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Lelong Numbers and Geometric Properties of Upper Level Sets of Currents on Projective Space

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ABSTRACT

Let T be a positive closed bidegree (p,p) current in $\mathbb{P}^n$. In this thesis, our goal is to understand more about the geometric properties of the sets of highly singular points of the current T. Lelong numbers will be the main tool used for determining how singular a point of a current is. For the first main result of this thesis, we let T be a positive closed current of bidimension $(1,1)$ with unit mass on the complex projective space $\mathbb{P}^2$. For $\alpha > 2/5$ and $\beta = (2-2\alpha)/3$ we show that if T has four points with Lelong number at least α, the upper level set $E^+_\beta(T)$ of points of T with Lelong number strictly larger than β is contained within a conic with the exception of at most one point.

Afterwards, we will let T be a positive closed current of bidimension (p,p) with unit mass on the complex projective space $\mathbb{P}^n$. Our aim here is to generalize some results of D. Coman as well as look at the result in the previous paragraph in a more generalized setting. For certain values of α and $\beta = \beta(p,\alpha)$ we show that if T has enough points where the Lelong number is at least α, then the upper level set $E^+_\beta(T)$ has certain geometric properties, in particular it will be contained in either a complex line L except for exactly p points of the upper level set that are not contained on the line, or the upper level set will be contained in a p-dimensional linear subspace.

Lelong Numbers and Geometric Properties of Upper Level Sets of Currents on Projective Space

by

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Chapter 1

Introduction

Our goal in this thesis is to investigate Lelong numbers of positive closed currents on projective space, and the geometric properties that the sets of highly singular points have. Before doing so, we must first motivate the problems, and even before that is the arduous slog through all of the necessary background information. We open with a discussion on the foundation upon which the main results of this thesis will be built. First and foremost is the notion of a plurisubharmonic function, which can be thought of as the higher dimensional analogy of a subharmonic function. Plurisubharmonic functions were birthed into the mathematical lexicon by Pierre Lelong and Kiyoshi Oka back in the early 1940's. These functions are the central object of study in pluripotential theory, and serve as the heart of our work, playing an important role as the common link between the various topics we will discuss. Stepping away from plurisubharmonic functions we will start to investigate the basics of currents, which are bidegree (p, q) differential forms with distribution coefficients and have an important connection to plurisubharmonic function, one such connection being that if u is psh then $dd^c u$ is a positive closed bidegree $(1, 1)$ current. A terse discussion on currents will lead us to the Monge-Ampère operator, that is the operator given by

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u,$$

which is a tool that will become pivotal for us when discussing the wedge products of currents. We will also use this operator in the definition of pluricomplex Green functions, i.e. functions that solve the Dirichlet problem for the Monge-Ampère equation on some domain Ω. That is, $(dd^c u)^n = 0$ on $\Omega \backslash S$, and has logarithmic poles in S.

After forming the foundation of our work, we will then recall some of the results of Jean

Pierre Demailly, Pierre Lelong, Yum-Tong Siu, John Erik Fornæss, Nessim Sibony, and Dan Coman. In this chapter we will start by first discussing what Lelong numbers are, laid out in the fashion originally presented to us by Pierre Lelong, and then generalized by Demailly. We can think of a Lelong number as a residue, a measurement of how singular a current is at a given point, specifically we have that the Lelong number of a positive closed bidegree (p, p) current T at the point a is given by

$$\lim_{r \to 0} \int_{\|z-a\| \leq r} T \wedge (dd^c \log \|z - a\|)^{n-p}.$$

We then investigate a plethora of properties of Lelong numbers, from some commonly known values of Lelong numbers (such as along the regular points of analytic varieties), to how Lelong numbers relate to the intersection number of varieties, a well known result of P. Thie, and theorems we can use to compare the values of Lelong numbers, which will allow us later to get bounds needed to prove some of the results in this thesis. We will also discuss generic Lelong numbers and a famous decomposition theorem of Siu, which gives us the structure of currents in terms of currents of integration along varieties and generic Lelong numbers. We then will investigate a theorem of Demailly that will allow us to consider smooth approximations of currents while retaining similar bounds on the Lelong numbers. Finally we close this third chapter by summarizing the works of Dan Coman that relate to the geometric properties of upper level sets (i.e., the sets of highly singular points of currents), and looking at some examples to see his theorems in action.

In chapter four, we take our look at the first main theorem of this thesis. We specifically work in $\mathbb{P}^2$, and we attempt to extend the work laid out by Coman by clearly establishing the

geometric properties of positive closed bidegree $(1,1)$ currents that have at least four points of "large" Lelong number, i.e., larger than $2/5$. We see that under these conditions we can find a conic that will contain (with possibly one exception) the upper level set E_β^+, where β is dependent on the smallest Lelong number for the four given points. After we prove this first result we will then work through some examples to demonstrate the sharpness of the assumptions of the main theorem of this section, by showing that the theorem fails if we have less than four points, and that the β value is sharp for this property.

This thesis will come to a close in chapter five, where we move our investigation of these geometric properties to $\mathbb{P}^n$. First we clean up some loose ends by generalizing a few results at the end of [2] from bidimension $(1,1)$ currents to bidimension (p,p). After that we look at some attempts to generalize the results from chapter four from $\mathbb{P}^2$ to $\mathbb{P}^n$. This unfortunately remains only a partial result, and we will close this thesis by discussing some of the obstacles in our way of making a succinct generalization of the main theorem from chapter four. We begin by starting with some preliminaries.

Chapter 2

Preliminaries

2.1 Plurisubharmonic Functions

We open by recalling the most basic of definitions:

Definition 2.1.1. [19][21] Let u be an upper semi-continuous function on a domain $\Omega \subset \mathbb{C}^n$ that is not identically $-\infty$ on any connected component of Ω. Then we say u is **subharmonic** if given $B(z, \rho) \subset \Omega$, we have

$$u(z) \leq \frac{1}{c_n} \int_{|\zeta|=1} u(z + r\zeta)\, d\sigma(\zeta)$$

for any $0 < r < \rho$ and where $c_n = \int_{|\zeta|=1} d\sigma(\zeta)$ is the Lebesgue measure on the sphere. If u is subharmonic on Ω we will write $u \in SH(\Omega)$.

Definition 2.1.2. [19] Let u be an upper semi-continuous function on a domain $\Omega \subset \mathbb{C}^n$ such that u is not identically $-\infty$ on any connected component of Ω. The function u is called **plurisubharmonic** (or **psh** for short) if for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $\zeta \to u(a + \zeta b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\zeta \in \mathbb{C} \mid a + \zeta b \in \Omega\}$.

In short, the above definition says that a function is plurisubharmonic if its restriction to any complex line L is subharmonic on $L \cap \Omega$. If u is plurisubharmonic on Ω we write $u \in PSH(\Omega)$. We will now review some basic properties of psh functions before moving on to how we intend to use them.

Example 2.1.3. Everyone's favorite plurisubharmonic function is surely $\log \|z\|$ for $z \in \mathbb{C}^n$. Some other easy to create psh functions would be $\log |f(z)|$ for any holomorphic function f.

Corollary 2.1.4. [19, Corollary 2.9.6] If Ω is an open subset of $\mathbb{C}^n$ then $PSH(\Omega) \subset SH(\Omega) \subset L^1_{\text{loc}}(\Omega)$.

Corollary 2.1.5. [19, Corollary 2.9.8] If $u, v \in SH(\Omega)$ and $u = v$ almost everywhere in Ω, then $u = v$ everywhere in Ω.

It is also well know that the maximum principle also applies to psh functions, i.e., we have the following

Corollary 2.1.6. [19, Corollary 2.9.9] If Ω is a bounded connected open subset of $\mathbb{C}^n$ and $u \in PSH(\Omega)$, then either u is constant or, for each $z \in \Omega$,

$$u(z) < \sup_{w \in \partial\Omega} \left\{ \limsup_{y \to w \; y \in \Omega} u(y) \right\}.$$

Proposition 2.1.7. [19, Proposition 2.9.23] If $u \in PSH(\mathbb{C}^n)$ and u is bounded above, then u is constant.

Theorem 2.1.8. [19, Theorem 2.9.14] Let Ω be an open subset of $\mathbb{C}^n$.

i) The family $PSH(\Omega)$ is a convex cone.

ii) If Ω is connected and $\{u_j\}_{j \in \mathbb{N}} \subset PSH(\Omega)$ is a decreasing sequence, then $u = \lim_{j \to \infty} u_j \in PSH(\Omega)$ or $u = -\infty$.

iii) If $u : \Omega \to \mathbb{R}$ and if $\{u_j\}_{j \in \mathbb{N}} \subset PSH(\Omega)$ converges to u uniformly on compact subsets of Ω, then $u \in PSH(\Omega)$.

iv) Let $\{u_\alpha\}_{\alpha \in A} \subset PSH(\Omega)$ be such that its upper envelope $u = \sup_{\alpha \in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization $u^* \in PSH(\Omega)$.

We also have means of constructing new plurisubharmonic functions out of old ones, as seen below.

Corollary 2.1.9. [19, Corollary 2.9.15] Let Ω be an open set in $\mathbb{C}^n$, and let ω be a non-empty proper open subset of Ω. If $u \in PSH(\Omega)$ and $v \in PSH(\omega)$, and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then the formula

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega \backslash \omega \end{cases}$$

defines a plurisubharmonic function in Ω.

Another useful result is the following, saying that if a psh function is bounded above off of some "small" set (we will address this in more rigor shortly), then the function can be extended across the small set to make a psh function on the whole domain. More precisely, we have the following.

Theorem 2.1.10. [19, Theorem 2.9.22] Let Ω be an open subset of $\mathbb{C}^n$ and let F be a closed subset of Ω of the form $F = \{z \in \Omega \mid v(z) = -\infty\}$ for some $v \in PSH(\Omega)$. If $u \in PSH(\Omega \backslash F)$ is bounded above, then the function $\tilde{u}$ defined by the formula

$$\tilde{u}(z) = \begin{cases} u(z) & z \in \Omega \backslash F \\ \limsup_{y \to z \; y \notin F} u(y) & z \in F \end{cases}$$

is plurisubharmonic in Ω.

We will now address the matter of the size of sets of the form F.

Definition 2.1.11. [19] A set $P \subset \mathbb{C}^n$ is said to be **pluripolar** (or **locally pluripolar**) if for each point $a \in P$, there is a neighborhood U_a of a and a function $v_a \in PSH(U_a)$ such that $P \cap U_a \subset \{z \in U_a \mid v_a(z) = -\infty\}$.

Corollary 2.1.12. [19, Corollary 2.9.10] Pluripolar sets have Lebsegue measure zero.

We call F in the above theorem a **complete pluripolar set** with $U_a := \Omega$ and $v_a = v$ for all $a \in P$, and as such is small in the sense that it has a zero Lebesgue measure. We now introduce a special type of "convexity". In doing so we first need to go through the following definition and theorem.

Definition 2.1.13. [16] Let δ be a continuous function on $\mathbb{C}^n$ such that

i) $\delta(z) \geq 0$ and $\delta(z) = 0$ iff $z = 0$

ii) $\delta(tz) = |t|\delta(z)$ if $t \in \mathbb{C}$

If $\Omega \subset \mathbb{C}^n$, $\Omega \neq \mathbb{C}^n$, we define the distance to the boundary by

$$d_\Omega(z) = \inf_{\zeta \in \mathbb{C}^n \setminus \Omega} \delta(z - \zeta),\ z \in \Omega.$$

Theorem 2.1.14. [16, Theorem 4.1.19] Let Ω be a domain in $\mathbb{C}^n$. Then the following are equivalent:

i) There is a plurisubharmonic function u in Ω, with $u \not\equiv -\infty$ in any component, such that

$$\{z \in \Omega; u(z) \leq t\} \subset\subset \Omega \text{ for every } t \in \mathbb{R}.$$

ii) If K is a compact subset of Ω then

$$\widehat{K} = \{z \in \Omega \mid u(z) \leq \sup_K u \text{ for all } u \in PSH(\Omega)\} \subset\subset \Omega.$$

iii) $z \to -\log d_\Omega(z)$ is a plurisubharmonic function in Ω for every distance function satisfying 2.1.13.

iv) $z \to -\log d_\Omega(z)$ is a plurisubharmonic function in Ω for some distance function satisfying 2.1.13.

Definition 2.1.15. [16, Definition 4.1.20] An open set $\Omega \subset \mathbb{C}^n$ is called **pseudo-convex** if the equivalent conditions of 2.1.14 are satisfied.

The importance of this notion of pseudo-convexity comes from the connection these domains have to domains of holomorphy. First we recall the definition of a domain of holomorphy.

Definition 2.1.16. [19, Definition 4.1.20] An open set $\Omega \subset \mathbb{C}^n$ is called a **domain of holomorphy** if there are no open sets Ω_1 and Ω_2 with the following properties:

i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$;

ii) Ω_2 is connected and $\Omega_2 \backslash \Omega \neq \emptyset$;

iii) for each $f \in \mathcal{O}(\Omega)$ there exists $\bar{f} \in \mathcal{O}(\Omega_2)$ such that $\bar{f}|\Omega_1 = f$.

