhaps especially important, as it is an abductive approach, rather than the inductive approach suggested in the research literature about structured exercises. According to Watson and Mason (2006), structured exercises should provide learners with opportunities to experience patterns and generalizations. In this approach, a number of cases reveals a pattern from which a generalization emerges. 'Special cases,' such as instances that show a degenerate relationship, can indicate possible structures from which students can make conjectures.

Watson and Shipman (2008) did not explicate exactly what they meant by teacher guidance, but the description of the teacher's usual teaching strategies suggest that this might include: (1) providing opportunities for various forms of mathematical reasoning, including adaptive reasoning, and deductive, inductive, and abductive approaches, (2) situating students as holders of knowledge within the classroom, (3) supporting students' procedural competencies in tandem with students' conceptual understandings, and (4) making mathematical strategies and thinking visible within the classroom.

Other studies have revealed additional pedagogical implications for teaching through LGEs, including in-the-moment decision making and example generation on the part of the teacher (Zaslavsky & Zodik, 2007, 2014; Zodik & Zaslavsky, 2008), the need for teacher actions to direct students' attention to relevant features (Arzarello et al., 2011), and the need to imbue examples with a purpose that match how students engage with them (Watson & Chick, 2011). Arzarello et al. (2011) emphasized the key role of the teacher in a study on introducing calculus in secondary schools through a graphical approach. They presented two case studies of grade nine (14-15 year old) students working in groups, who were asked to give an example of a function graph $f(x)$ with particular properties, such as maximums or minimums, and whose antiderivative also had certain properties. The authors proposed a cycle of example production and modification to describe the phases of example generation: (1) conjecturing, (2) rejecting, (3) iterating phases 1 and 2, and (4) concluding. The strategies of example generation, identified by Antonini (2011) and Hazzan and Zazkis (1999), appear to fit within Arzarello's et al. conjecturing phase. Examples, once generated by any means, are conjectured to fit the requirements; (potential) rejection results in an iteration of the first two phases until a suitable example is achieved. Arzarello et al.'s study differed from previous studies because the authors investigated a collective example space, generated by the participants in the classroom, rather than personal example spaces. In a whole group setting, the teacher intervened to direct "attention to discern a specific detail and then to recognise a relationship by perceiving a failure to instantiate a property" (p. 304). This served to bring about awareness of logical and theoretical requirements of the task to support structure in students' example spaces.

Watson and Chick (2011) analyzed four instructional episodes in which an experienced teacher used several sets of examples and discussed her pedagogical intentions with the authors. The authors recognized that the intention behind example use or generation was generally implicit, leading to discrepancies between the teacher's intention for the example and students' perception of it. This is in contrast to the teacher described by Watson and Shipman (2008), who often encouraged students to use the mathematical strategy of "looking at special cases," to conjecture about the structure of a concept. In this case, the purpose behind the generation of such "special cases" is made clear. Watson and Chick suggested that "Learning to learn mathematics includes learning what to do with examples, so that learners can choose from a range of ways to engage" (p. 293). This suggests that when using examples, whether teacher or student generated, it is important to imbue the set of examples with a purpose in order for students to discern mathematical ways to engage with examples and perceive the variation that the teacher had intended. This directly relates to making mathematical strategies and thinking visible within the classroom, as implied by the teacher's guidance in the study by Watson and Shipman (2008). In other words, the mathematical purpose of generating examples should be made clear by the teacher so that students have the opportunity to discern the various ways of engaging with examples to support their own mathematical reasoning and independence.

Implications for the Design of Tasks.

The combined research on exemplification, and specifically on teaching and learning through structured exercises and LGEs, suggests a number of principles for the design of tasks that incorporate structured exercises and LGEs. Tasks should:

- *should align with a clear purpose*, meaning that the task is reflective of an explicit purpose identified by the designer.
- *have balanced variation*, meaning that there is neither too much or too little variation in the relevant and irrelevant features of the examples or exercises. Too much variation can make it difficult, if not impossible, to discern the critical aspects of the object of learning, while too little variation can induce tedium.
- *promote mathematical sense-making* and offer opportunities for conjecture, verification, generalization and abstraction. This can be brought about by connections (either sameness or difference) between elements in a set, or instances of individual disturbance that fail to fit the learner's current sense of pattern.

Through optimal patterns of variation, students' attention may be drawn to the dimensions of possible variation and their associated ranges of permissible change, conveying both a sense of generality, that this example is a particular instantiation of a larger class of such examples, and means of generating new examples by varying the dimensions of variation along values within their range. While it is possible for students to make these perceptual shifts of generality and generation independently, research has shown that the teacher plays a critical role in students' learning from structured exercises and LGEs through the enactment of such tasks.

Implications for the Enactment of Tasks.

Recent research on teaching through structured exercises and LGEs suggests a number of implications for enacting tasks that incorporate structured exercises and LGEs. Because the teacher plays a critical role in students' learning from structured exercises and LGEs through the enactment of structured exercises and LGEs, there are parallels between the suggested implications for the design of such tasks and the enactment of such tasks. Hence, there is an interplay between the design and enactment of a task. In enacting these tasks, teachers should:

- *imbue the task with a purpose* that matches how students engage with the task.
- *provide scaffolding* for students that draws attention to the variance and invariance present in a set of exercises or examples and supports students in recognizing and expressing generalizations regarding the common structure of the elements in the set, particularly when students have reached an impasse.
- *encourage student independence* by pushing students to persist in their efforts to make sense of the set of exercises and examples. This includes allowing the time needed for students to develop a deep understanding, and encouraging discussion and debates among students with minimal interference.

The delicate balance between encouraging student independence, yet providing scaffolding to students as needed, is related to the responsive teaching required of the teacher when the lesson follows students' thinking. This necessarily requires in-the-moment decision making and careful attention to students' thinking on the part of the teacher. In the next sections, I review the literature about teacher learning, in particular teacher learning through the noticing of students' thinking and teacher learning through participation in lesson or learning study.

Teacher Learning

The research literature pertaining to LGEs clearly states that teachers, as well as students, have the opportunity to learn when teaching through LGEs. The mechanisms that bring about this learning for teachers, however, are less clear. What is it about accumulated teaching experience, or professional development experience, or enacting tasks involving LGEs that results in teacher learning? This section will discuss the research literature pertaining to teacher learning, specifically in regards to teacher noticing and participation in learning study or lesson study.

Teacher Learning Through Noticing

As the literature about teaching though LGEs suggested, teachers' in-the-moment decision making is of the utmost importance when a portion of the control for the direction that a lesson takes is given up to the students, and students' thinking becomes the central part of the lesson. In-the-moment decision making necessarily relies on what teachers are noticing about the events unfolding in the classroom and what possible responses or strategies become available to teachers within that moment. While some of the literature about noticing refers to aspects of the classroom that teachers notice, such as student behavior and thinking, I take a view of noticing as becoming aware of and developing one's own propensity to notice and modify one's own behavior. In a sense, this is akin to developing the accessible example spaces of teachers' pedagogical moves or responses. In order to change one's own behavior in the moment, moves or responses must become available (the development of one's own propensity to notice a particular response or move as appropriate in the given situation) in order to be enacted, and hence, act in a way that is different from how one might have responded previously. In this section I focus on literature related to teacher noticing in an effort to make sense of teachers' abilities to respond in real time to students' generated examples, conjectures, and reasoning.

Researchers have conceptualized what it means for a mathematics teacher to notice in many different ways. Some researchers take the view that the most important aspect of noticing is what teachers initially perceive when viewing a classroom lesson and what they miss (Star & Stickland, 2008; Star, Lynch & Perova, 2011). These authors argue that if aspects of classroom events do not catch teachers' attention in the first place, there is no opportunity for teachers to think about or respond to those events. The most crucial part of developing professional noticing is to develop the capability of seeing a multitude of classroom aspects. Other researchers include teachers' interpretations of what they perceive as a component of teacher noticing (Sherin, Russ, & Colestock, 2011; van Es, 2011; van Es & Sherin, 2008). This perspective is based on the assumption that teacher's expectations and knowledge influence what is perceived and how those perceived aspects are made sense of. Yet another conception of teacher noticing encompasses what teachers perceive, how they interpret it, and their intended response (Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). Jacobs et al. (2011) describe professional noticing of children's mathematical thinking as being comprised of three interrelated skills: (1) attending to children's strategies, (2) interpreting children's understandings, and (3) deciding how to respond on the basis of children's understandings (p. 99). According to Jacobs et al. (2010, 2011) these three skills occur almost simultaneously in the midst of teaching, and deciding how to respond is closely linked to attending to and interpreting students' understandings within teachers' in-the-moment decision making.

A number of researchers have commented on the highly variable nature of teacher noticing (Erickson, 2011; Jacobs et al., 2011; Miller, 2011). One of the most consistent findings is the difference between novice and experienced teachers in their abilities to notice. Jacobs et al. (2010) included four experience levels of participants in their study to better understand the development of teachers' abilities to notice children's mathematical thinking: preservice teachers, and experienced inservice teachers separated into three groups, based on their professional development experience: initial participants, advancing participants, and emerging teacher leaders. The focus of Jacob et al.'s study was about attending to children's mathematical thinking in order to teach in ways that build upon that thinking. The researchers hypothesized that teaching experience alone was not sufficient, and so separated teachers based on professional development experience with children's mathematical thinking, rather than number of years of teaching experience. The results showed that teaching experience alone can increase teachers' abilities to attend to children's strategies and interpret children's understanding, to some extent, as evidenced by the difference in the group means between preservice teachers and initial participants, who had yet to participate in professional development about children's mathematical thinking. Similar gains were not seen, however, for teachers' abilities to decide how to respond on the basis of children's understandings. Professional development about children's mathematical thinking, however, supported teachers in developing expertise in each of the component areas, and those teachers with sustained professional development (more than two years) continued to strengthen their expertise in interpreting children's understandings and using those interpretations to decide how to respond. Jacobs et al.'s study demonstrated the differences not just between novice and experienced teachers, but also differences in teachers' abilities to notice within the group of experienced teachers, based on their participation in professional development about children's mathematical thinking.

In addition to variability in teachers' abilities to notice based on teaching experience, and professional development experience, Erickson (2011) claimed that interpretations of what teachers notice differed based on the teachers' teaching philosophies. In other words, what teachers see and perceive is through their own lens of the world, or what Erickson called "pedagogical commitments." Despite one's own perspectives, however, Mason (2011) argued that if one can leave open her interpretations to allow for multiple possibilities, one can train oneself to be alert and intentional about particular things that are noticed so that in the future, one can act in news ways, rather than out of habit. Mason calls this the discipline of noticing. Teachers can prepare to notice in the moment by anticipating what might occur and actions they might like to come to mind, and teachers can "post-pare" by reflecting on recent teaching episodes to select what one wants to notice or be sensitized to in the future. Both preparing and "post-paring" are mental actions that can support teachers in noticing and acting in the moment, and bring about fresh, rather than habitual, actions.

While some research has used pre- and post-assessment tasks to measure changes in teachers' abilities to notice (Jacobs et al., 2010, 2011), qualitative analyses of the nature and development of teachers' noticing have begun to emerge to provide some frameworks and indicators of growth in teachers' abilities to notice. Jacobs et al. (2010) identified a number of shifts in teachers' professional noticing of children's mathematical thinking, including:

- A shift from general strategy descriptions to descriptions that include the mathematically important details;
- A shift from general comments about teaching and learning to comments specifically addressing the children's understandings;

- A shift from overgeneralizing children's understandings to carefully linking interpretations to specific details of the situation;
- A shift from considering children only as a group to considering individual children, both in terms of their understandings and what follow-up problems will extend those understandings;
- A shift from reasoning about next steps in the abstract (e.g. considering what might come next in the curriculum) to reasoning that includes consideration of children's existing understandings and anticipation of their future strategies; and
- A shift from providing suggestions for next problems that are general (e.g., practice problems or harder problems) to specific problems with careful attention to number selection. (p. 196)

Jacobs et al.'s growth indicators take teacher noticing to include the teacher's intended response, as reflected in the growth indicators that include reasoning and suggestions about next steps. The growth indicators tend to move from attending to features in general and reasoning in the abstract to more specific mathematical details and reasoning about next steps. Jacobs et al. caution that growth takes time and shifts may be minimal at first.

How do teachers develop their abilities to notice in the mathematics classroom? In my study, I am using a conception of noticing that includes all three interrelated components of (1) attending to students' mathematical thinking, (2) interpreting students' mathematical thinking, and (3) deciding how to respond on the basis of students' understanding. Jacobs et al. (2011) found that teaching experience, alone, is not sufficient for developing teachers' propensity to respond on the basis of students' understandings. In Jacob et al.'s study, professional development came in the form of five full days of workshops per year that drew heavily on the professional development project Cognitively Guided Instruction. What other forms could effective professional development take? In examining the development of expertise in other fields, Miller (2011) borrowed the term deliberate practice from a study on the development of expert musicians. “Deliberate practice requires (a) well-defined tasks at appropriate levels of difficulty, (b) informative feedback, and (c) opportunities for repetition and correction of error” (p. 57). Miller recognized that American teachers have few opportunities for deliberate practice. Mason (2011) suggested the strategies described in the discipline of noticing as a kind of deliberate practice that teachers can undertake on their own. Miller looked to video-based viewing tasks as opportunities for deliberate practice with preservice teachers, but also pointed out that Japanese lesson study methods can be seen as an example of deliberate practice. In the next section I examine lesson study and learning study as a means to support teachers in developing their abilities to notice through deliberate practice.

Deliberate Practice: Learning to Notice Through Learning Study

Learning study arose from the specific desire to help teachers put variation theory into practice (Lo, 2012). Inspired by both Chinese teaching studies (Gu, 1991, Ma, 1999) and Japanese Lesson Study (Lewis, 2002; Stigler & Hiebert, 1999), variation theory provides the theoretical framework for the design, implementation, and analysis of the lessons, while learning is studied on three levels: (1) student learning of a particular object of learning, (2) teacher learning related to how the enactment of the lesson opens or narrows opportunities for student learning, and (3) researcher learning in regards to helping teachers use variation theory as a pedagogical tool (Lo, 2012). Olander and Nyberg (2014) sum up the difference between lesson study and learning study:

> The latter [learning study] is focused on the content and its use of a learning theory, whereas lesson study also can be focused on methods or other parts of the lesson not concerning the content and does not need a theoretical framework as a guiding principle. (p. 239)

While lesson study is not necessarily devoid of a theoretical basis, the theory that underpins learning study is always variation theory.

Lo (2012) presented evidence to indicate the impact of learning studies from a three-year longitudinal study from the *Catering for Individual Difference – Building on Variation (CID(v)) Project*, conducted in Hong Kong from 2000-2003. Of 29 learning studies that had been carried out in mathematics, Chinese language, general studies, and English language in the two project schools, 27 had a complete data set of pre-tests and post-tests, and in 24 of those 27 learning studies, the research lesson had a positive effect on the performance of the whole group, with 25 of the 27 studies showing that the gap between the higher performing students and the lower performing students had narrowed. The impact of learning study on teachers' professional development was more difficult to measure, although questionnaires administered to all of the teachers and principals in the project were positive. Learning study can serve as a means of bringing novice, experienced, and expert teachers together to mutually engage in professional development focused on teaching and learning.

The teaching and learning of an object of learning are inextricably linked. Holmqvist, Gustavsson, and Wernberg (2008) stated that, "The enacted object of learning can be affected by what the teacher does or says, the students' own reflections, other students, or the learning materials" (p. 111). Learning study is focused on determining which of these factors matter and in what ways these factors matter in regard to a particular object of learning. In a learning study guided by variation theory, Olander and Nyberg (2014) conducted three cycles of learning study, each using a different lesson design for a single object of learning, informed by the previous implementations. In support of their theoretical position, the authors found that “certain patterns of variation seem to be more powerful than others in developing students’ knowledge” (p. 258). The “certain patterns” that Olander and Nyberg refer to, however, are not necessarily generalizable because content matters. In the case of children learning how to halve and double, the pattern of variation that produced the most significant results between pre- and post-test scores was when children simultaneously discerned the base value (what is being doubled or halved?) and the targeted amounts (either half or double). They found that a task cannot be separated from how it is enacted. Knowing what to say and do as the teacher should be grounded in an idea about what makes learning possible. To this end, variation theory can be used as a guide to help develop this skill, while learning study can be useful for both teachers’ professional development and the enhancement of students’ learning.

In terms of how a task is enacted by the teacher, learning study can provide a means to delve more deeply into the nuances in the communication between teacher and students and to understand ways in which what is said opens or narrows the space of learning (Olteanu, 2014). Olteanu (2014) suggests that communication succeeds when an understanding of a critical aspect of an object of learning is shared between speaker and hearer, whether this is communication between teacher and student or student to student. For complex concepts, in particular, such as those that appear in algebra at the secondary level, the success of communication depends on the opportunities to discern the meaning of the whole of the complex concept by knowing the meaning of the simple parts and the ways in which the simple parts are composed to instantiate the whole.

Since learning study is focused on student learning of a particular object of learning, it is important that teachers notice and respond to student thinking. The teacher in a learning study research lesson has the opportunity to attend to student strategies, interpret students' understandings, and decide how to respond on the basis of that understanding. Observers in a learning study research lesson have the opportunity to attend to student strategies, perhaps in more attentive ways, even, than the teacher herself, and interpret students' understandings. While observers do not have the opportunity to decide how to respond in real-time, they do have the opportunity to decide how they might respond. The alternative decisions that observers propose that they may have pursued, and the additional evidence provided by observers' close attention to student thinking provide rich opportunities for teacher discussion and, potentially, subsequent learning.

Questions for this Study

There are many open questions within the realm of teaching and learning through structured exercises and LGEs. The conception of structured exercises and LGEs as pedagogical tools is a relatively recent development in the mathematics education literature. There is sufficient evidence from the existing literature, however, that arranging examples and exercises with purposeful structure supports students in discerning critical aspects of mathematical concepts, and procedures (Petty & Jansson, 1987; Mason et al., 2009; Wilson, 1986). There is also sufficient evidence from the research literature, that example generation, rather than example presentation, better supports students in developing a deep and connected conceptual understanding (Dahlberg & Housman, 1997; Watson & Mason, 2002; Zaslavsky & Peled, 1996). Two lines of questioning could be further pursued – one directed at teachers' conceptualization of teaching through structured exercises and LGEs and her implementation of these constructs, and another directed at students' learning through structured exercises and LGEs. I have decided to pursue the former because students will not have opportunities to engage with and experience examples and exercises in these ways until teachers choose to incorporate them into their teaching.

While I am fairly certain that teachers generally have some rationale for their choice and sequencing of examples and exercises (Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008), it is unlikely that American teachers have heard of the notion of structured exercises and LGEs, explicitly. In fact, variation theory, which provides the bedrock for both structured exercises and LGEs, is relatively unheard of in the U.S. As such, I am not interested in the nature of teachers' knowledge about teaching through structured exercises and LGEs, but rather the development of that knowledge as in-service teachers transition from being introduced to these notions, to designing and implementing tasks that incorporate structured exercises and LGEs in collaboration with their colleagues.

Knowledge about teaching and learning through structured exercises and LGEs includes knowledge about task design and implementation. Task design encompasses teachers' choice of examples or exercises to include in a task, their sequencing and arrangement, prompts to elicit LGEs, and the overall structure of the lesson in which the task or prompts are embedded. Implementation encompasses the actions required by the teacher to advance students' awareness of mathematical structure through structured exercises and LGEs, including engaging students in the task, conveying the intended purpose of the task, and directing students' attention to critical aspects of the exercises and examples put forth for consideration. Thus, the development of teachers' knowledge about teaching and learning through structured exercises and LGEs suggests at least two lines of inquiry, one regarding pedagogical intent, which impacts on task design, and one regarding pedagogical implementation.

The goals of this study are to understand how teachers' knowledge about designing and enacting tasks that incorporate structured exercises and LGEs develops and seek to characterize this development. The following research questions will guide my study:

1. How do teachers conceptualize and develop their knowledge about task design that structures students' experiences of learning algebraic constructs?
2. How do teachers develop their knowledge about enacting tasks that incorporate structured exercises or learner generated examples (LGEs) in ways that support students in developing an awareness of algebraic structure?
3. What factors influence and shape teachers' conceptualization and implementation of structured exercises and learner generated examples (LGEs)?

CHAPTER 3 – METHODS

In this chapter, I discuss my research design, the participants, data collection, and analysis of the data. In brief, a team of four middle-grades mathematics teachers participated in four learning study cycles focused on the use of structured exercises and LGEs in their instruction. I sought to understand the development of teachers' knowledge about the design and enactment of tasks that incorporated structured exercises and LGEs in instruction and the factors that influenced that development. In summary, this is a case study about a team of middle grades mathematics teachers learning to design and implement sets of examples using specific patterns of variation through the process of learning study.

Research Design

To address my three research questions, I conducted a qualitative study to investigate how middle grades mathematics teachers developed their knowledge about teaching and learning through structured exercises and LGEs and the factors that influenced and shaped teachers' conceptualization and implementation of structured exercises and LGEs. I structured my work with the teachers using variation theory. My object of learning for the teachers was the design and implementation of structured exercises and LGEs. From the research literature and my own experiences, I identified two critical aspects of this object of learning: patterns of variation and the relevance structure. Critical features of the critical aspect patterns of variation included contrast, generalization, and fusion. The critical aspect relevance structure refers to the purpose for which a set of examples is being used. I wanted teachers to discern that sets of examples can have different relevance structures of purposes, and that learners should engage with the set of examples in ways that align with the relevance structure. The enactment of my plan for teachers' learning occurred through five hours of summer professional development and the series of four learning study cycles, which are discussed in the following sections. I analyzed multiple sources of data, including observations, interviews, and the collection of artifacts (such as the plan-to-guide learning, SMART Board files, class materials, etc.) created through the teaching and learning process, using variation theory as a framework for analysis. In a sense, my analysis of the four research lessons is an analysis of the enacted object of learning, and my analysis of the individual teachers' use and views of examples after the learning study intervention is an analysis of the lived object of learning. My research questions pertained to the development of teachers' knowledge over time, which resulted in a research design challenge about how to measure and describe teachers' learning. The theory of example spaces allowed me to describe changes in teachers' knowledge related to teaching and learning through structured exercises and LGEs by examining teachers' example spaces, consisting of their use and views of examples, in terms of population, generativity, connectedness, and generality over time.

As discussed in Chapter 2, Watson and Mason (2005) proposed the concept of example space and the subsequent development of a student's example space as an indicator of learning. Goldenberg and Mason (2008) defined an example space as "an experience of having come to mind one or more classes of mathematical objects together with construction methods and associations" (p. 189). Drawing on variation theory (Marton & Booth, 1997; Marton & Tsui, 2004), Watson and Mason contend that learning occurs as more dimensions of possible variation are discerned for a particular mathematical object and their associated ranges of permissible change are determined, expanding the scope and connectedness of the learners' example space. Dimensions of possible variation may or may not be critical aspects of a particular object of learning. Developing a particular way of seeing or perceiving of an object of learning requires that the learner separate critical aspects from non-critical aspects. A possible dimension of variation for structured exercises is the number of exercises within the set. The number of exercises within a set of structured exercises, however, is not a critical aspect of teaching and learning through structured exercises. A critical aspect of teaching and learning through structured exercises is the pattern of variation that learners have the opportunity to experience in working through the task. But the number of exercises is not quite irrelevant, as it relates to students' abilities to discern a pattern. A single exercise is unlikely to hold enough variation for students to discern the intended critical aspects. Likewise, too many exercises may induce a procedural orientation, and possibly tedium, shifting the purpose of the task from structural understanding to procedural fluency and memorization. The number of exercises in a set of structured exercises are particular values along the dimension of variation number of exercises. The particular values along that dimension instantiate the range of permissible change. While the range of permissible change is unlikely to include just one exercise, it may include two (carefully designed) exercises, up to any other number, although it seems likely that an upper limit could be suggested. Discerning dimensions of possible variation for a particular object of learning and their associated ranges of permissible change expands the scope of a learner's example space.

Zaslavsky and Zodik (2007) and Zodik and Zaslavsky (2008) extended the construct of example spaces to teachers' knowledge, suggesting that teachers' learning has to do with expanding their example spaces about the act of exemplification, including awareness about pedagogical potential and limitations in choice of examples. Based on the literature, I propose that a teachers' example space about learning and teaching through structured exercises and LGEs also encompasses knowledge of design, which includes example choice and awareness of pedagogical potential, and implementation, as various studies have found that enacting tasks involving examples and exercises in ways that match their intentions, or the relevance structure, is not trivial (Arzarello et al., 2011; Watson & Chick, 2011; Watson & Shipman, 2008). According to Sinclair et al. (2011), a personal example space (PES) includes a repertoire of available examples and methods of example construction. A PES around teaching and learning through structured exercises and LGEs would include examples of how structured exercises and LGEs can be used in teaching and learning and methods for constructing and designing sets of structured exercises and prompts for LGEs.

Sinclair et al. (2011) presented central characteristics about how a PES appears to be structured: (1) population, referring to the scarcity or availability of examples, (2) generativity, referring to the possibility of generating additional examples within the space, (3) connectedness, referring to examples being disconnected, loosely connected, or well-connected, and (4) generality, referring to the specificity of the example and its representativeness of a class of examples. These four aspects are not independent, however, and changes in one will likely be accompanied by changes at least one of the others. While the collection of particular artifacts (e.g., lesson plans, materials used in instruction), video recordings of lessons, and interviews indicated some of these characteristics of teachers' PES, it was important that the methodology made teacher thinking about designing and using examples visible to lead to a deeper understanding of the development of teachers' knowledge about teaching and learning through structured exercises and LGEs over time.

Lesson study results in a variety of artifacts (e.g. lesson plans, handouts, worksheets, teachers' observation notes, and student work), as well as opportunities for generating discussion among teachers related to ideas about teaching and learning, thereby making teachers' thinking visible (Lewis & Hurd, 2011). As discussed in Chapter 2, learning studies are a type of lesson study that takes variation theory as a theoretical framework for the design, implementation, and analysis of the lessons (Pang & Ling, 2012). Variation theory provides the theoretical underpinnings for structured exercises and LGEs as a means of supporting learners in becoming aware of mathematical structure. Thus, teachers used variation theory as a theoretical guide for the specific purpose of designing and implementing structured exercises and LGEs. Since variation theory serves as the common theoretical basis between structured exercises, LGEs, and learning study, learning study was an appropriate methodology for both data collection and supporting teachers in designing and implementing structured exercises and LGEs in their instruction.

In light of my research questions, I gave careful consideration to how the development of teachers' knowledge about teaching and learning through structured exercises and LGEs could be measured and described. My literature review revealed variation theory and the theory of example spaces as theoretical foundations for teaching and learning through structured exercises and LGEs. Variation theory was used to describe the characteristics of teachers' choices of examples and exercises in instruction (Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008), and was also suggested as a framework for the design of tasks that supported students in discerning mathematical structure (Mason, Stephens, & Watson, 2009). The theory of example spaces also draws on variation theory in terms of describing the scope and connectedness of a learner's example space. Example spaces arose as a construct to describe the space where LGEs came from. The development of one's example space is associated with learning, and was extended to the development of teachers' knowledge about exemplification (Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008). Drawing on these theoretical perspectives, I analyzed the development of teachers' knowledge about teaching and learning through structured exercises and LGEs. Variation theory provided a lens through which I could analyze the individual and collective space of learning generated by the learning study intervention and discern contrast within and between individual teachers' knowledge over time as the teachers sought to discern the critical aspects of designing sets of structured exercises and LGEs and teaching through variation. The theory of example spaces allowed me to characterize the individual teachers' PES as snapshots in time prior to and after the learning study intervention. The characterization of teachers' PES was evidenced by the teachers' thinking about individual and collective design and enactment of sets of examples as revealed through the learning study process and individual observations.

Participants

Tori, Robert, Lynn, and Shannon formed a team of seventh and eighth grade mathematics teachers at Augustus Middle School, a public school in the northeastern United States. I had a previous professional relationship with these four teachers, and the school district in which they worked, that allowed me access. In my previous relationship with the teachers, we had established a rapport that allowed for their willingness to work with me for this study. Because of the implementation of the Common Core State Standards in 2012-2013, and the inadequacies of available textbooks in terms of coverage and alignment to the standards, this team of teachers had been designing their own instructional materials for some time. Tori, Robert, and Lynn had previously collaborated on "unpacking" the standards, identifying the specific concepts and skills that were required of students at the seventh and eighth grade levels and supporting fourth through sixth grade teachers in the "unpacking" process and alignment to the middle grades standards. While they were mostly happy with the instructional materials that they had created over the previous few years, there were certain areas where they wanted to improve student learning outcomes, and they recognized that improvements in student learning outcomes stemmed from the design and implementation of their instructional materials. Because of these prior experiences, this particular team of teachers were interested in engaging in learning study to improve their design and practice for certain content areas.

Tori Goodman had seven years teaching experience, with the last four at Augustus Middle School. The entirety of her career had been at the middle school level. During this study, Tori taught four sections of seventh grade mathematics and one section of seventh grade accelerated mathematics. Robert Cavins had been teaching at Augustus Middle School for eight years. He had one year of previous teaching experience in a neighboring district, and so was beginning his tenth year of teaching at the time of this study. Robert taught four sections of Math 8 and one section of Algebra 1 to a class of accelerated eighth graders. Lynn Gray had 24 years of teaching experience at the time of this study. She had been teaching at Augustus Middle School for 13 years, ten of which were as a regular classroom teacher. During those ten years, she primarily taught eighth grade mathematics and Integrated Algebra for accelerated eighth grade students. Prior to that, Lynn taught at the middle school and high school level for 11 years at a neighboring school district, but primarily seventh grade mathematics for the last five years there. At the time of this study, Lynn taught seventh and eighth grade response-to-intervention (RTI) classes, a role she had held for three years. These classes were each comprised of two to seven students who had been identified as students in need of additional mathematics support through state and district testing, as well as teacher recommendation. Lynn taught five sections of Math 7 RTI and five sections of Math 8 RTI, seeing her students for 40 minutes every other day. She saw her role as supporting what was being taught in the regular seventh and eighth grade mathematics classrooms. She worked closely with the three regular classroom teachers, often drawing on their materials to work on the same procedures and concepts with students in her class and present and discuss them in a consistent manner. Shannon Edwards had been teaching at Augustus Middle School for three years, with one year of full-time teaching and two years of part-time teaching. Prior to that, Shannon served as a substitute teacher in a neighboring district for six years, with the last two years in long-term substitute positions. At the time of this study, Shannon was a part-time teacher and taught two sections of eighth grade mathematics and one section of seventh grade mathematics. This team of teachers had been colleagues for three years. None of the teachers had prior experience with structured exercises or LGEs, although all employed the use of examples and exercises in their instruction, as is often the case in the teaching of mathematics.

Setting

The middle school where the participants worked was part of a mid-size (<2000 students in K-12) public school district. The district was considered a suburban district situated on the fringe of a more rural area. This meant that there was a mix of students within the district, some of whom lived in typically suburban neighborhoods, and others who lived in the more rural farming areas of the district. The racial composition of the students who attended the school was predominantly White (97%), and about 16% of students were eligible for free or reduced-price school lunches. The student to teacher ratio was 15:1, and approximately 11% of the students had an individualized education plan (IEP) (U.S. Department of Education, 2013).

Collaboration among colleagues was encouraged and supported by the administration. A 40 minute common planning time for subject area teachers was built into the daily schedule. At the time of this study, the district offered small grants for teachers to support their individual and collaborative efforts to enhance students' learning or teachers' professional development. The district's support for teachers' action research was also evidenced by the publication of a monograph each year in which the results of teachers' research were shared throughout the district.

Data Collection

In the spring and early summer of 2015, I collected an initial data set to help me understand the teachers' use and views of examples prior to the learning study intervention. I observed three of the four teachers (as Tori was on leave), and I interviewed each of the four teachers. In August 2015, the teachers participated in five hours of professional development about structured exercises and LGEs. The purpose of the five hours of professional development was to support the teachers in discerning the critical aspects patterns of variation and relevance structure. I engaged teachers in a series of LGE activities, including "Give An Example Of…(another and another)", "Additional Conditions", and "Burying the Bone" (Bills et al., 2004). I intentionally used high school mathematics content, rather than middle school content, to engage the teachers in thinking about the task in the way a student might. For instance, I asked teachers to give an example of a periodic function, and then another, and then another, until a number of potential examples had been generated. From this set, I asked the teachers to verify which examples were, in fact, examples of periodic functions and which were not. I followed-up with a series of questions: What aspects of the examples could change while maintaining examplehood of a periodic function? What aspects must remain invariant? Based on this, how would they define what a periodic function was? After engaging in the task as learners, I led a discussion with the teachers about the pedagogical affordances of such a task. I asked the teachers how it differed from mathematics tasks they had previously experienced and what learning could arise from such a task. This experience with and discussion about an LGE, among other experiences during the summer professional development, were meant to contrast with previous ways in which the teachers and used and experienced examples in order to discern pattern of variation and relevance structure as critical aspects of sets of examples. Readings about using LGEs with learners (Zaslavsky & Zodik, 2014) and variation theory, and the subsequent discussions, supported these experiences. In addition, I gave each teacher a copy of *Thinkers* by Bills et al. (2004) a booklet that included a number of activities to stimulate mathematical thinking. I pointed out the activities that I had used with them, as learners, and suggested that they use it as a resource for design throughout the four learning study cycles. Lastly, since the team of teachers was new to learning study, we spent some time agreeing upon and establishing group norms, and I introduced them to the process of a learning study cycle.

In September 2015, the teachers and I began to meet during their daily common planning time twice a week to prepare for each of the four learning study cycles. On professional development days, we occasionally met for additional time. The focus of the learning study cycles was on supporting students in developing an awareness of algebraic structure through the use of structured exercises and LGEs. Figure 3.1 presents the procedure for conducting a learning study (Lo, 2012, p. 33). The teachers and I, as a participant observer, engaged in planning the research lesson, which included selecting a topic, identifying a tentative object of learning, diagnosing students' learning difficulties, confirming the object of learning and its critical aspects, and designing sets of examples. Each of the four teachers taught one of the research lessons. The other teachers, myself, and occasionally outside observers, observed the research lesson. One of those outside observers was Beth, a special education teacher. Beth worked predominantly with Lynn, but collaborated with each of the teachers to provide services for students with special needs. After observing the first research lesson, Beth chose to join the team of teachers and myself for the remaining three learning study cycles. In order to evaluate the learning outcomes of the learning study cycle (as shown in Fig. 3.1), we met for a debriefing session immediately after each research lesson to discuss the teachers' observations about student learning related to the design and enactment of the lesson (Lewis & Hurd, 2011). We generally met a few days after the research lesson, as well, to continue to evaluate the lesson and students' learning in light of the teachers' analysis of student work collected after the research lesson or observed in class. This secondary evaluation of the learning outcomes allowed the teachers to understand the lived object of learning. The lived object of learning is what the students did learn at the completion of the lesson, and beyond. The teachers often chose to give some sort of post-test, either as homework or during the next lesson. The results of this assessment, and what could be garnered about student learning in terms of the object of learning, could not be discussed at the debriefing session immediately after the research lesson, so a second evaluation meeting was necessary to develop a more complete picture of the lived object of learning. The secondary evaluation meeting also allowed for the teacher who taught the lesson to report back on how students' learning (or lack thereof) was reflected in the subsequent lesson. Taken together, the debriefing session and evaluation meeting was the basis for evaluating the overall impact of the learning study cycle among the teachers.

Data was collected from multiple sources, including observations of teaching, observations of planning and evaluation meetings, interviews, and artifacts created out of the learning study process. Each teacher was observed teaching in his or her classroom four to five times throughout the study: (1) an initial observation prior to the learning study intervention, with the exception of Tori, as discussed, (2) two observations during the learning study intervention, (3) an observation of the research lesson that the teacher taught, and (4) a post observation after participation in four learning study cycles (see Figure 3.2). Observing the

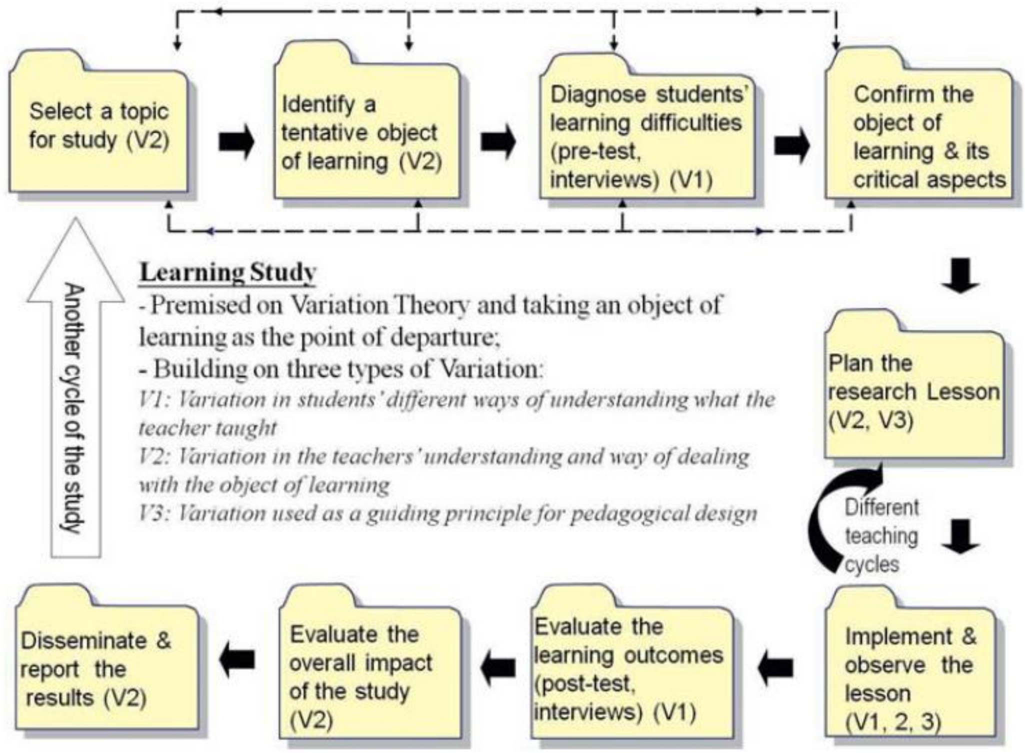


Figure 3.1. Procedure for conducting a learning study (Lo, 2012, p.33)

teachers prior, during, and after the intervention allowed me to observe changes in individual teachers' practice over the course of the study. When taking field notes for each observation, I focused on teachers' use of examples in instruction. The following questions, adopted from Watson and Chick (2011), guided my field notes:

- What examples are used?
- How many examples are used?
- Who provides the examples?
- How are examples introduced?
- How are examples discussed?
- What questions does the teacher ask?
- What features of examples are highlighted?

Artifacts associated with each observation were collected. Artifacts included materials for the observed lessons, such as handouts (e.g., note sheets, worksheets), electronic materials (e.g. SMART Board files), and homework. None of the teachers in this study wrote lesson plans. Instead, their SMART Board files served as their lesson plans. For each of the research lessons, the teachers wrote a Plan-to-Guide Learning (adapted from Lewis & Hurd, 2011, see Appendix A) to communicate their collaborative work and decisions amongst themselves, with me, and with outside observers.

Each planning meeting was audio recorded, and I took notes to capture the ideas pertaining to examples talked about by the teachers. For each week of planning meetings, I would listen to the recorded audio and create a detailed memo that combined the notes I had written during the planning meetings and transcriptions of episodes from the audio recordings that involved teachers' discussions pertaining to the purpose, design, or intended enactment of examples. Each of the four research lessons was video recorded, with a focus on the teacher. I took field notes of my observations of the four research lessons, using questions listed above to guide my observation. I audio recorded the debriefing meetings held immediately after each research lesson, and the evaluation meetings that were held a few days later. I also took notes during the debriefing and evaluation meetings to capture the ideas that teachers talked about regarding the design and implementation of examples. I collected copies of teachers' observation notes from each research lesson.

Each of the four teachers was interviewed twice, once prior to the learning study intervention, and once after the completion of the four learning study cycles. The interviews were semi-structured (see Appendix B for the interview protocol for the initial interview and Appendix C for the interview protocol for the post-interview) and focused on the teachers' use and view of exercises and examples in their instruction. Each interview lasted approximately 30 minutes to one hour, and was audio recorded and transcribed.

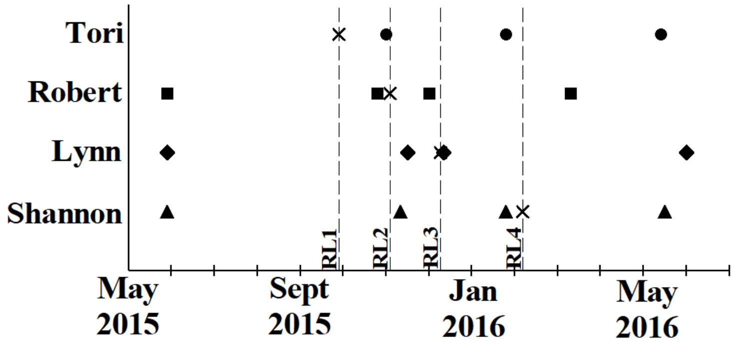


Figure 3.2. Timeline of each teachers' observations

Data Analysis

My analysis of the data was through the perspective of variation theory. Variation theory provides the theoretical underpinnings of learning study, the theory of example spaces, structured exercises, and LGEs. Through a lens of variation theory, instances of contrast, generalization, and fusion as patterns of variation were identified throughout the data sets. I used my own analytical framework to identify excerpts of the data that pertained to (1) the purpose, (2) the design, and (3) the enactment of examples, for each individual teacher and collectively for the group of teachers during the course of planning the four research lessons. Variation theory allowed me to analyze the individual and collective space of learning generated by the learning study intervention. After this analysis, I passed over the data a second time to characterize the teachers' PES to answer my research questions about the development of teachers' knowledge regarding task design and enactment. The theory of example spaces served as an analytical framework to provide snapshots in time of each teacher's PES of his or her use and views of examples (1) prior to the learning study intervention, and (2) after the learning study intervention. This framework included four characteristics of the structure of PES: (1)

population, (2) generativity, (3) connectedness, and (4) generality. For my analysis, I modified the descriptions of the four characteristics to be specific about the characteristics of teachers' personal example spaces about their use and views of examples:

- population: refers to the scarcity or density of available ways (design and enactment) and reasons (purpose) to use exercises and examples.
- generativity: refers to the possibility of generating new sets of examples in the form of structured exercises and LGEs as pedagogical tools.
- connectedness: refers to whether ways of viewing and using examples are disconnected, loosely connected, or well-connected.
- generality: refers to the extent to which a set of examples is specific or whether it is representative of a class of related sets of examples (i.e., structured exercises or LGEs).

I coded excerpts of the data that reflected each of these characteristics for each individual teacher. Teachers' design and enactment of examples, and their articulated purposes for using particular examples, comprised means of using examples that came to mind for the teacher and were made visible to me, as the researcher. Generativity, connectedness and generality are characteristics of the example space that intersected with the population and had the potential to show growth over time.

My first research question was: How do teachers conceptualize and develop their knowledge about task design that structures students' experiences of learning algebraic constructs? I coded each teachers' initial observation and interview using variation theory and the analytical frameworks described above. This served as a preliminary analysis of each teacher's PES of his or her use and views of examples prior to the learning study intervention. I then analyzed the set of four learning study cycles. The analysis of each learning study cycle included the follow data sets: (1) my memos from the planning meetings pertaining to the particular research lesson, merged with transcribed episodes from the audio of the planning meetings pertaining to the discussion of the purpose, design, and intended enactment of examples, (2) my field notes from the research lesson merged with transcribed episodes from the video recording of the research lesson pertaining to the implementation of examples, (3) my memo from the debriefing and evaluation meetings merged with transcribed episodes of the audio of the debriefing and evaluation meetings pertaining to the purpose, design, implementation, and evaluation of examples, and (4) artifacts created from the planning and teaching process, including the plan-to-guide learning, teacher created worksheets and handouts, and SMART Board files. This served as the first phase of analysis. In the second phase of analysis, I considered evidence of changes between individual teachers' preliminary PES and events that occurred during the first learning study cycle and classified these changes as characteristics of the teachers' newly expanded PES. I repeated this process for each of the four learning study cycles. Contrast was the most common pattern of variation that I discerned, and I took instances of contrast as evidence of a change in teachers' PES. Lastly, I coded each teachers' post observation and interview using variation theory and the analytical frameworks described above. Again, I considered evidence of changes between individual teachers' preliminary PES, the events of the four research lessons, and the events of the post-observation and interview. In order to answer this question, specifically, I considered the episodes coded for purpose and design under my own analytical framework of purpose, design, and enactment. Purpose and design pertain to the ways in which teachers used exercises, and so I specifically considered the ways in which the characteristics of teachers' PES changed over the course of the study.

My second research question was: How do teachers develop their knowledge about enacting tasks that incorporate structured exercises or learner generated examples (LGEs) in ways that support students in developing an awareness of algebraic structure? In contrast to the analysis regarding the first research question, data analyzed in order to answer the second research question needed to be about structured exercises and LGEs. The analysis only differed from the previous description in that I identified episodes in which structured exercises or LGEs were enacted or the enactment of structured exercises or LGEs was discussed, since this research question specifically was about the development of teachers' knowledge about enacting tasks that incorporated structured exercises or LGEs. These episodes served as the unit of analysis for answering this question and, as above, I considered the ways in which the characteristics of teachers' PES changed over the course of the study.

My third research question was: What factors influence and shape teachers' conceptualization and implementation of structured exercises and LGEs? This question allowed me to address potential reasons for differences in teachers' knowledge development about designing and enacting tasks that incorporate structured exercises and LGEs. Using a lens of variation theory and contrast, generalization, and fusion as patterns of variation, I compared the individual teachers' use and views of examples throughout the study and used emergent coding to identify possible factors for the differences in their take-up and use of structured exercises and LGEs.

My Role in the Research

The relationship between my participants and me posed potential ethical challenges. For instance, my participants may have felt compelled to participate, rather than participate out of a sincere desire to do so. I reiterated and reminded my participants that it was their choice whether or not they participated in the research study and reminded them that they could withdraw at any time without penalty. I was also cognizant of my positioning and the positioning of the teachers throughout the course of the study. I did not wish to establish myself as the holder of knowledge, but rather sought to learn from my participants. At the same time, I had to balance this with the fact that I had a broader theoretical knowledge base about variation theory, structured exercises, and LGEs from the literature and from my own attempts at the design and enactment of tasks. I provided instruction for the teachers about structured exercises, LGEs, and learning study from the literature, and so I have to acknowledge that in some ways, I was the expert in the room on these matters. I tried to keep from interjecting into the discussions about the planning of lessons, although there were times I felt it was necessary to remind the teachers of some aspect of variation theory, structured exercises, or LGEs. I continually worked to actively put aside my own experiences with incorporating structured exercises and LGEs in my own teaching to be open and receptive to their experiences. There were many instances where I questioned my own choice to insert my voice into the discussion or not. Ultimately, I was a participant observer and a member of the research team, not a silent observer, and so it was necessary to make decisions in the moment about contributing to the discussion or not. I tried to err on the side of giving voice to the teachers' ideas over my own, and most often offered my own thoughts into the discussion when the teachers seemed to have reached an impasse or were not considering a perspective borne out of variation theory.

CHAPTER 4 – RESULTS

In this chapter, I present the results from the analysis of my data with the intention of answering my research questions:

1. How do teachers conceptualize and develop their knowledge about task design that structures students' experiences of learning algebraic constructs?
2. How do teachers develop their knowledge about enacting tasks that incorporate structured exercises or learner generated examples (LGEs) in ways that support students in developing an awareness of algebraic structure?
3. What factors influence and shape teachers' conceptualization and implementation of structured exercises and learner generated examples (LGEs)?

This chapter is divided into three sections. In the first section, I describe each of the four teachers' initial use and views of examples in instruction in terms of purpose, design, and enactment. In the second section, I describe each of the four research lessons in terms of purpose, design, implementation, and evaluation. In the third section, I describe how each teacher's use and views of examples changed from the initial interview and throughout the course of the study.

Teachers' Initial Use and Views of Examples

In this section, I present each teacher's initial use and views of examples in terms of their purpose, design, and enactment, prior to the learning study intervention. The teachers' use and views of examples in terms of their purpose, design, and enactment comprised their initial (accessible) example space of the design and enactment of examples. I summarize each teacher's example space of the design and enactment of examples at the end of each subsection in terms of population, generativity, connectedness, and generality.

Tori Goodman

Tori Goodman had seven years teaching experience, with the last four at Augustus Middle School. Tori had taken a leave of absence at the end of the 2014-2015 school year, so I was not able to observe her prior to the learning study intervention. These findings were based on data collected prior to the intervention, which included the initial interview and class materials, including a video link, which she provided to me from her unit on percentages. Tori used a pedagogical model of a flipped classroom in her percentages unit. In that way, I was able to "observe" the lecture that students received through watching the video assigned for out-of-class viewing and discuss the examples and exercises that Tori chose in both the video and for class as a part of the initial interview. In this section, I present Tori's initial purposes for using examples and exercises, followed by her initial design and enactment of sets of examples.

Initial purposes for using examples. Tori's predominant purpose for using examples was exposure to variation in particular aspects of the examples. One of Tori's goals of teaching this unit was to expose students to a range of variations within word problems involving percentages. Her rationale for variation within the word problems was grounded in the need for students to develop the ability to approach novel situations.

Researcher: I'm hearing you say a lot about the word exposure, like exposure to different situations, exposure to different kinds of numbers, exposure to different orders or different arrangements of numbers, or where they could appear, like in an equation for example. So for you, these examples and exercises are about exposure?

Tori: Yeah. Because I don't want students to be like, well I only know how to solve it if it looks like *this.* In the real world, they want them to have a

> problem and be able to start manipulating it. I don't want them to be like, I've never seen anything like that before…The fact that they've been able to see things in various ways allows them to be more successful in manipulation later. (Preliminary Interview, Lines 464-477)

From Tori's perspective, students' experience of variation in the aspects of dollar amounts, percentage amounts, additional information, givens, and the quantity being asked for would allow them to approach similar situations involving percentages in their future mathematics courses and within real-life situations. Based on Tori's comment that, "they want them to have a problem and be able to start manipulating it," Tori appeared to focus on aspects related to algebraic manipulation. Tori sought to help students recognize that the equation "Original x Percent = Total" could be used throughout the given scenarios, where original represented the original cost, percent represented 100% plus the given percentage, and total represented the total amount including the percentage amount. Tori stated that, "the idea of how to find sales tax and gratuity is practically the same thing" (Preliminary Interview, Line 35). Hence, a secondary purpose of exposure and variation within examples was to convey a sense of generality. Tori wanted students to be able to approach novel situations involving percentages so she sought to convey the general relationship between the aspects in such examples in her design and enactment.

Within a larger purpose of using examples for exposure to variations in aspects, Tori sought to develop students' capabilities of comprehending contextual language through examples. One of her goals was for students to read a word problem and attend to the quantity being asked for and the contextual language. This purpose for examples was unique to Tori, although this may have been due to the contextual nature of the percentages unit. In discussing a word problem that asked for the hourly rate Tori said, "So they had one more reading step…Make sure you go back and answer the question being asked. So that was just one step further to show them the importance of reading the word problems" (Preliminary Interview, Lines 123-126). Tori sought to draw students' attention to the vocabulary associated with contextual language such as "with tax," "before tax," "including 6% tax," and "pay an additional 8% tax." Part of this was associated with the variation inherent in real life situations involving sales tax and gratuity. Tori described some of the variations associated with the context of percentages that she sought to expose students to:

> We see the tax in, when you purchase something, maybe a shipping fee, along with tax. And does the shipping fee come before the tax or after the tax? So it's mainly more about reading and when do we use tax? Is it something that is prior or afterwards, and exposing them to even those different vocabulary. (Preliminary Interview, Lines 70-73).

The variations in this case arose from Tori's desire to develop students' contextual language as well as the necessity of students needing to be able to comprehend a verbal problem situation involving percentages and approach it in an appropriate way. As before, the predominant purpose was exposure to variations in aspects of the class of examples about percentages, where the aspect was the contextual language used within word problems.

Tori's predominant purpose for using examples was exposure to variation in aspects, including: dollar amounts, percentage amounts, context, additional information (e.g. shipping fees), the given information, the quantity asked for, and the contextual language used in the word problem. She spoke at length about the desire to "expose" students to as wide a variation as possible of the problem situations they could encounter in order to develop a sense of generality about a class of examples and a means to approach novel examples. This main purpose influenced Tori's initial design and enactment of sets of examples.

Initial design of sets of examples. Tori's predominant purpose for using examples was exposure to variations in the aspects of the class of examples. Tori believed it was important to expose students to multiple variations in the aspects in order to develop students' ability to approach novel situations. The purpose of exposure to variations in aspects directly influenced her design, particularly in terms of variations within word problems. Within the class materials on percentages that Tori used, these variations included dollar amounts, percentage amounts, context, additional information (e.g. shipping fees), the given information, the quantity asked for, and the language used in the word problem.

Tori grouped the applications of sales tax and gratuity together to convey a sense of generality in the process for finding the total. This was contrasted with examples using the applications of sales tax and gratuity in which students were given the total and asked to find the original price. Table 4.1 summarizes the variations in the givens and quantity asked for in the set of examples that Tori used with her students. Tori was clear about her intention to reverse the givens and quantity asked for between lessons:

> The day before this, we had also worked on sales tax and gratuity, but it was this is the total bill. Now, with tax, what will be the total price? So the goal for today was to kind of work backwards. You know, what the total price is. What was it before tax? (Preliminary Interview, Lines 95-97)

Variation in the givens and quantity asked for created contrast between examples through reversal of the process (e.g. given the total, determine the original price), creating the opportunity for students to discern the givens and the quantity asked for as dimensions of variation of percentage word problems.

Table 4.1

Givens and Quantity Asked For in Sales Tax and Gratuity Word Problems, with Variations

Givens	Quantity asked for
Original price and percent tax or percent tip/gratuity Variations: • And shipping fee • Percent tax on a portion	Total Variations: • Given a rate (e.g. dollars per week), determine length of time (e.g. weeks)
Total and percent tax or percent tip/gratuity Variations: • And shipping fee	Original Price Variations: • Given a quantity (e.g. hours), determine a rate (e.g. cost per hour)

Because of the variation in real life contexts, the contextual language around sales tax and gratuity became a focal point of Tori's examples. Variation in the givens and the quantity asked for served Tori's purpose of developing students' academic language by the use of contextual language in the quantity asked for (see Table 4.1). Tori described how she was intentional about her arrangement of word problem examples in order to contrast the contextual language being used:

> With #3 and #4 [from the Sales Tax and Gratuity Classwork #2 worksheet (see Appendix D), I think just recognizing the different vocabulary for #3 and #4…I mean that's why I purposely put them together that way…mainly [as] a focal point of this says *including* 6% tax whereas this one says they had to *pay* an additional 8% tax. (Preliminary Interview, Lines 355-358)

From Tori's perspective, juxtaposing two examples allowed the differences in contextual language to be brought to the fore of students' attention. The juxtaposition also shed light on the relationship between the contextual language and the quantity asked for. A value that included sales tax would be the total and the original price would typically need to be determined, whereas a description of needing to pay an additional percentage typically indicated that the total needed to be determined.

One design strategy that Tori relied on was what she described as a small change from one example to the next. In the section on sales tax and gratuity, this often increased the complexity of the word problems. Tori described her sequence of examples for Sales Tax and Gratuity #2 worksheet (see Appendix E) as starting with straightforward, one-step problems, to more complex problems by way of "adding in one extra thing" (Preliminary Interview, Line 130). Tori said, "It's like ok, you should be able to do sales tax and gratuity. So now let's add in what happens if you have a shipping fee. What happens if I ask you for the hourly rate versus just the total?" (Preliminary Interview, Lines 130-132). Tori recognized and described the similarity in structure between these examples:

> So #4 was an extension in terms of same thing, tipping and gratuity, so it's very similar to problem #2, but then the question being asked is a little different. So, it's not just asking you what was the price before the tip. It was asking you what was the hourly rate. (Preliminary Interview, Lines 117-120)

Tori appeared to rely on proximity between examples via sequencing for students to notice structural similarities. With changes in multiple aspects (e.g., the context, the given dollar and percentage amounts, the quantity asked for), however, it is not clear if students recognized the increase in complexity as one additional step from the previous exercise and the structural similarity as Tori intended through her design.

In other instances, the one small change that Tori described served as a variation in only one of the aspects of the example. Tori talked about using this design strategy in topics that were dominated by practice:

> So I always make sure, especially during our equations and expressions unit, that every equation is a little different from the one before. So it's not just…for example, $2x + 8 = 15$, on every single one you're going to subtract this and then divide, and then they have eight of them that are like that. That's not the importance for me. It's ok, you can do this one, but what if it's $2x - 8 = 15$. Can you solve that problem? What if it's $-2x + 8 = 15$? What if it's $-2x - 8 = 15$? Um, what if it's $8 + 2x = 15$? So a lot of times I will give them the same exact numbers, but just rearrange the example problems so that they see when you do get the same values, when you don't get the same values. (Preliminary Interview, Lines 424-431)

In this sequence of exercises that Tori described, there was an opportunity for students to develop an awareness of the algebraic structure of linear equations through variance in the signs and location of numbers against a backdrop of invariance in the numbers and variables chosen. Furthermore, Tori's intention behind such a design was to allow students to recognize that structure for themselves:

> By changing something from a -8 to a +8, it's recognizing that it's similar in order of operations, addition and subtraction work the same, so you always need to make sure you address your addition and subtraction first, and that they're inverses of one another…I always throw in the negatives, like the -2 [as in $-2x + 8 = 15$], because all of a sudden,

> students will start thinking, ok, I'll just add two to both sides. And it's recognizing that that's a multiplication problem and not an addition problem. So, putting in just the slightest change of a negative, a lot of students might think is a totally different problem, so getting them familiar with, it's the same problem, just a different number. (Preliminary Interview, Lines 447-456)

From Tori's perspective, such a set of exercises was designed to allow students to discern both similarity and difference: the similarity in the processes of solving for either $2x + 8 = 15$ or $2x - 8 = 15$, and the difference between a negative on the constant terms and a negative on the *x* term.

Tori's initial design of her materials included variation in the dollar amounts, percentage amounts, context, additional information (e.g., shipping fees), the given information, the quantity asked for, and the language used in the word problems. This variation served Tori's predominant purpose for using examples for exposure to variations in aspects of the class of examples . While Tori gave careful consideration to her sequencing of examples and described her design as "one small change" between examples, with the intention of revealing structural similarities, it is not clear if the design itself would reveal those structural similarities to students with simultaneous variation in multiple aspects. Tori described one instance in which she designed a set of examples in which a change was made in only one aspect between examples in order to reveal the structure of two-step linear equations. Her description of the set of examples indicated that Tori had already, in fact, designed a set of examples that would be considered a structured exercise, prior to the intervention.

Initial enactment of sets of examples. While it was apparent that Tori attended to mathematical structure in her design, there was little evidence of Tori explicitly drawing attention to the structure during enactment. Within the Sales Tax and Gratuity video that students watched outside of the classroom, Tori was explicit about the goal of the lesson: “Our goal is to find the advertised price. So we’re trying to find what the original price was. So before tax was added on, what was the original price?” [Sales Tax and Gratuity video]. Tori had discussed how her design was purposeful in reversing the process from one day to the next. On the first day, the examples asked students to find the total bill given the original price and the tax. On the second day, the examples asked students to find the original price given the total bill and the tax. Despite her intentional reversal in the design, she does not explicitly contrast the second lesson’s goal with the first lesson’s goal. Tori considered reversal and contrast important aspects of the variation that students needed to experience, but was not explicit about this variation when enacting these examples.

Likewise, Tori carefully considered the structural similarities and differences between word problems when sequencing her examples, as described above, but it appeared as though she left students the responsibility to discern such structure through juxtaposition. I asked Tori if she did anything during enactment, or if she thought students recognized the similarity in structure between the pairs of examples. Tori did not describe any teacher actions to draw attention to the structural similarity, but rather suggested a rearrangement of the examples, specifically to place examples #2 and #4 one right after the other:

> Maybe switching #3 and #4 would have been better, and doing #2 then doing #4, because #2 is just like #4 except with that one extra step [finding the hourly rate]. So, maybe doing #2, and then having them do this one [#4] because it’s the same initial step, and then being like, notice how this is asking one further question. (Preliminary Interview, Lines 138-141)

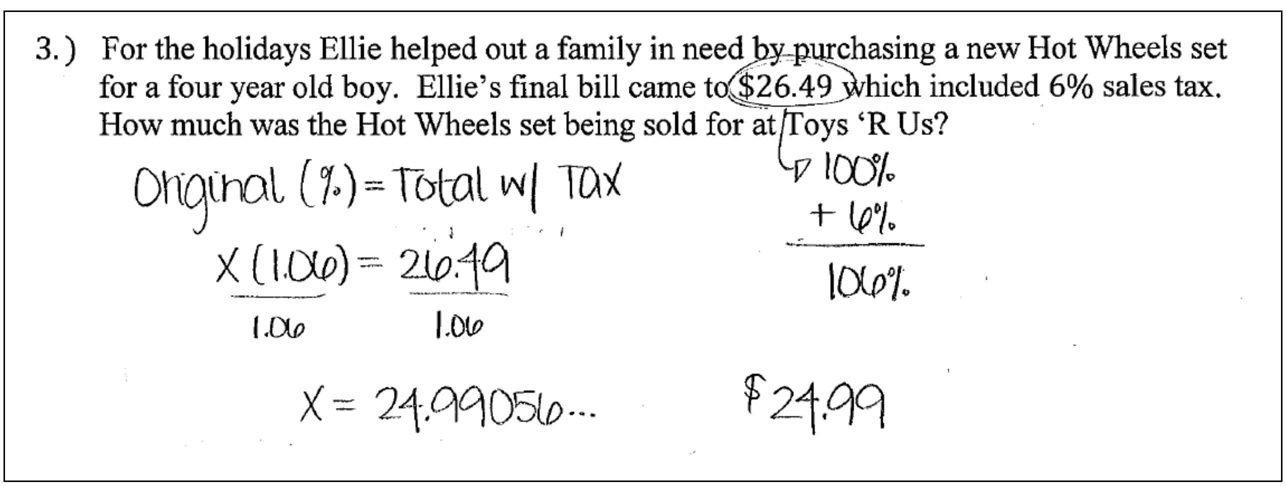
In addition to the rearrangement of examples, Tori mentioned a possible verbal indication of the similarity in structure between examples #3 and #4: “Notice how this is asking one further question.” Tori’s predominant strategy for drawing attention to structural similarities and differences was through a design strategy of sequencing of examples. Despite the careful consideration that Tori gave to the sequencing of examples in the design, the structure that she clearly noticed was left implicit for students.

In the only instance of Tori attending to the structure of the set of examples during enactment, she described how she intentionally included matched examples between the video worksheets and the practice worksheets to encourage students to seek out similarities in structure:

> I would try to make the questions as they were working on them similar to the ones that they had seen here [on Sales Tax and Gratuity #2]. I would tell them, you’re stuck on #3? See if you can find something similar to #3 in the ones that you did. (Preliminary Interview, Lines 192-196)

Based on Tori’s description, when Tori was explicit about directing students’ attention to mathematical structure between examples, it was often based on discerning aspects that were the same, particularly in terms of the givens and the quantity asked for. The context, numbers used, and sometimes, language used within the word problem, generally varied from example to example, while the structure of the problem, in terms of a representative algebraic equation and the types of givens and quantity asked for, was invariant.

As described in the section on Tori’s initial purposes for using examples and initial design, Tori sought to expose the generality of percent over various applications, often through variation in multiple aspects. Within the enactment, this appears to have occurred from a procedural perspective, rather than a perspective meant to develop conceptual understanding. Tori began with the same verbal equation in each worked example: Original (%) = Total [Sales Tax and Gratuity Classwork #2 Answer Key]. This written verbal equation was short-hand for the original price times the percent (100% plus either the tax or tip percentage) equals the total. Figure 4.1 shows the worked example #3 from the Sales Tax and Gratuity Classwork #2 worksheet. Within Tori's lessons, the original meant the original price of the item. The percent (%) meant the original whole (100%) plus the additional tax or tip to result in a percentage greater than 100. The total represented the total with tax or the total with gratuity. This written equation was used for every worked example throughout the enactment, conveying a sense of generality that these examples could all be approached and solved in the same manner. Students need only determine the given information and substitute them in to the written equation to solve for the quantity asked for, emphasizing the generality of the procedure, rather than a conceptual understanding of percentages.



