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Auslander-Reiten Sequences over Gorenstein Rings of Dimension One

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ABSTRACT

Let R be a complete local Gorenstein ring of dimension one, with maximal ideal $\mathfrak{m}$. We show that if M is a maximal Cohen-Macaulay R-module which begins an Auslander-Reiten sequence, then this sequence is produced by an endomorphism of $\mathfrak{m}$, which we call a Frobenius element, corresponding to a minimal prime ideal. We also observe that Frobenius elements can be easier to identify when R is a graded ring, instead of complete local. We give an example application, determining the shape of some components of Auslander-Reiten quivers, in Section 5.3. (An *Auslander-Reiten quiver* organizes the indecomposable maximal Cohen-Macaulay R-modules into a directed graph, with arrows corresponding to irreducible R-homomorphisms.)

In Chapter 4, we adapt results due to Zacharia and others, from the setting of Artin algebras. This allows us to list the potential shapes of the components of AR quivers in our setting. It also has an application to special cases of a well-known conjecture in commutative algebra (Section 4.2). The appendix contains some lemmas concerning connected graded rings of Krull dimension one, used in Chapters 2 and 5.

Auslander-Reiten Sequences over Gorenstein Rings of Dimension One

by

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Chapter 1

Introduction

Notation 1.0.1. Throughout this thesis, all rings are assumed noetherian. A ring which is described with any subset of the words {reduced, Cohen-Macaulay, Gorenstein, regular} is implicitly a commutative ring. By a *graded ring* we shall mean a $\mathbb{Z}$-graded ring, that is, a ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$ satisfying $A_i A_j \subseteq A_{i+j}$. If A has not been referred to as a graded ring, $\mathcal{J}(A)$ will denote the Jacobson radical of A; but if we have introduced A as a graded ring, then $\mathcal{J}(A)$ will denote the intersection of all maximal graded left ideals of A (in our situations this will always coincide with the intersection of all maximal graded right ideals). Similarly, but when A is commutative, if A is not given a grading then $Q(A)$ will denote the localization $A[\text{nonzerodivisors}]^{-1}$ (the total quotient ring of A), whereas if A is graded then we will set $Q(A) = A[\text{homogeneous nonzerodivisors}]^{-1}$. If A is any commutative ring, $\min A$ will denote its set of minimal primes.

If $A/\mathcal{J}(A)$ is a division ring, we will say that A is *local*, unless A is graded, in which case we will say that A is *graded-local*. By a *connected graded* ring we shall mean a commutative $\mathbb{N}$-graded ring $R = \bigoplus_{i \geqslant 0} R_i$ such that R_0 is a field. In this case $\hat{R}$ will denote the $\mathfrak{m}$-adic completion of R, where $\mathfrak{m} = \bigoplus_{i \geqslant 1} R_i$. If we introduce a local or graded-local ring as a pair $(R, \mathfrak{m})$ this will indicate that $\mathfrak{m} = \mathcal{J}(R)$, the unique maximal (graded) ideal of R.

We will say that an R-module M has rank (specifically, rank n), if $M \otimes_R Q(R)$ is a free

$Q(R)$-module (of rank n). We write $\overline{R}$ for the integral closure of R in $Q(R)$.

When R is Gorenstein (defined next section), and M is an R-module, we will use M^* to denote $\mathrm{Hom}_R(M, R)$.

1.1 The Basic Objects

Now we give a brief introduction to the objects studied in this thesis, namely maximal Cohen-Macaulay modules and Auslander-Reiten (AR) sequences of such.

1.1.1 Cohen-Macaulay modules and Gorenstein rings

Let $(R, \mathfrak{m})$ be a commutative local ring and M a finitely generated R-module. A sequence of elements $x_1 \dots, x_n \in \mathfrak{m}$ is called an *M-regular sequence* provided x_1 is a nonzerodivisor on M and for each $i \geqslant 2$, x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$. The length of the longest M-regular sequence is independent of choice of sequence and is called the *depth* of M denoted $\operatorname{depth}_R M$. A finitely generated module M is called *maximal Cohen-Macaulay* if $\operatorname{depth}_R M = \dim R$, the Krull dimension of R. A ring R is called *Cohen-Macaulay* if it is maximal Cohen-Macaulay as a module over itself.

If $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, let $\mathrm{CM}(R)$ denote the category of finitely generated maximal Cohen-Macaulay R-modules, and (following [3]) let $L_p(R)$ denote the full subcategory of $\mathrm{CM}(R)$ whose objects M have the property that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. If R is instead a Cohen-Macaulay connected graded ring, we define $\mathrm{CM}(R)$ and $L_p(R)$ in the same way except we restrict to graded modules.

Assume $(R, \mathfrak{m})$ is a commutative ring which is either local or connected graded. Then R is *Gorenstein* if and only if it is Cohen-Macaulay and $\dim_k(\mathrm{Ext}_R^{\dim R}(k, R)) = 1$. If R is Gorenstein, and $M \in \mathrm{CM}(R)$, then ([7, Theorem 3.3.10]): $\mathrm{Ext}_R^i(M, R) = 0$ for all $i \geqslant 1$, and the map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ given by $m \mapsto (f \mapsto f(m))$ is a natural isomorphism. We will denote $\mathrm{Hom}(M, R)$ by M^*.

We will have occasion to use the following basic lemma.

Lemma 1.1.1. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension one. Let $k = R/\mathfrak{m}$. Then the cokernel of the natural inclusion $R \hookrightarrow \mathfrak{m}^*$ is isomorphic to k.*

Proof. To begin with, we have a natural short exact sequence $0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0$. Applying $(\_)^*$, we get an exact sequence $0 \longrightarrow k^* \longrightarrow R^* \longrightarrow \mathfrak{m}^* \longrightarrow \operatorname{Ext}^1_R(k, R) \longrightarrow \operatorname{Ext}^1_R(R, R)$. Since $\operatorname{depth}_R R = 1$, $k^* = 0$, so this exact sequence can viewed as $0 \longrightarrow 0 \longrightarrow R \longrightarrow \mathfrak{m}^* \longrightarrow \operatorname{Ext}^1_R(k, R) \longrightarrow 0$. Moreover, $\dim_k(\operatorname{Ext}^1_R(k, R)) = 1$ since R is one-dimensional Gorenstein. $\square$

1.1.2 Auslander-Reiten sequences

In this subsection, assume that $(R, \mathfrak{m})$ is a complete (or graded-) local Cohen-Macaulay ring.

Definition 1.1.2. Let N be an indecomposable in $\mathrm{CM}(R)$. Then (cf. [27, Lemma 2.9′]) we may define an *Auslander-Reiten (AR) sequence starting from* N to be a short exact sequence

$$0 \longrightarrow N \xrightarrow{p} E \xrightarrow{q} M \longrightarrow 0 \tag{1.1.1}$$

in $\mathrm{CM}(R)$ such that M is indecomposable and the following property is satisfied: Any map $N \longrightarrow L$ in $\mathrm{CM}(R)$ which is not a split monomorphism factors through p. Equivalently, N is indecomposable and any map $L \longrightarrow M$ in $\mathrm{CM}(R)$ which is not a split epimorphism factors through q. The sequence (1.1.1) is unique (up to isomorphism of short exact sequences) if it exists, and is also called the AR sequence ending in M. Given an AR sequence (1.1.1), N is called the Auslander-Reiten translate of M, written $\tau(M)$; and $\tau^{-1}(N)$ denotes M.

Definition 1.1.3. A morphism $f : X \longrightarrow Y$ in $\mathrm{CM}(R)$ is called an *irreducible morphism* if (1) f is neither a split monomorphism nor a split epimorphism, and (2) given any pair of morphisms g and h in $\mathrm{CM}(R)$ satisfying $f = gh$, either g is a split epimorphism or h is a split monomorphism.

Irreducible maps are closely related to AR sequences: see Lemma 3.2.5.

Theorem 1.1.4. *(*[27, Theorem 3.4], [3, Theorem 3]*) Let $M \ncong R$ be an indecomposable in* $\mathrm{CM}(R)$*. Then $M \in L_p(R)$ if and only if there exists an AR sequence ending in M.*

Notice also that if R is Gorenstein, applying $(\_)^*$ shows that there exists an AR sequence ending in M if and only if there exists an AR sequence starting from M. The appendix of [1] contains a nice proof of Theorem 1.1.4 in a slightly different setting, but one which includes Gorenstein rings of dimension one.

Lemma 1.1.5. *Assume* $\dim R = 1$*, and let $N \in \mathrm{CM}(R)$. Then $N \in L_p(R)$ if and only if $N \otimes_R Q$ is a projective Q-module, where $Q = Q(R)$.*

Proof. The prime ideals of Q correspond to the prime ideals of R not equal to $\mathfrak{m}$. Now use the fact that, since Q is noetherian, a Q-module is projective precisely when it is free at all maximal ideals of Q (cf. [10, Exercise 4.11]). □

1.1.6. For the remainder of this section assume furthermore that R is Gorenstein of dimension one, and $M \ncong R$ is an indecomposable in $L_p(R)$. Then (ignoring a graded shift, in the graded case; it will not concern us) $\tau(M) = \mathrm{syz}_R(M)$ (cf. [27, 3.11]), where $\mathrm{syz}_R(M)$ denotes the syzygy module of M, which is defined to be the kernel of a minimal surjection onto M by a free R-module. The module $\tau^{-1}(M) = \mathrm{syz}_R^{-1}(M) \in L_p(R)$ is determined by $\mathrm{syz}_R(\mathrm{syz}_R^{-1}(M)) \cong M$, and can be computed via $\mathrm{syz}_R^{-1} M \cong (\mathrm{syz}_R(M^*))^*$.

Definition 1.1.7. Given a ring A, and A-modules X and Y, $\underline{\mathrm{Hom}}_A(X, Y)$ denotes $\mathrm{Hom}_A(X, Y)/\{\text{maps factoring through projective } A\text{-modules}\}$, and $\underline{\mathrm{End}}_A(X)$ denotes $\underline{\mathrm{Hom}}_A(X, X)$. An A-homomorphism is said to be *stably zero* if it factors through a projective A-module.

Lemma 1.1.8. [27, Lemma 3.8] *Let A be a commutative ring, and let X and Y be finitely generated A-modules. The sequence*

$$\mathrm{Hom}_A(X, A) \otimes_A Y \xrightarrow{\mu} \mathrm{Hom}_A(X, Y) \longrightarrow \underline{\mathrm{Hom}}_A(X, Y) \longrightarrow 0$$

is exact, where $\mu\colon \operatorname{Hom}_A(X,A)\otimes_A Y \longrightarrow \operatorname{Hom}_A(X,Y)$ *is given by* $f\otimes y \mapsto (x\mapsto f(x)y)$.

Lemma 1.1.9. $\underline{\operatorname{End}}_R(M)\cong \operatorname{Ext}^1_R(\operatorname{syz}^{-1}_R(M),M)$ *as left* $\operatorname{End}_R(M)$*-modules.*

Proof. Let $N=\operatorname{syz}^{-1}_R(M)$. By applying $\operatorname{Hom}_R(\_,M)$ to a short exact sequence $0\longrightarrow M\longrightarrow F\longrightarrow N\longrightarrow 0$ where F is free, we have an exact sequence $\operatorname{Hom}_R(F,M)\longrightarrow \operatorname{Hom}_R(M,M)\longrightarrow \operatorname{Ext}^1_R(N,M)\longrightarrow \operatorname{Ext}^1_R(F,M)=0$. It only remains to observe that the image of $\operatorname{Hom}_R(F,M)\longrightarrow \operatorname{Hom}_R(M,M)$ consists of all endomorphisms factoring through projectives, which simply follows from the definition of projective. $\square$

Remark 1.1.10. Let $M\in L_p(R)$ be a nonfree indecomposable. Then $\operatorname{End}_R M$ is a (graded-) local ring (cf. [3, Proposition 8]), and therefore so is $\underline{\operatorname{End}}_R M$. It follows from Lemma 1.1.9 and Theorem 1.1.4 that the ring $\underline{\operatorname{End}}_R M$ has a simple socle when considered as a left module over itself, and that if $h : M\longrightarrow M$ generates this socle then the AR sequence starting from M is obtained as the pushout via h of the short exact sequence $0\longrightarrow M\longrightarrow F\longrightarrow \operatorname{syz}^{-1}_R(M)\longrightarrow 0$ where F is free. In particular, if ι denotes the given injective map $M\longrightarrow F$, and $0\longrightarrow M\longrightarrow X\longrightarrow N\longrightarrow 0$ is the AR sequence starting from M, then $X\cong (M\oplus F)/\{(-h(m),\iota(m))|m\in M\}$.

1.2 Summary of Results

This thesis consists of five chapters, and an appendix. Our main setting has R as a complete local Gorenstein ring of dimension one, but R can be connected graded instead of complete local. The main goal of Chapter 2 is to show that there exists a set of elements (we call them Frobenius elements) of $\operatorname{End}_R \mathfrak{m}$, corresponding to the minimal primes of R, which produce the Auslander-Reiten (AR) sequences in a concrete way. In Section 2.3 we observe some added conveniences that arise when R is graded. In Chapter 3 we provide background about AR quivers. The AR quiver of R is essentially the directed graph with vertices the indecomposables in $L_p(R)$, and arrows corresponding to the irreducible maps. We also establish criteria for confirming that AR components have certain desirable properties– for example, properties which are hypotheses in classical theorems such as Riedtmann's Structure Theorem and the Brauer-Thrall Theorem. The main ideas in Chapter 3 come from [1], but we provide more details and correct an error.

In Section 4.1, we adapt results from Green-Zacharia [12] and Kerner-Zacharia [19], from the context of selfinjective Artin algebras, to our context of Gorenstein rings of dimension one. This allows us to list the potential shapes of the components of AR quivers in our setting, when we specialize slightly to assume that R is a complete intersection ring. We do not know if any result such as this had been previously known for nonartinian rings. The results of 4.1 also have applications to special cases of the Huneke-Wiegand conjecture, which we describe in 4.2.

In Chapter 5, we compute some 'Frobenius elements' (a pivotal concept in Chapter 2). In 5.3, we give an application of Theorem 2.2.14, to establish the shapes of some AR components (namely, so-called "tubes") over a graded hypersurface of the form $k[x,y]/((bx^p+y^q)f)$ where $f \in k[x,y]$ is an arbitrary homogeneous polynomial.

Chapter 2

AR sequences and Frobenius Elements

The main goal of this chapter is to prove Theorem 2.2.14, which gives a concrete description of how to compute AR sequences in the setting of a Gorenstein ring $(R, \mathfrak{m})$ of dimension one, using an element of $\operatorname{End}_R \mathfrak{m}$. In the case when R is reduced, we get a succinct definition of such an element, and we call it a *Frobenius element* for R (Definition 2.2.15). There are nice equivalent definitions when R is furthermore connected graded (Section 2.3).

Notation 2.0.1. In this chapter, an unadorned Q will only be used when we have introduced some ring R, and Q will always denote $Q(R)$, defined in Notation 1.0.1.

2.0.1 Trace lemmas

We establish some preliminary lemmas regarding trace. Observations of this general type have certainly been made before; see [2, Proposition 5.4]. First, we define the trace of an endomorphism of an arbitrary finitely generated projective module, as in [15].

Definition 2.0.2. Let A be a commutative ring, and let P be a finitely generated projective A-module. Then the map $\mu_P \colon \operatorname{Hom}_A(P, A) \otimes_A P \longrightarrow \operatorname{End}_A P$ given by $f \otimes x \mapsto (y \mapsto f(y)x)$ is an isomorphism, by Lemma 1.1.8. Let $\epsilon \colon \operatorname{Hom}_A(P, A) \otimes_A P \longrightarrow A$ denote the map given by $f \otimes x \mapsto f(x)$. For $h \in \operatorname{End}_A P$, we define $\operatorname{trace}(h) = \epsilon(\mu_P^{-1}(h))$. If $e_1, \ldots, e_n$ and $\varphi_1, \ldots, \varphi_n \in \operatorname{Hom}_A(P, A)$ are such that $\mu_P(\sum_{i=1}^n \varphi_i \otimes e_i) = \mathrm{id}_P$, then $\operatorname{trace}(h)$ furthermore equals $\sum_{i=1}^{n} \varphi_i(h(e_i))$. From this, and using that $P = \sum_i Ae_i$, it follows that trace is symmetric, in the sense that $\operatorname{trace}(gh) = \operatorname{trace}(hg)$ for all $g, h \in \operatorname{End}_A P$.

Remark 2.0.3. We can see that the above definition of trace specializes to the usual one when P is free, by taking the aforementioned $\{e_i, \varphi_i\}_i$ to be a free basis and the corresponding projection maps. If $A = k_1 \times \ldots \times k_s$ is a product of fields k_i, then by a similar argument we see that for any $h \in \operatorname{End}_A P$, we have $\operatorname{trace}(h) = (\operatorname{trace}(h \otimes_A k_1), \ldots, \operatorname{trace}(h \otimes_A k_s))$.

Recall that if R is an ungraded reduced ring, then Q is the product of fields $R_\mathfrak{p} = Q(R/\mathfrak{p})$ where $\mathfrak{p}$ ranges over $\min R$. In particular, each $R_\mathfrak{p}$ is an ideal of Q, and a Q-algebra.

Lemma 2.0.4. *Let R be a reduced ring (possibly graded), let M a finitely generated R-module such that $M \otimes_R Q$ is Q-projective, and let $h \in \operatorname{End}_R M$. Then* $\operatorname{trace}(h \otimes_R Q) \in \overline{R}$. *(In the ungraded case, the condition that $M \otimes_R Q$ is Q-projective is automatically satisfied, since Q is semisimple.)*

Proof. First suppose the graded case. Let $Q' = R[\text{nonzerodivisors}]^{-1}$; thus Q' is a localization of Q is a localization of R, and $R \subseteq Q \subseteq Q'$. As $M \otimes_R Q$ is Q-projective, there exists a finite set $\{e_i \in M \otimes_R Q\}_i$ and corresponding $\{\varphi_i \colon M \otimes_R Q \longrightarrow Q\}_i$ such that $y = \sum_i \varphi_i(y)e_i$ for all $y \in M \otimes_R Q$. Then the images of the e_i in $M \otimes_R Q'$ have the property that $y = \sum_i (\varphi_i \otimes_Q Q')(y)e_i$ for all $y \in M \otimes_R Q'$. Therefore $\operatorname{trace}(h \otimes_R Q) = \sum_i \varphi_i((h \otimes_R Q)(e_i)) = \sum_i (\varphi_i \otimes_Q Q')((h \otimes_R Q')(e_i)) = \operatorname{trace}(h \otimes_R Q')$. Since $\overline{R}$ is equal to the integral closure of R in Q' by [17, Corollary 2.3.6], we are thus reduced to the ungraded case.

Since $\overline{R} = \prod_{\mathfrak{p} \in \min R} \overline{R/\mathfrak{p}}$, we see by Remark 2.0.3 that it suffices to show $\operatorname{trace}(h \otimes_R R_\mathfrak{p}) \in \overline{R/\mathfrak{p}}$, for each $\mathfrak{p} \in \min R$. As $h \otimes_R R_\mathfrak{p} = (h \otimes_R R/\mathfrak{p}) \otimes_{R/\mathfrak{p}} R_\mathfrak{p}$, we may assume R is a domain. By [23, Theorem 2.1], h satisfies a monic polynomial with coefficients in R, say $f(X) \in R[X]$. Let $H = h \otimes_R Q$, and let $\mu(X) \in Q[X]$ denote the minimal polynomial of H. Let $\chi(X) \in Q[X]$ denote the characteristic polynomial of H, and take a field extension $L \supseteq Q$ over which $\chi(X)$ splits, say $\chi(X) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_s)$, $\alpha_i \in L$. Each α_i is also a root of $\mu(X)$, and therefore of $f(X)$. Therefore $R[\alpha_1, ..., \alpha_s]$ is an integral extension of R. Thus $\overline{R} \supseteq Q \cap R[\alpha_1, ..., \alpha_s]$, which contains the coefficients of $\chi(X)$. This proves the lemma, since $\operatorname{trace}(H)$ is the degree-$(s-1)$ coefficient of χ. □

Lemma 2.0.5. *In the situation of the previous lemma, assume that* $\dim R = 1$ *and that* $(R, \mathfrak{m})$ *is either a complete local ring or a connected graded ring. If some power of* h *lies in* $\mathfrak{m} \operatorname{End}_R M$*, then* $\operatorname{trace}(h \otimes_R Q) \in \mathcal{J}(\overline{R})$.

Proof. As $\mathcal{J}(\overline{R}) = \prod_{\mathfrak{p} \in \min R} \mathcal{J}(\overline{R/\mathfrak{p}})$, we may again assume R is a domain. In the connected graded case, we have $\mathcal{J}(\overline{\hat{R}}) \cap \overline{R} = \mathcal{J}(\overline{R})$ by Lemma 6.0.4, and we can therefore assume the complete local case. We can also assume $M \subseteq M \otimes_R Q$, i.e., replace M by its image in $M \otimes_R Q$. Now let $M\overline{R}$ denote the $\overline{R}$-module of $M \otimes_R Q$ generated by M. Note that $M\overline{R}$ is a free $\overline{R}$-module, since all torsion-free $\overline{R}$-modules are free. Since $\overline{R}$ is local, we can choose a basis for $M\overline{R}$ which consists of elements of M, say $e_1, ..., e_n$. (Indeed, setting $n = \operatorname{rank}(M\overline{R})$, Nakayama's lemma allows us to find a set $\{e_1, ..., e_n\} \subset M$ such that $M\overline{R} = \sum_i \overline{R} e_i$. Then we have a surjective endomorphism of $M\overline{R}$, equivalently an automorphism, mapping a basis onto $\{e_1, ..., e_n\}$.) By fixing this basis, we can identify $\operatorname{End}_R M$ as an R-subalgebra of the ring of $n \times n$ matrices $\operatorname{Mat}_{n \times n}(\overline{R})$, in the obvious way. By assumption on h, some power of h lies in $\mathfrak{m} \operatorname{Mat}_{n\times n}(\overline{R}) \subseteq \mathcal{J}(\overline{R}) \operatorname{Mat}_{n\times n}(\overline{R})$. Thus the image of h in $\operatorname{Mat}_{n\times n}(\overline{R}/\mathcal{J}(\overline{R}))$ is nilpotent. The lemma now follows from the fact that over a field, any nilpotent matrix has zero trace. □

2.1 Testing stable-vanishing with trace

In this section, let R simply be a commutative ring, and let M be a finitely generated R-module such that $M \otimes_R Q$ is a projective Q-module. Let $(\_)^*$ denote $\operatorname{Hom}_R(\_, R)$.

Notation 2.1.1. Given an R-algebra B, let $D_B(\_)$ denote $\operatorname{Hom}_R(\_, B)$. Let ν_B denote $D_B((\_)^*) = D_B \circ D_R(\_)$, and let λ_B denote the Hom-Tensor adjoint isomorphism

$$\lambda_B \colon D_B(M^* \otimes_R \_) \longrightarrow \operatorname{Hom}_R(\_, \nu_B M).$$

We also let μ_M denote the natural transformation $\mu_M\colon M^* \otimes_R \_ \longrightarrow \operatorname{Hom}_R(M, \_\,)$ given by $f \otimes x \mapsto (m \mapsto f(m)x)$. For future reference, we note that for a given R-module X, the map $\lambda_B \circ (D_B\mu_M)\colon D_B \operatorname{Hom}_R(M, X) \longrightarrow \operatorname{Hom}_R(X, \nu_B M)$ is given by the rule

$$[\lambda_B \circ D_B\mu_M](\sigma)(x)(f) = \sigma(\mu_M(f \otimes x))\,, \text{ for all } \sigma \in D_B \operatorname{Hom}_R(M, X),\ x \in X,\ f \in M^*. \tag{2.1.1}$$

Let $E = Q/R$. The exact sequence $0 \longrightarrow R \xrightarrow{\iota} Q \xrightarrow{q} E \longrightarrow 0$ induces the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D_R \operatorname{Hom}_R(M, _\,) \arrow[r] \arrow[d, "\lambda_R \circ D_R\mu_M"] & D_Q \operatorname{Hom}_R(M, _\,) \arrow[r, "q_*"] \arrow[d, "\lambda_Q \circ D_Q\mu_M"] & D_E \operatorname{Hom}_R(M, _\,) \arrow[d, "\lambda_E \circ D_E\mu_M"] \\
0 \arrow[r] & \operatorname{Hom}_R(_\,, \nu_R M) \arrow[r, "\iota_*"] & \operatorname{Hom}_R(_\,, \nu_Q M) \arrow[r] & \operatorname{Hom}_R(_\,, \nu_E M)
\end{tikzcd} \tag{2.1.2}$$

We now show that $D_Q\mu_M$ is an isomorphism on the category of finitely generated R-modules, so that the second map in the composable pair

$$D_R \operatorname{Hom}_R(M, \_\,) \xrightarrow{\lambda_R \circ D_R\mu_M} \operatorname{Hom}_R(\_\,, \nu_R M) \xrightarrow{q_*(\lambda_Q \circ D_Q\mu_M)^{-1}\iota_*} D_E \operatorname{Hom}_R(M, \_\,) \tag{2.1.3}$$

is well-defined.