It is a well known result of Oka, Bremermann, and Norguet that Ω is a domain of holomorphy if and only if Ω is pseudo-convex (see [16, Theorem 4.2.8]). So taking this in

connection with Theorem 2.1.14 part (i), we see that plurisubharmonic functions actually classify domains of holomorphy. Before leaving this topic of pseudo-convexity, we have one final definition that we will use in a later theorem:

Definition 2.1.17. [19] An open bounded set $\Omega \subset \mathbb{C}^n$ is called **hyperconvex** if it is connected and there is a continuous plurisubharmonic function $u_0 : \Omega \to (-\infty, 0)$ such that

$$\{z \in \Omega \mid u_0(z) < c\} \subset\subset \Omega$$

for each $c \in (-\infty, 0)$.

It is clear that every hyperconvex domain is pseudoconvex (see e.g., [11],[19]). However, the converse is not true, i.e., there are pseudoconvex domains that are not hyperconvex. An example can be found due to John Erik Fornæss in [11].

2.2 Currents

We now introduce currents. The subject of currents is a rich and detailed field, of which we will only give a terse treatment by covering the basic notions and getting right to the purpose they will serve in this thesis. We first let Ω be an open set in $\mathbb{C}^n$ and $\mathcal{D}_{p,q}(\Omega)$ be the $\mathcal{C}^\infty$ smooth forms of type (p, q) with compact support and the inductive limit topology (see e.g.[19], [16]).

Definition 2.2.1. [21, Definition 2.8] The elements of the dual space $\mathcal{D}'_{p,q}(\Omega)$ are the **currents** of bidimension (p, q).

If X is a complex manifold, then the same definition is used with X instead of Ω. In this thesis we will work only with currents that are in $\mathcal{D}'_{p,p}$, that is $p = q$.

Definition 2.2.2. [16, Definition 4.4.2] A current $T \in \mathcal{D}'_{n-p,n-p}(\Omega)$ is said to be a **positive** current if

$$\int_\Omega T \wedge \frac{i}{2}\lambda_{p+1} \wedge \bar{\lambda}_{p+1} \wedge \cdots \wedge \frac{i}{2}\lambda_n \wedge \bar{\lambda}_n \geq 0$$

for arbitrary $\lambda_{p+1} \ldots, \lambda_n \in \mathcal{D}_{1,0}(\Omega)$. We say a current is **closed** if $dT = 0$.

Recall the standard Kähler form β given by

$$\beta = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

and

$$\beta_p = \frac{1}{p!}\beta^p = \frac{1}{p!}\Big(\frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i\Big)^p,$$

and we now have the following.

Definition 2.2.3. [21, Definition 2.15] Let $T \in \mathcal{D}'_{p,p}(\Omega)$ and $\sigma_T = T \wedge \beta_p$. Then we call σ_T the **trace measure** of the current T.

The trace measure is useful as it bounds the coefficients of the current T [21, Theorem 2.16] and we will make use of the trace of a current in later topics. We note that we can write our currents as

$$T = \sum_{I,J} T_{I,J} dz^I \wedge d\bar{z}^J$$

where the $T_{I,J}$ are distributions, and I,J are increasing multi-indices with $|I| = |J| = n - p$. If $I = (I_1, I_2, \ldots, I_{n-p})$ then $dz^I = dz_{I_1} \wedge dz_{I_2} \wedge \cdots \wedge dz_{I_{n-p}}$. Thus T has bidegree $(n-p, n-p)$, which is the same as bidimension (p, p).

An interesting example would be as follows. Given a current $T \in \mathcal{D}'_{p,p}(\Omega)$ with L^1_{loc} coefficients, and a form $\varphi \in \mathcal{D}_{p,p}(\Omega)$, we have

$$\langle T, \varphi \rangle = \int_\Omega T \wedge \varphi$$

Proposition 2.2.4. [19, Proposition 3.3.4] Any positive current of bidegree (p, p) on $\Omega \subset \mathbb{C}^n$ has measure coefficients, i.e., the $T_{I,J}$ are measures.

We now look at another type of current we will make extensive use of in this thesis, the current of integration along a pure p-dimensional (where we use complex dimensions, so that is $2p$ real dimensions) analytic subvariety A of Ω. Recall pure p-dimensional means that the dimension of A is p at any point of A. Such a current T is denoted by $T = [A]$ and it behaves as follows:

$$\langle [A], \varphi \rangle = \int_{A_{\text{reg}}} \varphi$$

where A_{reg} is the set of regular points of A and $\varphi \in \mathcal{C}^\infty_{p,p}(\Omega)$, since A has dimension p, which means $[A]$ is a bidimension (p, p) (or bidegree $(n - p, n - p)$) current. We note that $[A]$ is always a positive current, and that $[A]$ is closed. We have the following results which will

make dealing with currents a bit easier.

Proposition 2.2.5. [16, Proposition 4.4.4] If T is the integration current on a p-dimensional analytic subvariety A of $\Omega \subset \mathbb{C}^n$, then the trace measure σ_T of T is the Euclidean surface measure on A.

Plurisubharmonic functions play an important role in the theory of currents, but first we should recall the exterior derivative $d = \partial + \bar{\partial}$. We will define $2\pi i d^c = (\partial - \bar{\partial})$ and then we get from these the operator dd^c (which will act on psh functions) given by

$$dd^c = \frac{-i}{2\pi}(\partial + \bar{\partial})(\partial - \bar{\partial}) = \frac{-i}{2\pi}(-2\partial\bar{\partial}) = \frac{i}{\pi}\partial\bar{\partial}$$

where choosing the $2\pi i$ coefficient will be a convenient choice as it clears out pesky 2π factors from our computations. With this knowledge, we now have a nice theorem that connects psh functions to currents.

Proposition 2.2.6. [19, Proposition 3.3.5] If $u \in PSH(\Omega)$, then $dd^c u$ is a closed positive bidegree $(1,1)$ current with measure coefficient.

A favorite current of any complex analyst would be $T = dd^c \log \|z - a\|$, or more generally $T = dd^c \log |f|$ for some holomorphic function f. On occasion, the converse is true, that is we can often represent a bidegree $(1,1)$ current locally as $dd^c u$, for some plurisubharmonic u. Specifically, we have the following, sometimes referred to as the dd^c theorem:

Theorem 2.2.7. [23, Theorem A.4.1] If T is a positive closed current of bidegree $(1,1)$, then for every $z_0 \in \Omega$ there exists a neighborhood U of z_0 and $u \in PSH(U)$ such that $T = dd^c u$ in U.

If we have that $u = \log|f|$ for some holomorphic function f, then we have the Lelong-Poincaré equation (see e.g. [9]) that states

$$dd^c \log|f| = [Z_f]$$

where Z_f is the zero set of f (and since f is holomorphic, Z_f is an analytic set).

2.3 Monge-Ampère Operator

Let T be a positive closed current, and we have the following:

Theorem 2.3.1. [9, Proposition 1.2] Let u be a locally bounded plurisubharmonic function. Then the wedge product $dd^c u \wedge T$ is a closed positive current and $dd^c u \wedge T = dd^c(uT)$.

Corollary 2.3.2. [9, Corollary 1.10] Let $u_1, , \ldots, u_q$ be locally bounded plurisubharmonic functions. Then the wedge product $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$ is symmetric with respect to the u_i, that is, we can interchange any u_i and u_j, where we inductively define $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q \wedge T)$.

Definition 2.3.3. [19] Let $u \in PSH(\Omega)$ be locally bounded. Then we define the operator

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u$$

to be the **generalized complex Monge-Ampère operator**, or just the **Monge-Ampère operator**. For any $k = 1, \ldots, n$, given $\{u_i\}_{i=1}^k \in PSH(\Omega)$ such that they are locally bounded, then we also call the operator

$$(u_1, \ldots, u_k) \to dd^c u_1 \wedge \cdots \wedge dd^c u_k$$

the Monge-Ampère operator.

The Monge-Ampère operator can give us a measure even if u_i are unbounded (e.g., see [7]). Of particular interest to us is using the Monge-Ampère operator in conjunction with closed positive currents. While the above results are a nice start, what we really want are some results that extend the above statements to unbounded plurisubharmonic functions. We let $u \in PSH(\Omega)$ and we let $L(u)$ be the set of points $x \in \Omega$ such that u is unbounded in every neighborhood of x. We call $L(u)$ the **unbounded locus** of u. We now have the following results that allow us to extend the above statements to unbounded psh functions provided that their unbounded loci are sufficiently small.

Corollary 2.3.4. [9, Corollary 2.10] Let T be a closed positive current of bidimension (p, p) and let u be a plurisubharmonic function on Ω such that $L(u) \cap \mathrm{Supp} T$ is contained in an analytic set of dimension at most $p - 1$. Then uT has locally finite mass, hence it is well defined and $dd^c u \wedge T = dd^c(uT)$.

Corollary 2.3.5. [9, Corollary 2.11] Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X such that $L(u_j)$ is contained in an analytic set $A_j \subset \Omega$ for every j. Then $dd^c u_1 \wedge \cdots \wedge dd^c u_q$ is well defined as soon as $A_{j_1} \cap \cdots \cap A_{j_m}$ has codimension at least m (or dimension at most $n - m$) for all choices of indices $j_1 < \cdots < j_m$ in $\{1 \ldots q\}$.

Example 2.3.6. From the previous theorems we see that $(dd^c \log \|z - a\|)^n$ is well defined and in fact we have

$$(dd^c \log \|z - a\|)^n = \delta_a$$

where δ_a is the Dirac point mass at a.

Proposition 2.3.7. [9, Proposition 2.12] Suppose that the divisors Z_j satisfy the above codimension condition and let (C_k) $k \geq 1$ be the irreducible components of the point set intersection $Z_1 \cap \cdots \cap Z_q$. Then there exist integers $m_k > 0$ such that $[Z_1] \wedge \cdots \wedge [Z_q] = \sum m_k [C_k]$. The number m_k is called the multiplicity of intersection of $Z_1, \ldots, Z_q$ along C_k.

Example 2.3.8. Let $[Z_1]$ and $[Z_2]$ both be 2-dimensional linear subspace such that $Z_1 \cap Z_2 = L$, where L is a complex line. Then

$$[Z_1] \wedge [Z_2] = [L]$$

We will make use of this property of intersections later when we start computing Lelong numbers.

2.4 Pluricomplex Green Functions

We now introduce a special type of plurisubharmonic function, the pluricomplex Green functions. Pluricomplex Green functions were introduced and studied in bounded domains in [8] , [18], [20], and [22]. Special cases were considered in [1] and [4]. To start, we let Ω be a connected domain in $\mathbb{C}^n$.

Definition 2.4.1. [18] We define the **pluricomplex Green function of Ω with a pole at** a to be

$$g_\Omega(z,a) = \sup\{u(z) \mid u \in PSH(\Omega,[-\infty,0)) \text{ and } u(z) \leq \log\|z-a\| + O(1) \text{ as } z \to a\}$$

We now have the following result of Demailly showing that this function satisfies the following properties:

Theorem 2.4.2. [8] [19, Theorem 6.3.6] If Ω is a bounded hyperconvex domain, then for any $a \in \Omega$, the function $u(z) = g_\Omega(z,a)$ is the unique function satisfying the following:

i) $u \in C(\Omega\backslash\{a\}) \cap PSH(\Omega)$

ii) $(dd^c u)^n = \delta_a$ in Ω where δ_a is the Dirac delta function at a.

iii) $u(z) = \log\|z-a\| + O(1)$ as $z \to a$

iv) $u(z) \to 0$ as $z \to \partial\Omega$

Now let $S = \{p_1,\ldots,p_k\} \subset \mathbb{C}^n$, and let $u \in PSH(\mathbb{C}^n) \cap L^\infty_{loc}(\mathbb{C}^n\backslash S)$ be such that $u = -\infty$ when restricted to S. Define γ_u as follows

$$\gamma_u := \limsup_{\|z\|\to+\infty} \frac{u(z)}{\log\|z\|}.$$

If γ_u is finite, we say u has logarithmic growth.

Definition 2.4.3. [4] If u is as above with γ_u finite and in addition u satisfies the Monge-Ampère equation $(dd^c u)^n = 0$ away from S, then u is an **entire pluricomplex Green function with logarithmic poles in** S.

If for $p_i \in S$ we have

$$u(z) - \gamma \log \|z - p_i\| = O(1) \text{ as } z \to p_i$$

then u has a logarithmic pole of weight γ at p_i. Further, let $\widetilde{E}(S) \subset PSH(\mathbb{C}^n) \cap L^{\infty}_{loc}(\mathbb{C}^n \backslash S)$ be the class of plurisubharmonic functions that have logarithmic poles of weight one at the points of S and logarithmic growth. We will use these in proving results in later chapters. We now end this section and move on to study some topics that will tie into the matter discussed in this chapter. In particular we will discuss the notion of Lelong numbers and upper level sets.

Chapter 3

Lelong Numbers and Upper Level Sets

3.1 Lelong Numbers

We will now introduce Lelong numbers as they were initially discussed in [21] by Pierre Lelong. To start, we let $\Omega \subset \mathbb{C}^n$ be an open set and X a complex manifold. Recall that $\mathcal{D}'_{p,p}(\Omega)$ are the bidimension (p,p) currents in Ω. We set

$$\alpha_a = dd^c \log \|z - a\|.$$

First we note that away from the singularity, this form is smooth and:

Proposition 3.1.1. [21, Proposition 2.21] $\alpha_a^n = 0$ in $\mathbb{C}^n \backslash \{a\}$.