3.) For the holidays Ellie helped out a family in need by purchasing a new Hot Wheels set for a four year old boy. Ellie's final bill came to $26.49 which included 6% sales tax. How much was the Hot Wheels set being sold for at Toys 'R Us?

Original (%) = Total w/ Tax

$$\frac{x(1.06)}{1.06} = \frac{26.49}{1.06}$$

$$x = 24.99056\ldots$$

$24.99

100%
+ 6%
106%

Figure 4.1. A worked example designed by Tori from her Sales Tax and Gratuity Classwork #2 worksheet, prior to the learning study intervention.

Despite the intentional structure that was clearly present in Tori's design of sets of examples, Tori did little by way of enactment to draw students' attention to that structure. Tori included matched examples between the video worksheets and the practice worksheets, and she encouraged her students to look for a similar problem when struggling with classwork. This was the only enactment strategy that Tori discussed regarding drawing students' attention to mathematical structure. This strategy, along with her consistency of using the written equation "Original(%) = Total" in each worked example, indicated the generality of the procedures used for solving word problems involving percentages, rather than drawing students' awareness to critical aspects of the class of examples, their ranges of permissible change, and the relationships among them.

Summary of Tori's initial purposes, design, and enactment. Tori's purpose for using examples, and her strategies for design and enactment of examples comprised the population of her initial example space of use and view of examples. Tori's predominant purpose for using examples was exposition, which stemmed from her desire to expose students to as much variation as possible within a class of examples. Tori believed that exposing students to such variation would support them in developing a sense of generality regarding the class of examples, and word problems involving percentages, in particular, and serve them well for approaching novel examples. This purpose influenced her initial design through the careful attention to variation in the given information, the quantity asked for, additional information (i.e. shipping fees) and contextual language. A key strategy that Tori used within her design was reversal. For instance, in the first lesson of the unit, students were given the original price and either a percent tax or tip and asked to find the total amount. In the second lesson of the unit, students were given the total amount and either a percent tax or tip and asked to find the original price. Tori relied on sequencing of the examples, particularly juxtaposition, to reveal the mathematical structure to students. The reversal and juxtaposition strategies provided an opportunity to create contrast between examples. With simultaneous changes in multiple aspects of the examples, it was not clear whether students saw the mathematical structure in the same way that Tori did. Tori did discuss a set of examples she had designed for a unit on solving two-step linear equations that involved only a single change within one aspect between examples and her desire for students to develop an understanding of how that single change affected the solution of the equation. This is an instance of Tori using restricted variation in her design in order to focus students' attention on particular aspects of a class of examples. Asking students if they could find a similar example from their homework was the only enactment strategy that Tori articulated to draw students' attention to the structure within the design.

Tori generated examples through her strategies of reversal and changes in the aspects of the problem, including one small change between examples, as she described for a set of exercises on solving linear equations, and adding in one extra thing to increase the complexity of percentage word problems. Within the topic of percentage word problems, Tori sought to connect sales tax and gratuity examples as structurally equivalent. Her use of the written equation "Original(%) = Total" for every worked example seemed to emphasize the generality of the procedure for solving percentage word problems, rather than emphasize discerning the critical aspects, their range of permissible change, and the relationships between them. Table 4.2 summarizes Tori's initial example space of use and view of examples.

Robert Cavins

Robert Cavins had been teaching at Augustus Middle School for eight years. He had one year of previous teaching experience in a neighboring district, and so was beginning his tenth year of teaching at the time of this study. Robert taught four sections of Math 8 and one section of Algebra 1 to a class of accelerated eighth graders. In this section, I present Robert's initial purposes for using examples and exercises, followed by his initial design and enactment of sets of examples.

Table 4.2

Tori's Initial Example Space of Use and View of Examples

Characteristic	Summative Description
Population	Predominant reason to use examples was exposure to variation in multiple aspects to convey generality. Designed for contrast between examples using strategies of reversal and juxtaposition. Described an instance where she used restricted variation, calling it "one small change" between examples. Scarce enactment strategies for drawing attention to mathematical structure. Asked, "Can you find something similar?" between in-class and homework examples.
Generativity	Use of restricted variation ("one small change" and "adding in one extra thing") prior to the learning study intervention, suggested the potential for generating sets of structured exercises using the same strategy. No suggestion of asking students to generate examples, but her use of a reversal strategy has the potential to be used for generating LGE prompts.
Connectedness	Predominant view of examples for exposure well-connected to simultaneous variation in multiple aspects. Variation in a single aspect ("one small change") was described, but not observed, suggesting a loose connection.
Generality	Saw sets of examples about sales tax and gratuity as representative of the related class of examples percentage word problems, consistently using the relationship that the original price multiplied by the quantity (1 + the percentage) equals the total. Unclear whether she saw sets of examples designed using reversal or "one small change" as representative of a general class of related sets of examples.

Initial purposes for using examples. Robert's purposes for using examples extended from his overall purpose for teaching mathematics; he wanted students to be able to apply skills and approach new problems using what they already knew. Robert described his desire for students' learning to be a controlled struggle. He wanted his students to be able to apply their mathematical skills and knowledge in novel situations:

> I want them to struggle a little bit. And it's…a controlled struggle. Cause you know, it can easily go into desperation, and anger, and things like that you know…You don't want them to be like wait – we didn't see an example like this…Use what you know. I would never give them an example that they can't handle. And I've had it happen in the past, but I've learned from that and seen, oh, we can't take that jump just yet, you know. Um, so, really, tiering, really making sure my examples aren't um, not too challenging, but at the same time, not too specific. Focusing on a bigger idea than just the very specifics. And then having them trust what they know from their past experiences to help them solve the problem. (Preliminary Interview, Lines 372-387)

Robert's initial purposes for examples included connection to students' knowledge ("trust what they know from their past experiences") and providing skills practice, which included attending to aspects of generality ("focusing on a bigger idea than just the very specifics"). I discuss each of these purposes in the upcoming sections.

Robert's first purpose was to use examples to connect students' prior knowledge to new concepts and procedures and to link forward to upcoming lessons, as a kind of foreshadowing. He described his use of an example on scientific notation at the beginning of class to connect to that day's lesson on multiplying by a monomial, "To transfer it [the process] from scientific notation to show them that multiplying…a monomial by a monomial, is the same procedure, and the same idea as scientific notation," (Preliminary Interview, Lines 20-22). This is an instance of connecting back to students' prior knowledge. Robert also used examples to foreshadow and connect from the current lesson to the next day's lesson: "These [exercises 8 and 9] were great because they are the ones that I used [in] the next lesson to link [to] order of operations, and what do we do first, how do we distribute, and then add and subtract" (Preliminary Interview, Lines

282-283). Robert saw conceptual connections between different classes of examples and regularly sought to convey those connections to his students via examples choices.

Robert's second predominant purpose for using examples was practice. While Robert intended that students develop fluency with procedures via practice, he also intended that students develop a sense of generality in understanding the range of permissible change of particular aspects of examples. Robert described the arbitrariness of constants in his chosen examples:

> It's not important what the numbers are. It's not important that they see, you know, very specific negative, positive [numbers]….As long as I believe it's a random, sort of conglomerate of all those, then I'll be happy with it…I do it for the fluency mostly. (Preliminary Interview, Lines 427-436)

From Robert's perspective, randomness was a desired characteristic of sets of examples and exercises chosen to develop students' fluency with skills, and Robert would often draw on outside resources for such skills practice (e.g., https://mathbits.com/ and https://www.kutasoftware.com/). Robert believed that the randomness of such sets of examples, particularly in terms of the coefficients and numbers used, was an asset for developing students' understanding of the range of permissible change in the coefficients and numerical values. Robert described how the random use of coefficients conveyed a sense of generality to the students:

> The lesson…[does not] focus on fractions or decimals at all. But throwing one [decimal or fraction coefficient] in here, randomly, throughout the year, is a- I've seen that, it just helps students refocus on *anything* could be multiplied by that. (Preliminary Interview, Lines 162-165)

From Robert's perspective, practice served to support students' development of fluency with skills, encouraged their discernment the range of permissible change in the numerical values that appeared in such examples, and through randomness in numerical values conveyed a sense of generality.

Robert's predominant purposes for using examples were connection and practice. The first purpose of connection was especially important to Robert in developing students' conceptual understanding of mathematics and using their previous knowledge to build new knowledge. From Robert's perspective, the connections within examples to prior knowledge supported a "controlled struggle" for students and tempered the conceptual and procedural leaps that Robert asked of his students. Practice was Robert's second predominant purpose for using examples. Randomness in the choice of numbers within the examples was meant to convey a sense of generality and draw students' awareness to the range of permissible change in the aspect number. These purposes influenced Robert's initial design and enactment of sets of examples.

Initial design of sets of examples. One of Robert's predominant purposes for using examples was connection. He wanted his students to be able to use their prior knowledge in approaching a novel example and challenge them while making sure that the conceptual and procedural leaps were not so great that they inhibited students' learning. The purpose of connection influenced Robert's desire to "tier" his examples in ways that balanced the tension between challenge and specificity. He intended for these examples to provide the right level of challenge, but also accessible to the students. From Robert's perspective, challenge and specificity was achieved through the careful choice and sequencing of examples. In designing such examples, Robert described a planning approach and an improvisational approach. I first discuss the planning approach, and then the improvisational approach.

Robert preferred to formally plan and design examples with topics for which he anticipated particular student difficulties or if the topic was exploratory with limited class lecture.

> When I know I can anticipate some problems from students, that's when I'll sit down. I want to make a great example so that this topic that students typically find challenging, they can find it [the topic] a little bit simpler…I'm more specific [in choosing examples] when they are the ones that are constructing the idea themselves. I am much more specific [in choosing examples] or I borrow from someone who was much more specific [in choosing examples], going through the step by step. So if they're [students are] exploring something on their own, as far as no lecture, or very limited class lecture, then I'm much more specific [in choices of examples]. (Preliminary Interview, Lines 551-562)

From Robert's perspective, topics that students historically struggled with, or lessons with limited class lecture, required careful planning of examples and exercises in order to support the connectivity of students' knowledge. Robert believed that the choice of examples could make a challenging topic easier for students to understand, and that the careful choice and sequence of examples could support students in discerning important mathematics in lessons that involved limited class lecture.

Robert's purpose of examples for connection shaped his design of sets of examples through restricted variation in the various aspects of the examples. Robert intended to extend the range of permissible change of some aspect of a previously studied class of examples. In the preliminary observation, Robert used an example about scientific notation to connect to that day's lesson on multiplying by a monomial.

> Using the fact that when we multiply [monomials] too, we keep the same base and add the exponents. So keeping them, that they know that rule about exponents, focusing on the very particulars keep the same ten, add the exponents, and now expanding into, well what if the bases are the same, but they're no longer ten anymore. (Preliminary Interview, Lines 25-28)

This example was intended to extend the range of permissible change of the base of an exponential expression from numerical values, specifically a base of ten, to algebraic bases. Robert further related this to the common structure between arithmetic and algebra: "Numbers behave just like polynomials. We just don't realize it a lot of times" (Preliminary Interview, Line 23). In order to draw his students' attention to similarity in mathematical structure between multiplying numbers in scientific notation and multiplying a monomial by a monomial, Robert restricted variation in other aspects of the examples: "Numbers that I picked are nice round numbers" (Preliminary Interview, Line 24). In explaining this choice, Robert said, "We're focusing on the real issue of adding the exponents" (Preliminary Interview, Lines 40-41). Thus, one way Robert used connection between examples was through the extension of the range of permissible change of an aspect of one class of examples to relate it to or include it within another class of examples, creating a sense of familiarity for students.

On the other hand, Robert relied on simultaneous variation in multiple aspects to convey a sense of generality, often within sets of examples intended for practice. Robert talked about choosing examples that were "typical" or "standard" of a class of examples, such as his choice of $3x(4x^2 + 2x - 4)$ in the class notes that he designed for multiplying a monomial and a polynomial (Preliminary Interview, Lines 130-132). Within the class notes, Robert included the instruction, "Remember to distribute to each term in the parentheses" [Multiplying Monomials by Polynomials notes], and demonstrated the procedure by drawing arcs from 3*x* to each of the terms inside the parentheses. Robert made this choice to enable students to apply the same procedure to new examples and support students' fluency with procedures. In talking about his choice of the homework assignment (see Appendix F), Robert said, "They're all standard, they're all questions my students can answer, and obviously they follow directly with the lesson that went [on] that day" (Preliminary Interview, Lines 206-207) Variation in the aspects within these examples was important for conveying the generality that Robert wanted his students to discern.

Variation in the complexity of terms was used to convey a sense of generality for multiplying a monomial by a polynomial. The following were the three practice exercises that Robert chose for the in class worksheet:

a.) $h^2(-2h+5)$

b.) $t^4(-3t^3+5t^2+t-8)$

c.) $-5x^2(3x^2+0.2x-20p)$

Robert described the variation in the terms for the purpose of students generalizing the meaning of term and to address a known misconception:

> I like some terms that don't have any variable with them [i.e., the 5 in exercise a.]. Um, so they have to just know that sort of it's thrown on at the end. Why is that thrown on like that? Um, throw in a random *t* here [exercise b.]. And I like this *t* – 8. For whatever reason, I think it's more a psychological thing. We see a lot of like, we think like this is one term just because there is, it's a simple term. Having students see that *t* is a term, but $-3t^3$ is also one term. (Preliminary Interview, Lines 145-149)

Robert had noticed from his experience that students often did not see complex single terms (that included exponents or multiple variables) as one term, and conversely, inappropriately saw simple expressions, such as $t - 8$ as consisting of a single term, rather than two terms. Because of this, Robert intentionally chose certain variations in the terms within the polynomials as a "tripwire" (A. Watson, personal communication, November 29, 2017) in order to extend students understanding of what is classified as a term. These "tripwires" appeared to serve as cues to Robert to discuss the meaning of term with the class, rather than an opportunity for students' to discern difference. Robert additionally described how practice exercise c in this set and practice exercise b in the previous set $[3xy(5x^2)]$ were included as "tripwires" to address the common student conflation of procedures for adding/subtracting polynomials and multiplying polynomials. Robert described his intention for students to recognize that, unlike adding polynomials, multiplying polynomials did not require like terms.

Researcher: [Things that you vary are] the number of variables, either one or two.

Robert: Yup. And what happens if they're not like terms. You know, like students saying well we can't multiply those, they're not like terms, starting to really dig into that, and how they don't have to be like terms. A few days later, you know, I had that on the board. You know, we can add only like terms. We can multiply any two polynomials.

Researcher: Right. Ok. So beginning to try to differentiate between addition and multiplication?

Robert: Definitely. Which to me is a big, a *big* idea of this unit, is when you start putting things together, as I say, that's where students, and I do see some students get confused. (Preliminary Interview, Lines 69-76)

In each of these examples, Robert included a multiplicand that contained a term that was not like terms with the multiplier. This was to elicit the student misconception that unlike terms could not be multiplied and to differentiate multiplying polynomials from adding polynomials through explaining that like terms are not needed for multiplication. As before, these “tripwire” examples served as a cue to Robert to explain. From Robert’s perspective, variation in certain aspects of the examples and exercises (number of variables, like or non-like terms, coefficients) served to establish the meaning of mathematical vocabulary and generalize it across forms (e.g., term), differentiate between and generalize procedures (e.g., addition and multiplication of polynomials), and convey a sense of the range of permissible change for a particular aspect of the class of examples (e.g., coefficients). For Robert, inclusion of examples that would potentially cause student difficulties would cue an opportunity for him to explain and thereby induce learning.

Robert viewed one of his own strengths as his ability to generate examples, as needed. This improvisational design style often intersected with Robert’s purpose of connection, as Robert described his ability to generate related examples during the course of teaching:

> One of my better qualities, I think, is coming up with problems on the spot…Just this morning…we were doing perimeter lengths with area. We started with four examples. They used string and did perimeter. We started with those four examples, then I [said]…what if I give you this random length and width? What’s the perimeter? What’s the area? Now what if I give you a length and an area? Well, I can make examples on the spot that flow together. (Preliminary Interview, Lines 467-474)

Within Robert’s description of his improvisational approach to designing sets of examples, he alluded to a sense of structure in which one of the givens was altered (length and width vs length and area). He also described generating examples on the spot to create contrast between areas of rectangles:

> And I remember being up there like, some kid gave me dimensions that the perimeter happened to be 20. And then I go, okay. And in my head real quick, ok, why don't I give you another example where the perimeter's 20, and have them be like, hup! Why's the perimeter the same?...So I quick, pick an example. What if it's eight and two? Well here you go, and then someone's like, oh – that perimeter's 20 also. Oh! The perimeter's 20 also. And then I'll ask a random question – which rectangle's bigger? Without us visualizing it at all. Which one's bigger? Oh! Well that one with the area. Oh, ok. It was a nice classroom discussion, but in my head it was like, I didn't plan that. I was just like, oh, let me pick one to a, give an example. And it worked great. (Preliminary Interview, Lines 478-487)

By choosing dimensions of a rectangle that yielded the same perimeter as the student's previous example, Robert held perimeter invariant, allowing for contrast to be created between the areas of the two rectangles.

In a later segment of the same lesson, Robert discussed saying to the students, "Here's a perimeter of 26. Well how many dimensions could we come up with? Oh, we could come up with this one, this one, let's see how many we can get for the same perimeter" (Preliminary Interview, Lines 502-504). This is an instance of Robert using an LGE prior to this study, as he asked students to come up with as many sets of dimensions as possible to yield a perimeter of 26. Robert improvised the use of this LGE to generalize perimeter over the critical aspect of dimensions. Thus, Robert's strength of generating examples in the moment allowed him to attend to contrast between examples, while his questioning in the moment (which resulted in the use of an LGE) allowed him to attend to mathematical structure via patterns of variance and invariance.

Robert used restricted variation in his design in order to make connections between prior knowledge and new material, and simultaneous variation in multiple aspects of examples to convey a sense of generality for the design of sets of examples for practice. From Robert's perspective, the restricted variation was related to tiering, in which Robert used examples for connection in ways that students could use previous knowledge and apply it to approach a novel example, and hence ease conceptual and procedural jumps for students. Robert used a planned design approach for topics he noticed students struggled or with exploratory tasks with limited class lecture. He preferred an improvisational approach for topics in which he felt more comfortable or was employing a direct teaching strategy. Within both design strategies, Robert recognized both the structural connections and structural generality, but it was unclear whether students attended to the variation in the same ways as Robert did.

Initial enactment of sets of examples. Robert's predominant purposes for using examples were connection and practice. For Robert, both of these purposes intersected with his desire to convey a sense of generality. An improvisational design with topics in which Robert felt more comfortable necessarily intersected with the enactment of such sets of examples. Examples were generated by Robert during the act of teaching, along with questioning, that both created opportunities for, and diminished, student discernment of critical aspects of a class of examples. In the episode that Robert described above, a student provided Robert with dimensions of a rectangle with a perimeter of 20 units. In the moment, Robert generated a set of different dimensions that also yielded a perimeter of 20 units. This choice created contrast and an opportunity for students to discern that two rectangles with the same perimeter need not have the same dimensions. Robert extended students' thinking by asking, "Which rectangle's bigger?" (Preliminary Interview, Line 484). This question created an opportunity for students to discern that rectangles with the same perimeter need not have the same area, fusing the relationship between those two critical aspects (perimeter and area) of a rectangle. Within that same segment, Robert extended a planned example with a perimeter of 26 by asking students to generate examples of dimensions that also yielded a perimeter of 26. While this created an opportunity for students to generalize a perimeter of 26 over many sets of dimensions, it is not clear if the enactment of the set of examples extended students' thinking regarding the relationship between the length and width in dimensions that yielded a perimeter of 26 (fusion) or the generalization that there exists an infinite number of sets of dimensions that yield the perimeter 26 (generalization). Because there was no discussion of the relationship between the aspects, or the generalization of perimeter over dimension, students may have interpreted Robert's posing of the question for the purpose of developing procedural fluency.

Some of Robert's choices of examples were aimed at developing students' conceptual understanding through connections made via the enactment. He often referred to these kinds of examples as "extensions." Robert said, "This example right here [$(3x^3)^2$] lets me know that they still understand exponents as repeated multiplication. That is huge to me" (Preliminary Interview, Lines 82-83). Within the enactment of this example in the preliminary observation, Robert draws attention to the exponent as repeated multiplication through notation:

> Then [Robert] asks, "What about $(3x^3)^2$?" Robert has written this below the original question with the 2 exaggerated as the exponent to the term in parentheses. Mitchell answers $9x^6$. Robert says, "Exactly right!" Then Robert writes out $= (3x^3)(3x^3)$ and

> says, “Because this is what to the second power means. This is repeated multiplication.”
>
> Then Robert writes $= 9x^6$. (Preliminary Observation, Lines 144-148)

The emphasis on the exponent due to the manner in how it was written, coupled with the expanded notation of the expression and Robert’s verbal explanation created an opportunity for connection between the exponentiation of an expression and multiplying by a monomial. Robert did not explicitly state that the exponentiation of an expression consisting of one term was another instance of multiplying a monomial by a monomial, hence it is unclear if students saw the exponentiation as a variation of multiplying monomials or a disconnected example.

Robert described how his teaching had changed over time to encompass a greater emphasis on developing students’ conceptual understanding. He described how his own teaching of examples in the unit on scientific notation had evolved over time, prior to the beginning of this study:

> I was probably more like, let’s just move the decimal point, count the zeros, don’t think about it. Let’s just do it. And now I’m more like, well, you’re multiplying by ten, dividing by ten. Repeated multiplication….Why are we moving the decimal point? Thinking more about it than just memorizing a procedure. (Preliminary Interview, Lines 318-322)

Through his years of experience, Robert had arrived at a point in his teaching where he was attending more to students’ conceptual understanding and the reasoning behind procedures than he previously had. He tended to do this through questioning (“Why are we moving the decimal point?”) during sets of examples, although as shown from the example enacted during the preliminary observation, above, Robert often appeared to be the one making the verbal statements regarding connections and the conceptual explanation behind procedures.

Robert's enactment of sets of examples, particularly his questioning, created some implicit opportunities for student discernment of critical aspects, generalization of those aspects over other critical aspects to expand the range of permissible change, and discernment of the relationships between critical aspects. Robert believed that his strength at improvising examples during the act of teaching supported opportunities for deep student thinking. From Robert's descriptions of teaching and what I observed, it was often Robert making the explicit verbal statements regarded connections and generalizations, rather than the students. Hence, it is not clear whether students saw the structure within the sets of examples in the same way that Robert did, even after having been told what it was they should attend to. Further, sets of examples for practice, such as asking students to generate dimensions of a rectangle that yielded a perimeter of 26, appeared to be taken up by students only as a command to complete a task, and the consideration of the range of permissible change of the dimensions and the relationship between the dimensions was left implicit.

Summary of Robert's initial purposes, design, and enactment. Robert's view that students needed to be able to approach novel situations using their prior knowledge greatly influenced his purposes for using examples. Robert used examples for the purpose of connecting students' prior knowledge to new knowledge and to connect current material to future lessons. Robert also used examples for the purpose of practice for students to gain fluency and to convey generality. Robert's design of examples for practice included simultaneous variation in multiple aspects. Robert's design was sometimes planned and sometimes improvisational. In both instances, he used restricted variation to help encourage students to make connections between prior knowledge and new knowledge and simultaneous variation in multiple critical aspects to encourage a sense of generality. While Robert recognized the structural connections and structural generality, the variation may not always have been restricted enough for students to discern the mathematical structure in the same ways that Robert did. Robert's questioning during enactment may have supported student discernment of structure in some ways, although it was often Robert who made the explicit verbal statements regarding structural connections and generalizations, making it unclear if the sets of examples, themselves, revealed the mathematical structure to the students. It was clear that Robert saw value in the careful planning of examples, particularly with topics in which he noticed students' struggling, but in some ways, his improvisational designs were more successful in attaining the restricted variation necessary for students to discern a particular critical aspect, generalize a critical aspect over others, and discern the relationship between critical aspects. This may have been born out of the necessity for Robert to change only a single critical aspect between examples in order to generate them in the moment. Thus, the population of Robert's example space of use and views of examples was comprised of examples used for connection and practice, with both improvisational and planned designs, and verbal, teacher-given explanations and questioning to draw attention to the mathematical structure in the sets of examples.

Robert indicated the improvisational use of an LGE prior to this study in which he asked, "How many ways can we…?" The generation of this LGE was borne out of restricting the perimeter of a rectangle and asking students for possible dimensions for that particular perimeter. Robert connected his tiering strategy when designing examples to students' opportunity to learn. In some ways, his tiering strategy also served to connect classes of examples for the purpose of conveying generality between concepts (e.g., scientific notation and multiplying by a monomial). From Robert's perspective, simultaneous variation and randomness within sets of examples for practice conveyed the generality of procedures. Table 4.3 summarizes Robert's initial example space of use and view of examples.

Table 4.3

Robert's Initial Example Space of Use and View of Examples

Characteristic	Summative Description
Population	Predominant use of examples was for connection to students' prior knowledge, and practice for fluency. Planned and improvisational design included tiering of examples to create connection between examples and increase complexity. Foreshadowing examples were used to connect to upcoming content. Some restricted variation was included in the design (e.g., restricting to whole numbers). For sets of examples used for practice, design included simultaneous variation in multiple aspects and occasional "tripwires". Randomness used to convey generality of various aspects of examples. Enactment included verbal, teacher-given explanations of mathematical structure and questioning strategies. Some questioning strategies, generally improvisational, resulted in LGEs.
Generativity	Use of tiering and the desire to create connections between examples prior to the learning study intervention, suggested the potential for generating sets of structured exercises using a similar strategy. Improvisational questioning that restricted variation and resulted in LGEs suggested the potential for a deliberate design of LGE prompts.
Connectedness	View of examples for connection loosely connected to a tiering design strategy and disconnected from improvisational design. View of examples for practice well-connected to simultaneous variation in multiple aspects and randomness to convey generality.
Generality	Sets of examples specific. Generality within sets of examples and across sets of examples for practice, in terms of randomness and simultaneous variation.

Lynn Gray

Lynn Gray had 24 years of teaching experience, with 13 years at Augustus Middle School. For the past three years, Lynn was teaching seventh and eighth grade response-to-intervention (RTI) classes. These classes were each comprised of two to seven students who had been identified as students in need of additional mathematics support through state and district testing, as well as teacher recommendation. Lynn saw her students for 40 minutes every other day. She saw her role as supporting what was being taught in the regular seventh and eighth grade mathematics classrooms. She worked closely with the three regular classroom teachers, often drawing on their materials to work on the same procedures and concepts with students in her class and present and discuss them in a consistent manner. In this section, I present Lynn's initial purposes for using examples and exercises, followed by her initial design and enactment of sets of examples.

Initial purposes for using examples. Lynn's predominant purpose for using examples was for assessment for instructional decision-making, which encompassed assessing students' prior knowledge and assessing students' learning to make next steps in the instructional trajectory. When I asked Lynn what purpose, for her, examples and exercises have in math instruction, Lynn said, "Whatever examples and exercises I choose to do with them [students], that allows me to see their knowledge" (Preliminary Interview, Lines 268-269). The feedback that Lynn garnered from the use of examples then influenced the trajectory of Lynn's instruction. Lynn described how the feedback garnered from students' completion of assigned examples could affect her instruction:

> And then the nice thing about RTI is you have that constant feedback. So I can change anything at any minute. I don't have to have it all set to the exact time and every example laid out perfectly. When I say that, I mean, you can kind of go with the flow. So if [I] see that they're really struggling with subtraction, for instance, then I would have probably thrown up a couple more subtraction problems. (Preliminary Interview, Lines 43-47)

For Lynn, who had previously been a regular mathematics classroom teacher, using examples for the purpose of assessment gained more significance with her RTI classes due to her increased ability to notice students' thinking and misconceptions with small class sizes. Lynn compared her work within her RTI classes to her previous work with regular mathematics classes:

> In this [RTI] type of position I'm in, I can see the misconceptions in one second, that I may not notice in a class of 25 kids…The fact is, it's so small that [I] can right away see what it is that they're mixed up on. (Preliminary Interview, Lines 284-290)

Lynn used examples for the purpose of assessing students' prior knowledge and learning, which often involved uncovering student misconceptions. She was then able to use this information to make choices about what examples she would use or assign next, affecting the trajectory of her instruction.

Lynn's second predominant purpose for the use of examples was for providing skills practice. Lynn seemed to have a skills-oriented approach to teaching mathematics, and believed that additional practice would improve students' skills.

> I feel like the more exercises you do, the better you get at a certain skill. Especially in math. There comes a point in time, like in this unit on polynomials that, the more you do of them, it's just going to come to you…They've had me, and they're in their math class everyday so the combination of the two definitely makes a difference after a few days of practicing the same type of problem. (Preliminary Interview, Lines 396-405)

From Lynn's perspective, repetitive practice served as the driving force behind students' ability to apply skills.

Lynn's predominant purposes for using examples were assessment of student understanding for instructional decision making and skills practice. Lynn assessed students' understanding based on initial examples and then used that assessment to make subsequent instructional decisions. Lynn's purposes for examples were largely influenced by her view of her role, as the RTI teacher, as support for the regular seventh and eighth grade mathematics classes. From Lynn's perspective, students were supported by additional practice and the re-teaching of skills in which students were making mistakes.

Initial design of sets of examples. Lynn's predominant purposes for examples as assessment of student understanding for instructional decision making and skills practice heavily influenced her design of sets of examples. A significant portion of each lesson was spent allowing students to work on sets of examples that allowed for both student practice and the opportunity for Lynn to discern student difficulties and misconceptions in order to address them. This was often accomplished through sets of random examples based within a limited number of classes of examples, such as a worksheet that combined adding, subtracting, and multiplying polynomials. Lynn described the purpose of a random collection of exercises as follows: "[A random collection of exercises] helps me see what skills they are good at and what skills they need to work on…[and] where their misconceptions are" (Preliminary Interview, Lines 280-281). From Lynn's perspective, the element of randomness in the design served her purpose of assessing students' understanding and drawing out their misconceptions.

Lynn discussed her observation of misconceptions arising when two or more classes of examples appeared together; students had difficulties differentiating among the learned procedures for each class of examples. In my preliminary observation of Lynn, she gave students two exercises as an assessment of their understanding at the end of the class. One of the exercises was about subtracting polynomials [$(3x^2 - 12) - (2x^3 - 4x^2 + 5x)$], while the other was multiplying a monomial by a polynomial [$4x^2(-2x^3 + x^2 - 3x)$].

By the end of class, learning these rules, right away it was like they didn't know whether they were going to combine like terms or they going to make this [multiplying $3x^2$ and $4x^2$ in the subtraction example] $12x^4$. And that's when they start to get confused. (Preliminary Interview, Lines 160-162)

Lynn anticipated students' conflation of rules between different classes of examples and intentionally designed sets of examples that provided such an opportunity. Lynn felt that students' ability to notice or discern differences between ideas or processes often came after periods of practice, when the two (or more) ideas or processes were combined. The conflation of two ideas or processes resulted in student errors, which alerted Lynn to draw attention to the differences between the ideas or processes within the enactment. Due to Lynn's observations of how misconceptions arise and students make sense of and differentiate between processes, simultaneous variation in multiple critical aspects was a critical element of Lynn's initial design of sets of examples for the purpose of assessing student understanding.

Lynn intentionally included an element of randomness in her designed tasks in order to address concerns she had about students' retention of material and application of skills. The randomness in her design was, in part, motivated by the course schedule, as Lynn felt that she had to address many topics within the short time that she saw her students (40 minutes; every other day). When I asked Lynn about how she chose and arranged examples and exercises, she said the following:

Lynn: It's just picking different types of problems, trying to tie in, because I only have them every other day, tying in what we worked on two days ago. Just trying to spiral stuff all the time.

Researcher: Can you talk a little bit more about what you mean by wide variety?

Lynn: I guess a lot of what I see with students at this level is that they'll learn something…and they'll be really good at it, and then three days later, they've learned three different new things in class. So then, they'll forget everything they learned before. And they just get really mixed up on it. So we were working on the angle unit. They were really good with complementary and supplementary angles, and then the minute we got into triangles and three angles is 180 [degrees], then it seemed like you gave them something from before and they completely forgot how to do it. So it's always trying to tie that other knowledge back in again. (Preliminary Interview, Lines 344-357)

From Lynn's perspective, a wide variety of examples and exercises meant presenting multiple classes of examples simultaneously that perhaps shared some common features (e.g., sums of angles). Lynn saw the effects of this design strategy as two-fold: (1) supporting student retention of previous materials, and (2) creating opportunities for students to discern the differences between classes of examples. Both effects were related to Lynn's purpose of assessing student understanding for instructional decision making, with the latter an intentional attempt to draw out student misconceptions, if present, and contrast between classes of examples and their associated procedures.

During lessons in which Lynn intended to actively teach or re-teach a skill or concept, she described her design of examples as slowly unfolding. Lynn's desire to slowly unfold new material was related to her notion of students' success:

> I don't want to overwhelm kids with new material and make them nervous about it. So I try to slowly unfold it. I don't want to shock them into something, that they go out of

class and they're like, "Oh my gosh, I don't understand this at all." (Preliminary Interview, Lines 57-60)

Lynn's conception of unfolding included starting with a related example from previous knowledge and material, increasing the difficulty, and a wide range of variation in aspects such as coefficients, variables, signs, and number of terms. Lynn described her design of the tasks she created for the preliminary observation (see Multiplying Monomials Worksheet in Appendix G):

Lynn: I try to pull from some previous knowledge before I introduced the whole idea of multiplying monomials. So I went back to our unit of laws of exponents just to see how much they remembered about it…Just to get them to remember the laws of exponents so that we could apply that to multiplying monomials.

…

Lynn: I tried to pick all different types here [referring to the Multiplying Monomials worksheet]. Starting out with just distributing one number, then I don't know. Just kind of switching it up. I just kind of threw together some problems. And then, again, just slowly, just working with one number there, and trying not to confuse them too much. And then getting a little bit more difficult.

Researcher: So I just heard you say getting more difficult. Is one of the ways you think about this as easier to more difficult?

Lynn: For instance, distributing just a number. Then distributing a variable. Then distributing a number and a variable. So that's sort of how I felt, yeah. And then working with negatives and distributing those.

Researcher: You said that you like to slowly unfold things. Is that what you're talking about now?

Lynn: Yeah, I try to scaffold the instruction so that it's not drastic for them because I just feel like, I don't want to give the kids anxiety, and sometimes, [students] can get turned off quickly if they feel like they don't get something. (Preliminary Interview, Lines 74-111)

From Lynn's perspective, the first two exercises on Multiplying Monomials (2) were meant to connect back to the examples of Laws of Exponents from Multiplying Monomials (1). Rather than draw an explicit connection between these exercises for students, however, Lynn used them to gauge students' previous knowledge prior to progressing with the lesson. Beginning with exercise #3 on Multiplying Monomials (2), Lynn used her unfolding strategy of distributing first a number, then a variable, and then a number and a variable, before including exercises in which a monomial is multiplied by a trinomial (#6 and #7). There was variation in multiple aspects across this set of examples, including the coefficients, numbers, signs, and degree of the multiplicand. The only aspect that Lynn held invariant was the variable.

Lynn's initial design of sets of examples drew on randomness as a design characteristic. Lynn chose examples from multiple classes of examples and spiraled previous content into current content. By presenting multiple classes of examples simultaneously, Lynn intentionally created opportunities for contrast and student discernment between classes of examples. Randomness also appeared as an aspect of tasks in which Lynn had intended to unfold the skill or concept, particularly as it related to the coefficients, variables, and structure of the examples chosen. As stated in the previous section, Lynn's notion of student success was tied to students' abilities to successfully apply procedures, and randomness as a design characteristic allowed for opportunities for both practice at applying procedures and assessment of students' abilities to successfully do so. Lynn's unfolding strategy also appeared to relate to her notions of student success in two ways. First, Lynn was concerned with, and seemed to have an awareness of, what students would be willing to do; she did not want to create anxiety for students around mathematical examples. Second, her unfolding strategy demonstrated Lynn's awareness of the need for incremental changes between examples within sets, rather than large leaps.

Initial enactment of sets of examples. Because students in her RTI sections may have had different classroom mathematics teachers or be in different class sections of the same mathematics teacher, Lynn often felt as though she was not able to teach a lesson straight through. While Lynn often started class with a common set of examples for all students, she described differentiating her instruction by allowing some students to work ahead, telling students to do different practice exercises, and choosing various practice exercises from her made-up packet, depending on how she assessed students' knowledge. Her instructional decisions appeared to be related to Lynn's reliance on randomness within practice examples. I observed Lynn allowing additional time for practice exercises for particular students and choosing different practice exercises during the preliminary observation:

> [Lynn] hands out [More Multiplying Monomials] worksheet to Shauna and Michael. Shauna and Michael go to the board. Caitlin sits at the desk in the second row. Shauna starts doing #1 on the board. Lynn stands to her left, looking at her work and then says, "When you have something just to the first power, you don't have to put the one. If it helps you to put it here when you're doing it, that's fine, but you don't have to put it here in your answer." Lynn then walks to her right to Michael: "Good!" Michael has done #1 and #2 on the board. Lynn circles back to Caitlin, talks to her (she's still working on the

> previous worksheet), and then gives her the sheet the other two are working on. Lynn tells Caitlin that she can skip around and try some other ones if she wants to. (Preliminary Observation, Lines 103-113)

The randomness inherent in Lynn's design of sets of examples complemented her choice to have students skip around, work ahead, or work on different sets of examples. Allowing students to work on different sets of examples was based on Lynn's purposes of assessment and practice. Lynn assessed students' understanding through practice examples and then chose additional practice examples based on that assessment. While this allowed for differentiation in terms of practice, if any mathematical structure was inherent within a set of examples, the opportunity for students to notice it was diminished through this instructional decision.

The emphasis in Lynn's RTI classes seemed to be on knowing rules and carrying out procedures. While speaking about the preliminary observation, Lynn stated, "My major goal is just to be able to differentiate between the different rules, between adding and subtracting versus multiplying" (Preliminary Interview, Lines 152-153). It is not clear from this statement if Lynn's goal was for students to simply understand the different rules or develop an understanding of when to apply a particular rule. The emphasis in her instruction remained on the differences between the procedures for each process, with little attention given to the differences between the structural features of the given examples that would alert students in recognizing an instance of a particular class of example. I asked if there were any important features that she tried to point out to students regarding the differences between adding/subtracting polynomials and multiplying a monomial by a polynomial as classes of examples. Lynn's response was regarding the methods students used to simplify such expressions:

> I have them do shapes…So they'll do triangles around the x^2 terms, and a circle around the x^3…But when I introduce it, that's what I do with them, is do the shapes. So a lot of them just continue to do that…For the multiplying, I feel like it's just they have to remember the rule [distribute and apply laws of exponents]…We did boxes too (see Figure 4.2)…especially when you are doing a trinomial times a binomial. (Preliminary Interview, Lines 168-181)

Lynn's instruction, as well as the sets of examples and exercises assigned in class, emphasized having rules and procedures and knowing how to apply them, with little emphasis or discussion on knowing when to apply them or why to do so.

	x^2	x	2
x	x^3	x^2	$2x$
-5	$-5x^2$	$-5x$	-10

Figure 4.2. Lynn's box method for multiplying a trinomial times a binomial as described in her initial interview.

As discussed in the section on Lynn's initial design of sets of examples, she occasionally used what she described as an unfolding strategy, such as on the Multiplying Monomials (1) worksheet (see Appendix G). While Lynn recognized connections in the mathematical structure between exercises such as $3x^2 \cdot 4x^3$ and $4x(2x^2 - 1)$ and structure within her arrangement of exercises, there is little evidence that this structure was conveyed to or recognized by students. After reintroducing the Law of Multiplication for Exponents and having students complete exercises one through four on Multiplying Monomials (1), Lynn said to the class, "Good. I wanted a little practice with that because we're doing something just like it; we're just going to increase the number of factors that we're working with" (Preliminary Observation, Lines 67-69). Lynn then revealed the rules for multiplying by a monomial on the SMART Board, which included the directive to distribute, with two bullet points underneath that read: “Multiply the coefficients,” and “Add the exponents on the variable” (Preliminary Observation, Lines 70-73, SmartNotebook file). After this, Lynn passed out the worksheet entitled Multiplying Monomials (2) and asked the students to get started. As she walked around the room, she repeated the procedure, “Multiply the coefficients and add the exponents” (Preliminary Observation, Lines 74-78). The only discussion about any relationship between the exercises regarded the coefficient of one in front of a variable when no coefficient is written:

> Lynn walks back to the SMART Board and says, “When there’s just an x, remember that this is a one x,” and writes a one in front of the x. Lynn continues, “Just like this [referring to exercise #1: $5x(2x)$]. We multiplied the coefficients and added the exponents. [Back to exercises #4: $x(3x + 1)$] One times three is three. And add these exponents. One plus one is two, so $3x^2$.” (Preliminary Observation, Lines 82-85)

Through enacting this set of exercises, Lynn drew attention to established notation for coefficients (i.e., a coefficient of one does not need to be written). Lynn repeated the procedure for exercise #1 and applied it to exercise #4 (for the first term), but otherwise does not draw attention to the similarities or differences in structure between the two examples. The variation in multiple aspects throughout the set of exercises may also have masked the scaffolding that Lynn intended. While students may have recognized an increase in level of difficulty, it was not clear if students made specific observations about what aspects made the exercise more difficult and the ways in which it was similar to and different from the previous exercises.

Summary of Lynn’s initial purposes, design, and enactment. Lynn’s predominant purposes for using examples were assessment and skills practice, which were closely linked. Lynn would often design sets of examples that included randomness as a characteristic, in terms of classes of examples (e.g., adding and subtracting polynomials with multiplying monomial by a polynomial) and in terms of the aspects (e.g., coefficients, degree, variable). Lynn's initial design strategies included (1) the presentation of multiple classes of examples simultaneously for assessment of student understanding and opportunities for contrast and student discernment between classes of examples, and (2) simultaneous variation in multiple aspects for assessment of student understanding and skills practice. For Lynn, randomness was an integral part of the design since it allowed her to assess students' understanding of when to apply particular procedures and created an opportunity for student conflation of procedures and misconceptions to arise. While enacting such sets of examples, Lynn would often make instructional decisions for individual students regarding which practice examples to do next. This created another element of randomness, as Lynn would often select different examples for different students. Because of these design and instructional decisions, the opportunity for students to discern algebraic structure within the sets of examples was often diminished. Differences between classes of examples were not generally made explicit. Rather, Lynn would notice students conflating rules and procedures for the classes of examples and address misconceptions that arose, occasionally in an attempt to separate the classes of examples, but more often through a re-teaching of the various procedures without explicit attention to the critical aspects that separated one class of examples from another.

When Lynn used her class time for more direct instruction, rather than practice and assessment, she described her design of sets of examples as slowly unfolding. This design strategy was related to her notions of student success, as she had an awareness of how large conceptual leaps between examples could be detrimental for students' learning. Lynn described the need for incremental changes between examples to alleviate student anxiety and create opportunities for students to notice the connection between examples, although sets of such examples were often enacted in ways that were more procedural, rather than focused on the incremental changes in aspects between examples that connected them to one another.

Assessment and practice were the primary purposes with the population of Lynn's initial example space of use and views of examples. Design elements within her example space included simultaneous variation, randomness, and an unfolding strategy generally based on an increase in the perceived level of difficulty of examples. Lynn generated sets of examples by mixing sets of different classes of examples and employing random variation. From her perspective, randomness within sets of examples connected to her dual purposes of practice and assessment and conveyed the generality of the procedures for a class of examples. Lynn described her unfolding strategy as a means of connecting to students' prior knowledge, but Lynn articulated no enactment strategies for drawing students' attention to the mathematical structure that she intended students to discern. Table 4.4 summarizes Lynn's initial example space of use and view of examples.

Shannon Edwards

Shannon Edwards had been teaching at Augustus Middle School for three years. At the time of this study, Shannon was a part-time teacher and taught two sections of eighth grade mathematics and one section of seventh grade mathematics.

Initial purposes for using examples. Shannon had a number of purposes for using examples, including explanation, exposure to variation in multiple aspects, providing skills practice, and assessment of student understanding for instructional decision making. Shannon used examples to explain mathematics to her students: "Examples in math is basically how I teach the kids," (Preliminary Interview, Line 8). She sought to expose students to variation in the

Table 4.4

Lynn's Initial Example Space of Use and View of Examples

Characteristic	Summative Description
Population	Predominant use of examples was for assessment for instructional decision-making and practice for fluency. Sets of examples included mixed classes of examples with simultaneous variation. Applying procedures for one class of examples to the other created an opportunity to contrast the classes of examples. Sets designed with an unfolding strategy to increase difficulty of examples and included a wide range of variation in aspects. Verbal enactment strategies for contrasting classes of examples, when student errors emerged. Scarce strategies for contrast between examples within a particular class of examples.
Generativity	Use of unfolding strategy suggested the potential for generating sets of structured exercises using a similar strategy.
Connectedness	View of examples for the use of assessment and practice for fluency well-connected mixed sets of examples and simultaneous variation.
Generality	Sets of examples specific. Generality within sets of examples used for assessment/practice, in terms of simultaneous variation and the use of mixed classes of examples.

aspects of examples based on what she perceived students would encounter in their mathematics career: "So I choose examples based on…here [are] the different ways you might see [the topic] in a question," (Preliminary Interview, Lines 8-9). Examples were used for skills practice as Shannon believed, "the more they do it [exercises], the more they get [understand], the more they'll understand the process," (Preliminary Interview, Lines 14-15). Lastly, Shannon used examples for assessing student understanding for instructional decision-making, in ways that overlapped with skills practice. She discussed including certain examples in her sets of examples in which the responsibility for completing the examples would shift to the students. Shannon would walk around to observe students' work: "And then, the second one, I have them do on their own...And [I] walk around to see, ok, did you really get this [angle relationships]? Are you understanding what this looks like and what this means?" (Preliminary Interview, Lines 59-61). Students' completion or attempt at examples independently provided feedback to Shannon about students' understanding and helped her to make instructional decisions. Based on the feedback, Shannon would often use examples at the beginning of class that addressed topics or procedures Shannon had noticed students struggled with. Shannon observed: "If I have seen something that kids are struggling with or the kids are making a mistake on…I'll have them do a warm-up [example to start the class] and go over something like that" (Preliminary Interview, Lines 164-166). Thus, examples used for the purpose of assessment could later lead to examples being used for the purpose of explanation as topics and procedures were re-demonstrated or re-taught.

Shannon suggested connection as another purpose of examples and exercises, although this was not observed in the preliminary observation. In discussing how she might revise the set of examples she had used for class, Shannon indicated that she might add a bridging example that served to connect numerical expressions to algebraic expressions:

> One of the things I maybe would do…[is] put in an example before the first one [on the Complementary and Supplementary Angles worksheet] where the expression is maybe one that has an angle that has a specific measurement…So if this was the 90 degree angle, so I would do something more like if this angle is 43 degrees and this angle is $x - 6$, how could we solve for that?...Okay, now take it to what if both angles have an expression? (Preliminary Interview, Lines 189-196)

Shannon showed an awareness of connection as a purpose for using examples. From Shannon's perspective, students demonstrated a strong understanding that complementary angles summed to ninety degrees, and they could easily give the complement to an angle if given an angle measure. She noticed that some students struggled, however, when the angle measures were abstracted as algebraic expressions. In this lesson, Shannon had abstracted both angle measurements using algebraic expressions. Her suggestion involved giving students one angle measure numerically and the second algebraically to focus students on solving for *x* before giving students both angles in terms of *x*. This purpose appeared to be related to her notions of student ability and success. Shannon stated, "That's a step that maybe some of those lower level kids needed" (Preliminary Interview, Line 199). From Shannon's perspective, it was students who she described as "lower level" that struggled with solving for the unknown variable and angle measures when both were given as algebraic expressions. Using a bridging example for connection between numerical expressions and algebraic expressions was a purpose that some students needed, but others did not. The purpose of connection was not drawn on as frequently as exposure to variation, explanation, skills practice, and assessment for instructional decision making, which all appeared to be more universal purposes, from Shannon's perspective.

Shannon drew on the following four purposes for using examples: (1) explanation (2) exposure to variation in aspects, (3) providing skills practice, and (4) assessment of student understanding for instructional decision making. A suggested fifth purpose was connection, which appeared related to notions of student success, with bridging examples needing to be used for the success of some students, but not others. Shannon had a general awareness of the variation in aspects throughout sets of examples for in class lecture, in class practice, and homework. Examples given for students' practice were also used by Shannon for the purpose of assessing students' understanding, hence these examples often served a dual role.

Initial design of sets of examples. One of Shannon's main purposes for using examples was exposure to variation in aspects. As a part-time teacher, Shannon often used the materials designed by Tori, the seventh grade mathematics teacher, or Robert, the eighth grade mathematics teacher. The lesson that I observed was a seventh grade lesson, hence Shannon's purpose of exposure to variation in aspects was likely influenced by Tori's purposes and design, as discussed in the previous section. Shannon saw examples as the main way of teaching and conveying the mathematics to students. The purpose of exposure to variation in aspects directly influenced her design, or her perception of the design, as she described that her choice of examples was based on the variations in the prompts that she believed was necessary for students to be exposed to:

> So I choose examples based on this is the content you need to know. Here's the different ways you might see it in a question. And so that these are the examples we are going to go over because you could see it this way or you could see it this way or you could see it this way or then they may see it in their homework in a way that we didn't go over. (Preliminary Interview, Lines 8-11)

The desire to expose students to variations in the prompt related to a class of examples was seen in the Preliminary Observation (May 29, 2015). In this lesson, Shannon was teaching students how to find angle measures based on supplementary and complementary angle relationships when given algebraic angle measures. Her examples included three in which the angles were presented pictorially and one that was presented verbally. Two of the examples exhibited a complementary relationship, and two exhibited a supplementary relationship. One of the examples included fractions as coefficients. All of the pictorial examples were presented as images in which the angles were oriented parallel to the bottom of the page (see Appendix H). This design provided opportunities for students to discern between complementary and supplementary angle relationships, generalize angle relationships over presentation (pictorial or verbal), and generalize angle measures over form (numerical or algebraic), including generalization of form over number (whole or rational). The set of examples as designed did not provide an opportunity for students to generalize angle relationships over orientation.

Shannon described her design as a sequence of examples that went from basic to complex. This was related to the purposes of exposure to variation in aspects and explanation. Shannon described her sequencing as follows:

> I'm going to choose the easiest example that I think possible so that they get the process and don't get hung up on fractions or decimals or something more complicated. So I'll choose a really straightforward [example] to show them the process…and then my examples will get progressively harder. For example, when we first introduced solving two-step equations we used whole numbers, then we'll use an example where maybe they have to distribute before they solve, and then an example with fractions. (Preliminary Interview, Lines 25-31)

From Shannon's perspective, complexity was equated with additional steps, such as distribution, or with fractional coefficients. These were variations in the prompts that she believed necessary to expose students to and provided opportunities for her to explain the mathematics through example. She placed an emphasis on the process as generalizable over each of these variations.

Shannon's use of examples for the purposes of explanation, skills practice, and assessment of student understanding for instructional decision making influenced her design through sequencing. While Shannon's sequencing went from basic to complex, as described above, examples tended to be arranged in (at least) pairs so that Shannon could explain a concept or demonstrate a procedure to the students, followed by students working on an example independently, or with support, as needed. Shannon described the intended enactment of her design as follows:

> After we talk about what they [complementary and supplementary angles] look like and what that means, then there's one example where we have a diagram and they name the different angle relationships with me. And then, the second one, I have them do on their own. (Preliminary Interview, Lines 57-59)

Shannon's description is that of an "I do, we do, you do" scaffolding strategy in which she explains a concept or process ("We talk about what they [complementary and supplementary angles] look like and what that means."), then she completes an example with input from students in the class ("They name the different angle relationships with me."). This is followed by students completing an example on their own ("I have them do on their own."). Her intention of using such a strategy during enactment heavily influenced her design, and intersected with her purposes for using examples as explanation, skills practice, and assessment of students' understanding.

Lastly, Shannon's design of sets of examples, and her choice of what example to start with, were tied to her notions of student success. Shannon stated:

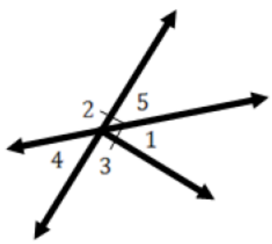
> I feel like if you start out too hard, I'm just going to lose them [students] right at the beginning. But then there's that fine line for those higher level kids. If I start out too easy then I'm going to lose them. (Preliminary Interview, Lines 240-243)

Shannon sought to find a balance between examples that could be perceived as too hard or too easy by students at either low or high levels, as she called it. Some of this was assuaged by enactment strategies that I will discuss below.

Shannon used many materials that had been designed by Robert, the 8th grade teacher, and Tori, the 7th grade teacher. This was in large part because Shannon was the newest addition to the seventh and eighth grade team of mathematics teachers, and a part-time teacher. While Shannon often discussed the materials as though she had designed them ("I'm going to choose the easiest example possible…"), more often she is describing her perception of the design in the materials. While she attended to the increase in complexity within the sets of examples and opportunities for assessing student learning and providing students with skills practice, there appeared to be contrast implicitly embedded in the design of the materials. Shannon did not show an explicit awareness of that contrast at the time of the preliminary interview and observation. The materials included a reversal in what was given and what was asked for, and contrast in the angle relationship. Figure 4.3 shows a pair of examples from the preliminary observation that demonstrated contrast through reversal. In example one, students are given a relationship (adjacent, supplementary, or complementary) and asked to name angles that have that relationship. In example two, students are given a pair of angles and asked to name the relationship. These materials had been designed by Tori, the 7th grade teacher. Recall from the previous subsection that contrast through reversal was an explicit part of Tori's design. Thus, Shannon did not express an awareness of the contrast and reversal of process intended by Tori in her design.

Shannon's perception of the design in her materials included variation in the prompts, and sequencing that went from basic to complex, for the purpose of exposure to variation in the aspects of the class of examples. Shannon sought a balance between examples that were perceived as too easy or too hard by the students, which was tied to her notions of student success. The sequencing in the design reflected an "I do, we do, you do" strategy through (at least) paired examples for the purposes of explanation, skills practice, and assessment of student understanding for instructional decision making. Shannon did not express an explicit awareness of the contrast and reversal implicitly embedded in the design. Rather, the reversal and contrast appeared to be perceived by Shannon as another variation in the prompt for the purpose of exposure to variation in aspects of the class of examples.

1.) Use the following set of angles to identify the following angles.

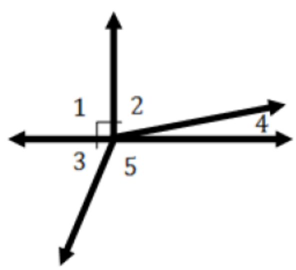


a.) Name a pair of adjacent angles.

b.) Name a pair of supplementary angles.

c.) Name a pair of complementary angles.

2.)



$\measuredangle 2$ and $\measuredangle 4$ are ____________________

$\measuredangle 3$ and $\measuredangle 5$ are ____________________

$\measuredangle 1$ and $\measuredangle 3$ are ____________________

If $\measuredangle 2 = 68°$, what is the measure of $\measuredangle 4$?

Figure 4.3. A pair of examples from Shannon's preliminary observation that demonstrated a reversal strategy. Since Shannon's examples were largely designed by Tori, this was likely reflective of Tori's use of the reversal strategy.

Initial enactment of sets of examples. Shannon's purpose of exposure to variation in the aspects of a class of examples and need for student success influenced her enactment through an emphasis on procedure and making sure that students were told how to complete a process prior to being asked to do so. This manifested itself as processes that were broken into steps and an enactment strategy that followed an "I do, we do, you do" model that played a role in Shannon's design. The following excerpt is Shannon's discussion of example 3 (see Appendix H) in the preliminary observation:

> Shannon continues, “Ok, so we’re going to take what we know about complementary and supplementary angles and solve equations with them. There’s three steps you need to follow. Step One is right here.” Shannon circles “Write an equation” in the directions and writes “Step 1” above it. Shannon says, “What do we know from the front? These angles are complementary, which means we have a sum of 90. So this angle plus this angle (Shannon draws in an angle symbol, $\measuredangle$, within each angle in the visual representation) added together has to be 90. So we have everything we need to write an equation. So this angle plus this angle equals 90.” As Shannon says this last sentence, she writes on the board: $(x - 2) + (7x + 20) = 90$. (Preliminary Observation Field Notes, Lines 117-124)

In this excerpt, Shannon took on the sole responsibility of writing an algebraic equation that could be used to determine the value of the variable and the missing angle measures. This was an enactment of the “I do” portion of the “I do, we do, you do” model. Shannon went on to describe Step Two as solving for the variable, garnering more student input into this portion of the solution, as students had prior experience combining like terms and solving two-step equations. Shannon identified Step Three as “find the actual measure of each angle” (Preliminary Observation Field Notes, Lines 141-142), and again, took on the responsibility herself for the process, including warning students of common mistakes and requiring that students show a check that their resultant angle measures did, in fact, add up to the required value of 90 degrees, in this case.

In the next example (see Appendix H, example 4), Shannon demonstrated the “we do” portion of the “I do, we do, you do” model:

> Shannon says, "Ok, let's go through another one. These angles are what kinds of angles? Supplementary. Supplementary angles add up to what sum? 180. Raise your hand if you know the equation to write to solve for *x* because that's our first step. *(Pauses for three seconds.)* I think some people who know the equation are moving on to solve for *y*. Mia, what is it?" Mia gives the equation: $3y - 1 + 7y - 19 = 180$. Shannon says, "Raise your hand if you know what the next step is…" (Preliminary Observation Field Notes, Lines 153-158)

Shannon identified the angle relationship displayed in the visual representation and answered her own question about the sum of the angle measures. In this example, she asked students to provide an equation that could be used to solve for *x*. She asked for further student input regarding the next step. While there was an opportunity for contrast present in the design of Examples 3 and 4 regarding the angle relationship (complementary and supplementary), Shannon, herself, took on the cognitive work of identifying the angle relationship within the enactment.

Without first identifying the angle relationship, Shannon asked students to complete Examples 5 and 6, demonstrating the "You do" portion of the "I do, we do, you do" model. Shannon asked students to consider Example 6 before asking students to provide a correct equation.

> "Read question 6. Raise your hand when you know what you have to do to solve #6."…Kurt is the only student with his hand up. Shannon nods to him. Kurt says, "You do $4k + 5 = 180$." Shannon says, "If I do that, one angle equals 180? Cherie?" Cherie says, "You do $4k + 5 + 14k - 5$." Shannon goes to the SMART Board and writes: $(4k + 5) + (14k - 5) = 180°$. Shannon says, "So now, you know what to do. You've

> done these. You're going to have to know how to solve these with no picture. This is important to have written down on your paper so that you know that angle one and angle two added together still equals 180." Shannon circles the work "two" in the question, underlines "are supplementary," and underneath this underlined portion, draws an arrow to 180° (i.e. → 180°). (Preliminary Observation Field Notes, Lines 184-193)

While Shannon left the discernment of the angle relationship up to the students in this example, it was only partially successful. This example provided an opportunity for generalization of angle relationships over form of the prompt (i.e. pictorial or verbal), but as before, the cognitive work was shifted away from the students by Shannon's declaration that, "You're going to have to know how to solve these with no picture. This is important to have written down on your paper so that you know that angle one and angle two added together still equals 180" (Preliminary Observation Field Notes, Lines 191-193). While Shannon seemingly associated such a verbal example with the previous supplementary pictorial example, there was no discussion of the relationship between the pictorial and verbal representation and how a drawing could be used in solving a question in which the information about the angle relationships was given verbally.

Shannon's concerns with student success created an awareness of examples that could be perceived as too hard or too difficult by students. As described in the section on Shannon's initial design of sets of examples, the sequencing of sets of examples often included an increase in complexity. Shannon tried to assuage students at all levels within the enactment by being explicit about the increase in complexity and focusing on the process in earlier, easier examples, which she discusses below:

> Sometimes everybody feels like I'm starting out with too easy of an example. But I make sure I say to them, listen, I know that this is really simple. I know you think you know

how to do this but just pay attention to this and the actual process that we're doing because when you see something harder, if you don't understand why we're doing what we're doing here, you're never going to be able to do the example that's coming up in three examples. You know so I try to get them to stay with me that way. (Preliminary Interview, Lines 245-250)

As described by Shannon's "I do, we do, you do" instructional strategy, she used initial, easier examples within the "I do" portion of her instruction to explain and to focus students' attention on the procedure for that day's lesson. From Shannon's perspective, her explicit warning that she would expect students to complete more difficult examples themselves served to hold higher level students' attention, while simultaneously focusing and holding lower level students' attention on the procedure at hand. In this sense, it appeared as though Shannon was attempting to express the generality of a particular procedure between examples, although it was not clear from the initial observation if such attempts were successful. Within the initial observation, the variation in the prompts and the angle type appeared to mask any generality students were meant to perceive.

Summary of Shannon's initial purposes, design, and enactment. Shannon's perception of what was necessary for student success directly affected how she viewed the purpose of examples, her design of sets of examples, and her enactment of examples. From Shannon's perspective, the purpose of examples was predominantly to explain and expose students to variation in aspects of a class of examples. Another important purpose from Shannon's perspective was the overlapping purposes of skills practice and assessment of student understanding to garner feedback and make instructional decisions. Connection was a suggested purpose that gained significance for students that Shannon identified as low level. The purposes of explanation, skills practice, and assessment of student understanding were manifested in a design and enactment that drew on an “I do, we do, you do” instructional model that seemingly moved the cognitive responsibilities for discernment of contrast and generalization away from the students. Because of her notions of student success, Shannon placed an emphasis on examples that were neither too hard, nor too easy, and the identification of steps for processes. She suggested the use of connecting examples to bridge from a known procedure to a new procedure, particularly for students that Shannon described as “low-level,” although this was not enacted. While Shannon saw some generalizable structure within the set of examples about complementary and supplementary angle relationships, her emphasis on presenting multiple ways in which students might see questions around a particular topic or class of examples created unsystematic simultaneous variation in a number of aspects (e.g., in angle relationships, coefficients, and angle presentation), that diminished opportunities for student discernment and generalization of critical aspects.

Shannon’s primary use of instructional materials designed by Tori and Robert made the generativity of Shannon’s own initial example space of use and views of examples difficult to discern. Drawing from her purposes of explanation and exposure to variation, Shannon’s main means of generating examples was to consider what aspects of the examples could change and providing as many of those variations as possible. Like Lynn, there was a strong connection between examples for practice and assessment. Shannon described the possible use of bridging examples, but connected the use of such examples to her perception of student ability and success. Table 4.5 summarizes Shannon’s initial example space of use and view of examples.

Table 4.5

Shannon's Initial Example Space of Use and View of Examples

Characteristic	Summative Description
Population	Predominant use of examples was for explanation, exposure to variations in a class of examples, practice for fluency, and assessment for instructional decision making Instructional materials predominantly designed by Tori or Robert. Shannon saw basic to complex sequencing as exposure to variation within the class of examples. Simultaneous variation in multiple aspects between examples for fluency and assessment. Enactment included an "I do, we do, you do" instructional model. Scarce strategies for drawing attention to mathematical structure within a set of examples.
Generativity	No indication of characteristics for designing sets of structured exercises or LGEs.
Connectedness	View of examples for explanation, practice, and assessment well-connected to "I do, we do, you do" instructional model. View of examples for exposure to variations within a class of examples well-connected to the instructional materials designed by Tori.
Generality	Generality of an "I do, we do, you do" instructional model across content, reflected in the way that Shannon viewed instructional materials.