Lemma 2.1.2. [1, Appendix]

(1) The map $D_Q\mu_M$ is an isomorphism on finitely generated R-modules, and the sequence 2.1.3 is exact.

(2) If R is Gorenstein of dimension one, and both M and the input module X lie in $\mathrm{CM}(R)$, then the image of $\lambda_R \circ D_R\mu_M$ consists of the stably zero maps $X \longrightarrow \nu_R M$.

Proof. Note that $\mu_M \otimes_R Q$ can be identified with $\mu_{M\otimes Q}\colon \operatorname{Hom}_Q(M \otimes_R Q, Q) \otimes_Q (\_ \otimes_R Q) \longrightarrow \operatorname{Hom}_Q(M \otimes_R Q, \_ \otimes_R Q)$, which is an isomorphism because $M \otimes_R Q$ is a projective Q-module. Thus $D_Q\mu_M$ is an isomorphism, since it can be viewed as $D_Q(\mu_M \otimes_R Q)$. The exactness of 2.1.3 is seen by chasing the diagram 2.1.2.

Now we assume the hypotheses of (2). Take a short exact sequence

$$0 \longrightarrow \mathrm{syz}_R(M) \longrightarrow F \xrightarrow{p} M \longrightarrow 0\,,$$

where F is a free R-module. Consider the commutative diagram

$$\begin{tikzcd}
(\mathrm{Hom}_R(F,X))^* \arrow[r] \arrow[d] & (F^* \otimes_R X)^* \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,F^{**}) \arrow[d] & \mathrm{Hom}_R(X,F) \arrow[l] \arrow[d] \\
(\mathrm{Hom}_R(M,X))^* \arrow[r, "D_R\mu_M"] & (M^* \otimes_R X)^* \arrow[r, "\lambda_R"] & \mathrm{Hom}_R(X,M^{**}) & \mathrm{Hom}_R(X,M)\,, \arrow[l]
\end{tikzcd} \tag{2.1.4}$$

where the vertical maps are induced by $p\colon F \longrightarrow M$, and the horizontal maps on the right are the isomorphisms induced by $M \cong M^{**}$ and $F \cong F^{**}$. It is easy to see that the image of the rightmost vertical map consists of the stably zero maps $X \longrightarrow M$, and it follows that the third vertical map consists of the stably zero maps $X \longrightarrow M^{**}$. Let H denote the map $\mathrm{Hom}_R(M,X) \longrightarrow \mathrm{Hom}_R(F,X)$ induced by p. Since the top row of diagram 2.1.4 consists of isomorphisms, establishing surjectivity of the leftmost vertical map, namely $D_R H$, is sufficient for proving (2). Let $N = \operatorname{cok} H$. By left-exactness of Hom, we have a left-exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,X) \xrightarrow{H} \mathrm{Hom}_R(F,X) \longrightarrow \mathrm{Hom}_R(\mathrm{syz}_R(M),X)\,,$$

and therefore N embeds into $\mathrm{Hom}_R(\mathrm{syz}_R(M),X)$. Thus $N \in \mathrm{CM}(R)$, so $\mathrm{Ext}^1_R(N,R) = 0$. Therefore the sequence $0 \longrightarrow N^* \longrightarrow \mathrm{Hom}_R(F,X)^* \xrightarrow{D_R H} \mathrm{Hom}_R(M,X)^* \longrightarrow 0$ is exact, so (2) is proved. $\square$

Lemma 2.1.3. *Assume R is Gorenstein of dimension one, and $M \in \mathrm{CM}(R)$. Then a given endomorphism $h\colon M \longrightarrow M$ is stably zero if and only if* $\mathrm{trace}(hg \otimes Q) \in R$ *for all $g\colon M \longrightarrow M$. (Recall the definition of trace, Definition 2.0.2.)*

Proof. Let η denote the isomorphism $\mathrm{End}_R M \longrightarrow \mathrm{Hom}_R(M,M^{**})$ induced by $M \cong M^{**}$, and let $\theta = (\lambda_Q \circ D_Q\mu_M)^{-1} \circ \iota_* : \mathrm{Hom}_R(M,M^{**}) \longrightarrow \mathrm{Hom}_R(\mathrm{End}_R M, Q)$. It follows from Lemma 2.1.2 that h is stably zero if and only if $[\theta(\eta h)](g) \in R$ for all $g\colon M \longrightarrow M$. So we aim to show that $[\theta(\eta h)](g) = \mathrm{trace}(hg \otimes Q)$. Let $\sigma : \mathrm{End}_R M \longrightarrow Q$ denote the map sending $g \in \mathrm{End}_R M$ to $\mathrm{trace}(hg)$. Thus, we wish to show $\theta(\eta h) = \sigma$; equivalently, $\iota_*(\eta h) = (\lambda_Q \circ D_Q\mu_M)(\sigma)$.

Take a finite collection of maps $\{\varphi_i : M \otimes_R Q \longrightarrow Q\}_i$ and elements $\{e_i\}_i \in M \otimes_R Q$, such that $w = \sum_i \varphi_i(w)e_i$ for all $w \in M \otimes_R Q$; thus $\mathrm{trace}(h \otimes Q) = \sum_i \varphi_i((h \otimes Q)(e_i))$, as we mentioned in Definition 2.0.2. Given $m \in M$, and $f \in M^*$, let g denote the endomorphism $\mu_M(f \otimes m)$. By equation 2.1.1, $(\lambda_Q \circ D_Q\mu_M)(\sigma)(m)(f) = \sigma(g) = \mathrm{trace}(hg) = \sum_i \varphi_i((h \otimes Q)((g \otimes Q)(e_i)))$. Now using firstly that $g \otimes Q$ is given by $w \mapsto (f \otimes Q)(w)m$, and then that $f \otimes Q$ and the φ_i's have output in Q, we have

$$\begin{aligned}(\lambda_Q \circ D_Q\mu_M)(\sigma)(m)(f) &= \textstyle\sum_i \varphi_i((h \otimes Q)((f \otimes Q)(e_i)m)) = \sum_i (f \otimes Q)(e_i)\varphi_i(h(m)) \\ &= (f \otimes Q)(\textstyle\sum_i \varphi_i(h(m))e_i) = f(h(m)) = \iota_*(\eta h)(m)(f).\end{aligned}$$

□

2.2 Main Result

In this section we prove Theorem 2.2.14, which is really a formula for the AR sequence ending in M, cf. Remark 1.1.10. For this section, let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay ring which is either a complete local ring or a connected graded ring. (We will assume R is Gorenstein from 2.2.6 onwards.) Throughout this section, $(\_)^*$ will denote $\mathrm{Hom}_R(, R)$.

Notation 2.2.1. For a commutative ring A, and A-modules $N \subseteq M$, we will sometimes use the standard notation $(N :_A M)$ to denote the ideal $\{a \in A | aM \subseteq N\} = \mathrm{Ann}_A(M/N)$.

Recall that for a commutative ring A, if M and N are finitely generated A-submodules of $Q(A)$, and M contains a faithful element w, i.e. $(0 :_A w) = 0$, then $\mathrm{Hom}_A(M, N)$ is naturally identified with $(N :_{Q(A)} M)$.

Notation 2.2.2. If R is reduced, let $\mathfrak{F}(R)$ denote $(R :_Q \mathcal{J}(\overline{R})) = \mathcal{J}(\overline{R})^*$. Let I^{cd} denote the conductor ideal, $I^{cd} = (R :_R \overline{R})$.

Lemma 2.2.3. *We have* $\operatorname{End}_R \mathfrak{m} \subseteq \overline{R}$. *Moreover,* $\operatorname{End}_R \mathfrak{m} = \mathfrak{m}^*$ *if* R *is not regular.*

Proof. It follows from [23, Theorem 2.1] that $\operatorname{End}_R \mathfrak{m} \subseteq \overline{R}$. For $\mathfrak{m}^* = \operatorname{End}_R \mathfrak{m}$, it suffices to show that every homomorphism $\mathfrak{m} \longrightarrow R$ has image in $\mathfrak{m}$, equivalently $\mathfrak{m}$ has no free summand (since any epimorphism onto R must split). But any proper direct summand of an ideal has nonzero annihilator; and if $\mathfrak{m}$ itself were free, then $\mathfrak{m} \cong R$ and R would be regular. □

Lemma 2.2.4. *Assume* R *is reduced. Then*

(1) $\mathfrak{F}(R) \subseteq \mathfrak{m}^*$. *If* R *is not regular, then* $\mathfrak{F}(R) \subseteq \operatorname{End}_R \mathfrak{m}$.

(2) Assume R *is not regular, and further that either (a)* R *is a domain, or (b)* R *is Gorenstein. Then* $\mathfrak{F}(R) \not\subseteq R$.

Proof. As $\mathfrak{m} \subseteq \mathcal{J}(\overline{R})$, we have $\mathfrak{F}(R) \subseteq \mathfrak{m}^*$, so we get (1) by Lemma 2.2.3.

Case 2a: R is a domain. Then $(\overline{R}, \mathfrak{m}_{\overline{R}})$ is a discrete valuation ring in the complete local case, and a polynomial ring over a field in the connected graded case (Lemma 6.0.5). Let π denote a generator for $\mathfrak{m}_{\overline{R}}$, and let n denote the positive integer such that $I^{cd} = \pi^n \overline{R}$. Then $\pi^{n-1}\overline{R} \subseteq \mathfrak{F}(R)$, while $\pi^{n-1}\overline{R} \not\subseteq R$.

Case 2b: R is Gorenstein. Consider the family of finitely generated R-submodules $X \subset Q$ such that X contains a faithful element. It is well-known and readily-checked that the application of $(\_\,)^*$ to such modules X is an inclusion reversing operation satisfying $X^{**} = X$. So, the observation that $R \not\subseteq \mathcal{J}(\overline{R})$ implies $\mathfrak{F}(R) \not\subseteq R$. □

Lemma 2.2.5. *Let* $I \subset R$ *be a (homogeneous) radical ideal of height zero. Then we have an* R*-algebra isomorphism* $Q/IQ \cong Q(R/I)$.

Proof. If $x \in R$ is a (homogeneous) nonzerodivisor, we have $R_x = Q$. To see this, it suffices to check that a given (homogeneous) nonzerodivisor $y \in R$ becomes a unit in R_x. As Ry is $\mathfrak{m}$-primary, we have $x^i = ry$ for some $i \geqslant 1$ and some $r \in R$. Therefore y is a unit in R_x; hence $R_x = Q$. As R/I is reduced, its associated primes are all minimal; namely they are the images of those minimal primes of R which contain I. So x remains a nonzerodivisor modulo I, since in general the set of zerodivisors equals the union of the associated primes. Therefore $Q(R/I) = (R/I)_x = R_x \otimes_R (R/I) = Q \otimes_R (R/I) = Q/IQ$. $\square$

2.2.6. For the remainder of this section, adopt the further assumption on R that it is Gorenstein and not regular. Let $I \subset R$ be a radical ideal of height zero, and assume I is homogeneous in the case that R is connected graded. Let $R' = R/I$, and $Q' = Q(R')$, which we identify with Q/IQ by Lemma 2.2.5. After the upcoming Proposition 2.2.8, we will assume the following condition, which is automatically satisfied if I is prime or R' is Gorenstein; see Proposition 2.2.8 and Remark 2.2.9.

Condition 2.2.7. *There exist (homogeneous) elements $z \in R$ and $\gamma' \in \mathfrak{F}(R') \setminus R'$ such that*

(1) $I = \operatorname{Ann}_R(z)$, and

(2) For some (equivalently, every) $\widetilde{\gamma}' \in Q$ such that $\widetilde{\gamma}' + IQ = \gamma'$, we have $\widetilde{\gamma}' z \notin R$.

Proposition 2.2.8. *If $I \in \min R$, then Condition 2.2.7 is satisfied.*

Proof. By Lemma 2.2.4, we can pick (homogeneous) $\gamma' \in \mathfrak{F}(R') \setminus R'$; any such γ' will do. Let ω denote the ideal $\operatorname{Ann}_R(I)$. Now $\omega \cong \operatorname{Hom}_R(R', R)$ is, up to a graded shift, a canonical module for R' ([7, Theorem 3.3.7] and [7, Proposition 3.6.12]), and therefore we have $\operatorname{End}_{R'} \omega \cong R'$ (cf. [7, Theorem 3.3.4] and the proof of [7, Proposition 3.6.9b]). We will also use that $I = \operatorname{Ann}_R(\omega) = \operatorname{Ann}_R(z)$ for each nonzero $z \in \omega$, which is true because all associated primes of R are minimal (since R is Cohen-Macaulay), so that any ideal strictly larger than I contains a nonzerodivisor.

Now let $\widetilde{\gamma}'$ be a lift of γ' to Q. Regarding ω as a subset of Q via $\omega \subset R \subset Q$, suppose that $\widetilde{\gamma}'\omega \subseteq \omega$. Then the action of $\widetilde{\gamma}'$ on ω agrees with the multiplication on ω by some $r \in R$, so $\widetilde{\gamma}' - r \in \operatorname{Ann}_Q(\omega) = IQ$. But then $\gamma' \in R'$ is a contradiction. So there must exist $z \in \omega$ such that $\widetilde{\gamma}' z \notin \omega$; any such z will do. As $\operatorname{Ann}_Q(I) \cap R = \omega$, we thus have $\widetilde{\gamma}' z \notin R$. $\square$

Remark 2.2.9. In Proposition 2.2.8, the hypothesis $I \in \min R$ can replaced by the assumption that R' is Gorenstein; the proof is similar. In this case $zR = \operatorname{Ann}_R(I)$, and one may obtain part (1) of Condition 2.2.7 by observing that, since $R' \in \operatorname{CM}(R)$ and R is Gorenstein, $\operatorname{Ann}_R(\operatorname{Hom}_R(R', R)) = \operatorname{Ann}_R(R') = I$.

Notation 2.2.10. For the remainder of this section we assume Condition 2.2.7, and fix such z, γ', and $\widetilde{\gamma}'$; and we set $\gamma = z\widetilde{\gamma}'$.

Lemma 2.2.11. *We have* $\gamma \in \operatorname{End}_R \mathfrak{m}$.

Proof. From Lemma 2.2.4 we have $\gamma'\mathfrak{m}R' \subseteq R'$. Since $zR \cong R'$, it follows that $\widetilde{\gamma}'z\mathfrak{m} \subseteq zR$, thus $\gamma \in \mathfrak{m}^* = \operatorname{End}_R \mathfrak{m}$ (Lemma 2.2.3). □

Lemma 2.2.12. *Let* $M \in L_p(R)$, *and* $h \in \operatorname{End}_R M$. *Then* $\operatorname{trace}(h \otimes Q') = \operatorname{trace}(h \otimes Q) + IQ$.

Proof. Take $\{\varphi_i : M \otimes_R Q \longrightarrow Q\}_i$ and $\{e_i\}_i \in M \otimes_R Q$ such that $w = \sum_i \varphi_i(w)e_i$ for all $w \in M \otimes_R Q$. If $\varphi_i' = \varphi \otimes_R R' \colon M \otimes_R Q' \longrightarrow Q'$ and e_i' denotes the image of e_i in $M \otimes_R Q'$, then $w' = \sum_i \varphi_i'(w')e_i'$ for all $w' \in M \otimes_R Q'$. Now $\operatorname{trace}(h \otimes Q') = \sum_i \varphi_i'((h \otimes Q')(e_i')) = \sum_i \varphi_i((h \otimes Q)(e_i)) + IQ = \operatorname{trace}(h \otimes Q) + IQ$. □

Notation 2.2.13. Assume $M \in L_p(R)$ has no free direct summands. Then there exists no surjection $M \longrightarrow R$, so $M^* = \operatorname{Hom}_R(M, \mathfrak{m})$, hence $M \cong \operatorname{Hom}_R(M, \mathfrak{m})^*$ is a module over the ring $\operatorname{End}_R \mathfrak{m}$. Therefore γ induces an endomorphism of M, by Lemma 2.2.11. Denote this endomorphism by γ_M. Denote by $[\gamma_M]$ the class of γ_M in the stable endomorphism ring.

Theorem 2.2.14. *Assume* $M \in L_p(R)$ *is a nonfree indecomposable. Then* $[\gamma_M] \in \operatorname{soc}(\underline{\operatorname{End}}_R M)$, *for* γ *as in Notation 2.2.10. Let* $M' = M \otimes_R R'$, *and suppose that either*

(1) M' *has rank (that is,* $M \otimes_R Q'$ *is* Q'*-free), and* $\operatorname{rank}(M')$ *is a unit in* R*; or*

(2) For some minimal prime $\mathfrak{p}$ *of* R', $\dim_{R'_{\mathfrak{p}}}(M \otimes_R R'_{\mathfrak{p}})$ *is a unit in* R, *and* $\mathfrak{p}\gamma' = 0$.

Then, $[\gamma_M]$ *generates* $\operatorname{soc}(\underline{\operatorname{End}}_R M)$. *Thus, it produces the AR sequence beginning in* M, *in light of Remark 1.1.10.*

Proof. First we show $[\gamma_M] \in \operatorname{soc}(\underline{\operatorname{End}}_R M)$, which by Lemma 2.1.3 is equivalent to having $\operatorname{trace}(\gamma h \otimes Q) \in R$ for an arbitrary nonisomorphism $h\colon M \longrightarrow M$. As $\operatorname{End}_R M/(\mathfrak{m} \operatorname{End}_R M)$ is an artinian local ring, there exists some $i \geqslant 1$ such that $h^i \in \mathfrak{m} \operatorname{End}_R M$, and thus $h^i \otimes_R R' \in \mathfrak{m} \operatorname{End}_{R'}(M')$. So $\operatorname{trace}(h \otimes Q') \in \mathcal{J}(\overline{R'})$, by Lemma 2.0.5. Now using Lemma 2.2.12, $\widetilde{\gamma'} \operatorname{trace}(h \otimes Q) + IQ \in \gamma' \mathcal{J}(\overline{R'}) \subset R + IQ$, whence $\gamma \operatorname{trace}(h \otimes Q) \in zR + zIQ = zR \subset R$.

It remains to show that γ_M is not stably zero. By Lemma 2.1.3, it suffices to show $\operatorname{trace}(\gamma_M \otimes Q) \notin R$. Assume condition (1). Then the desired statement is a consequence of Lemma 2.2.12 together with the observation that $\gamma IQ = 0$ (since $zI = 0$). Namely, we have $\operatorname{trace}(\gamma_M \otimes Q) = \gamma \operatorname{trace}(1_{M \otimes Q}) \in \gamma(\operatorname{rank}(M') + IQ) = \gamma \operatorname{rank}(M') \notin R$. Now, assume condition (2). Let $n = \dim_{R'_{\mathfrak{p}}}(M \otimes_R R'_{\mathfrak{p}})$, and let $\mathfrak{P} = \mathfrak{p} \cap R$ (standard notation for the preimage of $\mathfrak{p}$ with respect to $R \twoheadrightarrow R'$). Since $\mathfrak{p}\gamma' = 0$ by assumption, we have $\widetilde{\gamma'}\mathfrak{P} \subseteq IQ$, and therefore $\gamma\mathfrak{P} = z\widetilde{\gamma'}\mathfrak{P} \subset zIQ = 0$. The argument is finished as in the first case. □

Definition 2.2.15. If $(R, \mathfrak{m})$ is a reduced one-dimensional Gorenstein ring which is either a complete local ring or a connected graded ring, we will say that an element $\gamma \in Q(R)$ is a *Frobenius element* for R if $\gamma \in \mathfrak{F}(R) \setminus R$.

Note that a Frobenius element satisfies Notation 2.2.10, by Remark 2.2.9 (setting $I = 0$).

Example 2.2.16. Let k be a field and let R be a numerical semigroup ring, $R = k[t^{i_1}, ..., t^{i_n}]$ (or $R = k[|t^{i_1}, ..., t^{i_n}|]$). Let F denote the *Frobenius number* of the numerical semigroup $\mathbb{N}i_1 + \cdots + \mathbb{N}i_n$, which means $F = \max\{j \in \mathbb{N} | j \notin \mathbb{N}i_1 + \cdots + \mathbb{N}i_n\}$. (This definition is from the numerical semigroup literature.) Then t^F is a Frobenius element for R.

2.3 Frobenius elements in the graded case

In this section, assume $(R, \mathfrak{m})$ is a reduced connected graded Gorenstein (but not regular) ring of dimension one, and set $k = R_0$.

Lemma 2.3.1. *The set $\{i | R_i \neq \overline{R}_i\}$ is finite.*

Proof. It is enough to check that length$(\overline{R}/R) < \infty$. Equivalently, $\overline{R}/R$ is a finitely generated R-module annihilated by some power of $\mathfrak{m}$. We know that $\overline{R}$ (and thus $\overline{R}/R$) is finitely generated by Lemma 6.0.3; let $\{r_1/s_1, \ldots, r_n/s_n\}$ be generators (with r_i and $s_i \in R$, and each s_i a nonzerodivisor). Then $s = s_1 s_2 \cdots s_n$ is a nonzerodivisor such that $s(\overline{R}/R) = 0$. But as R is one-dimensional, we have $\mathfrak{m}^i \subseteq sR$ for some $i \geqslant 1$, and thus $\mathfrak{m}^i(\overline{R}/R) = 0$. □

Recall the notion of the graded-shift of a graded module M: For $i \in \mathbb{Z}$, the i-th shift $M[i]$ has $M[i]_j = M_{i+j}$ for all $j \in \mathbb{Z}$.

Definition 2.3.2. The a-invariant of R, denoted $a(R)$, is the integer such that $\operatorname{Ext}^1_R(k, R[a(R)]) \cong k$ (sitting in degree zero). (See Section 3.6 of [7].)

Proposition 2.3.3. *The a-invariant $a(R)$ equals* $\sup\{i | R_i \neq \overline{R}_i\}$.

Proof. Let $s = \sup\{i | R_i \neq \overline{R}_i\}$. As in Lemma 1.1.1, we have that $\mathfrak{m}^*/R \cong k[i]$ for some $i \in \mathbb{Z}$, and from the short exact sequence $0 \longrightarrow R \longrightarrow \mathfrak{m}^* \longrightarrow k[i] \longrightarrow 0$ we get that $i = -a(R)$. Since $\overline{R}_s \mathfrak{m} \subseteq \bigoplus_{i>s} \overline{R}_i \subset R$, we have $\overline{R}_s \subseteq \mathfrak{m}^*$. In particular, $(\mathfrak{m}^*/R)_s \neq 0$, and therefore $s = -i = a(R)$. □

Recall that $\min R$ denotes the set of minimal primes of R.

Proposition 2.3.4. *Let γ be a homogeneous element in $\overline{R}$. The following are equivalent:*

(1) γ is a Frobenius element for R;

(2) $\gamma \in \overline{R}_{a(R)} \setminus R_{a(R)}$;

(3) $\gamma \in \mathfrak{m}^* \setminus R$.

Proof. The implication (1) ⇒ (3) is immediate, since $\mathfrak{m} \subseteq \mathcal{J}(\overline{R})$ implies $\mathfrak{F}(R) \subseteq \mathfrak{m}^*$. We have (3) ⇒ (2) because $\mathfrak{m}^* \subseteq \overline{R}$ (see Lemma 2.2.3) and $(\mathfrak{m}^*/R)_i = 0$ for $i \neq a(R)$. Finally, for (2) ⇒ (1), we wish to show $\overline{R}_{a(R)} \subseteq \mathfrak{F}(R)$, i.e. $\overline{R}_{a(R)}\mathcal{J}(\overline{R}) \subseteq R$. In view of Proposition 2.3.3, it suffices to show $\mathcal{J}(\overline{R}) \subseteq \bigoplus_{i \geqslant 1} \overline{R}_i$. (In fact this holds with equality.) In the domain case, this follows immediately from Lemma 6.0.4 (a). The general case follows, since $\overline{R} = \prod_{\mathfrak{p} \in \min R} \overline{R/\mathfrak{p}}$ and $\mathcal{J}(\overline{R}) = \prod_{\mathfrak{p} \in \min R} \mathcal{J}(\overline{R/\mathfrak{p}})$. □

Remark 2.3.5. Proposition 2.3.4, together with Proposition 2.3.3, gives a criterion for determining the Frobenius number of a symmetric numerical semigroup $\sum_{i=1}^{e} d_i\mathbb{N}$, though it is presumably already known in some formulation. Namely, F is the Frobenius number of $\sum_{i=1}^{e} d_i\mathbb{N}$ if and only if $t^F \in \mathfrak{m}^* \setminus R$ for $(R, \mathfrak{m}) = (k[t^{d_1}, \dots, t^{d_e}], (t^{d_1}, \dots, t^{d_e}))$, where k is any field.