Now given a positive closed current T of bidimension (p,p), we set

$$\nu_T^a = T \wedge \alpha_a^p,$$

and it follows from Corollary 2.3.4 combined with Theorem 2.3.1 that ν_T^a is a positive measure (note that when $p = 1$ it is follows directly, then proceed by induction). Once again we recall the standard Kähler form β given by

$$\beta = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

and

$$\beta_p = \frac{1}{p!}\beta^p = \frac{1}{p!}\Big(\frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i\Big)^p,$$

and now we have the following

Theorem 3.1.2. [21, Theorem 2.23] Let $T \in \mathcal{D}'_{p,p}(\Omega)$ be a positive closed current and suppose $a \in \Omega$. Let $\sigma_T(a,r) = \int_{\|z-a\|\leq r} T \wedge \beta_p$ for $r \leq d_\Omega(a)$ and let ν_T^a be as before. Then $r^{-2p}\sigma_T(a,r)$ is an increasing function of r for $r < d_\Omega(a)$ and

$$\lim_{r\to 0} \frac{\sigma_T(a,r)}{\pi^p r^{2p}/p!} = \nu_T(a)$$

exists and is non-negative.

Definition 3.1.3. The **Lelong number** of T at a is $\nu_T(a)$.

Demailly showed that

$$\int_{\|z-a\|\leq r} \nu_T^a = \frac{\sigma_T(a,r)}{\pi^p r^{2p}/p!},$$

hence, using the notation of Demailly, we have

$$\nu(T,a) := \nu_T(a) = \lim_{r\to 0} \int_{\|z-a\|\leq r} T \wedge (dd^c \log \|z-a\|)^p = T \wedge (dd^c \log \|z-a\|)^p(\{a\}).$$

Example 3.1.4. [9, Remark 3.9] Suppose $T = [A]$, the current of integration along an analytic subvariety, where A has pure dimension p (i.e. the dimension at anny point of A is p). Note that

$$\lim_{r\to 0} \sigma_T(a,r) = \lim_{r\to 0} \int_{\|z-a\|\leq r} T \wedge \beta_p = \lim_{r\to 0} \pi^p r^{2p}/p,$$

and then for $a \in A_{\text{reg}}$ we get,

$$\nu(T,a) = \lim_{r\to 0} \int T \wedge (dd^c \log \|z-a\|)^p = \lim_{r\to 0} \frac{\sigma_T(a,r)}{\pi^p r^{2p}/p!} = 1$$

.

We now look at some basic results for Lelong numbers. Let φ be a plurisubharmonic function such that e^φ is continuous. Let $B(r) = \{x \in X | \varphi(x) < r\}$. We now have the following definitions.

Definition 3.1.5. [9, Definition 3.3] We say that φ is **semi-exhaustive** if there exists a real number R such that$B_\varphi(R) \subset\subset X$. Similarly, φ is said to be semi-exhaustive on a closed subset $A \subset X$ if there exists R such that $A \cap B_\varphi(R) \subset\subset X$.

Definition 3.1.6. [9, Definition 3.4] If φ is semi-exhaustive on Supp$T$ and if R is such that $B_\varphi(R) \cap \text{Supp}T \subset\subset X$, we set for all $r \in (-\infty, R)$

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge (dd^c\varphi)^p,$$

$$\nu(T,\varphi) = \lim_{r\to-\infty} \nu(T,\varphi,r).$$

The number $\nu(T,\varphi)$ will be called the **generalized Lelong number** of T with respect to the weight φ.

We consider now the positive measure μ_r given by

$$\mu_r = (dd^c \max\{\varphi, r\})^n - \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n, \quad r \in (-\infty, R).$$

and is discussed in more detail in [9]. Our reason for caring about this measure is that it

gives us the following theorem, the famous Lelong-Jensen Formula, and this formula will give us an important connection between the Lelong numbers and logarithmic poles.

Theorem 3.1.7. [9, Lelong-Jensen Formula (4.5)] Let V be any plurisubharmonic function on X. Then V is μ_r-integrable for every $r \in (-\infty, R)$ and

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n = \int_{-\infty}^{r} \nu(dd^cV, \varphi, t)\, dt.$$

Remark 3.1.8. We now let $\varphi = \log\|z-a\|$, and using the Lelong-Jensen formula we get

$$\mu_r(V) = \int_{B(r)} V(dd^c \log\|z-a\|)^n + \int_{-\infty}^{r} \nu(dd^cV, \log\|z-a\|, t)\, dt$$

which gives us

$$\mu_r(V) = V(a) + \int_{-\infty}^{r} \nu(dd^cV, \log\|z-a\|, t)\, dt$$

from which we can deduce (see [9, Example 4.9]) the following: V has a logarithmic pole of weight γ at a if and only if the Lelong number of the current dd^cV at the point a is γ, i.e., $\nu(dd^cV, a) = \gamma$. So γ is the largest such value satisfying $V(z) \leq \gamma \log|z-a| + O(1)$ near a and by definition $\nu(V, a) = \nu(dd^cV, a)$.

Example 3.1.9. As a simple example, consider $V(z) = \gamma \log\|z\|$, and note

$$\nu(dd^c\gamma\log\|z\|, 0) = \lim_{r\to 0} \int_{\|z\|<r} dd^c\gamma\log\|z\| \wedge (dd^c\log\|z\|)^{n-1} = \gamma \lim_{r\to 0} \int_{\|z\|<r} (dd^c\log\|z\|)^n = \gamma$$

Thus we see that the Lelong number is the weight of the logarithmic pole.

We now have the following theorem due to P. Thie [25, Theorem 5.1], but we will use Demailly's notation to remain consistent.

Theorem 3.1.10. [9, Theorem 5.8] Let A be an analytic set of dimension p in a complex manifold of dimension n. For every point $x \in A$, there exist local coordinates $z = (z', z'')$, $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ centered at x and balls $B' \subset \mathbb{C}^n$, $B'' \subset \mathbb{C}^{n-p}$ of radii r' and r'' in these coordinates, such that $A \cap (B' \times B'')$ is contained in the cone $|z''| \leq \frac{r''}{r'}|z'|$. The **multiplicity** of A at x is defined as the number m of sheets of any such ramified covering map $A \cap (B' \times B'') \to B'$. Then $\nu([A], x) = m$.

In particular we have that $\nu([A], x)$ will always be an integer. We now want to define the mass of a positive closed current T on $\mathbb{P}^n$. First we recall the Fubini-Study form ω on $\mathbb{P}^n$. To start we let $(z, t) \in \mathbb{C}^n \times \mathbb{C}\backslash\{(0,0)\}$, and consider the canonical projection

$$\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n.$$

So we have $\mathbb{C}^n = \{[1 : z_1 : \cdots : z_n] \in \mathbb{P}^n\}$, and the Fubini-Study form is given by

$$\pi^*\omega = dd^c \log \sqrt{|t|^2 + \|z\|^2},$$

and when restricted to $\mathbb{C}^n$,

$$\omega|_{\mathbb{C}^n} = dd^c \log \sqrt{1 + \|z\|^2},\ z \in \mathbb{C}^n,$$

and satisfies

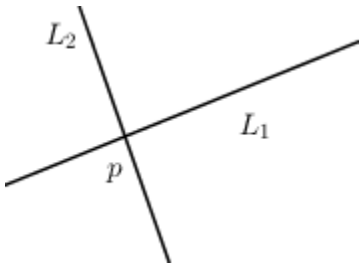
$$\int_{\mathbb{P}^n} \omega^n = \int_{\mathbb{C}^n} (dd^c \log \sqrt{1 + \|z\|^2})^n = 1.$$

The mass of a positive closed bidimension (p, p) current T is given by

$$\|T\| = \int_{\mathbb{P}^n} T \wedge \omega^p.$$

We now look at an example to see that these concepts can be relatively simple in practice.

Example 3.1.11. Let L_1 and L_2 be complex lines, p the point at which they intersect, and consider the current $T = [L_1] + [L_2]$. We will calculate $\|T\|$ and some Lelong numbers. Let $x_1 \in L_1$, and $x_2 \notin L_1 \cup L_2$.

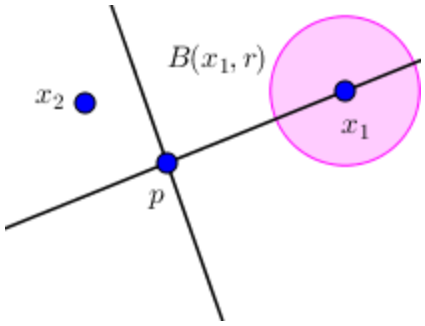


Computing the Lelong numbers at x_1 and x_2 we get:

$$\nu(T, x_1) = \lim_{r \to 0} \frac{\sigma_T(B(x_1, r))}{\pi r^2} = \lim_{r \to 0} \frac{\pi r^2}{\pi r^2} = 1$$

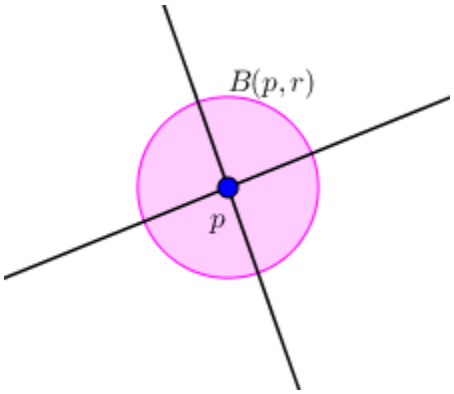
and

$$\nu(T, x_2) = \lim_{r \to 0} \frac{\sigma_T(B(x_2, r))}{\pi r^2} = \lim_{r \to 0} \frac{0}{\pi r^2} = 0$$



If we look look at the point of intersection p, we notice that any ball will intersect both lines, and we get:

$$\nu(T, p) = \lim_{r \to 0} \frac{\sigma_T(B(p, r))}{\pi r^2} = \lim_{r \to 0} \frac{\pi r^2 + \pi r^2}{\pi r^2} = 2$$



Alternatively we could just note that $\nu(T, p) = \nu([L_1], p) + \nu([L_2], p) = 2$ by linearity. Finally we compute $\|T\|$:

$$\|T\| = \int_{\mathbb{P}^2} ([L_1] + [L_2]) \wedge \omega = \int_{\mathbb{P}^2} [L_1] \wedge \omega + \int_{\mathbb{P}^2} [L_2] \wedge \omega = 1 + 1 = 2.$$

The following theorem of Fornæss and Sibony will help us compute masses of wedges of currents.

Theorem 3.1.12. [12, Theorem 4.4] Let T be a positive closed current of bidimension (p, p)

on $\mathbb{P}^k$. Let $R_1, \ldots, R_q$ be positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. Assume that $T \wedge R_1 \wedge \cdots \wedge R_q$ is well defined. Then

$$\|T \wedge R_1 \wedge \cdots \wedge R_q\| = \|T\|\|R_1\| \ldots \|R_q\|.$$

In particular $T \wedge R_1 \wedge \cdots \wedge R_q$ is non-zero and $\operatorname{supp}(T) \cap \operatorname{supp}(R_1) \cap \cdots \cap \operatorname{supp}(R_q) \neq \emptyset$.

We now look at a comparison theorem for Lelong numbers of which we will make use of in the proof of the main results of this thesis.

Theorem 3.1.13. [9, Corollary 5.10] If $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$ is well defined, then at every point $x \in X$ we have

$$\nu(dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T, x) \geq \nu(dd^c u_1, x) \ldots \nu(dd^c u_q, x)\nu(T, x).$$

Remark 3.1.14. Lelong numbers of plurisubharmonic functions (see Remark 3.1.8) may increase by restrictions to smaller spaces. More specifically if S is a p dimensional linear subspace of X and $a \in S$, then

$$\nu(dd^c V|_S, a) \geq \nu(dd^c V, a).$$

This follows since the Lelong number of V at a is the largest γ satisfying $V(z) \leq \gamma \log|z - a| + O(1)$, and when restricted to a smaller space, that inequality remains true.

Example 3.1.15. To see that we can actually get a larger Lelong number by restriction, consider the function $\varphi(z,w) = \max\{\log|z|, 2\log|w|\}$, $(z,w) \in \mathbb{C}^2$. Then we note by remark 3.1.8 that $\nu(\varphi, (0,0)) = 1$, however if $S = \{(0,w) \in \mathbb{C}^2\}$, then $\nu(\varphi|_S, (0,0)) = 2$.

Definition 3.1.16. If T is a closed positive current of bidimesion (p,p) on a complex manifold X, we call $E_c(T) = \{x \in X | \nu(T,x) \geq c\}$, $c > 0$ an **upper level set**.

Theorem 3.1.17. [24][9] If T is a closed positive current of bidimesion (p,p) on a complex manifold X, the upper level sets $E_c(T) = \{x \in X | \nu(T,x) \geq c\}$, $c > 0$, are analytic subsets of dimension $\leq p$.

Definition 3.1.18. If T is a closed positive current of bidimension (p,p) and A is an irreducible analytic set in X, we set

$$m_A = \inf\{\nu(T,x) | x \in A\}.$$

We call m_A the **generic Lelong number** of T along A.

Theorem 3.1.19. [9] If T is a closed positive current of bidimension (p,p) and A is an irreducible analytic set in X, then $\nu(T,x) = m_A$ for all $x \in A \backslash (\cup A_j')$, where $\{A_j'\}$ is a countable family of proper analytic subsets of A.

Proposition 3.1.20. [9, Proposition 6.18] Let T be a closed positive current of bidimension (p,p) and let A be an irreducible p-dimensional analytic subset of X. Then $\mathbb{1}_A T = m_A[A]$, in particular $T - m_A[A]$ is positive.