Summary of Initial Purposes, Design, and Enactment

All of the teachers had an awareness of variation in aspects of classes of examples within their design of sets of examples. For Tori and Shannon, variation was exposure for learners to experience ways in which the particular class of example might be presented. For Robert and Lynn, generality was conveyed through randomness, and simultaneous variation in multiple aspects was an important component of examples used for practice. In the dynamic of the four teachers, Shannon and Lynn often drew on materials created by Robert and Tori since Shannon taught part-time and Lynn taught the RTI sections. Shannon largely described her perception of the examples within materials she used that were designed by Tori, and did not articulate certain patterns of variance and invariance, such as implicit contrast, embedded in the design of sets of examples she used. Lynn's view of her role as the RTI teacher restricted her use of examples mainly to assessment for instructional decision making and practice. Robert and Tori, who more often designed their own sets of examples, articulated intentional purposes of the examples they used and rationale for their design. Robert used examples to connect students' prior knowledge to new knowledge under his overarching purpose of a focus on the generality of big ideas. Tori designed sets of examples with contrast between examples as a specific pattern of variation via reversal and juxtaposition. She also articulated a design strategy of one small change between examples for the purpose of exposure and generality. Tori described the design of what might be considered a structured exercise in her linear equations unit, where the numbers used in the examples remained invariant while the signs of the terms and the location of the numerical values changed between examples. Robert described the improvisational design of an LGE through questioning, where he asked students how many sets of dimensions for a rectangle they could generate that had a perimeter of 26 units. All of the teachers articulated some sort of restricted variation in the design of sets of examples (e.g., one small change, tiering, unfolding, bridging). Among the four teachers, there was a sense that an inclusion of particular kinds of examples, "tripwires", patterns of variation, such as contrast, and randomness within aspects, was sufficient for students' learning. The teachers articulated very few enactment strategies to support students in discerning the mathematical structure that they, themselves, saw within their designs of sets of examples.

Research Lessons

In this section, I present a description of each of the four learning study cycles in terms of purpose of the lesson, design of structured exercises and LGEs, implementation of structured exercises and LGEs, and evaluation of the lesson. While the learning study process appeared to proceed linearly (see Figure 3.1), there was a strong interaction between the purpose of the lesson and the design of sets of examples for this group of teachers. The process of designing examples clarified and refined the shared purpose and critical aspects of the object of learning for the lesson among the four teachers. Hence, the process was generally not linear, but rather included cycles of iteration between the preliminary phases presented in the learning study process, in which the critical aspects are identified, and the design of sets of examples in the planning. Each of the teachers taught one of the research lessons: Tori taught Research Lesson One, Robert taught Research Lesson Two, Lynn taught Research Lesson Three, and Shannon taught Research Lesson Four.

Research Lesson One

Tori taught the first research lesson on September 30, 2015 in her first period Math 7 class. Leading up to the first research lesson, the teachers had met for six 40-minute group planning meetings. Tori had volunteered to teach the first research lesson at the conclusion of our collective summer work on structured exercises, LGEs, and learning study. During that work, the teachers had decided to develop a research lesson that addressed the order of operations. Each of the learning study cycles generally followed the progression shown in Figure 3.1. I discuss each of the steps in the progression as they pertain to the purpose, design, and enactment of the lesson in the following sections.

Purpose of the Lesson. The first step of a learning study cycle is selecting a topic for study, which in this case was the order of operations. The second step is to identify a tentative object of learning. The object of learning includes both the direct object of learning, or the content of learning (i.e., the order of operations), and the indirect object of learning, or what learners are expected to be able to do with that content (Marton, 2017). The third step is to diagnose students' learning difficulties. The teachers conducted informal assessments with a subset of their students about the order of operations. The informal assessments consisted of a short set of examples, for which students wrote their responses. The teachers analyzed the informal assessment, leading to the fourth step: confirmation of the object of learning and its critical aspects. Critical aspects are aspects of the object of learning that the learner must notice, but is not yet able to. Hence, critical aspects are relative to both the object of learning and the learners. In aggregate, these first four steps of the learning study cycle explicate the purpose of the lesson.

For this lesson, the teachers identified the direct object of learning as writing equivalent expressions using properties of operations. They wanted students to understand that performing a correct step in evaluating an expression resulted in an expression that was equivalent to the first. They identified the indirect object of learning as correctly evaluating expressions containing rational numbers. The four critical aspects that the teachers identified were (CA1) the location of grouping symbols, (CA2) the value of the expression, (CA3) the role of grouping symbols, and (CA4) the maintenance of equivalency. Through noticing a critical aspect of an object of learning, it is necessary for the learner to become aware of some of the critical features. In this case, one critical aspect was the role of grouping symbols. Critical features of that aspect included the role of parentheses as (1) off-setting a number, (2) grouping numbers and symbols within an expression, and (3) indicative of multiplication. Table 4.6 lists the identified critical aspects and their associated critical features. The overarching purpose of this lesson was developing students' abilities to correctly evaluate expressions. The teachers chose to focus on the role of grouping symbols and, in particular, to distinguish among the roles of parentheses in order to serve the larger purpose of developing students' abilities to evaluate expressions correctly.

Table 4.6

Identified Critical Aspects for Research Lesson One

Critical Aspect	Critical Features
The location of grouping symbols (CA1)	The varied combinations of which characters of the expression the grouping symbols could be placed around
The value of the expression (CA2)	Correct numerical value; incorrect numerical values
The role of grouping symbols (CA3)	Off-setting a number; grouping numbers and symbols within an expression; indicative of multiplication
Maintenance of equivalency (CA4)	Equivalent expression; non-equivalent expression

The teachers identified the order of operations as the focus of the lesson because of their past experiences with students' struggles in correctly applying the order of operations to evaluate expressions. Tori and Shannon both indicated their concern with students' correct evaluation of expressions:

> *Tori:* The reason why order of operations stuck out in my head is that the kids struggled with more than two or three steps [in evaluating an expression]…For me, it's really a focus on can they [students] do three steps solidly? (RL1, Planning Meeting 1, 14:15 – 15:15)
>
> *Shannon:* My students struggled with…losing a sign of a number…I always tell them the sign in front of a number goes with the number. If they don't get that basic thing, they lose a sign somewhere. (RL1, Planning Meeting 1, 16:28-17:18)

For Tori and Shannon, the purpose of this lesson was to develop students' abilities to evaluate expressions using the order of operations correctly. Tori defined her purpose as the correct evaluation of expressions that involved only three steps, while Shannon attended to students' care with the use of negative signs. Robert, on the other hand, attended to students' conceptual understanding and flexible use of the order of operations. Robert recognized that the common acronym for order of operations, PEMDAS (Parentheses-Exponents-Multiplication-Division-Addition-Subtraction) was problematic:

> [In my Algebra 1 class] we really broke apart PEMDAS. I just destroy it. P can be [various grouping symbols other than parentheses]. I show the students different ways, like double distributing. There really *is* more than one way you can figure it out. (RL1, Planning Meeting 1, 16:03-16:25)

Robert described being explicit with his students about the problematic use of the acronym PEMDAS, including the restrictive use of P in PEMDAS for parentheses, when it in fact refers to a variety of grouping symbols. He emphasized the multiple approaches one could take to correctly evaluate an expression while still adhering to the order of operations. Robert pointed out that an expression such as $(3 + -5)(9 - 2)$ could be evaluated as $(-2)(7) = -14$ or as $27 - 6 - 45 + 10 = -14$ using distribution. While one is more efficient than the other, both approaches are correct. Lynn reiterated this point: "There are mathematically correct ways to solve a problem and mathematically incorrect ways to solve a problem and they need to be able to decipher the difference" (RL1, Planning Meeting 1, 22:30-22:40). For Lynn and Robert, the purpose of this lesson included developing students' abilities to distinguish between correct and incorrect mathematical moves.

In addition to the potentially misleading use of P in PEMDAS for grouping symbols that Robert had mentioned, Tori commented on the multiple meanings of parentheses notation in mathematics:

> What could you see? In [$20 \div 2(5 + 5)$] I think students see parentheses but don't recognize that parentheses in that case represents multiplication…For some of them it's

> seeing it in a different way…Seeing that the parentheses mean grouping, but also means multiplication. (RL1, Planning Meeting 1, 18:45-20:08)

Tori recognized that students needed to develop an awareness of the multiple roles that parentheses played within numeric and algebraic expressions. Students needed to be able to distinguish between the use of parentheses for grouping and the use of parentheses to indicate the operation of multiplication. Rather than articulate the critical aspects of writing equivalent expressions using the order of operations, the teachers articulated the relationships between critical aspects that they wanted students to discern, identifying the critical aspects in the process. The teachers wanted students to discern the following relationships:

- The location of the parentheses (CA1) matters to the value of the expression (CA2)
- The location of the parentheses (CA1) can alter its role in the expression (CA3)
- The role of the grouping symbol (CA3) must be interpreted correctly to maintain equivalency between expressions (CA4)
- The maintenance of equivalency (CA4) results in a correct numerical value for the expression (CA2)

The relationships between critical aspects that the teachers wanted the students to discern informed the design of their sets of examples.

Design of Structured Exercises and LGEs. The next step in a learning study cycle is planning the research lesson. Part of the activity of planning a lesson includes designing examples and materials to use within the lesson. In my analysis of the planning process, I focused on how teachers designed structured exercises and LGEs. Distinguishing among the roles of parentheses in expressions came to dominate the teachers' discussions throughout their planning of the lesson. Distinguishing among the roles of parentheses served the larger purpose of developing students' abilities to correctly evaluate expressions, and necessarily informed the design of the set of examples. In the following excerpt, the teachers discussed the roles of parentheses that they wanted students to develop an awareness of:

> Tori gave the example $3 + (-2)$. Does this mean you do parentheses first? No. This means that it's off-setting the -2. She suggested showing the students this and talking to the students about this. Lynn reminded the group that earlier Tori had talked about using "grouping symbols" instead of "parentheses". Lynn said that would avoid all of this. What does it mean to group versus just having a set of parentheses with a number in it? (RL1, Planning Meeting 4, Lines 26-30)

The discussion brought the multiple roles of parentheses within expressions to the fore of teachers' awareness. This included the role of off-setting a singular number, such as a -2 in the expression $3 + (-2)$ and the role of grouping expressions, as Lynn reminded the teachers. Shannon discussed the role of parentheses as indicating multiplication when positioned directly next to a number:

> Shannon shared an example of something on their 8th grade quiz: $4(x + 2)$. Students had to substitute in $x = -3$ resulting in $4(-3 + 2)$. Students correctly added the negative three and the positive two as negative one, but dropped the parentheses to get $4 - 1$, rather than $4(-1)$. Shannon said, "It's important that students know that once you combine what's in the parentheses, you can't drop them. Just combining what's in the parentheses doesn't make them go away. So that's an issue as well. So should we have a section that's about grouping, to address the brackets, absolute value, what do parentheses mean? What does it mean when you have one number in the parentheses? What happens when you combine what's in the parentheses, but you have a number

outside that's being multiplied by it? You *have* to keep the parentheses. So address all of those issues that come up with parentheses, which we've never really done, unless it comes up." Tori said she thought they'd need to address them with examples. Lynn agreed – "Otherwise, it'd be out of context for them." (RL1, Planning Meeting 4, Lines 31-41)

Shannon discussed an instance in which students did not recognize the role of parentheses as indicating multiplication. Because students did not recognize this role, Shannon described how many of them wrote an expression that was not equivalent. Recognizing the role of parentheses within an expression intersected with their proper use: "Just combining what's in the parentheses doesn't make them go away," (RL1, Planning Meeting 4, Line 34) except when it does, as in the case of $3 + (9 - 5)$. Teachers recognized that the distinctions among the roles and related uses of parentheses needed to be made explicitly for students – "which we've never really done, unless it comes up" (RL1, Planning Meeting 4, Line 39). This recognition indicated a change in teachers' awareness toward making their knowledge about the roles of parentheses explicit for the students. The lesson needed to include opportunities to make explicit distinctions among the roles of parentheses, rather than leave these distinctions implicit for the learners to discern.

The teachers decided that the design of the examples within the lesson needed to include contrast in order to explicitly distinguish among the various roles of parentheses within expressions. The teachers suggested two ways to create contrast through examples. Tori suggested the first way: "This is when we can go to the varied versus unchanged. All the numbers stay the same…What if I put parentheses here? What if I get rid of a plus sign and put parentheses here?" (RL1, Planning Meeting 4, Lines 42-44). Tori's suggestion created contrast between examples by varying the location of the parentheses and keeping the numbers used invariant. Shannon suggested the second way by drawing on an example from her eighth grade quiz: $4(x+2)-2x$. For this lesson, Shannon suggested the example $4(-3+2)-2(-3)$. Shannon's suggestion created contrast within an example by including two sets of parentheses within a single expression that had different roles. The first set of parentheses grouped the addition expression and indicated multiplication by the four, while the second set of parentheses off-set the negative three and indicated multiplication by two. Tori expressed concern that students would find the expression $4(-3+2)-2(-3)$ too difficult to evaluate correctly. I suggested taking Shannon's example and splitting it into two separate examples, where the numbers used remained invariant, per Tori's suggestion, but the role of the parentheses was different: $4(-2+3)$ and $4-2(3)$.

Before moving forward in the choice of examples, Tori spent a considerable amount of time within the group considering which numbers to use in the set of examples. Tori pointed out that if a student incorrectly wrote an equivalent expression to $4-2(3)$ as $2(3)$ because they erroneously did the subtraction before the multiplication, the student would get six as a result. Tori was concerned that she would see that as the student correctly performing the multiplication first, with a result of six, but erroneously dropping the four and the subtraction. Tori wanted to choose numbers that resulted in unique values for anticipated errors so that she could quickly ascertain students' thinking and the types of errors they were making.

> Tori suggested changing the 4 in $4-2(3)$ to an 8, but then changed her mind, because whether students did 8 – 2 first, or 2(3) first, they'd get 6 [for the first step]. She suggested $7-2(3)$, but then said no. If students subtracted 7 – 2 they'd get 5, but if they accidentally thought 2(3) was 2 + 3, they'd also get 5. So then Tori suggested 10: "So if I see an 8, I know they did 10 – 2 first." (RL1, Planning Meeting 4, Lines 76-80)

This excerpt consisted of Tori thinking out loud, with no input from the other teachers. While Tori seemed to be concerned with her ability, as a teacher, to quickly ascertain student thinking and recognize errors, the careful choice of numbers in the set of examples served students by avoiding confusion among the various parts of the expression and the results of their computations.

After deciding on the expression $10 - 2 - 3$ as the basis for the exercises, the teachers wrote a list of nine exercises that held the numbers and signs invariant, while varying the location (CA1), and hence the role (CA3), of the parentheses (see Figure 4.4). Robert suggested including the version of the expression without parentheses to show when the location of the parentheses (CA1) affected the value of the expression (CA2) and when it did not. Robert explained his rationale for including the expression $(10) - 2 - 3$:

> When you put parentheses around that one [the 10 in the expression $(10) - 2 - 3$], you think you're changing something, but you're not. Like the parentheses around -2 implies multiplication [$10(-2) - 3$], but in that one [$(10) - 2 - 3$] it doesn't change…Just showing that it didn't change at all…Why is it [the parentheses] important when you put it around the -2, but not around the 10? (RL1, Planning Meeting 4, Lines 125-130)

While the numbers and signs remained invariant in this set of examples, the location of the parentheses and the value of the expression varied. Robert recognized the importance of contrasting both the location of the parentheses and the resulting value to support students in discerning how the parentheses changed the value, if they changed the value at all. The contrast between the value of the original expression $10 - 2 - 3$ and the expressions with parentheses provided an opportunity for students to discern situations in which the parentheses can be dropped from the expression and when they cannot.

$$10 - 2 - 3 = 5$$
$$10 - 2(-3) = 16$$
$$10(-2) - 3 = -23$$
$$(10) - 2 - 3 = 5$$
$$10 - (2 - 3) = 11$$
$$10(-2 - 3) = -50$$
$$(10 - 2) - 3 = 5$$
$$(10 - 2)(-3) = -24$$
$$10(-2)(-3) = 60$$

Figure 4.4. The teachers wrote nine exercises in which the numbers and signs remained invariant and the location of the parentheses varied.

At this point, Tori voiced her concern about spending class time on too many exercises about parentheses in the expression: "My fear is that you spend all the time doing this and we haven't addressed absolute value, exponents, addition before subtraction, order of multiplication and division…I don't feel comfortable giving up an entire day not talking about [those things]" (RL1, Planning Meeting 4, Lines 136-141). Tori suggested cutting it down to two exercises. Shannon suggested five exercises:

> Shannon suggested writing $10 - 2 - 3$ five times on the paper "and as we go down, *we* insert the parentheses to show how the value of the expression changes. Now let's put parentheses around just the -3. Do we get the same answer? No. Somebody tell me why. Oh, because once you insert the parentheses that meant multiplying the two and the negative three, instead of just subtracting. Let's put [the parentheses] around the [quantity] negative two minus three. What does this give you?" (RL1, Planning Meeting 4, Lines 146-151)

Shannon's suggestion of making the insertion of parentheses teacher-led within the lesson made the differences in their location and their effect on the expression explicit. While the teachers agreed with Shannon's suggestion to insert the parentheses within the expressions during the lesson, Tori said, “How about choosing the three most important [exercises]?” (RL1, Planning Meeting 4, Line 157). The teachers had decided to take out the exercises that had more than one set of parentheses. Tori suggested not including $(10-2)-3$, $10(-2)-3$, and $(10)-2-3$ because she did not believe that students would make errors on these exercises: “[Students] know to do parentheses first. They’re going to get 8, subtract 3, and be fine. I don’t think anyone is going to get that wrong. So I don’t think that’s one that’s really important that we need to address” (RL1, Planning Meeting 4, Lines 158-160). Due to perceived time constraints, the teachers moved away from the systematic investigation of the effect of the placement of parentheses on an expression in favor of reducing the set of exercises to address perceived student misconceptions. The teachers decided on the following three exercises:

$$10(-2-3) \qquad\qquad 10-(2-3) \qquad\qquad 10-2(-3)$$

Tori, the lead teacher in this research lesson, would insert the parentheses into each expression during the lesson so that “each time, there’s one less thing in the parentheses,” (RL1, Planning Meeting 5, Line 204). In this set of exercises, the change in the placement of the parentheses was still systematic, but reduced the class time spent on the location of the parentheses and their effect on the expression. These three expressions made up a set of structured exercises that provided the opportunity to discern two relationships between the critical aspects: (1) the location of the parentheses (CA1) matters to the value of the expression (CA2), and (2) the location of the parentheses (CA1) alters its role in the expression (CA3) by comparing the role of parentheses as indicative of multiplication in the first and third exercises, and the role of parentheses as grouping in the second exercise. Because Robert’s suggestion was not taken up, the opportunity to discern whether parentheses could be dropped from the expression or not was diminished.

The teachers used a design principle of Explicit Contrast to address the need of differentiating among the roles of parentheses. The teachers used a strategy of "varied versus unchanged" within a set of structured exercises to make the location, and by relationship, the role, explicit to students. They struggled with deciding upon how many examples were enough. Eventually, the teachers decided on three examples in which the numbers and signs remained invariant, while the location of the parentheses changed between examples. Within their discussions regarding the planning and design of the set of structured exercises, suggestions regarding possible LGEs arose. While the teachers were discussing the variations in the location of parentheses and the resulting value of the expression, Robert suggested having students insert the parentheses to give them the challenge of figuring out where to put the parentheses to equal a certain value. Lynn agreed with Robert that this would be an opportunity for higher-level thinking, but his suggestion was not taken up by the group at this point.

Later, when deciding how many examples to use within the set of structured exercises, Robert again pushed for students to insert their own parentheses to obtain a particular value: "What if we did like three [exercises] with them and had them do three [exercises] in which one has to equal -50. One has to equal -24. Where would you put [the parentheses] there?" (RL1, Planning Meeting 4, Lines 152-153). Shannon took up Robert's idea, suggesting that it be included on the homework.

> Shannon said, "Even for their homework, we could put $10 - 2 - 3$ [twice] and tell students to insert one set of parentheses so in one you get 5, and in one you get -23. And just see if they can figure this out"…Tori said that was definitely going to be harder. She thought about putting it on the bottom of the homework. (RL1, Planning Meeting 4, Lines 164-169)

While Robert's earlier attempts at suggesting that students place the parentheses themselves largely went ignored, Tori responded to Shannon bringing up the idea by commenting that she though it would be difficult for students to do. Tori later suggested having students insert parentheses to obtain a different, rather than a specific, answer, saying that challenge was more at a seventh grade level. At this point, the teachers were conceptualizing LGEs as an opportunity for students to insert parentheses into an expression to make it equivalent to a particular value. Tori voiced concern over the perceived difficulty level of this activity, suggesting that having students insert parentheses to obtain any different value was at a more appropriate level of challenge than the suggestion to have the expression equal a particular value.

At the next planning meeting, Tori presented the three exercises the group of teachers had discussed as a structured exercise, followed by the expression $5 - 3^2 + 1$ being written three times across the page.

> Tori said in the first instance, they [students] would just solve it as is. In the next two, students could place one set of absolute value symbols wherever they want to, "but I was going to specify that they have to put it around at least two numbers." Tori said she anticipated the following three options: $|5 - 3^2| + 1$, $5 - |3^2 + 1|$, and $5| - 3^2 + 1|$. (RL1, Planning Meeting 5, Lines 206-216)

In this instance, Tori conceptualized LGEs as allowing students to insert their own absolute value symbols to obtain any value, with the requirement that the absolute value symbols had to be placed around at least two numbers. This eased Tori's concerns regarding the difficulty of the task and students' success. The requirement that the absolute value symbols had to be placed around at least two numbers served to restrict the possibilities of what students could obtain, making the task more manageable for Tori, the lead teacher.

Robert continued to suggest that students should generate expressions to equal a particular value. He suggested an LGE in which students choose two operations and write an expression equal to eleven: “So you tell them what it has to equal and they can come up with a different sort of variety. You could say it has to involve addition and multiplication” (RL1, Planning Meeting 5, Lines 278-280). In the earlier instance of an attempted LGE, students were choosing where to place parentheses or absolute value symbols. In this instance, Robert suggested having students generate an example of an expression equivalent to eleven. This indicated that Robert had changed his conceptualization of LGEs as the generation of examples by students, rather than student manipulation of a given example by choosing the location of parentheses.

The teachers began to discuss how open or restricted the LGE should be. This discussion was largely framed by the concern of perceived difficulty and the notion of student success:

> Lynn suggested giving them [the students] a choice of a few integers to use, but Robert said to make it more open than that. He thought it would be more of a challenge to restrict them to certain numbers. He said, “I’d be more for the creative, what are they gonna create?” Tori said, “So you’re saying, write an expression that is equivalent to eleven using three integers and two operations?” Robert anticipated that a potential problem would be students bypassing the need to think about order of operations…Then he suggested setting it up for them with blanks: ___ + ___ × ____ = 12…Lynn said, “I kind of like the structure with the blanks.” Tori said, “I think it has to be.” (RL1, Planning Meeting 5, Lines 289-310)

The teachers had begun to collectively conceptualize LGEs as students generating examples of a mathematical construct, which in this case, was an expression equivalent to eleven, later changed to twelve. The teachers grappled with what, if any, restrictions to place on the LGEs. They considered restricting the number of operations within the expression to two, restricting which operations could be used (e.g. addition and multiplication), and restricting the choice of integers to use. Due to concerns that students would bypass the need to consider the order of operations, they created a mathematical sentence frame, $\_\_\_ + \_\_\_ \times \_\_\_\_ = 12$, in which students had to fill in the blanks to make the equation true. This allowed for the choice of numbers for the blanks to remain open, while restricting which and how many operations, how many numbers, and the resulting value.

Teachers' perception of how difficult LGEs would be for students influenced the restrictions they chose to place or not place on the LGEs. Teachers' notions of student success with LGEs influenced who would have access to opportunities to experience LGEs. There was a disagreement between Tori and Robert about which students should complete the LGE:

> Tori said, "In my eyes, I foresee this as something I would give to kids who finish early. Like not make it an expectation of all." Robert pushed for it to be [an activity to end class]. "It might tell us their thinking." (RL1, Planning Meeting 5, Lines 303-305)

Tori's suggestion involved only requiring the LGE of those students who demonstrated their competency and ability in applying the order of operations in class through finishing the set of practice examples early. Robert, on the other hand, suggested that they, the teachers, could benefit from seeing how all of the students approached the LGE, not just the ones who had already demonstrated their facility with the order of operations. The notion of student success influenced both the design of the LGE, in terms of openness and restriction, and who would have access to the LGE. This suggested that while the teachers were beginning to conceptualize what an LGE was, they were still developing their understanding of the role LGEs played in teaching and learning. While Robert suggested that LGEs can tell the teachers something about students' learning, the role that LGEs have in students' learning was not discussed.

Within the design process of structured exercises, teachers developed an awareness of learners' needs. The teachers recognized that for students, the various roles of parentheses were undifferentiated, which led to errors in writing equivalent expressions. To address this need, the teachers constructed a set of structured exercises with 10 – 2 – 3 as the basis of the examples and included contrast in the design of the exercises. The numbers and symbols remained invariant among the set of examples, while the location of the parentheses varied between examples. The teachers considered how many examples were needed in the set – too many pushed time constraints and did not serve students' learning, while too few might not allow for students to discern the contrast the teachers had intended. In the planning of this lesson, the teachers initially grappled with understanding what constituted an LGE. At first, the teachers conceptualized of an LGE as manipulation of a given example. Robert presented his conceptualization of LGEs as the generation of examples by students, and then the teachers grappled with how restricted or open the LGE should be. Considerations of restriction and openness were influenced by teachers' notions of student success, which also served to potentially restrict which students would have the access and opportunity to experience LGEs.

Implementation of Structured Exercises and LGEs. The next step in a learning study progression is for one of the teachers to teach the lesson, while the other members of the research team observe the lesson. The observers do not focus on critiquing the teacher, but rather focus on the variation in students' different ways of understanding what the teacher taught. In this section, I describe the implementation of structured exercises and LGEs during the research lesson. The description is based on my field notes and my analysis of the videotape. The teachers' perspectives on the implementation of the lesson, specifically structured exercises and LGEs, is discussed in the next section on the evaluation of the lesson, as this is when their perspectives were shared.

Tori drew students' attention to the variance and invariance built into the design of the structured exercise as she introduced it, after spending time discussing the homework from the previous night:

> Tori says, "We have three expressions there. Each of those expressions are exactly the same. We're going to change them up by adding in some parentheses. And then we're going to see if adding parentheses and putting them in different places changes them or keeps them the same." The expression 10 – 2 – 3 is written out three times. Tori puts in the first set of parentheses as 10(-2 – 3). Tori continues: "Versus putting our parentheses here, around two and negative three." The second expression becomes 10 – (2 – 3). Continuing, Tori says, "And lastly, we're going to put in a set of parentheses just around that." Tori draws in parentheses in the third expression to change it to 10 – 2(-3). (RL1 Field Notes, Lines 113-121)

Tori enacted this set of exercises as the teachers had planned. The design of the structured exercise and the enactment gave students the opportunity to (1) discern the effect that the location of parentheses has on the value of the expression, and (2) discern the different roles of parentheses. In the first example, the parentheses grouped the -2 – 3 and indicated multiplication with its placement directly next to the 10. In the second example, the parentheses grouped the 2 – 3, and in the third, the parentheses offset the negative three and also indicated multiplication by their placement directly next to the two.

After allowing students to work for about four minutes, Tori asked the class, “Did you get the same thing?” to which the students respond, “No.” (RL1 Field Notes, Lines 133-134). Tori asked a student to explain what she noticed:

> The student says, “Well it started as ten minus two minus three [10 – 2 – 3]…When you put parentheses around the negative two and negative three, it changes it from subtracting them to multiplying them.” Tori says um, scrunches her face, and leans her head to the left, saying, “Careful.” The student says, “Instead of subtracting the two from the ten, it changes it to multiplying the ten by the negative two minus three [-2 – 3].” Tori says, “Good. So what Lucy had noticed is that it was ten minus two, and it became ten…” A number of students say “times.” Tori continues, “*Times* negative two. So the subtraction operation ended up changing into some type of multiplication *because* of those parentheses.” (RL1 Field Notes, Lines 140-148)

This student discerned the difference between the original expression, 10 – 2 – 3, and the first expression in which Tori inserted parentheses, 10(-2 – 3). The teachers had chosen to contrast the three expressions 10(-2 – 3), 10 – (2 – 3), and 10 – 2(-3). They did not choose to include the original expression 10 – 2 – 3. Students did not evaluate this expression, nor did Tori ask students to evaluate this expression in the enactment. This student’s discernment of the contrast in the operations between 10 – 2 – 3 and 10(-2 – 3), was not intended, but was taken up by Tori in the enactment when she asked the student to share what she had noticed. In this way, the contrast between the original expression, without parentheses, and the other expressions, with parentheses, which was left implicit in the design, was made explicit by Tori in the enactment.

In response to student errors Tori had noticed while students were working independently, she discussed the order in which the operations should be performed with the whole class:

> Tori says, "What operations do you see in that last expression [10 – 2(-3)]?" One student says subtraction. Tori says, "And what else?" Another student says multiplication. Tori points to the subtraction sign and the multiplication represented by the parentheses in the expression. Tori continues, "In your order of operations, what comes first? Subtraction or multiplication?" Students say aloud, "Multiplication." (RL1 Field Notes, Lines 166-169)

It is not clear from this excerpt if, while working independently, students had either (1) not recalled that multiplication comes before subtraction in the order of operations, or (2) not discerned the location of the parentheses as indicative of multiplication. Tori appeared to interpret the student errors as incorrectly applying the order of operations. The order in which operations are performed is necessarily linked to the location of the parentheses within the expression. Within this excerpt, Tori remained implicit about the relationship between the location of the parentheses (CA1) and their role as indicative of multiplication (CA3). For the remainder of this structured exercise, Tori did not refer back to the location of the parentheses within the expressions and the location's relationship to the role of the parentheses.

In the second set of expressions, consisting of $5 - 3^2 + 1$ as the basis of the examples, Tori evaluated the expression with the class, as it was, without inserting any grouping symbols. She then directed the students to place their own absolute value symbols around any two numbers in the next two expressions and evaluate them with their partner. Tori walked around the room, looking at students' work. After some time, Tori took two student papers and placed them side by side under the document camera. On one paper, the student had placed the absolute value symbols to give the expression $5|-3^2+1|$. The other student had placed the absolute value symbols to give the expression $5-|3^2+1|$. Tori asked the class if the absolute value symbols were in the same location, and the students responded no. This served to visually contrast the location of the absolute value symbols in the expressions. Then Tori asked, "Does it give you the same result?" A student responded no. Tori explicitly contrasted the effects the different location of absolute value symbols had on the expression:

> Tori said, "Let's talk about why. By putting the absolute value after the five, what operation comes into play?" Tori calls on a student who says, "Multiplication." Tori says, "Multiplication, yeah…But here [in the expression $5-|3^2+1|$], by moving your grouping symbol or absolute value, what operation comes into play, that was not in the first one? So we had multiplication. What operation do we now have?...Shout it out. What operation comes into play?" The students say subtraction. Tori continues, "So we got two different results based on where those absolute value symbols are located." (RL1 Field Notes, Lines 227-236)

Unlike in the previous set of structured exercises, Tori is explicit about contrasting the location of the absolute value symbols (CA1) and the effects on the expression, including which operations are indicated and the resulting value of the expression (CA2).

Early in the lesson, a student asked, "Is there any difference between parentheses and brackets, besides what they look like?" (RL1 Field Notes, Line 53). This resulted in an interesting discussion in which Tori seized an opportunity to enact contrast and students generated examples. A second student responded to the first by saying that there is a difference: with both parentheses and brackets, you perform any operations within the parentheses or brackets first, but parentheses can also mean to multiply. This created an opportunity for Tori to sort out what is the same and different about parentheses and brackets. This excerpt described how Tori responded to the first student's question and the second student's subsequent response:

> Tori draws [3+2] and (3+2) on the interactive whiteboard and then holds the marker out to the [second] student as she walks up to the interactive whiteboard. Tori asks the class if [3+2] and (3+2) mean the same thing. The class says yes. The [second] student comes up and writes a 13 to the right of the parentheses and says, "That would mean to multiply": (3+2)13. Tori writes a 13 to the right of [3+2]: [3+2]13. Another student asks if that is the same thing. Tori repeats, "Is that the same thing?" (RL1 Field Notes, Lines 57-63)

This demonstrated Tori noticing an opportunity to sort out what is the same and different between the use of parentheses and brackets in algebraic expressions, and choosing to use contrast as a pattern of variation to do so. Tori initiated contrast by choosing to write the same expression 3 + 2, changing only the brackets (or parentheses) placed around the expression. A student unconventionally writes 13 to the right of the expression in parentheses to indicate multiplication. Tori chose to preserve what the student wrote, keeping the numbers and their positions invariant, while only varying the types of brackets used. By making this choice, Tori intentionally varied only a single aspect between the expressions (3+2)13 and [3+2]13.

As the discussion continued, Tori again chose to enact contrast in the moment and varied only a single aspect between examples.

> Another student says they [brackets] are used for more than one grouping. Tori holds the marker out to him and asks him to come to the interactive whiteboard. The student comes up and writes $[8(9 \cdot 3)]$. Tori writes $(8[9 \cdot 3])$ and asks, "Is this the same thing?" The class was silent. One student said, "I don't know." Tori explains that parentheses and

brackets as grouping symbols are interchangeable. "They're just another grouping symbol to help us avoid this: $\left(\left((9 \cdot 3) \cdot 4\right)\right)5$. "Parentheses inside parentheses inside parentheses." (RL1 Field Notes, Lines 65-70)

The initial student's question, "Is there any difference between parentheses and brackets, besides what they look like?" (RL1 Field Notes, Line 53) indicated that at least one student had discerned difference between round brackets, or parentheses, and square brackets. Tori's take up of the initial student's question and how she chose to proceed, opened the classroom space to allow students to generate examples (e.g., (3+2)13 is an example in which the parentheses indicate multiplication; $[8(9 \cdot 3)]$ is an example in which different types of brackets are used for more than one grouping). Tori enacted contrast as a pattern of variation, varying only the type of bracket used between two examples, which provided an opportunity for other students in the class to discern the difference in symbols observed by the initial student. The discussion began to push toward generalization when Tori explained that parentheses and brackets are interchangeable within algebraic expressions. As Tori explained, the main purpose of different bracket symbols is to avoid the use of multiple sets of parentheses that may be hard to distinguish amongst.

The discussion pushed further toward generalization when another student asked, "Are they [braces] another grouping symbol?" (RL1 Field Notes, Line 73). After eliciting from students what braces look like {}, and instances in which they are used, Tori confirmed that braces are yet another grouping symbol and provided the example $\{3(3[4+2])\}$. Based on a student's suggestion that braces are also used for lists, Tori provided the example {3, 2, 7, 9}, indicating that while braces do act as a grouping symbol, like parentheses and brackets, they have additional mathematical meaning.

While parentheses and absolute value symbols as grouping symbols came up within the teachers' planning meetings, the issue of the differences amongst grouping symbols such as parentheses, brackets, and braces never emerged. It is possible that the teachers had generalized parentheses, brackets, and braces as equivalent mathematical symbols to the extent that the teachers no longer recognized that the type of bracket used could be a critical aspect that students might attend to. The unexpected need to distinguish amongst brackets and then generalize them as equivalent mathematical symbols provided Tori the opportunity to notice, in the moment, that she could use contrast between examples. Do to the length of this unexpected discussion, the LGE that the teachers had planned was not given to any of the students.

Tori enacted unintended contrast twice in response to student comments. The first time, Tori took up a student's question regarding the difference between parentheses and brackets. Tori enacted contrast in the moment by preserving the numbers and location of the numbers in an expression that a student had generated while varying only the grouping symbol used. The second time, Tori asked a student to explain what she noticed regarding the example 10(-2 – 3). The student explained that in the original expression, 10 – 2 – 3, the operation between the ten and the two had been subtraction, but that with the placement of parentheses, the operation between the ten and the two became multiplication. During the planning meetings, the teachers had decided to not include the example 10 – 2 – 3 in the set. This particular student drew on that example anyway, and Tori highlighted this students' thinking for the whole class. These two instances suggested that Tori was beginning to notice opportunities in the moment to enact contrast through changing one thing between examples. In the second set of structured exercises, those involving absolute value symbols, Tori compared an expression with the absolute value symbols to the original expression that the set of examples was based on that did not include absolute value symbols. This comparison provided her an opportunity to make the relationship between the location of the grouping symbols in the expression (CA1) and the resulting change in the operations and the value of the expression (CA3) explicit. This comparison was left largely implicit to students in the first set of structured exercises.

Evaluation of the Lesson. The last step in a learning study cycle is to evaluate the learning outcomes and the overall impact of the learning study cycle. Immediately after the research lesson was taught, the team of teachers and I met to discuss our observations of the lesson. We met again the next day, after Tori had given her students an informal assessment about the order of operations. The initial discussion immediately following the implemented research lesson followed a modified protocol borrowed from Japanese Lesson Study (Lewis & Hurd, 2011). First, the instructor shared her reflections on the lesson. Second, the team members presented and discussed the data from the research lesson (what they observed). Next, there was a general discussion focused on student learning and how specific elements of the design of the lesson promoted student learning or not.

Tori's first comments were about the discussion about parentheses and brackets. It was important for Tori to talk about what was unplanned. She commented on the time the discussion took and her subsequent choice to not give any students the LGE that had been planned. Tori said that she viewed the discussion as just as important as whether students could do the operations since "you first need to understand what you're being asked to do" (RL1 Debrief, Lines 8-9). Near the end of the debriefing session, I asked the teachers what new questions they had. The brackets discussion and whether it was enough for students to generalize the use of various forms of brackets remained a significant question for Tori: "Will students feel comfortable interchanging the parentheses, and the absolute values, and the brackets, and not get hung up on what's the difference? Recognize that they are the same; you just use them in different locations" (RL1 Debrief, Lines 206-209). Tori had recognized that an opportunity to discern the difference between types of brackets and generalize that most of them are interchangeable (besides absolute value symbols) was enacted. Tori does not know if this is what the students learned, or in other words, what the lived object of learning was. This was a significant learning event for Tori, in that she later anticipated this kind of discussion in her other classes. When we met the next day, however, Tori noted that the brackets discussion did not come up in any of her other classes.

The difference in the discussion between the set of examples using parentheses and the set of examples using absolute value symbols appeared to be a significant learning event for the team of teachers. Tori first discussed what had happened in the lesson:

> I'm not sure that we got out what we really wanted to with the moving of the parentheses. Yes, they all gave us different answers, but I don't know. I liked the conversation with the absolute value…That one had multiplication, but by moving it one spot over that brought in subtraction, so I definitely liked that. But I think I got that more when they [students] put them [absolute value symbols] in on their own. Where when I put them [parentheses] in, I'm not sure that was a conversation. (RL1 Debrief, Lines 35-40)

Tori noticed that the enacted contrast between examples, regarding the location of the parentheses or absolute value symbols, and the associated effects on the indicated operations in the expression, was more successful for the set of examples using absolute value symbols than for the set of examples using parentheses. Tori attributed the difference in success to students, rather than she, inserting grouping symbols (either parentheses or absolute value symbols) into the expressions.

Lynn, on the other hand, attributed the difference in success between the set of examples using parentheses and the set of examples using absolute value symbols to an explicit comparison between examples. “Maybe we need a more deliberate observation of what happens” (RL1 Debrief, Lines 43-44). Tori took up Lynn’s observation, and suggested a change to her remaining classes: “Maybe make the emphasis, that especially with the first two [examples], by moving that parentheses one spot over, how do the operations change, not just the answer?” (RL1 Debrief, Lines 46-47). Tori suggested comparing the first two examples: 10(-2 – 3) and 10 – (2 – 3) through questioning. Shannon suggested making this comparison explicit through a verbalization technique, and introduced the idea of comparing the examples to the original expression that the set of examples was based on, 10 – 2 – 3:

> You could have them read the expression to you. So it [the original expression] goes ten minus two minus three to [in the first example] ten *times* negative two minus three. And the next one [example] is ten *minus* two, in parentheses, two minus three. And then the last one is ten minus two *times* negative three. Those are three very different things when you read them out loud and say them out loud. (RL1 Debrief, Lines 47-50)

Shannon recognized the importance of the contrast between the expression 10 – 2 – 3, without parentheses, and the expressions with parentheses. This contrast would give students the opportunity to discern that the insertion of parentheses changed the operations within the original expression. Tori agreed with Shannon’s suggestion to begin with the original expression: “I guess that’s something I can start off with. All of these expressions are the same. What is 10 – 2 – 3? What would you get?...What if I put in parentheses? How does that change?” (RL1 Debrief, Lines 72-75). Tori continued to imply a questioning strategy, rather than Shannon’s suggested verbalization strategy, to draw students' attention toward the differences between the expressions and the resulting effects on the value of the expression and the operations.

During the teachers' meeting the next day, Tori talked about how she had students in her remaining classes first determine the value of the expression 10 – 2 – 3, before adding in any parentheses. She also took up Shannon's verbalization strategy by having a student read aloud the new expressions, with parentheses. "So I think that was helpful because looking back on it, they [students] didn't even realize that they were going to give you different answers because we never talked about what it [the expression, without parentheses] was originally" (RL1 Debrief, Lines 239-241). The teachers realized that the original expression was needed to give students the opportunity to discern that there was a difference in value between the original expression and the expressions that included parentheses. In the research lesson, the value of the original expression without parentheses was implicit for the students. The decision to revise the lesson and include the evaluation of the original expression for the remainder of the classes indicated a change in teachers' knowledge about the design and enactment of structured exercises: contrast between examples must be made explicit to support student discernment of the intended critical aspects and the relationships between the critical aspects.

Robert began thinking about how the explicit contrast between examples could support his eighth grade students and how he could incorporate the principle of Explicit Contrast into his own lesson design. Robert said, "I'm going to be more specific with that [explicit contrast]. Like what's the difference between squaring this [-3^2] and putting parentheses here and squaring it [$(-3)^2$]" (RL1 Debrief, Lines 104-106). The teachers' revision of the lesson to include explicit contrast, followed by the enactment of explicit contrast in Tori's remaining classes, and Robert's description of how he would use explicit contrast in another context suggested that the notion of explicit contrast between examples was beginning to emerge as a principle of task design and enactment for the teachers.

The principle of Explicit Contrast within task design manifested itself through the idea of a single change within one of the critical aspects between examples. Tori explained how small changes between examples helped students to discern the effects that change had on the expression. Tori began to extend the principle to other contexts, as Robert did:

> I liked the idea of having three of the exact same thing and altering it a tiny little bit for them to realize that that tiny little bit *changes everything*…I think that is something in math [we could] do easily all the time. [Referring to a suggestion by Robert for the topic of the next research lesson to be solving an equation for a variable] Like how easy would it be to just change where the *y* is? Where the *x* is? And then solve for the variable. That's so easy to do by keeping all the numbers the same and changing where the *x* and *y* are. (RL1 Debrief, Lines 285-293)

Within task design, a change within a single critical aspect between examples, while holding other aspects, such as numbers, operations, and variables invariant, allowed for that particular critical aspect to be discerned. By comparing examples with only a single difference, that difference becomes explicit to the learner. For the teachers, the principle of Explicit Contrast within task design included the notion of one little change between examples.

Because of the unplanned class discussion about brackets in the research lesson, Tori chose to not give the LGE to any of the students, due to time constraints. In a later class, however, Tori chose to give the LGE to some of her students. The prompt of the LGE asked students to fill in the blanks to make a true equation: $\_\_\_ + \_\_\_ \times \_\_\_ = 12$. During the teachers' meeting the following day, Tori described what happened:

> None of them used negative [numbers]. They were all positive [numbers]. And then it was just some type of multiplication of two numbers and then add whatever the difference was. But they got it all to work. Every single one of them. But at the same time, I only gave it to the kids that had already finished, so if they had already finished, and those were correct, they probably had a little better understanding anyway. (RL1 Debrief, Lines 253-257)

Tori seemed surprised that all of the students were able to generate a correct expression. Tori also acknowledged that only certain students, those who finished early, had access to the LGE. She appeared to imply that those students were able to generate a correct expression because they had already demonstrated an understanding of the content through finishing the order of operations exercises in class. Tori justified students' success with LGEs through their identification as higher-level learners.

Robert continued to push for the use of LGEs, but began to suggest, along the lines of Tori's reasoning, that LGEs were best suited for higher-level students:

> I would have liked to see, not in this lesson in particular, but in order of operations, more of those learner generated examples. I can see upper level students putting in [parentheses]; where do you want to put in these parentheses to make this number statement true? Give them a bunch of numbers, even added together, and vary that a little bit, make it subtraction, make it addition, throw in some multiplication. Where can you put the parentheses? I know I've done that with an accelerated pre-algebra group…You had to think four steps ahead and that's challenging. It's more for an accelerated student, but could be a way to differentiate also. (RL1 Debrief, Lines 303-314)

Within the debriefing, Robert seemed to retreat from his earlier assertion in the planning sessions that LGEs would be useful to give to all students, saying instead that they would be more appropriate for accelerated students or as a means of differentiating instruction for accelerated students. His description of a task in which students insert parentheses in order to make an equation true is also a departure from his prior conceptualization of LGEs, in this case, as examples of expressions equivalent to a given value. The teachers' comments in the debriefing suggested that their conceptualization of what LGEs are and the role they play in students' learning was still evolving.

The teachers had begun to articulate Explicit Contrast as a principle of both task design and enactment. Within task design, this principle manifested itself through a change in a single critical aspect between examples. Tori began to demonstrate this principal during the act of teaching by noticing an opportunity to preserve a student's given example with only a single change. During the debriefing, the teachers recognized the importance of including the original expression, without grouping symbols, as a source of comparison to make the contrast between the value of the expression with and without grouping symbols explicit. The teachers were still uncertain about the role of LGEs and continued to assert that while they may be appropriate for some students who were regarded as accelerated, they would be too challenging for many students.

Summary of Research Lesson One. The purpose of Research Lesson One was to develop students' abilities to write equivalent expressions using the order of operations. The teachers identified four critical aspects: (CA1) the location of grouping symbols, (CA2) the value of the expression, (CA3) the role of grouping symbols, and (CA4) the maintenance of equivalency. The teachers designed two sets of structured exercises. The first set had 10 – 2 – 3 as the basis for three examples in which the location of a set of parentheses was varied (CA1).

The second set had $5 - 3^2 + 1$ as the basis for three examples in which the location of a set of absolute value symbols was varied (CA1). During the design of these sets of structured exercises, the teachers questioned how many examples they needed. Too many examples might have induced tedium and used class time, while too few examples might not have provided the opportunity for students to discern the contrast between examples, regarding the location (CA1) and role of grouping symbols (CA3), as the teachers had intended. The teachers also designed an LGE in which the prompt asked students to fill in the blanks of ___ + ____ × ____ = 12 to generate a correct mathematical statement. The openness and restriction of the LGE was an issue for the teachers, which was linked to their notions of student success.

In the implementation of the lesson, Tori was explicit about the comparison between the examples in which absolute value symbols appeared and the original expression that the set of examples was based on, $5 - 3^2 + 1$. This provided an opportunity for students to discern that the location of absolute value symbols in an expression (CA1) can result in changes to the operations performed and the order in which operations are performed, affecting the resulting value of the expression (CA3). In the first set of structured exercises, a student suggested the contrast between the original expression 10 – 2 – 3 and 10(-2 – 3). This contrast was not intended, but taken up by Tori and enacted during the implementation of the lesson. Early in the lesson, Tori enacted contrast in the moment to provide students the opportunity to discern the difference between rounded and square brackets, and then generalized these grouping symbols as interchangeable. These instances suggested that Tori has begun to notice opportunities during the course of teaching in which she could use contrast through a change in only a single aspect between examples.

During the evaluation meetings, the teachers began to articulate Explicit Contrast as a principle of both task design and enactment. Within task design, this principle manifested itself through a change in a single critical aspect between examples. The teachers suggested verbalization and questioning as strategies to make the contrast between examples visible. The planned LGE was not given to any students during this research lesson, although Tori gave it to the students who finished early in one of her classes. She acknowledged that all of the students were successful at generating an example equivalent to twelve, given the constraints, but commented that because she only gave it to students who finished early, they likely had a good understanding of order of operations anyway. Robert continued to advocate for the use of LGEs, but acquiesced that they might be at a more appropriate level of challenge for advanced students. The teachers' notions of student success influenced how they viewed and used LGEs at this point in the study.

Research Lesson Two

Robert taught Research Lesson on November 5, 2015 in his eighth period Math 8 class. Prior to the research lesson, the teachers had met for eight 40-minute planning meetings and one longer (approximately two hours) meeting during a staff day. Robert had volunteered to teach the second research lesson at the debriefing of our first research lesson. The eighth grade teachers had taught point-slope form for an equation of a line for the first time the prior year. They had some questions about whether teaching point-slope form was effective in helping students become proficient in writing the equations of lines. Hence, writing equations of lines became the focus of the second research lesson.

In their rationale for the lesson, the teachers wrote: "We want to help with [students'] understanding of how to write equations of lines by giving them the experience of writing it in both ways (i.e. slope-intercept and point-slope form)," (RL2, Lesson Plan). This learning study cycle followed a similar progression as that described for the first research lesson. The teachers' work on this research lesson took nine planning meetings, which was quite a bit more time than the first research lesson, which took six planning meetings. The topic of writing equations of lines encompasses a broad range of inter-related concepts and procedures, and the teachers had extensive discussions about what the focus of this research lesson should be. The teachers decided that they wanted students to be able to flexibly use both slope-intercept form ($y = mx + b$) and point-slope form ($y - y_1 = m(x - x_1)$). Over the course of the planning meetings, the teachers discussed a number of concepts related to linear equations, including: (1) the equivalency of slope-intercept and point-slope form; (2) transforming equations written in point-slope form into slope-intercept form; (3) proportional versus linear relationships; (4) the given information (e.g., slope, point(s), *y*-intercept, rate of change) and its relationship to which form of an equation of a line might be more appropriate; (5) what was asked for in a question (e.g., write an equation, give the *y*-intercept, etc.) and its relationship to which form of an equation of a line might be more appropriate; and (6) the use of subscripts in mathematical notation in middle school, which was new for their middle grades students. In contrast to Research Lesson One on the order of operations, where there was little diversity in how the teachers applied and used the order of operations, there was considerable variation in how the teachers themselves wrote equations of lines.

The process of collectively planning a lesson can, at times, be messy, with the purpose informing the design, and conversely, decisions about the design changing or refining the purpose. In this research lesson, the teachers moved on to the planning phase before fully articulating a shared purpose for their work. They chose to design examples and then more fully explicate the purpose, making the process an iterative one. Thus, the steps in the learning study cycle did not always proceed linearly, particularly for the sections on purpose and design.

Purpose of the Lesson. As described earlier, in the beginning phase of a learning study, the research team (1) selects a topic for study, (2) identifies a tentative object of learning, (3) diagnoses students' learning difficulties, and (4) confirms the object of learning and its critical aspects. In aggregate, these first four steps of the learning study cycle articulate and clarify the purpose of the lesson. For this lesson, after discussing a number of concepts and procedures related to linear equations, the teachers identified the direct object of learning as writing equations of lines. They wanted students to understand that either slope-intercept or point-slope form may be used to write an equation of a line, but that one form may be more appropriate based on the given information or what the question asks for (RL2 Lesson Plan). The teachers settled on this topic and the direct object of learning at Planning Meeting Two. The indirect object of learning was writing a correct linear equation and having mathematical reasons for choosing either slope-intercept or point-slope form. The teachers identified three critical aspects of the object of learning: (CA1) the form of a linear equation, (CA2) the combinations of given information, and (CA3) what the question asks for. The teachers decided on the indirect object of learning and critical aspects much later (around Planning Meeting Seven) in the planning/design process, after sorting out their own diverse views on the topic.

When noticing a critical aspect of an object of learning the learner must become aware of some of the critical features. When noticing the first critical aspect (the form of a linear equation) the learner must become aware of at least two different forms of a linear equation, namely, slope-intercept form and point-slope form. If the learner believes there to be only one form, then the learner has not become aware of the critical aspect form of a linear equation. The various forms of a linear equation are called the critical features of the critical aspect form of a linear equation. In this case, the critical features included two forms: slope-intercept form and point-slope form. The teachers did not include standard form (i.e. $Ax + By = C$) as a critical feature. The teachers wanted students to discern that there were two correct forms for the equation of a given line. For the second critical aspect (the combination of given information) critical features included two points, a point and the slope, the slope and the *y*-intercept, or a point and the *y*-intercept. For the third critical aspect (what the question asks for) critical features included an equation of a line and the y-intercept. The teachers chose not to include questions that asked the students for the slope, so this was not a critical feature that needed to be discerned.

The overarching purpose of this lesson was to develop students' abilities to correctly write an equation of a line. The teachers chose to focus on how the given information could indicate that a particular form was more appropriate to use. For instance, if given a point and a slope, point-slope form would be more appropriate to correctly, and efficiently, obtain an equation of the line. Similarly, the teachers focused on how the quantity asked for could indicate that a particular form was more appropriate to use. When prompted to give a *y*-intercept, students should use slope-intercept form since the *y*-intercept can be obtained directly from the equation, without further manipulation. Students' discernment of which form of an equation of a line might be more appropriate in given situations supported the teachers' overarching purpose of correctly writing an equation of a line, although their meaning of appropriate evolved over the course of their planning.

The teachers identified writing equations of lines as the focus of the lesson because of their past experience with teaching point-slope form for the first time in the prior year. Robert explained the prior year's lesson:

> The teachers started the lesson with an initial example where students wrote the equation of a line given graphically using slope-intercept form. In the graphical representation, students could easily identify the *y*-intercept. Then the teachers asked the students the question, "What if we couldn't tell from the graph what the slope or *y*-intercept was?"…Robert said, "We tried to create the need for [point-slope form]. Instead of just giving them point-slope form, this is why we're using it" (RL1, Planning Meeting 1, Lines 73-77).

Robert explained that the teachers tried to motivate the need for point-slope form through contrasting an example in which the *y*-intercept could easily be determined from the graph with other examples, in which the value of the *y*-intercept could not easily be determined. From Robert's perspective, the inability to read the *y*-intercept from the graph created the need for point-slope form of a line. For each example, students were asked to write an equation of the line in both slope-intercept and point-slope form. In thinking about the prior year's lesson, however, Robert was concerned that the need for point-slope form was not conveyed to the students through the lesson:

> It turned out to be, in my opinion, students were like, well I like this way, and I like that way. And for me, that was okay, that's fine. Some kids really stuck to point-slope, other kids liked $y = mx + b$. For me, it was like, if you can do it one way…I'll be happy. But I'd rather do it better. (RL1, Planning Meeting 1, Lines 77-80)

Robert described how the lived object of learning for the students the previous year was that there were two ways to write an equation of a line. Most students chose one way and only used that form to find an equation of a line. Robert described how the objective of writing an equation of a line had been met, but that the usefulness of point-slope form, particularly in situations where the y-intercept could not easily be determined, had not been discerned by students. Which form of a line to use was based on students' preference, which often equated to their familiarity with the equation $y = mx + b$, rather than mathematical reasons for choosing one form or the other in various situations.

There was considerable variation in how the teachers themselves wrote equations of lines and thought about slope-intercept form and point-slope form. The diversity in their thinking appeared early on in the planning meetings, as the teachers discussed their thoughts on teaching point-slope form during the prior year:

> Robert said last year was the first year that they ever taught point-slope form. He said they liked that lesson and it went over well…Shannon was shaking her head no. Robert said, "You didn't like [teaching point-slope]?" Shannon said, "Mmm-mm (shaking her head left and right). My students hated it." (RL2, Planning Meeting 1, Lines 45-48)

Teaching point-slope form was not required at the eighth grade level. Robert, who also taught first year algebra to accelerated eighth grade students, was concerned about the connections, or lack thereof, that students were making between their prior knowledge and the knowledge built in future mathematics courses: "I do feel like sometimes they go to the high school, they learn a new, completely different way, and don't connect it to the way they did it before [slope-intercept form]," (RL2, Planning Meeting 5, Lines 343-345). Shannon, on the other hand, did not like teaching point-slope form, and indicated that the lesson did not go well in her classes – "My students hated it," (RL2, Planning Meeting 1, Line 48). With push-back from the students, and knowing that students had another method to write an equation of a line (slope-intercept form), the teachers briefly questioned whether point-slope form should be taught at all.

At Planning Meeting Three, I presented the teachers with a set of structured exercises that I had designed. Each of the exercises resulted in the same line. My intentions were (1) to try to get the teachers to be clearer about and articulate what it was they paid attention to in deciding which form of a line was more appropriate to use, and (2) to illustrate what a set of structured exercises about this topic could look like. This is the set of structured exercises I presented to the teachers:

1. Write an equation of a line with a slope of -3 and a *y*-intercept of 11.
2. Write an equation of a line that passes through (2, 5) with a *y*-intercept of 11.
3. Write an equation of a line with a slope of -3 that passes through (2, 5).
4. Write an equation of a line that passes through (2, 5) and (0, 11).

For each exercise, I asked the teachers, as a group, which form of an equation of a line they would use and why.

> For the first [exercise], [the teachers] said slope-intercept, because they were given slope and *y*-intercept. For #2, Robert said they could use either. Tori said she would use slope-intercept…For #3, they said point-slope. For #4, Robert said, "Now that one specifically, you'd hope slope-intercept, because you hope they identify the intercept. But if [the point] wasn't the intercept of (0, 11), then [point-slope] would be the easier one. (RL2, Planning Meeting 3, Lines 115-121)

The teachers all agreed that when given a slope and an intercept, slope-intercept form should be used; when given a point and a slope, point-slope form should be used. In all other cases, however, there was not a consensus about which form would be more appropriate. Robert defined more appropriate as easier, commenting that since the given point was a *y*-intercept, he would hope his students would use slope-intercept, but if the given point was not the *y*-intercept, then point-slope form would be easier to use.

At this point, the teachers decided that the *y*-intercept was a critical aspect, and that slope-intercept form was more appropriate to use anytime the *y*-intercept was known.

> Tori said, "So it seems like if you're given the *y*-intercept, immediately use $y = mx + b$." Lynn added, "Because you've already got one piece of your information."…Tori: "If given the *y*-intercept, use slope-intercept form." I said, "So a critical aspect then-?" Tori said, "Is whether the *y*-intercept is known." (RL2, Planning Meeting 3, Lines 131-135)

The teachers' asserted that slope-intercept form is more appropriate to use when the *y*-intercept is known because they viewed it as the easier, or more efficient method, since "you've already got one piece of your information" (RL2, Planning Meeting 3, Line 132). Because of this view, the teachers identified the *y*-intercept as a critical aspect and the ways in which it could be given in an exercise as critical features. They similarly identified the slope as a critical aspect. What the teachers viewed as the critical aspects, however, continued to evolve over the course of the next three planning meetings.

The teachers recognized in Planning Meeting Four that the combination of information given in an exercise was critical, rather than a singular piece of information.

> Tori said, "Right now we have [the critical aspects of *y*-intercept and slope] separate. It should be more like, given the *y*-intercept, and given two points, or as a point, and given [the value of the slope]." Robert said, "Yeah, it's the combination that's important." Tori replied, "They should really be grouped together, not separate." (RL2, Planning Meeting 4, Lines 132-135)

Tori and Robert came to a new understanding of the critical aspect as the combination of given information, not whether or not the *y*-intercept is given, and whether or not the slope is given. This shift in thinking about the critical aspects was related to the teachers' growing awareness of the variability in thinking and approaches for writing an equation of a line that had occurred within their small group.

In Planning Meeting Five, in order to make the teachers' thinking about writing equations of lines visible within the group, I wrote the points (0, -8) and (2, 3) on the board and asked each of the teachers to write out their work for how they would write an equation of a line through these two points. When they completed this example, I took their work and mine and summarized the different methods used:

> I said that I started like Tori did by calculating slope, but then I substituted into point-slope form, whereas Tori substituted into slope-intercept form. Shannon did not calculate the slope. She substituted the given information into slope-intercept [form], and solved for *m* in that equation. (RL1, Planning Meeting 5, Lines 258-261)

This exercise revealed the diversity of thinking within the group. Given the same information, "experts" chose three different methods to arrive at an equation of a line. Table 4.7 shows the procedures for each of the three methods described above. Earlier in the planning process, Robert had begun to define "more appropriate" as easier. This example revealed that one solution method could not always be considered easier than another. Both my solution and Tori's solution involved calculating the slope, and then substituting into one of the forms of a linear equation. The teachers suggested that "more appropriate" was related to efficiency. Each of the teachers took about the same amount of time to work out their solution. Robert commented: "Really, I don't think there is a real preference…[Students] could maybe do [one way] quicker, but not necessarily" (RL2, Planning Meeting 5, Lines 320-323). Hence, this exercise revealed that, at least in some cases, there is not always a clear or more appropriate method that should be used.

Table 4.7

Different Solutions for Writing an Equation of a Line with Given Points (0, -8) and (2,3)

My Solution	Tori's Solution	Shannon's Solution
$m = \frac{3-(-8)}{2-0} = \frac{11}{2}$ $y-(-8) = \frac{11}{2}(x-0)$	$m = \frac{3-(-8)}{2-0} = \frac{11}{2}$ $y = \frac{11}{2}x - 8$	$3 = m(2) - 8$ $11 = 2m$ $\frac{11}{2} = m$ $y = \frac{11}{2}x - 8$

The teachers continued to discuss the three methods that arose from the above exercise. Robert, again, tried to articulate the goal of the group:

> We understand the three ways that we did it…Is that our goal, for the students to understand the three different ways to do it?...That's what I sort of thought we were getting at. It's not just being able to do one way and stick with that one way, but to be fluent, in this case, in the three different ways that you could do that. (RL2, Planning Meeting 5, Lines 274-278)

Robert suggested that there is not necessarily a certain method that should be used to write an equation of a line, but rather that students should understand multiple solution methods. The teachers agreed and spoke briefly about ways that they could convey multiple solution methods, including (1) a direct instructional approach, where students are guided through the multiple solution methods, (2) a group activity in which each member of the group must write an equation of a line using a different method, and (3) a comparison of solution methods naturally generated in a group. The teachers' consideration of how to incorporate multiple solution methods into the lesson indicated that the teachers saw value in students having multiple solution methods. Tori commented on the variation in solution methods among the teachers:

> I wonder if our preference is because we don't [teach] at the high school level. We [teach] at the middle school level. So middle school level, I see no need for point-slope. High school, college level, you're like – why wouldn't I use point-slope? So I'm thinking it might be like a year [in school] kind of thing. (RL2, Planning Meeting 5, Lines 337-340)

Thus, while there was not necessarily agreement that point-slope form should be taught at the middle school level ("I see no need for point-slope"), nor necessarily agreement on which solution method or which form of an equation of a line should be used in various situations, there was a general agreement among the four teachers that there was value in having multiple solution methods.

The diversity of thinking within our group of six (myself, the team of four math teachers, and one special education teacher, Beth, interested in our work) made it difficult for the teachers to arrive at a shared goal and articulated purpose for the lesson. At Planning Meeting Five, Robert was still asking of the group, "Is that our goal?" (RL2, Planning Meeting 5, Lines 274-275). Before fully articulating a clear purpose for the lesson, the teachers had decided to proceed with designing the examples of the lesson. As they designed each example, they considered the opportunities for student learning that that example provided. In this sense, the process of this learning study was iterative in nature, with the design informing the purpose, and the increasingly articulated purpose informing the design. Through this process, the teachers ultimately arrived at the shared purpose that students should be able to write an equation of a line, understanding that either slope-intercept or point-slope form may be more appropriate depending on the given information or what the question asked for. In the design section, I discuss how the idea emerged that students should have mathematical reasons for their choice of which form to use, rather than choosing a form out of familiarity.

Design of Structured Exercises and LGEs. The teachers began to design examples during Planning Meeting Six. Robert said that he typically started class with a "warm-up," or an example meant to draw out and connect to prior knowledge. The teachers discussed what the warm-up example would be:

> Tori suggested asking students to just write the two forms of an equation of a line. Robert suggested giving two points. Tori clarified that she meant to have students write the formula for [both forms of an equation of a line], then give [students] two points to write an equation. Tori said, "They can choose which form to use, but at least they have them both written down." Lynn said, "That could kind of springboard the whole idea at that point. Okay, let's see how many people [used slope-intercept form], and how many people [used point-slope form], and lead into the lesson." (RL2, Planning Meeting 6, Lines 7-12)

This lesson was being taught after a lesson on slope-intercept form and another lesson on point-slope form. From the teachers' perspective, the warm-up exercise was meant to encourage students to recall the two forms for an equation of a line, and then choose one to write the equation of a line when given two points. The teachers decided that students should just be asked to write an equation of a line. The teachers would draw on the natural variation in solution methods to contrast the use of slope-intercept and point-slope form and use that variation to drive the lesson. This example connected to the form of a linear equation (CA1). Students should have discerned that there are two forms of a linear equation prior to this lesson. This example provided the opportunity for students to see that either form could be used for the same given information (CA2).