Proposition 2.3.6. *Assume R is generated, as a k-algebra, by graded nonzerodivisors. Then R is a semigroup ring $k[t^{i_1}, \dots, t^{i_n}]$ if (and only if) $R_{a(R)} = 0$.*

Proof. Let $a = a(R)$, and assume $R_a = 0$. We may assume $\gcd(\{i|R_i \neq 0\}) = 1$. Let $x_1, \dots, x_s$ be graded nonzerodivisors generating R as a k-algebra, and let $d_i = \deg x_i$, $i = 1, \dots, s$. Then $\{i|R_i \neq 0\} = \mathbb{N}d_1 + \cdots + \mathbb{N}d_s = \{i|(R/\mathfrak{p})_i \neq 0\}) = 1$ for each minimal prime $\mathfrak{p}$, and in particular $\gcd(\{i|(R/\mathfrak{p})_i \neq 0\}) = 1$. Therefore, we can apply Lemma 6.0.4 to each $R/\mathfrak{p}$, to see that the k-vector spaces $(\overline{R/\mathfrak{p}})_i$ are nonzero for each $\mathfrak{p}$ and each $i \geqslant 0$, and so $\dim_k \overline{R}_a$ is at least $|\min R|$, since $\overline{R} = \prod_{\mathfrak{p} \in \min R} \overline{(R/\mathfrak{p})}$. But when $R_a = 0$, we have $\dim_k(\overline{R}_a) = 1$ since $\overline{R}_a/R_a \subseteq \mathfrak{m}^*/R \cong k[-i]$, so $|\min R| = 1$, i.e. R is a domain. Moreover, $\overline{R}$ is isomorphic to the standard-graded polynomial ring over $\overline{R}_0$ (see Lemma 6.0.4), so that the condition $\dim_k(\overline{R}_a) = 1$ implies that $\overline{R}_0 = k$. Thus R is a graded k-subalgebra of a polynomial ring, i.e., is a semigroup ring over k.

Corollary 2.3.7. *Assume k is algebraically closed, and that R is generated, as a k-algebra, by graded nonzerodivisors. Then R is a semigroup ring over k $\Leftrightarrow$ $R_{a(R)} = 0$ $\Leftrightarrow$ R is a domain.*

Proof. Since k is algebraically closed, R is a domain if and only if it is a semigroup ring over k (see Remark 6.0.6). So the result is immediate from Proposition 2.3.6.

2.4 Syzygy of $[\gamma_M]$

In this section, assume R is a reduced complete local Gorenstein (but not regular) ring of dimension one, and fix a Frobenius element γ, and a module M satisfying the hypothesis of Theorem 2.2.14. Notice that syz_R gives us a well-defined isomorphism of R-algebras $\mathrm{syz} : \underline{\mathrm{End}}_R M \longrightarrow \underline{\mathrm{End}}_R(\mathrm{syz}_R M)$. So in view of Theorem 2.2.14, it may be natural to ask how exactly $\mathrm{syz}([\gamma_M])$ relates to $[\gamma_{\mathrm{syz}\, M}]$. We give an answer in Proposition 2.4.7. Let $M\overline{R}$ denote that $\overline{R}$-submodule of $M \otimes_R Q$ generated by M, and assume the following: $M\overline{R}$ is a free $\overline{R}$-module which possesses a basis consisting of elements in M. This is true if R is a domain, since $\overline{R}$ is in that case a DVR, and $M\overline{R}$ is a torsion-free $\overline{R}$-module.

Notation 2.4.1. Fix $\gamma \in \mathfrak{F}(R) \setminus R$, and fix elements $e_1, ..., e_n \in M$ forming a free $\overline{R}$-basis for $M\overline{R}$. Given $h \in \mathrm{End}_R M$, let $\overline{h}$ denote the unique $\overline{R}$-linear endomorphism of $M\overline{R}$ extending h. We regard $\overline{h}$ is an n-by-n matrix with entries in $\overline{R}$. Recall that I^{cd} denotes the conductor ideal, $(R :_R \overline{R})$.

Lemma 2.4.2. *We have $\gamma M\overline{R} \subseteq M$, and $I^{cd}(M\overline{R}) \subset \bigoplus_i Re_i$.*

Proof. As $\gamma\mathcal{J}(\overline{R})$ is an ideal of both R and $\overline{R}$, we have $\gamma\mathcal{J}(\overline{R}) \subseteq I^{cd}$. Therefore $(\overline{R}\gamma)\mathfrak{m} \subseteq (\overline{R}\gamma)\mathcal{J}(\overline{R}) \subseteq I^{cd} \subseteq \mathfrak{m}$, which says that $\overline{R}\gamma \subseteq \mathrm{End}_R \mathfrak{m}$. Since $M \cong \mathrm{Hom}_R(M, \mathfrak{m})^*$ is an $\mathrm{End}_R \mathfrak{m}$-module (cf. Notation 2.2.13), we obtain $(\overline{R}\gamma)M \subseteq M$, equivalently $\gamma M\overline{R} \subseteq M$. That $I^{cd}(M\overline{R}) \subset \bigoplus_i Re_i$ is clear, since $I^{cd}\overline{R}e_i = I^{cd}e_i \subset Re_i$. $\square$

We have the following immediate consequence.

Lemma 2.4.3. *Let $A \in \mathrm{End}_{\overline{R}}(M\overline{R})$, i.e. A is an $n \times n$ matrix with entries in $\overline{R}$ (recall that we have a fixed basis, $\{e_1, ..., e_n\}$). If each entry of A lies in $\gamma\overline{R}$, then A sends M into M. If each entry of A lies in I^{cd}, then $A|_M : M \longrightarrow M$ is stably zero.*

Lemma 2.4.4. *There exists $f \in \mathrm{End}_R M$ satisfying the following conditions:*

(i) $[f]$ generates $\mathrm{soc}(\underline{\mathrm{End}}_R M)$;

(ii) all nonzero entries of $\overline{f}$ lie in $\overline{R}\gamma$.

(iii) the first column of $\overline{f}$ is its only nonzero column.

(iv) $\overline{f}_{1,1} = \gamma$.

Proof. If we take an $n \times n$ matrix A with $A_{1,1} = \gamma$ and all other entries zero, then by Lemma 2.4.3, $A = \overline{h}$ for some endomorphism $h \in \operatorname{End}_R M$. As $\operatorname{trace}(h \otimes Q) = \operatorname{trace}(\overline{h}) = \gamma \notin R$, h is stably nonzero by Lemma 2.1.3. Therefore by essentiality of the socle of $\underline{\operatorname{End}}_R M$, there exists $g \in \operatorname{End}_R M$ such that $[gh]$ generates $\operatorname{soc}(\underline{\operatorname{End}}_R M)$. By Lemma 2.1.3, there exists $h' \in \operatorname{End}_R M$ such that $\operatorname{trace}(ghh' \otimes_R Q) \notin R$, i.e. $\operatorname{trace}(h'gh \otimes_R Q) \notin R$. As $h'gh$ is stably nonzero by Lemma 2.1.3 once more, $[h'gh]$ generates $\operatorname{soc}(\underline{\operatorname{End}}_R M)$. Let $f = h'gh$. Now $\operatorname{trace}(f \otimes_R Q) = \operatorname{trace}(\overline{f}) = \overline{f}_{1,1} \in (\gamma\overline{R}) \setminus R$. Therefore $\overline{f}_{1,1} = u\gamma$ for some unit $u \in \overline{R}$. Finally, replacing $\overline{f}$ by $u^{-1}\overline{f}$, the result still sends M into M, by Lemma 2.4.3. □

For the remainder, assume R is a domain, and assume $k = R/\mathfrak{m}$ is algebraically closed.

Proposition 2.4.5. *If $f \in \operatorname{End}_R M$ and $g \in \operatorname{End}_R(\operatorname{syz}_R M)$ are given such that $[f] \in \operatorname{soc}(\operatorname{End}_R M)$ and $[g] = \operatorname{syz}_R([f])$, then* $\operatorname{trace}\overline{f} + \operatorname{trace}\overline{g} \in R$.

Proof. By Lemma 2.1.3, trace induces well-defined maps $\underline{\operatorname{End}}_R M \longrightarrow \overline{R}/R$ and $\underline{\operatorname{End}}_R(\operatorname{syz}_R M) \longrightarrow \overline{R}/R$. As syz_R gives an isomorphism of R-algebras $\underline{\operatorname{End}}_R M \longrightarrow \underline{\operatorname{End}}_R(\operatorname{syz}_R M)$, it restricts to an isomorphism on socles, which are R-simple due to k being algebraically closed. Because of these remarks, we can take our pick of f and g, as long as $[f] \neq 0$ and $[g] = \operatorname{syz}_R([f])$; we will choose f as in Proposition 2.4.4. Let $n = \operatorname{rank}(M)$, and $\nu > n$ be the minimal number of generators of M. Let $\xi_1, ..., \xi_\nu$ be a set of generators for M, such that $\{e_1 = \xi_1, ..., e_n = \xi_n\}$ is an $\overline{R}$-basis for $M\overline{R}$. For each ξ_j we have an equation $\xi_j = \sum_{i=1}^{n} w_{i,j} e_i$, for $w_{i,j} \in \overline{R}$. Since $\overline{R} = R + \mathcal{J}(\overline{R})$ (due to k being algebraically closed), we may assume (after possibly modifying some of $\xi_{n+1}, \ldots, \xi_\nu$) that for each $j > n$, and for each i, we have $w_{i,j} \in \mathcal{J}(\overline{R})$ and therefore $w_{i,j}\gamma \in R$.

Take a free cover $\pi : F \longrightarrow M$ sending i-th basis element to ξ_i. Since $f \in \operatorname{End}_R M$ is as in Proposition 2.4.4, there is a $\nu \times \nu$ matrix $A : F \longrightarrow F$ such that $\pi A = f\pi$, with the following properties:

- Columns 2 through n of A are zero;
- $A_{ij} = w_{1j}\overline{f}_{i1}$ for $(i,j) \in \{1,...,n\} \times \{n+1,...,\nu\}$; and
- $A_{ij} = 0$ for $(i,j) \in \{n+1,...,\nu\} \times \{n+1,...,\nu\}$.

Set $N = \ker(\pi)$, and let $\vec{r} = [r_1,...,r_\nu]^T \in N$, that is, $\sum_{j=1}^{\nu} r_j\xi_j = 0$. Recalling that $M\overline{R}$ is free, and projecting onto the basis element e_1, we get $r_1 + \sum_{j=n+1}^{\nu} r_j w_{1,j} = 0$. If we set $\overline{f}_{i1} = 0$ for $i > n$, then by definition of A we have that the i-th entry of $A\vec{r}$ is $A_{i1}r_1 + \sum_{j=n+1}^{\nu} w_{1j}\overline{f}_{i1}r_j = A_{i1}r_1 + \overline{f}_{i1}\sum_{j=n+1}^{\nu} r_j w_{1,j}$, and by the above equation this equals $(A_{i1} - \overline{f}_{i1})r_1$. In other words, $A\vec{r} = r_1\vec{v}$ where $\vec{v} = [v_1,...,v_\nu]^T \in F$ is given by $v_i = A_{i1} - \overline{f}_{i1}$. So if we let $g \in \operatorname{End}_R N$ be the restriction of A, we see that the image of g has rank 1 (i.e., $\operatorname{im} g \otimes Q \cong Q$). We also see that the $A^2\vec{r} = A(r_1\vec{v}) = r_1v_1\vec{v}$, so that v_1, which equals $A_{1,1} - \gamma$, is an eigenvalue for g. Our goal is to show that $\operatorname{trace}(\overline{g}) + \gamma \in R$. Since $\operatorname{trace}(\overline{g}) = \operatorname{trace}(g \otimes_R Q)$ and $\operatorname{im}(g \otimes_R Q) \cong Q$, the following lemma finishes the proof. □

Lemma 2.4.6. *If $\varphi : F \longrightarrow F$ is an endomorphism of a free module over a domain D, with $\operatorname{im}(\varphi) \cong D$, and λ is an eigenvalue for φ, then $\lambda = \operatorname{trace}(\varphi)$.*

Proof. Let $\vec{x} = [x_1,...,x_s]^T \in F$ generate the image of φ. It is easily checked that $\varphi(\vec{x}) = \lambda\vec{x}$. Let $y_1,...,y_s \in D$ such that $\varphi_{\cdot,j} = y_j\vec{x}$. Then $\lambda\vec{x} = \varphi(\vec{x}) = \sum_{j=1}^{s} x_j\varphi_{\cdot,j} = \sum_{j=1}^{s} x_jy_j\vec{x}$. So $\lambda = \sum_{j=1}^{s} x_jy_j = \sum_j \varphi_{j,j} = \operatorname{trace}(\varphi)$. □

Proposition 2.4.7. *We have* $\operatorname{rank}(\operatorname{syz}_R M) \cdot \operatorname{syz}([\gamma_M]) + \operatorname{rank}(M) \cdot [\gamma_{\operatorname{syz} M}] = 0$.

Proof. Since $\operatorname{soc}\underline{\operatorname{End}}_R(M)$ is R-simple, the map $\operatorname{soc}\underline{\operatorname{End}}_R(M) \longrightarrow \overline{R}/R$, induced by trace, is injective. We also know that $[\gamma_M] \in \operatorname{soc}\underline{\operatorname{End}}_R(M)$, by Theorem 2.2.14. Therefore $[\gamma_M] = \operatorname{rank}(M) \cdot [f]$ if we take $f \in \operatorname{End}_R M$ as in Lemma 2.4.4; and Proposition 2.4.5 implies $[\gamma_{\operatorname{syz} M}] = -\operatorname{rank}(\operatorname{syz}_R M) \cdot \operatorname{syz}([f])$. The result follows. □

Corollary 2.4.8. *If* $\operatorname{rank}(\operatorname{syz} M)$ *is a unit, then* $\operatorname{syz}([\gamma_M]) = -\dfrac{\operatorname{rank}(M)}{\operatorname{rank}(\operatorname{syz} M)}[\gamma_{\operatorname{syz} M}]$.

Chapter 3

Background on AR Quivers

In this chapter we collect results on stable AR quivers of Cohen-Macaulay rings. Throughout this chapter, $(R, \mathfrak{m})$ will be assumed to be a Cohen-Macaulay complete local ring. But in the first section there will be no mention of rings, as we provide definitions and the classical background on stable translation quivers more generally. Then, largely following [1], we provide criteria for confirming that stable AR components are infinite (Lemma 3.2.22), and that there are no loops (Lemma 3.2.18). In Section 3.3 we give criteria for confirming that a component is a tube. This chapter is in general more detailed than [1], and in Lemma 3.2.18 we give a corrected version of [1, Lemma 1.23], the proof of which contained an error.

3.1 Stable translation quivers

The following definitions generally agree with [1] and [6], although the meaning of "valued" is different in [6].

Definition 3.1.1. A *quiver* is a directed graph $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices and Γ_1 is the set of arrows. A *morphism* of quivers $\varphi : \Gamma \to \Gamma'$ is a pair $(\varphi_0 : \Gamma_0 \to \Gamma'_0, \varphi_1 : \Gamma_1 \to \Gamma'_1)$ such that φ_1 applied to an arrow $x \to y$ is an arrow $\varphi(x) \to \varphi(y)$. For $x \in \Gamma_0$, x^- denotes the set $\{y \in \Gamma_0 | \exists \text{arrow } y \to x \text{ in } \Gamma_1\}$; and $x^+ = \{y \in \Gamma_0 | \exists \text{arrow } x \to y \text{ in } \Gamma_1\}$. Γ is *locally finite* if x^+ and x^- are finite sets for each $x \in \Gamma_0$. A *loop* is an arrow from a vertex to itself. A *multiple arrow* is a set of at least two arrows from a given vertex to another given vertex.

A *valued quiver* is a quiver Γ together with a map $v\colon \Gamma_1 \to \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 1}$. By a *graph* we mean an undirected graph. A *valued graph* is a graph G together with specified integers $d_{xy} \geqslant 1$ and $d_{yx} \geqslant 1$ for each edge x—y .

Definition 3.1.2. A *stable translation quiver* is a locally finite quiver together with a quiver automorphism τ called the *translation*, such that:

- Γ has no loops and no multiple arrows.
- For $x \in \Gamma_0$, $x^- = \tau(x)^+$.

Given a stable translation quiver (Γ, τ) and a map $v : \Gamma_1 \to \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, the triple (Γ, v, τ) is called a *valued stable translation quiver* if $v(x \to y) = (a, b) \Leftrightarrow v(\tau(y) \to x) = (b, a)$.

A stable translation quiver is *connected* if it is non-empty and cannot be written as disjoint union of two subquivers each stable under the translation.

Definition 3.1.3. Let C be a full subquiver of a quiver Γ which satisfies Definition 3.1.2 except possibly for the no-loop condition. We call C a *component* of Γ if:

(1) For all vertices $x \in C$, we have $\tau x \in C$ and $\tau^{-1}x \in C$.

(2) C is a union of connected components of the underlying undirected graph of Γ.

(3) There is no proper subquiver of C that satisfies (1) and (2).

Definition 3.1.4. By a *directed tree* we shall mean a quiver T, with no loops or multiple arrows, such that the underlying undirected graph of T is a tree, and for each $x \in T$, the set x^- has at most one element.

Given a directed tree T, there is an associated stable translation quiver $\mathbb{Z}T$ defined as follows. The vertices of $\mathbb{Z}T$ are the pairs (n, x) with $n \in \mathbb{Z}$ and x a vertex of T. The arrows of $\mathbb{Z}T$ are determined by the following rules: Given vertices $x, y \in T$, and $n \in \mathbb{Z}$,

- $(n, x) \to (n, y) \in \mathbb{Z}T \Leftrightarrow x \to y \in T \Leftrightarrow (n, y) \to (n-1, x) \in \mathbb{Z}T$;
- If $n' \notin \{n, n-1\}$, there is no arrow $(n, x) \to (n', y)$.

Remark 3.1.5. Let T be a valued quiver which is also a directed tree. Then there is a unique extension of v to $\mathbb{Z}T$ such that the latter becomes a valued stable translation quiver. Namely, if $v(x \to y) = (a, b)$, then $v((n, x) \to (n, y)) = (a, b)$, and $v((n, y) \to (n-1, x)) = (b, a)$.

Lemma 3.1.6. *Let T and T' be (valued) directed trees. Then $\mathbb{Z}T \cong \mathbb{Z}T'$ as (valued) stable translation quivers if and only $T \cong T'$ as (valued) graphs.*

Proof. See [6, Proposition 4.15.3]. □

A group Π of automorphisms (commuting with the translation) of a stable translation quiver Γ is said to be *admissible* if no orbit of Π on the vertices of Γ intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. The quotient quiver Γ/Π, with vertices the Π-orbits of Γ_0, and with the induced arrows and translation, is also a stable translation quiver. A surjective morphism of stable translation quivers $\varphi : \Gamma \to \Gamma'$ is called a *covering* if, for each $x \in \Gamma_0$, the induced maps $x^- \to \varphi(x)^-$ and $x^+ \to \varphi(x)^+$ are bijective. Note that if Π is an admissible group of automorphisms of Γ,

$$\text{the canonical projection } \Gamma \to \Gamma/\Pi \text{ is a covering.} \tag{3.1.1}$$

Theorem 3.1.7. (Riedtmann Structure Theorem*; see* [6, Theorem 4.15.6]) *Given a connected stable translation quiver Γ, there is a directed tree T and an admissible group of automorphisms $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}T)$ such that $\Gamma \cong \mathbb{Z}T/\Pi$. In particular, we have a covering $\mathbb{Z}T \to \Gamma$. The underlying undirected graph of T is uniquely determined by Γ, up to isomorphism.*

The underlying undirected graph of T is called the *tree class* of Γ.

Remark 3.1.8. Formally, the tree class T of Γ is constructed as follows (as in the proof of Theorem 3.1.7, which we will not reproduce here). Choose any vertex $x \in \Gamma$, and define the vertices of T to be the set of paths

$$(x = y_0 \to y_1 \to \cdots \to y_n) \quad (n \geqslant 0)$$

for which no $y_i = \tau(y_{i+2})$. The arrows of T are

$$(x = y_0 \to y_1 \to \cdots \to y_{n-1}) \longrightarrow (x = y_0 \to y_1 \to \cdots \to y_n) \ .$$

Remark 3.1.9. Suppose Γ is a valued stable translation quiver, and let $\varphi : \mathbb{Z}T \to \Gamma$ be a covering, which exists by the Theorem. Now $\mathbb{Z}T$ becomes a valued stable translation quiver, by setting $v(x \to y) = v(\varphi(x \to y))$. In particular, T becomes a valued quiver.

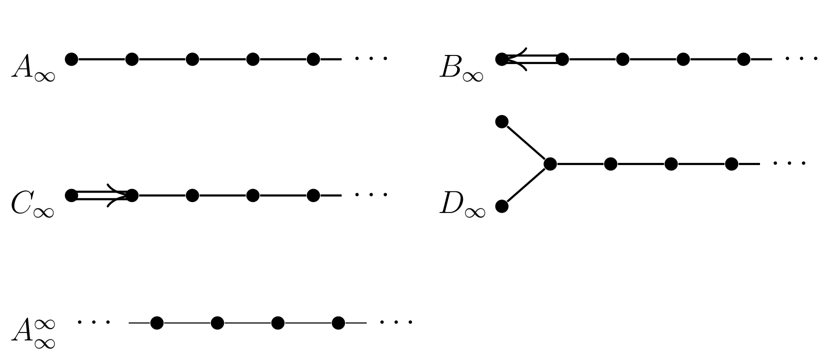
Definition 3.1.10. The *valued tree class* of a stable translation quiver Γ is a valued graph (T, v) where T denotes the tree class of Γ, and v: {edges of T} $\to \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is given as in Remark 3.1.9.

Definition 3.1.11. Let (Γ, v) be a valued, locally finite quiver without multiple arrows. For $x \to y$ in Γ, we write $v(x \to y) = (d_{xy}, d_{yx})$. If there is no arrow between x and y, we set $d_{xy} = d_{yx} = 0$. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers.

(i) A *subadditive function* on (Γ, v) is a $\mathbb{Q}_{>0}$-valued function f on the set of vertices of Γ such that $2f(x) \geqslant \sum_{y\in\Gamma} d_{yx} f(y)$, for each vertex x.

(ii) An *additive function* on (Γ, v) is a $\mathbb{Q}_{>0}$-valued function f on the set of vertices of Γ such that $2f(x) = \sum_{y\in\Gamma} d_{yx} f(y)$, for each vertex x.

Definition 3.1.12. The following valued graphs are called the infinite Dynkin diagrams:



In these pictures, the plain edges x—y indicate $d_{xy} = d_{yx} = 1$, and the edges $x \Rightarrow y$ indicate that $d_{xy} = 2$ and $d_{yx} = 1$.

Lemma 3.1.13. [6, Theorem 4.5.8] *Let (Γ, v) be a connected valued quiver without loops or multiple arrows. Suppose f is a subadditive function on Γ, and assume Γ has infinitely many vertices. Then:*

(1) The underlying valued graph of Γ is an infinite Dynkin diagram.

(2) If f is unbounded, or if f is not additive, then the underlying valued graph of Γ is A_∞.

3.2 The Cohen-Macaulay setting

For the remainder of this chapter, we assume $(R, \mathfrak{m})$ is a Cohen-Macaulay complete local ring. (But the same results hold when R is connected graded instead of complete local.)

Definition 3.2.1. If M and N are indecomposables in $\mathrm{CM}(R)$, let $\mathrm{Irr}(M, N)$ denote the module of nonisomorphisms $M \longrightarrow N$ modulo those which are not irreducible. Let k_M denote the division ring $(\mathrm{End}_R\, M)/\mathcal{J}(\mathrm{End}_R\, M)$. Thus $\mathrm{Irr}(M, N)$ is a right k_M-space, and a left k_N-space.

Definition 3.2.2. The Auslander-Reiten quiver of R is the valued quiver defined as follows:

- Vertices are isoclasses of indecomposables in $\mathrm{CM}(R)$.
- There is an arrow $M \to N$ if and only if there exists an irreducible morphism $M \to N$, i.e. $\mathrm{Irr}(M, N) \neq 0$. The value $v(M \to N)$ of the arrow $M \to N$ is (a, b) where a is the dimension of $\mathrm{Irr}(M, N)$ as a right k_M-space, and b is the dimension of $\mathrm{Irr}(M, N)$ as a left k_N-space.

Recall that we use τ to denote the AR-translate (defined at the end of Definition 1.1.2).

Lemma 3.2.3. *Let M and N be indecomposables in $L_p(R)$.*

(1) If $0 \to \tau N \to E \to N \to 0$ is an AR sequence, the number of copies of M appearing in a decomposition of E equals the dimension of $\mathrm{Irr}(M, N)$ as a right k_M-space.

(2) *If* $0 \to M \to E' \to \tau^{-1}M \to 0$ *is an AR sequence, then the number of copies of* N *appearing in a decomposition of* E' *equals the dimension of* $\operatorname{Irr}(M,N)$ *as a left* k_N*-space.*

Proof. See [27, Lemmas 5.5 and 5.6]. □

Remark 3.2.4. Suppose that $k = R/\mathfrak{m}$ is algebraically closed. Then in the notation of Lemma 3.2.3, we have $k = k_M = k_N$, and it therefore follows from Lemma 3.2.3 that the number of copies of N appearing in a decomposition of E' equals the number of copies of M appearing in a decomposition of E.