This brings us now to a very nice theorem of Siu's that looks at the structure of currents.

Theorem 3.1.21. [24][9] If T is a closed positive current of bidimension (p,p), then there is a unique decomposition of T as a (possibly finite) weakly convergent series

$$T = \sum_{j \geq 1} \lambda_j [A_j] + R, \quad \lambda_j > 0,$$

where $[A_j]$ is the current of integration over an irreducible $p-$dimensional analytic set $A_j \subset X$ and where R is a closed positive current with the property that $\dim E_c(R) < p$ for every $c > 0$.

We now look at a simple example to see this decomposition in action, as well as generic Lelong numbers and upper level sets.

Example 3.1.22. Let $\varphi = \log|z_1^2 z_2^3|$, and consider the bidegree $(1,1)$ current $T = dd^c\varphi$ on $\mathbb{C}^2$. We denote by A_1 the set $\{(0, z_2) \in \mathbb{C}^2\}$ and by A_2 the set $\{(z_1, 0) \in \mathbb{C}^2\}$. So we have that

$$T = dd^c\varphi = 2dd^c \log|z_1| + 3dd^c \log|z_2| = 2[A_1] + 3[A_2]$$

where the last equality comes from the Lelong-Poincaré equation. We have that the generic Lelong number along A_1 is $m_{L_1} = 2$ and that $m_{A_2} = 3$. We note that $\nu(T, x) = 2$ for every $x \in A_1$ with $x \neq 0$, and $\nu(T, 0) = 5$. Considering some of the upper level sets, if $c = 3$ then $E_3(T) = A_2$ and $\dim E_3(T) = 1$. If $c = 4$, then $E_4(T) = \{0\}$ and $\dim E_4(T) = 0$.

3.2 Regularization

In this short section we will look at a pivotal result of J.P. Demailly that helps us approximate a current with a better behaved current who's Lelong numbers approximate the ones of the current being approximated. This is a critical tool which will allow us to do some computations and get bounds on Lelong numbers. We let X be a compact complex manifold, but before we can dive into this result, we must introduce the following:

Definition 3.2.1. We say a function is **quasiplurisubharmonic** if it is locally the sum of

a plurisubharmonic functions and a smooth function.

Definition 3.2.2. [13],[3] Consider the set

$$PSH(X,\omega) := \{\varphi \text{ is quasiplurisubharmonic} \mid dd^c\varphi \geq -\omega\},$$

where ω is a closed real bidegree $(1,1)$ form. If $\varphi \in PSH(X,\omega)$ then we call φ **ω-plurisubharmonic** (or **ω-psh**).

Theorem 3.2.3. [6, Proposition 3.7] Let ψ be an ω-psh function on a compact complex manifold X such that $\frac{i}{\pi}\partial\bar{\partial}\psi \geq \gamma$ for some continuous $(1,1)$-form γ. Then there is a sequence of ω-psh functions ψ_m such that ψ_m has the same singularities as a logarithm of a sum of squares of holomorphic functions and

i) $\psi < \psi_m \leq \sup_{|\zeta - x|<r}\psi(\zeta) + C(\frac{|\log r|}{m} + r + m^{-1/2})$ with respect to coordinate open sets covering X. In particular, ψ_m converges to ψ pointwise and in $L^1(X)$ and

ii) $\nu(\psi,x) - \frac{n}{m} \leq \nu(\psi_m,x) \leq \nu(\psi,x)$ for every $x \in X$;

iii) $\frac{i}{\pi}\partial\bar{\partial}\psi_m \geq \gamma - \epsilon_m\omega$ with $\epsilon_m > 0$ decreasing to 0.

We say that the functions ψ_m have analytic singularities. Furthermore, we say that a positive closed bidegree $(1,1)$ current R has analytic singularities if $R = dd^c\varphi$ where φ has analytic singularities and R is smooth wherever R has generic Lelong number 0. We will also make use of the following proposition which follows from Demailly's regularization theorem 3.2.3.

Proposition 3.2.4. Let R be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^n$, $\nu(R, x_i) > a_i$, $i = 1, \ldots, N$ for $x_i \in \mathbb{P}^n$ and $a_i > 0$. Then there exists a positive closed bidegree $(1,1)$ current R' on $\mathbb{P}^n$ with analytic singularities such that $\|R'\| = \|R\|$, $\nu(R', x_i) > a_i$ for $i = 1, \ldots, N$, and $\nu(R', x) \leq \nu(R, x)$ for all $x \in \mathbb{P}^n$. In particular, R' is smooth in a neighborhood of every point where R has 0 Lelong number.

Proof. By the dd^c Theorem (Theorem 2.2.7), we can write $R = c\omega + dd^c\psi$, where $c = \|R\|$, for some $c\omega$-psh function ψ. By Demailly's regularization theorem 3.2.3, there exists a sequence of quasi-psh functions $\{\psi_m\}$ and $\epsilon_m \searrow 0$ such that $\{\psi_m\}$ have analytic singularities, $\nu(\psi, x) - \epsilon_m \leq \nu(\psi_m, x) \leq \nu(\psi, x) = \nu(R, x)$ and we have currents $R_m = (c + \epsilon_m)\omega + dd^c\psi_m$, which are positive by part (iii) of Demailly's result. Let $\eta_m := \|R\|/\|R_m\|$, and then since $\epsilon_m \searrow 0$ as $m \to \infty$, we have that $\eta_m \nearrow 1$ and $\nu(\psi_m, x) = \nu(R_m, x) \to \nu(R, x)$ from below. Since $\nu(R, x_i) > a_i$ we can find M such that for all $m > M$, $\nu(R_m, x_i) > a_i$. In particular we can find k large enough such that $\eta_k \nu(R_k, x_i) > a_i$ for $i = 1, \ldots, N$. Let $R' := \eta_k R_k$, then $\|R'\| = \|R\|$, $\nu(R', x_i) > a_i$ for $i = 1, \ldots, N$ and R' is smooth everywhere that R has Lelong number 0.

□

3.3 Geometric Properties

Let T be a positive closed current of bidimension (p, p) on $\mathbb{P}^n$ which has mass $\|T\| = 1$, where

$$\|T\| := \int_{\mathbb{P}^n} T \wedge \omega_n^p$$

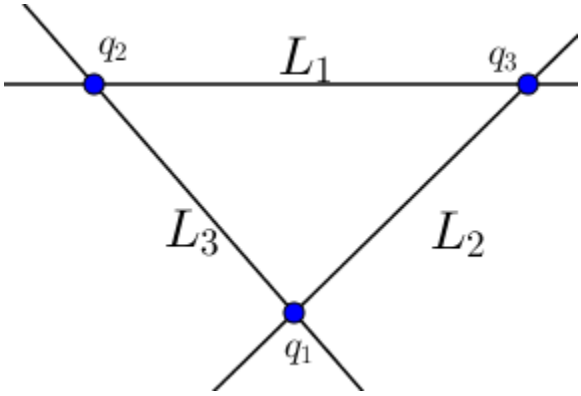
and ω_n is the Fubini-Study form on $\mathbb{P}^n$. We consider the following upper level sets of Lelong numbers $\nu(T,q)$ of the current T

$$E_\alpha(T) := \{q \in \mathbb{P}^n \,|\, \nu(T,q) \geq \alpha\},$$
$$E^+_\alpha(T) := \{q \in \mathbb{P}^n \,|\, \nu(T,q) > \alpha\}.$$

It has been shown by Siu [24] that $E_\alpha(T)$ is an analytic subvariety of dimension at most p when $\alpha > 0$. Our goal is to gain more understanding of the geometric properties of these upper level sets, and we start by first looking over some of the results proven by Coman.

Theorem 3.3.1. [2, Theorem 1.1] Let T be a positive closed current of bidimension $(1,1)$ in $\mathbb{P}^n$. If $\alpha \geq \frac{1}{2}$ then there exists a line L such that $|E^+_\alpha(T)\backslash L| \leq 1$. Moreover, if $\alpha \geq 2/3$ then $E^+_\alpha(T) \subset L$.

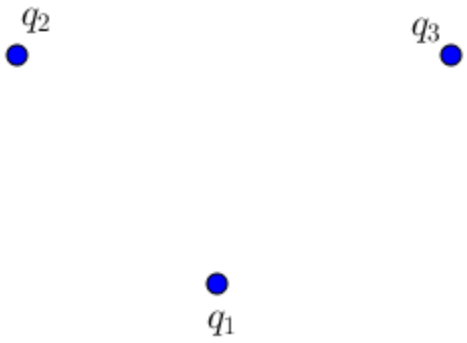
Example 3.3.2. For an example, consider three complex lines L_i, $i = 1,2,3$ in $\mathbb{P}^n$ and let $L_2 \cap L_3 = \{q_1\}$, $L_1 \cap L_3 = \{q_2\}$, and $L_1 \cap L_2 = \{q_3\}$.



Now consider the following current:

$$T = \frac{1}{3}\sum_{i=1}^{3}[L_i]$$

and we see that $\nu(T, q_i) = \frac{2}{3}$ for $i = 1, 2, 3$ and $\nu(T, x) = \frac{1}{3}$ for $x \in L_i$, $x \neq q_j$. Note that for $\alpha \geq \frac{2}{3}$, $E^+_\alpha(T) = \emptyset$, and if $\alpha \in [\frac{1}{2}, \frac{2}{3})$, then $E^+_\alpha(T) = \{p_1, p_2, p_3\}$.



For $\alpha \in [\frac{1}{2}, \frac{2}{3})$, then $|E^+_\alpha(T) \backslash L_i| = 1$ for all i. Thus allowing for the omission of one point of the upper level set is necessary.

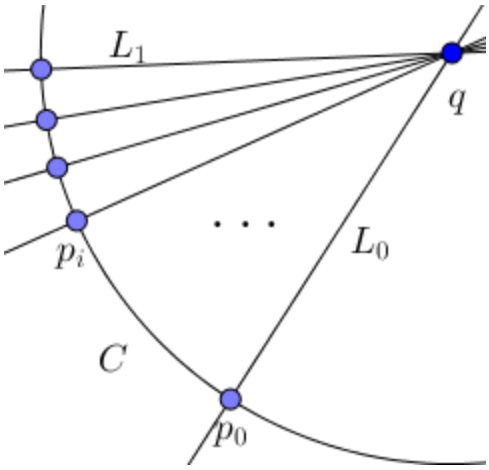
The next theorem Coman proves shows that we can contain the upper level set of a smaller α value in a degree two curve.

Theorem 3.3.3. [2, Theorem 1.2] Let T be a positive closed current of bidimension $(1, 1)$ in $\mathbb{P}^2$. If $\alpha \geq \frac{2}{5}$ then there exists a conic C (possibly reducible) such that $|E^+_\alpha(T) \backslash C| \leq 1$.

It is interesting to note that this theorem requires us to be in $\mathbb{P}^2$ specifically, as opposed to $\mathbb{P}^n$. The proof of the theorem relies on the fact that in $\mathbb{P}^2$, bidimension $(1, 1)$ is the same as bidegree $(1, 1)$, which is not the case in $\mathbb{P}^n$ for $n > 2$.

Example 3.3.4. [2, Example 3.9]

Let $C \subset \mathbb{P}^2$ be a conic and $\{p_i\}_{i=1}^\infty$ be points on C converging to some p_0 on C. Let q be a point off C and L_i the complex line containing q and p_i.



Let ϵ_i be such that $\sum_{i=0}^{\infty} \epsilon_i = \frac{1}{5}$. Consider the current:

$$T = \frac{2}{5}[C] + \sum_{i=0}^{\infty} \epsilon_i [L_i]$$

and observe that $\nu(T, q_i) > \frac{2}{5}$, and $E^+_{2/5}(T) = \{p_i\}_{i=0}^{\infty} \subset C$.

Coman then uses 3.3.1 to show that if we have two points where the current T has a large enough Lelong number, then we can contain a larger upper level set in a complex line. In particular, we have the following.

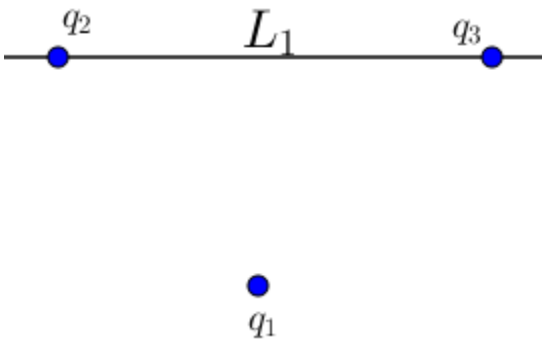
Theorem 3.3.5. [2, Theorem 3.10] Let T be a positive closed current of bidimension $(1,1)$ in $\mathbb{P}^n$. Assume that $\alpha > 1/2$ and there are points $q_1, q_2 \in \mathbb{P}^n$ so that $\nu(T, q_j) \geq \alpha$, $j = 1, 2$. If $\beta = (2-\alpha)/3$, then $|E^+_\beta(T) \backslash L| \leq 1$ for some complex line L.

We now look at an example to show that there do in fact exist situations in which $E^+_\alpha(T) \subset E^+_\beta(T)$.