Tori suggested thinking about choosing the two points on the warm-up so that choosing one form of an equation of a line was easier than the other. Robert said, “So is our goal, when we’re done with the warm-up, to discuss what method did you use? Did anyone do something different? And in that case, should we make the *y*-intercept not an integer?” (RL2, Planning Meeting 6, Lines 29-31). Drawing on the principle of Explicit Contrast, the teachers agreed to make the *y*-intercept a non-integral value in order to contrast two different solution methods and make it possible to discern that one solution method may be easier than the other. The teachers decided on giving the points (9, 3) and (-11, -2), which resulted in fractional values for both the slope and *y*-intercept. Robert further suggested asking students, “Why did you choose this method? So they have thoughts on paper to share” (RL2, Planning Meeting 6, Lines 39-40). The teachers decided that they were interested in helping students to make mathematically sound choices for choosing which form to use form. Lynn said that collecting students’ reasons for their choice, written on paper, could show growth in their reasoning from the beginning to the end of the lesson (RL2, Planning Meeting 5, Lines 60-61). The design of this example initiated the teachers’ shift in thinking from choosing a particular form based on given information, to choosing either form with supporting mathematical reasons.

For the next example (that the teachers called Example 1), Robert suggested contrasting two graphical representations of lines that had the same slope, but different *y*-intercepts. He suggested that one of the *y*-intercepts be an integer value, while the other be a fractional value. Robert described this as creating a need for point-slope form since the *y*-intercept could not easily be determined from the graph, which linked back to the lesson he used both the previous year and this year (prior to the research lesson) on point-slope form. Tori questioned why they should do this when the students had already experienced this need during the lesson on point-slope form. Robert responded:

> [The students] should have familiarity, but they'll be like wait – this one [with the integer *y*-intercept] was easy, and this one [with the fractional *y*-intercept], what do we do now? But maybe we'll keep it so the slope is easy to find…So just pick a point. I can put it right into point-slope form. I'm just thinking of starting to guide them to which method we think we want them to use in that type of problem. (RL2, Planning Meeting 6, Lines 70-74)

Robert seemed to recognize that one lesson on point-slope form likely was not enough for students to understand the usefulness of point-slope form for writing an equation of a line. His suggestion to contrast two lines in which one has a *y*-intercept with an integer value and the other has a *y*-intercept with a fractional value highlighted the usefulness of using point-slope form. Robert further suggested that the slope of the lines be easy to determine to support the ease of using point-form in the situation where the *y*-intercept could not easily be determined, but the slope and a point on the line could easily be determined. Lastly, Robert pointed out that in this exercise, they would be asking students to write an equation of a line, whereas asking them to find the *y*-intercept might cause the students to choose a different method. He suggested this be a different exercise.

The teachers chose the lines $y = \frac{2}{5}x + 3\frac{1}{5}$ and $y = \frac{2}{5}x - 2$, labeled them Line A and B, respectively, and positioned them on the same set of axes (see Figure 4.5). The teachers then discussed whether they wanted students to find equations for Lines A and B or to find the *y*-intercepts of Lines A and B. While the teachers had not yet articulated what the question asks for as a critical aspect, it was through this discussion that the teachers articulated this critical aspect and identified two of its critical features:

Robert: See, if the problem is just the equation. We were looking for the *y*-intercept. If it was just the equation, then it might be simpler to just-

Tori: Ok, so maybe the question is what is the *y*-intercept of line B?

Lynn: Instead of find the equation of the line?

Tori and Shannon: Yup.

Robert: For what purpose though?

Tori: Is it because the slope is easy to find?

Lynn: The main purpose would be that we couldn't find the *y*-intercept on the second one.

Tori: Right, just by looking at it. You would know that it would be between 3 and 4.

Robert: So what's the point? Is the point to write the equation of a line? Or is the point to find the *y*-intercept? So when you say to find the *y*-intercept, it is easier to use slope-intercept form, I think. And we're talking about easier. It's an opinion. But if I asked you to just write an equation, then point-slope definitely would be easier. So where are we trying to guide them to? Are we just trying to guide them to build a better understanding? (RL2, Planning Meeting 6, Lines 94-105)

When Robert asked, "For what purpose though?" he was trying to connect the design of the example back to the purpose of the lesson. Tori and Lynn interpreted Robert's question as, "Why would we ask for the *y*-intercept?" Their responses were based on the fact that within the design of the example, the slope of each line was fairly easy to determine, but it was the *y*-intercept that could not be easily read from the graph, since it was a fractional value. Robert was trying to be

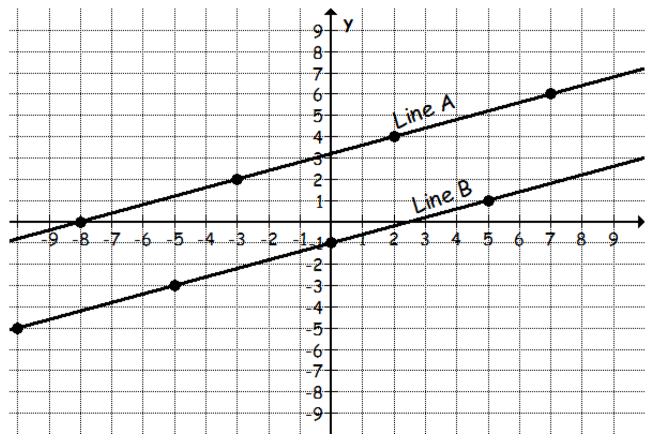


Figure 4.5. Teacher-designed graph presented with example 1 in Research Lesson 2 to contrast integral and fractional valued *y*-intercepts.

clear about and connect this exercise to the purpose of the lesson. If the purpose of the lesson was to choose which form of an equation of a line was more appropriate, or easier to use, then what was asked for in the question could affect which form of a line was easier to use. Robert pointed out that asking for the *y*-intercept might encourage students to use slope-intercept form, since they are looking for the *b* value. By asking the students to write an equation of a line, however, it would likely be easier to use point-slope form, since the slope and a point are relatively easy to find on the graph. Tori recognized Robert's argument, and exclaimed, "Oh! Which [form of a line] I prefer to use may depend on what we're looking for!" (RL2, Planning Meeting 6, Lines 105-106). The design of the exercises influenced the purpose of the lesson. As the teachers developed an awareness of and articulated that what the question asks for is a critical aspect of choosing which form of an equation of a line might be easier to use, the teachers clarified the critical aspects of the object of learning. Furthermore, the teachers identified two critical features of the critical aspect: what the question is asking (CA3): (1) the equation of a line, and (2) the *y*-intercept. Critical aspects for a particular object of learning and a particular set of learners can be hard to figure out (Marton, 2015). The iterative process of task design and consideration of the purpose of the task can help to clarify the critical aspects and critical features that are difficult to ascertain at the outset of a learning study.

Based on the above discussion, the teachers decided that it was important to ask both questions – What is an equation of the line? What is the *y*-intercept? – because what the question asks for was identified as a critical aspect. They decided that these two questions could not be asked within the same exercise, however. If the teachers were to ask for the *y*-intercept first, then once the *y*-intercept was determined, it could simply be substituted into slope-intercept form, and the students would not have the opportunity to discern that point-slope would be easier to use when the slope and a point could be seen on the graph. Similarly, if the teachers were to ask for an equation of a line first, students could simplify and transform an equation they had already obtained in point-slope form, diminishing the opportunity to see the relationship between what the question asks for (CA3) and which form of an equation of a line is easier to use (CA1). The teachers decided that in addition to the warm-up example, they would have one example where the *y*-intercept would vary and they would ask students to write an equation for each line (Example 1), and another example where the *y*-intercept would vary, but they would ask students to find the *y*-intercept (Example 2). Taken together, these two questions addressed the critical features that the teachers intended that students discern regarding the critical aspect, what the question asks for.

At the beginning of Planning Meeting Seven, I asked the teachers what it was they wanted students to discern through the exercises they had already designed. Robert: “One thing we’re addressing [is] that there are different forms of the same equation of a line” (RL2, Planning Meeting 7, Lines 116-117). This was the first critical aspect that the teachers identified, the form of a linear equation (CA1). This led to a discussion, again, about the purpose of the lesson. Shannon said, "So our bottom line is that [students] feel comfortable enough with both of the [forms of an equation of a line so] that they can figure out which equation is best to use, which form when." (RL2, Planning Meeting 7, Lines 148-150). Shannon referred to the familiarity and flexibility the teachers wanted students to have with both forms of an equation of a line. She said that this was so students "can figure out which equation is best to use." While earlier the teachers had talked about students choosing the more appropriate form to use in a given situation, as if there was a correct and incorrect choice, the teachers had began to shift to a consensus that while either form can be used at any time, depending on the given information, there might be a better choice, although it is not the only correct choice.

> I asked the teachers what they thought was meant by the words "better choice." Lynn said more accurate, fluency. Shannon said she thought their comfort level…She described that even if students are given a point and a slope, and we would like to see them use point-slope form, they might mess up signs in point-slope form…Robert said that depending on what is given, one way could be so much more work. Shannon suggested less steps…Lynn said that less steps allowed for less error. (RL2, Planning Meeting 7, Lines 161-165)

This discussion revealed that from the teachers' perspective, rather than the word "appropriate" referring to a singular choice, choosing an appropriate form was based on a number of factors, some of which were mathematical, and others of which were individual. Appropriate, in this sense, referred to the form in which students were more likely to obtain a correct equation of a line. This could be based on individual student tendencies ("they might mess up signs in point-slope form") and general student tendencies ("less steps allowed for less error"). Robert referred to appropriate as the form that was more efficient, in terms of written work. He earlier described this efficiency as "easier." Thus, "appropriate", for the teachers, encompassed aspects of efficiency and correctness, both mathematically and in terms of individual tendencies.

The teachers discussed Example 2 where they would ask students to find the *y*-intercepts of two distinct lines. They decided to hold the two lines from Example 1 invariant, but to take away the grid that appeared in Example 1, and instead indicate two points on each line. Robert suggested that the view of the graph presented to students be zoomed out, with points chosen beyond those that would have been seen on the graph in Example 1. In this way, it would not be immediately obvious that the two lines were the same as in the previous example, so students could not trivially determine the *y*-intercepts from the previous example. This choice provided the opportunity for students to discern that it might be easier to use one form of an equation of a line (CA1), depending on what the question asks for (CA3), which in this case, is the *y*-intercept. By holding the line itself invariant, students had the opportunity to discern that what matters is what the question asks for, not what the line is.

Example 3 was designed to reveal the relationship between the combination of given information (CA2) and the form of the linear equation (CA1). The teachers varied the given information (slope and *y*-intercept, given as a point; slope and a point other than the *y*-intercept; two points) while holding the line itself invariant:

> **Example 3:** Write an equation of the line with the information provided below. You may use either slope-intercept form or point-slope form.
>
> A.) A line with a slope of 3 that goes through the point $\left(0, 6\frac{1}{8}\right)$
>
> B.) A line with a slope of 3 that goes through the point $\left(-2, \frac{1}{8}\right)$
>
> C.) A line that goes through the points $\left(10, 36\frac{1}{8}\right)$ and $\left(13, 45\frac{1}{8}\right)$
>
> (RL2, Lesson Plan)

This structure was borrowed from the set of structured exercises that I had presented to the teachers in Planning Meeting Three. There was some discussion about enacting Example 3 collaboratively, in which particular students within the group must write an equation using a particular method. Shannon suggested having students write an equation for each exercise individually, then having them pair with a partner to discuss their methods, then pair with a second partner to discuss their methods. In response to Shannon's suggestion, Tori said, "Rather than the student to Robert conversation, or the Robert to student conversation, it kind of gets [the students] talking about which method [they used], and which they prefer" (RL2, Planning Meeting 8, Lines 45-47). The teachers had decided to rely on natural variation in the students' methods for writing an equation of a line to bring about the contrast needed to discern that (1) either form could be used to write an equation of a line (CA1), and (2) depending on the given information, one form of an equation of a line might be more appropriate, in terms of efficiency (i.e., the relationship between CA1 and CA2). For the teachers, the collaboration was necessary to create an opportunity for discernment through natural contrast.

The warm-up exercise and the three exercises discussed above constituted the set of structured exercises for this lesson. Shannon asked about what homework would be given with this lesson, and Robert suggested giving the students two points, asking them to write an equation of a line through the two points, and to write their reasons for why they chose that form for the equation of the line. He anticipated that this quick assignment would tell the teachers what the students took away from the lesson, or the lived object of learning. I suggested that they might try an LGE in which we ask students to go the other way: Given one of the forms, what is the given information?

> Robert said he worried about the wording of the question. Tori suggested, “Come up with two points where using point-slope [form] might be more useful.” Robert said, “This definitely seems like the highest level of thinking, to generate this. So this could be the second question”…Lynn said, “I don’t know if students are going to know that they can just put a *y*-intercept on a graph and count up and over any slope.” (RL2, Planning Meeting 8, Lines 142-152)

While the teachers decided to try using an LGE as a part of the homework assignment, they voiced concerns related to their notions of student success. As in Research Lesson One, the teachers viewed LGEs as requiring a higher level of thinking than the types of examples students are usually asked to do. Lynn voiced her concern that students would not be able to think of an approach to generate given information, referring to the particular approach of “just put a *y*-intercept on a graph and count up and over any slope.” It is not clear what Robert’s concern was about the wording of the question. It could have been related to students’ understanding of what the question was asking for. Robert’s concern also could have been more closely related to Lynn’s concern and students’ ability to find an approach to the problem. The teachers decided on the following question:

> Write an example of a problem in which you are asked to find the equation of a line, where it might be easier to use slope-intercept (alternatively, point-slope) form. (Hint: You might choose two points, you might draw a graph, or show a table of values, etc.) (RL2, Lesson Plan)

The teachers asked the question with a part a and a part b, so that one of the questions asked for an example of a problem in which slope-intercept form might be easier to use, and in the other, point-slope form might be easier to use. While the teachers asked the students to “write an example of a problem," they appeared to really be asking students to give an example of a set of given information, as indicated by the hint they provided. The teachers chose to include the hint because of their concern that students would not have a means to approach the problem. This concern seems to be related to the concern of openness versus restriction that appeared in Research Lesson One. The teachers seemed to believe, at this point, that presenting an LGE without a structure (as in Research Lesson One) or without a hint would impede students' success.

The set of structured exercises that the teachers designed included the warm-up exercise and the three exercises within the lesson. The warm-up exercise was intended to elicit the notion that either slope-intercept or point-slope form could be used to write an equation of a line (CA1) through natural variation in solution methods that the teachers anticipated would occur. The teachers applied the principle of Explicit Contrast through patterns of variance and invariance to create opportunities for the students to discern the critical aspects. In Example 1, the *y*-intercept varied between Line A and Line B so that one was an integral value and one was a fractional value. This contrast was intended to provide the opportunity for students to discern an instance when point-slope form might be easier to use. In designing this example, the teachers questioned whether they should ask students to write equations of the lines or find the *y*-intercepts. Their discussion about the design influenced the purpose through the clarification of one of the critical aspects. The teachers had already determined that a critical aspect was the combination of given information (CA2). Through the process of design, they determined that another critical aspect was what the question was asking for (CA3). Example 2 was intended to contrast with Example 1 by holding Lines A and B invariant and varying what the question asked for. Similarly, Example 3 had three parts in which the given information was varied, yet each set of given information produced the same line. The teachers decided that part of the enactment needed to include student collaboration as an opportunity to provide contrast through the natural variation in student thinking and solution methods. While the enactment of the set of structured exercises was intended to largely be teacher-led through the first two exercises, the teachers planned for students to partner and discuss solution methods with their classmates for the third exercise. Lastly, the teachers decided to include an LGE as a part of the set of exercises assigned for homework. The teachers included a hint within the question statement for the LGE to provide direction to students of a possible approach. The openness and restriction within the design of the LGE revealed teachers' concern with student success, similar to their attempt at an LGE in Research Lesson One. The teachers appeared to have associated the openness of LGEs with challenge and attempted to mitigate this challenge with framing (by including blank spaces and operation symbols in the Research Lesson One) and hints (in Research Lesson Two).

Implementation of Structured Exercises and LGEs. In this section, I describe the implementation of the set of structured exercises. The description is based on my field notes and my analysis of the videotape. As before, the teachers' perspectives on the implementation of the set of structured exercises and LGEs is described in the next section on the evaluation of the lesson.

Robert gave the students about eight minutes to work on the warm-up exercise. Then he instructed students to turn their paper over on to the back and describe why they chose to use either slope-intercept or point-slope form to write an equation of a line for the warm-up exercise. After allowing students two minutes to write, Robert elicited from the students the equations of the line that they had written: (1) $y = \frac{1}{4}x + \frac{3}{4}$, (2) $\frac{1}{4}x + 0.75$, (3) $y - 3 = \frac{1}{4}(x - 9)$. As intended in the design, the enactment of the warm-up exercise created an opportunity to discern the form of a linear equation (CA1). Through natural variation in students' solution methods, Robert elicited three equations of the line, two of which are written in slope-intercept form and one in point-slope form. Robert's enactment provided the opportunity for students to see that either form (CA1) could be used for the same given information (CA2). Then Robert initiated a discussion about the three equations:

Robert: What [are] the differences [between the equations]?...Are they all the same equation?

Ben: They pretty much are [all the same], just configured a little differently.

Robert: Okay. Expand on that. What do you think?

Vera: In point-slope form, you don't have to find *b* or write it.

Robert: Sure. What was all the work you had to do?

(RL2 Field Notes, Lines 41-48)

Robert asked the students, "What [are] the differences [between the equations]?" in order to contrast the two forms of a linear equation. He asked, "Are they all the same equation?" in order to provide the opportunity for students to see the equivalency between the equations – that they produce the same line. A student, Vera, suggested that writing the equation in point-slope form required less written work. Robert took up Vera's suggestion, and asked students to tell him their process for finding the equation of the line in slope-intercept form and point-slope form. The students' methods are summarized in Table 4.8. Both methods required calculating the slope of the line. Robert's students organized the given points in a T-table and calculated the change in *x* values and change in *y* values in that manner. Robert had not introduced the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to his students due to his past experience with students' difficulties with subscripts. In comparing the solution methods, Robert emphasized the efficiency of using point-slope form for

Table 4.8

Elicited Solutions for Writing an Equation of a Line in the Warm-Up Exercise

Slope-Intercept Form	Point-Slope Form

Shared Process:

x	y
9	3
-11	-2

Differences in x values is 20.
Differences in y values is 5.
Rate of change = $\frac{5}{20} = \frac{1}{4}$

Slope-Intercept Form	Point-Slope Form
$y = mx + b$	$y - y_1 = m(x - x_1)$
$3 = \frac{1}{4}(9) + b$	$y - 3 = \frac{1}{4}(x - 9)$
$3 = 2\frac{1}{4} + b$	
$3 - 2\frac{1}{4} = 2\frac{1}{4} - 2\frac{1}{4} + b$	
$\frac{3}{4} = b$	
$y = \frac{1}{4}x + \frac{3}{4}$	

this exercise. The following excerpt began just after a student had explained how to calculate the *y*-intercept when using slope-intercept form:

> Robert says, "Whoo! We're not done yet! Now we have to write it in [slope-intercept] form, $y = \frac{1}{4}x + \frac{3}{4}$. That sounds like a lot of work! Tell us why [slope-intercept form is] easier (said to a student). If you think it's easier, I guess." The student says that she picked point-slope form because she was given the *x* and *y* values, and all she had to do was find the slope. Robert says, "So you had to find the slope still; that stayed constant. Then what did you do?" The student says that she chose 9 and 3 and substituted in for x_1 and y_1. Robert says, "Yeah! Kelly did it that way too, live in front of us. She didn't [use

point-slope form] originally, but it was that easy for her to just plug in those points."

(RL2 Field Notes, Lines 64-70)

Robert contrasted the amount of written work necessary, for this exercise, to write an equation of the line in slope-intercept form with the amount of written work necessary when using point-slope form. He did this through a comparison of students' solution methods. He emphasized the efficiency and ease of using point-slope form for this exercise by referring to a student, Kelly, who wrote an equation of the line in point-slope form during the discussion.

Before having students start Example 1, Robert asked students to look at the given lines and think about what they noticed. He created an opportunity for students to think about how they would write an equation for each line in tandem with their peers:

> Robert says, "Put your pencils down. Before we start to dive into the problem…Look at Line A and Line B graphed on the coordinate grid below. I want you to turn to someone. What do you notice about the two lines? Looking at Line A and looking at Line B. And if we look at the question, it says write an equation that represents each line graphed. You may use either slope-intercept or point-slope form. Why don't you come up with a game-plan with somebody close by on how you would actually write the equation of the line?"
>
> (RL2 Field Notes, Lines 79-86)

Robert explicitly directed students' attention toward the lines and asked "What do you notice?" Robert did this to create an opportunity for students to discern the difference between the lines, and create contrast between the *y*-intercepts of the lines. Robert explicitly directed students' attention toward the question being asked. This was intended to set up the contrast with the next exercise. He reiterated students' choice between using slope-intercept and point-slope form in order to create a connection between what students noticed (which he intended was the varying *y*-intercepts, or the given information) and the form of a line that they chose to use. Robert also required students to discuss their solution method with a classmate, providing another opportunity to discern contrast or discern the relationship between the *y*-intercept (CA2) and form of a linear equation (CA1). The teachers referred to this as a "notice and focus" strategy. Just prior to the research lesson being taught, the teachers and I met as a group, and Robert shared some of his concerns of how the lesson had gone in his earlier classes. Robert said that he wanted to develop students' flexibility and their ability to stop and think before doing. He said, "They're not connecting…I have to put the pieces together for them" (RL2, Planning Meeting 9, Lines 256-257). Tori suggested using the notice and focus strategy. The teachers all appeared to be familiar with this strategy and described it as a strategy borrowed from English Language Arts. This was a strategy that Robert repeated for the second exercise, and intended to repeat for the third exercise, but did not due to time constraints.

After allowing about a minute and a half of discussion with a classmate, Robert asked the students to go ahead and write an equation for Lines A and B. Students worked for about three minutes before Robert got students' attention. By a show of hands, he asked how many students had chosen to use slope-intercept form, and then point-slope form, respectively, for Line A. Seven students used slope-intercept form, while eleven students used point-slope form. Robert then asked what students noticed about Line A:

> One student said that they could easily find the slope, and that it was two-fifths. Robert drew in a slope triangle on the line, saying up two, over five, and labeling the vertical and horizontal lengths…Then Robert asks, "What makes this problem more difficult than Line B?" A student says, "There's not an exact point where the line intercepts the *y*-axis." (RL2 Field Notes, Lines 108-112)

Robert's question, "What makes this problem more difficult than Line B?" explicitly created contrast between Line A and Line B. Students noticed that the slope could easily be determined from the graph, and that the *y*-intercept was a fractional value. After obtaining an equation in slope-intercept and point-slope form from the students, Robert again surveyed the students, by a show of hands, about which form of a line they chose to use for Line B. This time, 17 students chose slope-intercept form, and two students chose point-slope form. The survey strategy contrasted students' solution methods for each line. Robert asked, "Why would more people choose the slope-intercept form over the point-slope form? What would be the main reason for that?" (RL2 Field Notes, Lines 123-124). By asking this question, Robert created an opportunity for students to fuse the critical aspects of the combination of given information (CA2) and form of an equation of a line (CA1).

For Example 2, Robert again enacted the notice and focus strategy and then allowed students to work independently for about five and a half minutes. During this time, Robert walked around the classroom, listening to students' plans before they began, encouraging alternative strategies to find an exact value of the *y*-intercept, rather than an estimate, and for those students who finished, suggesting that they turn back to Example 1, saying, "Maybe you'll see some similarities" (RL2 Field Notes, Line 150). After getting the students' attention, Robert initiated a discussion with the whole group about the lines in Example 1 and Example 2:

> Robert asks, "What do we notice about Example 1, and what do we notice about Example 2?" Kelly says, "Line A in Example 1 and 2 have the same equation." Robert asks the class, "Does anyone want to expand on that?" Another student says, "They are the same line." Robert emphasizes, "They are the same line!" Robert asks, "Was it the same for

Line B? Is that the same also?" A student says yes. Robert says, "Same slope, same *y*-intercept. They're the same line." (RL2 Field Notes, Lines 154-158)

By establishing that lines A and B in Examples 1 and 2 are the same lines at the outset of the discussion, Robert set up the contrast between the graphical appearance of the lines and students' solution methods in Examples 1 and 2.

After asking about the similarities between Examples 1 and 2, Robert asked the class about the differences that they noticed. Lee responded, "They're showing a different part of the line" (RL2 Field Notes, Line 161). Andrea said, "It wasn't set up like our usual graph" (RL2 Field Notes, Line 161). Robert asked the class what was different about the appearance of the graph. Students said that the origin was not in the center and the axes were not labeled. The difference in the question being asked did not emerge. It was possible that students discerned that a critical aspect was the appearance of the graphical representation as different than usual, rather than the question being asked.

Robert continued, "'Let's take another tally here. How many here used slope-intercept form? Pretty much everybody! So let's think about that. Why did everyone use slope-intercept form?'...Lee says that point-slope form doesn't give them the *y*-intercept" (RL2 Field Notes, Lines 165-168). Lee referred to the question that students were asked in Example 2 – what is the *y*-intercept? – as the basis for using slope-intercept form to write an equation of the lines in Example 2. Because Robert had already established that the lines in Example 1 and Example 2 were the same, students had the opportunity to discern that the lines, themselves, were irrelevant. Lee's comment, that point-slope form doesn't give them the *y*-intercept, provided an opportunity for students to fuse the critical aspect of what the question asks for (CA3) with the critical aspect form of a linear equation (CA1), *if* students had discerned the critical aspect of what the question asks for. The class discussion, above, was about the differences in the graphical appearance of the lines between Examples 1 and 2, not the difference in what the question asked for. Only Lee's comment referred to the difference in what was being asked for between Examples 1 and 2. It is not clear if, like Lee, students discerned the difference in question and fused the critical aspects of what is being asked for (CA3) and form of an equation of a line (CA1), or if students relied on the familiarity of slope-intercept form for an example they viewed as different from usual.

By the time Robert got to Example 3, there were only three minutes remaining in class. Because of this, he directed students' attention to the relationship between the given information and which form of an equation of a line might be easier to use:

> Robert said, "Let's have a class discussion about which method might be easier if I ask you certain things. So let's read the question. Robert reads, "A line with a slope of three that goes through the point $\left(0, 6\frac{1}{8}\right)$. Who has an idea?" Shauna says slope-intercept, and Robert asks her why. Shauna says because the point it gives you tells you what the *y*-intercept would be…Robert asks, "Could someone argue that point-slope [form] was just as easy?" Students say yeah, and Robert asks why. Then he says (answering his own question), "Because it gives you a point!...Maybe either would be fine for that one." (RL2 Field Notes, Lines 189-192)

In the discussion of this example, Robert was explicit about the relationship between the form of an equation of a line (CA1) and the given information (CA2), although he said this as "if I ask you certain things." Robert's language seemed to refer to the critical aspect what the question asks for (CA3), when it was intended to refer to the critical aspect the combination of given information (CA2). This slip in language had the potential to conflate these two critical aspects for students, although Shauna interpreted Robert as intended and referred to the *y*-intercept being given as the basis for choosing slope-intercept form. Robert's enactment of this example also revealed that there were multiple correct solution methods. The relationship between the combination of given information and form of an equation of a line is not hard and fast, but rather a choice to be made based on mathematical reasons. Robert repeated this process for part B of Example 3, although he did not ask for students' reasons why point-slope form might be easier to use. He directed students to do part C on their own.

Within the implementation of Research Lesson Two, there were a number of instances where Robert enacted the principle of Explicit Contrast between and within examples. Explicit Contrast emerged as a principle of design and enactment during Research Lesson One. After significant amounts of discussion regarding the purpose of this lesson, the teachers designed the set of structured exercises using variance and invariance to create explicit contrast, as described in the previous section on the design of structured exercises and LGEs. In the implementation, Robert demonstrated the enactment of explicit contrast through a notice and focus strategy, in which Robert asked: What do you notice?; What is the same?; What is different? The notice and focus strategy, as enacted by Robert, provided opportunities for students to discern contrast within examples (e.g., the difference in the *y*-intercepts between Line A and Line B in Example 1) and contrast between examples (e.g., the difference in what is asked for and graphical representation between Examples 1 and 2). Robert also demonstrated the enactment of explicit contrast through a survey strategy, in which natural variation in students' solution methods created contrast between the form of an equation of a line and created opportunities for students to discern the relationship between critical aspects (e.g., the combination of given information and the form of an equation of a line; what the question asks for and the form of an equation of a line).

Robert's implementation of the lesson demonstrated that explicit contrast can be enacted between irrelevant aspects. The difference between the graphical representation of the lines between Examples 1 and 2 was discussed, while the difference in what was asked for between Examples 1 and 2 was not. While it is important for students to recognize that lines extend infinitely and that only a portion of them are viewed at any one time, the explicit contrast between representations may have inadvertently led to some students discerning the appearance of the graphical representation as critical for making a choice between which form of an equation of a line to use. The enactment of explicit contrast can support or impede students' understanding of the object of learning, depending on what aspects are brought to the fore of students' attention via that explicit contrast and whether those aspects are critical or irrelevant.

Evaluation of the Lesson. Immediately after the research lesson was taught, the team of teachers and I met to discuss our observations of the lesson. The discussion immediately following the research lesson followed a modified protocol borrowed from Japanese Lesson Study in which the instructor first shares his reflections, the team members discuss what they observed, and then a general discussion follows focused on how specific elements of the design of the lesson promoted student learning or not. We met again five days later during the teachers' planning period to discuss the homework assignment from the research lesson that Robert had collected.

Robert's first comments were regarding the strategy of notice and focus. This was a strategy that Robert had enacted only in the implemented research lesson and not in his three other classes that were held earlier in the day. Robert enacted this strategy based on Tori's suggestion at our meeting just before the implemented research lesson. Robert said, “The [notice and focus] was great. That, for me, was a big difference there. Instead of diving into the problem, coming up with a plan first. I walked around [the room] and heard a few [conversations] that were great” (RL2 Debrief, Lines 10-13). Robert noticed a contrast between the implementation of the research lesson and the implementation of the lesson in his earlier classes. The use of the notice and focus strategy resulted in student discussions about similarities and differences between the lines in Example 1. Robert also discerned contrast between his classes in which he used a survey strategy and those in which he did not:

> I didn’t [use a survey] in third period. I did do that fourth period. So there were times where I didn’t take a survey and did take a survey, and I thought fourth period went a little bit better with the survey than when I didn’t [take a survey in] third [period]. So that was interesting to see it in different classes when I did it differently. (RL2 Debrief, Lines 75-78)

Robert began to discern contrast between classes based on the enactment strategies that he chose to use. The notice and focus strategy and the survey strategy opened the space for students to discern explicit contrast, and Robert noticed that students were more likely to discern the contrast the teachers had intended in the design when it was supported by enactment strategies that opened the space for students to discern that contrast. As the teachers took up a design and enactment principle of Explicit Contrast, it appeared that Robert was also developing a greater awareness of contrast within his instruction.

In discussing the changes the changes they would potentially make to the lesson, the teachers suggested additional strategies for enacting explicit contrast. One change that Robert suggested was switching the labels for Line A and Line B: “I also maybe would have [switched]

Line A and Line B. Make Line A the easy [integral] *y*-intercept. Make Line B – so [students] are like this is easy! Oh! Why is this [Line B] harder?" (RL2 Debrief, Lines 193-195). Robert felt that students had been naturally conditioned to write an equation for Line A first, since A comes before B alphabetically. His suggestion to switch the labels so that students would write an equation for Line A first was based on what he anticipated would create contrast in difficulty for students, and then force students' noticing of the contrast in *y*-intercepts, as Line B then provided an additional challenge with a fractional *y*-intercept.

Another change was suggested regarding the method of contrasting student solution methods. Robert had collected the warm-up exercise where students had written an equation of the line given two points. Tori suggested an alternative way of contrasting students' solution methods for the warm-up exercise:

> Tori said, "This is a perfect example of when a [document camera] would be awesome. Here's Riley's paper, correct equation [using point-slope form]. Didn't need to do anything. Versus – look at all this [work]! And look at all the opportunities for a mistake!" Robert replied, "And then have that discussion, are they the same equation? Who did less work?" (RL2 Debrief, Lines 177-181)

Tori suggested that simply seeing two different student solutions juxtaposed, one using point-slope form and one using slope-intercept form, would create the contrast between methods in terms of the amount of mathematical work needed to be done. While Robert did this within the lesson, he took time in class to elicit the student methods and write them out on the SMART Board. Tori suggested that displaying students' actual work would save a significant amount of time and convey the contrast desired.

Robert mentioned in the previous excerpt the potential to have a discussion about the equivalency of the equations of the lines generated by students. One of the questions that emerged for the teachers was how to generalize the equivalency of the various forms of an equation of a line. The equivalency of the two forms of a linear equation was an aspect that was left implicit. Tori suggested that the time that potentially could have been saved by displaying students' solution methods for the warm-up exercise on a document camera could have been better spent on Example 3: "I think that would have helped [to say to students], you have these three equations [from parts a through c], can you now solve for *y* for all of them and prove they're all the same line or something" (RL2 Debrief, Lines 119-120). Tori suggested an algebraic approach to prove the equivalency of the various forms of an equation of a line. Lynn suggested an extension of this in which students graph each of the lines in order to visually see their equivalency. These approaches, however, generally required solving the equations for *y*, which the teachers had noticed was a misconception that students had developed – that they were not done, or the equation was not correct unless solved for *y*. Shannon hypothesized that students' difficulties with seeing various forms of an equation of a line as equivalent reflected students' concept of equivalency:

> I don't think [students] see point-slope [form] as a true equation of a line because it doesn't equal one thing. Or because it doesn't say *x* equals. It's like it doesn't equal anything, you know? So I think that's why, that's where [students] get hung up. Because when we do equations in seventh grade, it's a number equals a couple other things. When we solve equations, you get it down to a variable equals something. And so this is really [students'] first experience of this is an acceptable answer that isn't solved for one thing. I think that's what trips them up. Which [Robert] said – [students think] they now have to

> solve [point-slope form for *y*]. Why? I think that's the biggest thing. I think that's why they don't use point-slope [form]. (RL2 Debrief, Lines 157-164)

Shannon's hypothesis suggested that it may be necessary to find a way to extend the range of variation of objects that can be equivalent. Shannon observed that up until this point in students' mathematical education, solutions had always been given as single term as equal to something else (e.g., $x = 3$; $y = 2x - 5$). Instances in which one expression equaled another expression, such as $5x - 3 = -2x + 11$, was indicative of further mathematical work to be done to solve for x. The teachers identified that the task of generalizing the equivalent forms of an equation of a line possibly involved two components: (1) seeing that the forms of the equations of a line produced the same line via algebraic and graphical means, and (2) extending the range of variation of objects that can be equivalent and accepted as solutions.

Through the debrief discussion, the importance of variation emerged for the teachers. Shannon said, "I learned that the [students] are more apt to use slope-intercept [form] because that's what we focus on. I mean, I kinda knew that, but it's very obvious when you see a lesson like this" (RL2 Debrief, Lines 240-241). The lack of variation in form of an equation of a line that the teachers themselves used and emphasized in class extended to what the students used and viewed as acceptable in their own mathematical work. The teachers wondered aloud what their students would have the opportunity to learn had their unit been taught using a different variation:

> Shannon said, "I wonder if we taught this unit using only point-slope, well not only, but harping on point-slope instead of slope-intercept, and then – here's another way you can do it – slope-intercept. I wonder how that would be." Tori replied, "They would never [use slope-intercept] because they would see it as so much more work. They would be

> like why would I ever want to do that? Just always leave it in point-slope form. And now you're fighting [students] to show work." (RL2 Debrief, Lines 166-171).

Tori somewhat dismissed Shannon's suggestion because she believed that students would refuse to use slope-intercept form after seeing the ease with which they could use point-slope form. Tori's comment appeared to go against the purpose of developing students' flexible and intentional use of both forms. If, given point-slope form first, students would see slope-intercept form as useless, then why bother to teach it at all? Slope-intercept form serves the advantageous purpose of writing an equation of a line in a form in which the slope and *y*-intercept can directly be read from the equation. Slope-intercept form is also advantageous to graphing lines using technology or by hand. By the end of this discussion, however, Tori suggested that the best way to support students' flexible use of both forms was "spending the same amount of time on each form" (RL2 Debrief, Line 183) in their own teaching and instruction. The teachers' observations suggested that they recognized that learners do not discern that which does not vary, hence it was necessary to carefully consider the variation present within their design and instruction.

Students' awareness that Example 2 included a representation that did not have the "usual" graph sparked a discussion about the variation presented in graphical representations.

> Tori asks if Robert ever changes the increments on the *x* or *y*-axes. Robert says, "Yes, once or twice though. So a few, but not a lot." Tori had noticed that students did not seem to be paying attention to the scale. (RL2 Debrief, Lines 213-215)

Because Tori noticed that students "did not seem to be paying attention to the scale," she asked about the variations in scale that Robert had used in his past lessons. Tori appeared to realize that aspects that are not varied are not discerned by students. Learners discern through difference (Marton, 2015). If the only graphical representations presented to students appeared on a coordinate grid where the values on the *x* and *y* axes go from negative ten to positive ten in increments of one, students do not discern that aspects such as scale and viewing window can vary.

To support students in developing the scale as a critical aspect of graphical representations, Tori suggested comparing two graphs in which the lines appeared the same, but the scales were different:

> I've graphed the equation, like…$y = \frac{1}{2}x$, and then on a different graph, I graph $y = x$, but they look the same. So they're on two different graphs, but they look like the same graph, and they have different equations. And I ask [students] why? [The graphs] look exactly the same. Why are their equations different? And then it's because of the increments. It's because you went up by [increments] of one on the *y*-axis, but you went [over increments of] two on the *x*-axis, so in terms of the boxes, you go up one, over one, even though it's really over two on the grid. (RL2 Debrief, Lines 215-222)

Tori suggested holding the appearance of the line invariant through varying the increments of the scales on the *x* and *y* axes, and explicitly contrasting the different equations between the two graphs. Similar to Robert's earlier suggestion, the disturbance created when two graphs that appear identical but have different equations could be an effective means for students to discern other variations between the two graphs, namely the significance of the scale on the axes, and the insignificance of the grid lines drawn on the graph.

The teachers began to articulate the importance of careful planning to design certain patterns of variation and support students in discerning intended aspects. Robert stated that through the learning study process, he has learned about the importance of creating learning opportunities through planning:

> I learned that sometimes when we think things come naturally through conversation, it's not necessarily true. You can have those great "aha" moments with patches of kids in certain sections, but really creating a classroom moment like that takes a lot of planning…The assumptions that we make and then the reality that happens, requires more planning, not more work, but planning to set up those situations, instead of just here, do these 100 problems and see if you come up with that conclusion. Or here look at this one problem and see if you come up with that conclusion. More like a situation where you could strategically pick from and have [students] think about it. (RL2 Debrief, Lines 244-252)

Robert recognized that the intended object of learning is not always congruent with the lived object of learning ("The assumptions that we make and then the reality that happens"). Repetition alone, or exposure, alone, are not sufficient for students to see the object of learning in the way intended. Robert realized that students were more likely to see what was intended that they see when it was not left to chance (Ling and Marton, 2012). Hence, Robert had come to see the value and importance of careful planning, in terms of both design and intended enactment.

At our next meeting, five days after the implementation of the research lesson, Robert brought the completed homework that had been assigned at the end of the research lesson. Question 2 on the homework was an LGE that asked students to write an example of a problem that asks for an equation of a line in which it might be easier to use (a) slope-intercept form, and (b) point-slope form. The teachers decided to tally the information students gave in the problems that they wrote. Table 4.9 summarizes the combinations of information given by students in the completing the LGE and the frequency of each combination. Based on these results, Tori said, "Two points, or a slope and a point, are stronger for point-slope form" (RL2 Evaluation Meeting, Lines 13-14). The teachers thought that the students had a greater awareness of when point-slope form might be easier to use (when given either two points or a slope and a point), but students' awareness of when slope-intercept form might be easier to use was less convergent. The teachers were also curious as to why the students did not see the *y*-intercept, when given as a point, as a generic point on the line:

> Tori said, "When technically, slope and *y*-intercept (given for part a)…is the same as slope and a point, you know?" Robert said, "Yeah, that's interesting. I don't think [students] got that connection." Tori replied, "So interesting that the [students] see the *y*-intercept as the *y*-intercept and don't see it as a point that they could substitute into point-slope [form]." (RL2 Evaluation Meeting, Lines 14-17)

The teachers interpreted that students did not recognize the *y*-intercept, even when written as a point, as a generic point that could be substituted into point-slope form. While they commented that this was interesting, they did not offer any potential changes to their instruction or solutions for helping students to see that the *y*-intercept is a point like any other point. The teachers largely used this LGE as an assessment of students' understanding of when each form of an equation of a line might be easier to use, but it was not formative in the sense that it did not used to inform instruction in any way that was visible, at least to me.

The teachers had began to expand on the principle of Explicit Contrast in enactment by developing a repertoire of strategies for enacting explicit contrast within the lesson. The three strategies that the teachers used or suggested were (1) a notice and focus strategy, (2) a survey strategy, and (3) a comparison of student solution methods. Robert and Tori suggested a design strategy of creating intentional disturbance for students to notice contrast. A question arose regarding how the teachers could support students in generalizing that the various forms of a linear equation represent the same line. The teachers identified that the task of generalizing the equivalent forms of an equation of a line possibly involved two components: (1) experience of the equivalency of the lines via algebraic and graphical approaches, and (2) extending the range of what something can be equal to and what is acceptable as a solution. The teachers realized the importance of variation in their design and instruction, as they recognized that students do not discern that which does not vary. This idea is perhaps the cause and the solution to Shannon's hypothesis that students have difficulties with point-slope form because they do not often experience equations in which one side of the equation is not reduced to a single value or solved for a single variable. The LGE that the teachers assigned for homework was used by the teachers as an assessment that revealed some facets of students' understanding of writing linear equations. The LGE, however, was not used to inform instruction and was not used in other ways in service of students' learning.

Table 4.9

Frequency of Combinations of Information Given by Students in LGE

Part (a) Slope-Intercept Form		Part (b) Point-Slope Form	
Slope and *y*-intercept		Two points	10
Slope and *y*-intercept (given as a point)	4	Slope and a point	6
Slope and a point	1		
A point and *y*-intercept	1		
A point and *y*-intercept (given as a point)	1		
Two points	1		
Only *y*-intercept	2		

Summary of Research Lesson Two. The purpose of Research Lesson Two was to develop students' flexible use of both slope-intercept form and point-slope form, based on mathematical reasons, in order to write a correct equation of a line. Over nine planning meetings, the teachers had extensive discussions about a number of ideas related to linear equations and explored the variation in their own thinking about linear equations. Due to the diversity of thinking among the teachers, it took a significant amount of time to arrive at a shared purpose for the lesson. The teachers began designing examples for the research lesson in Planning Meeting Six, prior to fully articulating a shared purpose or solidifying the critical aspects. The design of the examples, however, allowed for greater articulation of the purpose and a refinement of the purpose through a clarification of the critical aspects. The teachers identified three critical aspects of writing an equation of a line: (CA1) the form of a linear equation, (CA2) the combinations of given information, and (CA3) what the question asks for.

The teachers designed a set of structured exercises to be implemented during the research lesson and two LGEs to assign for homework. Table 4.10 summarizes the exercises, the patterns of variation, and the critical aspect that each exercise was designed to address. The teachers designed two LGEs that asked students to write a problem that asked for an equation of a line in which (a) slope-intercept form might be easier to use, and (b) point-slope form might be easier to use. Like in Research Lesson One, teachers' notions of student success shaped the design of the LGEs. The teachers decided to provide a hint that students might give information (e.g., two points, draw a graph, or create a table of values). The teachers appeared to be associating students' success with the students' comfort in knowing what to do and having a means to approach the problem.

Just prior to the implementation of the research lesson, the teachers and I met again, during our normally scheduled meeting time. Robert had taught the lesson to three of his other classes at that point, and was concerned that students were not seeing the intended relationships between the critical aspects. Tori suggested that Robert enact a notice and focus strategy in which he asked students, with pencils down, to talk with a classmate and come up with an approach to the exercise before beginning it. The notice and focus strategy and a survey strategy opened up the space of learning to provide opportunities for students to discern contrast between the *y*-intercepts of Lines A and B, the graphical representations between Examples 1 and 2, and the solution methods used by students in both Examples 1 and 2.

Table 4.10

Summary of Patterns of Variation and Critical Aspects Addressed in Set of Structured Exercises

	Exercise	Critical Aspect to be Addressed
Warm-up	Write an equation of a line through the points	Either form of a linear equation may be used; Both forms produce equivalent equations/represent the same line
Example 1	Write an equation of Line A and Line B. Vary: *y*-intercept (integral and fractional) Invariant: slope	The combination of given information may make one form easier to use than the other.
Example 2	Find the *y*-intercept of Line A and Line B. Vary (as compared to Example 1): the given information; the graphical representation; the question being asked Invariant (as compared to Example 1): Lines A and B	What the question asks for can make one form easier to use than the other.
Example 3	Write an equation of the line for each set of given information. Vary: the given information Invariant: the line	The combination of given information may make one form easier to use than the other.

During the evaluation meetings, the teachers began to articulate strategies for enacting explicit contrast, including the notice and focus strategy, the survey strategy, and the strategy of comparing student solutions. This expanded on the verbalization strategies articled in the debriefing of Research Lesson One. The teachers began to hypothesize a design strategy of creating a disturbance to encourage students to notice explicit contrast. The teachers articulated that careful planning of variation and invariance was important in helping students discern what they wanted students to see. Shannon hypothesized that some of students' struggles with point-slope form may have reflected the lack of variation in their notions of equivalency up to that point in their mathematical education. Students had generally only experienced a single value or variable as equal to an expression, but in the case of point-slope form, an expression was written as equivalent to another expression, and students interpreted this as an indication to solve, usually for *y*. In reviewing the LGEs that students completed for homework, teachers were curious about students' seeming disassociation of the *y*-intercept as a generic point on the line, but they did not appear to use this facet of students' understanding formatively. The teachers appeared to view the LGEs as a means of assessing students' understanding of when each form of an equation of a line might be easier to use, but they not yet developed an awareness of a role that LGEs could play in the furthering of students' learning.

Research Lesson Three

Lynn co-taught Research Lesson Three on December 8, 2015 in her eighth period Math 7 RTI (Response-to-Intervention) class with Beth, a special education teacher. Beth had observed the first research lesson and began participating in the planning meetings for Research Lesson Two. Prior to the research lesson, the teachers met for five 40-minute planning meetings and one longer meeting (approximately two hours) during a staff day. They met again, for approximately 40-minutes, on the day of the research lesson. Lynn wanted to address challenges related to unit rates in this learning study cycle. During the first planning meeting, Lynn gave the following example to describe the difficulties her students had had with unit rates in the past:

> Lynn said, "If you can get four boxes for eleven dollars, what's the price per box? And they [the students] were just dividing"…Tori said, "So they didn't know to do four divided by eleven or eleven divided by four"…Lynn said she wants "them to be able to label what they're doing, so they know what the results are when they're getting it. Because they're getting 2.22. They don't know if that's two dollars and twenty-two cents [per box] or 2.22 boxes [per dollar]." (RL3, Planning Meeting 1, Lines 17-24)

Lynn's example of a typical student error suggested that she wanted to address students' conceptual understanding of a unit rate. A unit rate is a particular kind of rate that gives the amount per one unit of another quantity. Lynn's students appeared to be reducing the concept of a unit rate to a procedure, where many of them divided the two numbers in the problem statement in order of appearance (as in the above example, $4 \div 11$). Students' difficulties then arose in interpreting the resulting value of the quotient in the context of the problem statement.

Similar to Research Lesson Two, where a wide range of ideas about linearity were discussed, the teachers extensively discussed a number of ideas related to unit rates prior to deciding on the focus of this lesson. These ideas included: (1) the calculation of the value of a unit rate using proportions, (2) the interpretation of a calculated value as a unit rate in a context, (3) the comparison of two unit rates, and (4) the unit rate as the constant of proportionality in a proportional relationship. As in Research Lesson Two, the articulation and clarification of the purpose of the lesson was intertwined with the design of the set of examples within the lesson. I discuss the interplay between the design of the examples and the clarification of the purpose in the section on the design of the sets of structured exercises and LGEs.

This learning study cycle followed a progression similar to the two previous research lessons. The teachers moved quickly toward the design of the examples, once again before fully articulating a shared purpose for their work. Ultimately, the teachers identified the goal of the lesson as “students will be able to find a unit rate, interpret the unit of the unit rate, and determine how to make a comparison with it” (RL3, Lesson Plan). While the teachers wrote this as a single goal, there are three goals within this statement: (1) students will be able to calculate a value for the unit rate, (2) students will be able to interpret that value as a unit rate in context, and (3) students will be able to compare two unit rates. In contrast to Research Lessons One and Two, which each ultimately had a singular focus (i.e., writing equivalent expressions using the order of operations and writing linear equations), the multiple inter-related goals in this lesson became a source of difficulty in the design of the structured exercises. As discussed below, a factor that influenced both the design and the purpose was the nature of the class of students as an RTI class, where multiple skills and concepts introduced in students’ regular math classes were often re-taught and reinforced. Teachers’ notions of student success were also a factor in the design and implementation of the set of structured exercises.

Purpose of the Lesson

As described previously, in the beginning phase of a learning study, the research team (1) selects a topic for study, (2) identifies a tentative object of learning, (3) diagnoses students’ learning difficulties, and (4) confirms the object of learning and its critical aspects. In aggregate, these first four steps of the learning study cycle articulate and clarify the purpose of the lesson. For this lesson, the teachers identified the direct object of learning as interpreting the value of unit rates. As in Research Lesson Two, the teachers believed that in some contexts, one form of the unit rate was "more appropriate" than the other. The teachers discussed the two forms of the unit rate and the "appropriateness" of each:

> Tori talked about students deciding between cases per dollar or dollars per case. She said, "Which one makes more sense? When you're at the grocery store, do you care that you can buy point four [0.4] of a case for a dollar? No – because who buys point four of a case?" Lynn replied, "Or a box of pasta, like one point three [1.3] boxes [per dollar] – that's not going to make sense. You'll have spaghetti flying all over the place!" (RL3, Planning Meeting 1, Lines 65-69)

From the teachers' perspective, because purchasing a fractional value of a case would not make sense in the context, computing the amount of cases per dollar or boxes per dollar did not make sense. In this context, Tori and Lynn argued that computing the dollars per case or dollars per box was the more appropriate form of the unit rate to choose. In this case, the teachers used the word "appropriate" to mean useful or that the unit rate made sense for the context. Because of this perspective, the teachers wanted students to understand that in some contexts, either form of the unit rate could be appropriate, but in other contexts (such as the grocery store example, above), one form might be more practical than the other.

The teachers' sense of "appropriateness" was also associated with the role of the unit rate as the constant of proportionality in proportional relationships. Tori said, "The reason that this [the unit rate] is so important is that then we move into constant of proportionality, and that rate, that unit rate, dollars per box, is now the slope of the graph" (Research Lesson 3, Planning Meeting 1, Lines 83-84). Again, the teachers have a perspective that there is a preferred way to choose which variable is independent and which is dependent and, hence, there is a preferred way to choose to divide the two quantities to obtain the unit rate. However, the teachers are in agreement that the division of the quantities can be done either way:

> Lynn says, “Which runner finished the race with the fastest time per kilometer? So if they get their results for that, if they do it the right way, it’s time per kilometer, but if they do it the other way (Tori: “Which they can!), then it’s kilometers per minute, but they have to realize that the larger value is the one that’s running the fastest. So they have to be able to interpret what those numbers mean when they get them.” Tori suggested that a lot of it has to do with “understanding that the per, in this case, means division, or means a fraction”…Lynn says, “Whatever the per is, that’s your denominator. But then again, it doesn’t have to be the denominator.” (Research Lesson 3, Planning Meeting 1, Lines 74-82)

Both Tori and Lynn said that the division can be done with either quantity as the numerator or denominator. The interpretation of that resultant value as a unit rate in context is what is important. The interpretation included understanding the value as a quantity with units and understanding the meaning of larger and smaller values in the context. In particular, interpretation of the quantity involved an understanding of the word “per” as meaning for one unit of the denominator.

Lynn explained that in the prior school year she taught a lesson on unit rates with her RTI students immediately after the lesson on unit rates in students’ regular mathematics classes. Lynn said, “I thought they would automatically – (Tori: “Feel comfortable?”). Yes. And the minute students started doing [the exercises], they didn’t transfer [the skills and concepts from class]. And so that’s when we went back and we had to go through [unit rates]” (Research Lesson 3, Planning Meeting 1, Lines 86-88). Lynn described the need to unexpectedly reteach unit rates in her RTI classes before talking about comparing unit rates. At this point, the teachers considered potential examples for the lesson in order to address some of the issues from the prior year, including students' abilities to correctly calculate a value for the unit rate and interpreting that value as a unit rate within the context. Tori first suggested starting with a typical unit rate example, such as dollars per box, then using the unit rate to determine the cost for three boxes. Tori commented that students could multiply by three or set up a proportion to solve. This example constructs a unit rate as a constant of proportionality. For instance, suppose the cost per box is \$0.70. Thus, the unit rate is 0.70 dollars per box. One could write an equation for a proportional relationship as $y = 0.70x$, where x is the number of boxes and y is the total cost. 0.70 is the constant of proportionality. In this equation, one would multiply 0.70 times three to get the total cost for three boxes. This example suggests to students how a unit rate can be used to calculate amounts for a larger value of the unit.

Tori then suggested a comparison of both forms of a unit rate, such as the dollars per box and the number of boxes per dollar:

> Tori says, "Or would it be better to do both ways with them? Part (a) could be what is the cost per box. Part (b), what is the number of boxes per dollar? So that [students] see both of them." Lynn replied, "If they see both of them, it's going to help them understand the difference…I think that with this lesson, we should do it both ways, and then talk about it. Because that's where the real understanding is coming in…The answer doesn't make sense the other way for what you're looking for." (Research Lesson 3, Planning Meeting 1, Lines 95-102)

The suggestion of this example indicated teachers' continued application of the principle of Explicit Contrast. The teachers recognized that contrasting the two forms of the unit rate (dollars per box and boxes per dollar) could provide an opportunity for students to discern that there are two forms of a unit rate for a given situation, and the differences between them, including the practicality of one form over the other in some contexts. Tori referred back to Research Lesson Two and students' discernment of when each form might be more useful:

> Maybe it's going back to what we did with Robert [in Research Lesson Two], which one [form of a linear equation and form of the unit rate, respectively] is more appropriate? And how do you know? You're talking about boxes. I can't buy 0.3 of a box. People are curious about how many gallons per mile they're getting, but I may also want to know how many miles per gallon I'm getting. (Research Lesson 3, Planning Meeting 1, Lines 103-106)

As in Research Lesson Two, Tori's suggestion regarded students' abilities to make sense of and reason about mathematical situations. In Research Lesson Two, the teachers' view of which form of an equation of a line was more "appropriate," was based on efficiency, with a dependence on the given information. In this research lesson, the teacher's view of which form of a unit rate was more "appropriate" was based on making sense of the context and the units and determining which form, if either, was more practical in the context. In both lessons, the teachers were aware that either choice (i.e., either form of an equation or a line and either form of a unit rate) was correct, but due to reasons of efficiency or practicality, one form could be a better choice. Tori suggested that this lesson may parallel Research Lesson Two in their purpose of helping students to discern which choice might be better, when both are correct.

Lynn supplied the group with her set of examples from the prior year as a starting point for the discussion. I pointed out that some questions asked for a specific unit rate (e.g., What is the cost per box?), in which case there was a correct form of the unit rate to choose. There were other questions that asked “What’s the better deal?”, in which either form of the unit rate could be chosen. Shannon replied that the value students calculated was their choice, and described student misconceptions she had noticed:

> It’s up to [the students]. Can they interpret how they chose their unit rate correctly? We have a lot of kids who will find the amount per dollar and they’ll say that the lower one is the better deal because they’re not thinking I’m getting more for my dollar. They’re just thinking, oh it costs less. It’s a lower number. (Research Lesson 3, Planning Meeting 1, Lines 108-112)

Shannon’s observation suggested that students needed to correctly calculate a value of a unit rate, interpret that value as a unit rate in context and make sense of the unit rate in context in order to decide whether one would want to choose the smaller or larger value. The discussion suggested that the purpose for this lesson had not been fully explicated, as the suggested examples addressed a range of ideas related to unit rates. The first suggested example conceptualized of the unit rate as a constant of proportionality and required students to scale up to find amounts needed for larger values of the independent variable. The second suggested example contrasted the two forms of a unit rate with a focus on determining which form, if any, would be more practical in various contexts. The third suggestion was examples that asked “Which is the better buy?” These types of examples appeared in Lynn’s example set from the previous year, and Shannon pointed out students’ misconceptions with these kinds of exercises, related to students’ interpretation of the calculated values as unit rates in context. The teachers’ goal for the lesson was, “students will be able to find a unit rate, interpret the unit of the unit rate, and determine how to make a comparison with it” (RL3, Lesson Plan). The second and third suggested examples related to the interpretation and comparison of unit rates, respectively. The first suggested example was related to determining a value of a unit rate, but extended beyond calculating that value to using that value to scale up to larger amounts of the quantity. Despite not yet establishing a shared purpose for the lesson, the teachers began to move into the design phase and considered specific examples for the lesson. In Research Lesson Two, the design of the examples led to the articulation and clarification of the critical aspects for the research lesson. In this lesson, the design of the examples led to the articulation of the indirect object of learning and what the teachers called the critical aspects. Because of this, I discuss the object of learning and the critical aspects within the section on design as they arose within the design process.

Design of Structured Exercises and LGEs

Unlike the previous two research lessons, the teachers began to plan examples during Planning Meeting One. After considering a number of contexts for the examples, including motion, pulse, and flow of water, the teachers decided on a baking context. The teachers felt that a baking context would interest their students and provide the opportunity to use manipulatives. Tori suggested a series of questions:

> For every three marshmallows, there's one Hershey bar. What if I gave you six marshmallows? How many Hershey bars would you need? What if I gave you 12 marshmallows? How many Hershey bars should I give you? What if I only gave you one marshmallow? (RL3, Planning Meeting 1, Lines 147-150)

Tori suggested giving students the unit rate (marshmallows per Hershey bar) and then using that unit rate to scale up to find the amount of Hershey bars needed for larger amounts of marshmallows, followed by scaling down to arrive at the other form of the unit rate (Hershey bars per marshmallow). The teachers discussed starting with two given amounts of marshmallows and Hershey bars, as in a recipe, and having students calculate the unit rates of marshmallows per Hershey bar and Hershey bars per marshmallow, since students would have already studied proportions. While the teachers decided that the direct object of learning was unit rates, they had yet to articulate and clarify the indirect object of learning, or what the learners were expected to become able to do with the content identified in the direct object of learning (Marton, 2015). Tori sought clarification from the group about the goal of the lesson:

> Tori asked the group, "So once you find the unit rate, is the goal then to go the other way? What if I only had one marshmallow?" Lynn replied, "I was thinking a larger amount. Like what if you had 55 marshmallows, how many Hershey bars? So they'd *have* to set [the proportion] up…" Tori said she thought that [the use of proportions to find a larger amount] should come before unit rate. I said that they could get the number of Hershey bars for a larger amount of marshmallows using a proportion, or they could use the unit rate and scale up…They [the teachers] talked about showing this both ways. First, using a proportion, then having students get the unit rate and use it to scale up. (RL3, Planning Meeting 1, Lines 167-179)

The teachers had not yet developed a clear, shared goal for this lesson. Tori wanted to compare the two forms of a unit rate. Lynn wanted to use the unit rate within a proportion to compute unknown amounts. The teachers decided to address both of these goals through the set of structured exercises shown in Figure 4.6.

The teachers created a hot cocoa recipe that involved two snack-size Hershey bars and six marshmallows. The teachers intended to give students actual snack-size Hershey bars and marshmallows as manipulatives to assist students in thinking through these exercises. Exercise 1 was: "If I gave you 27 marshmallows, how many Hershey bars would I need to give you?" (see Figure 4.6).

$$\frac{6 \text{ marshmallows}}{2 \text{ Hershey bars}} = \frac{27 \text{ marshmallows}}{x \text{ Hershey bars}}$$

$$6x = 54$$

$$x = 9 \text{ Hershey bars}$$

Exercise 1 was intended to review setting up proportions and cross-multiplying, which students studied in their regular mathematics course. Exercise 2 was, “What if you only had one Hershey bar?” This exercise elicited the idea of a unit rate (amount of marshmallows per one Hershey bar). The teachers intended on students approaching this exercise with manipulatives, visually separating the ingredients in their recipe, and through setting up a proportion. Exercise 3 was, “If

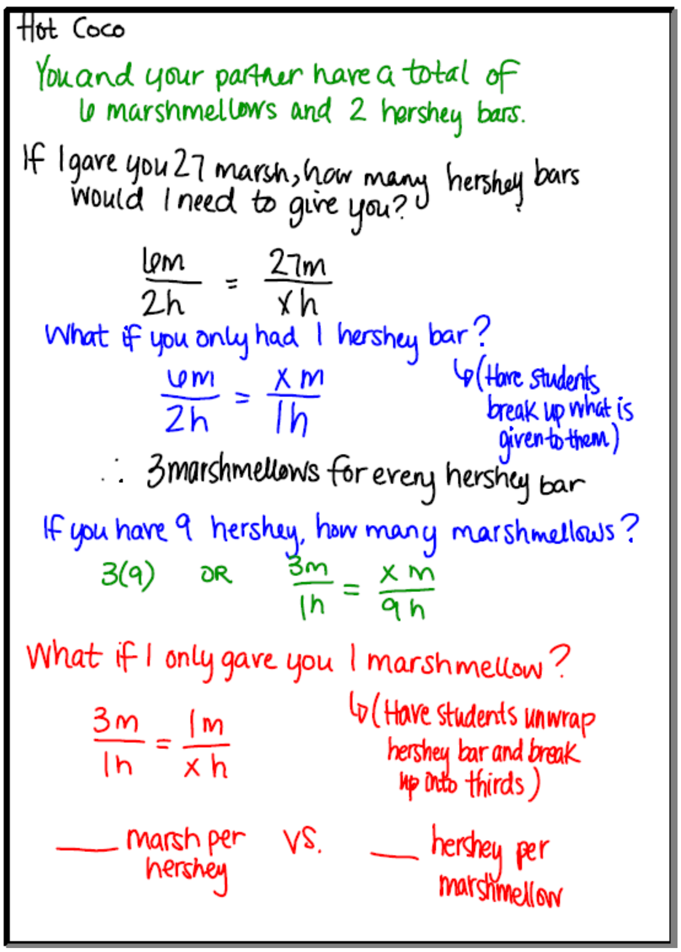


Figure 4.6. Teacher-designed working set of structured exercises in Research Lesson 3 to contrast (1) methods for finding an unknown amount (setting up a proportion and cross multiplying; scaling up from a unit rate), and (2) forms of a unit rate.

you have nine Hershey [bars], how many marshmallows [do you need]?" This exercise was meant to encourage students to use the unit rate to scale up to find an unknown amount for a larger value of the unit. It also paralleled Exercise 1, as it resulted in the same equivalent ratio (27 marshmallow: 9 Hershey bars). Exercise 3 made the scale factor visible (see Figure 4.7). One must multiply one Hershey bar by nine to obtain nine Hershey bars, and hence three marshmallows must also be multiplied by nine. This is what the teachers referred to as "scaling up." In Exercise 1, a scale factor between six and 27 to obtain an equivalent ratio is not immediately obvious, since six does not evenly divide 27. Using the unit rate made the scale factor needed to obtain the equivalent ratio more visible. The teachers intended for students to see the usefulness of setting up a proportion and cross-multiplying (as in Exercise 1) and the usefulness of scaling up from a unit rate (as in Exercise 3). Exercise 4 was, "What if I only gave you one marshmallow?" This exercise elicited the other form of a unit rate, and the teachers intended on contrasting the two forms within the lesson.

$$\boxed{\begin{array}{r} \dfrac{3\text{ marshmallows}}{1\text{ Hershey bar}}\dfrac{\times 9}{\times 9} = \dfrac{x\text{ marshmallows}}{9\text{ Hershey bars}} \\ x = 27\text{ Hershey bars} \end{array}}$$

Figure 4.7. Solution to Exercise 3

Near the end of the first planning meeting, I asked the teachers to articulate an object of learning for this lesson:

> Shannon said it [the object of learning] could be two different things: "It could be [calculating] a unit rate. This whole first example is all about being given a ratio of things…ultimately we end up with a unit rate…both forms of the unit rate. So it's either…showing that unit rate can be shown in two different ways, or them taking the next step from that and figuring out which is more appropriate to use in certain situations." (RL3, Planning Meeting 1, Lines 222-227)

Shannon saw the direct object of learning as calculating both forms of a unit rate and the indirect object of learning as choosing which is more "appropriate" or practical in particular contexts. Shannon talked about the object of learning as being "two different things." The teachers' uncertainty about the object of learning was indicative of the teachers' learning about variation theory while simultaneously participating in learning study. Tori echoed Shannon's understanding of the direct and indirect object of learning, and suggested that the following examples differ in context:

> So maybe, in terms of this list of examples, if you then move to a cooking/baking [context], either [form of the unit rate] is appropriate. If you use gallons per minute, either [form of the unit rate] is appropriate. And then maybe the last one is boxes and pasta. Which [form of the unit rate] is better for interpretation? Clearly dollars per box because you can't have boxes per dollar. (RL3, Planning Meeting 1, Lines 229-232)

Tori suggested varying the context from exercise to exercise, with the last exercise creating a disturbance for the learners as a context in which one form of the unit rate was more practical. From Shannon's and Tori's perspective, the goal of the lesson was to calculate both forms of a unit rate across various contexts and decide which form was more useful for the context, if either. Lynn's perspective of the goals of the lesson, however, differed from Tori and Shannon:

> Lynn said that she wanted students to "know when to do what." I suggested and clarified back what I had heard them talking about – "You want students to use the appropriate unit rate in appropriate situations." Shannon replied, "Yeah." Lynn said, "I also…want to springboard off of that into which is the better buy, and kids picking the least expensive one." While Lynn was talking, Tori said she thought that would be the next day. Lynn

> continued, “Or the runner. Which one ran faster?” (RL3, Planning Meeting 1, Lines 249-255)

From Lynn’s perspective, the goal, or indirect object of learning, of the lesson was comparing unit rates (e.g., which of two purchases is the better buy or who is the faster runner). The form of the unit rate was a critical aspect. The context was a critical aspect. The relationship between these two critical aspects (i.e., form and context) was the practicality of the form of the unit rate within the context. These would be the same critical aspects that Shannon and Tori were attending to, and perhaps the same indirect object of learning, although Shannon and Tori thought that the goal of this particular lesson should focus on making the decision as to which form of the unit rate was more practical, before using the unit rates to make comparisons.