Lemma 3.2.5. *Let* M, N *be indecomposables in* $L_p(R)$*, and let* $0 \longrightarrow M \xrightarrow{f} X \xrightarrow{g} \tau^{-1}M \longrightarrow 0$ *and* $0 \longrightarrow \tau N \xrightarrow{h} Y \xrightarrow{k} N \longrightarrow 0$ *be AR sequences. Given* $\theta \in \operatorname{Hom}_R(M,N)$*, the following are equivalent:*

- θ *is irreducible;*
- *there exists a split epimorphism* $p \in \operatorname{Hom}_R(X,N)$ *such that* $\theta = pf$*;*
- *there exists a split monomorphism* $q \in \operatorname{Hom}_R(M,Y)$ *such that* $\theta = kq$*.*

Proof. See [27, Lemma 2.13]. □

Notationally, we allow τ to be a partially-defined morphism on the AR quiver of R; τx is defined precisely when the vertex x corresponds to a non-projective module in $L_p(R)$, by [27, Theorem 3.4]. The following fact is used in [1], and the proof essentially follows that of [4, VII 1.4].

Lemma 3.2.6. *Let* $x \to y$ *be an arrow in the AR quiver of* R*, and let* $(a,b) = v(x \to y)$*. If* τy *is defined, then* $v(\tau y \to x) = (b,a)$*. If* τx *and* τy *are both defined, then* $v(\tau x \to \tau y) = v(x \to y)$*.*

Proof. We need not prove the last sentence, as it follows from the previous. Let M and $N \in \mathrm{CM}(R)$ be the modules corresponding to x and y respectively. We first show k_N and

$k_{\tau N}$ are isomorphic k-algebras, where $k = R/\mathfrak{m}$. Let $0 \longrightarrow \tau N \xrightarrow{p} E \xrightarrow{q} N \longrightarrow 0$ be an AR sequence. Given $h \in \operatorname{End}_R N$, there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tau N \arrow[r, "p"] \arrow[d, "h'"'] & E \arrow[r, "q"] \arrow[d] & N \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \tau N \arrow[r, "p"] & E \arrow[r, "q"] & N \arrow[r] & 0 \end{tikzcd} . \tag{3.2.1}$$

Indeed, note that hq is not a split epimorphism, because if h is surjective, then h is an isomorphism, and thus hq is not a split epimorphism because q is not. Therefore, by Definition 1.1.2, there exists $u : E \longrightarrow E$ such that $hq = qu$, and the existence of h' follows.

By the dual argument, any given $h' \in \operatorname{End}_R(\tau N)$ can be fit into a commutative diagram of the same form.

We wish to show that $h \mapsto h'$ induces a well-defined map $k_N \to k_{\tau N}$. If so then it is a surjective ring map from a division ring, hence an isomorphism, so we will be done. It suffices to show that, given any commutative diagram 3.2.1 such that h is a nonisomorphism, it follows that h' is also a nonisomorphism. Suppose, to the contrary, that h is an nonisomorphism and h' is an isomorphism. We may assume h' is the identity map, since we could certainly compose the diagram 3.2.1 with a similar diagram which has $(h')^{-1}$ on the left. As h is not a split epimorphism, it factors through q. But then the top sequence in 3.2.1 splits, cf. [21, Ch. III, Lemma 3.3]; and this of course is a contradiction. Thus $k_N \cong k_{\tau N}$ as k-algebras.

In particular, $\dim_k(k_N) = \dim_k(k_{\tau N})$. As $\dim_{k_M} \operatorname{Irr}(M, N) = \dim_{k_M} \operatorname{Irr}(\tau N, M)$ is an immediate consequence of Lemma 3.2.3, our aim is to show $\dim_{k_N} \operatorname{Irr}(M, N) = \dim_{k_{\tau N}} \operatorname{Irr}(\tau N, M)$. By the former, we have $\dim_k \operatorname{Irr}(M, N) = \dim_k \operatorname{Irr}(\tau N, M)$. Thus, $\dim_{k_N} \operatorname{Irr}(M, N)$ $= \dim_k \operatorname{Irr}(M, N)/ \dim_k(k_N) = \dim_k \operatorname{Irr}(\tau N, M)/ \dim_k(k_{\tau N}) = \dim_{k_{\tau N}} \operatorname{Irr}(\tau N, M)$. □

Definition 3.2.7. If R is Gorenstein, the *stable Auslander-Reiten (AR) quiver* of R is the valued quiver defined as in Definition 3.2.2, except that the vertices are only the isoclasses of nonfree indecomposable modules $M \in L_p(R)$. By a *stable AR component*, we shall mean a component (Definition 3.1.3) of the stable AR quiver.

Definition 3.2.8. Let (Γ, τ) be a translation quiver, and x a vertex of Γ. If $x = \tau^n(x)$ for some $n > 0$, we say that x is *τ-periodic*. A module $M \in \mathrm{CM}(R)$ is said to be τ-periodic if it corresponds to a τ-periodic vertex in the AR quiver of R, i.e., $M \cong \tau^n M$. When R is Gorenstein of dimension one, we will omit the prefix "τ-" and just say M is periodic.

The following is well-known.

Lemma 3.2.9. *If (Γ, τ) is a connected translation quiver containing a τ-periodic vertex, then all of its vertices are τ-periodic.*

Proof. If x is a vertex in Γ and $\tau^n x = x$, then τ^n induces a permutation on the finite set x^-, and so some power of τ^n stabilizes x^- pointwise. Thus each vertex in x^- is τ-periodic; likewise for x^+, so every vertex in Γ is τ-periodic by induction. □

Definition 3.2.10. We say that a connected translation quiver is *periodic* if one, equivalently all, of its vertices is τ-periodic.

A so-called "tube" is a common example of a periodic translation quiver:

Definition 3.2.11. A valued stable translation quiver Γ is called a *tube* if $\Gamma \cong \mathbb{Z}A_\infty/\langle\tau^n\rangle$ for some $n \geqslant 1$. If $n = 1$, Γ is called a *homogeneous* tube.

Remark 3.2.12. Let Γ be a connected periodic stable translation quiver, and suppose the valued tree class of Γ is A_∞. Then Γ is a tube. To see this, let Π be an admissible group of automorphisms of $\mathbb{Z}A_\infty$ such that $\Gamma \cong \mathbb{Z}A_\infty/\Pi$. Note that every automorphism of the stable translation quiver $\mathbb{Z}A_\infty$ is of the form τ^n for some $n \geqslant 0$. Thus $\Pi = \langle\tau^n\rangle$ for some $n \geqslant 0$; and the periodicity implies $n \geqslant 1$.

Notation 3.2.13. If R is Gorenstein of dimension one, and M is an indecomposable in $L_p(R)$, define an R-module $\mathrm{push}(M)$ as follows. If M is free, let $\mathrm{push}(M) = 0$. Otherwise let $\mathrm{push}(M)$ denote the unique module (up to isomorphism) such that there exists an AR sequence $0 \longrightarrow M \longrightarrow \operatorname{push}(M) \longrightarrow \operatorname{syz}_R^{-1}(M) \longrightarrow 0$. More generally, if $M = \bigoplus_{i=1}^n M_i$ with each M_i in $L_p(R)$, then we set $\operatorname{push}(M) = \bigoplus_{i=1}^n \operatorname{push}(M_i)$.

Notation 3.2.14. (See, e.g., [23, 14.1-14.6].) For an R-module M, let $e(M)$ denote the multiplicity of M. This can be defined as $e(M) = \lim_{n\to\infty} \frac{d!}{n^d} \operatorname{length}(M/\mathfrak{m}^n M)$, where $d = \dim R$, but the reader may ignore this definition; we use only the following properties:

- If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact, then $e(M) = e(M') + e(M'')$.
- For all $M \in \mathrm{CM}(R)$, $e(M)$ is a positive integer.

Notation 3.2.15. Define a function e_{avg} from τ-periodic maximal Cohen-Macaulay R-modules to $\mathbb{Q}_{>0}$ as follows: If M is τ-periodic of period n, let $e_{\text{avg}}(M) = \frac{1}{n}\sum_{i=0}^{n-1} e(\tau^i(M))$.

Lemma 3.2.16. *Assume R is Gorenstein of dimension one, and $M \in L_p(R)$ is indecomposable and periodic. If $\operatorname{push} M = X \oplus F$ where X has no free direct summands and F is a (possibly zero) free module, then X is periodic, and $e_{\text{avg}}(\operatorname{push} M) \leqslant 2e_{\text{avg}}(M)$.*

Proof. We know X is periodic from Lemma 3.2.9. Note that if $N \in \mathrm{CM}(R)$ is periodic, then for any $j \in \mathbb{Z}$, and $n \in \mathbb{N}$ a multiple of the period of N, $\sum_{i=j}^{n+j-1} e(\tau^i N) = ne_{\text{avg}}(N)$. For each integer i, we have by Lemma 3.2.6 an AR sequence $0 \longrightarrow \tau^{i+1}M \longrightarrow F_i \oplus \tau^i X \longrightarrow \tau^i M \longrightarrow 0$, where F_i is a (possibly zero) free module. So $e(\tau^i X) \leqslant e(\tau^{i+1}M) + e(\tau^i M)$, hence $\sum_{i=1}^n e(\tau^i X) \leqslant \sum_{i=1}^n e(\tau^{i+1}M) + \sum_{i=1}^n e(\tau^i M)$ for each $n \in \mathbb{N}$. This inequality gives the desired result by taking n to be a common multiple of the periods of M and X, and dividing both sides by n. $\square$

The following goes back at least to [14] (in a slightly different setting).

Lemma 3.2.17. *Let C be a connected τ-periodic valued stable translation quiver which is a (not necessarily full) subquiver of the stable AR quiver of R. Then the valued tree class of C admits a subadditive function (Definition 3.1.11).*

Proof. Let T denote the valued tree class (Definition 3.1.10) of Γ. By definition of T, we have a value-preserving covering $\varphi\colon \mathbb{Z}T \to C$. Define a function $f\colon \mathbb{Z}T \to \mathbb{Q}_{>0}$ by the rule $f(x) = e_{\text{avg}}(\varphi(x))$. We claim that the restriction of f to T is a subadditive function. That is, $2f(x) \geqslant \sum_{y\in T} d_{yx}f(y)$, for each vertex x of T. By Lemma 3.2.6, $d_{yx} = d_{(\tau^{-1}y)x}$ for all $x, y \in C$, hence for all $x, y \in \mathbb{Z}T$. In what follows, for any $x \in T$, the sets x^- and x^+ will always be taken with respect to $\mathbb{Z}T$; to signify the predecessors of x with respect to T we can use $x^- \cap T$. If $x \in T$, then x^+ equals the disjoint union of $x^+ \cap T$ and $\tau^{-1}(x^- \cap T)$. Now, we have

$$\begin{aligned}\sum_{y\in T} d_{yx}f(y) &= \sum_{y\in x^-\cap T} d_{yx}f(y) + \sum_{y\in x^+\cap T} d_{yx}f(y)\\ &= \sum_{y\in x^-\cap T} d_{\tau^{-1}yx}f(\tau^{-1}y) + \sum_{y\in x^+\cap T} d_{yx}f(y) = \sum_{y\in x^+} d_{yx}f(y).\end{aligned}$$

So subadditivity of f is equivalent to $2f(x) \geqslant \sum_{y\in x^+} d_{yx}f(y)$. Since φ is a covering, $\sum_{y\in x^+} d_{yx}f(y) = \sum_{y\in\varphi(x)^+} d_{y\varphi(x)}e_{\text{avg}}(y)$, which is bounded by $2e_{\text{avg}}(\varphi(x))$ by Lemma 3.2.16. So f is subadditive. $\square$

Lemma 3.2.18. *Assume R is Gorenstein, let $M \in L_p(R)$ be a nonfree indecomposable, and suppose there exists an irreducible map from M to itself. Let C denote the component of the stable AR quiver containing M, and assume C is infinite. Then C is a homogeneous tube with a loop at the end:*

$$\begin{tikzcd} M = X_0 \arrow[loop left] \arrow[r, shift left] & X_1 \arrow[l, shift left] \arrow[r, shift left] & X_2 \arrow[l, shift left] \arrow[r, shift left] & X_3 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \end{tikzcd}$$

In particular, $\tau X_i \cong X_i$ for all $X_i \in C$.

Proof. First we show that $M \cong \tau M$. If not, then the AR sequence ending in M is $0 \longrightarrow \tau M \longrightarrow M \oplus \tau M \oplus N \longrightarrow M \longrightarrow 0$ for some $N \in \mathrm{CM}(R)$. Then $e(N) = 0$, hence $N = 0$. Now Miyata's Theorem [24, Theorem 1] says that the given AR sequence splits, which is a contradiction. So $\tau M \cong M$.

Since C has a loop, it does not satisfy the definition of stable translation quiver (Definition 3.1.2). But removing the loops in C (and keeping all vertices and all non-loop arrows), we get a τ-periodic connected stable translation quiver; call it Γ, and let T denote valued tree class of Γ. Now T admits a subadditive function given by e_{avg}, as in the proof of 3.2.17. From the fact that Γ is not a full subquiver of the AR quiver of R, it follows that e_{avg} is strictly subadditive (i.e., not additive). As Γ is infinite and τ-periodic, T must be infinite. Therefore $T \cong A_\infty$ by Lemma 3.1.13, and $\Gamma \cong A_\infty/\langle\tau\rangle$, by Remark 3.2.12. So Γ has the form

$$X_0 \rightleftarrows X_1 \rightleftarrows X_2 \rightleftarrows X_3 \rightleftarrows \cdots .$$

Suppose $M = X_i$ for some $i > 0$. Then we have an AR sequence $0 \longrightarrow X_i \longrightarrow X_i \oplus X_{i-1} \oplus X_{i+1} \oplus F \longrightarrow X_i \longrightarrow 0$, for some free module F. Then $e(X_i) \geqslant e(X_{i-1}) + e(X_{i+1})$. But consideration of the AR sequences ending in X_{i-1} and X_{i+1} yields $2e(X_{i+1}) > e(X_i)$ and $2e(X_{i-1}) \geqslant e(X_i)$. These inequalities contradict the previous one, so $M = X_0$. $\square$

The following "Maranda-type result" corresponds to Lemma 1.24 in [1]. In our setting, namely that of a Cohen-Macaulay complete local ring, this result is well-known (but possibly has only been stated for the case when the ring is an isolated singularity). The following proof can be found, for example, in [20, Proposition 15.8] and its corollaries.

Lemma 3.2.19. *Let M and N be nonisomorphic indecomposables in $L_p(R)$, and let $x \in \mathfrak{m}$ be a nonzerodivisor. Then there exists $i \geqslant 1$ such that M/x^iM and N/x^iN are nonisomorphic indecomposable modules.*

Proof. Since M lies in $L_p(R)$, $\mathrm{Ext}^1_R(M,N)$ has finite length (since for any nonmaximal prime $\mathfrak{p}$, we have $0 = \mathrm{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{Ext}^1_R(M,N)_\mathfrak{p}$). Therefore we may assume, after replacing x by a suitable power of itself, that $x\,\mathrm{Ext}^1_R(M,N) = 0$. By applying $\mathrm{Hom}_R(M, \_)$ to the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "x^2"] \arrow[d, "x"'] & N \arrow[r] \arrow[d, equal] & N/x^2N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "x"] & N \arrow[r] & N/xN \arrow[r] & 0
\end{tikzcd},$$

we obtain a commutative exact diagram

$$\begin{tikzcd} \operatorname{Hom}_R(M,N) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(M,N/x^2M) \arrow[r] \arrow[d] & \operatorname{Ext}^1_R(M,N) \arrow[d, "x"] \\ \operatorname{Hom}_R(M,N) \arrow[r] & \operatorname{Hom}_R(M,N/xM) \arrow[r] & \operatorname{Ext}^1_R(M,N) \end{tikzcd} . \tag{3.2.2}$$

Consider the maps $\theta\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M/xM,N/xN)$ and $\theta_2\colon \operatorname{Hom}_R(M/x^2M,N/x^2N) \to \operatorname{Hom}_R(M/xM,N/xN)$ given by tensoring all maps with $R/(x)$. Notice that in diagram 3.2.2, the horizontal and vertical maps into $\operatorname{Hom}_R(M,N/xM)$ can be identified with θ and θ_2 respectively, while the rightmost vertical map is zero. Therefore a diagram chase yields

$$\operatorname{im}(\theta) = \operatorname{im}(\theta_2). \tag{3.2.3}$$

We claim $i = 2$ will suffice. Suppose M/x^2M is not indecomposable. Then there exists a nontrivial idempotent $e \in \operatorname{End}_R(M/x^2M)$. Consider the equation 3.2.3 in the case $M = N$; now θ and θ_2 are of course ring homomorphisms. Since $\operatorname{End}_R M$ is (noncommutative-) local, so is $\operatorname{im}\theta$, and therefore $\theta_2(e)$ is either 0 or 1. Since $1 - e \in \operatorname{End}_R(M/x^2M)$ is also a nontrivial idempotent, we may assume $\theta_2(e) = 0$, i.e. $\operatorname{im} e \subseteq xM/x^2M$. But then $e^2 = 0$ is a contradiction.

Now suppose $\varphi\colon M/x^2M \longrightarrow N/x^2N$ is an isomorphism, with inverse $\psi\colon N/x^2N \longrightarrow M/x^2M$. By 3.2.3, there exist $\widetilde{\varphi}\colon M \longrightarrow N$ such that $\widetilde{\varphi} \otimes_R (R/x) = \varphi \otimes_R (R/x)$, and $\widetilde{\psi}\colon M \longrightarrow N$ such that $\widetilde{\psi} \otimes_R (R/x) = \psi \otimes_R (R/x)$. By Nakayama's Lemma, $\widetilde{\varphi}$ and $\widetilde{\psi}$ are surjective. Thus $\widetilde{\psi}\widetilde{\varphi}$ is a surjective endomorphism, equivalently, an isomorphism; and thus $\widetilde{\varphi}$ is an isomorphism. □

Lemma 3.2.20. *Assume* $\dim R = 1$*, and let* M *be an arbitrary indecomposable in* $\operatorname{CM}(R)$*. Then there exists an irreducible morphism* $M \longrightarrow R$ *if and only if* M *is isomorphic to a direct summand of* $\mathfrak{m}$*. If* R *is Gorenstein, then there exists an irreducible morphism* $R \longrightarrow M$ *if and only if* M *is isomorphic to a direct summand of* $\mathfrak{m}^*$.

Proof. Write $\mathfrak{m}$ as a direct sum of indecomposables, $\mathfrak{m} = \bigoplus_i \mathfrak{m}_i$. Let ι_i denote the inclusion map $\mathfrak{m}_i \hookrightarrow R$. To see that ι_i is irreducible, take a factorization $\iota_i = hg$ in $\mathrm{CM}(R)$, where h is not a split epimorphism. Then h is not onto, so $\operatorname{im} h \subseteq \mathfrak{m}$. Then if h' denotes the map into $\mathfrak{m}$ given by $x \mapsto h(x)$, and p_i denotes the projection $\mathfrak{m} \twoheadrightarrow \mathfrak{m}_i$, we have that $p_i h' g = 1_{\mathfrak{m}_i}$, so g is a split monomorphism; hence ι_i is irreducible. Now let M be an indecomposable in $\mathrm{CM}(R)$ and let $f : M \longrightarrow R$ be an irreducible morphism. Let ι denote the inclusion map $\mathfrak{m} \hookrightarrow R$. Since f is not a split epimorphism, $\operatorname{im} f \subseteq \mathfrak{m}$, hence $f = \iota g$ for some $g : M \longrightarrow \mathfrak{m}$. As ι is certainly not a split epimorphism, g is a split monomorphism.

For the last sentence of the statement, note that the irreducible maps from R are obtained by dualizing the irreducible maps into R. □

We recall the Harada-Sai Lemma:

Lemma 3.2.21. [4, VI. Cor. 1.3] *Let Λ be an artin algebra (e.g. a commutative artinian ring). If $f_i \colon M_i \to M_{i+1}$ are nonisomorphisms between indecomposable modules M_i for $i = 1, ..., 2^n - 1$ and $\mathrm{length}(M_i) \leqslant n$ for all i, then $f_{2^n-1} \cdots f_1 = 0$.*

Lemma 3.2.22. [1, Proposition 1.26] *Assume R is Gorenstein of dimension one, and $\mathfrak{m}$ is indecomposable; and suppose R has a stable AR component C which is finite. Then C consists of all isoclasses of non-projective indecomposables in $\mathrm{CM}(R)$.*

Proof. As C is finite, Lemma 3.2.19 implies that we can take $x \in \mathfrak{m}$ such that for each pair $M \ncong N$ in C, M/xM and N/xN are nonisomorphic indecomposable modules.

We may assume R is not regular, and therefore $\mathfrak{m}$ is not free. Now first we show $\mathfrak{m} \in C$. Suppose not; then there are no irreducible maps to R from any module in C (Lemma 3.2.20). Therefore if $N \in C$ and $N \to N'$ is any irreducible map in $\mathrm{CM}(R)$, N' must lie in C (since $L_p(R)$ is closed under syz_R^{-1}, and therefore under irreducible maps by consideration of AR sequences). Pick a module $M \in C$. By replacing x by a power of itself if necessary, we can choose $f \colon M \to R$ such that $f(M) \nsubseteq xR$, i.e. $f \otimes_R (R/x) \neq 0$. Since f is not a split monomorphism, and there exists an AR sequence beginning in M, f equals a sum of maps of the form gh, where h is an irreducible map between modules in C. Since $g \in \mathrm{Hom}_R(N,R)$ for some $N \in C$, g is not a split monomorphism, and can in turn be written as a sum of maps of the form kl where l is an irreducible map in C; now $f = \sum klh$. Continue this process until we have written f as a sum $\sum_i g_i h_{2^{n-1},i} \cdots h_{1,i}$ where each $h_{j,i}$ is an irreducible map in C, and $n = \max\{\mathrm{length}(N/xN) | N \in C\}$. Note that each $h_{j,i} \otimes_R (R/x)$ is a nonisomorphism by our assumption on x together with Lemma 3.2.18. Therefore, Lemma 3.2.21 implies $f \otimes_R (R/x) = 0$, contradiction. Thus $\mathfrak{m} \in C$.

Now just suppose C omits some indecomposable nonfree $M \in \mathrm{CM}(R)$. Again choose $f \colon M \to R$ such that $f \otimes_R (R/x) \neq 0$. Note that any map to R which is not a split epimorphism factors through $\mathfrak{m}$. Whereas in the previous paragraph we reached a contradiction via Lemma 3.2.21, by "stacking irreducible maps while moving forwards through C", we now obtain a contradiction by "stacking irreducible maps while moving backwards through $C \cup \{R\}$". □

Remark 3.2.23. Assume R is Gorenstein and let C be a stable AR component without loops. Then C is a valued stable translation quiver (by Lemma 3.2.6) and therefore has a valued tree class T (Definition 3.1.10). Then T carries the information of how many nonfree direct summands $\mathrm{push}(M)$ and $\mathrm{push}(\mathrm{push}(M))$ (in general, $\mathrm{push}^i(M)$) have for modules $M \in C$. Let us explain further. Let x be the vertex in C corresponding to M, and let $n = \sum_{(x \to y) \in C} d_{yx}$. Then n is the number of nonfree summands in $\mathrm{push}(M)$; that is, $\mathrm{push}(M) = F \oplus \bigoplus_{i=1}^n X_i$ where F is a (possibly zero) free module, and the X_i are (not necessarily nonisomorphic) nonfree indecomposables in $L_p(R)$. We have a value-preserving covering $\varphi \colon \mathbb{Z}T \to C$, and after possibly composing φ with a power of τ, we have $x \in \varphi(T)$, say $x = \varphi(u)$. Since $\varphi \colon \mathbb{Z}T \to C$ is a covering, $\sum_{(x \to y) \in C} d_{yx} = \sum_{(u \to w) \in \mathbb{Z}T} d_{wu}$, and by definition of $\mathbb{Z}T$ this equals $\sum_{w \in T} d_{wu}$. Thus $n = \sum_{w \in T} d_{wu}$. Likewise, $\sum_{w,z \in T} d_{zw} d_{wu}$ is the number of nonfree direct summands in $\mathrm{push}(\mathrm{push}(M))$.