Example 3.3.6. Using L_i and q_i as in 3.3.2, consider the current:

$$T = \frac{1}{2}[L_1] + \frac{1}{4}[L_2] + \frac{1}{4}[L_3]$$

Computing the Lelong Numbers, we see that $\nu(T, q_i) = \frac{3}{4}$ for $i = 2, 3$ and $\nu(T, q_1) = \frac{1}{2}$. If $\alpha \in (\frac{1}{2}, \frac{3}{4})$, then $\beta = (2 - \alpha)/3 < \frac{1}{2}$, so $E^+_\beta(T) = \{L_1 \cup \{q_1\}\}$



and observe $E^+_\alpha(T) = \{q_2, q_3\} \subset E^+_\beta(T)$, showing the containment is proper, and $|E^+_\beta(T) \backslash L_1| = 1$, satisfying the conclusion of the theorem.

In [5], Coman then generalizes some of the above results to bidimension (p, p) currents on $\mathbb{P}^n$, and we get the following.

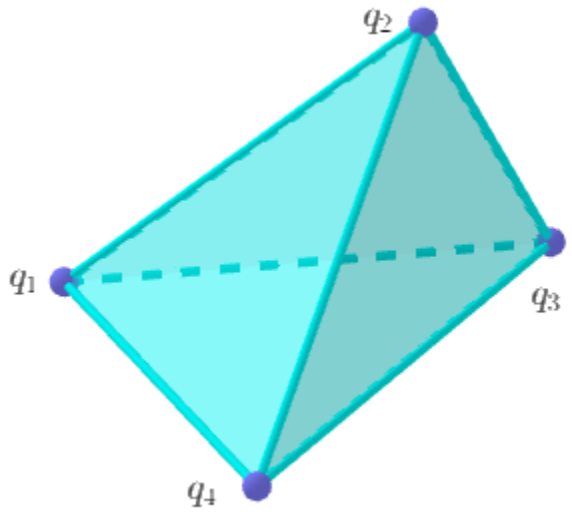
Theorem 3.3.7. [5, Theorem 1.2] If T is a positive closed current of bidimension (p, p) on $\mathbb{P}^n$, $0 < p < n$, with $\|T\| = 1$, then the set $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ is either contained in a p-dimensional linear subspace of $\mathbb{P}^n$ or else it is a finite set and $|E^+_{p/(p+1)}(T, \mathbb{P}^n) \backslash L| = p$ for some line L.

Theorem 3.3.8. [5, Theorem 1.3] Let T be a positive closed current of bidimension (p, p) on $\mathbb{P}^n$ such that $1 < p < n$, $\|T\| = 1$, and the set $E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n)$ is not contained in a $p-$dimensional linear subspace of $\mathbb{P}^n$. If $W = \text{Span } (E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n))$, then dim W = $p+1$ and there exist plane conics $C_j \subset W$ and points $z_j \in W$, $1 \leq j \leq N_p$, where $N_p = \binom{p+2}{3}$, such that z_j lies in the plane containing C_j and

$$E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n) \subset C_1 \cup \cdots \cup C_{N_p} \cup \{z_1, \ldots, z_{N_p}\}.$$

We will close this section by looking at an one last example, of 3.3.7.

Example 3.3.9. Let $q_1, q_2, q_3, q_4 \in \mathbb{P}^n$, $n > 2$, be linearly independent points. A_i, $i = 1, 2, 3, 4$ be 2-dimensional linear subspace of $\mathbb{P}^n$ such that $\{q_i\}_{j=1, j\neq i} \subset A_i$.



Consider the bidimension $(2, 2)$ current T given by

$$T = \frac{1}{4}\sum_{i=1}^{4}[A_i]$$

and we see that $\nu(T, q_i) = \frac{3}{4}$, and for $x \neq q_i$, $\nu(T, x) \leq \frac{1}{2}$. Thus since $p = 2$, $p/(p+1) = 2/3$, and we see that $E^+_{2/3}(T) = \{q_1, q_2, q_3, q_4\}$. Since the q_i are in general position, they cannot all be contained in a 2-dimensional linear subspace, and any line L will only be able to contain two of these points. But note for any such line that is containing two of these points, call it L, that $|E^+_{2/3}(T)\backslash L| = 2 = p$.

We have now covered quite a substantial amount of background information! With all of this in mind we are now ready to advance on the main results of this thesis containing in the following two chapters. Our first result lies in 2-dimensional complex projective space.

Chapter 4

Properties of Bidegree $(1,1)$ Currents on $\mathbb{P}^2$

4.1 Introduction

Let T be a positive closed current of bidimension $(1,1)$ in $\mathbb{P}^2$ with unit mass. Our goal is to establish a result analogous to Coman's result 3.3.5 for conics, i.e. to find β in terms of α such that given a few points in $E_\alpha(T)$, we can find a conic that either contains $E_\beta^+(T)$ or at most omits one point of $E_\beta^+(T)$. The results of this chapter are contained within [14]. Coman showed that we needed two points of "large" Lelong number in his result, and that it fails if we have less than two such points. Since two points uniquely define a complex line, one may suspect initially that we would need five points in general position with "large" Lelong number to make an analogous result for conics, as five points in general position define a unique conic. However it turns out that we only need four such points, and that the four points can be in any position. Specifically, we want to prove the following:

Theorem 4.1.1. [14, Theorem 1.1] Let T be a positive closed current of bidimension $(1,1)$ in $\mathbb{P}^2$, $\|T\| = 1$, $\alpha > 2/5$ and $\beta = \frac{2}{3}(1-\alpha)$. Let $\{q_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that $\nu(T, q_i) \geq \alpha$. Then there exists a conic C (possibly reducible) such that $|E_\beta^+(T) \backslash C| \leq 1$.

After proving this, we will look at several examples to establish that each assumption is necessary, and that β is sharp for this property. We will also need to use entire pluricomplex Green functions that we covered in the second chapter in the upcoming result, but for convenience let us recall the definition. Let $S = \{p_1, \ldots, p_k\} \subset \mathbb{C}^n$, and let $u \in PSH(\mathbb{C}^n) \cap L^\infty_{loc}(\mathbb{C}^n \backslash S)$ be such that $u = -\infty$ when restricted to S. Define γ_u as follows

$$\gamma_u := \limsup_{\|z\| \to +\infty} \frac{u(z)}{\log \|z\|}.$$

If γ_u is finite, we say u has logarithmic growth. If in addition u satisfies the Monge-Ampère equation $(dd^c u)^n = 0$ away from S, then u is an entire pluricomplex Green function. If for $p_i \in S$ we have

$$u(z) - \alpha \log \|z - p_i\| = O(1) \text{ as } z \to p_i$$

then u has a logarithmic pole of weight α at p_i. With this information, we have the following two propositions by Coman that we will need:

Proposition 4.1.2. [2, Proposition 2.1] Let $S = \{p_1, \ldots, p_k\} \subset \mathbb{C}^n$ and let T be a positive closed current of bidimension (l, l) on $\mathbb{P}^n$. If $u \in PSH(\mathbb{C}^n)$ has logarithmic growth, it is locally bounded outside a finite set, and $u(z) \leq \alpha_i \log \|z - p_i\| + O(1)$ for z near p_i, where $\alpha_i > 0$, $1 \leq i \leq k$, then

$$\sum_{i=1}^{k} \alpha_i^l \nu(T, p_i) \leq \gamma_u^l \|T\| \, .$$

We define $m_j(S) := \max\{|S \cap C| : C \text{ an algebraic curve}, \deg C = j\}$, i.e. the maximum number of points of S contained on a degree j algebraic curve.

Proposition 4.1.3. [2, Proposition 2.3] Let $S \subset \mathbb{C}^2$ be such that $|S| = 7$ and $m_2(S) = 5$. Then S has an entire pluricomplex Green function u with $\gamma_u = 4$, such that u has logarithmic poles of weight 2 at 3 of the points of S, and of weight 1 at the remaining 4 points of S.

Proposition 4.1.4. [2, Proposition 2.4.(i)] Let $A \subset \mathbb{C}^2$ with $|A| = 7$, $m_1(A) \leq 3$, $m_2(A) = 6$, and let Γ be the conic such that $|A \cap \Gamma| = 6$. Let $q \notin A \cup \Gamma$. If $m_1(A \cup \{q\}) \leq 3$, then there

exists $u \in PSH(\mathbb{C}^2)$ with $\gamma_u = 3$ such that u is locally bounded outside a finite set, and $u(z) \leq \log \|z - p\| + O(1)$ near each $p \in A \cup \{q\}$.

4.2 Setting the Stage

First we prove the following lemmas that will be quite useful to us in the upcoming proofs. They show that for T, a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^2$, T cannot have small mass if the points of T with large Lelong number have certain configurations.

Lemma 4.2.1. [14, Lemma 3.1] Let T be a positive closed current of bidimension $(1,1)$ in $\mathbb{P}^2$, $\alpha > 2/5$ and $\beta = \frac{2}{3}(1-\alpha)$. Assume that $\{q_i\}_{i=1}^4$ are points in $\mathbb{P}^2$ such that $\nu(T, q_i) \geq \alpha$ and $\{p_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that $\nu(T, p_i) > \beta$, let $\{x_i\}_{i=1}^8$ be a relabeling of $\{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^4$. Assume $x_1, \ldots, x_4 \in L_1$, where L_1 is a complex line, and either

i) there exist complex lines L_2 and L_3 such that $\{x_1, x_5, x_6\} \in L_2$ and $\{x_2, x_7, x_8\} \in L_3$,

or

ii) there exists an irreducible conic Γ such that $x_1, x_2, x_5, x_6, x_7, x_8 \in \Gamma$.

Then $\|T\| > 1$.

Proof. Suppose for contradiction that $\|T\| \leq 1$. Since we already have points where T has non-zero Lelong number, $T \neq 0$. Note that the current $S := T/\|T\|$ has mass 1, and if $\nu(T, x) > c$, then $\nu(S, x) > c$, so we may assume that $\|T\| = 1$.

(i) By Siu's decomposition theorem 3.1.21, the current T can be decomposed as follows

$$T = a[L_1] + b[L_2] + c[L_3] + R$$

where R is a positive closed current of bidimension $(1,1)$, i.e. bidegree $(1,1)$, on $\mathbb{P}^2$, R has generic Lelong number 0 along each L_i, and $0 \leq a, b, c \leq 1$ are the generic Lelong numbers along L_1, L_2, L_3 respectively. Thus we now have

$$R = T - a[L_1] - b[L_2] - c[L_3].$$

Choose α' such that $\alpha > \alpha' > 2/5$ and $\nu(T, p_i) > \frac{2}{3}(1-\alpha') = \beta' > \beta$. Let $\{x_i\}_{i=1}^8$, be as they are in the assumptions. Using this new information, we have the following:

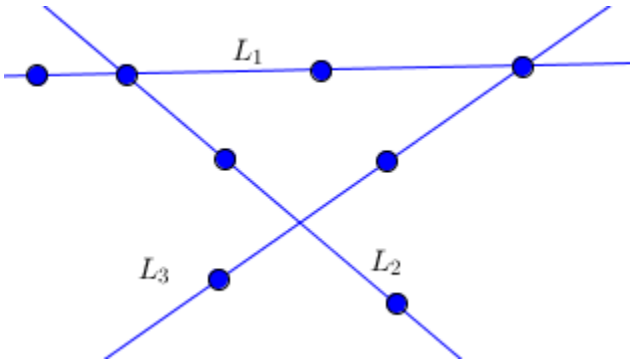
$$\nu(R, x_1) = \nu(T, x_1) - a - b, \quad \nu(R, x_2) = \nu(T, x_2) - a - c$$

$$\nu(R, x_3) = \nu(T, x_3) - a,\ \nu(R, x_4) = \nu(T, x_4) - a,\ \nu(R, x_5) = \nu(T, x_5) - b$$

$$\nu(R, x_6) = \nu(T, x_6) - b,\ \nu(R, x_7) = \nu(T, x_7) - c,\ \nu(R, x_8) = \nu(T, x_8) - c$$

which gives us that

$$\sum_{i=1}^{8} \nu(R, x_i) > 4\alpha' + 4\beta' - 4a - 3b - 3c$$



By 3.2.4, we have a current R', such that $\|R'\| = \|R\|$, R' preserves the above inequality,

and R' is smooth wherever R has Lelong number 0. Since the set of singularities of R' is analytic, and R' is smooth at generic points of L_i, 2.3.4 tells us that $R' \wedge [L_i]$, $i = 1, 2, 3$ are well defined measures. Let $S := ([L_1] + [L_2] + [L_3])$, and thus $R' \wedge S$ is well defined. We now have

$$3(1 - a - b - c) = \int_{\mathbb{P}^2} R' \wedge S \geq \sum_{i=1}^{8} R' \wedge S(\{x_i\})$$

$$\geq \sum_{i=1}^{8} \nu(R', x_i) > 4\alpha' + 4\beta' - 4a - 3b - 3c$$

where the first equality comes from 3.1.12 and the second inequality comes from the comparison theorem for Lelong numbers 3.1.13, since

$$\int_{\mathbb{P}^2} R' \wedge S \geq \sum \nu(R' \wedge S, x_i) \geq \sum \nu(R', x_i)\nu(S, x_i)$$

and $\nu(S, x_i) \geq 1$. So we now have

$$3(1 - a - b - c) > 4\alpha' + 4\beta' - 4a - 3b - 3c \implies a > \frac{4\alpha' - 1}{3}.$$

Consider now just the current $R_a := T - a[L_1]$, and $S_a := \frac{R_a}{1-a}$, note that $\|S_a\| = 1$ and for $x_i \notin L_1$ we have either

$$\nu(S_a, x_i) = \frac{\nu(R_a, x_i)}{1 - a} > \frac{\alpha'}{1 - \frac{4\alpha' - 1}{3}} = \frac{3\alpha'}{4 - 4\alpha'} > \frac{1}{2}$$

or

$$\nu(S_a, x_i) = \frac{\nu(R_a, x_i)}{1-a} > \frac{\beta'}{1-\frac{4\alpha'-1}{3}} = \frac{2(1-\alpha')}{4(1-\alpha')} = \frac{1}{2}$$

so by 3.3.1, $m_1(\{x_5, x_6, x_7, x_8\}) \geq 3$, which is a contradiction since $m_1(\{x_5, x_6, x_7, x_8\}) = 2$.