Through planning the exercises, the teachers tried to converge on a shared purpose for the lesson. Tori suggested that rather than have exercises that involved scaling up to larger amounts, they get right to the two forms of the unit rate. Then, the teachers decided to use the context of Hershey bars and marshmallows to ask which mixture was more chocolatey:

> They [the teachers] discussed giving each group a different set of marshmallows and chocolate. Tori suggested having students calculate the unit rate both ways (Hershey bars per marshmallow and marshmallows per Hershey bar)…They discussed possible student misconceptions [when comparing their mixtures]: looking at the difference between the number of Hershey bars and marshmallows, or just looking at the amount of chocolate. (RL3, Planning Meeting 1, Lines 289-293)

The ratio of Hershey bars to marshmallows (or marshmallows to Hershey bars) is a critical aspect when comparing which mixture is chocolatier. While the teachers did not explicitly identify the ratio as a critical aspect when making comparisons, they did identify what is not critical: the amount of chocolate and the difference between the number of Hershey bars and marshmallows. In order to make a comparison between two ratios, it is useful to have either common denominators or common numerators. The teachers decided that they would have students calculate both forms of the unit rate for their mixture and then use those unit rates to make comparisons between the mixtures. Students could use either unit rate (i.e., marshmallows per Hershey bar or Hershey bars per marshmallow) to make a comparison, recognizing that the comparison between mixtures depends upon the units within the context:

> Lynn said, "I think that's a great bridge into making sense of your unit rate because you could look at it and say, well which one would make more sense to look at, your chocolate per marshmallow ratio or your marshmallow per chocolate [ratio]? Either way is fine, but your larger number would be your chocolate per marshmallow, but if it was marshmallow per chocolate, it would be the smaller number you would go for." Tori replied, "Right, because you want the least number of marshmallows [per Hershey bar]." Lynn said, "Yeah, and we could talk about the meaning behind unit rates." (RL3, Planning Meeting 1, Lines 289-303)

While Lynn was not explicit, she described how the magnitude of the ratio was a critical aspect of comparing unit rates. Critical features of the critical aspect magnitude that students would need to discern were less than, equal to, or greater than. Developing the ability to make comparisons between mixtures depends upon understanding the ratios in context. Hence, it is necessary for learners to see the relationship between the critical aspects magnitude and context.

At Planning Meeting Two, I asked the teachers what the critical aspects were when comparing unit rates:

> Lynn said, “That there’s no incorrect way to do it, but it all depends on what you need to be looking for.” I asked what they [the teachers] wanted students to get out of this particular part of the lesson [comparing mixtures]. Tori said, “That it’s more chocolate per marshmallow, but less marshmallow per chocolate…and in the end, it’s still the same group of people that have the best [more chocolatey].” (RL3, Planning Meeting 2, Lines 97-102)

The teachers articulated that students need to be able to interpret the unit rate in context in order to make a correct comparison between the unit rates. They did not, however, explicitly identify the specific critical aspects that students needed to discern in order to be able to make comparisons between unit rates. Critical aspects are the specific aspects of an object of learning that the learners must discern in order to develop the intended capability. The teachers wrote in the lesson plan that the first critical aspect (CA1) was “The quantities being compared in a unit rate can be interchanged (i.e., dollars per box or boxes per dollar” (RL3 Lesson Plan). This is the critical aspect Form (CA1), which has two critical features: (1) the amount of Quantity A per one unit of Quantity B, and (2) the amount of Quantity B per one unit of Quantity A. As Lynn said, “there’s no incorrect way to do it,” and so it was critical that students see both forms of the unit rate as correct rates.

The teachers identified two other critical aspects, Context (CA2) and Magnitude (CA3). In articulating these critical aspects, the teachers attended to the relationships between each of these critical aspects, respectively, and Form (CA1):

- In some situations, either form of unit rate is appropriate, but in other situations, one might be more appropriate than the other (relationship between CA1 and CA2)

- When you use one form of the unit rate, you want more (e.g., boxes per dollar), but when you use the other form, you want less (e.g., dollars per box). This still results in the same "optimal" ratio (relationship between CA1 and CA3). (RL3 Lesson Plan)

It is the relationships amongst the critical aspects, in addition to the critical aspects themselves, which students must discern in order to make comparisons between unit rates. In terms of variation theory (Marton, 2015), discerning the relationship between critical aspects is the pattern of variation fusion, although the teachers did not use this language themselves. Through contrast, students might discern that there are two forms of a unit rate for a given ratio and that these forms are different. Again, through contrast, one might discern that there are different contexts in which different unit rates are useful. The recognition, however, that in some contexts, one form of the unit rate is preferred or more practical, while in other contexts either form is practical, is an instance of fusion, as one has discerned the relationship between two or more critical aspects. Table 4.11 summarizes the critical aspects that the teachers identified for this lesson and their associated critical features.

Table 4.11

Identified Critical Aspects for Research Lesson Three

Critical Aspect	Critical Features
(CA1) Form	Either form of the unit rate (i.e., boxes per dollar, dollars per box)
(CA2) Context	Contexts in which either form of the unit rate is useful, and contexts in which one form of the unit rate is preferred
(CA3) Magnitude	Less than, equal to, greater than

The teachers may have assumed that the students had already discerned the critical aspects Context and Magnitude, and hence it was the relationship between those two critical aspects and Form that their students needed to discern. The teachers were in the process of developing their knowledge of variation theory. Their understanding of critical aspects, at this moment in time, was what they wanted students to see or discern in order to develop the capability of understanding and using unit rates. What the teachers wanted the students to discern was the relationships between certain critical aspects and, hence, this was what they called the critical aspect. As this was the third learning study cycle, I was trying to remove myself from the discussion as much as possible. The teachers' lack of a clear articulation of the critical aspects likely lies in their unfamiliarity with variation theory and the fact that the teachers themselves did not yet discern variation theory as a critical aspect of learning study. As the only source of knowledge in the room on variation theory, my desire to remove myself from the discussion impeded teachers' abilities to apply variation theory to the planning of the lesson through the identification of critical aspects. Since much of my work with the teachers had focused on structured exercises and LGEs, the examples became the focus of their planning, rather than the framework of variation theory. What the framework seeks to establish for learning study, however, is intentional purpose for the design. Variation theory posits that it is necessary to first address what is to be learned, in relation to the particular learners, rather than how it is to be learned (Marton et al., 2004).

For this lesson, the teachers stated that they wanted students to learn (1) how to calculate a value for the unit rate through setting up proportions and cross-multiplying, (2) to interpret that value as a unit rate in context, (3) to determine the usefulness of the form of a unit rate in context, and (4) to compare unit rates. This was in contrast to the previous two research lessons that had singular goals. In planning this lesson, the teachers talked about *how* this lesson would be taught before clearly articulating *what* would be taught. The teachers talked about using manipulatives, grouping students, and particular examples. In Research Lesson Two, the teachers also began designing exercises before fully articulating the purpose. In that lesson, however, the design of the exercises provided a talking point from which the teachers clarified the critical aspects of the object of learning. The design of the exercises in Research Lesson Three did not serve to clarify the critical aspects of the object of learning, but rather extended the object of learning to include multiple goals related to unit rates. This may have been a consequence of this particular class of students as an RTI class. The teachers viewed RTI classes as support for regular mathematics classes, in which students are re-taught skills and concepts from class and provided with additional practice. Lynn taught this class of students only every other day. The RTI class schedule meant that the same content needed to be covered in half the time. Lynn, and the other teachers, had grown accustomed to addressing multiple objectives in a single lesson.

The teachers' focus on *how* the lesson would be taught, rather than *what* would be taught, impeded the development of a shared purpose and understanding of the lesson amongst the teachers. Robert, in particular, was concerned about how the teachers' goal of students' making sense of the context in order to compare unit rates would be met with the hot cocoa context that the teachers had chosen:

> Robert asked what was meant by more chocolaty? He said, "Obviously the one with more chocolate will be more chocolaty. Will [the students] understand chocolate compared to the marshmallow?"…Robert pointed out, "[In examples from previous lessons] The apple [juice concentrate] was compared to water, so that would make it more appley. So what makes this more chocolaty?"…Robert said in his mind, marshmallow would just make it more sweet, not take away chocolate flavor…Robert suggested switching it to milk, but [the other teachers] said they wanted the hands on aspect with the chocolate and marshmallow…Robert questioned whether students would understand what was being asked. He said, "I would want the one with five chocolates more than the others." [The

other teachers] eventually decided that they thought students would understand [what was meant by more chocolaty]. (RL3, Planning Meeting 2, Lines 83-95)

Robert was concerned about students' ability to make sense of the context, a critical aspect of comparing unit rates. This context, unlike others they had used, did not include a quantity, such as water, that would diminish the flavor of the mixture. Robert cautioned that, like him, students might interpret "more chocolaty" to mean a greater amount of chocolate within the mixture, rather than a greater ratio of chocolate to marshmallow within the mixture. He suggested changing the quantity of marshmallows to milk. The other teachers said that that would not allow for the use of manipulatives like they wanted. Thus, while Robert appeared to be pressing the group about *what* students would have the opportunity to learn through this context, the other teachers were focused on *how* the content would be delivered (i.e., using manipulatives). After the planning for the lesson shifted in this direction, Robert came to the planning meetings but did not contribute more comments to the planning process.

The teachers' insistence that the students have access to marshmallows and chocolate as manipulatives was related to the teachers' notions of student success. The teachers believed that the manipulatives would be of interest to students and allow for visualization of the ratios being discussed. The teachers' beliefs about how students could be successful strongly influenced both the design and the enactment of this lesson, as I discuss in the section on the implementation of the structured exercises. For instance, the teachers decided during their planning meetings that students should set up a proportion and cross multiply for every exercise.

> Lynn said, "If they're calculator usage students, they're going to have to set [up proportions to calculate unit rates], and I think that's the only way it's going to work for a lot of these kids…I want to envision that whatever result we're getting from this next

> section is because they're setting [the proportion] up correctly and cross multiplying." Tori said that [at the time of the research lesson, it] would be about the tenth day of [students] working on setting up and solving proportions. Lynn said she couldn't see a lot of students just scaling up from the unit rate, and Tori agreed. Lynn said, "They have to have something that will work for every [exercise] and this [setting up a proportion and cross-multiplying] will." (RL3, Planning Meeting 1, Lines 270-278)

Lynn described the desire for her students to have a singular method of working with rates "that will work for every [exercise]." This was in contrast to Research Lesson Two, in which the teachers desired that students should develop flexible approaches to writing an equation of a line, based on the given information. It appeared that what the teachers meant by success, in this case, was obtaining a correct answer and being comfortable with knowing how to approach the exercise. Lynn's desire for students to have a single, universal approach to working with rates stemmed from students' documented academic accommodations. For instance, most of the students in Lynn's classes were granted calculator use as an accommodation for documented disabilities. Students' calculator use influenced the teachers' decision to set up a proportion and cross-multiply for each exercise:

> Lynn said, "I just want to preface this with the fact that most of our students in this class are using calculators, so for them, they may not even get the idea that it's easier to just multiply by nine…For students who use calculators, every number is the same to them…Things we expect them to see, they may not, because of their number sense." (RL3, Planning Meeting 1, Lines 180-188)

Lynn's assertion that "For students who use calculators, every number is the same to them," seemed to directly stem from her initial concerns about unit rates and students' difficulties in interpreting the value of the unit rate. Rather than see values within the exercise as quantities with a value and a unit in context, from Lynn's perspective, students only saw numbers which were largely devoid of meaning. There were a number of potential approaches to addressing students' struggle with number sense. The approach at this school, within this group of teachers, was to move forward in the curriculum, at the same pace as the regular mathematics classes, with additional support within RTI classes. The support often consisted of re-teaching and diminishing the cognitive load, which could be seen in the teachers' desire for students to set up a proportion and cross-multiply for every instance of rates that students encountered. To support students' in understanding the meaning of the numerical values, the teachers stressed labels on the numbers to represent the units. For instance, when the teachers wrote out the proportions they wanted students to set up for the exercises, they included the labels m and h on the numbers as shorthand for marshmallows and Hershey bars, respectively (see Figure 4.6). While this was done with the intention of supporting students in seeing the numerical values within the proportion as quantities, and hence the resulting rate as a quantity, it had the potential to inadvertently contribute to students' development of alternative conceptions of variables.

For most of the remaining five planning meetings, the teachers discussed a pre- and post-assessment to understand students' abilities to compare unit rates, the grouping of students, the layout of the SMART Board file they would be using, the layout of papers and worksheets for the students, and the flow of the implementation of the lesson. The teachers made only minor changes to the exercises that had been designed in the first two planning meetings. Possibly due to Robert's suggestion, Lynn changed the hot cocoa context for the examples to hot fudge. She hoped that a visualization of melting the two ingredients together would better support students in making sense of what "more chocolatey" meant in the context. The final set of structured exercises is given in Table 4.12. For each exercise within the set, I identified the critical aspects that were addressed within that exercise, and I identified aspects that varied and remained invariant between exercises.

Table 4.12

Summary of Patterns of Variation and Critical Aspects Addressed in Set of Structured Exercises

	Exercise	Critical Aspects to be Addressed[a]
Warm-up	Given a certain amount of marshmallows (Hershey bars), how many Hershey bars (marshmallows) would you need?	None. Reviews procedure for setting up a proportion and cross multiplying.
	Vary: unknown quantity (Hershey bars and marshmallows) Invariant: ratio of marshmallows to Hershey bars	
Exercise 1	Calculate how many marshmallows are needed for one Hershey bar (with manipulatives)	(CA1) The quantities being compared in a unit rate can be interchanged (i.e. dollars per box or boxes per dollar).
	Vary (as compared to the Warm-up): the calculation of a unit rate Invariant: ratio of marshmallows to Hershey bars	
Exercise 2	Calculate how many Hershey bars are needed for one marshmallow (with manipulatives)	(CA1) The quantities being compared in a unit rate can be interchanged (i.e. dollars per box or boxes per dollar).
	Vary (as compared to Example 1): form of the unit rate Invariant: ratio of marshmallows to Hershey bars	
Exercise 3	Given a new recipe, calculate both forms of the unit rate. (Teacher-led)	(CA1) The quantities being compared in a unit rate can be interchanged (i.e. dollars per box or boxes per dollar).
	Vary: ratio of marshmallows to Hershey bars Invariant: the calculation of unit rates	
Exercise 4	Given a new recipe, calculate both forms of the unit rate. (With a partner)	(CA1) The quantities being compared in a unit rate can be interchanged (i.e. dollars per box or boxes per dollar).
	Vary: ratio of marshmallows to Hershey bars (across the partnerships) Invariant: the calculation of unit rates	

	Exercise	Critical Aspects to be Addressed[a]
Exercise 5	Which new recipe is more chocolatey? Vary: the magnitude of the unit rates (calculated in Exercise 4); the form of the unit rate Invariant: Which recipe is more chocolatey	When you use one form of the unit rate (CA1), you want more, but when you use the other form, you want less (CA3).
Exercise 6	Which store is offering the cheapest bag of marshmallows? Vary: the context, the ratios, the magnitude of the unit rates; the form of the unit rate Invariant: the calculation of unit rates	In some situations (CA2), either form of unit rate (CA1) is appropriate, but in other situations, one form might be more appropriate than the other.

[a] Critical aspects, as written by the teachers. The critical aspect CA1 is Form, CA2 is Context, and CA3 is Magnitude.

The first two examples served as a review of setting up proportions and cross multiplying in order to calculate an unknown value. This was the procedure that the teachers intended that the students would use for the remaining exercises to calculate the unit rate. The teachers intended that the two forms of the unit rate would be contrasted between Exercises 1 and 2 (shown in Table 4.12) and again within Exercises 3 and 4 when partnerships were given a new recipe and were instructed to calculate both forms of the unit rate. In Exercise 5, students would be asked to compare their individual recipes using the unit rates to determine which recipe was more chocolaty. This addressed the relationship between Form (CA1) and Magnitude (CA3).

Students would need to make sense of the unit rates in context in order to decide whether more or less of the unit rate would result in a more chocolaty mixture (i.e., more Hershey bars per marshmallow or less marshmallows per Hershey bar). Exercise 6 varied the context, the ratios given within the problem statement, the magnitudes of the unit rates, and depending on how the students approached the problem, possibly the form of the unit rate used. The teachers intended that the process used to compare the rates would remain invariant. In particular, the teachers intended that the context of Exercise 6 would contrast with the context of Exercise 5. In the last exercise, the teachers believed that the dollars per bag of marshmallows was the preferred or more practical unit rate because one could not purchase a partial bag of marshmallows. In the previous context (Exercise 5), however, either form of the unit rate could be considered practical since marshmallows and Hershey bars could be broken into partial amounts. The teachers intended that this contrast would allow students to discern the relationship between the critical aspects of Context (CA2) and Form (CA1). Like in Exercise 5, the students would need to interpret their calculated unit rates in context and determine whether to choose the greater or lesser rate. From Exercises 5 and 6, the teachers intended that students would discern the relationship between the critical aspects of Form (CA1) and Magnitude (CA3).

LGEs were not suggested or discussed during the planning of this lesson. In the next section, I discuss the implementation of this set of structured exercises and the enacted object or learning, or what the students had the opportunity to discern through the implementation.

Implementation of Structured Exercises and LGEs

In this section, I describe the implementation of the set of structured exercises from my perspective. The description is based on my field notes and my analysis of the videotape. The teachers' perspectives on the implementation of the set of structured exercises are discussed in the section on the evaluation of the lesson.

The lesson was 45 minutes long. The first thirteen minutes of the lesson were spent on the Warm-up Exercises, which were meant to be a review of setting up proportions and cross-multiplying to find an unknown value. Lynn guided the students to the method of setting up a proportion and cross-multiplying:

Lynn says, “Any ideas of how you would figure out how many Hershey bars you would need if you have 36 marshmallows? What do you think? Bailey?” Bailey says, “Eight divided by two.” Lynn: “Eight divided by two? Okay. That would give you some idea, yeah. And then you’d have to do another step after that. Ok, what do you think? Luke?” Luke says to cross-multiply. Lynn replies, “You want to cross multiply. Ok. If you’re going to cross-multiply, what do you need to set up?” (RL3 Field Notes, Lines 23-27)

Lynn initially opened the space of learning by asking the students for their ideas about how they would solve the problem. While she acknowledged that Bailey’s solution could be a correct pathway, she decided not to pursue Bailey’s line of reasoning as a possible strategy. The other step after Bailey’s suggestion of dividing eight by two is never explicated. Rather, Lynn chose to ask for another student suggestion. Luke said to cross-multiply. It was this solution strategy that Lynn decided to pursue. The teachers had decided that the only solution strategy they wanted students to use was setting up a proportion and cross-multiplying. Bailey’s suggestion, however, may have led directly into a discussion about unit rates, as dividing eight marshmallows by two Hershey bars would have resulted in the unit rate of four marshmallows per Hershey bar.

For Exercises 1 and 2, Lynn distributed baggies containing two Hershey’s fun-size chocolate bars and eight mini marshmallows to each student. Lynn asked the students to figure out how many marshmallows they would need if they only had one Hershey bar using the items in their baggie. Students were to work with their assigned partner. After a couple of minutes, Lynn elicited from the students that they would need four marshmallows. She asked Gabriel and Holden to explain what they did:

Holden says, “We divided it in half…but I took one Hershey bar away, and half of the marshmallows away.” Lynn is nodding and says, “Beautiful!...You took your recipe and divided it in half, right? Perfect! So let's set this up as a proportion on our papers, just to prove that we got the right answer, okay? So what would I set up?" (RL3 Field Notes, Lines 160-164)

Holden's solution method of dividing the number of Hershey bars in half and the number of marshmallows in half stemmed from the recognition that there were two Hershey bars and that they wanted half of that amount. Thus, they also needed half of the amount of marshmallows. Numerically, this could be shown like so: $\frac{8\text{ marshmallows}}{2\text{ Hershey bars}}\frac{\div 2}{\div 2} = \frac{4\text{ marshmallows}}{1\text{ Hershey bars}}$. This is different from Bailey's suggestion at the outset of the lesson to divide eight marshmallows by two Hershey bars, which is the standard method for calculating a unit rate. Bailey's method shares the set of eight marshmallows equally between the set of two Hershey bars to arrive at an answer of four marshmallows per one Hershey bar. Holden's method resulted in the *amount* of marshmallows needed for one Hershey bar, obscuring that four was also the value of the unit rate. Holden's method had the potential to demonstrate that the rate of eight marshmallows per two Hershey bars was equivalent to the unit rate of four marshmallows per Hershey bar. Lynn, however, was focused on the goal of having students set up proportions and cross multiply to calculate the unit rate. The teachers had decided that these students should have one method of calculating unit rates and determining unknown values that would work across a number of exercises involving proportions.

Lynn then suggested to the class that they set up the proportion and cross-multiply to verify their result. One of the students suggested a correct proportion, and Lynn wrote on the SMART Board:

$$\frac{8\text{ m}}{2\text{ h}} = \frac{x\text{ m}}{1\text{ h}}$$

Lynn prompted students what to do next to solve the proportion and wrote out:

$$2x = 8$$

$$x = 4$$

Lynn said, “Four! And didn’t we figure that out when we did it? So four marshmallows, exactly” (RL3 Field Notes, Lines 180-181). The results were verified as being the same, and Lynn demonstrated that the procedure of setting up a proportion and cross multiplying resulted in a correct answer. Setting up a proportion and cross multiplying diminished an opportunity for students to connect the numeric or algebraic mathematical solution methods to their physical actions. While a procedure for solving for an unknown value using proportions was generalized, the connection between the physical and mathematical processes was lost.

Lynn then asked students to use their items to figure out how many Hershey bars would be needed if they only had one marshmallow (Exercise 2). When Lynn asked for a student to explain their solution method, the student had foregone using the manipulatives and instead set up a proportion. With the students’ input, Lynn went through the steps of solving the proportion to arrive at a value for the answer, “Okay, so you’re going to divide both sides by?...Eight. Beautiful. Now, what is two eighths in simplest form? One fourth, right? So that’s how Luke determined his answer” (RL3 Field Notes, Lines 214-216). Within this explanation, Lynn did not maintain the labels on the numerical values. There was no connection to the context about what the value eight represented, or the value two eighths, or the value one fourth. This was in contrast to the planning of this lesson in which the teachers intended to stress labels on numerical values. The teachers had identified the interpretation of calculated values within the context as a source of difficulty. Also, by this time in the lesson, Luke had discerned that an algebraic solution was the acceptable solution method, as he immediately used this approach, despite being told to solve the problem using the manipulatives.

Lynn did try to connect the result of one fourth to the manipulatives. Unlike Holden's solution method from the previous exercise, in which he took away half of the items in each group, Lynn described the process of sharing four marshmallows amongst a single candy bar. Because the candy bar was pre-scored into four sections, each marshmallow could be lined up with one of the pre-scored sections of the candy bar.

> …And then it's easy once we break our candy bars apart, right? Because each little piece (of the Hershey bar) matches up with one marshmallow. But what does each little piece [of the Hershey bar] represent? What does that one little piece represent out of a whole Hershey? It represents…? Bailey told me earlier. What does that one little piece represent out of the whole Hershey bar? One fourth, right? Make sense? (RL3 Field Notes, Lines 216-219)

As before, the connection between the physical and mathematical processes was not made clear. The fact that one fourth is a rate was also obscured. In the algebraic solution, the answer was left unlabeled as one fourth. In the physical solution, one fourth of the Hershey bar was obtained. This highlighted the numerical value of one fourth as an amount rather than a unit rate. Lastly, the contrast between the values obtained in Exercises 1 and 2, four and one fourth, respectively, was left implicit. There was no discussion about the two forms of the unit rate, nor were these values discussed as unit rates.

In the next part of the lesson, each pair of students was to be given a new recipe to calculate the two forms of the unit rate for. The teachers had decided to do an example as a group first with a new recipe for Mrs. E and Mrs. G (Exercise 3). The new recipe in this example was six marshmallows to two Hershey bars. Lynn elicited student suggestions for determining the number of marshmallows per Hershey bars and the number of Hershey bars per marshmallow.

As for Exercise 1, Holden suggested dividing the number of Hershey bars and marshmallows in half:

> Holden says, “Divide them in half. Three marshmallows [indecipherable] and one Hershey.” Lynn replied, “Awesome. So Holden said I’m going to just divide these by two, and I’m going to get three marshmallows for every one Hershey [bar]. Make sense? Okay? So, three marshmallows per Hershey [bar]. We’ve got our answer for that one. Now we could have done what? If we couldn’t see that we could have divided them both by two, what could we have done to figure this out? What have we been doing all along? What always works for us? What do we do? Can anybody tell me? We set up? A proportion, right? And cross multiply? Could we have done that? (RL3 Field Notes, Lines 235-241)

Despite the teachers’ desire for students to use a proportion and cross multiply, many of the students continued to use alternative methods of solving the exercises. Lynn acknowledged Holden’s solution method, but pushed the class to use the solution method of setting up a proportion and cross multiplying. Holden also gave his answer as amounts (“Three marshmallows…and one Hershey”), while Lynn said Holden’s answer as a unit rate (“three marshmallows per Hershey bar”). Lynn did not make it explicit, however, that the values that students were calculating were rates, or in particular, unit rates.

Lynn asked the students for suggestions to determine how many Hershey bars per marshmallow. Gabriel recognized a reciprocal relationship between the two forms of the unit rate, despite the relationship earlier being left implicit:

> *Gabriel*: Three over one.
>
> *Lynn*: What’s that?

Gabriel: Three over one.

Lynn: Three over one. You're getting close, but three over one would be…(as she writes 3/1 on the SMART Board).

Gabriel: Or one over three.

Lynn: One over th- Wow! I love it! Awesome, Gabriel! So one third of a Hershey is for every marshmallow, okay? Now let's set this one up and cross multiply because I want people to see how he's getting one third, okay? Do you want to explain how you got it or do you want us to set it up?

Gabriel: I just went with marshmallows and Hershey [bars].

Lynn: So you just alternated it and switched it around. One over three. Beautiful. I love it! Okay, let's set it up and see how we got one third solving a proportion, alright?

(RL3 Field Notes, Lines 243-250)

Gabriel initially said it would be three over one Hershey bars per marshmallow. After Lynn wrote $\frac{3}{1}$ on the SMART Board, Gabriel said "Or one over three." This was indicative of Gabriel's discernment of the reciprocity of the two forms of the unit rates. Lynn suggested setting up a proportion to show the rest of the class how to obtain $\frac{1}{3}$ as the result and asked Gabriel if he wanted to explain. Gabriel said, "I just went with marshmallows and Hershey [bars]." Gabriel had not set up a proportion. Gabriel determined the number of marshmallows and the number of Hershey bars and understood that these values would be written as a fraction. It seemed as though Gabriel exhibited the difficulties the teachers wanted to address at the outset of this lesson; he did not see the resulting value as a unit rate in context. It was not clear for Gabriel whether he should have $\frac{3}{1}$ or $\frac{1}{3}$. Lynn praised Gabriel because he just "alternated it and switched it around," but the opportunity to contrast these two quantities and their meanings in context was not taken up by Lynn.

Lynn asked students to figure out the number of Hershey bars per marshmallow and the number of marshmallows per Hershey bar for their own recipes. Lynn walked over to Dylan and Bailey. Bailey was having difficulty setting the proportion up correctly. Lynn guided her through the procedure:

> Lynn says, "…So your recipe is twelve marshmallows for five Hershey [bars], so you want to set that up…Okay. Equals…So now you want to label the other fraction marshmallows over Hershey [bars]. Okay? And then you have to think about, if you're looking for how many marshmallows *per* Hershey [bar], that's one, so you are going to put a one where?" Bailey says, "Here," and points to the numerator of the second fraction. Lynn replies, "It says marshmallows *per* Hershey [bar]." Bailey says, "Oh, right here," and writes a one in the denominator of the second fraction…Lynn slides over to her right in front of Dylan's desk and says, "So what did you end up getting for this one then?" Dylan says, "One third." Lynn looks at Dylan's work:
>
> $$\frac{12\text{ m}}{5} = \frac{1\text{ m}}{x\text{ h}}$$
>
> $$\frac{12x}{12} = \frac{5}{12}$$
>
> $$x = 0.4167$$
>
> Lynn says, "So if you set it up twelve over five, one marshmallow. So how many Hershey [bars] per marshmallow?...So why don't we just leave it as a fraction?...So can

you write this $\left[\frac{5}{12}\right]$ in your answer blank for that [Hershey bars per marshmallow]. (RL3 Field Notes, Lines 294-305)

Bailey's confusion with where to place the one in the proportion, and Dylan's confusion about his result indicated students' continued struggle to make sense of the numbers as quantities, with units, within a context. While Dylan set up a correct proportion to calculate the number of Hershey bars per marshmallow, Lynn talked with Bailey, his partner, about setting up a proportion to calculate the number of marshmallows per Hershey bar. It was not clear whether Dylan's set up of the proportion to calculate this value was intentional, or if this was the same error Bailey made (i.e., putting one in the wrong location). Lynn interpreted the value Dylan obtained as Hershey bars per marshmallow, told him to leave it in fraction rather than decimal form, and told him where to write his answer on his worksheet. It was Lynn who did the cognitive work of understanding the meaning of the calculated value in context, and it was not clear from this exchange what Dylan or Bailey understood about the values obtained.

Lynn then asked two pairs of students to write their calculated values for the unit rates on the board (see Figure 4.8). While writing the unit rates as proper fractions may have aided students in comparing two marshmallows per Hershey bar with $2\frac{2}{5}$ marshmallows per Hershey bar, the reciprocity between the two forms of the unit rates was obscured. Lynn had earlier asked Dylan to write his result, 0.4167, as the fraction $\frac{5}{12}$ rather than a decimal. This meant that students were left to compare the unit rates $\frac{1}{2}$ Hershey bars per marshmallow with $\frac{5}{12}$ Hershey bars per marshmallow. Students' weak number sense, as Lynn described during the planning of this lesson, potentially made this comparison difficult. Students would need to either get common denominators to compare these values, or rewrite both values in their decimal form.

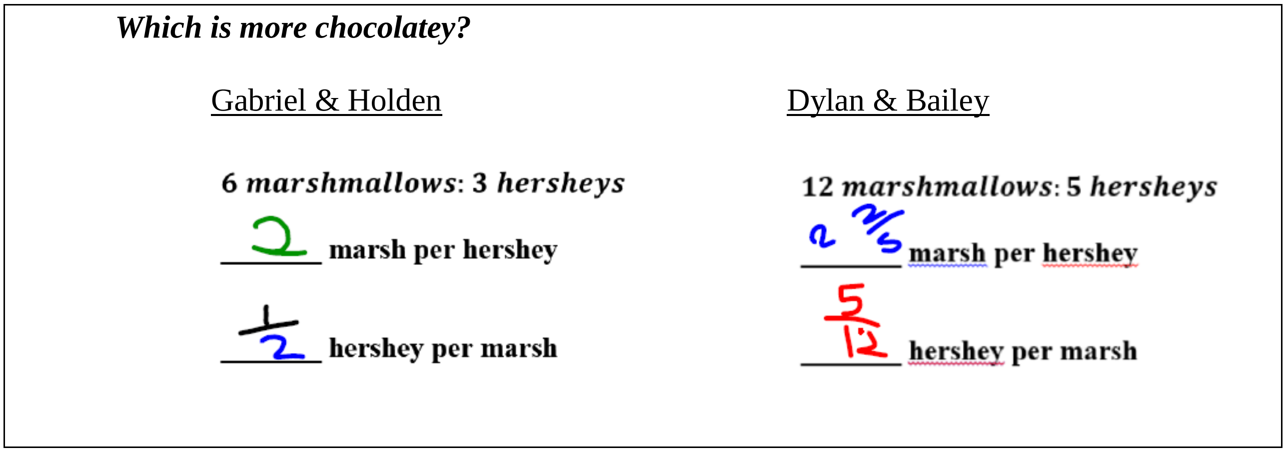


Figure 4.8. The SMART Board display from Research Lesson 3 for students to compare two of the fudge recipes.

Lynn wanted the students to compare these two recipes and determine which recipe was more chocolatey. Three of the students thought that Gabriel and Holden had the recipe that was more chocolatey, while the other three students thought that Dylan and Bailey had the recipe that was more chocolaty. Holden explained why he thought his and Gabriel's recipe was more chocolatey:

> It would be ours because there are less marshmallows, but I get how it could have been Dylan's and Bailey's because they got more Hershey [bars] than we do. They got more marsh [sic] per Hershey [bar]. We've got more Hershey [bars] per marsh [sic] than they do so it could have been us. (RL3 Field Notes, Lines 343-346)

Holden tried to contrast the chocolaty-ness of the two recipes. His language made it unclear whether he was attending to the calculated unit rates or the amounts given in the recipes. Holden said, "It would be ours because there are less marshmallows." Two marshmallows per Hershey bar is a lesser rate than $2\frac{2}{5}$ marshmallows per Hershey, but six marshmallows, as given in Gabriel's and Holden's recipe, was also less than the twelve marshmallows given in Dylan's and Bailey's recipe. Holden then said, "But I get how it could have been Dylan's and Bailey's because they got more Hershey [bars] than we do." Holden referred to the amount of Hershey bars here to justify why Dylan and Bailey may have had the more chocolaty recipe. This was the misconception that Robert had anticipated would arise from this context – that a greater amount of chocolate would indicate more chocolaty, rather than a greater unit rate of chocolate per marshmallow. In Holden's next two statements, he explicitly referred to and correctly compared the unit rates between the two recipes, but he was still uncertain about what was meant by "more chocolaty," as indicated by his language that "it could have been us."

Unlike the previous two research lessons, where the teachers designed the exercises for explicit contrast, in this lesson, explicit contrast was designed but not enacted. Despite this, some students recognized the reciprocity between the two forms of the unit rate. The teachers' (except for Robert) insistence that students set up proportions and cross multiply in order to calculate unit rates did not appear to support students in making sense of the numerical values in the context. While students included labels on the values as they were initially written into the proportion, the labels were not carried through the process of cross multiplication. The result was inserted back into the original proportion to obtain the correct label, but this result was an amount (of either Hershey bars or marshmallows), rather than a rate. Since the result that students obtained from solving a proportion was an amount, this possibly added to students' conflation of the amounts of a quantity and the unit rates.

Because of time constraints, Lynn did not get to the exercise in which the context was changed. Thus, students did not have the opportunity to discern that in some contexts one form of the unit rate could be more practical than the other. Within the chocolate fudge recipe, either Hershey bars per marshmallow or marshmallows per Hershey bar was useful in determining which mixture was more chocolaty. The teachers' goal that was most emphasized was setting up a proportion and cross multiplying in order to calculate a unit rate. In the next section, I discuss the evaluation of the research lesson from the teachers' perspectives.

Evaluation of the Lesson

Immediately after the research lesson was taught, the team of teachers and I met to discuss our observations of the lesson. We met again two days later during our regularly scheduled meeting time to finish our discussion. The discussion immediately following the research lesson followed a modified protocol borrowed from Japanese Lesson Study in which the instructor first shares her reflections, the team members discuss what they observed, and then a general discussion follows focused on how specific elements of the design of the lesson promoted student learning or not. At our second meeting, I recapped the big ideas that were discussed during the first meeting and invited the teachers to share other thoughts on those ideas or other ideas that they thought about or noticed.

One of the first things that Lynn commented on was the pacing of the lesson. The Warm-up examples took much longer than she had anticipated. Lynn also said that she felt as though the transition from Exercise 2 to Exercise 3 was not clear; there was not enough emphasis on unit rates and the procedure for calculating a unit rate:

> And I felt like I didn't do enough of a discussion about it [unit rate] on our example [Example 3] because Holden did it in his head. So I didn't want to be like ok, now we're going to write all this down, so I thought, oh, we'll just do it for the second one [calculating the other form of the unit rate], but I don't think it was enough to just do it for that second example because then Gabriel got it in his head! So then I was like, okay…you know, so there wasn't a lot of talk about the whole per, like the unit, and how to set that up [in a proportion], and I think it just didn't give them enough help to do those

> on their own… Like I didn't do enough work with it once we got it, our results…So when they were off to do their own, they were kind of fumbling. (RL3 Debrief, Lines 33-40)

In Examples 1 and 2, the students were asked to find how many marshmallows (or Hershey bars) were needed *for one* Hershey bar (or marshmallow). The word "per" was not used in the examples. In Examples 3 and 4, students were given a new recipe. Below that were two blank lines that students were expected to fill-in. One was for the number of marshmallows per Hershey bar, and the other was for the number of Hershey bars per marshmallow. Lynn pointed out that "there wasn't a lot of talk about…per…the unit, and how to set that up [in a proportion]." Lynn suggested that students' difficulties stemmed from not understanding how to set up a correct proportion, specifically because of students' incomplete understanding of the word per as implying for one of the quantity. Tori echoed Lynn's interpretation of the students' difficulties by suggesting that the implicit one in the word "per" needed to be emphasized as a part of the discussion, and that students needed to be directed to write down the one within their proportions:

> I think we needed to add in, 'So what does per Hershey [bar] mean? How many Hershey [bars] do I have? Oh, one? So let's have everyone write down that it's per *one* Hershey [bar]. Let's put a one in [the proportion]. Per marshmallow? Or that's really just per *one* marshmallow. (RL3 Debrief, Lines 47-49)

For this particular group of students, the teachers emphasized developing students' understanding of a generalized procedure for calculating a unit rate. For the use of any method, students need to understand the word *per* as meaning for one of a particular quantity. For the teachers, *per* is beginning to emerge as a critical aspect. For Lynn and Tori, the importance of understanding the concept of *per* as meaning one of a particular quantity is so students can correctly substitute a value of one into a proportion in order to calculate a value for the amount of the other quantity associated with one of the quantity of interest.

The teachers thought that this lesson may have been improved had they omitted the Warm-up Exercises and instead started with Exercise 1:

> Tori said, "If the object of the lesson was really to work on comparing [unit rates], I know [the Warm-up Exercises] were nice because it kind of led them [students] into [using] proportions, but I feel like it took so much time with that, that maybe we needed to start right off with giving them the Hershey bars [and marshmallows]…and get them right into how much [marshmallows] would I need for one Hershey bar?" Robert said, "And setting up the proportions right there with that would have been the warm-up into proportions. To get to the focus of comparing the unit rates." (RL3 Debrief, Lines 60-65)

Tori connected her suggestion to omit the Warm-up Exercises to the purpose of the lesson, and the indirect object of learning – comparing unit rates. The time gained from omitting the Warm-up Exercises would have given an opportunity for an explicit discussion of *per* and its connection to the procedure for calculating a unit rate using a proportion and cross multiplying. Robert pointed out that the teachers could have used the method of setting up a proportion and cross multiplying to calculate the unit rate without the Warm-up Exercises. The teachers recognized that this change would have supported the purpose of the lesson (i.e., comparing unit rates), and the goal for students to use proportions and cross multiplying to calculate unit rates.

The teachers considered their choice of numbers in the exercises. Lynn suggested that their choice of numbers in Exercise 3 was a possible factor for the method for calculating a unit rate not being clear to the students:

> [The numbers in Example 3] were easily divisible by two. So the whole method of finding the unit rate [through setting up a proportion and cross multiplying] got lost, I think because of the easy numbers we used. So I think that either we use all easy numbers in that situation or we use a difficult one for that one so that I could have modeled how to set [the proportion] up. (RL3 Debrief, Lines 98-101)

Alternative methods were more efficient than setting up a proportion and cross multiplying for Exercise 3. This conflicted with the teachers' goal of having students set up a proportion and cross multiply to calculate the unit rate. Lynn suggested they eschew the single method approach and choose easier numbers (i.e., numbers that are divisible by a common factor) so that an alternative method can be both modeled for and used by students, or that the numbers chosen for Exercise 3 be ones in which the unit rate cannot obviously be determined through dividing out a common factor from the amounts of Hershey bars and marshmallows.

On the other hand, values for the number of Hershey bars and marshmallows in the recipes that resulted in fractional values for the unit rates caused unanticipated challenges when comparing the unit rates to determine which recipe was more chocolaty. Shannon said, "One half of a Hershey per marshmallow compared to five twelfths of a Hershey per marshmallow – like that's *hard* for students to *really* understand unless you have common denominators" (RL3 Debrief, lines 76-78). Lynn was clear during the planning of this lesson that these particular students struggled with number sense. The difficulty, then, that students would have comparing $\frac{1}{2}$ Hershey per marshmallow to $\frac{5}{12}$ Hershey per marshmallow, was compounded with the students' still fragile understanding of unit rate. An additional step was required in the procedure to get common denominators in order to make the comparison between unit rates. Shannon suggested that numbers that resulted in whole unit rates may have been better for supporting students' understanding of the comparison of fractional unit rates: “I wonder if whole unit rates would help lead them into then comparing fractional unit rates” (RL3 Debrief, Lines 69-70). Through the process of redesign brought up in the discussion, the significance of the numbers chosen in the exercises and students’ potential ways of reasoning about them were brought to the fore of the teachers’ awareness. The suggested design of the exercises identified Value as another critical aspect of unit rate with two critical features: (1) Whole and (2) Fractional. Shannon recognized that it was important for students to discern a unit rate as a mathematical object, rather than a computation, and wondered whether whole unit rates would support students in seeing the calculated value as a quantity with mathematical and practical meaning. This, however, would have been in conflict with the teachers’ goal of having students use a proportion and cross multiply to calculate unit rates. When the resulting unit rate is a whole number, other methods are likely more efficient.

Shannon also questioned the students’ use and understanding of the Hershey bars and marshmallows as manipulatives with the fractional unit rate:

> I don’t know that they really got the whole one quarter of a Hershey bar because as soon as they [students] broke it [Hershey bar] up, they were like here’s one piece, two piece, three pieces, four pieces, you know? So then did they see one piece of Hershey to one marshmallow? (RL3 Debrief, Lines 135-138)

Shannon wondered whether students understood that that value $\frac{1}{4}$ referred to $\frac{1}{4}$ of a Hershey bar, or one piece of the Hershey bar out of four total pieces in a whole Hershey bar, for *each* marshmallow. Thus, Shannon identified that an additional critical aspect necessary to discern in order to understand fractional unit rates is the Whole of the quantity. Alternatively, Tori suspected that perhaps Bailey saw the one in $\frac{1}{4}$ as referring to the one marshmallow, while the four referred to the four pieces in the Hershey bar. These possible alternative conceptions suggested by the teachers, indicated they noticed that the students did not relate the physical manipulatives with the abstract mathematics correctly. Coupled with the students' struggles with number sense, the procedure for calculating and comparing unit rates likely continued to appear opaque to many of the students.

Robert suggested that students might have benefitted from more opportunity within the lesson to use the manipulatives, without the initial example.

> I thought it would have been nice to have two groups working at a table where they can sort of spread out and lean over, and think about how to divvy up [the Hershey bars and the marshmallows]…A little less [teacher] talking, and more, ok let's get in to them [students] experimenting [with their different recipes]. Just more of them working together I think gives richer conversations. (RL3 Debrief, Lines 103-124)

Rather than model how to calculate the unit rate using a proportion and cross multiplying, Robert thought that using manipulatives could support students in discovering procedures for calculating unit rates. Robert also referred to "richer conversations" that could arise from students' collaborative work. Lynn echoed the idea that students seem to learn through talking about their ideas:

> In [one of my other classes], they all picked the wrong one [recipe]. They picked the one that was least chocolaty…But it's neat. Sometimes, when you ask them to explain why they picked it, that's when they realize that they're wrong. Because then they look at it and they go, I picked it because, and when they start to explain it, they're like oh wait! That's what happened with Holden. Holden started explaining it [his answer] and then realized, oh! I gotta change my answer. I think that's good. (RL3 Debrief, Lines 106-111)

The idea that students learn through the process of communicating their thinking appeared to be emerging for Lynn. She recognized students' abilities to learn through giving an explanation for their answer in both the research lesson and one of her other classes. Robert, however, appeared to have previously recognized the opportunity to learn through the communication that collaborative work can provide. He suggested modifying the lesson in a way that allowed for more student exploration and collaboration at a table setting with the intent that this would generate more student conversations. In Research Lesson 2, which was taught by Robert, student provided explanations were an important component of generating explicit contrast between solution methods.

The teachers' redesign of the exercises in the lesson within the evaluation meetings indicated some ways in which the teachers developed their knowledge about designing and enacting structured exercises. Within the evaluation of the lesson, the teachers identified additional critical aspects that pertained to the concept of unit rate that had not emerged during the planning of the lesson. These critical aspects were *per* and Value. Having observed how their initial design played out in implementation, the teachers articulated ways in which the exercises could be redesigned and alternative enactments that would bring the identified critical aspects more to the fore of the learners' awareness. The teachers identified missed opportunities for explicit contrast between the forms of the unit rate that could be attended to through a redesign of the exercises. Student talk began to emerge as a strategy for designing for explicit contrast. Robert suggested that extending students' collaborative work with the manipulatives would generate richer class discussions, and Lynn recognized the learning opportunities inherent in students' formulation of mathematical explanations. I call this a design strategy because the teachers need to plan for opportunities for students' talk and collaboration. The teachers' discussion about the quantities chosen for the exercises, however, and thinking about how students' might respond to particular values of those quantities, suggested that an important aspect of the design is anticipating what students might notice and planning ways to respond to that student thinking during enactment. The teachers began to recognize that there are opportunities for contrast and fusion that can be brought about through student talk, but there remained work to do in order to make that contrast and fusion explicit during the course of the enactment.

Summary of Research Lesson Three

The object of learning for Research Lesson Three was comparing unit rates. The teachers identified three critical aspects that pertained to comparing unit rates, relative to their students: (CA1) Form, (CA2) Context, and (CA3) Magnitude. In articulating the critical aspects, the teachers focused on the relationships between the critical aspects that they wanted students to discern. Discerning the relationship between critical aspects supposes that the critical aspects, themselves, have already been discerned by the learners, and the relationships between the critical aspects fuses them together in important ways. According to Marton (2015), this is the pattern of variation of fusion.

As in Research Lesson Two, the design of the examples led to the articulation of the indirect object of learning and the critical aspects. This approach in Research Lesson Two focused the teachers' design on patterns of variance and invariance that would allow for contrast and student discernment of the critical aspects. In this research lesson, however, this approach led to an expansion of the goals of the lesson. The first two research lessons had singular goals (i.e., writing equivalent expressions using the order of operations and writing an equation of a line, respectively). This research lesson had three goals: (1) the calculation of a value of the unit rate through setting up a proportion and cross multiplying, (2) the interpretation of that calculated value as a unit rate in context, and (3) the comparison of two unit rates. In contrast to the previous two research lessons, when planning this lesson, the teachers attended more to *how* students would learn, rather than *what* students would learn. This influenced the design of the structured exercises because the teachers sought a context that would allow for students' use of manipulatives, rather than focusing on whether exercises within that context would provide opportunities for students to discern the critical aspects of the object of learning and the relationships among those aspects. In Research Lesson One, the teachers considered how many exercises were enough to allow for student discernment of the critical aspects. In this research lesson, the teachers placed a greater emphasis on considering the number of opportunities students would have to practice the skill of setting up a proportion and cross multiplying to calculate a unit rate.

The teachers believed that students in an RTI class would be most successful if they had a single method for approaching these problems and all exercises involving proportions. The teachers chose to stress the method of setting up proportions and cross multiplying to calculate unit rates. Within the implementation of the lesson, this manifested itself through Lynn's take-up and explanation of solutions that involved this approach and an insertion of this approach to verify the results of students who used other approaches. The strategies for enacting explicit contrast that were generated in the previous two research lessons were absent from this enactment.

Through the process of evaluating the lesson, the teachers developed their knowledge about the design and enactment of structured exercises. The teachers discussed potential redesigns of the set of structured exercises and alternative methods of implementation. After observing the implementation of their initial design, the teachers thought more deeply about and identified additional critical aspects of comparing unit rates. The teachers continued to talk about the principle of Explicit Contrast in both the design and enactment and considered ways in which explicit contrast could bring the critical aspects more to the fore of learners' awareness. One way of designing for explicit contrast, and possibly fusion, included planning for opportunities for student talk. Lynn recognized the value in students providing explanations, while Robert suggested that more collaborative student work with the manipulatives could generate discussions that brought out contrast and fusion amongst the critical aspects. While student talk began to emerge as a design strategy for generating explicit contrast, strategies for eliciting student thinking and making the contrast and fusion explicit to the whole group during the enactment was yet to be considered.

This research lesson was heavily influenced by the teachers' notions of student success. A model for RTI classes had previously been established at this school, and the teachers largely felt compelled to adhere to it. This particular model of RTI meant that a number of learning goals needed to be covered in a single lesson with a focus on developing students' facility with a specific procedure. This largely stemmed from the pressure that the teachers' felt from high-stakes testing and their belief that having a single procedure that worked for all exercises involving proportional relationships would be less confusing for their students. Because of the teachers' emphasis on setting up a proportion and cross multiplying, the teachers spent little time in the planning of the lesson anticipating what students would do and how they would think about the exercises. Without an anticipatory set, Lynn had no strategies in the enactment for responding to student thinking and approaches rather than to redirect to the one approach that had been discussed in the planning. The planning of this lesson did not go into the level of detail that would provide Lynn with strategies or means of responding to variations in student thinking. Despite a number of difficulties in the planning and implementation, in the evaluation of the lesson, the teachers identified additional critical aspects, revisions to the design of the set of structured exercises to provide a greater focus on the critical aspects, student talk as a strategy for designing for explicit contrast, and the necessity to make both contrast and fusion explicit in the implementation of the lesson.

Research Lesson Four

Shannon taught Research Lesson Four on February 5, 2016, in her eighth period Math 8 class. Prior to the research lesson, the teachers met for six 40-minute planning meetings and one longer meeting (approximately two hours) during a staff day. They met again, for approximately 40 minutes, on the day of the research lesson to review their plan. Because of when the teachers wanted to teach this research lesson, Shannon knew the topic had to be on either exponents or scientific notation. Lynn suggested they teach the lesson on adding and subtracting numbers written in scientific notation. The teachers discussed what students tended to struggle with when adding and subtracting numbers in scientific notation:

> Tori asked where they [students] tend to struggle most – converting to standard form to add and subtract [see Figure 4.9, Method 1] or trying to do it "the quick way" [see Figure 4.9, Method 2]…Shannon said, "They remember that you have to have the same power in order to add and subtract, but then after that, it's awful! They don't know how to move their decimal, and half of them try to use shortcuts that sometimes work, or they add and move the decimal the wrong way, but then change the power. And they *really* struggle when it's a negative exponent. *Really, really* struggle." Robert said, "I would try to avoid rules at all cost…because they don't understand what they're doing." Tori replied,

"Sometimes I don't! It's like, I need to rethink about this." Robert said, "Exactly! So to just memorize rules, they usually just forget those. I think focusing on the exponent rules, those rules, is a little bit better." (RL4, Planning Meeting 1, Lines 13-30)

The teachers talked about two methods they had used for adding and subtracting numbers written in scientific notation (see Figure 4.9). For Method 1, each number is converted into standard form and then the numbers are added. In Method 2, one of the addends is rewritten to have the same power of ten as the other addend. The coefficients of the addends are added together and the sum is multiplied by the appropriate power of ten. For the sake of efficiency, the teachers wanted to avoid Method 1, but students had difficulty in the past with Method 2. The teachers believed that the students did not understand the mathematical basis for the procedural "rules" or "shortcuts" they were attempting to apply. Robert suggested that the basis for this work should stem from the exponent rules, specifically the product rule for exponents, $a^m \cdot a^n = a^{m+n}$.

Add: $3.9 \times 10^3 + 1.4 \times 10^5$

Method 1	Method 2
$3.9 \times 10^3 = 3{,}900$	$1.4 \times 10^5 = 140 \times 10^3$
$1.4 \times 10^5 = 140{,}000$	$3.9 \times 10^3 + 140 \times 10^3 = 143.9 \times 10^3$
$3{,}900 + 140{,}000 = 143{,}900$	$143.9 \times 10^3 = 1.439 \times 10^5$
$143{,}900 = 1.429 \times 10^5$	

Figure 4.9. The two methods the teachers discussed for adding numbers written in scientific notation.

In contrast to Research Lessons Two and Three, the teachers spent a significant amount of time identifying the object of learning and its critical aspects before designing the structured exercises and LGEs for the lesson. For the first time, the teachers incorporated an LGE into the in-class portion of the research lesson. Research Lesson One included an LGE that was intended to be given to students who finished early, but was not given due to time constraints. Research

Lesson Two included an LGE for homework. Research Lesson Three did not include an LGE at all. The teachers decided that the topic of the lesson would be writing equivalent expressions using a different power of ten. The teachers had never taught a lesson specifically on this concept. Rather, it was incorporated as a part of the procedure for adding and subtracting numbers in scientific notation. The teachers hoped that a day spent on this concept and the development of students' conceptual understanding of the product rule for exponents and equivalency would mean they would need less time for the later lesson on adding and subtracting numbers in scientific notation. The teachers thought carefully about how this lesson would fit into the larger unit on exponents, and extensively discussed a number of ideas related to students' understanding of exponents. These ideas included: (1) the movement of the decimal place in an expression when multiplying and dividing by ten, (2) negative exponents, and (3) multiplying exponential expressions. While the discussion of related ideas in Research Lesson Three served to expand the goals of the research lesson, in this lesson, as in Research Lessons One and Two, the teachers returned to a singular goal. The goal of this lesson was "to have students write equivalent expressions using various powers of ten" (RL4, Lesson Plan).

Purpose of the Lesson

As described previously, in the beginning phase of a learning study, the research team (1) selects a topic for study, (2) identifies a tentative object of learning, (3) diagnoses students' learning difficulties, and (4) confirms the object of learning and its critical aspects. In aggregate, these first four steps of the learning study cycle articulate and clarify the purpose of the lesson. For this lesson, the teachers identified the direct object of learning as applying the product rule for exponents. They identified the indirect object of learning as rewriting numbers in scientific notation using powers of ten. The teachers identified three critical aspects: (CA1) the value of the power of ten, (CA2) the value of the coefficient, and (CA3) the maintenance of equivalency. As in Research Lesson Three, the teachers had difficulty articulating the critical aspects they attended to. CA1 and CA2 were mostly identified through the process of designing the set of structured exercises. In the initial planning, the maintenance of equality (CA3) dominated the teachers' discussion, and the teachers articulated the critical features of the maintenance of equivalency (CA3) they wanted students to discern.

The teachers spent a significant amount of time discussing the scope of the unit on exponents in its entirety. The teachers decided they were not going to have students simplify expressions such as $3x^5 \cdot 8x^4$ in this lesson. The teachers discussed the student errors they had noticed in the past:

> Robert said that two years ago, the teachers had used a bunch [of examples similar to $3x^5 \cdot 8x^4$]. "In my head, I thought we were preparing them for Algebra, and then they [students] just started confusing bases. [Students] were able to evaluate some very complicated expressions, but then when it came down to $10^8 \times 10^3$, it was 100^{11} every single time. And I started to think about why were they doing that?" Shannon said, "They weren't recognizing a number as a base." Robert replied, "We never focused in on what's the difference between a base and a coefficient. We needed a lesson where we were just comparing the two." (RL4, Planning Meeting 1, Lines 193-199)

Robert had begun to apply the principle of Explicit Contrast to other lessons as he thought about them. Robert interpreted students' struggles as being related to students not seeing the contrast between the base of an exponential expression and a coefficient. Whereas a coefficient is always a number, the base of an exponential expression can be a number or a variable. Shannon said, "They weren't recognizing a number as a base." From Shannon's perspective, students had discerned the difference between a base and a coefficient, but students had not generalized the meaning of base to include numbers. The students seemed to discern that coefficients are numbers and when multiplying exponential expressions, the numbers are multiplied. The teachers identified the base of an exponential expression as a critical aspect. While not a focus of this lesson, the teachers realized that students needed to discern the critical features numerical values and variables of the critical aspect base of an exponential expression. Further, students needed to discern that when exponential expressions with the same base are multiplied, the base stays the same, whether it is a numerical value or a variable. The teachers recognized that students' discernment of the base of an exponential expression, its associated critical features and its relationship to the critical aspect operation was a prerequisite for this lesson. The teachers' recognition of the base of an exponential expression as a critical aspect was indicative of the teachers' increasing awareness of critical aspects for objects of learning attended to in other lessons. In this lesson, the product rule for exponents would need to be applied in a novel way for students to determine the missing exponent in order to write an equivalent expression with a different base. For instance, the teachers had included the following exercise on their pre-test: $10^8 \cdot 10^{\square} = 10^{12}$. Students needed to recognize that eight and the exponent of the second expression must sum to twelve.

Most of the teachers' discussions were about the maintenance of equivalency (CA3). The teachers wanted students to discern that when rewriting exponential expressions with a different power, the expressions remain equivalent. The teachers articulated a number of critical features of this aspect that would allow for the equivalency to be maintained between expressions:

- when multiplying exponential expressions with the same base, the base of the exponent stays the same and the exponents are added together (i.e. the product law of exponents),
- the associative property of multiplication, and
- to expand a number written in scientific notation into standard form, a negative power of ten moves the decimal point of the coefficient to the left; a positive power of ten moves the decimal point of the coefficient to the right (i.e., multiplication by a power of ten).

Thinking about their end goal, Shannon said they wanted students to be able to write a number with at least three different powers of ten. Lynn said, "And understand that they all represent the same number, because I think that [equivalency] is something that they [students] don't get at all" (RL4, Planning Meeting 1, Lines 45-46). The teachers had not been explicit about a common theme or aspect amongst the four research lessons. Each of the four research lessons, however, attended to the aspect of equivalency. Unlike Research Lessons Two and Three, where the aspect of equivalency was left implicit and not identified as critical, the equivalency of expressions arose to the fore of the teachers' awareness in Research Lessons One and Four as a critical aspect.

Tori and Shannon recalled a method Robert had shown them for rewriting an expression using a different power of ten, which highlighted the aspect of equivalency through the application of the product law for exponents:

> Tori went up to the board to explain. She has 3×10^4 written. She says, "I want it to be ten to the third, what does this (*pointing to the* □ *symbol*) have to be?" Now she has written $3 \times 10^{\square} \times 10^3$. Tori continued, "Well that has to be a one. One and three is four.

What is three times ten to the first? 30." Shannon says, "Kids get stuck on that because when we introduce scientific notation in *standard* form, [the coefficient has to] be between one and…less than ten, and so then [students] are like, you can't write it like that!"…Robert replies, "And then in the same task we'll say 'in scientific notation' (*doing air quotes with his hands*), when what we mean is standard scientific notation. So it's like, which one of those [expressions] is scientific notation?" Tori says, "This [30×10^3] is not in scientific notation, but it's the understanding that these [3×10^4 and 30×10^3] are still equivalent." (RL4, Planning Meeting 1, Lines 62-70)

The teachers realized that they needed to make explicit contrast between a number written in scientific notation and an equivalent number written using a different power of ten. The teachers wanted students to understand that the numbers were equivalent, despite the difference in the way they appeared.

In order to generalize the aspect of equivalency to include exponential expressions that appear differently, Tori thought it was important to be explicit about applying the associative property and maintaining equivalency from one step of the procedure to the next. For Tori, the maintenance of equivalency was critical, and the associative property of multiplication was a critical feature of the aspect maintenance of equivalency that students needed to discern. Tori went to the board and showed the following example: Given 3×10^4, rewrite it with 10^{-3} as the power of ten. Tori wrote out the following steps:

$$3 \times 10^4 \quad \text{Want } 10^{-3}$$

(1) $3 \cdot (10^{?} \cdot 10^{-3})$

(2) $3 \cdot (10^{7} \cdot 10^{-3})$

(3) $(3 \cdot 10^{7}) \cdot 10^{-3}$

In Step (1), 10^4 is rewritten as a product of two powers of ten, in which one factor was the target power of ten, 10^{-3}. In Step (2), the value of the unknown exponent (?) in Step (1) was computed to be seven by applying the product law of exponents. In Step (3), the associative property of multiplication was applied, maintaining equivalency and grouping the factors that make up the coefficient of the rewritten expression. Because one would need to divide by ten seven times (10^{-7}) to get from 10^4 to 10^{-3}, the expression must also be multiplied by ten seven times (10^7) to maintain equivalency.

One of the critical features of the critical aspect maintenance of equivalency that the teachers articulated was multiplication by a power of ten. To expand a number written in scientific notation into standard form, a negative power of ten moves the decimal point of the coefficient to the left; a positive power of ten moves the decimal point of the coefficient to the right. Multiplication by a power of ten requires discerning the relationship between the value of the power of ten (CA1) and the value of the coefficient (CA2). The teachers discussed the importance of students developing a conceptual understanding of multiplying and dividing by ten and the resulting movement of the decimal place. "Lynn said, "[Students] need to understand which direction they are going to be moving their decimal based on the exponents."…Shannon said, "Which really goes back to making the number [coefficient] larger or smaller. Do they really understand that?"" (RL4, Planning Meeting 1, Lines 175-180). The teachers had explained to their students that a positive exponent makes the number [coefficient] larger, while a negative exponent makes the number [coefficient] smaller.