Proposition 3.2.24. *(cf.* [1, Lemma 1.23 and Theorem 1.27]*) Assume that R is Goren-*

stein of dimension one, $\mathfrak{m}$ *is indecomposable, and* $\mathrm{CM}(R)$ *has infinitely many isoclasses of indecomposables. Let* C *be a periodic component of the stable AR quiver of* R*, and suppose that either* R *is a reduced hypersurface and* C *has no loops, or that there exists some* $M \in C$ *such that* $\mathrm{push}(\mathrm{push}(M)) = X \oplus Y \oplus F$ *for some indecomposables* X *and* Y*, and some possibly-zero free module* F*. Then,* C *is a tube.*

Proof. If C has a loop, then by Lemma 3.2.18, for every $M \in C$, the module $\mathrm{push}\, M$ has two nonfree indecomposable summands, and therefore $\mathrm{push}(\mathrm{push}(M))$ has four. So we may assume C has no loops. Thus C is a valued stable translation quiver, and we have a valued directed tree T and a value-preserving covering $\varphi\colon \mathbb{Z}T \to C$. Let the function $f\colon \mathbb{Z}T \to \mathbb{Q}_{>0}$ be given by $f(x) = e_{\mathrm{avg}}(\varphi(x))$. As seen in Lemma 3.2.17, f restricts to a subadditive function on T. Since φ is surjective, every vertex of C lies in the τ-orbit of a vertex in $\varphi(T)$. Note also that C has infinitely many vertices, by Lemma 3.2.22. Therefore T is infinite, so it is an infinite Dynkin diagram by Lemma 3.1.13. If R is a reduced then $\{e(M) | M \in C\}$ is unbounded (see [27, Theorem 6.2]); and so if R is a reduced hypersurface (and thus all modules in C have period 2) then f is unbounded. Then $T \cong A_\infty$, by Lemma 3.1.13. If the alternate condition holds, we get $T \cong A_\infty$ by eliminating the other infinite Dynkin diagrams (which are pictured in Definition 3.1.12), in light of Remark 3.2.23. Thus C is a tube, by Remark 3.2.12. $\square$

Chapter 4

AR Quivers over C.I. rings of dimension one

In this chapter we assume that $(R, \mathfrak{m})$ is a complete (or graded-) local complete intersection ring of dimension one, and let $k = R/\mathfrak{m}$. (Recall that complete intersection implies Gorenstein.) The unadorned symbols syz and syz^{-1} will stand for syz_R and syz_R^{-1}, respectively.

4.1 AR quivers and syz- and cosyz-perfect modules

In this section, we adapt results from Green-Zacharia [12] and Kerner-Zacharia [19]. In particular, we will see that the tree class of any stable AR component must be Dynkin or Euclidean, and further prune down this list of possibilities when the modules in a given component are "eventually cosyz-perfect" (Definition 4.1.11).

Definition 4.1.1. Let $\cdots F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a minimal free resolution of a finitely generated R-module M. Then the i-th *Betti number* $\beta_i(M)$ denotes the rank of F_i. We say that *the complexity of M is at most n*, and write $\mathsf{cx}\, M \leqslant n$, if there exists $b \in \mathbb{Q}_{>0}$ such that $\beta_i(M) \leqslant b i^{n-1}$ for all $i \gg 0$. We say that *the complexity of M is n*, and write $\mathsf{cx}\, M = n$, if $\mathsf{cx}\, M \leqslant n$ and $\mathsf{cx}\, M \nleqslant n-1$.

Remark 4.1.2. Several of the lemmas in this section do not require that R be a complete intersection, but we use it to prove the main statements. The properties of complete intersections which we use are as follows: If M is a finitely generated R-module, then

(a) $\beta_n(M) \leqslant \beta_{n+2}(M)$ for all $n \gg 0$ (see [9, 3.1]);

(b) $\{\beta_n(M)\}_{n\geqslant 0}$ is unbounded, provided M is not eventually periodic ([9, 4.1]);

(c) $\mathsf{cx}\, M < \infty$ ([5, Theorem 8.1.2]).

Notation 4.1.3. (1) Let M be an indecomposable in $L_p(R)$, and consider the AR sequence ending in M, $0 \longrightarrow \operatorname{syz} M \longrightarrow \bigoplus_{i=1}^n X_i \oplus F \longrightarrow M \longrightarrow 0$ where F is free, and each X_i is indecomposable and nonfree. Then we define $\alpha(M)$ to be n.

(2) If C is a component of the stable AR quiver of R, define

$$\alpha(C) = \sup\{\alpha(M) | M \in C\}.$$

Notation 4.1.4. Let $M, N \in \mathrm{CM}(R)$, and $f \in \operatorname{Hom}_R(M, N)$. By extending f to a map between the minimal free resolutions of M and N, we get induced maps $\operatorname{syz}^n f \colon \operatorname{syz}^n M \longrightarrow \operatorname{syz}^n N$. These are not uniquely determined, but in the stable category they are; i.e., $[\operatorname{syz}^n f] \in \underline{\operatorname{Hom}}_R(\operatorname{syz}^n M, \operatorname{syz}^n N)$ is well-defined.

Definition 4.1.5. We say that a module M is *stable* if M has no free direct summands.

Lemma 4.1.6. [25, Proposition 2.8] *Let M and N be R-modules, and assume M is stable. Let $\Lambda = \operatorname{End}_R M$.*

(a) If $f \colon M \longrightarrow N$ is stably zero, then $f(M) \subseteq \mathfrak{m}N$.

(b) If $g \in \operatorname{End}_R M$ satisfies $g(M) \subseteq \mathfrak{m}M$, then $g \in \operatorname{rad} \Lambda$.

Proof. (a) Since M is stable, any homomorphism $h \colon M \longrightarrow F$ must satisfy $h(M) \subseteq \mathfrak{m}F$ if F is free. Part (a) follows.

(b) Given such g, together with any $h \in \operatorname{End}_R(M)$, we have that $(\mathrm{id}_M - hg) \otimes_R k = \mathrm{id}_{M \otimes k}$, and therefore $\mathrm{id}_M - hg$ is surjective by Nakayama's Lemma, and therefore it is in fact an isomorphism. This proves (b). □

The following proof is essentially from [25, Theorem 3.1], but we give a simpler and more direct version.

Lemma 4.1.7. *Let M and N be stable modules in* $\mathrm{CM}(R)$*, and let $f \in \mathrm{Hom}_R(M, N)$. If f is irreducible, then so is any choice of* $\operatorname{syz} f$*.*

Proof. It suffices to show that a map $g \in \mathrm{Hom}_R(M, N)$ is split mono (resp. epi) if and only if every choice of $\operatorname{syz} g$ is split mono (resp. epi). By Gorenstein duality it is therefore enough to show that g being split mono implies every choice of $\operatorname{syz} g$ is split mono. Take $p\colon N \longrightarrow M$ such that $pg = \mathrm{id}_M$. Let g' and p' be choices for $\operatorname{syz} g$ and $\operatorname{syz} p$, respectively. Then, as $\mathrm{id}_{\operatorname{syz} M}$ and $p'g'$ are valid choices of $\operatorname{syz} \mathrm{id}_M$, we have that $\mathrm{id}_{\operatorname{syz} M} - p'g'$ is stably zero, and therefore lies in $\operatorname{rad} \mathrm{End}_R(\operatorname{syz} M)$, by Lemma 4.1.6. Therefore $p'g'$ is an isomorphism, and g' is split mono. □

Lemma 4.1.8. *If $f\colon M \longrightarrow N$ is an irreducible map in* $\mathrm{CM}(R)$*, then f must be either a monomorphism or an epimorphism.*

Proof. Since $\dim R = 1$, a submodule of a Cohen-Macaulay R-module is again Cohen-Macaulay. If $f\colon M \longrightarrow N$ is neither a monomorphism nor an epimorphism, then the factorization $M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$ shows that f is not irreducible. □

Lemma 4.1.9. *If $0 \longrightarrow X \xrightarrow{[f_1, f_2]^T} Y_1 \oplus Y_2 \xrightarrow{[g_1, g_2]} Z \longrightarrow 0$ is any short exact sequence of abelian groups, then f_1 is an epimorphism if and only if g_2 is an epimorphism. If it is an AR sequence in* $\mathrm{CM}(R)$*, then f_1 is a monomorphism if and only if g_2 is a monomorphism.*

Proof. The first statement is straightforward, and the second statement then follows from Lemma 4.1.8. □

The following is an important notion in [12] and [19], where it is called Ω-perfect.

Definition 4.1.10. Given $M, N \in L_p(R)$, an irreducible map $f\colon M \longrightarrow N$ is said to be *syz-perfect* if M and N are stable and $\operatorname{syz}^n f$, for $n \geqslant 0$, are either all monomorphisms or all epimorphisms. If M is a nonfree indecomposable in $L_p(R)$, then M is called syz-perfect if every irreducible map $X \longrightarrow M$ and every irreducible map $\operatorname{syz} M \longrightarrow Y$ are syz-perfect. M is called *eventually* syz-perfect if $\operatorname{syz}^n M$ is syz-perfect, for some $n \geqslant 0$.

We will also use the dual notion:

Definition 4.1.11. Given $M, N \in L_p(R)$, an irreducible map $f\colon M \longrightarrow N$ is *cosyz-perfect* if M and N are stable and $\operatorname{syz}^n f$, for $n \leqslant 0$, are either all monomorphisms or all epimorphisms. If M is a nonfree indecomposable in $L_p(R)$, then say that M is cosyz-perfect if every irreducible map $M \longrightarrow X$ and every irreducible map $Y \longrightarrow \operatorname{syz}^{-1} M$ are cosyz-perfect. Call M eventually cosyz-perfect if $\operatorname{syz}^n M$ is cosyz-perfect, for some $n \leqslant 0$.

Our arguments in this section are essentially those given in [12] and [19]; the adjustments are relatively minor, Lemma 4.1.14 being an exception. The notion of cosyz-perfect seems better suited to proving results about possible shapes of AR components; see Theorem 4.1.30, which we do not have a proof for if we assume syz-perfect instead. (Although most of our arguments remain valid when dualized, Lemma 4.1.29 does not.) In Section 4.2 we use the notion of syz-perfect as well.

Our first goal here is to prove Proposition 4.1.12. We will later address the case when all modules in a given stable AR component are eventually cosyz-perfect.

Proposition 4.1.12. *(cf.* [19, Theorem 2.11]*) Assume $M \in L_p(R)$ is a nonfree, nonperiodic indecomposable which either fails to be eventually* syz-perfect*, or fails to be eventually* cosyz-perfect*. Then the stable AR component containing M admits an additive function, and R has some ideal which is a periodic module.*

Lemma 4.1.13. *Let* $0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0$ *be a short exact sequence in* $\mathrm{CM}(R)$*. Then* $\operatorname{syz} g$ *is an epimorphism if and only if* $\dim_k(Y/\mathfrak{m}Y) = \dim_k(X/\mathfrak{m}X) + \dim_k(Z/\mathfrak{m}Z)$*.*

Proof. By taking free modules F and H of ranks $\dim_k(X/\mathfrak{m}X)$ and $\dim_k(Z/\mathfrak{m}Z)$ respectively, the Horseshoe Lemma gives a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{syz} X \arrow[r, "f'"] \arrow[d] & P \oplus \operatorname{syz} Y \arrow[r, "g'"] \arrow[d] & \operatorname{syz} Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F \oplus H \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.1.1}$$

for some (possibly zero) free module P. Note, furthermore, that the map $g' \colon P \oplus \operatorname{syz} Y \longrightarrow \operatorname{syz} Z$ has the form $g' = [l \;\; \operatorname{syz} g]$ for some $l \in \operatorname{Hom}_R(P, \operatorname{syz} Z)$. Write $f' = [f'_1 \;\; f'_2]^T$ for some $f'_1 \colon \operatorname{syz} X \longrightarrow P$ and $f'_2 \colon \operatorname{syz} X \longrightarrow \operatorname{syz} Y$. Now suppose $\operatorname{syz} g$ is an epimorphism. Then by Lemma 4.1.9 f'_1 is also an epimorphism, so that P is isomorphic to a summand of $\operatorname{syz} X$, and therefore $P = 0$, which implies $\dim_k(Y/\mathfrak{m}Y) = \dim_k(X/\mathfrak{m}X) + \dim_k(Z/\mathfrak{m}Z)$. Conversely, if $\dim_k(Y/\mathfrak{m}Y) = \dim_k(X/\mathfrak{m}X) + \dim_k(Z/\mathfrak{m}Z)$ then $P = 0$ and $\operatorname{syz} g = g'$ is onto. $\square$

Lemma 4.1.14. *(cf.* [12, Lemma 2.1]*) Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in $\mathrm{CM}(R)$ *with* g *irreducible, and suppose that* $\operatorname{syz} g$ *is a monomorphism. Then* X *is isomorphic to an ideal of* R.

Proof. It is part of the general (Auslander-Reiten) folklore that if $f' \colon X \longrightarrow Y'$ is any map in $\mathrm{CM}(R)$, then either f factors through f' or f' factors through f. To see this, assume that f' does not factor through f. This says that the pushout of $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ by f' does not split. Therefore, the irreducibility of g implies that the middle map in the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "f'"'] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & W \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

is a split monomorphism, and it follows that f factors through f'.

Now assume that X is not isomorphic to an ideal. Then if $f' \in \operatorname{Hom}_R(X, \mathfrak{m}^*)$, f' cannot be injective (since $\mathfrak{m}^*$ may be viewed as a finitely generated submodule of $Q(R)$, and therefore embeds into R), and therefore f does not factor through f'. So any $f'\colon X \longrightarrow \mathfrak{m}^*$ factors through f, by the above. In other words, $\operatorname{Hom}_R(f, \mathfrak{m}^*)\colon \operatorname{Hom}_R(Y, \mathfrak{m}^*) \longrightarrow \operatorname{Hom}_R(X, \mathfrak{m}^*)$ is surjective. Using the surjection $\mathfrak{m}^* \longrightarrow k$ from the exact sequence $0 \longrightarrow R \longrightarrow \mathfrak{m}^* \longrightarrow k \longrightarrow 0$ (Lemma 1.1.1), we have a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_R(Y, \mathfrak{m}^*) \arrow[r] \arrow[d] & \operatorname{Hom}_R(Y, k) \arrow[d] \\
\operatorname{Hom}_R(X, \mathfrak{m}^*) \arrow[r] & \operatorname{Hom}_R(X, k)
\end{tikzcd} . \tag{4.1.2}$$

Note that the horizontal maps in 4.1.2 are surjective since $\operatorname{Ext}^1_R(Y, R) = \operatorname{Ext}^1_R(X, R) = 0$. Therefore, the right-hand vertical map is surjective since the left-hand map is. Therefore $\dim_k(\operatorname{Hom}_R(Y, k)) = \dim_k(\operatorname{Hom}_R(X, k)) + \dim_k(\operatorname{Hom}_R(Z, k))$. But for any R-module M, we have $\operatorname{Hom}_R(M, k) = \operatorname{Hom}_k(M/\mathfrak{m}M, k)$ and $\dim_k(M/\mathfrak{m}M, k) = \dim_k(M/\mathfrak{m}M)$. Now Lemma 4.1.13 finishes the proof. □

Recall that multiplicity, $e(\_)$, is additive along short exact sequences.

Lemma 4.1.15. *For $M \in \mathrm{CM}(R)$, we have $\mu(M) \leqslant e(M) \leqslant \mu(M)e(R)$, where $\mu(\_)$ denotes minimal number of generators.*

Proof. The inequality $\mu(M) \leqslant e(M)$ is well-known. For $e(M) \leqslant \mu(M)e(R)$, note that the short exact sequence $0 \longrightarrow \operatorname{syz} M \longrightarrow R^{(\mu(M))} \longrightarrow M \longrightarrow 0$ implies $e(M) = \mu(M)e(R) - e(\operatorname{syz} M)$. □

Lemma 4.1.16. *(cf.* [20, 15.25]*) For any irreducible map $X \longrightarrow Y$ between indecomposable modules, we have that $e(X)e(R) \geqslant e(Y)$ and $e(Y)e(R) \geqslant e(X)$.*

Proof. By consideration of the AR sequence ending in Y, we have $e(X) \leqslant e(Y) + e(\operatorname{syz} Y)$, which in turn equals $\mu(Y)e(R) \leqslant e(Y)e(R)$. To get the other direction, we may dualize

$X \longrightarrow Y$ to get an irreducible map $Y^* \longrightarrow X^*$, and use that $e(X) = e(X^*)$ and $e(Y) = e(Y^*)$. □

Lemma 4.1.17. *(cf. [12, Proposition 2.4]) Let* $0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0$ *be a short exact sequence in* $\mathrm{CM}(R)$*, such that g is irreducible.*

(a)If g is not eventually cosyz*-perfect, there exists $n \leqslant 0$ such that* $\ker(\operatorname{syz}^n g)$ *is isomorphic to a periodic ideal.*

(b)If g is not eventually syz*-perfect, there exists $n \geqslant 0$ such that* $\ker(\operatorname{syz}^n g)$ *is isomorphic to a periodic ideal.*

Proof. (a) For each $i \leqslant 0$ we can apply a dualized Horseshoe Lemma to obtain a short exact sequence $0 \longrightarrow \operatorname{syz}^i X \longrightarrow P^i \oplus \operatorname{syz}^i Y \xrightarrow{[l, \operatorname{syz}^i g]} \operatorname{syz}^i Z \longrightarrow 0$ for some free module P^i, and $l \in \operatorname{Hom}_R(P^i, \operatorname{syz}^i Z)$. When $\operatorname{syz}^i g$ is surjective, we see as in the proof of Lemma 4.1.13 that P^i must be zero. Now since g is not eventually cosyz-perfect, Lemma 4.1.14 shows that there exist infinitely many negative values of i such that $\operatorname{syz}^i g$ is surjective and $\ker(\operatorname{syz}^i g) \cong \operatorname{syz}^i X$ is isomorphic to an ideal. For each such i, we have $\beta_i(X) \leqslant e(R)$ by Lemma 4.1.15, and therefore $\beta_{i+1}(X) \leqslant (e(R))^3$ by Lemma 4.1.16. Noting that $\beta_i(M) = \beta_{-i}(M^*)$ for all $i \in \mathbb{Z}$, we see that $\{\beta_i(X^*)\}_{i \geqslant 0}$ is bounded, by Remark 4.1.2 (a). Therefore X^* is eventually periodic, by Remark 4.1.2 (b). But in the setting of Cohen-Macaulay modules over a Gorenstein ring, this is the same as saying that X^* is periodic, and the same as saying that X is periodic. Part (a) follows.

(b) By making the dual argument, we see that there exist infinitely many positive values of i such that $\operatorname{syz}^i g$ is surjective and $\ker(\operatorname{syz}^i g) \cong \operatorname{syz}^i X$ is isomorphic to an ideal; and $\{\beta_i(X)\}_{i \geqslant 0}$ is bounded, and X is periodic. □

For $W \in \mathrm{CM}(R)$, define $d_W \colon \mathrm{CM}(R) \longrightarrow \mathbb{N}$ by $d_W(M) = \dim_k \underline{\operatorname{Hom}}_R(M, W)$. The following lemma is dual to [11, Lemma 3.2]. As the proof is also simply the dual, we omit it.

Lemma 4.1.18. *Let* $0 \longrightarrow \operatorname{syz} M \longrightarrow E \longrightarrow M \longrightarrow 0$ *be an AR sequence.*

(a) If M is not a summand of W then $d_W(M) + d_W(\operatorname{syz} M) \geqslant d_W(E)$.

(b) If in addition $\operatorname{syz} M$ *is not a direct summand of* W*, then equality holds in (a).*

Proof of Proposition 4.1.12. Let C denote the stable AR component containing M. In order to show that the tree class of C admits an additive function, it now suffices to find $W \in \mathrm{CM}(R)$ such that: (1) $d_W(M) = d_W(\operatorname{syz} M)$ for all $M \in C$, (2) no direct summand of W occurs in C, and (3) d_W is not zero on C.

By Lemma 4.1.17 and the assumption that C contains a module which is either not eventually syz-perfect, or not eventually cosyz-perfect, we can find an irreducible epimorphism $g\colon Y \longrightarrow Z$ such that $\ker g$ is a periodic ideal and either Y or Z lies in C (the other then being a direct sum of modules in C). Now let $W = \bigoplus_{i=0}^{n-1} \operatorname{syz}^i(\ker g)$ where n is the period of $\ker g$ (actually $n = 2$ since R is a complete intersection). Recall that $\underline{\operatorname{Hom}}_R(X', Y') \cong \underline{\operatorname{Hom}}_R(\operatorname{syz} X', \operatorname{syz} Y')$ for all $X', Y' \in \mathrm{CM}(R)$. Therefore W satisfies (1), since $W \cong \operatorname{syz} W$. Note that W satisfies (2) since we are assuming C is not periodic. Lastly, (3) follows from the identity $\underline{\operatorname{Hom}}_R(X', Y') \cong \operatorname{Ext}^1_R(X', \operatorname{syz} Y')$ and the nonsplit extension $0 \longrightarrow \ker g \longrightarrow Y \longrightarrow Z \longrightarrow 0$. □

In Theorem 4.1.30 we will address the case when all modules in a given stable AR component are eventually cosyz-perfect. We first set about proving Lemmas 4.1.20 and 4.1.21.

Notation 4.1.19. In order to avoid some repetitious verbiage, let us for the remainder of this section use C to denote a nonperiodic stable AR component such that every module in C is eventually syz-perfect, and use C' to denote a nonperiodic stable AR component such that every module in C' is eventually cosyz-perfect.

Lemma 4.1.20. *(cf.* [19, Proposition 2.2]*) Assume* $M \in C'$ *is* cosyz*-perfect. (a) If* $\alpha(M) = 1$ *or* 2*, then the AR sequence beginning in* M *has one of the following shapes:*

$$\begin{tikzcd}[column sep=small] & X \arrow[dr, two heads] & \\ M \arrow[ur, hook] & & \operatorname{syz}^{-1} M \end{tikzcd}$$

(1)

$$\begin{tikzcd}[column sep=small] & X \arrow[dr, two heads] & \\ M \arrow[ur, hook] \arrow[dr, two heads] & & \operatorname{syz}^{-1} M \\ & Y \arrow[ur, hook] & \end{tikzcd}$$

(2a)

$$\begin{tikzcd}[column sep=small] & X \arrow[dr, hook] & \\ M \arrow[ur, hook] \arrow[dr, hook] & & \operatorname{syz}^{-1} M \\ & Y \arrow[ur, hook] & \end{tikzcd}$$

(2b′)

(b) If $\alpha(M) = 3$ or 4, then the AR sequence beginning in M has one of the following shapes:

$$\begin{tikzcd}[row sep=tiny] & X \arrow[dr, hook] & \\ M \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook] & \operatorname{syz}^{-1} M \\ & Z \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & X \arrow[dr, hook] & \\ M \arrow[ur, hook] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook] & \operatorname{syz}^{-1} M \\ & Z \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & W \arrow[ddr, hook] & \\ & X \arrow[dr, hook] & \\ M \arrow[uur, two heads] \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook] & \operatorname{syz}^{-1} M \\ & Z \arrow[ur, hook] & \end{tikzcd}$$

(3a) **(3b′)** **(4)**

Dually,

Lemma 4.1.21. *Assume $M \in C$ is* syz*-perfect. (a) If $\alpha(M) = 1$ or 2, then the AR sequence ending in M has one of the following shapes:*

$$\begin{tikzcd}[row sep=tiny] & X \arrow[dr, two heads] & \\ \operatorname{syz} M \arrow[ur, hook] & & M \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & X \arrow[dr, two heads] & \\ \operatorname{syz} M \arrow[ur, hook] \arrow[dr, two heads] & & M \\ & Y \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & X \arrow[dr, two heads] & \\ \operatorname{syz} M \arrow[ur, two heads] \arrow[dr, two heads] & & M \\ & Y \arrow[ur, two heads] & \end{tikzcd}$$

(1) **(2a)** **(2b)**

(b) If $\alpha(M) = 3$ or 4, then the AR sequence beginning in M has one of the following shapes:

$$\begin{tikzcd}[row sep=tiny] & X \arrow[dr, hook, two heads] & \\ \operatorname{syz} M \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook, two heads] & M \\ & Z \arrow[ur, hook, two heads] & \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & X \arrow[dr, two heads] & \\ \operatorname{syz} M \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook, two heads] & M \\ & Z \arrow[ur, hook, two heads] & \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] & W \arrow[ddr, hook, two heads] & \\ & X \arrow[dr, hook, two heads] & \\ \operatorname{syz} M \arrow[uur, two heads] \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook, two heads] & M \\ & Z \arrow[ur, hook, two heads] & \end{tikzcd}$$

(3a) **(3b)** **(4)**

These will be proven via 4.1.15- 4.1.28.

Remark 4.1.22. We mentioned earlier, but re-emphasize, that not all statements in $\mathrm{CM}(R)$ "are dualizable"; see Lemma 4.1.29, for example.

Lemma 4.1.23. *(cf.* [12, Proposition 3.2] *and* [12, Lemma 3.4]*) Assume that $M \in C'$ is* cosyz*-perfect, and let* $0 \longrightarrow M \xrightarrow{[f_1, f_2, \ldots, f_r]^T} \bigoplus_{i=1}^r E_i \xrightarrow{[g_1, g_2, \ldots, g_r]} \operatorname{syz}^{-1} M \longrightarrow 0$ *be an AR sequence where each E_i is nonzero but not necessarily indecomposable. Suppose $r \geqslant 3$. Then*

(a) At most one of the f_i's is mono.

(b) If f_i is mono, then g_i is mono.