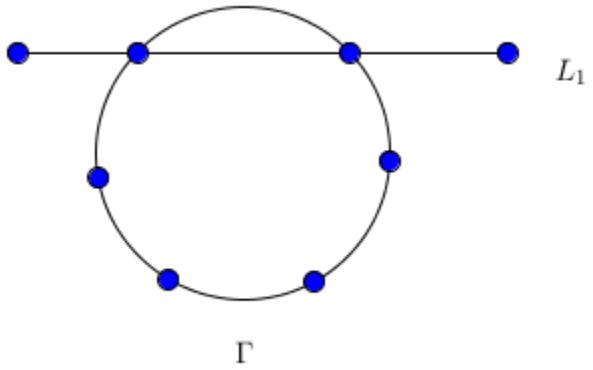
(ii) Let b be the generic Lelong number of Γ. We use the same argument as above, and consider the measures $R' \wedge [L_1]$ and $R' \wedge [\Gamma]$ to get

$$3(1-a-2b) = \int_{\mathbb{P}^2} R' \wedge [L_1] + \int_{\mathbb{P}^2} R' \wedge [\Gamma] \geq \sum_{i=1}^{8} \nu(R', x_i) > 4\alpha' + 4\beta' - 4a - 6b$$

which again gives

$$a > \frac{4\alpha' - 1}{3} .$$

Now considering R_a gives us the same contradiction.



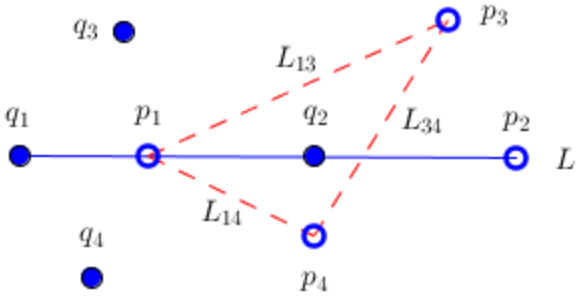
□

Assumptions (i) and (ii) are unfortunately restrictive and can seemingly limit the situations in which we can use the result. However, if L_1 contains one, two or three of the $q_i \in E_\alpha(T)$, then we can drop the assumptions (i) and (ii) of the previous lemma, which will simplify arguments in the later proofs.

Lemma 4.2.2. [14, Lemma 3.2] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^2$, $\alpha > 2/5$, $\beta = \frac{2}{3}(1-\alpha)$, $\{q_i\}_{i=1}^4$ and $\{p_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that $\nu(T, q_i) \geq \alpha > 2/5$ and $\nu(T, p_i) > \beta$. Assume there exists a complex line L containing either exactly $\{q_1, q_2, p_1, p_2\}$, exactly $\{q_1, p_1, p_2, p_3\}$, or exactly $\{q_1, q_2, q_3, p_1\}$ and the four points not on L are in general position. Then $\|T\| > 1$.

Proof. Arguing as we did at the start of the previous lemma, we may assume $\|T\| = 1$. We will show that we can construct a conic satisfying the hypothesis of 4.2.1, and then we are done as 4.2.1 says $\|T\| > 1$. Suppose L is a complex line containing $\{p_1, p_2, q_1, q_2\}$, and we will let $B = \{q_3, q_4, p_3, p_4\}$. Then by the hypothesis, $m_1(B) = 2$. Let α' be such that $\alpha > \alpha' > 2/5$ and $\nu(T, p_i) > \frac{2}{3}(1-\alpha') > \beta$. Note that either $m_1(\{p_1, p_3, p_4\}) = 2$ or $m_1(\{p_2, p_3, p_4\}) = 2$, and w.l.o.g. say that p_1, p_3, p_4 are in general position. We will let L_{jk} be the line containing p_j and p_k, and consider the current given by

$$R = \frac{5\alpha' - 2}{15\alpha'}([L_{13}] + [L_{14}] + [L_{34}]) + \frac{2}{5\alpha'}T$$



and note $\|R\| = 1$. We have the following inequalities:

$$\nu(R, q_i) \geq \frac{2}{5\alpha'}\alpha > \frac{2}{5}\,, i = 1, 2, 3, 4$$

and

$$\nu(R, p_i) > \frac{10\alpha' - 4}{15\alpha'} + \frac{4 - 4\alpha'}{15\alpha'} = \frac{2}{5}\,, i = 1, 3, 4.$$

Thus by 3.3.3, there is a conic Γ containing at least six of $\{q_i\}_{i=1}^4 \cup \{p_1, p_3, p_4\}$. Note that Γ cannot contain all seven points, otherwise L would be a component of Γ, which would mean that Γ is a reducible conic and thus that $m_1(B) > 2$ since the points off of L must also be collinear. Likewise, the point Γ must omit is one of the points on L, i.e. it must omit one of q_1, q_2 or p_1. If Γ is irreducible, then we are done. If not, then note Γ must be a reducible conic consisting of two lines, say $\Gamma = L_1 \cup L_2$. Since Γ contains all four points of B, it must be the case that each line L_i contains exactly two points of B (since $m_1(B) = 2$), and as no points of B are on L, we have that each L_i also contains a point of $L \cap \Gamma$. Finally note that since Γ contains six points, L_1 and L_2 cannot share the same point on L, i.e. $L_1 \cap L_2, \cap L = \emptyset$. So we now have all of the hypotheses of 4.2.1 satisfied, and thus $\|T\| > 1$, a contradiction.

If we have that L contains $\{q_1, p_1, p_2, p_3\}$, and $B = \{q_2, q_3, q_4, p_4\}$ is such that $m_1(B) = 2$, then using the current given by

$$R = \frac{5\alpha' - 2}{15\alpha'}([L] + [L_{14}] + [L_{24}]) + \frac{2}{5\alpha'}T,$$

we can argue as we did above to get a conic Γ containing six of the points $q_1, q_2, q_3, q_4, p_1, p_2, p_4$ satisfying the conditions of 4.2.1, and we are done.

Finally if we have that L contains $\{q_1, q_2, q_3, p_1\}$, and $B = \{q_4, p_2, p_3, p_4\}$ is such that $m_1(B) = 2$, then using the current given by

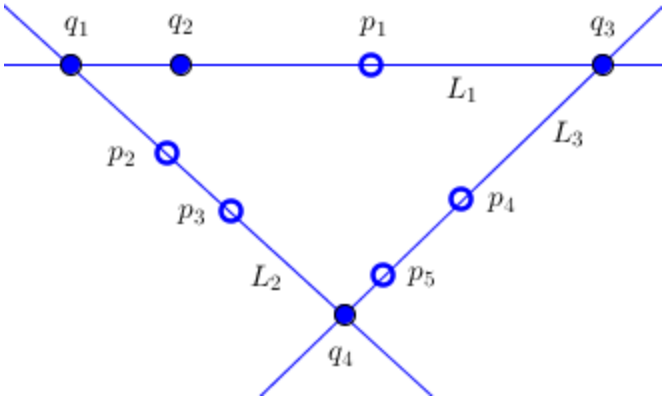
$$R = \frac{5\alpha' - 2}{15\alpha'}([L_{23}] + [L_{24}] + [L_{34}]) + \frac{2}{5\alpha'}T,$$

we can argue as we did above to get a conic Γ containing six of the points in $q_1, q_2, q_3, q_4, p_2, p_3, p_4$ satisfying the conditions of 4.2.1, and again, we are done.

□

Lemma 4.2.3. [14, Lemma 3.3] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^2$, $\alpha > 2/5$, $\beta = \frac{2}{3}(1-\alpha)$, $\{q_i\}_{i=1}^4$ and $\{p_i\}_{i=1}^5$ be points in $\mathbb{P}^2$ such that $\nu(T, q_i) \geq \alpha > 2/5$ and $\nu(T, p_i) > \beta$. Assume there exist three distinct complex lines L_1, L_2, and L_3 containing exactly $\{q_1, q_2, q_3, p_1\}$, $\{q_1, q_4, p_2, p_3\}$, and $\{q_3, q_4, p_4, p_5\}$, respectively. Then $\|T\| > 1$.

Proof. Suppose for contradiction that $\|T\| = 1$. We attack this situation in cases, depending on how the points p_1, p_2, p_3, p_4, p_5, q_2 (i.e. the points not on the intersections of the three lines) fall. First note that $m_1(\{p_2, p_3, p_4, p_5\}) = 2$. We now break this into cases.



Case 1: Suppose that $m_1(\{p_2, p_3, p_4, p_5, q_2\}) = 2$. Then consider the points q_1, q_2, p_3, p_4, p_5, noting that they are in general position, so there is an irreducible conic γ_1 containing them. Now consider the current $R = T - a[L_1] - b[L_2] - c[\gamma_1]$, where $0 \leq a, b, c \leq 1$ are the generic Lelong numbers of T along L_1, L_2, γ_1 respectively. Let $\alpha' \in (2/5, \alpha)$ be as before, i.e. $\nu(T, p_i) > \frac{2}{3}(1 - \alpha') = \beta' > \beta$. Then by using 3.2.4 as we did in Lemma 4.2.1, there is a current R' such that $\|R'\| = \|R\|$, R' maintains the same lower bounds, and 2.3.4 gives us that $R' \wedge [L_i]$ and $R' \wedge [\gamma_1]$ are well defined measures. Define $S := ([L_1] + [L_2] + [\gamma_1])$, and now we have

$$4(1 - a - b - 2c) = \int_{\mathbb{P}^2} R' \wedge S \geq \sum \nu(R', x_i)\nu(S, x_i) \geq$$

$$2\nu(R', q_2) + \nu(R', q_3) + \nu(R', q_4) + \sum_{i=1}^{5} \nu(R', p_i) + \nu(R', p_3)$$

$$> 4\alpha' + 6\beta' - 4a - 4b - 6c.$$

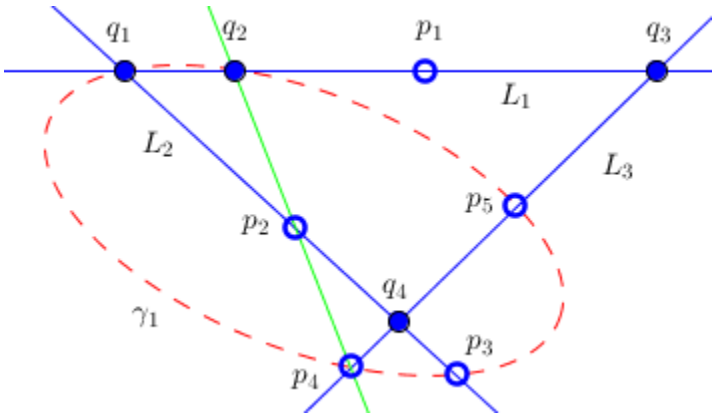
Now using the above inequality we get

$$4 - 2c > 4\alpha' + 6\beta' = 4\alpha' + 4(1 - \alpha') = 4$$

which is a contradiction as $c \geq 0$. We will use similar techniques to handle the remaining cases.

Case 2: We have $m_1(\{p_2, p_3, p_4, p_5, q_2\}) = 3$. That means q_2 is on a line with two p_i, one of the p_i is on L_2 and one on L_3, say w.l.o.g. $m_1(\{q_2, p_2, p_4\}) = 3$.

Case 2a: If $m_1(\{q_2, p_3, p_5\}) = 2$, then the same argument as above gets us to a contradiction.



Case 2b: We have $m_1(\{q_2, p_3, p_5\}) = 3 = m_1(\{q_2, p_2, p_4\})$ and $m_1(\{p_1, p_2, p_3, p_4, p_5\}) = 2$. Then note $m_1(\{q_2, q_4, p_1, p_2, p_5\}) = 2 = m_1(\{q_2, q_4, p_1, p_3, p_4\})$ and there are irreducible conics γ_1 and γ_2 containing $\{q_2, q_4, p_1, p_2, p_5\}$ and $\{q_2, q_4, p_1, p_3, p_4\}$ respectively. Define a current $R = T - a[\gamma_1] - b[\gamma_2]$, let α' be as before, and then once again proposition 3.2.4 and 2.3.4 gives a current R' such that $\|R'\| = 1 - 2a - 2b$ and

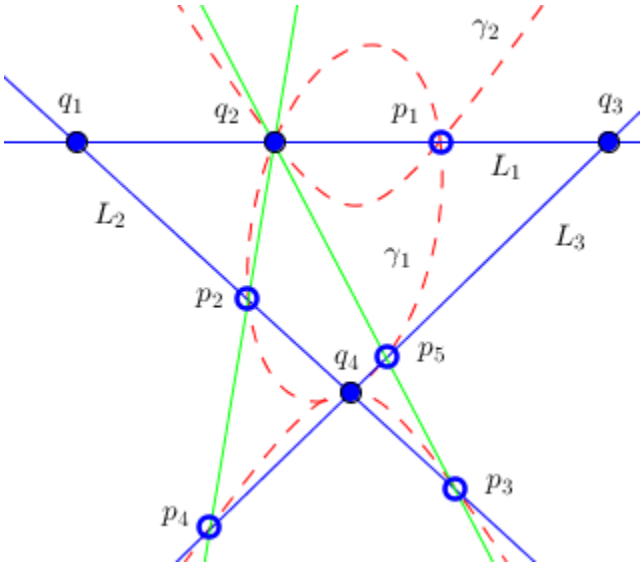
$$4(1 - 2a - 2b) = \int_{\mathbb{P}^2} R' \wedge ([\gamma_1] + [\gamma_2]) \geq$$

$$2\nu(R', q_2) + 2\nu(R', q_4) + \nu(R', p_1) + \sum_{i=1}^{5} \nu(R', p_i)$$

$$> 4\alpha' + 6\beta' - 8a - 8b$$

$$\implies 4 > 4\alpha' + 6\beta' = 4$$

again giving us a contradiction.