When designing examples for the lesson, the teachers considered asking the students how they would rewrite 3600×10^1 as 3.6×10^4. Tori and Lynn were concerned about students relying on a process without understanding:

[Lynn said,] "But is that going to lead to, well I have to move the decimal place three spots [places to the left] here [to get from 3600 to 3.6], so I just add three to the exponent [$10^{1+3} = 10^4$]? I'm afraid it will"…Tori replied, "They're [Students are] just going to model the one that's there without really thinking about why they're doing it." (RL4, Planning Meeting 4, Lines 287-292)

The teachers were concerned that the students would conjecture about a relationship within the example and develop a procedure for writing an expression with a different power of ten, but would not develop a conceptual understanding of why it was mathematically valid to do such a procedure. Understanding the movement of the decimal place in relation to multiplication, division, and place value was a priority for the teachers over a rule for the procedure. Hence, the teachers wanted students to discern the relationship between all three critical aspects: the value of the power of ten (CA1) and its relationship to the value of the coefficient (CA2) in terms of the movement of the decimal place, and the relationship of that movement to the maintenance of equivalency (CA3).

In the early stages of planning this lesson, the teachers had considered a lesson on adding and subtracting numbers written in scientific notation. The teachers identified the power of ten as a critical aspect of that object of learning (i.e., adding and subtracting numbers written in scientific notation). Because of this, the teachers chose to focus this lesson on writing numbers written in scientific notation with a different power of ten and carefully considered the critical aspects and features of that object of learning. The teachers identified three critical aspects: (CA1) the value of the power of ten, (CA2) the value of the coefficient, and (CA3) the maintenance of equivalency. Table 4.13 summarizes the identified critical aspects for Research Lesson Four and their associated critical features.

In the next section, I discuss how the identification of these critical aspects influenced the design of the structured exercises and LGEs in this lesson and, in turn, how the design of the structured exercises and LGEs provided opportunities for the teachers to clarify and refine the critical aspects, and identify additional critical features of the critical aspects.

Table 4.13

Identified Critical Aspects for Research Lesson Four

Critical Aspect	Critical Features
Value of the power of ten (CA1)	Integral values less than the given power of ten, greater than the given power of ten, negative, and zero
Value of the coefficient (CA2)	Multiples of powers of ten less than the initial coefficient; multiple of a power of ten greater than the coefficient
Maintenance of equivalency (CA3)	Product law of exponents, associative property of multiplication, expansion of a number written in scientific notation into standard form

Design of Structured Exercises and LGEs

This was the first research lesson in which the teachers chose to use an LGE within the in-class portion of the lesson. Robert suggested an LGE to generalize the critical aspect equivalency for numbers that appeared in different ways:

> Robert suggested an LGE where they ask: How many ways could you write 36,000? He anticipated that many students would jump right to scientific notation because that's what they have been doing [in class], but then could also go to the prime factorization, or writing it as "36 times this, 360 times this, extend to 3 times 12,000, so we could get them to generate lots and lots of examples from one expression." (RL4, Planning Meeting 3, Lines 69-74)

Robert said that he saw the purpose of this LGE as eliciting the idea of equivalent expressions: "Just getting out the idea that we can write numbers in *so many* different ways. We're going to focus on this way [written with a power of ten] today" (RL4, Planning Meeting 3, Lines 110-111). The LGE was meant to elicit students' concepts of equivalency, and would serve to begin the lesson on equivalent expressions written with a power of ten.

Three challenges emerged for the teachers as they designed the LGE to be used in the lesson: (1) the wording of the prompt, (2) students who struggle with the task, and (3) teachers' acting in-the-moment. The teachers wanted students to write down various numerical representations of 36,000. They anticipated that students would write 36,000 in scientific notation, and possibly in prime factorization, since those two topics were recently covered in class. I suggested that students might also write 36,000 as a sum of numbers. I asked the teachers what kinds of expressions they wanted students to generate:

> Shannon said that she wanted to stick to multiplication, and Tori and Robert agreed to avoid expressions such as 35,999 + 1. I asked, "So how can we ask [the question] to get what we want out?" Shannon said, "Couldn't you just say using multiplication?" Tori suggested, "Using multiplication, how many ways can you write the product 400?" I pointed out that by asking "How many ways?" the answer would be infinite. I suggested, "Using multiplication, write 36,000 in a different way." Robert added on, "Then another. Then another." (RL4, Planning Meeting 3, Lines 117-123)

The teachers realized the significance of the wording of the prompt. Asking the question, "How many ways?" motivated a numerical response, which the teachers did not want. Rather, the teachers wanted students to write down equivalent numerical expressions of the given number. The teachers decided that they wanted to restrict the range of variation by including the condition "using multiplication." Robert suggested asking the students to write 36,000 in a different way again, and again. This was the strategy of "Give An Example Of…(another and another)" (Bills et al., 2004) from a text we had used in the summer professional development session on provoking mathematical thinking and generalizing usefully in mathematics. The prompt to write an equivalent numerical expression for 36,000 two additional times challenged students to expand upon the range of permissible change for numerical representations equivalent to 36,000 that they initially considered.

In addition to the wording of the prompt, the teachers carefully considered which number to use in the prompt. Tori suggested, “Using multiplication, how many ways can you write the number 400?” (RL4, Planning Meeting 3, Lines 120-121). The teachers had discussed using the numbers 400, 17,000, and 36,000. The numbers 400 and 17,000 were considered to narrow the range of variation since they have fewer factors than 36,000. When the teachers could not decide on which number, I suggested thinking about which choice would help to drive the lesson forward. The teachers wanted the LGE to elicit some different powers of ten. Tori said, “So if [we choose] a smaller number, will these options [different powers of ten] come out? Will it be harder to get options if you constrain them [the options]?” (RL4, Planning Meeting 4, Lines 226-228). Tori recognized how opening up the example space, rather than constraining it, was important for the design of the lesson. Designing for greater variation in the generation of students’ examples created opportunities to move the lesson forward.

One of the teachers’ concerns with using LGEs was the variation in how their students would respond to an LGE. Robert said, “It’s tough to start. You don’t want to have a one group struggling and struggling the whole time, and another group that’s excited about it [the task] and ready to do it. You kind of want to bring them all together” (RL4, Planning Meeting 3, Lines 102-104). The teachers were challenged to consider how they might contend with the variation in their students' thinking that the LGE revealed. Tori suggested modeling a similar example for students prior to the LGE:

> My fear…is that the kids are going to come in, and they're going to write [36,000] as 3.6×10^4, and then just be like, why do I have to write it any other way? So maybe if you format it, like using one of those other numbers. Four hundred. I could write [400] as two times two times two hundred. I could write that as twenty times twenty. (RL4, Planning Meeting 4, Lines 216-220)

By modeling an example for the students prior to the LGE, Tori intended to spark students' thinking about the many equivalent ways that a number can be written. She also wanted to preemptively diminish the likelihood of a student giving up in the face of a struggle because they were unsure of how to proceed. Robert was more optimistic that students would rise to the challenge:

> I don't think they'd give up right in that moment. And in fact if we *don't* say the word scientific notation or allude to that at all, they [students] might be drawn to it, but it might open that [example space] up. They can multiply by anything. (RL4, Planning Meeting 4, Lines 222-224)

Robert thought that modeling an example prior to the LGE would restrict, rather than open, the example space. He also was not convinced that students would immediately consider writing 36,000 in scientific notation. Robert wanted to avoid influencing the examples that students' generated and allow for their creativity in the example space.

As the teacher of this lesson, Shannon needed a strategy for addressing those students who were struggling to write 36,000 in a different way. I suggested that she could begin to ask those students who *had* written 36,000 in a numerically equivalent way to write that representation on the board. By making the example space visible to the whole class, students could be attuned to notice dimensions of variation that they had not recalled or considered previously and generate new examples based on tweaking a dimension of variation, perhaps with a different value within the range of permissible change. For instance, if a student displayed their example of 36×1000, that might spur another student to consider the example 360×100, or other variations. Thus, one strategy that I suggested to teachers for addressing the challenges associated with variation in student thinking when using LGEs was to make that student thinking visible for whole class consideration.

After students' generation of examples, the teachers wanted to focus in on the examples that had been written with a power of ten as a factor to drive the lesson forward. Considering which examples students might generate and the teacher's potential response emerged as a design principle for LGEs. I will call this design principle Attention to Generation and Response. The teachers were confident that 3.6×10^4 would emerge as an example because of the students' previous work with scientific notation. The teachers hoped to elicit other variations of 36,000 written with powers of ten. Because the teachers could not be certain of which examples the students would generate, it would be necessary for Shannon to respond in-the-moment to students' thinking. The need to notice, process, and respond in-the-moment presented a challenge to the teachers. The teachers tried to alleviate the pressure on Shannon to act in-the-moment by building a possible list of expressions equivalent to 36,000, using powers of ten:

$$
\begin{aligned}
&360{,}000 \times 10^{-1} \\
&36{,}000 \times 10^{0} \\
&3{,}600 \times 10^{1} \\
&360 \times 10^{2} \\
&36 \times 10^{3} \\
&3.6 \times 10^{4} \\
&0.36 \times 10^{5}
\end{aligned}
$$

While the teachers did not anticipate that the students would generate this list in its entirety, they intended on building this list from the expressions that students *did* generate. A question that emerged for the teachers was: How much is enough? How many of the examples on this list did students need to generate in order to drive the lesson forward? This was reminiscent of the teachers' consideration of how many examples were enough in Research Lesson One. The teachers decided that three of the expressions would be enough to move forward and that those expressions need not necessarily be written with an exponent. For instance, a student might generate the example 360×100. The teachers decided that they could use this and ask students how they would write the expression with an exponent. From there, the teachers intended that Shannon would ask students if they could write 36,000 with the various powers of ten that had yet to be generated. Tori suggested to Shannon that she have the completed list written out and accessible during the lesson "so that she didn't have to think too much about it in the midst of [teaching]" (RL4, Planning Meeting 4, Line 271). Having the list of equivalent expressions written with various powers of ten available would allow more of Shannon's attention to be directed toward the examples that students *did* generate and deciding her next moves for getting students to generate the remainder of the list. Hence, the design principle of Attending to Generation and Response for LGEs that emerged for the teachers included teachers' attentiveness toward what examples students might generate, potential teacher moves to respond to those examples, and the creation of supportive tools (e.g., a list of equivalent expressions written with various powers of ten) to support the teacher in acting in-the-moment.

Despite the challenges of incorporating LGEs that the teachers identified, the teachers recognized the contrast that student generation of examples could reveal. The teachers considered a set of student generated examples that could arise and planned their response (see

Figure 4.10). After rewriting expressions that were multiplied by a power of ten with an exponent, the teachers intended to ask the students: “What is the same? What is changing?” The teachers wanted students to notice that the base of ten remained invariant, while the value of the coefficient (CA2) of the expression changed as the power of ten (CA1) changed. By holding the value of the expression invariant (at 36,000) and the base (of ten) invariant, the teachers designed an opportunity for students to discern the value of the coefficient and the value of the power of ten as critical aspects. The teachers designed for explicit contrast between the examples generated by students and planned a verbalization strategy for the implementation, asking, “What is the same? What is changing?” to draw students’ attention to the patterns of variation within the set of examples.

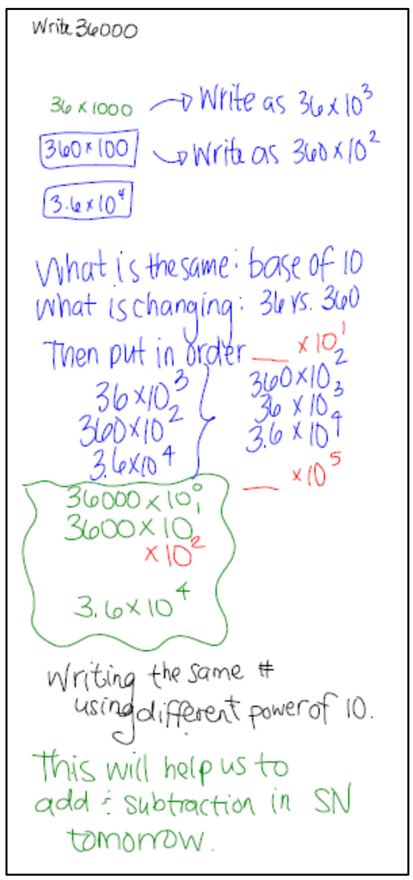


Figure 4.10. The record of teachers’ thinking as they anticipated students’ generated examples for an LGE and the teacher’s planned response.

The teachers then considered a set of structured exercises that would develop students' understanding of the procedure for rewriting a number given in scientific notation with a different power of ten. The teachers had identified the value of the power of ten as a critical aspect (CA1). The teachers wanted students to discern four critical features of the critical aspect, value of the power of ten: integral values (1) less than the given power of ten, (2) greater than the given power of ten, (3) negative, and (4) zero. As in Research Lesson One, the teachers considered how many exercises were enough. The teachers decided that it wasn't necessary to rewrite each power of ten in the sequence they had generated (i.e., from 10^{-1} to 10^{5}). The teachers chose four exercises that aligned with the four critical features that they wanted students to discern. Figure 4.11 shows the first set of four exercises that the teachers designed. The teachers chose to hold the given expression, 3.6×10^{4}, as invariant. They chose this number to extend from the LGE and because students would be rewriting numbers written in scientific notation the next lesson, in order to add and subtract them. The first three exercises had target powers of ten that were less than the given power of ten. When decomposing 10^{4} into factors of powers of ten, this would result in the unknown factor being a positive power of ten. Within these first three exercises, the first exercise had a positive target power of ten, the second had a target power of ten to the zero, and the third had a negative target power of ten. In the fourth exercise, the target power of ten was greater than the given power of ten. This was a reversal of the third exercise, where the target power of ten was negative and a positive power of ten was the unknown. In the fourth exercise, the target power of ten was positive, and a negative power of ten would need to be determined. For each of these exercises, the teachers planned to draw a box around 10^{4} as the given power of ten and draw a box around its decomposition into factors of powers of ten directly below it (as shown in Figure 4.11). The teachers intended that this would draw explicit attention to the contrast in how the value 10^4 was written while also generalizing the critical aspect maintenance of equivalency across representations via the critical feature the product law of exponents.

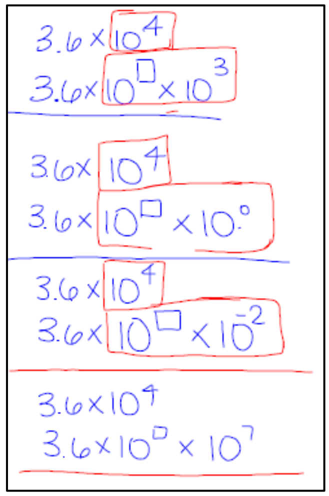


Figure 4.11. The first iteration of the set of structured exercises in Research Lesson Four.

The second iteration of the set of structured exercises was a set of six exercises. This set included three exercises (numbered one through three) that were intended to be teacher led and three exercises (numbered four through six) intended for students to complete individually (see Figure 4.12). In Exercises four through six, the value of the given number was changed to 2.5×10^5. In Exercise 4, the target power of ten was less than the given power of ten. In Exercise 5, the target power of ten was greater than the given power of ten. This structure paralleled Exercises 1 and 3, respectively. Due to time constraints, the teachers cut the second exercise from the first iteration that included a target power of 10^0. They instead included a target power of 10^0 within Exercise 6. The teachers considered a negative power of ten and a zero power of ten as special cases; the students would rarely be asked to rewrite numbers with a negative power or zero power of ten at this grade level. The teachers chose to include them as exercises to expand students' notions of the range of permissible change for the target power of ten. The teachers chose to make the last exercise, Exercise 6, the special case where the target power of ten is 10^0 because they felt comfortable cutting it from the lesson, if needed, due to time constraints.

1.) How would you write 3.6×10^4 so it has a power of 10^3?
2.) How would you write 3.6×10^4 so it has a power of 10^{-2}?
3.) How would you write 3.6×10^4 so it has a power of 10^7

4.) How would you write 2.5×10^5 so it has a power of 10^2?
5.) How would you write 2.5×10^5 so it has a power of 10^8?
6.) How would you write 2.5×10^5 so it has a power of 10^0?

Figure 4.12. The second iteration of the set of structured exercises in Research Lesson Four.

The teachers decided that they would go through the first three exercises in two phases (see Table 4.14). In the first phase, the students would only be decomposing the given power of ten into factors of powers of ten, one of which was the target power of ten. This would focus students' attention on the power of ten as a critical aspect (CA1). In the second phase, students would expand the coefficient of each of the four exercises, leaving the target power of ten invariant. This would draw explicit contrast between the original coefficient of the expression, 3.6, and the coefficient of the expression rewritten with the target power of ten. This was intended to allow an opportunity for students to discern the value of the coefficient (CA2) as a critical aspect with two critical features: (1) a multiple of a power of ten less than the initial coefficient, and (2) a multiple of a power of ten greater than the coefficient. Within both phases, the teachers wanted to generalize the critical aspect maintenance of equivalency across representation. The teachers intended on being explicit about how equivalency was being maintained between representations through the symbolic use of parentheses or boxing in and verbal explanations.

Lastly, the teachers created a second set of structured exercises to give to students as a homework assignment (see Figure 4.13). This homework assignment provided the opportunity to fuse the three critical aspects discussed above: the value of the power of ten (CA1), the value of the coefficient (CA2), and the maintenance of equivalency (CA3). The value of the expressions within each row would remain invariant (maintenance of equivalency). The first column would be the set of expressions written with a power of ten one less than the given power of ten. The third column would be the set of expressions written with a power of ten one more than the given power of ten. The arrangement of this structured exercise provided an opportunity for students to look "With and Across the Grain," a strategy from the text used in the summer professional development sessions (Bills et al., 2004), to fuse the relationship between the value of the coefficient and the value of the power of ten as both changed. Table 4.15 summarizes the patterns of variation and critical aspects addressed across both sets of structured exercises.

Table 4.14

Intended Phases of Completion and Maintenance of Equivalency for the Set of Structured Exercises

Exercises	Written Expressions	Maintenance of Equivalency
Phase One		
Exercise 1: How would you write 3.6×10^4 so that it has a power of 10^3?	$3.6 \times (10^4)$ $3.6 \times (10^1 \times 10^3)$	Product law of exponents –
Exercise 2: How would you write 3.6×10^4 so that it has a power of 10^{-2}?	$3.6 \times (10^4)$ $3.6 \times (10^6 \times 10^{-2})$	Decomposition of the given power of ten
Exercise 3: How would you write 3.6×10^4 so that it has a power of 10^7?	$3.6 \times (10^4)$ $3.6 \times (10^{-3} \times 10^7)$	
Phase Two		
Exercise 1: How would you write 3.6×10^4 so that it has a power of 10^3?	$(3.6 \times 10^1) \times 10^3$ 36×10^3	Associative property of multiplication
Exercise 2: How would you write 3.6×10^4 so that it has a power of 10^{-2}?	$(3.6 \times 10^6) \times 10^{-2}$ 3600000×10^{-2}	Expansion of the coefficient into standard form
Exercise 3: How would you write 3.6×10^4 so that it has a power of 10^7?	$(3.6 \times 10^{-3}) \times 10^7$ 0.0036×10^7	

The teachers designed an LGE and two sets of structured exercises to use in Research Lesson Four. This was the first lesson in which the teachers chose to use an LGE as a part of the in-class lesson. As in Research Lessons Two and Three, there was an interplay between the design of the lesson and the identification of critical aspects and features. The object of learning was rewriting numbers in scientific notation using powers of ten. The teachers identified three critical aspects for the object of learning: (CA1) the value of the power of ten, (CA2) the value of the coefficient, and (CA3) the maintenance of equivalency. The teachers drew on suggested classroom activities from the text discussed during the summer professional development sessions (Bills et al., 2004) for the design of the LGE and the set of structured exercises assigned for homework. The LGE used the strategy of "Give an Example Of…(another and another)" to elicit equivalent expressions written with various powers of ten. Attending to Generation and Response emerged as a design principle for LGEs, as the teachers carefully anticipated the examples the students might generate and the teachers' subsequent response to those examples. The teachers realized that the use of an LGE put increased pressure on the teacher to notice, process, and respond in-the-moment. The anticipation of generated examples and possible responses allowed for a number of strategies for the teacher to draw on. The teachers developed a list of expressions equivalent to 36,000 written with powers of ten as a tool to allow for more of the teacher's attention to be directed toward noticing and responding in-the-moment. The teachers continued to employ the design principle of Explicit Contrast through patterns of variance and invariance. The explicit contrast that emerged from the set of LGEs served to begin the lesson on rewriting a number written in scientific notation with a different power of ten. Within the first set of structured exercises, the teachers intended to explore the structure of the equivalent expressions that emerged from the LGE through a focus on the value of the power of ten (CA1) in Phase One, the value of the coefficient (CA2) in Phase Two, and the maintenance of equivalency (CA3) throughout. The design of the second set of structured exercises used the strategy of “With and Across the Grain” (Bills et al., 2004) to support the fusion of the three critical aspects.

Rewrite each of the given numbers using one less and one more power of ten.

	One power of ten less		One power of ten more
1.)	______	3.72×10^4	______
2.)	______	3.7×10^8	______
3.)	______	4×10^7	______
4.)	______	6.4×10^3	______
5.)	______	2.56×10^{-5}	______
6.)	______	1.78×10^{-9}	______

Figure 4.13. The second set of structured exercises that the teachers assigned for homework.

Table 4.15

Summary of Patterns of Variation and Critical Aspects Addressed in Set of In-class Structured Exercises

		Exercise	Critical Aspects to be Addressed
Structured Exercises Set 1 (In-class)	Exercise 1	How would you write 3.6×10^4 so that it has a power of 10^3?	Phase One: value of the power of ten (CA1), maintenance of equivalency (CA3)
	Exercise 2	How would you write 3.6×10^4 so that it has a power of 10^{-2}?	
	Exercise 3	How would you write 3.6×10^4 so that it has a power of 10^7?	Phase Two: value of the coefficient (CA2), maintenance of equivalency (CA3)
		Vary: the target power of ten Invariant: the value of the number; the critical aspect attended to (CA1 in Phase One, CA2 in Phase Two)	
	Exercise 4	How would you write 2.5×10^5 so that it has a power of 10^2?	Fusion of the critical aspects: value of the power of ten (CA1), value of the coefficient (CA2), maintenance of equality (CA3)
	Exercise 5	How would you write 2.5×10^5 so that it has a power of 10^8?	
	Exercise 6	How would you write 2.5×10^5 so that it has a power of 10^0?	
		Vary: the target power of ten Invariant: the value of the number	

		Exercise	Critical Aspects to be Addressed
Structured Exercises Set 2 (outside of class)		Rewrite each of the given numbers using one less and one more power of ten.	Fusion of the critical aspects: value of the power of ten (CA1), value of the coefficient (CA2), maintenance of equality (CA3)
	Exercise 1	3.72×10^{8}	
	Exercise 2	3.7×10^{4}	
	Exercise 3	4×10^{7}	
	Exercise 4	6.4×10^{3}	
	Exercise 5	2.56×10^{-5}	
	Exercise 6	1.78×10^{-9}	
		Vary: the value of the number Invariant: the target powers of ten (one less and one more)	

Implementation of Structured Exercises and LGEs

In this section, I describe the implementation of the set of structured exercises from my perspective. The description is based on my field notes and my analysis of the videotape of the research lesson. The teachers' perspectives on the implementation of the set of structured exercises are discussed in the section on the evaluation of the lesson.

As the teachers had planned, Shannon started the lesson with the LGE using the strategy "Give an example of…(another and another)" (Bills et al., 2004). Shannon's language hinted at scientific notation, and reframed the question to ask "how many" ways 36,000 could be rewritten using multiplication:

We've been doing scientific notation. Today we're going to focus on powers of ten and rewriting expressions using powers of ten. Can you come up with one way right now, to write 36,000, using multiplication. Using multiplication, can you write 36,000 in a

> different way?...So once you have one, can you write another way?...Always using multiplication. Once you have two can you come up with another way?...Can you come you come up with another way? So ultimately my question is, how many ways can you come up with [a number equivalent to 36,000 using multiplication]? (RL4 Field Notes, Lines 9-13)

Despite the teachers' discussion about the prompt for the LGE in the planning meetings, Shannon's language suggested potential examples that she was looking for and reframed the question. The teachers had discussed during the planning meetings their desire to leave the prompt open, at least initially, and narrow as needed, with questions such as, "Can you write [an example of an expression equivalent to 36,000] with an exponent?" Shannon began with the statement "We've been doing scientific notation," and told the students that the focus of the lesson was rewriting expressions using powers of ten. Her delivery of the prompt hinted that Shannon was looking for specific types of examples from the students, and possibly diminished the range of equivalent expressions that students may have generated without an implicit nudge in a particular direction. Shannon's language reframed the question from an action ("Using multiplication, write 36,000 in a different way"), to a yes or no question ("Can you write 36,000 in a different way?"), to a question that asked for a numerical answer ("How many ways can you write 36,000?"). The reframing of the prompt had the potential to disrupt the generation of examples that the teachers had intended in the design of the LGE.

At the time of the enactment, I was concerned, as an observer, that Shannon's statement about the focus of the lesson on powers of ten and rewriting expressions using powers of ten, might restrict the range of variation of examples that students generated. As students worked to generate examples, Shannon walked up and down the rows of desks, observing what students were writing down and she asked some students to write their expressions on the SMART Board. The following expressions were recorded on the SMART Board by Shannon's students:

$$36{,}000 \times 1$$
$$3.6 \times 10^4$$
$$36 \times 1000$$
$$18{,}000 \times 2$$
$$360 \times 10^2$$
$$90 \times 400$$

Shannon said, "Just because these are the ones written on the board doesn't mean that these are the only ones there are….I want to focus on a few of them" (RL4 Field Notes, Lines 22-23). Shannon moved the expressions $18{,}000 \times 2$ and 90×400 to the right side of the SMART Board, leaving the following expressions written on the right side of the SMART Board:

$$36{,}000 \times 1$$
$$3.6 \times 10^4$$
$$36 \times 1000$$
$$360 \times 10^2$$

Shannon asked, "How could I write this [$36{,}000 \times 1$] using a power of ten?" A student responded with "Thirty-six thousand times ten to the zero [$36{,}000 \times 10^0$]," (RL4 Field Notes, Line 25). Shannon similarly asked the class how 36×1000 could be written using a power of ten. Shannon then arranged the four examples in order from the least to greatest exponent so that the list of examples appeared as follows on the SMART Board:

$$36{,}000 \times 10^0$$

$$360 \times 10^2$$
$$36 \times 10^3$$
$$3.6 \times 10^4$$

Shannon intentionally left a gap between $36{,}000 \times 10^0$ and 360×10^2. Shannon then asked the class, "What's missing?" (RL4 Field Notes, Line 26). The students suggested that 3600×10^1 and 0.36×10^5 were missing. Shannon added these two expressions to the list of examples:

$$36{,}000 \times 10^0$$
$$3600 \times 10^1$$
$$360 \times 10^2$$
$$36 \times 10^3$$
$$3.6 \times 10^4$$
$$0.36 \times 10^5$$

Shannon quickly asked the following series of questions and a few students answered "yes" each time: "Could you keep going down?...Could you keep going up?...Do these all equal 36,000?" (RL4 Field Notes, Lines 27-28). As indicated by her hand gestures, Shannon's meaning of "going down" and "going up" was in the directional sense of the list of examples that had been generated on the SMART Board. "Going down" referred to increasing the value of the exponent while decreasing the value of the coefficient, while "going up" referred to decreasing the value of the exponent while increasing the value of the coefficient. It was not clear which aspects of the expressions students were attending to at this point in the lesson, or which portion of the students recognized the equivalency amongst the expressions. Shannon asked, "Which one [example] is in proper scientific notation?" (RL4 Field Notes, Line 28). A student responded that 3.6×10^4 was in proper scientific notation. Shannon's use of the word "proper" suggested that numbers written with a power of ten were also in scientific notation, albeit not in "proper" scientific notation. Shannon's question provided the opportunity for contrast between expressions written in scientific notation and those examples which were not, but there was no further discussion about the aspects that separated numbers written in scientific notation from those that were not. It was not clear if the learners had discerned what aspects of the representation constituted the difference between numbers written in scientific notation and equivalent numbers written with a power of ten.

Shannon copied the six examples to the next SMART Board slide, which read: "All of these are ***Equivalent Expressions***, they all equal 36,000!" Shannon asked the class, "What's staying the same for all of those numbers, besides the fact that they're equivalent to 36,000?" (RL4 Field Notes, Lines 29-30). One student said that the base was always ten, and another student said that each expression included multiplication. Shannon asked, "What's changing?" (RL4 Field Notes, Line 32). One student said that the exponent was changing, while another student said that the coefficient was changing. Shannon's questioning – What stayed the same? What's different? – provided an opportunity for contrast amongst the expressions, as the students articulated which aspects of the set of examples varied and which remained invariant. Shannon's enactment opened the space of learning and provided students with an opportunity to discern the value of the exponent and the value of the coefficient as critical aspects (CA1 and CA2, respectively).

Shannon passed out the students' note page (see Figure 4.14). The teachers had discussed enacting this set of structured exercises in two phases (see Table 4.14). Shannon implemented this set of structured exercises, however, in three phases: (1) the decomposition of 10^4 into a product of powers of ten, one of which was the target power of ten, (2) the regrouping of the expression using the associative property, and (3) the evaluation of the coefficient of the new expression written with the target power of ten. To begin Phase One of the implementation of the set of structured exercises, Shannon said:

> We are going to look at three point six times ten to the fourth power [3.6×10^4], which is thirty-six thousand in scientific notation. And the first thing we want to do is write this using a power of ten to the third…On the second line [of the students' note page], I've broken up the power of ten. One of them is what I want to change it to. I'm going to ask myself, "Using the law of exponents for multiplication, can I figure out what I would

write in that box?" So I know that whatever I put in that box, plus three, has to equal four. (RL4 Field Notes, Lines 33-37)

Shannon drew students' attention to the value of the power of ten (CA1) and the maintenance of equivalency (CA3) through the decomposition of the given power of ten into a product of powers of ten using the product law of exponents. She then did the same with Exercise 2 and Exercise 3. In Table 4.16, the portion of the exercise completed in each respective phase appears in red text.

Unit 8: Scientific Notation - Changing Powers Date: ____________

How can we write equivalent expressions in scientific notation?

1.) How would you write 3.6×10^4 so that it has a power of 10^3 ?

3.6×10^4

$3.6 \times 10^{\square} \times 10^3$

2.) How would you write 3.6×10^4 so that it has a power of 10^{-2} ?

3.6×10^4

$3.6 \times 10^{\square} \times 10^{-2}$

3.) How would you write 3.6×10^4 so that it has a power of 10^7 ?

3.6×10^4

$3.6 \times 10^{\square} \times 10^7$

Figure 4.14. An excerpt of the first three exercises from the students' note page from the first set of structured exercises in Research Lesson Four.

In Phase Two of the implementation of the set of structured exercises, Shannon drew students' attention to the maintenance of equivalency (CA3) via the critical feature the associative property of multiplication:

> Now let's go back to the first one [Exercise 1]. I'm going to rewrite this whole line, including the exponent, and I'm going to move my parentheses there [around 3.6×10^1]. Can I do that? Why can I do that mathematically? What [property] says I can move my parentheses and change the grouping? (RL4 Field Notes, Lines 43-45)

In Phase Two, Shannon drew students' attention toward the maintenance of equivalency (CA3) between the expressions by applying the associative property of multiplication. Unlike the critical aspects of the value of the power of ten (CA1) and the value of the coefficient (CA2), which needed to be discerned by students in this lesson, and hence, contrasted against a background of sameness, the teachers believed that students had already discerned the critical aspect maintenance of equivalency (CA3). The focus in this lesson for CA3 was the generalization of the critical aspect to include decomposition of a power of ten, the associative property, and expansion as critical features that all maintained equivalency between expressions. Shannon completed Phase Two of each example before proceeding to Phase Three.

In Phase Three of the implementation of the set of structured exercises, Shannon drew attention to the value of the coefficient (CA2) in the rewritten expression and attended to the maintenance of equivalency (CA3) of the expression throughout the procedure:

> Shannon says, "Then back to Exercise 1…I have [the expression] written with ten to the third power. I'm going to take what's in scientific notation and write it in standard form. How do I write three point six times ten to the first power in standard form? It's 36."

Shannon writes 36×10^3. Shannon asks, “How do I know that I’ve gotten this right? Zain?” Zain says, “They both equal thirty-six thousand.” Shannon replies, “The number that I started with and the number that I ended with are both equivalent to thirty-six thousand.” (RL4 Field Notes, Lines 50-54)

The focus of Phase Three was on the value of the coefficient (CA2) and the maintenance of equivalency (CA3) via the critical feature the expansion of a number written in scientific notation into standard form. Shannon drew explicit attention to the maintenance of equivalency between the initial form of the number and the final form of the number, implying the contrast in form. The expansion of the coefficient in Phase Three was enacted with Exercises 2 and 3.

Table 4.16

Implemented Phases of Completion and Critical Aspects Attended to in Each Phase

	Exercise 1	Exercise 2	Exercise 3	Critical Aspect Attended To
Phase One	$3.6 \times (10^4)$ $3.6 \times (10^1 \times 10^3)$	$3.6 \times (10^4)$ $3.6 \times (10^6 \times 10^{-2})$	$3.6 \times (10^4)$ $3.6 \times (10^{-3} \times 10^7)$	The value of the power of ten (CA1) / The maintenance of equivalency (CA3)
Phase Two	$3.6 \times (10^4)$ $3.6 \times (10^1 \times 10^3)$ $(3.6 \times 10^1) \times 10^3$	$3.6 \times (10^4)$ $3.6 \times (10^6 \times 10^{-2})$ $(3.6 \times 10^6) \times 10^{-2}$	$3.6 \times (10^4)$ $3.6 \times (10^{-3} \times 10^7)$ $(3.6 \times 10^{-3}) \times 10^7$	The maintenance of equivalency (CA3)
Phase Three	$3.6 \times (10^4)$ $3.6 \times (10^1 \times 10^3)$ $(3.6 \times 10^1) \times 10^3$ 36×10^3	$3.6 \times (10^4)$ $3.6 \times (10^6 \times 10^{-2})$ $(3.6 \times 10^6) \times 10^{-2}$ 36×10^3	$3.6 \times (10^4)$ $3.6 \times (10^{-3} \times 10^7)$ $(3.6 \times 10^{-3}) \times 10^7$ 36×10^3	The value of the coefficient (CA2) / The maintenance of equivalency (CA3)

Exercises 4, 5, and 6 were designed by the teachers as exercises for student practice and teacher assessment of student learning. The teachers called these “Show Me” examples:

Shannon says to the class, “In the ‘Show Me,’ I want you to show me that you understand. We can work through the first one [Exercise 4: How would you rewrite 2.5 ×

10^5 so that it has a power of 10^2?] together. Elyssa, where am I going to start?" Elyssa says, "I would move my decimal place three places to the right." Shannon asks, "Why?" Elyssa replies, "So I would get two thousand five hundred times ten to the second power." Shannon says, "Can you tell me the steps you used to get there?" Elyssa replies, "No." (RL4 Field Notes, Lines 60-62)

Elyssa obtained an expression equivalent to 2.5×10^5 written with a power of 10^2. Despite not verbalizing her procedure, Elyssa must have had some procedure for obtaining her result. Elyssa may have noticed the relationship between the value of the exponent and the value of the coefficient, in terms of the movement of the decimal place, but may not have been able to articulate the relationship that she noticed. It is not clear whether Elyssa had developed a procedure without understanding the mathematical validity of the procedure or if Elyssa did not articulate her understanding in words at the time. The separation of the first three exercises in the set of structured exercises into phases may have impeded the cohesiveness of the procedure for students. Without a cohesive procedure, Elyssa may have noticed and modeled the differences between the given and final expressions either due to a perception of efficiency or confusion regarding the intermittent steps that maintained equivalency.

Shannon asked for other student input and elicited from students a procedure akin to the three phrases presented in the first three exercises. Shannon then asked students to complete Exercises 5 and 6 with all work shown. In discussing Exercises 5 and 6 after a few minutes of students working independently, Shannon again reiterated the need for students to show work: "You need to show how you split up the powers of ten. You can skip the third line [regrouping with parentheses], but you can't go from the first line to the last line without any work" (RL4 Field Notes, Lines 81-82). Shannon did not elicit from students the relationship they were noticing between the critical aspects value of the power of ten (CA1) and value of the coefficient (CA2) that enabled the students to go from "the first line to the last line without any work." Shannon may have shied away from eliciting the relationship and fusing CA1 and CA2 due to the teachers' concerns that emerged in the planning meetings of students memorizing rules without understanding.

Shannon typically allowed time for students to start their homework in class. She assigned the second set of structured exercises as homework with about five minutes remaining in the lesson. For Exercise 2, Shannon clarified that one less power of ten meant that students would write an equivalent expression with a power of 10^3. One more power of ten meant that students would write an equivalent expression with a power of 10^5. Despite this clarification, the students continued to struggle with the assignment. After a few minutes of independent work and Shannon helping students individually, Shannon spoke to the whole class to clarify the assignment again. Many students were not writing an equivalent numerical expression with one less and one more power of ten. Some students were only writing the power of ten that was one more and one less, omitting the coefficient. Some students were writing the power of ten that was one more and one less and leaving the coefficient unchanged. Other students were writing the number in standard form.

> This [column to the left] is one less [than the given power], so I want something on this line times ten to the third that is equivalent to three point seven two times ten to the fourth [3.72×10^4]. Write this [3.72×10^4] in standard form so you know. So whatever I put on this line [in the left column] has to be equal to thirty-seven thousand two hundred. Whatever is on this line [in the right column] has to have a power of ten to the

fifth. Does that help ease your confusion?...You have to show your work. (RL4 Field Notes, Lines 87-91)

Shannon clarified twice what one power of ten less and one power of ten more meant, including the power of ten that students would need to use for each in the example. Shannon drew explicit attention to the maintenance of equivalency amongst the expressions in each row of the assignment. Again, Shannon insisted on students' work being shown regarding how they were rewriting the expression with a different power of ten through the decomposition of the given power of ten.

Evaluation of the Lesson

Immediately after the research lesson was taught, the team of teachers and I met to discuss our observations of the lesson. We met again six days later during our regularly scheduled meeting time to finish our discussion. The discussion immediately following the research lesson followed a modified protocol borrowed from Japanese Lesson Study in which the instructor first shares her reflections, the team members discuss what they observed, and then a general discussion follows focused on how specific elements of the design of the lesson promoted student learning or not. At our second meeting, the teachers discussed how the focus on changing the power of ten in the research lesson supported students in understanding the lesson on adding and subtracting numbers written in scientific notation, which was taught the following class. They also discussed the results of the post-assessment Shannon had given to her students two days after the research lesson.

While the research lesson had taken place in Shannon's eighth period Math 8 class, Shannon had taught the same lesson to her seventh period class immediately prior to the research lesson. Because I happened to be at the school for the research lesson, I asked Shannon if I could observe her implementation of the lesson in the seventh period class, and she agreed. Much of Shannon's observations were drawn from comparing the implementations of the lesson between the two classes. Shannon immediately said, "I was surprised at how much better eighth period went than seventh period" (RL4 Debrief, Line 5). Tori asked her to elaborate on what went better, and Shannon talked about the flow of the lesson and how she felt more comfortable with the technology. Having taught the lesson in seventh period, Shannon was more confident in what to expect from the students and her role in engaging the students in the lesson for the implementation during eighth period.

I asked Shannon how she had felt about using LGEs since it was the first time they had been used in a research lesson. Shannon said, "…I think if I'm going to use LGEs more, I need to be able to figure out what to do with the ones I don't want better…[In the seventh period class] I just didn't know what to do with [them]!" (RL4 Debrief, Lines 45-47). This was an instance of Shannon needing to process the examples that were generated and make a decision about how to respond in the moment. One difference that I had noticed and pointed out between the implementation of the lesson in seventh period and eighth period was her treatment of the LGEs. Students had written various expressions equivalent to 36,000 on the SMART Board. One of the features of the SMART Board is that what is written is treated as an object, so it can be moved or copied. The teachers had intended that Shannon would copy the expressions that contained a power of ten and insert them onto the next SMART Board screen to focus on just those expressions (and ignore other expressions, such as $3 \times 12{,}000$). In the seventh period class, Shannon copied the expressions and inserted them onto the next SMART Board, as had been planned. In the eighth period class, Shannon asked students to rewrite those expressions that contained a power of ten with an exponent *before* she moved them to the next SMART Board screen. For instance, a student had generated the expression 360×100. Shannon asked students how 100 could be written with an exponent and rewrote this expression as 360×10^2. Shannon's decision to rewrite these expressions in-the-moment made it clearer why she chose the expressions she did to move to the next SMART Board screen. Lynn commented, "[You] just organized it before you moved it" (RL4 Debrief, Line 56). Shannon said, "Which made it easier to then look at the questions of what's staying the same?...Now that we have these examples organized and nice, those questions flowed better." (RL4 Debrief, Lines 56-58). Shannon had noticed how a small adjustment to the enactment of the lesson – rewriting the expressions with a power of ten before copying them to the next SMART Board screen – allowed for a better transition to the questions regarding the patterns of variance and invariance in the set of LGEs that she wanted students to discern. It was not clear when Shannon made the decision to make this change, or even if it was a conscious choice. The comparison between the two implementations created contrast for Shannon that allowed her to discern a critical aspect of implementing LGEs, how to focus on the examples of interest, that was then shared with the other teachers via the discussion.

The teachers realized that using LGEs revealed to them aspects that students were attending to and students' awareness, within the moment, of the range of permissible change for particular aspects. Tori asked Shannon if she had noticed any student generated examples that included a negative power. Shannon said she had not. The teachers discussed what they had noticed about the examples that students both generated and did not generate:

> Tori said, "That means no one had [for example] 36,000,000 times .001. So that's also interesting. Why are they [students] not feeling as comfortable with that?" Shannon replied, "Yeah, it was all making the coefficient smaller and making the power of ten

> bigger"…Tori said, "Yeah. So I wonder what makes these students kind of side toward – I like bigger powers of ten!...Why aren't our kids feeling comfortable having a .0001?" Beth said, "It's not a familiar number." Shannon said, "I think they get that if they multiply by ten or 100 or 1000 I'm just making the number [coefficient] bigger by one place value. I don't think they go the opposite way. I don't think they get that point one makes it smaller by [a factor of] ten. I don't think they can go that way." Beth replied, "That's like unchartered territory I think for a lot of students." (RL4 Debrief, Lines 75-83)

The teachers wondered about students' tendency to use positive powers of ten. They surmised that students were more comfortable with positive powers of ten that are equivalent to large numbers, rather than negative powers of ten that were equivalent to decimal values. Shannon thought that perhaps students' conception of place value did not include division by ten. The teachers felt that the students tended to use numbers and values that felt more familiar. Tori wondered if it would be possible to push students to consider values for the power of ten beyond what they felt comfortable with:

> Tori told Robert that when he does this lesson on Monday to push the students there. She wondered what would happen. Tori said, "What if you pushed them to go back to ten to the negative third?" I suggested that he could also ask, "Could you write one with a negative exponent?" (RL4 Debrief, Lines 83-86)

Tori considered how it might be possible to expand students' notions of the range of permissible change for the value of the power of ten. Tori's suggestion indicated that she believed that students had knowledge and awareness of negative powers of ten, but that the LGE as prompted did not draw them out. She realized that students needed to be encouraged to go beyond the values of powers of ten that felt comfortable or familiar to them.

Shannon said she was surprised about students' difficulty on the homework, and the teachers considered what the sources of difficulty might have been:

> Shannon said, "I was really surprised at some of the work I was seeing on the homework. I mean, just all over the place! I don't know if that was me not being clear enough about *this* is the answer we want to end up with or-" Robert interjected, "The instructions. One more and one less." Shannon replied, "Yeah, which I thought I had alleviated any confusion by explaining to them"…Lynn said we did not do many examples where the exponent was changing by only one, so she wondered if the students' results confused them because "it didn't look right." Tori said, "I think it was the way we had it [the task] laid out." Shannon replied, "Yeah, it looked totally different than the other ones [examples]." (RL4 Debrief, Lines 16-25)

Shannon initially considered her implementation of the lesson as the source of difficulty for the students. She questioned her own clarity in presenting the lesson. Robert, Lynn, and Tori, however, each considered the design as problematic. Robert suggested that the instructions were not clear, while Tori conjectured that the layout of the exercises was a source of difficulty. Lynn looked back toward the design of the lesson and suggested that they did not include enough examples within the lesson itself in which the value of the power of ten was being changed by only one.

The teachers considered two options for revising the design of the set of structured exercises used for homework: (1) Word the prompts as they had been worded in the in-class set of structured exercises (i.e., "Write 3.72×10^4 so it has a power of 10^5."), and (2) A fill in the blank structure, such as $3.72 \times 10^4 = \_\_\_\_\_\_\_\_\_\_ \times 10^5$. While Tori and Robert conjectured that the design of the set of structured exercises was problematic, Lynn suggested that the choice of examples within the lesson, rather than the task, was the source of difficulty. Lynn's suggestion reflected the view that success was devoid of struggle - had the teachers included more examples within the lesson in which the value of the power of ten was only changed by one, students would know what they should do in the homework. As was seen in each of the prior research lessons, the teachers' notions of student success largely meant that students' had comfort in knowing what procedures to perform. The teachers' concern for student success, in terms of students knowing what to do, inhibited their willingness to try an LGE in Research Lesson One, led to including a hint for the LGE designed for homework in Research Lesson Two, and influenced the sequencing of examples in Research Lesson Three, particularly their choice to model an example, prior to asking students to complete such an example independently. Tori and Robert's suggestion to revise the layout and prompt of the set of structured exercises, rather than replicate the exercises within the lesson, suggested a shift in their thinking regarding what students were capable of, and their notions of student success. The source of difficulty for students was within what the teachers had designed, and if redesigned might allow for students to approach a novel problem.

During the planning of the lesson, the teachers had discussed avoiding "rules" for rewriting an expression with a different power of ten. One of the purposes of a set of structured exercises, however, is to lead to generalization about mathematical structure. The teachers had not discussed in the planning how Shannon might respond to students who inevitably noticed the relationship between the changes in the power of ten and the changes in the coefficient. The teachers discussed how Shannon chose to respond:

> Robert said, "When [Elyssa] was like, "Oh, you can just move the decimal over," I thought you handled that great because that was the only time I heard anyone say anything about moving the decimal…Her statement was correct, but…then you did a good job focusing on the process because later she didn't know that trick…What she said was correct, but she wasn't really demonstrating that she understood." Tori said, "Tyler also made a comment in the very beginning about moving the decimal and changing the power." Shannon replied, "I couldn't figure out how it [Tyler's comment] related, so I was like okay? I didn't really know how to respond to that." (RL4 Debrief, Lines 88-97)

Avoidance was the only strategy that Shannon had readily available to her to approach students' attempts at generalization in-the-moment. The teachers were concerned that an articulation of the relationship between the changes in the power of ten and the changes in the coefficient would lead to a rule or a "trick" that students would attempt to memorize and apply without understanding. Despite Shannon's avoidance of the generalization, and insistence that the steps be written out, a number of students continued to use "shortcuts" that they saw to their advantage, albeit many in unsuccessful ways. Shannon said, "The kids who are very attentive to, okay, I'm going to do it like the notes…did it correctly...But then the kids who like shortcuts and didn't have it written out were a mess" (RL4 Debrief, Lines 36-38). The teachers needed strategies to support students in articulating generalizations that they saw without reducing them to rules or "tricks." The teachers also needed strategies for supporting students in evaluating the correctness of the results they obtained, based on their generalizations, and making adjustments, as needed.

The teachers had noted they, themselves, struggled with the language of generalizing the relationship between the power of ten and the coefficient. They wanted to stay away from the language of “moving the decimal place,” but they did not feel as though they had language to replace it.

> Robert said, “Don’t think about moving the decimal.” Tori suggested, “Is it [the coefficient] getting smaller or bigger?” Shannon said, “I hate it when I’m like, okay, let’s move the decimal point because that doesn’t sound right, but I don’t know how else to-” Tori replied, “You could say, let’s make the number smaller. So this is ten to the negative third. So let’s make the number smaller. So I’m going to make the number smaller and just move it.” Robert said, “But be specific, it’s the coefficient that’s getting smaller.” (RL4 Debrief, Lines 98-104)

Tori’s suggestion to use the language of getting bigger or smaller did not address either how much bigger or smaller to make the coefficient or why they would be making the coefficient bigger or smaller. The teachers realized the lack of language that they had regarding the generalization about the relationship between the power of ten and the value of the coefficient, which contributed to their strategy of avoidance within the implementation. Not only was there a need for strategies to support students’ articulation of generalizations, the teachers needed support, themselves, in articulating generalizations, particularly in the language of middle grades students.

The purpose of Research Lesson Four was to have students write equivalent expressions using various powers of ten. The teachers had identified the power of ten as a critical aspect of adding and subtracting numbers in scientific notation, a skill that students had struggled with. The teachers intended that a lesson focused solely on writing an equivalent expression with a different power of ten would support students’ understanding of adding and subtracting numbers in scientific notation. At the second evaluation meeting, held six days after the research lesson, the teachers discussed how the lesson on writing equivalent expressions with a different power appeared to be influencing students' abilities to add and subtract numbers written in scientific notation:

> Tori asked Shannon, "So overall, is it helping with adding and subtracting?" Shannon said, "They are not making the bridge. I wonder if it's because I wasn't there on Tuesday. So, Friday we did the changing powers. Monday we did adding and subtracting, and we did the warm-up where we reviewed changing the powers." I said, "And you thought it went well because you sent me an email saying it went great!" Shannon replied, "Yeah! Great, great, great! Then I wasn't there on Tuesday where they [students] had to do just all word problems. Then yesterday [students] were like, "We don't know how to add and subtract in scientific notation!"…[Students] were telling me that [they] didn't know how to add and subtract in scientific notation, but the hardest part of it [changing the powers], [they] nailed on [their] quiz! What am I missing?" (RL4 Debrief, Lines 169-180)

The teachers recognized that students' continued struggle with adding and subtracting numbers in scientific notation meant that there was some critical aspect both they and their students had yet to recognize and attend to ("What am I missing?"). Shannon said, "I think we [her and the students] came up with the conclusion that when you're adding or subtracting in scientific notation the first question in your head should be, how do I write 1.6×10^{7} as a power of whatever?" (RL4 Debrief, Lines 185-188). Shannon identified that the values of the power of ten of the two addends was a critical aspect for adding and subtracting numbers in scientific notation. Having discerned that critical aspect, students needed to be able to choose an appropriate strategy for writing an equivalent expression for either one or both of the expressions so that the powers of ten were the same. As a result, the teachers concluded that the research lesson itself was successful, but new questions arose regarding students' transfer of writing equivalent expressions to adding and subtracting numbers in scientific notation.

Summary of Research Lesson Four

The object of learning for Research Lesson Four was writing an expression equivalent to a number given in scientific notation using a different power of ten. This object of learning was borne out of teachers' experiences with students struggling to add and subtract numbers in scientific notation. They identified writing an equivalent expression with a different power of ten as a critical aspect of being able to add and subtract in scientific notation and decided to make that critical aspect the object of learning of the research lesson with its own associated critical aspects and critical features. The teachers identified three critical aspects of the object of learning: (1) the value of the power of ten, (2) the value of the coefficient, and (3) the maintenance of equivalency.

This was the first research lesson in which the teachers chose to incorporate an LGE into the lesson. The LGE was used to elicit the concept of equivalency between various representations of a number. The design of an LGE and consideration of how it would be implemented introduced a number of challenges for the teachers, including the wording of the prompt to elicit examples of the intended type, and challenges related to variation in student thinking and how to respond in-the-moment. The principle of Attending to Generation and Response emerged for the teachers to approach these challenges. This principle says that teachers should carefully consider what examples the learners might generate given the prompt and plan, to the best of their ability, their response to those examples. Since the examples that learners will generate can never be fully known ahead of time, there will always be an element of the teacher needing to notice which examples are generated, deciding how to proceed to drive the lesson forward, and responding to student thinking in-the-moment. The development of tools, such as a list of examples intended to move the lesson forward, may be useful to support the teacher in attending to the generated examples and acting in-the-moment.

As in the previous three research lessons, the teachers attended to the principle of Explicit Contrast in the design of the prompt and anticipatory set for the LGE, and the teacher-designed sets of structured exercises. The LGE was designed so that each student generated example would be equivalent to 36,000. The teacher would focus on those examples written with a power of ten and then have students consider what stayed the same and what was changing. They intended that explicit contrast would emerge through the patterns of variance and invariance. Within the first set of structured exercises that the teachers designed, an visual instructional strategy of boxing in the power of ten and its decomposition into a product of powers of ten using the law of exponents for multiplication drew students' attention to the explicit contrast between the representation of equivalent values.

The language used by the teachers and the students emerged as a source of challenge at multiple points within Research Lesson Four. Teachers had spent time carefully considering how to word the prompt of the LGE. Despite their careful planning, Shannon delivered the prompt in the implementation in ways that reframed the question in the language the teachers had rejected during the planning. Despite teachers' best intentions in planning, the complex act of teaching still made it easy to slip into using unintended language in the enactment. The issue of language emerged again, as the teachers struggled with the appropriate language to use to make generalizations about the relationships between the critical aspects without reducing the generalizations to a set of rules. They rejected the language of "moving the decimal place," but struggled to find the language to replace it to convey what was intended. As a result, the opportunities provided by the LGEs and structured exercises for generalization were not taken up by Shannon in the lesson.

Lastly, the teachers' use of an LGE provided the teachers with an opportunity to learn that their students only generated expressions with positive powers of ten. The learners' restriction of the exponent to positive powers of ten raised questions for the teachers about why their students did not consider negative powers of ten and decimal coefficients to write an expression equivalent to 36,000. The teachers' observation provided them information about their students' mathematical knowledge that they likely would not have noticed without having asked the students to generate examples, and created more questions for the teachers about their students' learning.

Summary of the Research Lessons

Each of the four research lessons addressed an aspect of equivalency, despite the variation in the content they addressed. Research Lesson One was about writing equivalent expressions using the order of operations. Research Lesson Two was about writing an equation of a line using slope-intercept or point-slope form and implicitly understanding the equivalency of these two forms. Research Lesson Three was about calculating a value for both forms of a unit rate, interpreting that value as a quantity in context, and comparing unit rates. While the values of the unit rates were not equivalent, the relationship between the quantities that either form of the unit rate describes is equivalent. Research Lesson Four was about writing an equivalent expression for a number given in scientific notation with a different power of ten. Without intending to at the outset of the study, the aspect of equivalency served as a common theme throughout the four learning study cycles.

The Principle of Explicit Contrast emerged as a principle of both design and enactment. The teachers employed a design strategy they called "varied versus unchanged" to draw students' attention toward the aspects of a set of examples that were changing. The teachers also used intentional disturbance created through patterns of variance and invariance to design for explicit contrast between examples. The teachers used verbal and visual enactment strategies to make the contrast within the design explicit, and strategies to compare student solution methods. Verbal strategies for enacting explicit contrast included: (1) asking "What is the same? What is changing?", (2) reading out loud to emphasize the difference between two examples, and (3) notice and focus, borrowed from English Language Arts. A visual strategies for enacting explicit contrast was boxing in equivalent expressions with contrasting representations. Strategies to compare student solution methods were: (1) student-to-student talk, (2) juxtaposition of student solution methods, and (3) a survey of student solution methods.

The teachers did not use the language of critical aspects in the same way that it is used in variation theory. When the teachers referred to critical aspects, they were more often referring to the *relationship* between critical aspects that they wanted students to discern. While the teachers identified critical aspects through their articulation of the relationships they wanted to encourage students to see, they had difficulty articulating the specific critical aspects of the objects of learning. In order to articulate the relationships they wanted students to discern about a particular object of learning, and implicitly identify the critical aspects, the teachers often had to carefully consider and deconstruct their own ways of thinking about the object of learning to uncover what it was that they, themselves, attended to.

A shared understanding of *what* was being taught, rather than *how* it was being taught, was important for establishing a focus for the design of examples. A shared understanding of the content and the purpose of the lesson often developed in tandem with the design of the examples. Throughout the four learning study cycles, the purpose served the design, and the design typically served to clarify or refine the purpose, including the critical aspects. In Research Lesson Three, the design of the examples did not serve to clarify the critical aspects of the object of learning, but rather extended the object of learning to include multiple goals related to unit rates. This may have been related to the class as an RTI class and established norms, or the difficulty of ascertaining the critical aspects of an object of learning.

Collectively, the teachers fairly easily took up structured exercises and used contrast as a pattern of variation through a strategy of "varied versus unchanged." While it was the relationships between critical aspects that the teachers articulated that they wanted students to discern, the teachers were concerned about the introduction of "rules" that would reduce the mathematics to something to memorize rather than understand. The teachers needed strategies to support students in articulating generalizations about the relationships that they saw among critical aspects without reducing them to rules or "tricks." In some instances, the teachers realized the lack of language that they had, themselves, regarding the generalizations about the relationships they wanted to bring to the fore of students' awareness. This indicated that the teachers needed support in articulating generalizations of relationships among critical aspects, particularly in the language of middle grades students.

Lastly, the teachers' notions of student success largely influenced their design and use of LGEs. The teachers did not choose to incorporate an LGE into the in-class portion of a research lesson until the fourth learning study cycle. Within that cycle, the design Principle of Attending to Generation and Response emerged. Teachers realized the necessity of attending to what examples might be generated and how to respond to those examples. This included attending to the openness/restriction of the prompt for the LGE, and the development of tools, such as a list of anticipated examples that students would generate, that would support the teacher in attending to student thinking and responding in-the-moment to the examples that students' did generate.

Changes in Teachers' Use and Views of Examples

In this section, I describe changes in each teachers' use and views of examples from prior to the learning study intervention to after the learning study intervention. This analysis is chronological, as noted in my methodology, to describe changes over time and the interplay between purpose, design, and enactment. Evidence for changes in teachers' use and views of examples, and structured exercises and LGEs in particular, was drawn from my analysis of each teacher's individual observations and interviews, and contributions to the group during the four learning study cycles. Based on this evidence, I revised the table that summarized each teacher's example space of use and view of examples. These tables serve as a visual representation of the dynamic nature of each teacher's example space of use and view of examples, but the tables are only representative of the aspects of their example spaces that were made visible to me.

Tori Goodman

Prior to the learning study intervention, Tori had a keen awareness of the variation she included in her sets of examples. Her choice of variation was drawn from her desire to expose students to different ways in which an example could be asked, or different features that an example might include, within a class of examples. Tori believed that exposing students to such variation would support them in developing a sense of generality regarding the class of examples. Tori's key strategies for her design of sets of examples was reversal and juxtaposition. While Tori saw the mathematical structure within her sets of examples, with simultaneous variation in multiple aspects of the examples, it was not clear if students were attending to the aspects and mathematical structure in the same ways that Tori did.

In Research Lesson One, Tori took up a design strategy of varied versus unchanged to create contrast through examples: “This is when we can go to the varied versus unchanged. All the numbers stay the same…What if I put parentheses here? What if I get rid of a plus sign and put parentheses here?” (RL1, Planning Meeting 4, Lines 42-44). The use of *invariance* was a significant change in Tori’s use of examples. Prior to the learning study intervention, Tori had described one of her design strategies as “one small change,” which most often meant a change in the quantity being asked for. For instance, one example might ask students to find the total cost of an item with tip. The next question might ask students to find the cost per hour. Tori saw this as “one small change” because students would need to use the same procedure to find the total cost and then take one additional step to find the rate. The context and numbers used in the example, however, would also vary, making it unclear whether students saw the mathematical structure between the examples as Tori did. Tori’s new use of invariance in the design highlighted the “one small change” that Tori intended.

Tori extended her understanding of patterns of variance and invariance to other contexts. In Research Lesson Two, Tori demonstrated her awareness that aspects that are not varied are not discerned by students when she noticed that Robert’s students did not seem to pay attention to the scale on the graphs: “Tori asks if Robert ever changes the increments on the x or y-axes. Robert says, “Yes, once or twice though. So a few, but not a lot” (RL2 Debrief, Lines 213-215). Tori suggested a pair of examples in which two graphs appeared exactly the same, but had different equations. She intended that this would generate a disturbance for students who would naturally look for the difference between the two graphs that caused the difference in equations and find that the scale was different for the two graphs. Creating a disturbance through patterns of variance and invariance was a new strategy for Tori for the design of examples that encouraged students to seek out the particular aspect Tori intended they discern.

In my observation of Tori after the four learning study cycles, Tori had created a set of exercises for the homework that used patterns of variance and invariance to contrast the operation used in exponential expressions. The following were the first three exercises on the Laws of Exponents Review Homework (see Appendix I), assigned during Post-Observation 3 (May 11, 2016):

1.) $x^6 \cdot x^2$ 2.) $x^6 \div x^2$ 3.) $(x^6)^2$

In these exercises, Tori held the variable and the exponents invariant while varying the operation (multiplication, division, repeated exponentiation). Tori described her design choice:

> So, just starting off one, two, and three, very basic. Notice how the bases stayed the same. All the exponents stayed the same. It was just the operation that changed. So that was also a big kind of thing that I know we had talked about in our previous [research] lessons – like what remains unchanged, what stays the same. So my focus here was that it's really the operation that's most important. (Post-Interview Transcript, Lines 672-677)

Tori drew on Explicit Contrast as a design principle within her individual teaching after the four learning study cycles, and demonstrated a strategy of varying only the aspect that she wanted to bring to the fore of students' awareness.

Prior to the learning study intervention, Tori gave strong consideration to the examples she included in her lessons and had strong rationales for how she designed them. She readily took up aspects of structured exercises and the principle of Explicit Contrast in her design, but exhibited inhibitions about the use of LGEs. In Research Lesson One, Tori restricted which students had access to the LGE by giving it only to those students who had finished the order of operations assignment early. She was concerned about students' abilities to be successful with a task in which they, rather than she, generate examples. She saw, however, that each of the students she gave the LGE to were able to generate a correct expression equivalent to twelve. In Research Lesson Four, Tori wanted to either constrain the LGE they were designing in order to control the range of possible examples students would generate or demonstrate an example of thinking through a similar LGE before having students do it themselves. Her notions of student success continued to hinder her use and conceptualization of LGEs. During Research Lesson Four, however, Tori recognized how opening up the example space, rather than constraining it, was important for the design of the lesson. Designing for greater variation in the generation of students' examples created opportunities to move the lesson forward.

In the implementation of Research Lesson Four, Tori saw how quickly students were able to generate expressions using multiplication equivalent to 36,000, and in the debriefing meeting after the lesson, Tori wondered if it would be possible to push students to consider values negative values for the power of ten:

> Tori told Robert that when he does this lesson on Monday to push the students there. She wondered what would happen. Tori said, "What if you pushed them to go back to ten to the negative third?" I suggested that he could also ask, "Could you write one with a negative exponent?" (RL4 Debrief, Lines 83-86)

Tori's suggestion to encourage students to expand beyond their initial conceptualization of the range of permissible change for the value of the power of ten indicated a significant shift in her own use and conceptualization of LGEs. Rather than consider how students might struggle with an LGE, Tori's suggestion indicated that she believed that students had knowledge and awareness of negative powers of ten, but that the LGE as prompted did not draw them out. She realized that students needed to be encouraged to go beyond the values of powers of ten that felt comfortable or familiar to them, and she considered ways to do that.

In a post-observation following the four learning study cycles, Tori designed a set of prompts for LGEs (see Appendix J) that asked students to generate exponential expressions using multiplication, division, and repeated exponentiation, respectively, that were equivalent to a given expression. Tori enacted the first prompt (#16) with the whole class, asking students to generate expressions equivalent to 8^8.