Proof. (a) Suppose f_1 and f_2 are both monomorphisms. Since f_1 is mono, so is $[g_2, \ldots, g_r]\colon E_2 \oplus \cdots \oplus E_r \longrightarrow \operatorname{syz}^{-1} M$, by Lemma 4.1.9. Therefore $e(\operatorname{syz}^{-1} M) \geqslant e(E_2) + \cdots + e(E_r) \geqslant e(E_2) + e(E_3) \geqslant e(M) + \frac{1}{e(R)} e(M)$, using Lemma 4.1.16. Then since M is cosyz-perfect, induction gives $e(\operatorname{syz}^{-n} M) \geqslant e(M)(1 + \frac{1}{e(R)})^n$ for all $n \geqslant 1$. This implies $\operatorname{cx} M^* = \infty$ (in view of Lemma 4.1.15), which is a contradiction to the assumption that R is a complete intersection.

(b) By considering the AR sequences beginning at M and $\operatorname{syz}^{-1} M$, we see that $\sum_{j=1}^{r}(e(E_j) + e(\operatorname{syz}^{-1} E_j)) = e(M) + 2e(\operatorname{syz}^{-1} M) + e(\operatorname{syz}^{-2} M)$. On the other hand, consideration of the AR sequence beginning at the summands of the E_j's gives inequalities $e(E_j) + e(\operatorname{syz}^{-1} E_j) \geqslant e(\operatorname{syz}^{-1} M)$ for each j, so we have $\sum_{j=2}^{r}(e(E_j) + e(\operatorname{syz}^{-1} E_j)) \geqslant (r-1)e(\operatorname{syz}^{-1} M) \geqslant 2e(\operatorname{syz}^{-1} M)$. Therefore, $e(E_1) + e(\operatorname{syz}^{-1} E_1) \leqslant e(M) + e(\operatorname{syz}^{-2} M)$. This implies $e(\operatorname{syz}^{-1} E_1) \leqslant e(\operatorname{syz}^{-2} M)$, provided $f_1\colon M \longrightarrow E_1$ is mono. But then $\operatorname{syz}^{-1} g_1\colon \operatorname{syz}^{-1} E_1 \longrightarrow \operatorname{syz}^{-2} M$ cannot be epi, so $\operatorname{syz}^{-1} g_1$ is mono, and therefore g_1 is mono (since g_1 is cosyz-perfect).

□

Lemma 4.1.24. *Assume that $M \in C$ is* syz*-perfect, and let*
$0 \longrightarrow \operatorname{syz} M \xrightarrow{[f_1, f_2, \ldots, f_r]^T} \bigoplus_{i=1}^{r} E_i \xrightarrow{[g_1, g_2, \ldots, g_r]} M \longrightarrow 0$ *be an AR sequence where each E_i is nonzero but not necessarily indecomposable. Suppose $r \geqslant 3$. Then*

(a) At most one of the g_i's is epi.

(b) If g_i is epi, then f_i is epi.

Proof. Dual to the proof of Lemma 4.1.23. □

Lemma 4.1.25. *(cf. [12, Lemmas 3.4 and 3.5]) Assume that $M \in C'$ is* cosyz*-perfect, and let* $0 \longrightarrow M \xrightarrow{[f_1, f_2, \ldots, f_r]^T} \bigoplus_{i=1}^{r} E_i \xrightarrow{[g_1, g_2, \ldots, g_r]} \operatorname{syz}^{-1} M \longrightarrow 0$ *be an AR sequence where each E_i is nonzero but not necessarily indecomposable. If $r \geqslant 4$, then each f_i is epi.*

Proof. Let $B = E_4 \oplus \cdots \oplus E_r$, and let $f'^T = [f_4, \ldots, f_r]^T\colon M \longrightarrow B$ and $g'\colon B \longrightarrow \operatorname{syz}^{-1} M$ be the induced irreducible maps. Suppose that f_1 is a monomorphism. Then so is

$[f_1, f_2]^T \colon M \longrightarrow E_1 \oplus E_2$. Then $[g_1, g_2] \colon E_1 \oplus E_2 \longrightarrow \operatorname{syz}^{-1} M$ is a mono by Lemma 4.1.23 (b), which in turn implies that $[f_3, f']^T$ is mono (Lemma 4.1.9). So $e(M) \leqslant e(E_3) + e(B)$.

But on the other hand, since f_1 is mono, so is $[g_2, g_3, g']$, so that $e(E_2) + e(E_3) + e(B) \leqslant e(\operatorname{syz}^{-1} M)$. Putting these inequalities together, and employing Lemma 4.1.16, we get $(1 + \frac{1}{e(R)})e(M) \leqslant e(\operatorname{syz}^{-1} M)$. Then induction gives $(1+\frac{1}{e(R)})^n e(M) \leqslant e(\operatorname{syz}^{-n} M)$ for each $n \geqslant 1$, which (in view of Lemma 4.1.15) implies $\operatorname{cx} M^* = \infty$, contradiction. □

Lemma 4.1.26. *Assume that $M \in C$ is* syz*-perfect, and let*
$0 \longrightarrow \operatorname{syz} M \xrightarrow{[f_1,f_2,\dots,f_r]^T} \bigoplus_{i=1}^r E_i \xrightarrow{[g_1,g_2,\dots,g_r]} M \longrightarrow 0$ *be an AR sequence where each E_i is nonzero but not necessarily indecomposable. If $r \geqslant 4$, then each g_i is mono.*

Proof. Dual to the proof of Lemma 4.1.25. □

Proposition 4.1.27. *(cf.* [12, Lemmas 3.4 and 3.5]*) For $M \in C$, we have $\alpha(M) \leqslant 4$.*

Proof. We may assume $M \in C$ is syz-perfect. Let

$$0 \longrightarrow \operatorname{syz} M \xrightarrow{[f_1,f_2,\dots,f_r]^T} \bigoplus_{i=1}^r E_i \xrightarrow{[g_1,g_2,\dots,g_r]} \operatorname{syz}^{-1} M \longrightarrow 0$$

be the AR sequence ending in M, each $E_i \neq 0$, and assume to the contrary that $r \geqslant 5$. Let $B_1 = E_1 \oplus E_2$ and $B_2 = E_3 \oplus \cdots \oplus E_r$ and rewrite the AR sequence as

$$0 \longrightarrow \operatorname{syz} M \xrightarrow{[h_1,h_2]^T} B_1 \oplus B_2 \xrightarrow{[k_1,k_2]} M \longrightarrow 0$$

where $h_1 = [f_1, f_2]$, $h_2 = [f_3, \dots, f_r]$, $k_1 = [g_1, g_2]$ and $k_2 = [g_3, \dots, g_r]$. Since B_2 has at least 3 direct summands, Lemma 4.1.26 implies that k_1 is a mono. If k_2 is also mono then so is h_1^T, which we can compose with k_1 to get a monomorphism $\operatorname{syz} M \hookrightarrow M$. Therefore we get an infinite chain of monomorphisms $\dots \hookrightarrow \operatorname{syz}^2 M \hookrightarrow \operatorname{syz} M \hookrightarrow M$, implying $e(M) \geqslant e(\operatorname{syz} M) \geqslant e(\operatorname{syz}^2 M) \geqslant \dots$. Then Lemma 4.1.15 implies that $\operatorname{cx} M \leqslant 1$, contrary to assumption. So k_2 must be epi. But since k_1 is a mono, so is h_2^T, and we get a contradiction to Lemma 4.1.24, since B_1 has at least 2 direct summands. □

Proposition 4.1.28. *For $M \in C'$, we have $\alpha(M) \leqslant 4$.*

Proof. May be proved as the dual of the proof of Proposition 4.1.28. (One has the option of slightly shortening the argument by using Lemma 4.1.29 instead of appealing to complexity.) □

Proof of Lemma 4.1.21. Let us write the AR sequence as

$$0 \longrightarrow \operatorname{syz} M \xrightarrow{[f_1, f_2, \ldots, f_r]^T} \bigoplus_{i=1}^r E_i \xrightarrow{[g_1, g_2, \ldots, g_r]} M \longrightarrow 0\,,$$

where the E_i's are nonzero indecomposables, and $r \leqslant 4$ by Proposition 4.1.27. There is nothing to prove if $r = 1$. Assume $r = 2$. By Lemma 4.1.9, it suffices to show that the maps f_1 and g_1 cannot both be mono. But if they were, we would have an infinite chain of monomorphisms $\ldots \hookrightarrow \operatorname{syz}^2 M \hookrightarrow \operatorname{syz} M \hookrightarrow M$, and a contradiction as in the proof of Proposition 4.1.27.

Now assume $r = 3$. By Lemma 4.1.24 (a), at most one of the g_i's is epi. Note that f_i is epi whenever g_i is mono, as we saw in case $r = 2$. So the classification of the $r = 3$ case is finished by Lemma 4.1.24 (b). In the case $r = 4$, all g_i's are mono by Lemma 4.1.26, and therefore all f_i's epi. □

Proof of Lemma 4.1.20. May be proved as the dual of the proof of Lemma 4.1.21. □

Note that the dual of the following lemma does not hold. (The dual proof fails because, while $\mathrm{CM}(R)$ is closed under kernels, it is not closed under cokernels.)

Lemma 4.1.29. *Any chain of epimorphisms $X^0 \twoheadrightarrow X^1 \twoheadrightarrow \ldots \twoheadrightarrow X^n \twoheadrightarrow \ldots$ in $\mathrm{CM}(R)$ must eventually terminate.*

Proof. Recall $e(X) > 0$ for every $X \in \mathrm{CM}(R)$. Therefore, the additivity of $e(\,)$ along short exact sequences implies that any epi $X \twoheadrightarrow Y$ in $\mathrm{CM}(R)$ which is not an isomorphism must satisfy $e(X) > e(Y)$. □

Our next goal is the following.

Theorem 4.1.30. *(cf.* [19, Theorem 2.1]*) Let C' be a nonperiodic stable AR component of R such that every module in C' is eventually* cosyz-*perfect, assume also that k is algebraically closed. Then C' is of type $\mathbb{Z}\Delta$, where Δ is either a Euclidean diagram of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_i$ $(i = 6, 7, 8)$ or a Dynkin diagram of type E_i $(i = 6, 7, 8)$, A_∞, A_∞^∞ or D_∞.*

Lemma 4.1.31. *(cf.* [12, Lemma 2.6]*) Let $f\colon X \longrightarrow Y$ be an irreducible epimorphism, where X and Y are indecomposables in $L_p(R)$. If X is* cosyz-*perfect, then so is Y.*

Proof. Let $0 \longrightarrow Y \xrightarrow{f} \operatorname{syz}^{-1} X \oplus Z \xrightarrow{g} \operatorname{syz}^{-1} Y \longrightarrow 0$ be the AR sequence beginning at Y. If $Z = 0$, then f and g are cosyz-perfect since X is; and then Y is cosyz-perfect. So assume $Z \neq 0$, and write $f = [f_1\ f_2]^T$ and $g = [g_1\ g_2]$. As $\operatorname{syz}^n g_1$ is epi for each $n \leqslant 0$, so is $\operatorname{syz}^n f_2$ (Lemma 4.1.9). If f_1 were epi, then the compositions $(\operatorname{syz}^n g_1)(\operatorname{syz}^n f_1)$ would give an infinite sequence $Y \twoheadrightarrow \operatorname{syz}^{-1} Y \twoheadrightarrow \ldots \twoheadrightarrow \operatorname{syz}^n Y \twoheadrightarrow \operatorname{syz}^{n-1} Y \twoheadrightarrow \ldots$ of epimorphisms, contradicting Lemma 4.1.29. So f_1 is a mono, and therefore g_2 is a mono, by Lemma 4.1.9. Let Z' be a direct summand of Z. Then since $g_2\colon Z \longrightarrow \operatorname{syz}^{-1} Y$ is mono, so is the induced map $Z' \longrightarrow \operatorname{syz}^{-1} Y$. Likewise, since f_2 is epi, so is the induced map $Y \longrightarrow Z'$. Since our hypotheses are preserved by syz^{-1}, the maps $Z' \longrightarrow \operatorname{syz}^{-1} Y$ and $Y \longrightarrow Z'$ are thus cosyz-perfect. It remains to consider irreducible maps involving $\operatorname{syz}^{-1} X \oplus Z'$. Now since $g_1\colon \operatorname{syz}^{-1} X \longrightarrow \operatorname{syz}^{-1} Y$ is not mono, the map $\operatorname{syz}^{-1} X \oplus Z' \longrightarrow \operatorname{syz}^{-1} Y$ induced by g must also not be mono; so it is epi, and thus cosyz-perfect (again, because our hypotheses are preserved by syz^{-1}). Since $f_1\colon Y \longrightarrow \operatorname{syz}^{-1} X$ is not epi, the map $Y \longrightarrow \operatorname{syz}^{-1} X \oplus Z'$ induced by f must also not be epi; it is mono, and cosyz-perfect. □

Lemma 4.1.32. *(cf.* [19, Lemma 2.3]*) Let $0 \longrightarrow M \xrightarrow{[f_1, f_2]^T} X \oplus Y \xrightarrow{[g_1, g_2]} \operatorname{syz}^{-1} M \longrightarrow 0$ be the AR sequence beginning at a* cosyz-*perfect module M, and assume that the module X is indecomposable and f_1 is an epimorphism. Then X is* cosyz-*perfect, and either $\alpha(X) = 1$, or $\alpha(X) = 2$ and the AR sequence beginning with X is of type* **(2a)**.

Proof. X is cosyz-perfect by Lemma 4.1.31, and we have an irreducible epimorphism $\operatorname{syz}^{-1} M \twoheadrightarrow \operatorname{syz}^{-1} X$. So we are done by consideration of the list in Lemma 4.1.20. □

Lemma 4.1.33. *(cf.* [19, Cor. 2.4]*) If the AR sequence beginning at a* cosyz*-perfect module* M *has the form* $0 \longrightarrow M \xrightarrow{[f_1,f_2]^T} X_1 \oplus Y \xrightarrow{[g_1,g_2]} \operatorname{syz}^{-1} M \longrightarrow 0$ *with* X_1 *indecomposable, and* f_1 *an epimorphism, then there exists a finite chain of irreducible epimorphisms* $X_1 \twoheadrightarrow X_2 \twoheadrightarrow \ldots \twoheadrightarrow X_r$ *with* $r \geqslant 1$, $\alpha(X_r) = 1$, *and* $\alpha(X_i) = 2$ *for all* $1 \leqslant i < r$.

Proof. The existence of the chain $X_1 \twoheadrightarrow X_2 \twoheadrightarrow \ldots$ follows from Lemma 4.1.32, and it terminates by Lemma 4.1.29. □

Recall that if k is algebraically closed, the AR quiver of R is symmetric in the sense of Remark 3.2.4.

Proposition 4.1.34. *(cf.* [19, Prop. 2.5]*) Assume* k *is algebraically closed, and let* $0 \longrightarrow M \xrightarrow{[f_1,f_2,\ldots,f_r]^T} \bigoplus_{i=1}^r X_i \xrightarrow{[g_1,g_2,\ldots,g_r]} \operatorname{syz}^{-1} M \longrightarrow 0$ *be the AR sequence beginning at module* $M \in C'$, *where each* X_i *is a nonfree indecomposable (which is automatic if* M *is* cosyz*-perfect). Then, for all* $i \neq j$, $X_i \not\cong X_j$, *unless* $r = 2$ *and* C' *is of type* $\mathbb{Z}\widetilde{A}_1$.

Proof. For simplicity, assume $X_1 = X_2$, and call it simply X. By multiplicity considerations, either f_1 and f_2 are both mono or they are both epi. But we may also assume that $\operatorname{syz} X$ is cosyz-perfect. Then by consideration of the AR sequence beginning at $\operatorname{syz} X$, in the context of Lemma 4.1.20, we see that f_1 and f_2 are both mono. But we can also assume M is cosyz-perfect, and therefore the AR sequence beginning at M must be of type **(2b′)**, and $r = 2$. Since we have irreducible maps $g_1 \colon X \longrightarrow \operatorname{syz}^{-1} M$ and $g_2 \colon X \longrightarrow \operatorname{syz}^{-1} M$, X satisfies exactly what we assumed of M at the outset of this proof, so these are the only irreducible maps coming from X. Therefore the tree class (recall Remark 3.1.8) of C' is just a single arrow connecting two vertices, i.e. A_2. Then $C' = \mathbb{Z}A_2$ when we ignore multiple arrows; and if we take multiple arrows into account then we clearly just need to double all arrows in $\mathbb{Z}A_2$, which gives $\mathbb{Z}\widetilde{A}_1$. □

Lemma 4.1.35. *(cf.* [19, Lemma 2.7]*) Assume* k *is algebraically closed, and* $\alpha(C') = 2$. *If there exists a* cosyz*-perfect module* $M \in C'$ *with* $\alpha(M) = 2$ *and AR sequence*

$0 \longrightarrow M \xrightarrow{[f_1,f_2]^T} X_1 \oplus Y_1 \xrightarrow{[g_1,g_2]} \operatorname{syz}^{-1} M \longrightarrow 0$ *such that* f_2 *is an epimorphism, then* C' *is of type* $\mathbb{Z}A_\infty$. *If there does not exist such* M, *then* C' *is of type* $\mathbb{Z}A_\infty^\infty$ *or* $\mathbb{Z}\widetilde{A}_n$ *for some* $n \geqslant 1$.

Proof. Assume there exists such M. By Lemma 4.1.33, we have a finite chain of irreducible epimorphisms $Y_1 \twoheadrightarrow Y_2 \twoheadrightarrow \ldots \twoheadrightarrow Y_r$, where $\alpha(Y_r) = 1$ and $\alpha(Y_i) = 2$ for all $1 \leqslant i < r$. Since there is an irreducible mono $M \hookrightarrow X_1$, we must have $\alpha(X_1) \geqslant 2$, and therefore $\alpha(X_1) = 2$ since we are assuming $\alpha(C') = 2$. Since $\operatorname{syz}^n g_1 \colon \operatorname{syz}^n X_1 \longrightarrow \operatorname{syz}^{n-1} M$ is an epimorphism for all $n \leqslant 0$, the AR sequence beginning in $\operatorname{syz}^n X_1$ is of type (**2a**) for all $n << 0$ (namely those n for which $\operatorname{syz}^n X_1$ is cosyz-perfect). Therefore if $0 \longrightarrow X_1 \longrightarrow X_2 \oplus \operatorname{syz}^{-1} M \longrightarrow \operatorname{syz}^{-1} X \longrightarrow 0$ is the AR sequence beginning at X_1, we must have $\alpha(X_2) = 2$, and the AR sequence beginning in $\operatorname{syz}^n X_2$ is of type (**2a**) for all $n << 0$. By induction, we obtain an infinite chain of irreducible maps $M \xrightarrow{h_1=f_1} X_1 \xrightarrow{h_2} X_2 \xrightarrow{h_3} X_3 \longrightarrow \ldots$ (where each h_i is "eventually mono" in the sense of $\operatorname{syz}^n(h_i)$ being mono for all $n << 0$). Therefore the tree class of C' is A_∞. Since any proper admissible quotient of $\mathbb{Z}A_\infty$ is periodic (namely a tube), C' is of type $\mathbb{Z}A_\infty$.

Now assume there does not exist M as above, but take $M \in C'$ such that $\alpha(M) = 2$. We may take M to be cosyz-perfect, and the sequence beginning at M is of type (**2b′**). So we have irreducible monomorphisms $M \hookrightarrow X_1$ and $M \hookrightarrow Y_1$ and $\alpha(X_1) = \alpha(Y_1) = 2$, and the AR sequences beginning at X_1 and Y_1 are also of type (**2b′**) (after sufficient application of syz^{-1}). Arguing as above, we therefore get infinite chains $M \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$ and $M \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \ldots$, and the tree class of C' is A_∞^∞. Now, C' has the form $\mathbb{Z}A_\infty^\infty/G$ where G is an admissible group of automorphisms of $\mathbb{Z}A_\infty^\infty$. Note that any automorphism $\rho \colon \mathbb{Z}A_\infty^\infty \longrightarrow \mathbb{Z}A_\infty^\infty$ is determined by the image of a vertex and one of its immediate successors, and it follows that ρ is either a translation or a translation followed by a reflection.

To finish the proof, it suffices to rule out the latter case. Suppose to the contrary that G contains $\rho = rt$ where r is a reflection and t is a translation. We naturally draw $\mathbb{Z}A_\infty^\infty$ so that syz-orbits form horizontal lines extending infinitely in both directions, and the immediate successors y and z of a vertex x lie on horizontal lines immediately (let us say "by one unit") above and below that of x. So we might write height(y) =

height$(x) + 1$ and height(z) = height$(x) - 1$. Since we can choose where height = 0, we can assume that r simply "negates heights". Now if t has vertical component v_t, we have height$(\rho(x)) = -(\text{height}(x) + v_t)$, hence height$(x) - \text{height}(\rho(x)) = 2\,\text{height}(x) + v_t$. So we can choose x such that height$(x) - \text{height}(\rho(x)) \in \{0, 1\}$. Now it follows that x is a successor of $\tau^{-1}x$ in $\mathbb{Z}A_\infty^\infty/\langle\rho\rangle$ and thus that there is a chain of irreducible maps $\operatorname{syz}^{-1} M \longrightarrow \ldots \longrightarrow M$ in C', for M in C' corresponding to x. However, we may assume M is cosyz-perfect, and therefore that we have an infinite sequence of irreducible monomorphisms $\ldots \hookrightarrow \operatorname{syz}^n M \hookrightarrow \ldots \hookrightarrow \operatorname{syz}^{n+1} \hookrightarrow \ldots \hookrightarrow \operatorname{syz}^{-1} M \hookrightarrow \ldots \hookrightarrow M$. Then $e(M) \geqslant e(\operatorname{syz}^{-1} M) \geqslant \ldots$, which implies that $\mathsf{cx}\, M^* = 1$ (in view of Lemma 4.1.15). But then M^* is periodic, and then M is periodic, contrary to assumption. $\square$

Lemma 4.1.36. *(cf.* [19, Lemma 2.8]*) Assume k is algebraically closed, and $\alpha(C') = 3$. Let $M \in C'$ be* cosyz*-perfect with AR sequence*

$$0 \longrightarrow M \xrightarrow{[f_1,f_2,f_3]^T} X \oplus Y \oplus Z \xrightarrow{[g_1,g_2,g_3]} \operatorname{syz}^{-1} M \longrightarrow 0\,.$$

(a) If f_1, f_2, f_3 are epimorphisms, then C' is of type $\mathbb{Z}E_i$ or $\mathbb{Z}\widetilde{E_i}$, $i \in \{6, 7, 8\}$.

(b) If f_1 is a mono, then C' is either of type $\mathbb{Z}D_\infty$, or of type $\mathbb{Z}\widetilde{D}_n$ for some $n \geqslant 5$.

Proof. (a) By Lemmas 4.1.20 and 4.1.33, the tree class of C' is a rooted tree T with three finite branches. It is either Dynkin or Euclidean: see Section 1 of [19]. Now it only remains to remark that if G is a nontrivial group of automorphisms of $\mathbb{Z}T$, then $\mathbb{Z}T/G$ is periodic. Let ρ be a nontrivial automorphism and let $x \in T$ be the root of T, i.e., take x corresponding to M. Clearly, $\rho(x) = \tau^n x$ for some integer n, and since C' is nonperiodic we may therefore assume that $\rho(x) = x$. But it is easy to check that any admissible automorphism fixing a vertex is the identity map.

(b) Now the AR sequence beginning in M is of type **(3b′)**. Applying Lemma 4.1.33 to $Y_1 = Y$ and $Z_1 = Z$, we obtain chains of irreducible epimorphisms $Y_1 \twoheadrightarrow Y_2 \twoheadrightarrow \ldots \twoheadrightarrow Y_r$ and $Z_1 \twoheadrightarrow Z_2 \twoheadrightarrow \ldots \twoheadrightarrow Z_s$, with $\alpha(Y_r) = \alpha(Z_s) = 1$, and $\alpha(Y_i) = \alpha(Z_j) = 2$ for $1 \leqslant i < r$ and $1 \leqslant j < s$. First we show that $r = s = 1$. We may assume that X is cosyz-perfect. Let

$$0 \longrightarrow X \longrightarrow \operatorname{syz}^{-1} M \oplus V \longrightarrow \operatorname{syz}^{-1} X \longrightarrow 0,$$

$$0 \longrightarrow Y_1 \longrightarrow Y_2 \oplus \operatorname{syz}^{-1} M \longrightarrow \operatorname{syz}^{-1} Y_1 \longrightarrow 0,$$

and

$$0 \longrightarrow Z_1 \longrightarrow Z_2 \oplus \operatorname{syz}^{-1} M \longrightarrow \operatorname{syz}^{-1} Z_1 \longrightarrow 0$$

be AR sequences, and let us show that $Z_2 = 0$. By consideration of these AR sequences as well as those beginning at M and $\operatorname{syz}^{-1} M$, we can obtain $e(M)+e(\operatorname{syz}^{-2} M) = e(\operatorname{syz}^{-1} M)+e(V)+e(Y_2)+e(Z_2)$ and in particular

$$e(M) + e(\operatorname{syz}^{-2} M) > e(\operatorname{syz}^{-1} M) + e(V) + e(Z_2). \tag{4.1.3}$$

Note that $X \longrightarrow V$ is a monomorphism since $\operatorname{syz}^{-1} M \longrightarrow \operatorname{syz}^{-1} X$ is such. So we have monomorphisms $M \hookrightarrow X \hookrightarrow V$ and $M \hookrightarrow X \hookrightarrow \operatorname{syz}^{-1} M$ (since the AR sequence beginning in M is of type **(3b′)**), and therefore $e(M) \leqslant e(V)$ and $e(M) \leqslant e(\operatorname{syz}^{-1} M)$. Now the inequality 4.1.3 gives $e(\operatorname{syz}^{-2} M) > e(M) + e(Z_2)$. Then we have $e(\operatorname{syz}^{-2} M) > e(M)(1 + (\frac{1}{e(R)})^2)$ by Lemma 4.1.16. But then $e(\operatorname{syz}^{-2n} M) > e(M)(1 + (\frac{1}{e(R)})^2)^n$ for all $n \geqslant 1$, by induction. Then $\operatorname{cx} M^* = \infty$, which is a contradiction. So we have $r = s = 1$.