Case 2c: We have $m_1(\{q_2, p_3, p_5\}) = 3 = m_1(\{q_2, p_2, p_4\})$, and $m_1(\{p_1, \ldots, p_5\}) = 3$, so either $m_1(\{p_1, p_2, p_5\}) = 3$ or $m_1(\{p_1, p_3, p_4\}) = 3$. Suppose $m_1(\{p_1, p_2, p_5\}) = 3$ and $m_1(\{p_1, p_3, p_4\}) = 2$ then note $m_1(\{q_2, q_4, p_1, p_3, p_4\}) = 2$ and there is an irreducible conic γ_1 containing $\{q_2, q_4, p_1, p_3, p_4\}$. Let l_1 be the line containing p_1, p_2, p_5 and l_2 be the line containing q_2, q_4. Note that by construction, none of the p_i can fall on l_2 and $p_2, p_5 \notin \gamma_1$, otherwise either L_2 or L_3 would be a component of γ_1, which cannot be as γ_1 is irreducible. Define a current $R = T - a[\gamma_1] - b[l_1] - c[l_2]$, let α' be as before, and then 3.2.4 and 2.3.4 gives a current R' such that $\|R'\| = 1 - 2a - b - c$ and

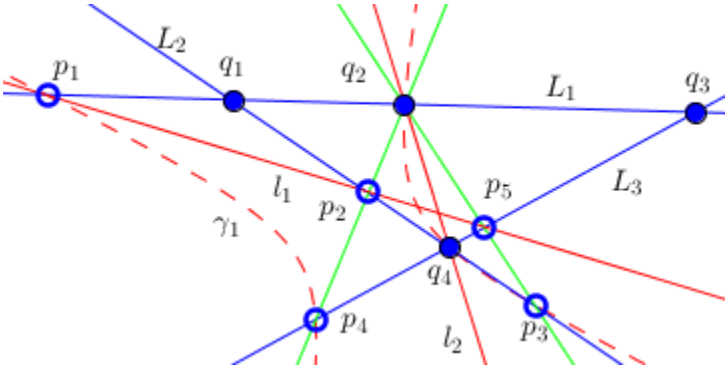
$$4(1 - 2a - b - c) = \int_{\mathbb{P}^2} R' \wedge ([\gamma_1] + [l_1] + [l_2]) \geq$$

$$2\nu(R', q_2) + 2\nu(R', q_4) + \nu(R', p_1) + \sum_{i=1}^{5} \nu(R', p_i)$$

$$> 4\alpha' + 6\beta' - 8a - 4b - 4c$$

$$\implies 4 > 4\alpha' + 6\beta' = 4$$

again giving us a contradiction. If $m_1(\{q_2, p_3, p_5\}) = 3 = m_1(\{q_2, p_2, p_4\})$, $m_1(\{p_1, p_2, p_5\}) = 2$ and $m_1(\{p_1, p_3, p_4\}) = 3$, a similar argument gives us a contradiction.

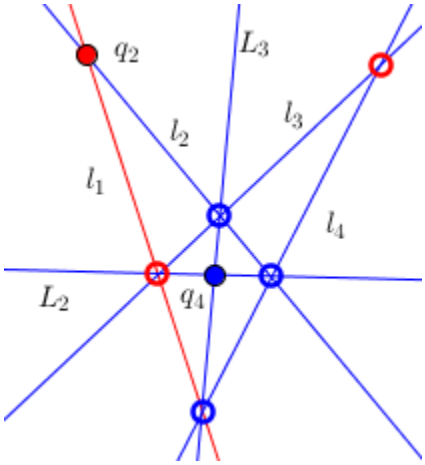


Case 2d: Finally $m_1(\{q_2, p_3, p_5\}) = 3 = m_1(\{q_2, p_2, p_4\})$, $m_1(\{p_1, p_2, p_5\}) = 3$ and $m_1(\{p_1, p_3, p_4\}) = 3$. Consider the seven points subset $\{q_2, q_4, p_1, p_2, p_3, p_4, p_5\}$, and note that we have $\{q_4, p_2, p_3\} \in L_2$, $\{q_4, p_4, p_5\} \in L_3$, and we also have lines l_1, l_2, l_3, l_4 containing $\{q_2, p_3, p_5\}$, $\{q_2, p_2, p_4\}$, $\{p_1, p_3, p_4\}$, and $\{p_1, p_2, p_5\}$ respectively.

Note that $m_2(\{q_2, q_4, p_1, p_2, p_3, p_4, p_5\}) = 5$, so we can apply 4.1.3, so there exists an entire pluricomplex Green function u with $\gamma_u = 4$, and u has weight two logarithmic poles and three of the seven points, and weight one at the remaining four. First note that we cannot have weight two poles at both q_2 and q_4, for if we do, then we also have a weight two pole at say p_1, and 4.1.2 gives us that

$$4 = \gamma_u \|T\| \geq 2\nu(T, q_2) + 2\nu(T, q_4) + 2\nu(T, p_1) + \sum_{i=2}^{5} \nu(T, p_i) > 4\alpha + 6\beta = 4$$

a contradiction. So since u cannot have a double pole at both q_2 and q_4, at least one of the l_i or L_i will have two points such that u double poles at both points and a third where u has a single pole, say w.l.o.g. we have l_1 with this property, where u has double poles at $x_1, x_2 \in l_1$and has a single pole at $x_3 \in l_1$. But now applying 4.1.2, we get



$$4 \geq \int_{\mathbb{C}^2} [l_1] \wedge dd^c u \geq 2\nu([l_1], x_1) + 2\nu([l_1], x_2) + \nu([l_1], x_3) = 2 + 2 + 1 = 5$$

an obvious contradiction. However now we have ruled out all of the possible ways in which p_1, p_2, p_3, p_4, p_5, q_2 fall, thus it must be the case that$\|T\| > 1$. □

4.3 Proof of the Main Result

We now prove the main result. This is done by proving a few propositions which consider the various cases that can occur depending on how the four points are positioned. For the remainder of this section, assume that T is a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^2$ with $\|T\| = 1$. We review some basic notions before we proceed. Consider $A = \{x_1, \ldots, x_{p+1}\}$, $x_i \in \mathbb{P}^n$. By the Span (A), we mean the smallest linear subspace

$V \subset \mathbb{P}^n$ that contains A. If $p \leq n$ and span(A) is a p-dimensional space, then we say $\{x_i\}_{i=1}^{p+1}$ are linearly independent. If we have $p > n+1$ points, then we say they are in general position if any $n+1$ of them are linearly independent.

We also remind the reader the definition of upper level sets:

$$E_\alpha(T) := \{q \in \mathbb{P}^n \,|\, \nu(T,q) \geq \alpha\},$$
$$E_\alpha^+(T) := \{q \in \mathbb{P}^n \,|\, \nu(T,q) > \alpha\},$$

and we begin by looking at the first of our three cases.

Proposition 4.3.1. [14, Proposition 3.4] Let $\{q_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that they are in general position and $\nu(T,q_i) \geq \alpha > 2/5$. Let $\beta = \frac{2}{3}(1-\alpha)$. Then there exists a conic C (possibly reducible) such that $|E_\beta^+(T) \backslash C| \leq 1$.

Proof. Let $\{q_i\}_{i=1}^4$, be as above and let $p_1 \in E_\beta^+(T)$, $p_1 \neq q_i$ (noting that if no such p_1 exists then we are done). Since the q_i are in general position, we let Γ_1 be the unique conic defined by the q_i and p_1. If Γ_1 satisfies the conclusion, then we are done. If not then we can find two points, p_2 and p_3 such that $p_2, p_3 \in E_\beta^+(T) \backslash \Gamma_1$. Let α' be such that $\alpha > \alpha' > 2/5$ and $\nu(T,p_i) > \frac{2}{3}(1-\alpha') > \beta$. If the p_i are in general position, we will let L_{jk} be the line containing p_j and p_k. Define a current R as follows:

$$R = \frac{5\alpha' - 2}{15\alpha'} \sum_{1 \leq j < k \leq 3} [L_{jk}] + \frac{2}{5\alpha'} T$$

and note $\|R\| = 1$. We have the following inequalities:

$$\nu(R, q_i) > \frac{2}{5\alpha'}\alpha > \frac{2}{5}$$

and

$$\nu(R, p_i) > \frac{10\alpha' - 4}{15\alpha'} + \frac{4 - 4\alpha'}{15\alpha'} = \frac{2}{5}.$$

If instead the p_i are all on a line L, then we use the current

$$R = \frac{5\alpha' - 2}{5\alpha'}[L] + \frac{2}{5\alpha'}T$$

and get the same inequalities as above. In either case, by 3.3.3, there is a conic Γ_2 containing at least six of the $\{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^3$. As Γ_1 is uniquely defined by the q_i and p_1, Γ_2 must omit one of the seven points, and the point omitted must be one of the q_i or p_1, else $\Gamma_1 = \Gamma_2$, which means one or both of p_2, p_3 would be on Γ_1, which is a contradiction. If Γ_2 satisfies the conclusion, then we are done. So suppose Γ_2 does not satisfy the conclusion of our proposition, and then there is $p_4 \in E_\beta^+(T) \backslash \Gamma_2$.

We will let $A = \{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^3$, and we will note that $|A| = 7$, $m_2(A) = 6$, $|A \cap \Gamma_2| = 6$ and $p_4 \notin A \cup \Gamma_2$. We will make use of these observations shortly. Define $S = A \cup \{p_4\}$. We now consider the following possibilities for S: $m_1(S) \leq 3$, $m_1(S) = 4$, and $m_1(S) \geq 5$.

Suppose $m_1(S) \leq 3$. Then this means that $m_1(A) \leq 3$ and by the above observations about A, we can apply 4.1.4, i.e. there exists $u \in PSH(\mathbb{C}^2)$ such that $\gamma_u = 3$, u is locally bounded outside of a finite set, and u has logarithmic poles of weight one at each point in S. Now by 4.1.2, we have that:

$$3 = \gamma_u \|T\| \geq \sum_{i=1}^4 \nu(T, q_i) + \sum_{i=1}^4 \nu(T, p_i) > 4\alpha + 4\beta = \frac{4}{3}\alpha + \frac{8}{3} > 3.$$

This is a contradiction, thus we cannot have $m_1(S) \leq 3$.

Suppose $m_1(S) \geq 5$. Let L be the line such that $|S \cap L| \geq 5$. If L contains $\{p_i\}_{i=1}^4$ and one of the q_i, then Γ_2 is reducible (as regardless of which point Γ_2 omits, it still contains at least three points on L), and L is a component which implies that $p_4 \in \Gamma_2$, which is impossible. As the q_i are in general position, L contains three of the p_i and two of the q_i. If $p_1 \in L$ then we have L is a component of Γ_1 and at least one of p_2 or p_3 is on L, which is a component of Γ_1, and thus impossible as $p_2, p_3 \notin \Gamma_1$. So $p_1 \notin L$, but now L contains p_4 and at least three points of Γ_2, so L is a component of Γ_2, which means $p_4 \in \Gamma_2$, another contradiction. As the q_i are in general position, this covers all the possible ways that $m_1(S) \geq 5$.

So if there is $p_4 \in E_\beta^+(T) \backslash \Gamma_2$, it must be the case that $m_1(S) = 4$. So there is a line L containing exactly four points of S. This decomposes into a few more cases depending on what four points the line L contains. The first and easiest is if L contains $\{p_i\}_{i=1}^4$ (which means that none of the q_i lie on L as $m_1(S) = 4$). Then consider the current

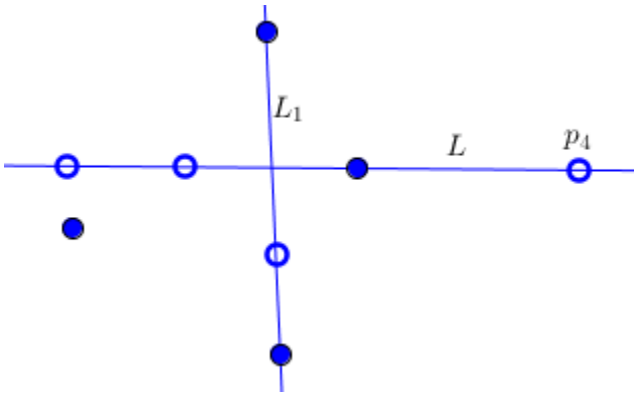
$$R = \frac{5\alpha' - 2}{5\alpha'}[L] + \frac{2}{5\alpha'}T.$$

Routine calculations show that $\|R\| = 1$, $\nu(R, p_i) > \frac{2}{5}$, and $\nu(R, q_i) > \frac{2}{5}$, so by 3.3.3, we have that there is a conic containing at least seven points of S, which means L is a component of this conic, which implies that at least three of the q_i are collinear as L cannot contain more than four points, which is a contradiction.