> Tori says, "So this time, I'm giving you the answer, and I want to know the expression that it could have come from…[A] student gives, "$8^2 \times 8^6$." Tori says yes, then asks, "Can we come up with one using negative exponents?" A student gives $8^{-2} \cdot 8^{-6}$. Tori asks, "Does this equal 8^8?" Another student says no, that this gives 8^{-8}…Tori then says, "How about one with repeated exponentiation?" A student gives $(8^2)^4$. Tori asks for "One more, both [exponents] positive."…Jon gives $(8^{16})^{\frac{1}{2}}$. Tori says, "Whoa! So we can have fractions here too!" (Post-Observation 3, Lines 122-144)

Reversal was a design strategy that Tori applied to her examples prior to the learning study intervention. Tori applied reversal as a design strategy to design the set of LGEs. Students were typically given an expression and asked to simplify it. Here, Tori gave the students an exponential expression and asked students to provide an exponential expression using various operations that was equivalent. Tori took up multiple LGEs from students, verifying their accuracy with the help of the class ("Does this equal 8^8?"), and correcting them, if necessary. Tori took up her own suggestion that she gave to Robert in Research Lesson Four and encouraged students to push beyond the range of permissible change for exponents that they were comfortable with by asking students for expressions that involved negative exponents. The range of permissible change was extended even further to include rational numbers when a student gave a fractional value for an exponent (“Whoa! So we can have fractions here too!”). By the end of the research study, Tori appeared to conceptualize of LGEs as useful in revealing students’ thinking and using them to extend the range of permissible change for a particular aspect.

Tori did not anticipate students generating examples that included fractional exponents. During the post-interview, I showed Tori a compilation of student responses from the homework assigned after the post-observation. She noticed the student misconception that $y^4 \div y^4 = y^1$. Tori described asking two or three students to share the examples they had generated and, at the time, Tori did not notice the misconception.

> Now that I have seen all of them together, I was just like, oh my god – like so many of them did it wrong! But no one that shared did it wrong! But it could have been, maybe, the first person shared, and I’m like, yes! Because remember, when I subtract these, I get one. So it could have just been I said that and then these kids were like, ugh! But then they didn’t fix it. You know, they realized that it was wrong, but then didn’t fix it. I don’t know. But clearly with that division one, a lot of them had it wrong, but then didn’t fix it. (Post-Interview Transcript, Lines 732-738)

Tori’s realization that there were student misunderstandings that presented themselves through LGEs, that she did not notice at the time, prompted her to suggest various strategies for the follow-up of LGEs.

Tori's suggestions for following up LGEs included verification of student responses by students. During post-observation 3, Tori had students check each other's work, a decision she made in-the-moment in response to some students completing the assignment sooner than others.

> When I was noticing more and more kids were finishing early, I was like, why don't you go see what somebody else did? Because I think that that's really valuable as well, for them to see what somebody else was thinking because maybe with the kids that did fractions, maybe somebody else would have checked there's and would have been like, I would have never thought of doing fraction! Just so they could see multiple different ways. (Post-Interview Transcript, Lines 619-623)

Tori recognized the value in students sharing the LGEs they had generated with other students, as well as the value of verification that the generated examples met the required characteristics. Because this involved predominantly peer-evaluation, Tori missed the opportunity, in-the-moment, to view the set of generated examples and garner important insights from them about students' learning. A similar phenomenon happened during the follow-up of the LGEs generated for homework. While Tori asked a few students to share, the students that volunteered to share had generated examples that met the required qualifications. Tori was not able to notice, in-the-moment, the misconceptions that students continued to demonstrate regarding laws of exponents.

In the set of LGEs, Tori attended to generalization as a pattern of variation both during the whole class enactment, and when she prompted students to check each other's work (e.g., "This equals 8^8, and so does this one, and so does this one!"). She elicited multiple examples for each case – multiplication, division, and repeated exponentiation – which generalized that there were many such examples that simplified to the given expression. Tori, however, did not push students to generalize the relationship between the exponents in each case of generated examples algebraically. After the first example, Tori asked students to complete the remaining prompts on their own and generate only a single example for each case. Opportunity for generalization potentially was diminished by the decision to have students generate only a single example for each case. As students finished the assignment, Tori asked them to go check another students' work. Through this, students had an opportunity to verify other students' generated examples, and potentially generalize the relationship between the exponents for each prompt, but students' verifications and generalizations were not made explicit during the class.

At the beginning of the lesson, Tori used simultaneous variation to fuse the relationship between the base of an exponential expression and two powers of a power raised to a power. Tori displayed three examples across the SMART Board and asked students how they would expand and simplify each one:

Tori…[shows] the three examples:

1.) $(3^2)^3$ 2.) $(5^4)^2$ 3.) $(x^{-3})^3$

Tori asks of #1, "How could I expand this?"…Tori calls on a student who says, "Three squared times three squared, times three squared." Tori writes this on the SMART Board, then asks how this could be simplified. Another student says 3^6. Nina says, "Couldn't you just multiply the two exponents?" Tori responds, "Hold on, hold on." Nina says more quietly, almost to herself, "Oh, that's the quick way." Tori asks for someone to expand #2…Meg says, "Couldn't you just…" Tori cuts her off, saying "Hold on," and pushing her palms down toward the floor…There are more students echoing, "Couldn't you just-" Then Tori taps below the three exercises, making the directive appear: "Describe a short way you could simplify the above without writing out the entire problem." Many students start to say, "Multiply the exponents." Tori says, "Wait. You need to keep the…?" Some

> students then say base. Tori nods, continuing, "And…?" Many students respond, "Multiply the exponents." (Post-Observation 3, Lines 59-75)

In contrast to Research Lesson Four, Tori intended that the set of exercises presented here created an opportunity for fusion and a generalization of the relationship between the critical aspects.

> *Researcher:* What role did these exercises have for students' learning for you?
>
> *Tori*: The most important part was the expanding piece of it. What would this be if I write it out, so then we can fun a quicker way, or a more mathematically sound way we could evaluate these…So the purpose of the first three is really just to show them that relationship between a power raised to a power. (Post-Interview Transcript, Lines 162-169)

Tori did not enact contrast in this lesson to reveal the critical aspects of exponential expressions raised to a power, but it may have been that Tori either assumed or knew that the critical aspects had previously been discerned. Her goals here were the generalization of the relationship between the base of the exponential expression and the two exponents, and that the generalization was grounded in mathematically valid reasons. Students readily tried to articulate the relationship they were noticing. Nina whispered, "Oh, that's the quick way," suggesting that the "quick way" as a generalization had previously been used in Tori's class. While Tori had spoke about using variation in multiple aspects to convey generality prior to the learning study intervention, the only articulation of a generalization between aspects of a set of examples that I observed was that given by Tori as Original (%) = Total and used non-standard mathematical notation. After the learning study intervention, Tori opened the space of learning to students' articulation of the relationship between aspects in an example.

Tori's use of LGEs in post-observation 3, in conjunction with the benefits she recognized, opened up a pathway for the use of LGEs in her future lessons, and opportunities for extending the practice to other teachers.

> So something like this [LGEs used in post-observation 3] was not a big time commitment. It did not take me very long to come up with an idea of how I could have them come up with an example, and then how I could easily use that in my class…It could be a quick, how could you get kids to jot some information down about what they're thinking. And that could take five minutes. It doesn't have to be a full blown 40 minutes lesson that you're reconstructing. (Post-Interview Transcript, Lines 872-884)

Tori realized, through enactment of LGEs, that much could be learned, by the teacher, about students' thinking, with a minimal time commitment. Tori further identified two strategies for designing LGEs: (1) start with an example in which there are no known misconceptions, and (2) have an example available to reference. Tori had started the LGE in post-observation 3 by asking students to generate an expression using multiplication that resulted in 8^8. She noticed that some students were making the common mistake of multiplying the bases and creating expressions such as $4^4 \cdot 4^4$. Because this was a known misconception, Tori said, "I think that it would be better for them to start with something that just has a variable…If I had done #17 [x^{16}] first, I wouldn't have had any of those issues," (Post-Interview Transcript, Lines 366-370). The first student to respond to Tori's LGE prompt gave the expression $8 \times 8 \times ... \times 8$, which was not the structure that Tori wanted students to generate. Tori cited the lack of having an example to reference, indicating what was meant by an exponential expression using multiplication:

> Maybe I just need to have something on the side, or have their homework ready to pull up…If I'm looking at their homework problems, something like this [$u^5 \cdot u^{-4}$]. To be

like, okay, so our final answer was u^1. So that came from this. So I'm giving you x^{16}. Where did it come from, in regards to multiplication? I think if I did something more like that, that would've led them to more success with that. (Post-Interview Transcript, Lines 389-393)

When attempting to incorporate LGEs into lessons for the first time, Tori recognized the importance of being clear about the prompt, an important aspect of the design of LGEs that emerged during the learning study cycles. Tori suggested referencing an example that had a structure consistent with the examples that she expected students to generate. In addition, Tori suggested starting with an example in which known student misconceptions would not interfere with the task as intended, while simultaneously allowing for multiple valid examples to be generated.

Summary of Changes in Tori's Use and Views of Examples. Like prior to the learning study intervention, Tori described her main purpose for using examples as exposure to variation in multiple aspects. After the four learning study cycles, Tori's main purpose for using examples continued to be exposure, but her consideration of the variations between examples became strongly based on patterns of variance and invariance and students' opportunities to discern contrast. Prior to the learning study intervention, Tori already gave strong consideration to each of the examples she chose to include within tasks that she designed. In contrast to her initial designs, which were largely based on variation in multiple aspects for the sake of exposure, within and after the learning study intervention, Tori used variation in a particular aspect against a backdrop of invariance in other aspects in order to bring that particular aspect to the fore of learners' awareness. While she continued to use reversal as a means of creating contrast between examples, she also intentionally created disturbance for learners to draw their attention and create opportunities for discerning difference between examples. Tori used simultaneous variation in critical aspects, as in the post-observation lesson, to provide an opportunity for students to generalize relationships between critical aspects. Tori was initially skeptical about the use of LGEs due to her notions of student success. By the end of the learning study intervention, however, Tori saw the potential of LGEs for expanding students' conceptualization of the range of permissible change of an aspect of a class of examples.

While Tori articulated few enactment strategies for drawing attention to mathematical structure prior to the learning study intervention, Tori developed a number of strategies throughout the four learning study cycles that she continued to use beyond the learning study intervention. Tori began asking students, "What's the same? What's different?" and she suggested a Notice and Focus strategy for Research Lesson Two. In Research Lesson Four, Tori suggested an enactment strategy for the teacher to encourage students to push beyond the boundaries they had constructed for themselves (e.g., asking for an example with additional conditions), and used this strategy in the post-observation after the four learning study cycles.

Tori saw the cross-content potential of both structured exercises and LGEs and began to view both as small changes within her task design that need not be time consuming, but had the potential to reap a number of benefits for students' learning. She continued to generate examples, and in particular sets of structured exercises, by employing a single change in a critical aspect between examples. She generated other examples, including LGEs, using reversal. For instance, rather than ask students to simplify an expression using the laws of exponents, in the post-observation lesson, Tori asked students to give an example of an expression that could have yielded the simplified expression. Tori generalized "Give an example of…" as a type of LGE that could be used across content. Likewise, she generalized "one small change," and an awareness of variance and invariance across content. Table 4.17 summarizes Tori's example space of use and view of examples after the learning study intervention. Changes to her example space prior to the learning study intervention (see Table 4.2) are bolded.

Robert Cavins

Prior to the learning study intervention, Robert's main goal for teaching mathematics was to develop students' ability to approach new problems through the application of skills and concepts that they already knew, and this goal shaped his use and views of examples. His primary purposes for examples prior to the learning study intervention were connection and skills practice, which included developing fluency and attending to aspects of generality. Throughout the learning study intervention, Robert placed an emphasis on developing students' conceptual understanding and mathematical flexibility.

In Research Lesson One, Robert wanted students to understand that expressions can be evaluated in multiple ways, despite the rigidity that the acronym for order of operations, PEMDAS (Parentheses-Exponents-Multiplication-Division-Addition-Subtraction), conveys. Robert described being explicit with his students about the problematic use of the acronym PEMDAS, including the restrictive use of P in PEMDAS for parentheses, when it in fact refers to a variety of grouping symbols. He emphasized the multiple approaches one could take to correctly evaluate an expression while still adhering to the order of operations.

Robert strongly pushed for the use of LGEs in Research Lesson One, but was met with skepticism about their use due to the teachers' notions of student success. At first, Robert thought of the insertion of parentheses into a given expression in order to obtain a particular value as an LGE. He later suggested using an LGE to open the space of variation by asking students to give an example of an expression equal to a given value using particular operations

Table 4.17

Tori's Example Space of Use and View of Examples after the Learning Study Intervention

Characteristic	Summative Description
Population	Predominant reason to use examples was, and is still, described as exposure in multiple aspects. Designed for contrast between examples using strategies of reversal and juxtaposition. Described an instance where she used restricted variation, calling it "one small change" between examples. Now explicitly using patterns of variance and invariance, describing it as "varied versus unchanged". Uses creating a disturbance as a means of creating contrast between examples. Simultaneous variation in critical aspects is used to support students in generalizing relationships between critical aspects (i.e., fusion). Designed LGEs using reversal. Enactment strategies, that were not evident prior to the learning study intervention, include: (1) What's the same? What's different? (2) Notice and focus, (3) Asking for an example with additional conditions to expand students' notions of the range of permissible change.
Generativity	Has strategies for generating sets of structured exercises across content using patterns of variance and invariance (i.e., "varied versus unchanged"). Uses reversal as a strategy for generating prompts for LGEs in conjunction with "Give an example of…(and another, and another)".
Connectedness	Viewing examples for the purpose of exposure is loosely connected to designing examples using patterns of variance and invariance to bring certain aspects to the fore of learners' awareness. There appears to be a greater purpose than just exposure that is not articulated. Asking for an example with additional conditions to expand students' notions of the range of permissible change is more well-connected to the view of examples for exposure, but with students, rather than the teacher, driving that exposure to variation in a particular aspect.
Generality	Used patterns of variance and invariance across content (e.g., order of operations, linear equations, and exponential equations). Discussed "Give an example of…" as a type of LGE that could be used across content.

and a certain amount of numbers. While there was take up of his idea, the teachers restricted the LGE in various ways due to their notions of student success. The teachers discussed restricting the numeric values to only integers, and ultimately, the teachers decided to use a sentence frame: ____ + ____ × ____ = 12. The sentence frame restricted the operations students could use and the amount of numbers. The sentence frame also forced the students to consider the order of operations since multiplication would need to be performed prior to addition. Robert thought that the LGE should be given to all of the students, but he again faced concern and reluctance from the other teachers that all students could be successful with such a task.

Robert began considering early on in the learning study intervention how ideas related to structured exercises, LGEs, and the design principle of Explicit Contrast could be incorporated into his own lessons, outside of the learning study cycles. In my second observation of Robert, which occurred during the learning study intervention, he applied the principle of Explicit Contrast in his design and enactment of sets of examples. Robert designed a set of examples that included two graphs: one of a linear, proportional relationship, and one of a linear relationship. Between the two graphs, Robert held the slope invariant and varied the y-intercept (see Appendix K). He asked students to list similarities and differences between the two graphs, to list out points on each of the two graphs, to determine the y-intercept of each graph, and to think about which one represented a proportional relationship. Robert's choice of this design drew from his initial purpose of connection. Students had studied proportional relationships in seventh grade. Robert wanted to connect students understanding of proportional relationships to linear relationships, and he chose to do so through explicit contrast in a graphical representation of two such functions. In my initial observation of Robert, prior to the learning study intervention, Robert, rather than the students, made explicit verbal statements regarding structural connections and generalizations. In my second observation of Robert, which occurred during the learning study intervention, Robert opened the space of learning for student talk:

> Jenna says they are both increasing. Robert says "So what does that mean?" Kate says, "They're both positive." Robert questions, "What do you mean by positive?" Kate replies, "Positive slope." Gregory says, "They both have the same rate of change"…[A] student says, "Up one over two"…Another student says, "Both go through the y-axis," while another says, "Both are linear." Then Robert asks students about how the functions are different. Kate says, "The b slope [referring to Graph B] is further up on the y-axis…Nadia says, "They have different starting points"…Liam says, "The y-intercepts are different"…Another student says, "They have different points." (Observation 2, Lines 90-109)

Robert elicited students' observations of the similarities and differences between the two graphs. He questioned students to explain what they meant by descriptions such as increasing and positive. Robert's questioning about the similarities and differences, and his questioning to encourage articulation of what students meant by their descriptions, provided an opportunity for students to discern the critical aspects of linear and proportional functions through explicit contrast.

> Robert then asks the last question, "Which graph represents a proportional relationship?" A student says Graph A, because it crosses at (0,0). Heidi says because *y*/*x* is constant. Robert writes *y*/*x* on the SMART Board and asks, "What is this?" A student says that it is the constant of proportionality, so Robert writes $k = \frac{y}{x} = \frac{3}{6} = \frac{5}{10} = \frac{1}{2}$ [where x, y represents points on the line of the proportional relationship]. Robert also writes: $\frac{2}{-2} \neq \frac{4}{2} \neq \frac{5}{4}$ [where x, y represents points on the line of the linear relationship]…Robert then says, "But we said they have the same rate of change, so how come they don't have the

same constant of proportionality?" Kate says that one is not proportional because they have different y-intercepts. (Observation 2, Lines 125-132)

Robert questioned students as to why the two graphs would have the same rate of change but one would have a constant of proportionality and one would not. Robert's strategy of focusing students on the similarities and differences between the two graphs and questioning created explicit contrast to provide an opportunity for students to discern the difference between linear and proportional relationships, and further to fuse the constant of proportionality and a *y*-intercept of zero with proportional relationships. Unlike in my initial observation, prior to the learning study intervention, students, rather than Robert, were making statements regarding the structural similarities and differences and generalizations about proportional and linear relationships.

Beginning in Research Lesson Two and continuing on for the remainder of the study, Robert focused on designing and enacting sets of examples meant to develop students' awareness of methods and which might be best in a given situation. In Research Lesson Two, the teachers' goal was to develop students' understanding that either slope-intercept or point-slope form may be used to write an equation of a line, but that one form may be more appropriate based on the given information or what the question asks for. In Observation 3, during the learning study intervention, Robert wanted students to discern when to use various methods for solving a system of linear equations. In the post-observation, Robert wanted students to discern when to use various methods to solve a quadratic equation. In each case, Robert constructed his lessons by enacting a set of structured exercises in class, followed by assigning an LGE for homework in order to assess students' understanding.

Whereas Robert was likely to improvise examples prior to this study, he described the value in carefully planning and structuring his examples for the purpose of helping students make discernments about which solution method to use in particular situations:

> Examples are something that you can determine before class. You don't have to come up with them on the spot, which is usually my strength, but like, planning is much better…I'd rather have an example ready for them that I know is perfect. And perfect, what does that mean? Good for them to solve in that sort of method that I really want to push them in. So I see examples used for things where they have choice of how to solve it. (Post-Interview, Lines 611-618)

Robert's description of a "perfect" example meant that students could discern the difference in critical aspects between one example and another that indicated that a particular solution method should be used. He described the role of structured exercises as allowing students to recognize similarities and differences (Post-Interview, Lines 667-668). In this case, the differences were about understanding when and why a particular solution strategy might be easier to use, while the similarities were about recognizing how the form of the solution might affect which strategy can be used (e.g., completing the square must be used when the solutions are irrational). Robert generalized how he could design sets of structured exercises for the purpose of understanding when and why a particular solution method should be used:

> So we've definitely seen through my two [lessons], how different methods, a lot of things were optional. Point-slope form, slope-intercept form, factoring, completing the square. Things where kids have that choice in how to solve it. Examples can be the best use that I've seen. So I can extend that to systems of equations. I can extend that to linear functions, in general. Like, the implications are vast. (Post-interview, Lines 592-597)

From Robert's perspective, students' ability to recognize structure within examples allowed them to make a choice about which solution method to apply when, and to justify that choice. Further, Robert recognized how similar sets of structured exercises could be designed across various mathematical contexts. This indicated that Robert had a basis for the future design of sets of structured exercises.

Robert recognized the value of LGEs for assessing student thinking and uncovering misconceptions. Within the post-observation, Robert assigned an LGE for homework in which students were asked to create three quadratic equations that had different constraints in terms of their solution (rational/irrational) and method of solution (factoring/completing the square). Robert said, "I wanted to see if they could show me that they understood what makes them solvable by the different methods" (Post-Interview, Lines 475-476). Hence, this was an assessment of whether students were attending to the critical aspects of quadratic equations and their solvability by various methods. Through this process, Robert recognized the value of LGEs in uncovering student misconceptions: "But then you go and actually have [students] create it, and you find that there are some misunderstandings there," (Post-Interview, Lines 398-399). From Robert's perspective, using LGEs provided a means to make student thinking about a certain class of examples visible.

Robert most often employed an LGE after a set of structured exercises, for the purpose of assessment. LGEs proved to be more challenging than structured exercises for Robert to implement, as evidenced by the lack of use of them during observations. One LGE was assigned for homework during Research Lesson 2, which had been designed by the group of teachers collectively. Robert only independently designed and used an LGE as a part of the homework assignment during the post-observation. Despite the lack of use of LGEs throughout the study, Robert discussed at length his views of the affordances and challenges of LGEs, in the context of the LGE that he assigned for homework in the post-observation, which indicated factors for their use and important developments in Robert's knowledge about enacting tasks that incorporated LGEs.

Robert's view of the affordances and challenges of LGEs were related to his willingness to implement them in his classroom. Robert recognized the value of LGEs for assessing student thinking and uncovering misconceptions and the potential for student generation of new mathematics. Through implementing LGEs, Robert realized challenges related to the task as it appeared and as it was interpreted by students, although he decided that these were challenges he could address through revisions in implementation. The pressure of high-stakes testing caused tension and affected Robert's willingness to take up and follow through with the new mathematics that arose through LGEs.

Through using LGEs, Robert was able to notice aspects of examples that students were attending to. When Robert asked students to create quadratic equations that could be solved using various methods and with either rational or irrational solutions, a number of students were eager to make the most challenging examples. Through the examples that students generated, Robert observed that students predominantly attended to the form of the coefficients as a source of challenge: "What made it the most challenging [to the students]? – different coefficients. Like coefficients were decimals…is that really that challenging?...And some kids just used large numbers and other kids were like, no, it doesn't make it that challenging" (Post-Interview, Lines 422-427). During the course of the post-interview, Robert and I were looking over the equations that students had created. Within the set of LGEs, we noticed five equations that students had generated that had a leading coefficient other than one (see Figure 4.15). The quadratic equations that appear in Group A have a constant (two) that can be factored out to reveal a quadratic equation with a leading coefficient of one. Those in Group B, however, still have a leading coefficient other than one, even after constant terms are factored out. It was these equations that Robert wondered about after looking at the LGEs for a second time, noticing that this was a part of students' thinking that he had not attended to in the moment. Robert said, "See, this is where you'd want to understand, see where they got that. How did they get that? What were their thought processes, how they got a leading coefficient other than one, and it was factorable" (Post-Interview, Lines 522-524). While Robert had noticed the form of the coefficients (e.g., decimal values, large numbers, etc.) in the moment, it was not until later when Robert noticed that a leading coefficient other than one was significant. This was similar to Tori's recognition, after the lesson had passed, that students continued to demonstrate a misunderstanding that $y^4 \div y^4 = y$.

Group A	Group B
$2x^2 - 8x - 24 = 0$ $2x^2 + 10x - 28 = 0$ $2x^2 + 14x + 12 = 0$	$8x^2 - 10x - 12 = 0$ $28x^2 - 41x + 15 = 0$

Figure 4.15. Student generated factorable quadratic equations with a leading coefficient other than one from Robert's Post-Observation 4.

Robert realized the potential for LGEs in bringing about new mathematics, describing how some students had realized that they could start with their choice of roots for the quadratic equation and "work backward" to build up the equation.

> We never did examples like [these]…Where like, oh, there's the roots. Write a quadratic equation that could have that…We never went through a process where ok, you solve

these for zero…and now it's just (x – 6)(x + 2). I thought by doing this task [the LGE assignment] it brought that out naturally. (Post-Interview, Lines 437-441)

While Robert did not use LGEs for the intended purpose of generating new mathematics, his observation revealed the potential for LGEs to be used in ways that drive the mathematical goals of the lesson forward organically.

While Robert recognized a number of affordances of using LGEs, he cited a number of challenges, including issues related to the prompt, its enactment by the teacher, and its interpretation by students. The prompt for an LGE affects what students generate. In the post-observation, Robert implemented a task where students would generate three quadratic equations with different conditions. He called this the "Creator's Task." The created quadratic equations were then meant to be given to another student, "The Solver," and the "Solver's Task" was to solve each of the equations for the purpose of verification and practice. Robert presented the Creator's Task to students written on a worksheet (see Figure 4.16). Despite the written and verbal indication that the equations could be written in any order, several students generated equations in the specific order that they were presented (i.e., the first equation given by students had rational solutions and could be solved by factoring). This made the Solver's task less demanding, as the student who was solving knew which method to use on each equation and whether the solutions were rational or irrational, without having to analyze the quadratic equation for various aspects. Robert noted that he would be clearer on these directions in revising this task so that the solver could not assume a method of solution for each equation.

Despite the written and verbal indication that the equations could be written in any order, several students generated equations in the specific order that they were presented (i.e., the first equation given by students had rational solutions and could be solved by factoring). This made the Solver's task less demanding, as the student who was solving knew which method to use on each equation and whether the solutions were rational or irrational, without having to analyze the quadratic equation for various aspects. Robert noted that he would be clearer with the directions in revising this task so that the solver could not assume a method of solution for each equation.

<u>Creator's Task</u>: Create three quadratic equations, written in standard form, $0 = ax^2 + bx + c$, where a, b, and c are integers. Use the following conditions to write the quadratic equations. DO NOT solve the equations on this sheet, you may check your answer on a scrap sheet of paper.

First equation: Create a quadratic equation that has a rational solution and can be solved by <u>factoring</u>.

Second equation: Create a quadratic equation that has a rational solution that can be solved by <u>completing the square</u>.

Third equation: Create a quadratic equation that has an irrational solution that can be solved by <u>completing the square</u>.

Write the equations below in any order. Again, do not solve the equations on this sheet.

Figure 4.16. The Creator's Task, as presented by Robert from Post-Observation 4.

Some students did not interpret the task as Robert had intended: finding quadratic equations, rather than creating them, or creating quadratic equations that appeared difficult without verifying that they met the criteria. When Robert had asked a student how he knew that the solution to his quadratic equation was rational, the student explained that he had gone back and found an example that was factorable from past materials: "They found a problem. But they found one that fit the criteria" (Post-Interview, Line 462). For the quadratic equation with irrational solutions, some students generated quadratic equations with coefficients that they decided looked difficult, and assumed that the solution would be irrational, without verification:

> They just wrote one that looked really tough. And then I go, what's the solution to that? Oh, I don't know…I'm gonna make one that's nasty-…and then taking a good bet that it was going to be an irrational solution. (Post-Interview, Lines 463-473)

From Robert’s perspective, these students’ interpretation of the task evaded the purpose of recognizing the salient features of quadratic equations that made them solvable by factoring or completing the square.

In order to address the challenges above regarding how the task appeared and how it was interpreted by students, Robert suggested implementing the task during class time, rather than assigning it for homework:

> I really would have rather had it done in class than the whole for homework type of thing, so we could have those discussions on how you do that and everything…How we’re actually generating these problems. Because I think a lot of kids did maybe look back and pick an example that they’d already done. So maybe if I was there, I know they wouldn’t have done that. (Post-Interview, Lines 555-564)

Robert recognized how enacting the task during class time, rather than for homework, could diminish some of the challenges related to students’ interpretation of the task. It would also allow him to provide an opportunity to make the class example space visible, as in Research Lesson Four, and provide for opportunities to facilitate a discussion about processes for generating such equations.

The pressure to devote class time to the mathematics that is likely to appear on high stakes testing was a challenge for implementing and following up with LGEs. I had asked Robert if he had had a follow-up discussion with the class about how quadratic equations with various features could be generated, given his recognition that such a method arose naturally through implementing the Creater’s/Solver’s Task.

> No, it didn’t lead to that too much, ‘cause, um, I don’t know the specific need for that…Those were old Course 1 type questions [previous state assessment]. And they just

> don't have to do that anymore. Or [students] could always answer the question in a different way. [There are] multiple choice questions that say…which equation has these roots? – you don't have to go that way. (Post-Interview, Lines 529-534)

Later on in the interview, however, Robert had stated that he would rather have had students complete the task in class than for homework. One of his reasons for this was to facilitate a discussion about methods of generating quadratic equations with particular features. This suggested the tension between taking up and following through with the mathematics that arose from LGEs and the mathematics that students are asked to do on high-stakes assessments. From Robert's perspective, it was a challenge to devote class time to methods of generating quadratic equations, despite this arising naturally from the LGE, as students are not asked to generate quadratic equations with particular features on the state's high-stakes assessment.

Robert's main use of LGEs was for assessment and revealing student thinking. He also recognized the potential for LGEs to generate new mathematics, but appeared unsure of how and whether he should take this up in the classroom. This uncertainty arose from the tension between the new mathematics that students' generated and the content that appeared on the state's high-stakes test. There were other challenges that arose through the implementation of an LGE, such as the diminished level of student thinking and student interpretations that side-skirted the goals of the task. Robert realized, however, that these challenges could be met and possibly avoided by implementing the LGE in class, as opposed to homework, and also allowed for the potential take-up of the newly generated mathematics, despite the tension with high-stakes tests.

Robert continued to rely on external resources for the development of students' procedural fluency with skills, although he appeared to attend to the potential for connections, generality, and tiering of examples within these external resources in ways that he had not prior to the study. His continued use of external resources was due to practicality, but also, in part, to trepidation about design. Robert was drawn to the connections among mathematical ideas and concepts that external resources were trying to make apparent. For instance, Robert described how a homework set from emathematics instruction (https://emathinstruction.com/) showed that completing the square was useful for both solving quadratic equations and writing the equation in vertex form, and the associations between vertex form and the graph of the quadratic. Robert described how the connections supported students in making sense of mathematics: “They could even graph it and…say that, oh this is one, two, three, four point something, and is that what I get when I solve the quadratic? I think making those connections, giving kids those other…avenues to look at” (Post-Interview, Lines 63-67). The connections that Robert referred to in this instance were between the graphical representations of quadratic functions and the algebraic solution for the roots of quadratic equation, and how understanding the connection supported students in making sense of and evaluating their own result.

Rather than improvise examples, as Robert was likely to do prior to this study, Robert improvised questioning that tended to draw students’ attention to differences between examples via explicit contrast, or evoke the generality of an example. In an example from the homework in the post-observation (from https://emathinstruction.com/), students were asked to use the method of completing the square to find the zeros of $f(x) = 2x^2 + 12x + 5$. Robert asked, “What made this one more challenging?” Robert’s intention in asking this question was that students would discern that the leading coefficient was something other than one:

> I knew it would be more challenging…because of the leading coefficient. Obviously being greater than one, that added a step to the work, and just from past experience,

> knowing that that added step…could make things more confusing. (Post-Interview, Lines 142-145)

In another instance of enacting explicit contrast, Robert asked students, "What can you tell immediately from the function when it's written in standard form?" (Post-Interview, Lines 166-167). This served to draw contrast between standard form and vertex form, in which the *y*-intercept can immediately be read from the former, but not the latter. In a later example from the homework assignment, a student said, "The answer's irrational so it's unfactorable." In response, Robert asked the class, "Is this always true?" Robert explained this choice, "It's extending it from this example to, can we generalize just based on this one example?" (Post-Interview, Lines 182-183). His question was meant to evoke the generality of unfactorable quadratic equations in relation to their roots as irrational. While the external resource, itself, was not designed in ways that clearly encouraged students to discern structure, Robert found opportunities to enact explicit contrast and generalization through his questioning.

> Robert expressed some trepidation about taking on design in an individual capacity: "If I were creating my own, maybe I wouldn't think of having them look at simplest radical form and rounding to the nearest hundredth in the same lesson. Maybe I would have been focused more on just completing the square, and yes, connecting it to old material. Maybe I wouldn't have been interested in looking at the graphing aspect…This is an example of a great lesson I use because it has all things included that maybe on my own I wouldn't have necessarily though of to use all in one lesson. (Post-Interview, Lines 103-109)

While external resources, such as emath instruction (https://emathinstruction.com/), may not have revealed structure to students because of variation in multiple aspects, Robert recognized these sets as better than random and they alleviated some of his concerns about what might be left out by relying on his own designs. For instance, in the example discussed previously, in which students were asked to use the method of completing the square to find the zeros of $f(x) = 2x^2 + 12x + 5$, Robert had asked, "What made this one more challenging?" with the intention that students would discern that the leading coefficient was something other than one. In class, however, a student said, "Because there's a plus five at the end, and two doesn't go into five evenly." Robert realized, then, the variation in multiple aspects within this example, "We would [factor] out a two normally…I forgot about that too; the five is what makes it more challenging…it has a leading coefficient, and then, they're not all even numbers," (Post-Interview, Lines 160-163). Alluding to the connections and the structure that Robert discerned in these sets, and the multiple ways of asking a question and for forms of the solution, Robert said, "So instead of just having a page where it's all these quadratic equations, solve them all by completing the square, us thinking that they [students] would now really understand, or they'll [students] really be able to complete the square," (Post-Interview, Lines 133-135). Robert had discerned that leading coefficient of two as challenging, but a student discerned the constant of five challenging because a common factor could not be factored out evenly. Variation in multiple aspects allowed for multiple aspects to be discerned, including those that were intended and those that were not. Robert saw this type of external resource as better than a random collection of exercises, as it addressed conceptual understanding, in addition to procedural fluency, in some ways, but the resource also required additional teacher questioning to enact the Principle of Explicit Contrast and generalization that was not readily apparent.

Summary of Changes in Robert's Use and Views of Examples. Like Tori, Robert's purposes for using examples in instruction remained invariant from before to after the learning study intervention. Robert developed new strategies, however, for the design and enactment of sets of examples that continued to align with his purposes. Robert found that careful design of the examples used in structured exercises allowed students to make choices about what solution method to use in particular situations. He recognized that a perfect example clearly led to one solution method over another to allow for student students' discernment of the differences between the examples. Robert developed his knowledge about enacting tasks that incorporated structured exercises and LGEs through implementation. For instance, through implementing an LGE as a homework assignment in the post-observation, several challenges arose. Robert described how these challenges could be diminished by changes in the design (e.g., wording of the prompt, layout) and the enactment (e.g., in class versus as a homework assignment). Robert also recognized the potential for LGEs in generating new mathematics naturally, and the possibility of taking this up in future lessons. Because Robert only used LGEs as homework assignments in this study, I cannot determine how Robert developed his knowledge about how to take up and use the examples generated by students in the classroom. He likely had some ideas about how to do so (e.g., discussion about the methods of generating quadratic equations with particular features), but it was not clear from this study how that would actually happen or how it would affect Robert's knowledge about the design and enactment of LGEs. Robert had a strong desire prior to the learning study intervention to develop both students' conceptual understanding and procedural fluency. This may have been related to the potential he saw of structured exercises in helping students to discern when and why particular solution methods might be used. Once Robert had some success with structured exercises in allowing students to discern the salient features of examples that pointed to one solution method over another (Research Lesson Two), Robert continued to use structured exercises in this way throughout the study (Observation

3 and Post-Observation 4).

Robert's view of the affordances and challenges of LGEs influenced how he chose to use them. His view that LGEs were particularly useful for assessment and making students' thinking visible to him, influenced his choice to use LGEs as homework assignments. Robert struggled with the tension between taking up "new mathematics" generated through the use of LGEs and adhering to material that appeared on the state's high-stakes testing. Because of this, Robert appeared less sure of enacting LGEs during class, although he recognized how doing so could address other challenges that he faced. In particular, Robert recognized how student verification that the conditions of LGEs had been met could service his purpose of developing fluency and conveying generality.

Robert generated sets of structured exercises across content that allowed for the comparison of various solution methods through restricting variation. He generated LGEs of the type "Give an example of…(another and another)," often using a reversal strategy of examples typically asked in class, as Tori did. Robert also generated LGEs of the type "Additional Conditions," a variation on "Give an example of…(another and another)." Robert was outspoken about lessening the restrictions connected to variation placed on LGEs in order to open the example space of what could be made available for students' consideration. He generalized the relationship between the prompt and examples generated by students and began to consider how the timing of the LGE, and whether it was given during or outside of class time, influenced what was made available to the learners and to himself, as the teacher. Table 4.18 summarizes Robert's example space of use and view of examples after the learning study intervention. Changes to his example space prior to the learning study intervention (see Table 4.3) are bolded.

Table 4.18

Robert's Example Space of Use and View of Examples after the Learning Study Intervention

Characteristic	Summative Description
Population	Predominant use of examples for connection and practice for fluency remained invariant. Shift to placing more value in planned designs that continue to use tiering, foreshadowing, and restricted variation. Use of restricted variation expanded to include contrast using a strategy of "varied versus unchanged," particularly to convey flexibility between solutions methods (i.e., one problem multiple solutions). Suggestion of verification of LGEs as a means of practice for fluency. LGEs predominantly used for assessment, as a way of making student thinking visible to the teacher. Potential use of LGEs for generating new knowledge. Enactment strategies shifted to include student-given explanations of similarities and differences between examples and observed relationships among critical aspects. Questioning strategies to enact explicit contrast and evoke generality.
Generativity	Generated multiple sets of structured exercises for the comparison of various solution methods (e.g., same quadratic function, presented in varying forms to link to varying solution methods: graphical, factoring, completing the square). Generation of prompts for LGEs using reversal of examples asked in class with "Additional Conditions" and "Give an example of…(another and another)".
Connectedness	View of examples for connection well-connected to design of structured exercises for discernment of relationship between given form and solution method. View of LGEs well-connected to assessment and loosely connected to practice. Disconnected to a purpose of connection and using LGEs to explore the range of permissible change of critical aspects and relationships among critical aspects.
Generality	Generalized sets of structured exercises to compare solution methods across content. Generalized LGE prompts "Give an example of…(another and another)" and "Additional Conditions".

Lynn Gray

Prior to the learning study intervention, Lynn primarily used examples for practice and assessment, uses that were closely linked within her RTI classes. Lynn described unfolding as a design strategy that she used for sets of examples prior to the learning study intervention. Over the course of the study, Lynn equated her conception of structured exercises with her conception of unfolding or scaffolding sets of examples:

Researcher: What do you see as the role of a set of structured exercises?

Lynn: I feel first of all that it lays it out for them slowly. It allows them to be confident, which is a big piece of a student's willingness to take risks. I think you have to slowly unfold things for them, especially at the RTI level.

…

Researcher: Could you talk a little bit about the difference, if there is a difference for you, between structured and scaffolded?

Lynn: I feel like they're the same. Because I feel like the scaffolding is the structure of it. Like how is this going to make the most sense to them? (Post-Interview, Lines 336-346)

Lynn's description of structured exercises as scaffolded exercises or examples was consistent with how she described her design of sets of examples through unfolding prior to the learning study intervention. It appeared as though Lynn did not perceive contrast between sets of scaffolded examples and sets of structured exercises, and so her conception of structured exercises did not differ from her design strategies at the outset of this study.

While Lynn wanted to develop her students' conceptual understanding, her notions of student success trumped Lynn's willingness to take risks within her sets of examples. Her notions of student success included students' feeling comfortable knowing what they had to do for a given example and their confidence that they could complete the example. Lynn's notions of student success were largely influenced by her role as the RTI teacher and how her teaching assignment had been constructed by the school as a class for re-teaching, reviewing, and practice. In contrast to Robert, who sought opportunities for students to develop flexibility with different solution methods, Lynn often presented a single step-by-step solution method to students because she believed it would ease students' confusion and build their confidence. This instructional method, however, created conflict with some of Lynn's intentions, as it did not appear to support the development of students' conceptual understanding.

Lynn's conception of LGEs involved student generation of word problems based on given conditions. In Observation 2, Lynn gave students the following task:

> Given the linear equation: $y = 10x + 100$
>
> Describe a relationship that could be represented by the above equation.

Lynn told the students, "Think of a real-life example that could be represented by the equation. Think about the examples we've done – could give you an idea" (Observation 2, Lines 159-160). The students struggled with the task, voicing their confusion over what to do. Lynn ultimately completed the task for the students:

> Lynn says, "So we need to think about a real life situation where we had a rate of change of ten and an initial value of 100." Lynn asks if anyone has a gym membership. The students say no. She starts to talk about the example of a gym membership – say you have to pay $100 to become a member at the gym. Then Kevin says, "Costco." Lynn says, "Okay, maybe you have to pay $100 to become a member at Costco. Then what could the 10 represent?" Aidan says, "Ten points per purchase?" Lynn responds, "Ten

points per purchase? Ok. How about $10 per month to keep your membership?"

(Observation 2, Lines 162-170)

The only aspect allowed to vary in this case was the context, which is not a critical aspect of linear equations. Lynn may have wanted to generalize the use of linear equations across contexts through connecting them to students' real-life experiences. It was likely, however, that her eighth grade students were lacking in real-life experiences that they would recognize and interpret as linear equations, particularly ones in which 100 as a *y*-intercept and ten as a rate of change would make sense. Similarly, in Post-Observation 4, Lynn asked students to create a real-world problem in which students would need to find the volume of a given three-dimensional figure. As before, the only aspect that was allowed to vary was the context of the word problem. Lynn seemed to be mostly concerned with students' abilities to connect the mathematics they were learning in the classroom to real-life situations. In this sense, Lynn's use of tasks, which she perceived as LGEs, did not allow for student discernment of mathematical structure. Due to Lynn's notions of student success, she equated students' confusion around this example with their lack of success, and associated this with her students' difficulty with LGEs.

Lynn's view of the affordances of structured exercises and LGEs was consistent with her view of examples as assessment for instructional decision making. The strength of using structured exercises and LGEs was the information that could become available to the teacher through their use:

Researcher: What do you feel like you learned about structured exercises and LGEs?

Lynn: Well both of them provide a lot of feedback as a teacher, to be able to assess what your students are learning. The more deliberate you are about what examples you're giving, the more specific…the information you get back

> from a student, based on how you set it up. Instead of just setting up random examples of something, and then waiting until there's a pattern of some sort that's really obvious, that you can then understand as a misconception. (Post-Interview, Lines 475-481)

Lynn recognized how structured exercises and LGEs could be intentionally designed. From Lynn's perspective, the intentional design served the purpose of providing information for the teacher, in terms of instructional decisions. When I asked Lynn what she saw as the role of LGEs, she said:

> It's a formative assessment to see how much they know about whatever topic you're working on, and then to help me decide where to go from there with them. What misconceptions need to be addressed? Do I need to spend more time on this topic? Or can I move on? (Post-Interview, Lines 363-366)

This view of LGEs appeared in conflict with the way in which Lynn attempted to incorporate LGEs into her lessons as student-generated word problems based on given information and contextualized in real-world situations. Students' ability to generate a word problem did not necessarily give any helpful feedback on the mathematical nature of their understanding of a particular topic, nor did it reveal misconceptions related to mathematical structure. It may be that Lynn held this particular view of LGEs, but had not yet developed a means to incorporate them into her lessons as formative assessments that provided useful feedback.

Lynn expressed a number of fears regarding using LGEs and structured exercises in her classroom, some of which were associated with her own beliefs about her capacity as a teacher and others associated with her notions of student success. One of Lynn's fears of using LGEs was knowing whether or not students were correct in the moment. For the word problem task given in Post Observation 4, Lynn had contemplated giving students a figure with no dimensions and allowing students to choose their own dimensions. In discussing her choice to use the task that included dimensions, Lynn discussed her fear of not being able to assess the correctness of students' work.

> I guess my other fear was that if they had put their own measurements on, I didn't know what the correct answers were for that, and I didn't know, if each one of them had different answers, how would I help them, like make [sure] that they were doing it correctly? (Post-Interview, Lines 114-116)

Lynn immediately offered a possible solution to this fear, however, recognizing that LGEs could be taken up and used in a subsequent class.

> If I didn't need to correct it right then, we could do something with it the following day…So, it sort of gives you a springboard for your lesson, and then you get to assess, what's the best way to go about given them feedback on this, or taking it another step? (Post-Interview, Lines 117-125)

Lynn's view was that she had to assess the correctness of students' responses and through that assessment, provide students with feedback or make instructional decisions. Her fear at not being able to assess students' correctness in the moment influenced her hesitancy in incorporating LGEs into her lessons.

Lynn's fears associated with using LGEs also regarded the mathematical trajectory of the lesson and her ability to respond in the moment. Lynn said, "It makes me a little nervous to do the LGEs because you have to think real quick, and you might not know what they're [students are] thinking when they do it [generate an example]" (Post-Interview, Lines 154-155). Lynn was concerned about her ability to notice important mathematics and respond in-the-moment. Her earlier suggestion, that LGEs could be collected and followed up with at a later time, could serve as one strategy to alleviate that fear.

Lynn's notions of student success shaped her willingness to try LGEs within her lessons, her design of the prompt for the LGE, and her implementation of the LGE.

> My students didn't have a lot of experience with creating things on their own in class. I just haven't done a lot of that with them. The nature of my class, a lot of times, is just the practice piece and identifying misconceptions and fixing those with them to help them be more successful. Not that I don't like doing something like [LGEs], I just don't feel like I've had enough opportunities to really try it out. And I'm glad that I was able to [try an LGE], and I want to incorporate more of that…So I tried to make it [constricted]. I wanted to lay it out pretty solid so that there weren't a lot of open ends for [the students] to have to be creative at this point, just to start out. So that's why I gave them the dimensions and tried to give them just a little bit of an idea of what I wanted. (Post-Interview, Lines 88-97)

Citing students' lack of experience with providing examples, or in this case, writing word problems, Lynn equated a task being constricted with student success. She believed that openness and the need to be creative could potentially be detrimental to student success. As Lynn explained, "The minute that [the students] start to feel they're confused, they shut down or get nervous about it" (Post-Interview, Lines 354-355). This was in contrast to Robert's view of LGEs and the discussion that the teachers had in Research Lesson Four, where they decided that greater restriction on the conditions for the LGE would be detrimental to the variation in examples that students generated and would not drive the lesson forward. Lynn continued to rely on the view demonstrated in Research Lesson Three that for her students, success was associated with lack of struggle. Lynn did not have strategies for implementing structured exercises and LGEs that balanced the mathematical challenge and the need for support. In addition to collecting LGEs at the end of a lesson and figuring out how to follow-up with them for the next lesson, Lynn needed other, low-risk, strategies for her early attempts at incorporating LGEs into her lessons. Lynn needed structured exercises and LGEs that would give her and her students not just experience, but successful experience.

While Lynn did not conceive and implement structured exercises and LGEs in the intended ways, she increasingly made attempts to enact explicit contrast. In Observation 2, asked students, "What's different between this problem and the other ones?" (Observation 2, Line 139). Her question drew attention to the differences in the given information (a starting value and rate versus two points) in word problems that asked students to write linear equations that modeled the situation. In Research Lesson 3, Lynn opened the space for student talk regarding their observation of rates and unit rates. In Observation 3, which was a continuation of Research Lesson 3, Lynn again opened the space for student talk, which provided an opportunity for explicit contrast between the two forms of the unit rate and the magnitude of the unit rates in order to make a comparison. The class started with the previously calculated unit rates displayed on the board (see Figure 4.17).

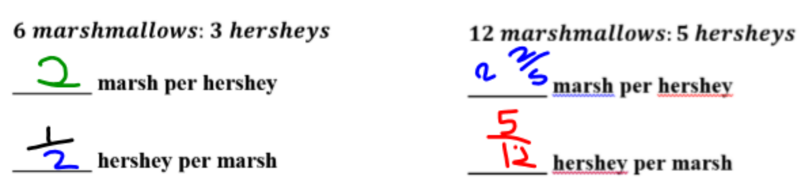


Figure 4.17. SMART Board display of unit rates at the beginning of Observation 3

Lynn asked the students to decide which recipe was more chocolaty. Her student, Holden, explained:

"If you look at [marshmallow] to [Hershey], there's two [marshmallow] per Hershey, and if you look at the other [recipe], it's $2\frac{2}{5}$, and $2\frac{2}{5}$ is bigger than ours [2 marshmallows per Hershey]. So that could mean their [recipe] is more marshmallowy, and if you look at the bottom [unit rate], our number is bigger." (Observation 3, Lines 30-32)

Contrast was enacted in this situation through Holden's observation that he could determine which recipe was more chocolaty by comparing either the number of marshmallows per Hershey bar or the number of Hershey bars per marshmallow. He also recognized that a larger number of marshmallows per Hershey bar indicated that the recipe was more marshmallowy, while a greater numbers of Hershey bars per marshmallow indicated that the recipe was more chocolatey.

Summary of Changes in Lynn's Use and Views of Examples. Lynn's predominant use of examples and exercises for assessment and practice remained largely unchanged throughout this study. This was, in part, influenced by her notions of student success and the nature of her classes as support classes for RTI students. Lynn's use of structured exercises and LGEs appeared to be influenced by her conception of what structured exercises and LGEs were. Lynn saw structured exercises as examples that had been scaffolded, in the sense of unfolding. For Lynn, this often meant either an increase in difficulty level or a pattern of enactment in which Lynn modeled an example followed by students trying an example and assessing them on their performance. Because Lynn did not discern contrast between sets of structured exercises and sets of scaffolded examples, Lynn likely assumed that this was something that she was already incorporating into her lessons.

Lynn conceived of LGEs as learner generated word problems based on particular givens that were contextualized in real-world situations. While this may have served to encourage students to make connections between mathematics in the classroom and mathematics in their own life, it did little to shed light on the mathematical structure of classes of examples. Lynn's fears associated with enacting LGEs played a factor in her willingness to implement them in instruction. She worried about her ability to respond in the moment and to assess the correctness of students' responses. Lynn suggested assigning an LGE toward the end of class to alleviate some of her fears. As she described, this would allow her an opportunity to make sense of students' thinking and make decisions about either providing feedback or the next instructional steps. This suggests that Lynn had developed a possible strategy for beginning to incorporate more LGEs into her lessons going forward. While Lynn struggled in her attempts to incorporate structured exercises and LGEs into her lessons, there was evidence to suggest that she began to enact Explicit Contrast within her lessons through questioning and opening up the space of learning for student talk. Hence, while there was little evidence of changes in the population of Lynn's example space of use and views of examples in terms of purpose and design, there were some changes in strategies for enactment. Because Lynn did not progress to designing or using sets of structured exercises or LGEs in her lessons, there is no evidence of changes to the generativity, connectedness, or generality of her example space. Table 4.19 summarizes Lynn's example space of use and view of examples after the learning study intervention. Changes to her example space prior to the learning study intervention (see Table 4.4) are bolded.

Shannon Edwards

As a part-time teacher, Shannon generally used materials that had been designed by either Robert (for her eighth grade mathematics class) or Tori (for her seventh grade mathematics classes). Prior to the learning study intervention, Shannon talked about the role of examples as her means of explaining mathematics to students. She interpreted the variation in sets of

Table 4.19

Lynn's Example Space of Use and View of Examples after the Learning Study Intervention

Characteristic	Summative Description
Population	Predominant use of examples for assessment for instructional decision-making and practice for fluency unchanged. Continued use of sets of examples that included mixed classes of examples with simultaneous variation. Unfolding strategy equated with scaffolding and structured exercises. Designed contextual-based LGEs (i.e., generation of word problems) with many restrictions on the space of variation. Some strategies for enactment of explicit contrast, including questioning and some elicitation of student talk.
Generativity	Lack of contrast between scaffolded and structured exercises indicated that the generation of new sets of structured exercises was unlikely. Possible generation of contextually-based LGEs for assessment, but no indication of the possibility of generating LGEs for the exploration of mathematical structure.
Connectedness	View of examples for the use of assessment and practice for fluency well-connected mixed sets of examples and simultaneous variation. Use of examples for developing students' conceptual understanding loosely connected to some sets of scaffolded examples.
Generality	Over-generalization of sets of examples that are unfolded or scaffolded and structured exercises. Over-generalization of LGEs as learner generated contextually-based word problems.

examples that she used as exposure to the variations that students would see within a class of examples, and often enacted them using an "I do, we do, you do" instructional model. Throughout the learning study cycles, Shannon made a number of insightful comments that recognized the need for students to discern particular critical aspects of a mathematical concept. In Research Lesson One, Shannon commented on the need for students to discern between expressions in which grouping symbols did matter and expressions in which grouping symbols did not matter and when they could be dropped. In Research Lesson Two, Shannon suggested that students' struggles with point-slope form of a line might be attributed to students' conceptualization of equivalency. Students needed to generalize that an expression can be equivalent to other expressions, as well as single numerical values or a single variable. In Research Lesson Three, Shannon recognized that it was important for students to discern a unit rate as a mathematical object, rather than a computation, and wondered whether whole unit rates would support students in seeing the calculated value as a quantity with mathematical and practical meaning. In the discussion about Research Lesson Four, Shannon suggested that students had not generalized the meaning of base to include numbers. The students seemed to discern that coefficients are numbers, and when multiplying exponential expressions, the numbers are multiplied. Further, throughout the learning study cycles, Shannon suggested strategies for enacting Explicit Contrast: (1) reading aloud to emphasize the difference in expressions depending on the placement of parentheses, (2) student-to-student talk about solution methods, and contributed significantly toward the design of the sets of examples. Despite Shannon's contributions to the design of examples and insight about the discernment of critical aspects, Shannon struggled to incorporate structured exercises and LGEs into her lessons.

Shannon's purpose for using examples became more focused on the goal and purpose of the lesson. She talked about how her use of examples had become more mindful over the course of the learning study cycles:

> I think I consider [examples] differently…The learning study really helped push me away from, we just need to go through examples in class. Every kind of example that [students] could possibly see so that they know everything. And it's not that. It's giving specific examples that focus on what you want them to know…I think every example and structured exercise that we do needs to have a goal in mind…It's what do I want this

> example to show them? What do I want this example to teach them? What do I want them to get out of this example? So it's more of the pre-work that determines what comes out of it. (Post-Interview, Lines 274-296)

Shannon recognized the benefit of purposefully planning examples to bring particular aspects of a class of examples to the fore of learners' awareness. By the end of four learning study cycles, Shannon no longer viewed the purpose of examples as exposure to all of the variations of a particular class of examples that students might see, as she did prior to the learning study intervention.

Shannon continued to use materials designed by Tori and Robert, so there was little observational evidence for changes to her design of sets of examples. She continued to describe a variation and sequencing strategy of basic to complex and connections to previous content for the purpose of explaining mathematics:

> Examples are a way to scaffold their learning. Start with an easy one, especially with a new concept or maybe bring in a bridge from an old concept to a new concept. I don't know if scaffold is the right word. But kind of build upon, here's an easy one, now let's look at a complicated one, and maybe give them examples of, in general, questions they might see. (Post-Interview, Lines 186-190)

Shannon recognized the importance of incremental changes between examples so that conceptual leaps between examples were not large. Shannon's design strategy for sets of examples seemed to parallel the "Pointing Toward Generality (particular, peculiar, general)" activity from the Thinkers text (Bills et al., 2004). Rather than the students generating the examples, however, Shannon did so. While Shannon either did not recognize or did not articulate the alignment of her description of example design with the "Pointing Toward Generality," strategy, Shannon intended to convey the generality of the set of examples to all such examples of the class that students might encounter ("Give them examples, of, in general, questions they might see"). After the learning study intervention, Shannon considered how the generality of a class of examples could be conveyed through the design of the set of examples.

Like Lynn, Shannon's conception of structured exercises seemed to be equated with her conception of scaffolded exampled. I asked Shannon what she saw as the role of structured exercises. Shannon said, "So I see [structured exercises] as more higher-level thinking. We're going to start here and then slowly build upon the complicated-ness…and push the kids to do what they thought they couldn't to expand their level of thinking" (Post-Interview, Lines 198-201). Shannon located the variation in a set of structured exercises within the complexity of the exercises, rather than within the critical aspects. The only difference that Shannon alluded to between structured exercises and scaffolded examples was the "higher-level thinking," required of structured exercises, which may have been in reference to the generalization of relationships among critical aspects that sets of structured exercises typically are meant to convey.

Despite Shannon's insight into the intentional purpose and pre-planning of examples, evidence of Shannon's take-up of ideas related to variance and invariance emerged during her in-the-moment decisions during teaching. In order to address a misconception that students had, Shannon produced a set of examples, in-the-moment, that used patterns of variance and invariance for the purpose of fusion, or discerning the relationship between two critical aspects. As students were working independently on a set of examples to practice simplifying exponential expressions that were being divided, Shannon noticed that some students were incorrectly simplifying 6^0 as zero. Shannon addressed this misconception through a list of examples:

Shannon says, "If I had 6^3, what is that?"…A student says 216. Shannon asks, "What is 6^2? 6^1?" and gets student responses. Shannon has the following list written out on the side of the SMART Board:

$$6^3 = 216$$

$$6^2 = 36$$

$$6^1 = 6$$

$$6^0 = 1$$

Shannon asks, "How do I go from six to 36?" (Students say, "Times six.") And 36 to 216? ("Times six.") So let's go back the other way." Many students loudly go, "OH!" "We're understanding!" Sidra says, "Mind blown!" Shannon says, "So any number to the zero power is one, because of this. When you go up, you multiply by that number. When you go down, you divide by that number, so when you divide a number by itself, you get one." (Observation 3, Lines 94-103)

Shannon generated a set of examples in-the-moment, which could be considered a set of structured exercises. Within this set of examples, the base of the exponential expression, six, remained invariant, while the exponent decreased by one from an exponent of three to an exponent of zero. While the exercises were completed "with the grain" ("If I had 6^3, what is that?"), Shannon's encouragement for students to look "across the grain" (Bill et al., 2004) revealed the mathematical structure of exponents. Increasing the exponent by one is multiplying by the base. Decreasing the exponent by one is dividing by the base. Thus, Shannon designed a set of examples in-the-moment that attended to patterns of variance and invariance and enacted them using a new strategy of looking "across the grain," between examples, to reveal the mathematical structure.

In the post-observation, after the learning study intervention, Shannon had asked students during the prior class period to write a word problem that could be solved using a linear inequality. Unlike Lynn's similar attempt at using an LGE, Shannon designed the prompt to be open to allow for variations in the context, the numbers, and the solution. Shannon expected that this would generate a set of practice examples and she decided to collect the student generated word problems and redistribute them to students as a practice example to start class the next day. As students began working, students' lack of understanding about the structure of word problems was unexpectedly revealed. I heard a student say, "This doesn't even have a question!" (Post-Observation 4, Line 26), and three other students came up to Shannon saying, "Mine doesn't ask a direct question" (Post-Observation, Lines 28). It was evident that students' had struggled to write word problems with necessary information, but this was only revealed as other students attempted to solve them. Shannon discussed a few of the student generated inequalities with the class:

> Shannon reads a problem: "Ms. M gets paid $100 plus a $10 bonus. She wanted to reach at least $250." Shannon asks the class, "What's missing?" A student says, per hour, or…" Shannon says, "Right. Lucy said to me, where am I supposed to put my variable?" Shannon reads another example: A family has a budget of $150. They are going to an amusement park and have to pay $25 to get in, and it's $10 per ticket pack. What was good about this?" Jared says that it has a budget. He also says that it tells them how much to get in and how much per ticket pack. Shannon says, "Right. It's simple, but it has everything it needs"…Shannon reads another example, "A songwriter makes 70 songs per week, plus an additional 15 songs per album written. This week, the songwriter set a goal of writing at most 100 songs"…Shannon says, "What didn't it ask?" A student says,

> "How many." There is some discussion about whether the question needs to ask "how many" of something. Shannon at first says yes, but then refers to the homework questions saying they didn't ask how many." (Post-Observation 4, Lines 48-69)

It was clear that Shannon struggled with how to use this set of contextualized LGEs or word problems to move the lesson forward or make a mathematical point. She contradicted herself by saying that one example "has everything it needs," but then saying that the next example did not ask a question, such as "How many….?" Within the set of homework examples, the information related to the scenario was presented first, and then part (a) of the prompt indicated the quantity of interest ("how many buckets") for the inequality (see Appendix L). For instance, a prompt on the homework read, "Write an inequality to show how many buckets, *b*, of fish the whale needs to be fed to meet its quota for the day". Like Lynn, Shannon conceived of LGEs as student generation of contextualized word problems. Shannon did not anticipate that the openness of the prompt would provide an opportunity for students to discern the structure of word problems, including necessary and given conditions and the conclusion. The openness of Shannon's LGE was in contrast to the restrictedness of Lynn's LGE. Unlike Lynn's attempt at an LGE, Shannon's attempt opened the space of learning, albeit in unexpected ways.

During the Post-Interview, I asked Shannon about the set of word problems that students had generated for this observation. Shannon referred to her disappointment about the lack of information that students included in their word problems:

> My disappointment was the lack of information they gave. Some of them didn't give enough information at all. I had been hoping through the examples we had already done in class, they would kind of know what information was important, and some of them left

some information off. And it made it difficult to solve them. (Post-Interview, Lines 84-88)

Shannon assumed that students would independently discern the critical aspects of a word problem about linear equalities through completing a number of examples of word problems about linear inequalities. Shannon went on to identify critical aspects of word problems about inequalities as (1) the initial or constant value, (2) the rate, (3) the question statement that defines the unknown variable, and (4) the given conditions. Shannon considered how she could bring the critical aspects of linear inequality word problems to students' awareness:

> So [the question statement] would be a critical aspect that I guess when I go over examples, if I'm going to have them do these learner-generated, pointing out, okay this is important. It's important to ask what exactly are we trying to find? Maybe one of the ways to help that is to tell [students] to define their variable. (Post-Interview, Lines 101-104)

Shannon suggested using a verbalization strategy to explain to the students why the question within a word problem is a critical aspect. She also suggested revising the prompt to include the directive for students to fine their variable to give some indication of the unknown value the solver of the word problem was intended to find.

Shannon's use of LGEs was for assessing students' understanding of the concept. Shannon said, "If [students] can write an inequality story then they understand how to break down a word problem [and] how to write an inequality from a work problem. So it gives me a good picture of their understanding" (Post-Interview, Lines 269-271). Like Tori, reversal was a strategy for designing prompts for LGEs. Since students had been given word problems in class to solve, the reversal of this process was to ask students to write a word problem. Shannon's conceptualization of LGEs positioned students in the role of teacher:

> Have you learned enough to make your own [example]? Like I always think the best way to show me that you've learned is to teach someone else. Because if you know enough to teach someone else, then you've understood the concept. [LGEs] show [students'] level of understanding. (Post-Interview, Lines 204-207)

Shannon conceived of LGEs for the purpose of assessing students' understanding, implicating that LGEs should be used after instruction. Shannon, however, recognized this limitation in her own conceptualization of LGEs:

> I need to figure out how to fit [LGEs] into the lesson better…Maybe figure out how to use LGEs before a concept. Because I feel like right now, my idea of LGEs where I can use them is, okay, now that they've learned how to read a word problem and solve an inequality, now they can write it. But then I have all these LGEs, but we're moving on to the next thing, so where do they fit in? (Post-Interview, Lines 301-307)

Despite having used LGEs to motivate Research Lesson Four, Shannon struggled with how she could incorporate LGEs into her lessons other than as a means of assessing students' understanding of a concept.

Because Shannon predominantly used materials designed by Tori and Robert, there was more observational evidence of changes to her enactment of sets of structured exercises and LGEs than her design. In Observation 3, during the learning study intervention, Shannon enacted a set of examples designed by Tori. Shannon displayed the following three examples on the SMART Board:

$$1.)\ 2^2 \cdot 2^3 \qquad 2.)\ 5^3 \cdot 5 \qquad 3.)\ y^4 \cdot y^2$$

Shannon completed the first example with the students, writing out $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$. She told students to complete the next two examples, including the expansion and the simplification. After allowing the students a couple of minutes to complete Examples 2 and 3, Shannon asked two students to provide answers, and then prompted students to generalize the relationship between the factors and the product:

> Shannon says, "From these examples, is there a shortcut?" Kurt says, "You can take the two exponents and simplify them." Shannon says, "Okay. What do you notice about the bases?" A student says that the base stays the same, and Shannon gets another student to clarify that the exponents are added, rather than 'simplified.' Shannon writes on the SMART Board: ★SAME BASE…KEEP base, add exponents. (Observation 3, Lines 25-29)

Shannon used this set of examples to enact fusion. Critical aspects included the exponents and base of the factors and the exponent and base of the product. Students initially discerned what changed – the exponents, and articulated that the exponents of the two factors are added to obtain the exponent of the product. Shannon had to prompt students to consider the relationship between the base of the factors and the base of the product ("What do you notice about the bases?"). This set of examples, however, did not enact generalization as a pattern of variation. Generalization is a pattern of variation that separates critical aspects from non-critical aspects, attuning students to notice the aspects that can vary and the range of that variance while still maintaining examplehood (Marton, 2015). One student, Nicole, said that in the given examples all of the bases were prime, and she asked if they had to be prime. Shannon said no, and used a verbal explanation to convey the generality of Example 3: "*y* is just a variable that represents any number" (Observation 3, Line 31). While the set of examples allowed for fusion and the discernment of the relationships between the critical aspects, it did not provide an opportunity for students to generalize the base over the set of real numbers. Shannon attempted to convey the generality of a single example through the strategy of a verbal explanation.