Let $X_1 = X$ and let $0 \longrightarrow X_1 \longrightarrow \operatorname{syz}^{-1} M \oplus X_2 \longrightarrow \operatorname{syz}^{-1} X_1 \longrightarrow 0$ be the AR sequence beginning at X_1. If $\alpha(X_1) = 3$ then both summands of X_2 have $\alpha = 1$, by the above. In this case the tree class T of C' is $\widetilde{D}_5$ (six vertices). Now assume the other case: $\alpha(X_1) = 2$. We may assume that X_1 and X_2 are cosyz-perfect. Lemma 4.1.20 implies that the AR sequence beginning in X_1 is of type **(2b′)**, and that the AR sequence beginning in X_2 is either of type **(2b′)** or **(3b′)**. If the latter then T is $\widetilde{D}_6$. By continuing this process, we see that T is either D_∞ or $\widetilde{D}_n$ for some $n \geqslant 5$. If $T = D_\infty$, then we see that $C' = \mathbb{Z}T$ for the same reasons used in part (a). If $T = \widetilde{D}_n$, then we may argue as follows. If T has an even number of vertices, then there are two vertices, x and y, which are closest to the center of T (as opposed to the ends of T, where the vertices have $\alpha = 1$). Now any automorphism ρ of $\mathbb{Z}T$ must send x

to $\tau^n x$ or $\tau^n y$ for some integer n. But if $\rho \neq 1$ and $\mathbb{Z}T/\langle\rho\rangle$ is not periodic, then we must have $\rho(x) = \tau^n y$ (some n). We may assume that $x \to y \in T$. Then $\rho(y) = \tau^{n-1}(x)$, and $\rho^2(x) = \tau^{2n-1}x$ implies that $\mathbb{Z}T/\langle\rho\rangle$ is periodic, since $2n - 1 \neq 0$. So $C' = \mathbb{Z}T$. If T has an odd number of vertices then we have the easier argument as in (a). □

Lemma 4.1.37. *(*[19, Lemma 2.9]*) Assume that k is algebraically closed, and that C' contains a module M such that $\alpha(M) = 4$. Then C' is of type $\mathbb{Z}\widetilde{D}_4$.*

Proof. By Lemmas 4.1.20 and 4.1.33, the tree class T of C' is a finite rooted tree with four arms. But it is either Dynkin or Euclidean, by Section 1 of [19], so it must be $\widetilde{D}_4$. Thus $C' = \mathbb{Z}\widetilde{D}_4$. □

Proof of Theorem 4.1.30. The theorem follows from Lemmas 4.1.35, 4.1.36, and 4.1.37. □

4.2 Application to the Huneke-Wiegand Conjecture

Conjecture 4.2.1. *(*[18]*) Let D be a Gorenstein local domain of dimension one and M a nonzero finitely generated torsionfree D-module, that is not free. Then $M \otimes_D M^*$ has a nonzero torsion submodule.*

As shown in [16, Theorem 5.9], the above condition on $M \otimes_D M^*$ may be replaced by the condition that $\operatorname{Ext}^1_D(M, M) \neq 0$. In turn, this is equivalent to $\underline{\operatorname{Hom}}_D(\operatorname{syz}_D M, M) \neq 0$.

We continue to assume R is a complete (or graded-) local complete intersection ring dimension one. We can confirm special cases of the conjecture, as follows.

Proposition 4.2.2. *Let $M \in L_p(R)$ be a nonfree indecomposable with $\alpha(M) \geqslant 3$, which is either eventually* syz*-perfect or eventually* cosyz*-perfect. Then* $\operatorname{Ext}^1_R(M, M) \neq 0$.

Proof. Recall that $\operatorname{Ext}^1_R(M, M) \cong \underline{\operatorname{Hom}}_R(\operatorname{syz} M, M) \cong \underline{\operatorname{Hom}}_R(\operatorname{syz}^{i+1} M, \operatorname{syz}^i M)$ for all $i \in \mathbb{Z}$. In particular, we may replace M by some $\operatorname{syz}^n M$ to assume that M is syz- or cosyz-perfect.

By Lemma 4.1.6 (a), it suffices to find a map $M \longrightarrow \operatorname{syz}^{-1} M$ whose image is not contained in $\mathfrak{m} \operatorname{syz}^{-1} M$. Suppose first that M is cosyz-perfect, and let

$$0 \longrightarrow M \xrightarrow{[f_1, f_2, \dots, f_r]^T} \bigoplus_{i=1}^r E_i \xrightarrow{[g_1, g_2, \dots, g_r]} \operatorname{syz}^{-1} M \longrightarrow 0$$

be the AR sequence beginning in M. Since $[g_1, g_2, \dots, g_r]$ is surjective, there exists some i such that $\operatorname{im}(g_i) \nsubseteq \mathfrak{m} \operatorname{syz}^{-1} M$. Therefore if each f_i is an epimorphism, some composition $g_i f_i \colon M \longrightarrow \operatorname{syz}^{-1} M$ has the desired property. Therefore we are done if M has AR sequence of type **(3a)** or **(4)**. Similarly, we are done if M is syz-perfect and has AR sequence of type **(3a)**, **(3b)**, or **(4)**.

To finish this proposition, it remains to consider the case when M is cosyz-perfect with AR sequence of type **(3b′)**; now $r = 3$, and f_2 and f_3 are epimorphisms. As we saw in Lemma 4.1.36, we have in this case $\alpha(E_2) = \alpha(E_3) = 1$. Now we have an AR sequence $0 \longrightarrow E_2 \xrightarrow{g_2} \operatorname{syz}^{-1} M \xrightarrow{p} \operatorname{syz}^{-1} E_2 \longrightarrow 0$. By applying syz^{-1} if necessary, we may assume $\mathfrak{m} \ncong \operatorname{syz} E_2$, so that the free module does not occur in the AR sequence beginning in $\operatorname{syz} E_2$, and therefore the application of syz to the latter sequence yields again an AR sequence. In particular, $\operatorname{syz} p$ is epi, and Lemma 4.1.13 thus implies that $g_2(E_2) \nsubseteq \mathfrak{m} \operatorname{syz}^{-1} M$. The map $g_2 f_2 \colon M \longrightarrow \operatorname{syz}^{-1} M$ satisfies the desired property $g_2 f_2(M) \nsubseteq \mathfrak{m} \operatorname{syz}^{-1} M$. □

Proposition 4.2.3. *Let M be nonfree indecomposable in $L_p(R)$, with $\alpha(M) = 2$. Suppose that either M ends an AR sequence of type* **(2b)** *or that M is* cosyz-*perfect and begins an AR sequence of type* **(2a)**. *Then* $\operatorname{Ext}^1_R(M, M) \neq 0$.

Proof. In case **(2b)** we have a surjection $\operatorname{syz} M \twoheadrightarrow M$, which is stably nonzero by Lemma 4.1.6, so we are done.

Let

$$0 \longrightarrow M \xrightarrow{[f_0, g_0]^T} X_1 \oplus Y \xrightarrow{[g_1, f_0']} \operatorname{syz}^{-1} M \longrightarrow 0$$

be the AR sequence beginning in M, with f_1 an epimorphism. By Lemma 4.1.33 there exists a finite chain of irreducible epimorphisms $X_0 = M \twoheadrightarrow X_1 \twoheadrightarrow X_2 \twoheadrightarrow \dots \twoheadrightarrow X_r$ with $r \geqslant 1$,

$\alpha(X_r) = 1$, and $\alpha(X_i) = 2$ for all $0 \leqslant i < r$. In fact we can take irreducible epimorphisms $f_i \colon X_i \twoheadrightarrow X_{i+1}$, $0 \leqslant i < r$, and $f_i' \colon \operatorname{syz}^{-1} X_{i-1} \longrightarrow \operatorname{syz}^{-1} X_i$ $(1 \leqslant i \leqslant r)$, together with irreducible monomorphisms $g_i \colon X_i \longrightarrow \operatorname{syz}^{-1} X_{i-1}$, to form AR sequences

$$0 \longrightarrow X_i \xrightarrow{[f_i, g_i]^T} X_{i+1} \oplus \operatorname{syz}^{-1} X_{i-1} \xrightarrow{[g_{i+1}, f_i']} \operatorname{syz}^{-1} X_i \longrightarrow 0\ ,$$

for $1 \leqslant i < r$. (Given such an AR sequence for a given i, we know that we have an AR sequence beginning with $0 \longrightarrow X_{i+1} \xrightarrow{[f_{i+1}, g_{i+1}]^T} \cdots$ for some f_{i+1} by Lemma 3.2.5.)

As argued in case **(3b′)**, $g_r(X_r) \nsubseteq \mathfrak{m} \operatorname{syz}^{-1} X_{r-1}$. As f_{r-1} is epi and $g_r f_{r-1} = -f_{r-1}' g_{r-1}$, we get $g_{r-1} \nsubseteq \operatorname{syz}^{-1} X_{r-2}$. Continuing in this way, we see that $g_1(X_1) \nsubseteq \mathfrak{m} \operatorname{syz}^{-1} M$, and by composing with the epi f_0 we get a stably nonzero map $g_1 f_0 \colon M \longrightarrow \operatorname{syz}^{-1} M$, as desired. $\square$

Chapter 5

Examples

The basic example of a Frobenius element is Example 2.2.16. In the following two sections, we compute some other examples. Then we calculate the shape of some components of AR quivers.

5.1 Frobenius elements for hypersurfaces

In this section, let $(R, \mathfrak{m})$ be a domain hypersurface ring $R = k[|x, y|]/(f)$ over an algebraically closed field k. We assume f is an irreducible power series lying in $(x, y)^2$. Let $S = k[|x, y|]$, and let $\mathfrak{m}_S$ denote the maximal ideal of S. Let m and n denote the integers such that f is regular in x (see below) of order m and regular in y of order n.

Definition 5.1.1. For $f \in S$, we say that "f is regular in x of order m" if m is the smallest among those integers j satisfying: f has a term of the form ax^j where $a \in k^{\times}$. We have the analagous definition for y.

Note 5.1.2. We will use without proof ([13, Theorem 3.3], or [8]), the following facts: the integral closure of R is a power series ring $k[|t|]$, and $\underline{v}(x) = n$, $\underline{v}(y) = m$, where $\underline{v}$ denotes the valuation (on $k[|t|]$).

The following lemma does not require that f be irreducible, only that $y \nmid f$.

Lemma 5.1.3. *We have $x^{m-1}/y \in \overline{R} \setminus R$.*

Proof. We can write $f = ux^m + yg$, for some unit $u \in S$ and some $g \in \mathfrak{m}_S$ (since we assume $f \in \mathfrak{m}_S^2$). In R we have $(x^{m-1}/y)x = x^m/y = -gu^{-1} \in \mathfrak{m}$ and $(x^{m-1}/y)y = x^{m-1} \in \mathfrak{m}$, so $x^{m-1}/y \in \mathrm{End}_R(\mathfrak{m}) \subseteq \overline{R}$. That $x^{m-1}/y \notin R$ is also clear (that is, there exists no $h \in S$ such that $x^{m-1} - hy \in fS$).

Proposition 5.1.4. *Assume* $\gcd(m,n) = 1$. *Then the conductor ideal* $(R :_R \overline{R})$ *equals* $t^{(m-1)(n-1)}\overline{R}$, *using the notation of Note 5.1.2.*

Proof. It is well known (see for instance, [17, Example 12.1.1]) that $(m-1)(n-1)-1 = \max\{i \in \mathbb{Z} | i \notin \mathbb{N}m + \mathbb{N}n\}$ is the Frobenius number of the semigroup $\mathbb{N}m + \mathbb{N}n$. Let $\underline{v}(R)$ denote the value semigroup of R, that is, $\underline{v}(R) = \{\underline{v}(r) | r \in R \setminus 0\}$. Since $m = \underline{v}(y)$ and $n = \underline{v}(x)$ are elements of $\underline{v}(R)$, we have that $\underline{v}(R) \supseteq \{i \in \mathbb{Z} | i \geqslant (m-1)(n-1)\}$, and it follows that $t^{(m-1)(n-1)}\overline{R} \subseteq R$. It remains to check that $(m-1)(n-1)-1 \notin \underline{v}(R)$.

Fix $g \in R$ and an expression for g, $g = \sum_{i,j} g_{i,j}x^iy^j$ where $g_{i,j} \in k$. Note that $\underline{v}(x^iy^j) = in + jm$. Notice also that if $in + jm = i'n + j'm < mn$, then $i = i'$ and $j = j'$; to see this, use the equation $(i-i')n = (j'-j)m$, and recall that $\gcd(m,n) = 1$ by assumption. Therefore if $g_{i,j} \neq 0$ for some pair (i,j) satisfying $in + jm \leqslant (m-1)(n-1)-1$, then among the nonzero terms $g_{i,j}x^iy^j$ of g, there is a *unique* term of minimal valuation. In this case, $\underline{v}(g) = \min\{\underline{v}(x^iy^j) | g_{i,j} \neq 0\} < (m-1)(n-1)-1$. On the other hand, if $g_{i,j} = 0$ for all pairs (i,j) satisfying $\underline{v}(x^iy^j) \leqslant (m-1)(n-1)-1$, then $\underline{v}(g) > (m-1)(n-1)$.

Corollary 5.1.5. *Assume* $\gcd(m,n) = 1$ *for* f. *Then* x^{m-1}/y *and* y^{n-1}/x *are Frobenius elements for* R.

Proof. For $z \in \mathcal{J}(\overline{R})$, we have $\underline{v}(z) \geqslant 1$, and therefore $\underline{v}(zx^{m-1}/y) \geqslant 1 + (m-1)n - m = (m-1)(n-1)$, and therefore zx^{m-1}/y is contained in the conductor ideal, and in particular in R. So $x^{m-1}/y \in \mathcal{J}(\overline{R})^* = \mathfrak{F}(R)$. Moreover, $x^{m-1}/y \notin R$ by Lemma 5.1.3. Of course the same reasoning applies to y^{n-1}/x. □

Remark 5.1.6. In Corollary 5.1.5, the condition that $\gcd(m, n) = 1$ cannot be ommitted. Consider the map $\varphi : k[|x, y|] \longrightarrow k[[t]]$ sending $x \mapsto t^4 + t^{17}$, $y \mapsto t^6$, and let $R = \operatorname{im}\varphi$. We know from Note 5.1.2 that the generator f of $\ker\varphi$ has $m = 6$, $n = 4$, hence $\underline{v}(x^{m-1}/y) = 14$. If $\gamma = x^{m-1}/y$, then the conductor ideal would be $t^{15}\overline{R}$. But clearly $t^{15} \notin R$.

5.2 Frobenius elements for some binomial rings

Next we consider some graded reduced complete intersections. For some $n \geqslant 2$, choose integers $a_2, ..., a_n$ and $b_2, ..., b_n$, such that $a_i \geqslant 2$ and $b_i \geqslant 2$ for each i. Let k be a field of characteristic zero, and let $A = k[t_1, ..., t_n]/(t_1^{a_2} - t_2^{b_2}, ..., t_1^{a_n} - t_n^{b_n})$. Let $d_1, ..., d_n$ be positive integers such that setting $\deg(t_i) = d_i$ results in each binomial $t_1^{a_1} - t_i^{b_i}$ being homogeneous; this choice is unique if we insist that $\gcd\{i|A_i \neq 0\} = 1$.

Lemma 5.2.1. *A is a complete intersection, and each t_i is a nonzerodivisor on A.*

Proof. Let s denote the sequence $t_1^{a_2} - t_2^{b_2}, ..., t_1^{a_n} - t_n^{b_n}$. Recall that the sum of a nonzerodivisor and a nilpotent is a nonzerodivisor (since the nonzerodivisors are precisely the elements contained in no associated primes, and the nilpotent elements are precisely the element contained in all associated primes). From this it is easy to see that t_1, s is a regular sequence, and that for each $j \in \{2, ..., n\}$ there is a shuffling s'_j of s (namely, move $t_1^{a_j} - t_j^{b_j}$ to the beginning) such that the sequence t_j, s'_j is a regular sequence. The desired results follow from permutability of regular sequences.

Lemma 5.2.2. *A is reduced.*

Proof. Let $I = (t_1^{a_2} - t_2^{b_2}, ..., t_1^{a_n} - t_n^{b_n})$. Since I is a binomial ideal such that each t_i is a nonzerodivisor on the quotient $k[t_1, ..., t_n]/I = A$, I is a so-called *lattice ideal* ([26, Theorem 8.2.8]). Futhermore, in characteristic zero, every lattice ideal is radical ([26, Theorem 8.2.27]). $\square$

Proposition 5.2.3. *The element $\frac{\prod_{i=2}^n t_i^{b_i-1}}{t_1} \in Q(A)$ is a Frobenius element for A.*

Proof. Let $\gamma = \frac{\prod_{i=2}^n t_i^{b_i-1}}{t_1}$. By Proposition 2.3.4, it suffices to show that $\gamma \in \mathfrak{m}^* \setminus R$. It is clear that $\gamma t_i \in R$ for $i = 1, \ldots, n$, which says that $\gamma \in \mathfrak{m}^*$. If we work in the polynomial ring $k[t_1, ..., t_n]$, the monomial $\prod_{i=2}^n t_i^{b_i-1}$ does not appear in any polynomial of the form $t_1 f$ or $\sum_{i=2}^n (t_1^{a_i} - t_i^{b_i}) f_i$, and it follows that $\prod_{i=2}^n t_i^{b_i-1} \notin t_1 R$, i.e. $\gamma \notin R$. □

Corollary 5.2.4. *A is a semigroup ring over k if and only if $-d_1 + \sum_{i=2}^n d_i(b_i - 1) \notin \sum_{i=1}^n \mathbb{N} d_i$. (These conditions are equivalent to A being a domain if k is algebraically closed, see Remark 6.0.6.)*

Proof. We have $a(A) = -d_1 + \sum_{i=2}^n d_i(b_i - 1)$ by Propositions 2.3.3 and 5.2.3. Since $\{i | A_i \neq 0\} = \sum_{i=1}^n \mathbb{N} d_i$, the result follows from Proposition 2.3.6. □

Here is another example. Let $\alpha_1, ..., \alpha_{n-1}$ and $\beta_2, ..., \beta_n$ be integers $\geqslant 2$ such that $\alpha_i \geqslant \beta_i$ for all $i \in \{2, ..., n-1\}$. Let I be the ideal of $S = k[t_1, ..., t_n]$ generated by $\{t_i^{\alpha_i} - t_{i+1}^{\beta_{i+1}}\}_{i=1}^{n-1}$, and set $B = S/I$. Arguing as before (and keeping the assumption char $k = 0$), B is a reduced complete intersection of dimension 1.

Proposition 5.2.5. *The element $\frac{\prod_{i=2}^n t_i^{\beta_i-1}}{t_1} \in Q(B)$ is a Frobenius element for B, and B is a semigroup ring if and only if $-d'_1 + \sum_{i=2}^n d'_i(\beta_i - 1) \notin \sum_{i=1}^n \mathbb{N} d'_i$, where $d'_i = \deg(t_i)$.*

Proof. Similar to the previous proposition and corollary. □

5.3 A quiver computation

In this section, we apply Theorem 2.2.14 and Proposition 3.2.24 to determine the shape (namely, a tube) of some components of the Auslander-Reiten quiver of the hypersurface ring $\hat{R}$ defined in 5.3.4, below.

5.3.1. Let S be a regular (graded-) local ring, and $f \in S$ a nonzero element. Let $R = S/fS$. A *matrix factorization* of f is a pair of matrices (φ, ψ), with entries in S, such that $\varphi\psi =$

$\psi\varphi = f\,\mathrm{id}_{l\times l}$ for some $l > 0$. As consequences of the definition, we have $\operatorname{cok}\varphi \cong \operatorname{cok}(\varphi \otimes_S R)$, and ([27, 7.2.2])

$$\operatorname{im}(\varphi \otimes_S R) = \ker(\psi \otimes_S R) \text{ and } \operatorname{im}(\psi \otimes_S R) = \ker(\varphi \otimes_S R). \tag{5.3.1}$$

In particular, $\operatorname{cok}\varphi$ and $\operatorname{cok}\psi$ are periodic R-modules, of period two.

Remark 5.3.2. Let (φ, ψ) and (φ', ψ') be matrix factorizations of f. Let n_1 and n_2 be the integers such that φ is n_1-by-n_1 and φ' is n_2-by-n_2. Given $h\colon \operatorname{cok}\varphi \longrightarrow \operatorname{cok}\varphi'$, there of course exist $\alpha\colon S^{(n_1)} \longrightarrow S^{(n_2)}$ and $\beta\colon S^{(n_1)} \longrightarrow S^{(n_2)}$ making the diagram

$$\begin{tikzcd} S^{(n_1)} \arrow[r, "\varphi"] \arrow[d, "\beta"'] & S^{(n_1)} \arrow[r] \arrow[d, "\alpha"] & \operatorname{cok}\varphi \arrow[r] \arrow[d, "h"] & 0 \\ S^{(n_2)} \arrow[r, "\varphi'"] & S^{(n_2)} \arrow[r] & \operatorname{cok}\varphi' \arrow[r] & 0 \end{tikzcd} \tag{5.3.2}$$

commute. Then $\left(\begin{pmatrix} \varphi' & -\alpha \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} \psi' & \beta \\ 0 & \varphi \end{pmatrix} \right)$ is a matrix factorization of f.

If (φ, ψ) is a matrix factorization such that φ and ψ each contains no unit entry, then it is called a *reduced* matrix factorization. If (φ, ψ) is a reduced matrix factorization, then neither $\operatorname{cok}\varphi$ nor $\operatorname{cok}\psi$ contains a free summand (cf. [27, 7.5.1]).

Let us from now on assume, furthermore, that $\dim R = 1$, and that R is either a complete local ring or a connected graded ring.

5.3.3. Let (φ, ψ) be a reduced matrix factorization for f, and let γ be as in Notation 2.2.10. Let $M = \operatorname{cok}\varphi$, and pick α and β lifting $\gamma_M \in \operatorname{End}_R M$ in the sense of Remark 5.3.2. One may check that the valid choices for α are precisely those choices such that $\psi\alpha = \gamma\psi$ after passing to R. Now assume M satisfies the conclusion of Theorem 2.2.14, namely that $[\gamma_M]$ generates the socle of $\underline{\operatorname{End}}_R M$. By Remark 1.1.10, $\operatorname{push}(M) \cong (\operatorname{im}(\psi \otimes_S R) \oplus R^{(n)})/\{(-\gamma c, c) | c \in \operatorname{im}(\psi \otimes_S R)\}$, where n denotes the side length of the matrices φ and ψ. Then we see that

$$\mathrm{push}(M) \cong \mathrm{cok} \begin{pmatrix} \varphi & -\alpha \\ 0 & \psi \end{pmatrix}.$$

5.3.4. Let k be a field, of characteristic not equal to 2, and let us set up a connected graded hypersurface $(R, \mathfrak{m})$ as follows. Let p and q be relatively prime integers $\geqslant 3$, and let $S = k[x, y]$ be the graded polynomial ring such that $S_0 = k$, $\deg x = q$, and $\deg y = p$. Let $f \in S$ be a homogeneous polynomial which is not divisible by x. Let $g = (bx^p + y^q)f$, where $b \in k$, and b is allowed to be zero. Now, let $R = k[x, y]/(g)$. The $\mathfrak{m}$-adic completion of R is $\hat{R} = k[|x, y|]/(g)$. Let $v = \deg(f)/p$, which is an integer because $x \nmid f$. We assume that $f - y^v \in xS$. Lastly, assume that there are infinitely many isoclasses of indecomposables in $\mathrm{CM}(R)$.

Now fix an ideal of R of the form $I = (x^m, y^n)$, where $1 \leqslant m < p - 1$ and $2 \leqslant n < q$. We will show that stable AR component containing $\hat{I}$ is a tube, by showing that $\mathrm{push}(\mathrm{push}(\hat{I}))$ has only two indecomposable summands, and applying Proposition 3.2.24. However, we will work over R:

Remark 5.3.5. Let C be a component of the stable AR quiver of R. Now consider the valued translation quiver C' obtained from C by identifying vertices x and y when they correspond to modules which are merely graded-shifts of one another. (We defined "graded-shift" above Definition 2.3.2.) By [3, Theorem 3], C' is naturally identified with a component of the stable AR quiver of $\hat{R}$. Therefore we might as well work over R, and just not try to keep track of the grading on M and the grading on $\mathrm{push}(M)$ simultaneously.

Notation 5.3.6. Let $\gamma = y^{q-1}f/x \in Q$. If $b \neq 0$, set $R' = S/(bx^p + y^q)S$; if $b = 0$, set $R' = S/yS \cong k[x]$. In either case, R' is a domain:

Lemma 5.3.7. *If $b \neq 0$, then $S/(bx^p + y^q)S$ is a domain.*

Proof. As S is factorial, it suffices to show $bx^p + y^q$ is irreducible. Since a product ss' fails to be homogeneous if either s or s' does, $bx^p + y^q$ is either irreducible or equal to a product of homogeneous nonunits. Let s and s' be homogeneous elements satisfying $ss' = bx^p + y^q$, and

s a nonunit. Then s has a term of the form αx^i for $\alpha \in k \setminus \{0\}$, so that $q | \deg s$. Likewise $p | \deg s$, and thus $\deg s = \deg(bx^p + y^q)$, hence $\deg s' = 0$, so s' is a unit. $\square$

We will use the following piece of arithmetic several times. We omit the easy proof.

Lemma 5.3.8. *If $b_1 < q$ and $b_2 < 0$, or if $b_1 < 0$ and $b_2 < p$, then $b_1 p + b_2 q \notin p\mathbb{N} + q\mathbb{N}$.*

Lemma 5.3.9. *Let (φ, ψ) be a reduced matrix factorization of g and such that each indecomposable direct summand of $\operatorname{cok} \varphi$ has rank, and char k does not divide any of these ranks. Let α be a matrix such that $\psi\alpha = \gamma\psi$ after passing to R. Then,* $\operatorname{push}(\operatorname{cok} \varphi) = \operatorname{cok} \begin{pmatrix} \varphi & -\alpha \\ 0 & \psi \end{pmatrix}$.

Proof. By 5.3.3 we only need to check that γ satisfies agrees with Notation 2.2.10, and the indecomposable summands of $\operatorname{cok} \varphi$ satisfy the hypotheses in Theorem 2.2.14. If $b \neq 0$ then $R' = S/(bx^p + y^q)S$ and we set $z = f$ (see Remark 2.2.9 and $\gamma' = y^{q-1}/x$, which lies in $\operatorname{Hom}_{R'}(\mathfrak{m}_{R'}, R')$. As $\deg(y^{q-1}/x) = p(q-1) - q \notin p\mathbb{N} + q\mathbb{N}$ by Lemma 5.3.8, we have γ' is a Frobenius element for R' by Proposition 2.3.4. So $\gamma = y^{q-1}f/x$ agrees with Notation 2.2.10. If $b = 0$, then $R' = S/yS$ and we set $z = y^{q-1}f$ and $\gamma' = 1/x \in (R' :_{Q'} \mathcal{J}(\overline{R'})) \setminus R'$. Again $\gamma = y^{q-1}f/x$ agrees with Notation 2.2.10. It only remains to note that $M \otimes_R Q'$ is a free Q'-module of rank equal to that of $M \otimes_R Q$, by Lemma 2.2.5. $\square$

In preparation for what immediately follows, let us observe that $g - y^{q+v} \in x^m S$. Indeed, we have by assumption $f - y^v \in xS$, and $\deg f = \deg(y^v) = pv$. So if $x^i y^j$ is a monomial occurring in $f - y^v$, then we have $i > 0$, and $qi + pj = pv$. Since $\gcd(p, q) = 1$, i is therefore a positive multiple of p; in particular, $i > m$. Thus, if $\equiv$ denotes congruence modulo x^m, we have $f - y^v \equiv 0$, and $g - y^{q+v} = (bx^p + y^q)f - y^{q+v} \equiv y^q(f - y^v) \equiv 0$.

Let

$$\varphi = \begin{pmatrix} (g - y^{q+v})/x^m & -y^n \\ y^{q+v-n} & x^m \end{pmatrix}, \text{ and } \psi = \begin{pmatrix} x^m & y^n \\ -y^{q+v-n} & (g - y^{q+v})/x^m \end{pmatrix}; \tag{5.3.3}$$

then $I \cong \operatorname{cok}\varphi$, and (φ, ψ) is a matrix factorization of g. Let

$$\alpha = \begin{pmatrix} 0 & -bx^{p-m-1}y^{n-1}f \\ x^{m-1}y^{q-n-1}f & 0 \end{pmatrix}, \tag{5.3.4}$$

and note that $\psi\alpha = \gamma\psi$ after passing to R. Therefore if we let $\xi = \begin{pmatrix} \varphi & -\alpha \\ 0 & \psi \end{pmatrix}$, it follows from Lemma 5.3.9 that $\operatorname{cok}\xi = \operatorname{push}(I)$.

By Remark 5.3.2, we can pick a matrix β, with entries in S, such that

$$\alpha\varphi = \varphi\beta. \tag{5.3.5}$$

In fact

$$\beta = \begin{pmatrix} y^{q-1}(f-y^v)/x & -x^{m-1}y^{n-1} \\ y^{q-n-1}(bx^{p-m-1}y^v f + (f-y^v)(g-y^{q+v})/x^{m+1}) & -y^{q-1}(f-y^v)/x \end{pmatrix}.$$

We will never need to refer to the actual entries of β, though we will use that β has no unit entries. By equation 5.3.5, the pair

$$(\xi, \eta) \text{ forms a matrix factorization of } g, \text{ where } \xi = \begin{pmatrix} \varphi & -\alpha \\ 0 & \psi \end{pmatrix}, \text{ and } \eta = \begin{pmatrix} \psi & \beta \\ 0 & \varphi \end{pmatrix}. \tag{5.3.6}$$

Furthermore, (ξ, η) is a reduced matrix factorization.

——Regarding all of these matrices, and all other matrices to be introduced in this section, we from now on abuse notation: we will always take the entries as living in R rather than S, unless stated otherwise**!!**——

The reader can check directly that $\alpha\varphi = -\gamma\varphi$. In other words,

$$\varphi\beta = -\gamma\varphi. \tag{5.3.7}$$

Definition 5.3.10. We choose a matrix W such that $\eta W = \gamma\eta$. Such W exists by 5.3.3. Let Z and Z' be 2-by-2 matrices such that $W = \begin{pmatrix} \alpha & Z' \\ 0 & -\beta + \psi Z \end{pmatrix}$.

We explain why W can be chosen to be of this form. To begin with, let W be an arbitrary matrix such that $\eta W = \gamma\eta$, and let A, B, C and D be 2-by-2 matrices such that $W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The equation $\begin{pmatrix} \psi & \beta \\ 0 & \varphi \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \gamma\psi & \gamma\beta \\ 0 & \gamma\varphi \end{pmatrix}$ implies $\varphi D = \gamma\varphi$, which equals $-\varphi\beta$ (equation 5.3.7). Therefore $\varphi(D + \beta) = 0$, and this implies $D + \beta = \psi Z$ for some matrix Z. That we may choose $\begin{pmatrix} A \\ C \end{pmatrix}$ to be $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ follows from the equation $\psi\alpha = \gamma\psi$.

Now, let θ denote the 8-by-8 matrix $\theta = \begin{pmatrix} \xi & -W \\ 0 & \eta \end{pmatrix}$. As $\operatorname{rank}(\operatorname{cok}\eta) = \operatorname{rank}(\operatorname{cok}\xi) = \operatorname{rank}(\operatorname{push}(I)) = 2$, Lemma 5.3.9 gives $\operatorname{cok}\theta = \operatorname{push}(\operatorname{cok}\xi) = \operatorname{push}(\operatorname{push}(I))$. By Proposition 3.2.24, in order to show the stable AR component containing I is a tube, it suffices to show that $\operatorname{cok}\theta = X \oplus Y \oplus F$, for some indecomposables X and Y and some possibly-zero free module F. It suffices to do this for $\operatorname{im}\theta$ instead of $\operatorname{cok}\theta$. The term "indecomposable" is unambiguous:

Lemma 5.3.11. [3, Lemma 1] *Given an indecomposable N in $L_p(R)$ (i.e., N has no proper graded direct summand), we have that $\hat{N}$ is indecomposable in $L_p(\hat{R})$. In particular, N is indecomposable as an R-module.*

We state the above discussion as a lemma.

Lemma 5.3.12. *In order to establish that the component of the AR quiver containing $\hat{I}$ is a tube, it suffices to show that $\operatorname{im}\theta = X \oplus Y$ for some graded modules X and Y each having no proper graded direct summand.*

We begin by multiplying θ on the left and on the right by invertible matrices. Let id denote the 2-by-2 identity matrix, and let $H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$. Let P' denote the 8-by-8 matrix

$$P' = \begin{pmatrix} 0 & \mathrm{id} & -\mathrm{id} & 0 \\ \mathrm{id} & 0 & 0 & 0 \\ 0 & H & H & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}, \text{ which is invertible with inverse } \begin{pmatrix} 0 & \mathrm{id} & 0 & 0 \\ \frac{1}{2}\mathrm{id} & 0 & \frac{1}{2}H^{-1} & 0 \\ -\frac{1}{2}\mathrm{id} & 0 & \frac{1}{2}H^{-1} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}; \text{ and}$$

let

$$P = \begin{pmatrix} 0 & \mathrm{id} & 0 & 0 \\ \frac{1}{2}\mathrm{id} & 0 & \mathrm{id} & 0 \\ -\frac{1}{2}\mathrm{id} & 0 & \mathrm{id} & -Z \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}, \text{ which is invertible with inverse } P^{-1} = \begin{pmatrix} 0 & \mathrm{id} & -\mathrm{id} & -Z \\ \mathrm{id} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\mathrm{id} & \frac{1}{2}\mathrm{id} & \frac{1}{2}Z \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

$$\text{Now } P'\theta = \begin{pmatrix} 0 & \mathrm{id} & -\mathrm{id} & 0 \\ \mathrm{id} & 0 & 0 & 0 \\ 0 & H & H & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \varphi & -\alpha & -\alpha & -Z' \\ 0 & \psi & 0 & \beta - \psi Z \\ 0 & 0 & \psi & \beta \\ 0 & 0 & 0 & \varphi \end{pmatrix} = \begin{pmatrix} 0 & \psi & -\psi & -\psi Z \\ \varphi & -\alpha & -\alpha & -Z' \\ 0 & H\psi & H\psi & H(2\beta - \psi Z) \\ 0 & 0 & 0 & \varphi \end{pmatrix},$$

and

$$P'\theta P = \begin{pmatrix} 0 & \psi & -\psi & -\psi Z \\ \varphi & -\alpha & -\alpha & -Z' \\ 0 & H\psi & H\psi & H(2\beta - \psi Z) \\ 0 & 0 & 0 & \varphi \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} & 0 & 0 \\ \frac{1}{2}\mathrm{id} & 0 & \mathrm{id} & 0 \\ -\frac{1}{2}\mathrm{id} & 0 & \mathrm{id} & -Z \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & \varphi & -2\alpha & \alpha Z - Z' \\ 0 & 0 & 2H\psi & 2H(\beta - \psi Z) \\ 0 & 0 & 0 & \varphi \end{pmatrix}.$$

Let c_j denote the j-th column of $P'\theta P$, $j = 1, ..., 8$. and let $M = \sum_{j=3}^{8} Rc_j$. It remains to show that M is an indecomposable module.

The module M is graded if we take the following degrees for its generators. We omit the slightly tedious justification.

$$\deg(c_3) = (v-n)p - mq,\ \deg(c_4) = -pq,\ \deg(c_5) = (v-n-1)p - q,$$

$$\deg(c_6) = (v-1)p - (m+1)q,\ \deg(c_7) = (2v-n-2)p - (m+2)q + pq,$$

$$\deg(c_8) = (v-2)p - 2q.$$

Assume $M = M' \oplus M''$ for some nonzero graded summands M' and M''; now, by Lemma 5.3.12, producing a contradiction will complete our overall argument. Note that $\deg c_4 < \deg c_j$ for all $j \geqslant 3$ different from 4. Since R is connected, it follows that c_4 lies in either M' or M''; let us assume $c_4 \in M'$. Let $\pi\colon M \oplus (Rc_1 + Rc_2) \to M''$ denote the projection map onto M'', with the goal of showing that $\pi = 0$. We have $\pi(c_1) = \pi(c_2) = \pi(c_4) = 0$. Also immediate is $\pi(c_3) = 0$ since $x^m c_3 = y^{q+v-n} c_4$ and x is a nonzerodivisor.

Let $r_3, ..., r_8 \in R$ be homogeneous elements such that $\sum_{j=3}^{8} r_j c_j = \pi(c_5)$ and $\deg(r_j) = \deg(c_5) - \deg(c_j)$. Then each of $\deg(r_6) = -np + mq$, $\deg(r_7) = (-v+1)p + (m+1-p)q$, and $\deg(r_8) = (-n+1)p + q$ does not lie in $\mathbb{N}p + \mathbb{N}q$ by Lemma 5.3.8, and so $r_6 = r_7 = r_8 = 0$.

For a brief moment let us consider matrices with entries in S. Namely let $\widetilde{W}$ denote a "lift to S" of the matrix W, and let $\widetilde{\theta}$ be the lift of θ, $\widetilde{\theta} = \begin{pmatrix} \xi & -\widetilde{W} \\ 0 & \eta \end{pmatrix}$. By the same reasoning used for the matrix factorization (ξ, η), we know that $\widetilde{\theta}$ is part of a matrix factorization $(\widetilde{\theta}, \widetilde{\theta}')$ where $\widetilde{\theta}' = \begin{pmatrix} \eta & \widetilde{W}' \\ 0 & \xi \end{pmatrix}$ for some 4-by-4 matrix $\widetilde{W}'$. Let $\theta' = \widetilde{\theta}' \otimes_S R$.

As $\theta\theta' = 0$, each column of matrix $P^{-1}\theta'$ is a syzygy relation for the columns of $P'\theta P$. We can compute that the last four entries of the column $P^{-1}\theta'_{\cdot,4}$ are, in order, $-\frac{1}{2}y^n, \frac{1}{2}x^m, 0, 0$. Therefore $\frac{1}{2}x^m c_6 \in \frac{1}{2}y^n c_5 + \sum_{j=1}^{4} Rc_j$. Then, $\pi(c_6) = \frac{y^n}{x^m}\pi(c_5) = \sum_{j=3}^{5}(y^n/x^m) r_j c_j$. In particular R must contain the fourth entry of this column: $\frac{y^n}{x^m}(r_3 y^{q+v-n} + r_4 x^m - 2r_5 x^{m-1} y^{q-n-1} f) \in R$. Therefore, since $y^{q+v}/x^m \in R$, we have $2r_5 y^{q-1} f/x \in R$. Since $r_5 \in k$ and we are assuming char $k \neq 2$, this implies that either $r_5 = 0$ or $y^{q-1}f/x \in R$. If the latter were true, then $rx = y^{q-1}f$ for some $r \in R$, and lifting r to a preimage $s \in S$ we would have $sx - y^{q-1}f \in gS$. But $sx - y^{q-1}f$ has nonzero y^{q+v-1}-term, whereas $\deg g = \deg y^{q+v} > \deg y^{q+v-1}$, so this is a contradiction. Hence $r_5 = 0$. Therefore $\pi(c_5) = r_3c_3 + r_4c_4 \in \ker(\pi)$, hence $\pi(c_5) = 0$ as π is idempotent; and $\pi(c_6) = (y^n/x^m)\pi(c_5) = 0$.

Now we simply repeat the argument in order to show that $\pi(c_8) = \pi(c_7) = 0$. For $r'_3, ..., r'_8 \in R$ homogeneous such that $\sum_{j=3}^{8} r'_j c_j = \pi(c_8)$ and $\deg(r'_j) = \deg(c_8) - \deg(c_j)$, each of $\deg(r'_5) = (n-1)p - q$, $\deg(r'_6) = -p + (m-1)q$, and $\deg(r'_7) = (-v+n)p + (m-p)q$

does not lie in $\mathbb{N}p + \mathbb{N}q$ by Lemma 5.3.8, so $r'_5 = r'_6 = r'_7 = 0$. The last two entries of $P^{-1}\theta'_{\cdot,7}$ are x^m and $-y^{q+v-n}$, so we obtain $x^m c_7 \in y^{q+v-n}c_8 + \sum_{j\leqslant 6} Rc_j$, and therefore $\pi(c_7) = (y^{q+v-n}/x^m)\pi(c_8) = (y^{q+v-n}/x^m)(r'_3c_3 + r'_4c_4 + r'_8c_8)$, whose fifth entry is

$-r'_8(y^{q+v-n}/x^m)W_{34}$. If $r'_8 = 0$ then $\pi(c_7) \in \ker\pi$ whence $0 = \pi(c_7) = \pi(c_8)$; so, showing $r'_8 = 0$ is the last step. If $r'_8 \neq 0$ then it is a unit, and therefore $(y^{q+v-n}/x^m)W_{34} \in R$. Then the lemma below would imply $y^{q+v-1}/x \in R$, and the reader can check that this is false.

Lemma 5.3.13. *W_{34}, the (3,4)-th entry of the matrix W, lies in $kx^{m-1}y^{n-1} \setminus \{0\}$.*

Proof. Recall that $\eta W = \gamma\eta$ by definition of W. As $\eta_{4,4} = x^m$, we get $\gamma x^m = \eta_{4,\cdot}W_{\cdot,4} = y^{q+v-n}W_{34} + x^mW_{44}$. As x is a nonzerodivisor and $\gamma \notin R$, we have $W_{34} \neq 0$. We naturally choose W so that $\deg\eta_{ij} + \deg W_{jj'} = \deg(\gamma\eta_{ij'})$ for each i, j, j'. Setting $i = 4$, $j = 3$, $j' = 4$, we have $\deg(W_{34}) = \deg(\gamma\eta_{4,4}) - \deg\eta_{4,3} = \deg(\gamma x^m) - \deg(y^{q+v-n}) = \deg(y^{q-1}fx^{m-1}) - \deg(y^{q+v-n}) = (n-1)p + (m-1)q$. Since p and q are coprime, it follows that $W_{34} \in kx^{m-1}y^{n-1}$. □

Thus $\pi = 0$, so that M is indecomposable and the given AR component is a tube by Lemma 5.3.12.

Chapter 6

Appendix

In this appendix we record some lemmas for reduced connected graded rings of dimension one. The following theorem is well-known.

Theorem 6.0.1. *Let B be one-dimensional, noetherian, local domain with integral closure $\overline{B}$ and $\mathfrak{m}_B$-adic completion $\hat{B}$. Then the following are equivalent.*

(1) $\hat{B}$ is a domain. ("B is analytically irreducible.")

(2) $\overline{B}$ is local and $\hat{B}$ is reduced. ("B is unibranched and analytically unramified.")

(3) $\overline{B}$ is local and finitely generated as a B-module.

Notation 6.0.2. If R is a connected graded ring, let $\hat{R}$ denote the completion of R with respect to its graded maximal ideal, $\mathfrak{m}$.

Lemma 6.0.3. *Let R be a reduced connected graded ring. Then:*

(1) The integral closure of R in $R[nonzerodivisors]^{-1}$ coincides with the integral closure of R in $Q = R[graded\ nonzerodivisors]^{-1}$, our definition of $\overline{R}$. Moreover, $\overline{R} = \bigoplus_{i\geqslant 0} \overline{R}_i$ is an $\mathbb{N}$-graded subring of Q.

(2) We have $\hat{R} = \prod_{i\geqslant 0} R_i$.

(3) The completion, $\hat{R}$, is also reduced. If R is a domain, then $\hat{R}$ is a domain.

(4) The integral closure, $\overline{R}$, is finitely generated as an R-module.

(5) The integral closure of the completion, $\overline{\hat{R}}$, is finitely generated as an $\hat{R}$-module.

Proof. Statement (1) is [17, Corollary 2.3.6]. Statement (2) can be checked by noting that $\{\mathfrak{m}^i\}_i$ is cofinal with $\{\bigoplus_{j\geqslant i} R_j\}_i$, and checking that the completion of R with respect to the latter filtration is isomorphic to $\prod_{i\geqslant 0} R_i$. From (2) we see that $\hat{R}$ is reduced, resp. a domain, if R is such. As R is a finitely generated algebra over the field R_0, (4) is a consequence of [22, Theorem 72]. The last assertion is a consequence of Theorem 6.0.1 (alternatively, it follows from (4)). □

Lemma 6.0.4. *Let D be a connected graded domain of dimension one, and let $\mathfrak{q} = \bigoplus_{i\geqslant 1} D_i$, and $\mathfrak{n} = \bigoplus_{i\geqslant 1} \overline{D}_i$. Then*

(a) $\overline{D}_0$ is a field, and

(b) $\prod_{i\geqslant 0} \overline{D}_i = \hat{\overline{D}}^{\mathfrak{n}} = \hat{\overline{D}}^{\mathfrak{q}} = \overline{\hat{D}}$.

Proof. The notation $\overline{D}_i$ means $(\overline{D})_i$, and makes sense due to Lemma 6.0.3, as does $\overline{\hat{D}}$. Since $\overline{D}$ is an $\mathbb{N}$-graded domain, $\mathfrak{n}$ is a prime ideal, and is thus maximal since $\dim \overline{D} = \dim D = 1$. So $\overline{D}_0$ is a field. Now $\prod_{i\geqslant 0} \overline{D}_i = \hat{\overline{D}}^{\mathfrak{n}}$ by Lemma 6.0.3. Note that $X_{\mathfrak{n}} \neq 0$ for all graded $\overline{D}$-modules $X \neq 0$. Now $\overline{D}_{\mathfrak{n}}/(\mathfrak{q}\overline{D}_{\mathfrak{n}})$ is an artinian local ring, so there exists $i \geqslant 1$ such that $((\mathfrak{n}^i + \mathfrak{q}\overline{D})/\mathfrak{q}\overline{D})_{\mathfrak{n}} = 0$, hence $(\mathfrak{n}^i + \mathfrak{q}\overline{D})/\mathfrak{q}\overline{D} = 0$. Thus $\{\mathfrak{n}^i\}_i$ and $\{\mathfrak{q}^i\overline{D}\}_i$ are cofinal, so $\hat{\overline{D}}^{\mathfrak{n}} = \hat{\overline{D}}^{\mathfrak{q}}$. Lastly we show $\hat{\overline{D}}^{\mathfrak{q}} = \overline{\hat{D}}$. Note that $\overline{D} \hookrightarrow \overline{\hat{D}}$, and since $\overline{\hat{D}}$ is complete by Lemma 6.0.3, we have $\overline{\hat{D}} \supseteq \hat{\overline{D}}^{q} \supseteq \hat{D}$. It remains to observe that $\hat{\overline{D}}^{q}$ is normal. But any I-adic completion of an excellent, normal ring, such as $\overline{D}$, is normal ([22, Theorem 79]). □

Lemma 6.0.5. *Let D be a connected graded domain of dimension one , and let $l = \min\{i > 0 | \overline{D}_i \neq 0\}$. Let t be any nonzero element of $\overline{D}_l$. Then $\overline{D} = \bigoplus_{i\geqslant 0} \overline{D}_0 t^i$ is the polynomial ring over the field $\overline{D}_0$ in the variable t; and $\hat{\overline{D}} = \prod_{i\geqslant 0} \overline{D}_i t^i$ is the power series ring.*

Proof. By the previous lemmas, $\overline{D}$ is connected graded, so we may assume $D = \overline{D}$ to improve notation. Then the previous lemma also shows that $\hat{D} = \prod_{i\geqslant 0} D_i$ is a normal domain. Thus it is a DVR; let $\pi \in \hat{D}$ be a uniformizing parameter. So $\pi\hat{D} = \prod_{i\geqslant 1} D_i$. Then $t = u\pi^i$ for some unit $u \in \prod_{i\geqslant 0} D_i$, and it follows that $i = 1$, hence t is a uniformizing parameter for $\hat{D}$. It follows that $D_i = 0$ for $i \notin \mathbb{N}l$, and $D_i = D_0 t^{i/l}$ for $i \in \mathbb{N}l$. The lemma follows. $\square$

Remark 6.0.6. Note that if D_0 is algebraically closed, Lemma 6.0.4 shows that D is just a semigroup ring $k[t^{i_1}, ..., t^{i_n}]$.

Lemma 6.0.7. *Let R be a reduced connected graded ring which is integrally closed. Then R is isomorphic to a polynomial ring over R_0.*

Proof. As $R = \overline{R} = \prod_{\mathfrak{p}} \overline{R/\mathfrak{p}}$, where $\mathfrak{p}$ ranges over the minimal primes of R, the assumption that R is connected implies that R has only one minimal prime, and therefore R is a domain (since we are assuming it is reduced). Now apply Lemma 6.0.5. $\square$

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Employment

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