Shannon told the class they were then going to focus on dividing exponential expressions. She did Example 1 with the class: v^3/v^2:

> Shannon asks the class, "What is v^3? What is v^2?" Shannon shows the expansion: $\frac{v \cdot v \cdot v}{v \cdot v}$. Then she crosses out pairs of vs in the numerator and denominator using red, which leaves v. She asks students to do Example 2: $\frac{b^5}{b^2}$. Taylor puts her hand up and says, "I notice something!" Shannon goes through the second problem with the class, expanding, canceling like factors, and simplifying: $\frac{b^5}{b^2} = \frac{\cancel{b \cdot b} \cdot b \cdot b \cdot b}{\cancel{b \cdot b}} = b^3$. Shannon says to the class, "Can we describe a short way to simplify?" Taylor says, "Subtract the exponents." Shannon whispers to Taylor, "What about the base?" Taylor says, "Keep the base." (Observation 3, Lines 58-65)

Again, Shannon enacted fusion as a pattern of variation with a set of examples. In this set of examples, however, a student had already began seeking out a generalization of the relationship between critical aspects ("I notice something!"). As in the previous set of examples, Shannon prompted the students to draw awareness to the base as a critical aspect that remains invariant between the factors and the product.

Summary of Changes in Shannon's Use and Views of Examples. By the end of four learning study cycles, Shannon no longer viewed the purpose of examples as exposure to all of the variations of a particular class of examples that students might see, as she did prior to the learning study intervention. Shannon appeared to place a stronger emphasis on strategic choices of examples to impart a sense of generality to students regarding the class of examples. Like

Lynn, Shannon struggled to incorporate structured exercises and LGEs into her lessons, but unlike Lynn, Shannon appeared to have a strong conceptualization of structured exercises and LGEs as useful for conveying the generality of a class of examples. Shannon conceived of structured exercises as sets of examples that increased in complexity and conveyed generality. Her description of a sequence of examples was reminiscent of the "Pointing Toward Generality (particular, peculiar, general)" activity from the *Thinkers* text (Bills et al., 2004). Shannon primarily conceived of LGEs for the purpose of assessment and designed a contextual-based LGE that she intended to use as practice with her students. Unlike Lynn's design, which was restricted, Shannon's open design allowed for the space of variation to be open, revealing students' lack of discernment of the critical aspects of a word problem for linear inequalities. Changes to Shannon's example space of her use and views of examples were most often revealed during enactment, as she continued to use materials designed by Tori and Robert. She suggested enacting the Principle of Explicit Contrast through a verbal, reading aloud, strategy in Research Lesson One. In an observation during the learning study intervention, Shannon generated a set of structured exercises in-the-moment and enacted a strategy of asking students to look "With and Across the Grain" (Bills et al., 2004) to discern the mathematical structure via fusion.

Shannon talked about, but did not demonstrate, designing sets of examples using basic to complex sequencing that reflected the activity "Pointing Toward Generality (particular, peculiar, general)" (Bills et al., 2004). Like the other teachers, a strategy for generating LGEs was through a reversal of the typical questions asked, including asking students to give an example of a word problem. Despite Shannon's recognition that examples should be imbued with a purpose, her primary means of generating structured exercises was through improvisation, like Robert prior to the learning study intervention. Shannon had generalized LGEs to be a type of assessment. Like

Lynn, Shannon needed more design and implementation strategies for structured exercises and LGEs, and needed opportunities to build her confidence around designing and enacting tasks that incorporated structured exercises and LGEs. Table 4.20 summarizes Shannon's example space of use and view of examples after the learning study intervention. Changes to her example space prior to the learning study intervention (see Table 4.5) are bolded.

Summary of Changes in Teachers' Use and Views of Examples

I identified changes in each of the teachers' examples spaces of their use and views of examples from prior to the learning study intervention to after. For all of the teachers, the changes were primarily within the population of their example spaces, as they considered how structured exercises and LGEs could be designed and implemented. By the end of the study, Tori and Robert had begun to design and implement both structured exercises and LGEs in their own lessons. Furthermore, they could articulate strategies for designing other sets of structured exercises and LGEs across various content, which spoke to the generativity of their example spaces. Robert demonstrated and spoke of sets of structured exercises to compare multiple solution methods, and both Tori and Robert saw "Give an Example of…(another and another)" as a type of LGE that could be applied across various content, which indicated the generality of their example spaces.

Lynn and Shannon, on the other hand, did not tend to incorporate planned sets of structured exercises, and their attempts at LGEs were prompts asking students to generate contextualized word problems. Prior to the learning study intervention, Tori and Robert were already taking on a large part of the design of tasks for themselves. As the RTI teacher, Lynn typically used random collections examples as she saw them as aligned to her goals of re-teaching and reviewing several topics simultaneously. As a part-time teacher, Shannon typically used the materials that had been designed by Tori or Robert. The teachers' varying degrees of experience with task design may have contributed to their take up of structured exercises and LGEs.

Table 4.20

Shannon's Example Space of Use and View of Examples after the Learning Study Intervention

Characteristic	Summative Description
Population	Shift in predominant view of examples from explanation, exposure, practice, and assessment to the conveyance of generality of a class of examples. Predominantly used instructional materials designed by other teachers, but described the design of a set of structured exercises as basic to complex or "Pointing Toward Generality (particular, peculiar, general)." Designed contextual-based LGEs (i.e., generation of word problems), but opened the space of variation to consider critical aspects of the class of examples. Instance of a set of structured exercises improvised during enactment with verbal indications to students to look "With and Across the Grain" to notice and generalize relationships among critical aspects, enacting fusion as a pattern of variation.
Generativity	Described design strategies including "Pointing Toward Generality (particular, peculiar, general)", reversal for the generation of a context-based LGE, and improvisation of a set of structured exercises using "With and Across the Grain" indicated the potential for generating future sets of structured exercises and LGEs.
Connectedness	View of examples as for the conveyance of generality loosely connected to Shannon's design practice. Her view seems well-connected to her descriptions of design of sets of examples, but did not progress to her observed designs.
Generality	Views structured exercises as having the purpose of conveying generality, which appeared to theoretically extend across context, but was not observed. Generalization of LGEs as a type of assessment.

Lynn's conception of LGEs was strongly connected to her notions of student success. By the end of the study, Lynn continued to equate restrictions on an LGE with success. She thought that providing less opportunities for variation within the prompt would lead toward greater student success in generating what was intended. So, while both Lynn and Shannon asked students to write a word problem, the prompt that Shannon used provided some opportunity for students to discern the structure of a word problem involving linear inequalities, while the prompt that Lynn used could only serve to generalize the volume of a particular cylinder and rectangular prism over context. While Lynn and Shannon both struggled to design and implement structured exercises and LGEs, there was some take up of the Principle of Explicit Contrast through patterns of variance and invariance within their implementation of examples, and Shannon carefully began to consider the purpose of each example.

All of the teachers conceptualized of LGEs as a means to assess student understanding. Lynn and Shannon largely saw this as an assessment that came after a period of learning. Robert, on the other hand, viewed the assessment as making student thinking about a certain class of examples visible. He was more likely to use an LGE at the beginning of a period of learning or within to reveal the aspects of a class of examples that students were attending to. Tori began to consider ways in which she could push students beyond the familiar and comfortable range of change for particular critical aspects to expand students' horizons of the range of permissible change and evoke the generality of particular class of examples.

CHAPTER 5 – DISCUSSION AND CONCLUSIONS

The goal of this study was to examine how teachers' knowledge about designing and enacting tasks that incorporated structured exercises and LGEs developed. The participants were a team of four middle grades mathematics teachers who were observed and interviewed about their use and views of examples prior to the learning study intervention. The four teachers participated in five hours of professional development over the summer to introduce them to learner generated examples (LGEs), structured exercises, variation theory, and lesson/learning study. The team of four teachers and myself met, in general, twice a week to plan and implement a series of four learning study cycles about (1) the order of operations, (2) writing linear equations, (3) calculating, interpreting, and comparing unit rates, and (4) writing a number in scientific notation with a different power of ten. In each of the learning study cycles, the teachers attempted to incorporate sets of examples designed using specific patterns of variation. Each of the teachers was observed twice during the learning study intervention and observed and interviewed about their use and views of examples after the learning study intervention. The data collected over the course of the study were analyzed to answer the research questions:

1. How do teachers conceptualize and develop their knowledge about task design that structures students' experiences of learning algebraic constructs?
2. How do teachers develop their knowledge about enacting tasks that incorporate structured exercises or learner generated examples (LGEs) in ways that support students in developing an awareness of algebraic structure?
3. What factors influence and shape teachers' conceptualization and implementation of structured exercises and learner generated examples (LGEs)?

Discussion of Findings

In this section, I discuss the major findings about the development of teachers' knowledge about the design of sets of examples, the enactment of tasks that incorporate structured exercises or LGEs, and the factors that shaped how structured exercises and LGEs were thought about and used. The teachers thought about and developed their knowledge of task design through (1) careful consideration of their own thinking about a class of examples and the aspects of that class of examples they attended to, (2) the deliberate practice of task design and revision, and (3) attending to patterns of variation before attending to particular types of tasks, such as structured exercises or LGEs. The teachers developed their knowledge about enacting tasks that incorporated structured exercises or LGEs through (1) the deliberate practice of implementing such tasks and considerations for revision, and (2) through the exchange of ideas for enactment with colleagues. Factors that influenced and shaped teachers' conceptualization and operationalization of structured exercises and LGEs included (1) teachers' perceptions of control, (2) teachers' notions of student success, and (3) teachers' prior opportunities and experience with task design.

Research Question 1 – Knowledge About Task Design

Consideration of Aspects Teachers Attend to

The teachers spent a significant amount of time thinking about their own thinking, and deconstructing their own knowledge to discover the specific aspects of a class of examples that they, themselves, attended to as doers of mathematics. Because teachers' knowledge was generally quite fused, in the sense of variation theory and understanding mathematical structure as a set of relationships instantiated in particular situations, they were not always conscious of the specific aspects they attended to when thinking about a class of examples. Teachers needed to deconstruct their own knowledge to uncover the aspects of mathematical structure that should be forefronted in the design of tasks for learners. Differences in expert versus novice views of examples is well documented in the exemplification literature (Mason, 2006; Mason & Pimm, 1984; Sinclair et al., 2011), in particular with regard to generality and conceiving of a particular example as representative of an entire class. While the teachers often viewed an aspect of a class of examples as general, such as seeing that the base of an exponential expression could take on any real number, students often viewed such an aspect as particular. For instance, one of Shannon's students asked whether the base of exponential expressions must be prime. Teachers needed to become attuned to the aspects of a class of examples they had fused and generalized in order to consider ways to bring them to the fore of learners' awareness.

The process of attuning oneself to one's own thinking and deconstruction of knowledge was the means through which the teachers articulated the relationships among critical aspects that they wanted students to discern and identified the critical aspects of an object of learning. Marton (2015) claimed that critical aspects and critical features can be hard to discover. One way that the teachers searched for and found critical aspects and critical features was through the deconstruction of their own knowledge about an object of learning. The teachers' collaborative efforts supported the deconstruction of their own knowledge, particularly when there was contrast between their ways of thinking, such as their different ways of thinking about when to use either slope-intercept or point-slope form. Through contrast in the ways that the teachers thought about a particular object of learning, the aspects they attended to could be discerned.

Because teachers needed to think about their own thinking regarding a class of examples, there was a reciprocal relationship between the design of the sets of examples and the purpose for the set of examples, as defined by the object of learning and its critical aspects and features.

The purpose, including the object of learning, informed the design of examples, but the design of examples served to clarify and refine the critical aspects and features of the object of learning as teachers considered their own thinking about the examples.

Deliberate Practice of Task Design

To develop their knowledge about task design that structured students' experiences of learning algebraic constructs, teachers needed to engage in the deliberate practice of task design, implementation, and consider possible revisions of the task design. The process of learning study provided a means of deliberate practice for the teacher in which (1) their task of planning a lesson, including the examples to be used within the lesson, was well-defined and at the appropriate level of difficulty, (2) the observation of the research lesson and collaborative consideration of student learning in relationship to the design and enactment of examples provided informative feedback, and (3) teachers had an opportunity to consider revisions to the design for potential future implementations of the tasks. Miller (2011) recognized that American teachers have few opportunities for deliberate practice and pointed out that Japanese lesson-study methods can be seen as an example of deliberate practice. This study confirmed learning study as a form of deliberate practice for the development of knowledge and expertise about task design.

Attending to Patterns of Variation

Teachers in this study attended to patterns of variation both separate from and within the design of structured exercises and LGEs. By the end of this study, Lynn and Shannon struggled to incorporate structured exercises and LGEs into their lessons, yet attended to patterns of variation, particularly contrast, in the design of sets of examples. This suggested that a landmark along the trajectory of the development of teacher knowledge about task design that structured students' experiences of learning algebraic concepts was the development of an awareness and understanding of the patterns of variation, especially contrast.

The Principle of Explicit Contrast emerged in the first learning study as a principle of task design. The teachers employed two design strategies to draw students' attention toward the aspects of a set of examples that were changing: (1) "varied versus unchanged", and (2) the intentional creation of disturbance. These two strategies were intertwined as students considered why particular aspects between two examples remained invariant while others had varied. The "varied versus unchanged" strategy confirmed Mason's (2006) suggestion to constrain the variation in a set of exercises to one or two dimensions of possible variation to increase the likelihood that students will notice the critical aspects and the structure that the teacher had intended. The intentional creation of disturbance within the design of sets of exercises echoed the definition of structured exercises from the research literature (Rowland, 2008; Watson & Mason, 2006; Watson & Shipman, 2008) and incited the search for relationships between critical aspects of a set of examples.

The Principle of Attending to Generation and Response emerged in Research Lesson Four as the teachers considered the design of an LGE. Prior to Research Lesson Four, the teachers had predominantly used LGEs as an assessment tool. The teachers' use of LGEs in Research Lesson Four marked a shift toward using LGEs as a pedagogical tool. The teachers realized that they should attend to the variation in examples that students might generate, given the wording of the prompt, and plan how to respond to those examples. While a number of studies have considered the implications of teaching with LGEs (Arzarello et al., 2011; Watson & Chick, 2011; Zaslavsky & Zodik, 2007, 2014; Zodik & Zaslavsky, 2008), I did not find any studies that considered the design of LGEs or planning for LGEs.

A design dilemma arose for the teachers as they attended to patterns of variation within both teacher- and student-generated sets of examples: How many examples is enough? A number of researchers discussed the problem of the optimal amount of variation in both relevant and irrelevant attributes of a class of examples (Charles, 1980; Fischbein, 1993; Kellogg, 1980; Mason & Pimm, 1984; Petty & Jansson, 1987; Wilson, 1986) in addition to the sequencing, representations of examples, and attention to student errors (Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2008). The teachers discovered that the number of examples within a set, and which particular examples, were critical to conveying the contrast they intended students to discern. Of course, a practical concern was time. As time is a precious resource within the current system of schooling, the teachers were concerned with using just enough examples to allow for students' discernment of the contrast between examples and the relationships between critical aspects that was intended. For instance, in Research Lesson One, the teachers decided upon a set of three examples to contrast the location of a set of parentheses within the examples and the discernment of the relationship between the location of the parentheses and the value of the expression. The teachers found, however, that it was necessary to include the original expression without parentheses to contrast the value of that expression with the ones that included parentheses.

The experience of the teachers in my study would support the assertion made by Watson and Mason (2006) that "constructing tasks that use variation and change optimally is a design project in which reflection about learner responses leads to further refinement and further precision of *the number of examples included* [italics mine], example choice and sequence." Through deliberate collaborative practice, the teachers considered which aspects of a class of examples they, themselves, were attuned to notice and carefully designed sets of examples that encouraged students to attend to the aspects they had identified to develop their understanding of mathematical structure.

Research Question 2 – Knowledge About Enacting Structured Exercises and LGEs

Deliberate Practice of Implementation

Teachers developed their knowledge about enacting tasks that incorporated structured exercises and LGEs in much the same way that they developed their knowledge about task design – through deliberate practice. In this case, it was the deliberate practice of implementation, as teachers may have engaged in the deliberate practice task design, as in the design of an LGE for Research Lesson One, but without implementation, little can be learned about how to enact such tasks. As the Principle of Explicit Contrast emerged for the teachers in the design of tasks, so too did the Principle of Explicit Contrast emerge for enactment. Watson and Chick's (2011) assertion that the purpose and intention of examples is often left implicit for the learners was reflected in the teachers' enactment of examples prior to the learning study intervention. A significant element of teachers' learning was the recognition that the contrast intended in the design of examples be made explicit for learners during the enactment. Throughout the learning study intervention, teachers developed strategies to bring the contrast intended in the design of tasks to the fore of learners' awareness during enactment. Verbal strategies included (1) questioning ("What's the same? What's different?", (2) reading out loud, and (3) notice and focus. Boxing in, as in Research Lesson Four, was used as a visual strategy to contrast the representation of equivalent expressions. Strategies to compare student solution methods included: (1) student-to-student talk, (2) juxtaposition of student solution methods, and (3) a survey, by a show of hands, of student solution methods.

In Research Lesson Four, the teachers included a power of 10^0 as a special case in their set of structured exercises. The special case of 10^0 arose in the set of examples generated by learners when a student generated $36{,}000 \times 10^0$ as an expression equivalent to $36{,}000$. Tori wondered in the debrief meeting after the lesson if Shannon had seen any students generate negative powers of ten, which she had not. Tori suggested to Robert that for his class, he might consider pushing students beyond positive powers of ten and a power of zero. She suggested adding an additional condition to the prompt: "Now write one that has a negative exponent." Zasklavsky and Zodik (2014) similarly suggested that the teacher should persist and push students to generate more and more examples that are different from the previous ones in order to encourage students to go beyond their accessible example spaces and access their potential example space. They noted that the place just beyond the reach of students' accessible example spaces was where learning occurs. Tori enacted her own suggestion during my observation of her after the four learning study cycles. Tori prompted students to give examples of exponential expressions using multiplication that simplified to 8^8. Tori prompted for students to give additional examples using a negative exponent, but was surprised when a student gave an example with a rational exponent. Through deliberate practice, Tori discovered for herself how pushing students to go beyond the ranges of permissible change that they are comfortable with can generate new insights. Tori's experience confirmed Bill's et al. (2004) observation that, "Typically, learners play safe with ranges of change, not going beyond the familiar unless encouraged and supported in doing so" (p. 3). Teacher's knowledge about enacting structured exercises and LGEs seemed to develop in the same fashion – teachers played safe with their enactment of examples, tending toward how they had experienced and enacted examples in the past. The learning study intervention provided encouragement for enacting structured exercises and LGEs, and supported teachers, through deliberate practice in the collective space, for doing so.

The teachers in this study used a subset of LGEs as an object itself, as a set of structured exercises might be used, that through its consideration, was meant to support students in learning a new concept. Watson and Shipman (2008) found that students can learn new concepts from LGEs, and cited the importance of the availability of the collective example space generated by the class of learners and the teachers' guidance. In Research Lesson Four, Shannon chose particular students to write their generated example on the board, and from that set, chose examples to focus on to drive the lesson forward toward the mathematical goal of writing equivalent expressions using a different power of ten. Shannon's enactment of the LGE in Research Lesson Four was the first time that an LGE had been enacted during a research lesson, and the first time that it was used for pedagogical purposes (i.e., driving the lesson toward the mathematical goal), rather than for the purpose of assessment, as they had been used in Research Lessons One and Two.

Through the enactment of the LGE in Research Lesson Four, Shannon asked students to consider what was remaining the same and what was changing within the set of examples to reveal the critical aspects and underlying mathematical structure to the students through the search for patterns. Watson and Shipman (2008) noted that while noticing patterns can help the learner discern something about the structure, conjecturing about the relationship within and across examples is a non-trivial shift of perception that allows one to see both the dimensions of possible variation and the range of permissible change within those dimensions. However, while the teachers wanted students to discern the relationship between critical aspects in the set of examples, they struggled with how they and students should articulate the relationship without reducing it to a rule or a "trick." Bills et al. (2004) asserted that "[learners] may need support in articulating their generalisations verbally or expressing them in writing, diagrams or symbols" (p. 3). The teachers were concerned about students' application or rules without understanding, but found it difficult, themselves, to articulate important relationships in student-friendly language. In part due to the challenge of language, the teachers lacked strategies for supporting students' generalizations of the relationships among critical aspects.

Much of the research literature about LGEs suggested that consideration of the validity of generated examples was an important component of inducing a learning event (Antonini, 2011; Arzarello et al., 2011; Hazzan & Zazkis, 1999; Zaslavsky & Zodk, 2014). In this study, the only time that LGEs were offered for verification was when Tori made the decision, in-the-moment, to have students' check each other's work. What the students may have noticed or learned through this process, however, was never made visible. In my post-interviews, I presented each of the teachers with the set of examples that students had generated during the observation. The teachers' consideration of the validity of the examples, in those moments, led to some insights about how LGEs could be used in their future teaching. Zaslavsky and Zodik (2014) found that disagreements among learners about the validity of suggested examples led to learning events. My study suggests that incongruencies between the beliefs that teachers held about their students' learning and what was revealed through a set of LGEs led to learning events about enacting both structured exercises and LGEs.

Exchange of Ideas with Colleagues

Many of the strategies for enacting explicit contrast arose from the exchange of ideas with colleagues. Tori suggested a notice and focus strategy to Robert after he talked about earlier implementations of the lesson, prior to the research lesson. The notice and focus strategy was borrowed from English Language Arts and applied in the mathematics classroom. Shannon suggested reading out loud as a strategy to Tori to convey the contrast in operations between two otherwise identical expressions with parentheses placed in different locations. While the teachers may have developed some of these strategies individually, the exchange of ideas within the process of learning study supported the teachers in developing their knowledge about enacting structured exercises and LGEs.

Research Question 3 – Factors that Shape Teachers' Use and View of Structured Exercises and LGEs

Teachers' Perceptions of Control

The teachers, collectively, were more apt to use structured exercises than LGEs. The teachers viewed structured exercises as more consistent with what they would consider a typical activity of a mathematics classroom and afforded them a greater sense of control as the provider of the example set. The teachers considered LGEs to have greater challenges than structured exercises, due to their perception of the lack of control they would have regarding the examples students would generate. The teachers also felt challenged by the demands of responding in-the-moment to students' generation of examples, which teachers perceived as more demanding than the challenges presented by structured exercises to respond in-the-moment. While a number of researchers have acknowledged the additional demands on teachers to act in-the-moment when using LGEs (Arzarello et al., 2011; Watson & Chick, 2011; Zaslavsky & Zodik, 2007, 2014; Zodik & Zaslavsky, 2008), none have offered strategies for either mitigating or navigating these demands. The Principle of Attending to Generation and Response that emerged for the teachers provides a step toward acknowledging and mitigating the demands of acting in-the-moment during the implementation of LGEs.

Notions of Student Success

As teachers' familiarity with teacher-provided examples and their perception of control influenced their take-up of structured exercises and LGEs, their notions of student success, which were related to students' lack of familiarity with learner-provided examples, strongly influenced teachers' take-up of LGEs. The teachers grappled with restriction versus openness in the design of the LGEs. In the teachers' first attempts at designing prompts for LGEs, in Research Lessons One and Two, they included a mathematical sentence frame and hints to the students about what they expected students to generate, restricting the range of variation in the examples that students were likely to generate. The teachers initially associated greater restriction with student success, as they associated student success with knowing what to do and obtaining a correct answer. In Research Lesson Four, Tori initially wanted to model generating examples of expressions equivalent to a given number, prior to asking students to do so. After deciding not to model the generation of examples, the teachers carefully considered for which number the generated expressions should be equivalent. There was a shift in teachers' perception as they realized that a smaller number with less factors would restrict the range of examples that students' generated. In order to create more assurance that students would generate the kinds of examples needed to move the lesson forward, the teachers' opted to choose a larger number with more factors and open the space of learning for greater variation in the set of LGEs.

For Research Lesson Three, the teachers did not attempt to design an LGE. Research Lesson Three was taught to Lynn's students in the RTI class. The teachers' notions of student success for this particular group of students led them to disregard the use of LGEs. For this group of learners, the teachers wanted students to experience success through knowing what to do and obtaining correct answers. They decided to demonstrate only a single method of calculating a value for a unit rate, as they perceived that multiple solution strategies would confuse students. The use of LGEs conflicted with a sense of one correct answer and students' comfort in knowing exactly what to do. Lynn continued to associate restriction with student success for this group of learners after the four learning study cycles. In her attempt of an LGE, Lynn prompted students to generate a contextualized word problem to find the volume of either a cylinder or a rectangular prism. Lynn restricted the shape and dimensions of the solid, allowing only the context to vary. The restrictions that she placed on the prompt due to her notions of student success served only to generalize that particular example over context, rather than reveal any mathematical structure.

There was evidence that the learning study work amongst the teachers about designing and enacting structured exercises and LGEs served, in some ways, to disrupt teachers' notions of student success. Robert had the most optimistic notions of student success, as evidenced by his continued suggestions to open the space of variation when considering the design of LGEs. The apparent conflict between Robert's notions of student success and the other teachers' notions became strongly evident in Research Lesson Three, when Robert's voice about the design of the set of examples was effectively silenced. The implementation of the lesson revealed that students did, in fact, have prior knowledge about and multiple ways of thinking about proportional relationships, and within the evaluation meetings, the teachers discussed possible redesigns of the set of examples and ways of responding to student thinking. The teachers' shift to using an LGE as a pedagogical tool in Research Lesson Four to drive the lesson toward the mathematical goal also provided evidence that some of their notions about student success had been disrupted. They had enough confidence in students' abilities to generate examples of the kind being asked for, and that certain examples would emerge that were needed to drive the lesson forward.

Perceiving of LGEs as a pedagogical tool, rather than as a tool for assessment, was a non-trivial shift that suggests an expansion of teachers' notions of student success.

Opportunities and Experience with Task Design

Prior to the learning study intervention, Tori and Robert were already taking on a large part of the design of tasks for themselves. Augustus Middle School had not adopted a textbook for their mathematics courses, so teachers were left to design their own instructional materials. The responsibility for the design of instructional materials fell largely to Tori and Robert as full-time seventh and eighth grade mathematics teachers. As the RTI teacher, Lynn typically used random collections of examples. She saw such sets random sets of examples as aligned to her goals of re-teaching and reviewing several topics simultaneously. As a part-time teacher, Shannon typically used the materials that had been designed by Tori or Robert. The teachers' varying degrees of experience with task design contributed to their take up of structured exercises and LGEs. While it was borne of necessity, Tori and Robert's engagement in the design of instructional materials was a form of deliberate practice. With experience having designed, implemented, and revised tasks, Tori and Robert appeared more willing than Lynn and Shannon to take on the risks of trying new types of tasks in their design and enactment. Experience with task design and enactment also seemed to be associated with teacher confidence. Lynn and Shannon talked about the value they saw in tasks that incorporated structured exercises and LGEs but were less confident in their abilities to design and implement them in their individual classrooms.

Implications

Examples have been and will continue to be used in the teaching of mathematics. Given the importance of algebra to students' future opportunities and understanding mathematical structure as a basic tenet to understanding algebra, using sets of examples in ways that support students' discernment of mathematical structure could have a significant bearing on students' success in algebra and beyond. Consideration of the optimal variation in sets of examples for the discernment of mathematical structure across various algebraic topics is a design project worthy of time and thought. This study suggests potential landmarks along the way of the trajectory of individual teachers' learning about designing and enacting tasks using variation.

The fact that at various points throughout this study, each of the four teachers attended to patterns of variation in design and enactment, particularly contrast, without designing or incorporating structured exercises or LGEs, suggests that developing teachers' capacities to attend to variation in the design and enactment of tasks would be a useful step toward developing students' capacities to notice and use mathematical structure. This suggests that a landmark along the pathway for the development of teacher knowledge about task design that structures students' experiences of learning algebraic concepts is the development of an awareness and understanding of the patterns of variation, and in particular contrast, before attending to specific types of tasks in design, such as structured exercises and LGEs.

Given my finding that opportunities and experience with task design was an influencing factor in teachers' take-up and use of structured exercises and LGEs, attending to variation in the design and enactment of tasks as a component of preservice teachers' work does not seem to be a useful path to pursue. Rather, in-service professional development about attending to variation in the design and enactment of tasks seems to be a more fruitful endeavor for the development of teacher knowledge. Earlier experience with task design may have provided a necessary contrast for teachers between task design using patterns of variation and task design not using patterns of variation.

In-service professional development about attending to variation in the design and enactment of tasks would need to have collaborative and individual components and be sustained. In my study, it took a period of meeting twice a week for twenty-two weeks and four learning study cycles for teachers to attempt to implement an LGE into the research lesson. The development of teacher knowledge through deliberate practice takes a significant amount of time. The exchange of ideas among colleagues was an important finding about how teachers develop their knowledge about the enactment of tasks that incorporate structured exercises and LGEs. This suggests that professional development about attending to variation in the design and enactment of tasks would need to have a collaborative component to open the space of learning about enactment strategies. In my study, the learning study intervention supported teachers in their design and enactment of structured exercises and LGEs in their collective work, but individual teachers did not take up all of the developments of knowledge about designing and enacting structured exercises and LGEs within their own teaching. This implies that teachers needed individual encouragement and support for making changes in their own practice. Varying sub-components of professional development could be offered based on teachers' experience with curriculum and task design, which appeared to be a factor that influenced teachers' confidence in taking up and implementing structured exercises and LGEs.

The most significant contribution of this study is the detail it provides of teachers' work as they learn about designing and enacting tasks using variation. The detail of the teachers' work contributes to the field's knowledge bases about exemplification and the process of teacher learning about designing and enacting tasks using variation, particularly in the United States. I am not aware of any other study that puts the work of designing and enacting tasks through variation in the hands of practicing teachers to begin to understand the development, and the detail within the development, of the teachers' knowledge and ensuing changes to practice.

Regarding exemplification, this study begins to detail the design and enactment of structured exercises and LGEs in practice. This study confirmed some aspects of designing and enacting structured exercises and LGEs that were already known or suggested by the research literature, such as the creation of disturbance to support student discernment of critical aspects and the role of the teacher in encouraging and supporting students in pushing beyond the boundaries of their accessible example space. Issues that were addressed, including attending to patterns of variation for the discernment, generalization, and fusion of critical aspects of an object of learning, have implications beyond the specific design and enactment of structured exercises and LGEs, and could be a useful consideration for the design of any task meant for learning. Further, the discernment, generalization, and fusion of critical aspects is not unique to mathematics. As considering patterns of variation in design and enactment was useful across middle grades mathematics topics in this study, so too could they be useful across elementary and high school mathematics topics, and likely for other subject areas.

Furthermore, this study builds on and supports studies about teacher learning. Much of what is reported in this study details the complex process of teacher learning within the deliberate practice of learning study. The development of teacher knowledge is slow and can be transient without sustained efforts. Understanding the nature of the development of teacher knowledge may be helpful in the design of professional development systems that support the growth of teacher knowledge, and by extension, greater student success. More specifically, this study begins to identify landmarks along the trajectory of the development of teachers' knowledge about designing and enacting tasks using variation. The significance of contrast within the design of sets of examples and during the enactment of those sets of examples was identified as a principle of design and enactment by the teachers. Just as teachers identified the Principle of Explicit Contrast for the benefit of student learning, so too does the Principle of Explicit Contrast apply to teachers' learning about the design and enactment of such tasks.

Limitations

At any particular moment in time, only a person's accessible example space can possibly be displayed, and because it is accessible does not necessarily mean that it will be expressed. Goldenberg and Mason (2008) point to the adage *absence of evidence is not evidence of absence* (p. 189) to caution that care must be taken when using example spaces as a research tool for revealing understanding. The absence of evidence of particular ways of using and implementing examples and exercises in instruction implied only that the participant had not perceived a reason to express that knowledge, not the absence of that knowledge. This indicates a second limitation regarding the comparison of example spaces over time: Was the appearance of some new aspect of the example space evidence of learning or evidence of earlier knowledge that simply was not expressed? A benefit of my research design, however, was multiple forms of data and opportunities to verify findings with other data sources. In my analysis I looked for patterns in the knowledge expressed by teachers and included data directly from the teachers about their perspectives of their own learning.

Other limitations of this design were related to the nature of American schooling, in which second opportunities for teaching the same lesson, after teachers have had time to reflect on it and revise it, are just not available. As such, my design included four distinct learning study cycles in terms of content, but linked learning study cycles in terms of focus. The teacher, however, implemented the planned research lesson multiple times for each of his or her sections of seventh or eighth grade mathematics. In one instance, I was able to observe changes related to enactment of a particular lesson when Shannon taught the planned Research Lesson Four in back-to-back periods. For the other three research lessons, the teacher described other implementations outside of the research lesson itself, what they noticed about the design and enactment, and changes that they decided to make either in-the-moment or between implementations. Observing changes across a number of implementations of the same lesson may provide additional insight into generalizable teaching and design principles that teachers hold on to and incorporate into practice. This study was, of course, also limited by a short time frame. Rather than tell the whole story of teachers' development of knowledge about the design and enactment of structured exercises and LGEs, I take this study to only be accessing the beginning of that journey.

I cannot make claims about causality for teacher learning. As Watson and Mason (2005) claimed, exemplification and perceptions of generality are individual and situational, indicating that the development of teachers' example spaces related to their use and views of examples are also individual. A particular instance might have served as a learning event for one or more of the teachers, but not all. Because of this, I cannot claim that certain events caused teacher learning, but only that a particular event appeared to result in an opportunity for teacher learning.

The generalizability of the results of this study is constrained by the small number of participants. The characteristics of the participants and characteristics of the setting were also unique. The four teachers had been colleagues for two years at the time of this study, and the school district was supportive of teacher collaboration. The naturalistic setting supports generalizability to other settings with other groups of teachers, at least those that have a similar level of support for teacher collaboration.

Questions for Further Study

Issues about communication and language arose in a number of instances throughout this study. In some instances, the teachers realized the lack of language that they had regarding the generalizations about the relationships they wanted to bring to the fore of students' awareness. This indicated that the teachers needed support in articulating generalizations of relationships among critical aspects, particularly in the language of middle grades students. How can teachers develop their mathematical communication on the basis of clarity and mathematical validity? How can teachers develop strategies for supporting students in articulating their generalizations in various ways? Furthermore, how can teachers support students in making generalizations about relationships among critical aspects without reducing them to rules?

There is still much to be explored regarding the optimal variation for a set of examples for particular objects of learning across mathematical topics and grade levels. As Marton (2015), noted critical aspects are difficult to ascertain. Studies that seek to discern the critical aspects and critical features of particular objects of learning in mathematics would be beneficial for task and curriculum design, and for teachers to attend to within enactment. Because critical aspects are always relative to the learners, to what extent can tasks be generalized as useful for student learning across locales? What strategies, other than the ones identified in this study, do teachers have for drawing attention to the critical aspects of an object of learning within enactment?

Lastly, this study found that teachers' opportunities for and experience with task design influenced their take-up and use of structured exercises and LGEs. My analysis of teachers' example spaces about their use and views of examples has begun to detail the development of teacher knowledge about examples, but a number of questions remain. What are preservice teachers' conceptions of the use and views of examples? How do teachers' conceptions of their use and views of examples change as they transition from preservice to in-service teachers? How do their conceptions of use and views of examples change over a sustained period of time, possibly over a career? What do teachers do with particular examples or sets of examples in which they observe student success? What are features of these examples or sets of examples? What are features of teachers' enactment of these examples and sets of examples? Teachers hold an incredible amount of knowledge about teaching built over years of deliberate practice within their classrooms. Finding more opportunities to access and share their knowledge can support significant growth for the field of teacher learning.

Conclusions

The purpose of this study was to investigate how teachers develop knowledge about designing and enacting tasks that incorporated structured exercises and LGEs, and factors that shaped the development of that knowledge and the variation in teachers' use of structured exercises and LGEs. To that end, I recruited a team of four middle grades mathematics teachers who were interested in exploring ideas related to examples and exercises through the process of learning study. The team of teachers participated in four learning study cycles in which they designed and implemented tasks involving either structured exercises or LGEs. I interviewed each teacher about his or her use and views of examples prior to the learning study intervention and after the learning study intervention. Each teacher was observed four to five times throughout the course of the study, and all planning, debrief, and evaluation meetings for each of the four research lessons were audio-recorded. The four research lessons were video recorded. The data was analyzed through the lens of variation theory and two analytical frameworks were used to analyze teachers' example spaces of their use and views of examples over time and the purpose, design, and enactment of examples.

A significant contribution of this study is in detailing the processes of the development of teacher knowledge about the design of tasks that attend to patterns of variation. From the teachers' collective work, principles of task design and enactment emerged, along with associated strategies. The Principle of Explicit Contrast emerged as a principle of both design and enactment. The teachers employed a design strategy they called "varied versus unchanged" to draw students' attention toward the aspects of a set of examples that were changing. The teachers also used intentional disturbance created through patterns of variance and invariance to design for explicit contrast between examples. The teachers used verbal and visual enactment strategies to make the contrast within the design explicit, and strategies to compare student solution methods. Verbal strategies for enacting explicit contrast included: (1) asking "What is the same? What is changing?", (2) reading out loud to emphasize the difference between two examples, and (3) notice and focus, borrowed from English Language Arts. A visual strategy for enacting explicit contrast was boxing in equivalent expressions with contrasting representations. Strategies to compare student solution methods were: (1) student-to-student talk, (2) juxtaposition of student solution methods, and (3) a survey of student solution methods. The Principle of Attending to Generation and Response emerged for the design of LGEs. Teachers realized the necessity of attending to what examples might be generated and how to respond to those examples. This included attending to the openness/restriction of the prompt for the LGE, and the development of tools, such as a list of anticipated examples that students would generate, that would support the teacher in attending to student thinking and responding in-the-moment to the examples that students' did generate.

The teachers thought about and developed their knowledge of task design and enactment through deliberate practice that included careful consideration of their own thinking about a class of examples and the aspects of that class of examples they attended to, the collaborative exchange of ideas with colleagues, the collective and individual design of sets of examples using patterns of variation, including structured exercises, and LGEs, and the revision or potential revision of such tasks. Factors that influenced and shaped teachers' conceptualization and implementation of structured exercises and LGEs included teachers' perceptions of control of the examples, or lack thereof, teachers' notions of student success, and teachers' prior opportunities and experience with task design.

APPENDIX A

Plan-to-Guide Learning

Research Lesson Team: AnnMarie O'Neil, Shannon Edwards, Robert Cavins, Lynn Gray, Tori Goodman
Instructor:
Class to be Taught/Grade Level:
Date:

1.) Title of the Lesson:
2.) Research Theme, Goals: Our research theme is to design and structure exercises and examples in ways that promote students' abilities to discern important algebraic ideas and concepts. We will do this by considering specific patterns of variation (what varies, what remains invariant) within sets of structured exercises and tasks that involve learner generated examples.

Broad Subject Matter Goals:

Lesson Goals:

3.) Object of Learning:
   Direct –
   Indirect –
      a.) Critical Aspects:

4.) Relationship of the lesson to CCSS

5.) Rationale: Why did we choose to focus on the particular object of learning that we did and identify the particular critical aspects that we did? What was our rationale for our instruction? Why did we choose the exercises and examples the way we did, and so on….

6.) Pre-test / Post-test:
   Pre – test:
   Post – test:

7.) Students' thinking process on the topic (inf. student interviews/pretest/past experience):
   a.) Type A Students:
   b.) Type B Students:
   c.) Type C Students:
   d.) Anticipated Errors:
      1.)

8.) Patterns of Variation to help discern each Critical Aspect of the research lesson

Critical Feature to be Discerned	Varied	Unchanged

9.) Planned Structured Exercises or Learner Generated Examples:

10.) Lesson Procedure:

Learning Activities, Teacher's Questions and Expected Student Reactions	**Teacher's Support**	**Points of Evaluation**
1.		

11.) Evaluation

a. Were students able to?

APPENDIX B

Initial Interview Protocol

I am interested in learning more about teachers' use and views of examples and exercises in instruction. I am interviewing you today since you are interested in learning study to explore some ideas about examples and exercises further. Today, I am interested in your use and views of examples and exercises before we begin the learning study process.

I'm going to start off asking you some questions about the observation I completed ________.

1.) When I was here to observe, you used these particular examples: (*insert*). What role did these examples have for students' learning?
    - How did you choose those particular examples?
    - How did you decide to sequence those examples?
    - How did those choices of examples support student learning?

2.) When I was here to observe, you had the students complete these particular exercises in class or for homework: (*insert*). What role did these exercises have for students' learning?
    - How did you choose those particular exercises?
    - How did you decide to sequence those exercises?
    - How did those choices of exercises support student learning?

3.) Reflecting back on it, how would you evaluate the examples and exercises that you chose to use that day?

Now, I'm going to ask you about your use of examples and exercises in instruction, in general.

4.) What purpose, for you, do examples and exercises have in mathematics instruction?
    - What do you see as the role of a random collection of exercises?

5.) Talk about how you make decisions about what examples and exercises to use in a lesson or for homework and how you sequence them.
    Possible prompts:
    - How do you decide where to start?
    - Have you ever used particular examples or exercises to:
        - Draw attention to a particular aspect?
        - Address misconceptions?
        - Convey a sense of generality?
        - Talk about uncommon cases?
    - What other features of examples and exercises might you attend to?

6.) Could you please share with me a lesson plan that shows how you use examples or exercises? (*Teachers will be asked to prepare this ahead of the interview.)
    - Describe this lesson and the role that examples and exercises played in it.

APPENDIX C

Post-Interview Protocol

I am interested in learning more about teachers' use and views of examples and exercises in instruction and about how teachers' knowledge about designing and enacting tasks with examples and exercises develops. I am interviewing you today since you participated in learning study this year. Today, I am interested in your use and views of examples and exercises and your perspectives on what you may have learned through participation in learning study.

I'm going to start off asking you some questions about the [post] observation I completed ___.

1.) When I was here to observe, _________. What role did these exercises have for students' learning?

- How did you choose those particular examples?
- How did you decide to sequence those examples?
- How did those choices of examples support student learning?
- In what ways did you expect that this particular arrangement or sequencing of exercises would support students in developing their understanding of ____?
- Would you make any revisions or changes to this assignment, based on your experiences?
- What do you recall about students' difficulty with ____?

2.) When I was here to observe, you [used this particular handout or note page].

- What were your thoughts as you were creating and planning this?
- What was the object of learning, or what was the goal, in having students complete this structured exercise/LGE?
  - Were there any mathematical goals that you wanted this structured exercise/LGE to drive forward?
- What did you see or hear to let you know that students were making sense of the mathematical structure?
- Here is a partial list of the examples that students generated in class. Thoughts and reactions to this?

3.) Reflecting back on the lesson, and the structured exercise/LGE, in particular, how might you revise this lesson for future years?

Now, I'm going to ask you about your use of examples and exercises in instruction, in general.

4.) What purpose, for you, do examples and exercises have in mathematics instruction?

- What do you see as the role of a random collection of exercises?
- What do you see as the role of a set of structured exercises?
- What do you see as the role of learner generated examples?

5.) Talk about how you make decisions about what examples and exercises to use in a lesson or for homework and how you sequence them.

- How do you decide where to start?
- Have you ever used particular examples or exercises to:
    - Draw attention to a particular aspect?
    - Address misconceptions?
    - Convey a sense of generality?
    - Talk about uncommon cases?
- What other features of examples and exercises might you attend to?

6.) Can you discuss another lesson (something I haven't seen) where you chose to use either structured exercises or learner generated examples?

- Describe this lesson and the role that examples and exercises played in it.

Now I'm going to ask you about your experience participating in learning study, particularly around the use of examples and exercises.

7.) What are three things you learned by participating in learning study over the course of the last year?

- How do you think that learning came about?
- Can you describe the moment where you had that realization?

8.) Do you think that you consider and use examples and exercises in any ways that are different than before you went through learning study?

9.) What, specifically, do you feel like you learned about structured exercises and learner generated examples?

- Do you see yourself incorporating structured exercises and learner generated examples into your lessons? In what ways?

10.) What do you still feel like you need to learn about structured exercises and learner generated examples?

APPENDIX D

Tori's Sales Tax and Gratuity Classwork #2 Worksheet for the Preliminary Interview

1.) Josh ordered a pizza last night from Pete's Pizzeria. Even though he decided to pick up his own pizza, Josh was gracious enough to leave a 5% tip. Which of the following expressions could be used to determine the total amount that Josh paid?

A $p + 0.5p$ **B** $1.5p$ **C** $p(1 + 0.05)$ **D** $1 + 0.05p$

2.) Ilah went to Royalty Nail Spa to get a manicure and gave her manicurist a 20% tip. Which of the following expressions could be used to determine Ilah's total bill?

A $m + 1.20m$ **B** $m(1 + 0.20m)$ **C** $m + \frac{1}{5}m$ **D** $\frac{1}{5}m$

3.) For the holidays Ellie helped out a family in need by purchasing a new Hot Wheels set for a four year old boy. Ellie's final bill came to $26.49 which included 6% sales tax. How much was the Hot Wheels set being sold for at Toys 'R Us?

4.) Alaura went to the movie theater to see the new movie Frozen. She paid $11 to see the movie and then had to pay 8% tax on the small soda and popcorn for $7.75. How much did Alaura spend in total at the movie theater?

5.) When purchasing the new Call of Duty, Jordan was charged 8% for tax and then an additional $3.99 for shipping and handling. If Jordan's total bill came to $61.18, at what price was Call of Duty being advertised for on Amazon.com?

6.) At the end of the year, Vinny realized that he had misplaced his science book and couldn't find it. Whenever a book is misplaced, the school charges a fee equal to 25% of the original cost of the book. Vinny received a notice stating that he owed the school $60 for the lost book. Knowing that this price includes the 25% fee, how much must the school have purchased the book for?

7.) Tyler purchased a new Burton snowboard for $139.95. Tyler had to pay an additional 6.5% for sales tax along with $6.25 for shipping and handling which was applied after sales tax. If Tyler had been saving $20 a week for his new snowboard, how long did it take him to pay for his snowboard?

8.) In our stockings this year, my mom gave all four of us a bag of Lindt Chocolate that contained five different kinds of candy. If my mom's credit card was charged $54, which includes an 8% sales tax, what must be the price of one bag of chocolate?

APPENDIX E

Tori's Sales Tax and Gratuity #2 Worksheet for the Preliminary Interview

1.) With some of the money that Tristan received over the holidays, Tristan bought a new pair of soccer cleats for $77.97. If this price includes the state tax of 4%, what was the advertised price of the soccer cleats?

2.) For New Year's Eve, Lauren's family went out to dinner at The Blue Springs in Bolton. Including an 18% tip, their total bill came to $74.71. Determine what the bill was prior to tip being added to the bill.

3.) Student Council held a fundraiser selling Augustus Flyers gear. An order was placed for one sweatshirt and one long sleeve shirt totaling $51.85. This price included state tax of 8% and a shipping fee of $3.25, which was applied after tax. What must have been the cost of the sweatshirt and long sleeved shirt?

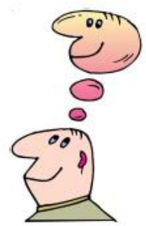
4.) For Kelsy's wedding she hired a limo driver to bring her from her house to the church. With a 16% tip, Kelsy's bill came to $44.37. If Kelsy rented the limo for three hours, what is the hourly rate that the limo was rented at?

APPENDIX F

Adding Polynomials Worksheet (https://mathbits.com/) from Robert's Initial Observation

Adding Polynomials Name ____________________

Add or subtract the following polynomial expressions.



1. $(-3x^2+5x-19)+(4x^2-2x)$

2. $(3a^2-4a)-(-2a^2+5a-8)$

3. $(x^4+x^3+6)-(4x^3-3x-1)$

4. $(2y^2+3y+6)+(3y^2+4y-1)+(-2y^2-4y-6)$

5. $(5x^2+12x-4)-(2x^2+x-3)+(3x^2-6x-4)$

6. $(3x^2-5x-4)+(x^3-7x^2-5)+(-x^2+4x-2)$

7. $(-a^2+5a^3+1)+(-4a^3-2a+3)-(-a^3+2a^2+7)$

8. $(-2x^3-x^2-10)-(4x^3-7x-9)-(3x^3-2x^2-8)$

9. Subtract $2y^2+5y+8$ from $6y^2-2y+3$.

10. From the sum of $2x^2-3x-1$ and x^2+5x, subtract $5x^2+2x+6$.

11. Subtract $4x^4+2x^2+5$ from the sum of $2x^4-4x^2-3$ and $3x^4-3x^2+9$.

12. Write the polynomial which represents the perimeter of a triangle whose sides are represented by $2x^2+3x$, $5x^2-1$, and $3x^2+7x-1$.

13. Represent the perimeter of a square whose side is $4x^2+3x-1$.

14. Express the perimeter of a rectangular pasture whose length is $3x^2-2x+1$ and whose width is x^2-3x-6.

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APPENDIX G

Lynn's Multiplying Monomials Tasks for the Preliminary Observation

<u>Multiplying Monomials (1)</u>

Recall the Laws of Exponents:

$y^4 \cdot y^2$

Describe a short way you could simplify the above without writing out the entire problem.

- **Multiplying Exponents**

1.) Keep the base.

2.) _______ the exponents.

- **Simplify the examples below using the exponent rules.**

1.) $x^3 \cdot x^4 \cdot x^{-5}$
2.) $3x^2 \cdot 4x^3$
3.) $a^4x^2 \cdot a^2x^3$
4.) $4y^8 \cdot 3y$

<u>Multiplying Monomials (2)</u>

1.) $5x(2x)$

2.) $-4x^3(3x^5)$

3.) $-3(2x - 4)$

4.) $x(3x + 1)$

5.) $4x(2x^2 - 1)$

6.) $10(2x^2 + x - 3)$

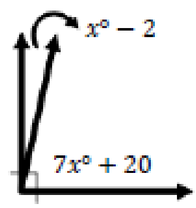
7.) $5x(x^2 - 7x + 1)$

APPENDIX H

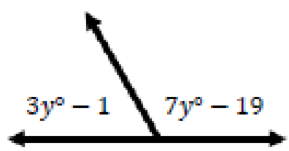
Shannon’s Complementary and Supplementary Angles In-class Examples

Write an equation, solve for the variable, and then find the actual measure of each angle.

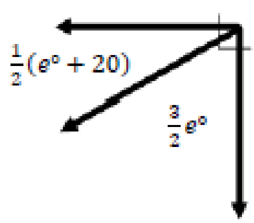
3.)



4.)



5.)



6.) The following two angles are supplementary. Determine the measure of $\measuredangle 1$ if

$\measuredangle 1 = 4k^\circ + 5$

$\measuredangle 2 = 14k^\circ - 5$

APPENDIX I

Laws of Exponents Review Homework from Tori's Post-Observation 3

Simplify the following expressions:

1.) $x^6 \bullet x^2$	2.) $x^6 \div x^2$	3.) $(x^6)^2$
4.) $5^3 \bullet 5^{-9}$	5.) $\dfrac{m^{10}}{m^{15}}$	6.) $(8^{-3})^3$
7.) $y^{-4} \div y^{-1}$	8.) $(b^4w^3)^6$	9.) $(12d^{-1})^{-2}$
10.) $2g^2 \cdot 3g^3$	11.) $-9x^5y^8 \div 3x^{-4}y^4$	12.) $\dfrac{-5c^4}{-15c}$

APPENDIX J

LGE designed by Tori for Post-Observation 3

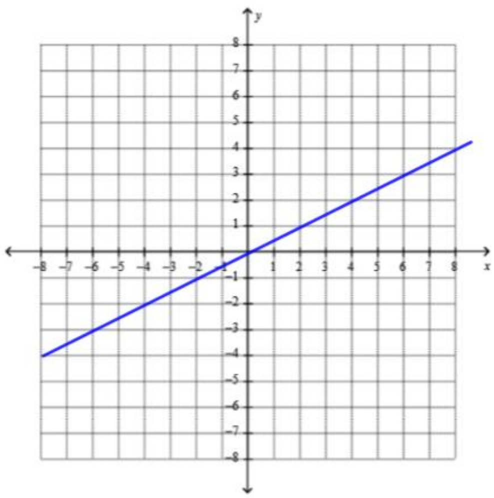
* Provided with the following expressions, determine an equivalent expression involving multiplication, division or repeated exponentiation that may have yielded in the original expression.

16.)	8^8	Multiplication:	Division:	Repeated Exponentiation:
17.)	x^{16}	Multiplication:	Division:	Repeated Exponentiation:
18.)	a^6b^{-8}	Multiplication:	Division:	Repeated Exponentiation:
19.)	5^{-7}	A combination of multiplication and division		
20.)	y^2	A combination of multiplication and repeated exponentiation		

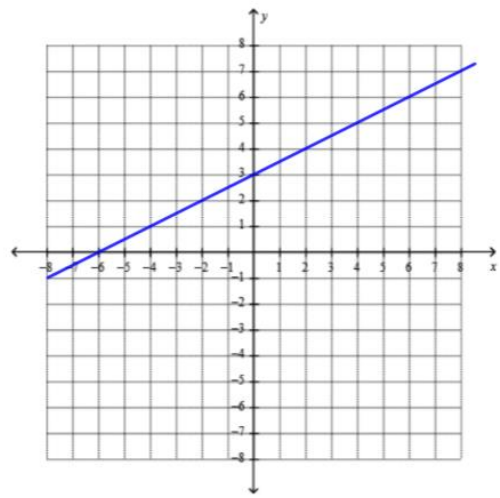
APPENDIX K

Robert Applying Explicit Contrast in Design in Observation 2 (October 22, 2015)

Look at the two function graphs below.



Graph A



Graph B

a.) How are the function graphs similar: ____

____

How are the function graphs different:____

____

b.) Fill in the table of values above with ordered pairs from each graph.

x				
y				

x				
y				

c.) What point is the y-intercept of Graph A?: ______

d.) What point is the y-intercept of Graph B?: ______

e.) Which graph from above represents a <u>proportional relationship</u>? How do you know?

APPENDIX L

The Set of Examples Given for Homework Prior to Shannon's Post-Observation 4

1.) Write and solve your own two-step (at least!) inequality.
   a.) Write a "story" inequality problem like the ones we looked at in class.
   b.) Write and solve the inequality that goes with the "story" you wrote above.

2.) A killer whale has eaten 75 pounds of fish already today. It needs to eat at least 140 pounds of fish each day to reach its quota. The buckets that Sea World uses to hold the fish they feed the killer whale hold 15 pounds of fish.
   a.) Write an inequality to show how many buckets, b, of fish the whale needs to be fed to meet its quota for the day.
   b.) How many more buckets of fish does will the killer what eat today?

3.) A drive-in movie theater charges $3.50 per car to see a movie. The drive-in has already admitted 100 cars. They need to admit enough cars to earn at least $500.
   a.) Write and solve an inequality to show how many car, c, the drive-in still needs to admit.

4.) One line of text typed on a page takes up about $\frac{3}{16}$ of an inch. There are 1 inch margins at the top and bottom of a page. How many typed lines can you fit on a page that is 11 inches long?
   a.) Write and solve an inequality to show how many typed lines, t, will fit on the page.

5.) Solve and graph the following inequality.

$$7b - 12b + 1.4 > 8.4$$

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CURRICULUM VITAE

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EDUCATION

Ph.D. in Mathematics Education, Syracuse University, June 2018. Dissertation: *Middle Grades Mathematics Teachers' Learning Through Designing Structured Exercises and Learner Generated Examples.* Thesis Advisor: Dr. Helen M. Doerr.

Master of Science in Teaching and Curriculum, Mathematics Education, Syracuse University, 2009.

B.S. Mathematics, Mathematics Education, Summa Cum Laude, Syracuse University, 2008.

New York State Professional Teaching Certificate, Secondary Mathematics, 2013-2018.

Florida State Professional Teaching Certificate Eligibility, Mathematics 6-12, DOE #1334750

PROFESSIONAL EXPERIENCE

August, 2017 to present
Mathematics Professor, Valencia College, Orlando, Florida

Responsible for undergraduate mathematics instruction, including Statistical Methods, Precalculus, and College Trigonometry. Teaching responsibilities include planning lessons as the instructor of record, maintaining student engagement hours to support students' learning outside of class, writing and administering assessments, and recording student grades. Participation in the Teaching and Learning Academy for tenure-track faculty.

August, 2013 to May, 2017
Teaching Assistant, Syracuse University, Syracuse, New York

Responsible for undergraduate/graduate secondary mathematics methods instruction and supervision of candidates enrolled in candidacy student teaching. Teaching responsibilities include modeling effective teaching practices informed by current research, building students' knowledge and understanding of the purpose and requirements for edTPA, and preparing student teacher candidates to effectively plan, implement, and reflect on lessons and assessments through discussion of readings, task design and modification, peer-teaching opportunities, and analytic writing.

Taught Foundational Mathematics via Problem Solving I and II for elementary education majors, and introductory statistics. Teaching responsibilities included planning lessons as the instructor of record, maintaining office hours to support students' learning outside of class, writing and administering assessments, and recording student grades. Met weekly with other instructors to discuss the implementation of the course-sequence and expand the course sequence from two courses to three courses.

July, 2010 to July, 2017
Summer Start Program Instructor, Syracuse University, Syracuse, New York

Worked closely with Prof. Helen Doerr to design a six-week summer course for incoming engineering students organized around the concept of rate of change, with the goal of improving students' success in their first semester mathematics course. Served as primary instructor, co-planning the course with Prof. Doerr summer 2010-2014. Supervised three graduate teaching assistants providing support for the course in summer 2015-2016.

September, 2010 to June, 2013
Mathematics Teacher, Marcellus Central School District, Marcellus, New York

Taught seventh grade mathematics for three years at C. S. Driver Middle School to 105 students in five sections, created a safe and comfortable learning environment that encouraged active student participation and collaboration, and guided and engaged students in communicating mathematically, problem solving and mathematical reasoning. Taught Integrated Algebra, Business Math and Precalculus at Marcellus High School for one year.

September, 2009 to June, 2010
Mathematics Teacher, Roxboro Road Middle School, Mattydale, New York

Taught seventh grade mathematics to 94 students in five sections. Piloted *IMPACT Mathematics* for a potential textbook adoption.

August, 2008 to May, 2009
Teaching Assistant, Syracuse University, Syracuse, New York

Responsible for undergraduate mathematics instruction in Precalculus for engineering majors and Foundational Mathematics via Problem Solving II for elementary education majors. Infused algebra into precalculus for a section of under-prepared first-year engineering students who otherwise would have placed into college algebra. The success of the students participating in this course resulted in the regularization of algebra infused precalculus as a course offering in the mathematics department.

Recent Professional Development Workshops for K-12 Teachers

November 8, 2016 *It's Go Time: Seeing the Future Through the NEW NYS Science Standards Conference*, Onondaga-Cortland-Madison BOCES, Roxboro Road Middle School, Mattydale, New York
Facilitated an introduction and debriefing to a live research lesson for fifth grade students on the impact that humans have on water. This included introducing the purpose of lesson study, expectations for observers, and facilitating the debriefing discussion between the teaching team and observers after the live lesson.

May 24, 2016 *Weaving Math Practices into Practice*, Onondaga-Cortland-Madison BOCES, Liverpool campus, Liverpool, New York
Co-taught with Dr. Krystal Barber from the State University of New York at Cortland, for approximately 30 K-12 elementary and mathematics teachers from schools across the county. We emphasized task selection and modification and orchestrating mathematical discussion as a means of incorporating the Common Core Standards for Mathematical Practice into instruction on a consistent basis.

Publications

Peer reviewed journals.

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2013). A modeling perspective on interpreting rates of change in context. *Mathematical Thinking and Learning*, *15*(4), 314-336.

Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2013). Teaching practices and exploratory computer simulations. *Computers in the Schools, 30*(1/2), 102-123.

Doerr, H. M., Meehan, D. J., & O'Neil, A. H. (2012). A natural approach to the number e. *Mathematics Teacher*, *105*(6), 432-439.

Peer reviewed conference proceedings.

O'Neil, A. H., & Doerr, H. M. (2015). Using variation theory to design tasks to support students' understanding of logarithms. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of European Research in Mathematics Education* (pp. 433-439). Prague, Czech Republic: Charles University.

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Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2013). Teaching practices and modelling changing phenomena. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth*

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Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2013). Interpreting and communicating about phenomena with negative rates of change. In *American Society of Engineering Education Annual Conference and Exposition, Conference Proceedings.*

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2012). Interpreting average rate of change in context. In L. R. VanZoest, J.J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 965 – 968). Kalamazoo, MI: Western Michigan University.

Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2012). Teaching practices, technology and student learning. In L. R. VanZoest, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1065 – 1072). Kalamazoo, MI: Western Michigan University.

Doerr, H. M., Ärlebäck, J. B., & O'Neil, A. H. (2012). An integrated modeling approach to a summer bridge course. In *American Society for Engineering Education Annual Conference and Exposition, Conference Proceedings.*

Doerr, H. M., Staniec, A.C., & O'Neil, A. H. (2012). Designing for improved success in first year mathematics. In *American Society for Engineering Education Annual Conference and Exposition, Conference Proceedings.*

Doerr, H. M., & O'Neil, A. H. (2011). A modelling approach to developing an understanding of average rate of change. In M. Pytlak, T. Rowland, & W. Swoboda (Eds.), *Proceedings of the Seventh Conference of European Research in Mathematics Education* (pp. 937-946). Rzeszów, Poland: University of Rzeszów.

CONFERENCE PRESENTATIONS

Doerr, H. M., & O'Neil, A. H. (2012, April). *The development of students' understandings of average rate of change*. Paper presented at the research pre-session of the Annual Meeting of the National Council of Teachers of Mathematics, Philadelphia, PA.

GRANTS

Syracuse University, School of Education Research and Creative Project Grant award for "*Middle Grades Mathematics Teachers' Learning about Structure in Exercises and Examples*." This award was used to fund my dissertation data collection from Aug., 2015 to May, 2016. Total award, $700.

Syracuse University, Mathematics Department Travel Grant to attend the Ninth Congress of European Research in Mathematics Education in Prague, Czech Republic, Feb., 2015. Total award, $1000.

Honors and Awards

Certificate in University Teaching (CUT), Future Professoriate Program, The Graduate School, 2016.

Recipient of the Outstanding Teaching Assistant Award, Syracuse University, 2015. This award recognizes the top 4% of teaching assistants campus wide for teaching excellence.

Recipient of the 2012 Mathematics Division Best Paper Award of the American Society for Engineering Education for my paper "Designing for Improved Success in First-year Mathematics," co-authored with Helen M. Doerr and Andria Costello Staniec. This paper was based on my work in the SummerStart Program and the algebra infused precalculus course.

Robert M. Exner Award, for exceptional achievement in mathematics education, Syracuse University, 2009.

Association of Mathematics Teachers of New York State Scholarship, 2007.

Remembrance Scholar, Syracuse University, 2007. Awarded based on distinguished academic achievement, citizenship, and service to the community.

Professional Activities and Service

November 2017 to March 2018, Statistics Textbook Search Committee, Secretary, Valencia College, East Campus.

August 2015 to May 2016, Participant in the Women in Science and Engineering (WiSE) Future Professoriate Program, Syracuse University.

August 2014 to May 2016, Participant in the Future Professoriate Program, Syracuse University.

Fall 2014 and 2010, Manuscript Reviewer for Biennial Meeting, Congress of European Research in Mathematics Education.

February 2007 to present, Member of the National Council of Teachers of Mathematics.