We will now assume that $m_1(A) \leq 3$, and consider the remaining cases. Then later we will consider them for when $m_1(A) = 4$.

If L contains three p_i and one q_i then note that since $m_1(A) \leq 3$ it must be the case that

$p_4 \in L$. Suppose that the four points not on L are not in general position so there is a line, say L_1 containing three of the points not on L, and they must be two q_i and one p_i (as the three q_i not on L are in general position), and $L \cap L_1 \cap A = \emptyset$ as $m_1(A) \leq 3$. Noting that $|\Gamma_2 \cap (L \cup L_1)| \geq 5$, one of L or L_1 is a component of Γ_2 by Bezout's theorem. As L contains p_4, it must be the case that L_1 is a component of Γ_2. But since L_1 contains only three points of Γ_2, and at least two points of Γ_2 are on L, it must be the case that $\Gamma_2 = L \cup L_1$, but this means $p_4 \in \Gamma_2$, which is a contradiction. So the four points not on L must be in general position.

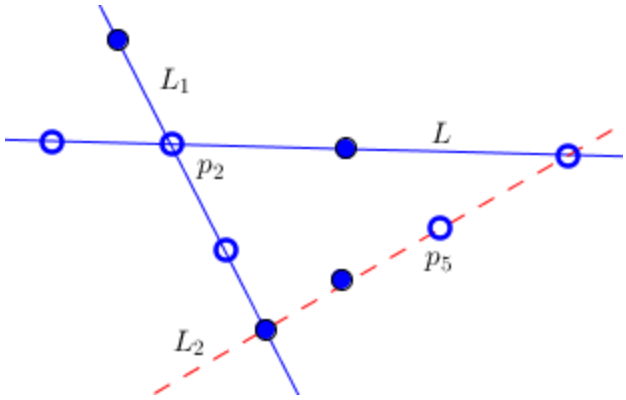


Note that since the four points off of L must be in general position, and L contains one of the q_i and three of the p_i, we have satisfied all of the hypotheses of 4.2.2, and thus $\|T\| \neq 1$, which is a contradiction.

If L contains two p_i and two q_i, we let B be the four point set consisting of the two p_i and two q_i not contained on L. Since $m_1(A) \leq 3$, it must be the case that $p_4 \in L$, and that $m_1(B) \leq 3$. If $m_1(B) = 3$ then we can argue as we did above to get that Γ_2 is reducible, and it contains L as a component, but then $p_4 \in \Gamma_2$, which is impossible. So it must be the case that $m_1(B) = 2$ and again we can apply 4.2.2 to get a contradiction. This finishes the case where L contains two p_i and two q_i, and also finishes the case $m_1(S) = 4$ when $m_1(A) \leq 3$.

So far we have shown that if there is in fact a point $p_4 \in E_\beta^+(T)\backslash\Gamma_2$, then it must be the case that $m_1(S) = 4 = m_1(A)$. It only remains to consider the cases where L contains one q_i and three p_i or two q_i and two p_i. We will first consider when L contains three p_i, and let $B = S\backslash(S \cap L)$, noting that $m_1(B) < 4$ as the q_i are in general position. If $m_1(B) = 2$ then by Lemma 4.2.2, $\|T\| > 1$, a contradiction.

Thus $m_1(B) = 3$ and then $m_2(S) = 7$. After reindexing (if necessary) say that $p_1 \in L$. Let $C = L \cup L_1$ where L_1 contains the three collinear points in B (noting that L_1 contains two q_i and one p_i, and say q_4 is the point of$B$ not on L_1). We will show that C is the desired conic satisfying the conclusion of the proposition. If not, assume for contradiction there exists $p_5 \in E_\beta^+(T)\backslash C$. If $L \cap L_1 \cap S = \emptyset$, then set $A' = S\backslash\{p_1\}$ and $S' = A' \cup \{p_5\}$. So note that $|A'| = 7$, $m_1(A') = 3$, $m_2(A') = 6$ (since if $m_2(A') = 7$, either all four points in B are collinear or one point of B is on L and neither of those can happen), $|A' \cap C| = 6$, $p_5 \notin A' \cup C$, and $m_1(S') \leq 4$. If $m_1(S') = 3$ then we can use 4.1.2 and 4.1.4 as before to get a contradiction. If $m_1(S') = 4$ then since $p_5 \notin C$, there is a line L_2 containing p_5 and three other points from A'. By construction, L_2 must contain q_4 as well as one point of $L \cap S'$ and one point of $L_1 \cap S'$. However, L_2 contains at least one q_i and $m_1(S'\backslash L_2) = 2$ so we can apply 4.2.2 and thus $\|T\| > 1$. If $L \cap L_1 \cap S \neq \emptyset$ then the intersection must be one of the points contained on L, since otherwise if the intersection was a point on L_1, then $|L \cap S| = 5$, a contradiction. Further, it must be one of the p_i,w.l.o.g. say $p_i = p_2$, as the q_i are in general position. We set $A' = S\backslash\{p_2\}$, and argue the same way to get a contradiction. We have shown that if $m_1(A) = m_1(S) = 4$ and there is a line containing three of the p_i and one q_i, then there can be no such p_5 and C is the desired conic that satisfies the conclusion.

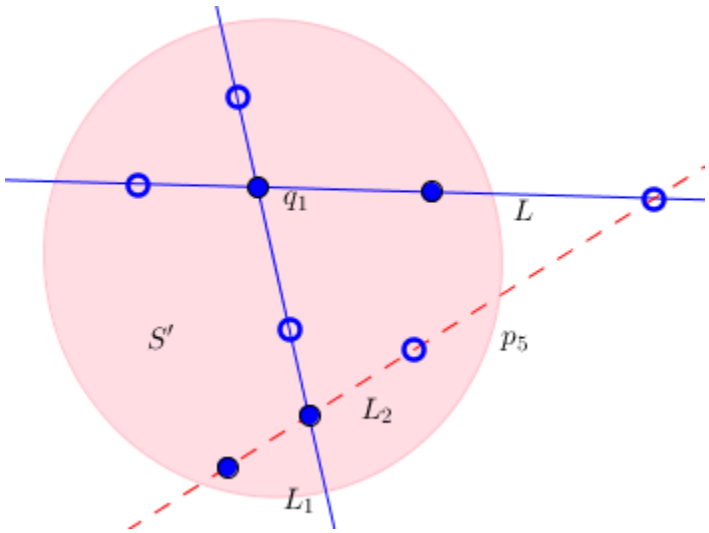


Finally we consider when L contains two p_i, two q_i, and $m_1(A) = 4$. Again we let $B = S \backslash S \cap L$ and note that $m_1(B) \neq 4$ or else we get that $\Gamma_1 = \Gamma_2$. Furthermore, if $m_1(B) = 2$, we can apply 4.2.2 to get a contradiction. Our only remaining consideration is when $m_1(B) = 3$. Let L_1 be the line containing three points from B. We will re-index our points so that $\{q_1, q_2, p_1, p_4\} \in L$ and $B = \{q_3, q_4, p_2, p_3\}$.

Let $C = L \cup L_1$, and again we will show this is the desired conic. Suppose for contradiction that $p_5 \in E_\beta^+(T) \backslash C$. Assume $L \cap L_1 \cap S = \emptyset$. Let $A' = S \backslash \{p_1\}$ (recalling $p_1 \in L$), $S' = A \cup \{p_5\}$, and note that $|A'| = 7$, $m_1(A') = 3$, $m_2(A') = 6$, $|A' \cap C| = 6$, $p_5 \notin A' \cup C$, and $m_1(S') \leq 4$. If $m_1(S') = 3$ then we can use 4.1.2 and 4.1.4 as before to get a contradiction. If $m_1(S') = 4$ then since $p_5 \notin C$, there is a line containing p_5 and three other points from A', but now we argue as before using 4.2.2 to reach a contradiction. If instead $L \cap L_1 = \{p_i\}$, then note it must be some $p_i \in L$ (otherwise $m_1(S) > 4$), we set $A' = S \backslash \{p_i\}$ and the same argument shows that C is the desired conic.

Suppose $L \cap L_1 = \{q_i\}$, and w.l.o.g. say that point is $q_i = q_1$. Then $C = L \cup L_1$ omits $q_k \in B$, (as the q_i are in general position), say that omitted point is q_4. We will let L_2 be the line that contains q_4 and p_5. If $L_2 \cap C \cap S = \emptyset$, then we can set $B' = \{q_3, q_4, p_3, p_5\}$, note that $m_1(B') = 2$, and apply 4.2.2 using L and B' to get a contradiction. If L_2 hits exactly

one point on $L \cap S'$ and no points on $L_1 \cap S'$, then again we can let $B' = \{q_3, q_4, p_3, p_5\}$ and again use 4.2.2. If L_2 hits exactly one point on $L_1 \cap S'$ and no points on $L \cap S'$, then we can let $B' = \{q_2, q_4, p_1, p_5\}$ and again use 4.2.2. If L_2 hits two points on $C \cap S'$, then note at least one of those two points must be a p_i (as the q_i are in general position) w.l.o.g. say it is p_1 on L, and we can set $B' = \{q_2, q_4, p_4, p_5\}$, again $m_1(B') = 2$. Now using L_1, which contains $\{q_1, q_3, p_2, p_3\}$ (i.e. two q_i and two p_i) and B', we argue as before using 4.2.2 to get a contradiction. This resolves the case of L containing two p_i and two q_i, the case of $m_1(A) = m_1(S) = 4$, and thus we have finished the proof.



□

Proposition 4.3.2. [14, Proposition 3.5] Let $\{q_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that q_1, q_2, q_3 lie on a line L_1 and q_4 does not fall on L_1. In addition, $\nu(T, q_i) \geq \alpha > 2/5$. Let $\beta = \frac{2}{3}(1 - \alpha)$. Then there exists a conic C (possibly reducible) such that $|E_\beta^+(T) \backslash C| \leq 1$.

Proof. Let $\{q_i\}_{i=1}^4$, be as described in the assumptions, and let $p_1 \in E_\beta^+(T) \backslash L_1$, with $p_1 \neq q_4$ (noting that if no such p_1 exists then we are done). We will let l_1 be the line that connects p_1 and q_4 and let $\Gamma_1 = L_1 \cup l_1$. Now there exist points $p_2, p_3 \in E_\beta^+(T) \backslash \Gamma_1$, else we are done.

Before moving on, we will show that we can assume that $m_1(\{p_1, p_2, p_3\}) = 2$. For suppose that all three p_i lie on a line, say l_2, then $L_1 \cup l_2$ gives us a conic containing six of the seven points. Then there is a $p_4 \in E_\beta^+(T) \backslash (L_1 \cup l_2)$. If $p_4 \notin l_1$ then note $\{p_1, p_2, p_4\}$ are in general position. If $p_4 \in l_1$, then note $\{p_2, p_3, p_4\}$ are in general position. Either way, we will reindex the set and call the points $\{p_1, p_2, p_3\}$ where p_1 is the point on Γ_1. Let α' be such that $\alpha > \alpha' > 2/5$ and $\nu(T, p_i) > \frac{2}{3}(1 - \alpha') > \beta$ and let L_{jk} be the containing p_j and p_k. Define a current R as follows:

$$R = \frac{5\alpha' - 2}{15\alpha'} \sum_{1 \leq j < k \leq 3} [L_{jk}] + \frac{2}{5\alpha'} T$$

and note $\|R\| = 1$. We have the following inequalities:

$$\nu(R, q_i) > \frac{2}{5\alpha'} \alpha > \frac{2}{5}$$

and

$$\nu(R, p_i) > \frac{10\alpha' - 4}{15\alpha'} + \frac{4 - 4\alpha'}{15\alpha'} = \frac{2}{5}.$$

Thus by 3.3.3, there is a conic Γ_2 containing at least six of the $\{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^3$. As Γ_1 is uniquely defined by the q_i and p_1, Γ_2 must omit one of the seven points, and the point omitted must be one of the q_i or p_1, else $\Gamma_1 = \Gamma_2$, which means one or both of p_2, p_3 would be on Γ_1, which is a contradiction. If Γ_2 satisfies the conclusion, then we are done. So suppose Γ_2 does not satisfy the conclusion of our proposition, and then there is $p_4 \in E_\beta^+(T) \backslash \Gamma_2$.

We will let $A = \{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^3$, and we will note that $|A| = 7$, $m_2(A) = 6$, $|A \cap \Gamma_2| = 6$

and $p_4 \notin A \cup \Gamma_2$. We will make use of these observations shortly. Define $S = A \cup \{p_4\}$. We now consider the following possibilities for S: $m_1(S) \leq 3$, $m_1(S) = 4$, and $m_1(S) \geq 5$.

Suppose $m_1(S) \leq 3$. Then this means that $m_1(A) \leq 3$ and so we can apply 4.1.4, i.e. there exists $u \in PSH(\mathbb{C}^2)$ such that $\gamma_u = 3$, u is locally bounded outside of a finite set, and u has logarithmic poles of weight one at each point in S. Now by 4.1.2, we have that:

$$3 = \gamma_u \|T\| \geq \sum \nu(T, q_i) + \sum \nu(T, p_i) > 4\alpha + 4\beta = \frac{4}{3}\alpha + \frac{8}{3} > 3.$$

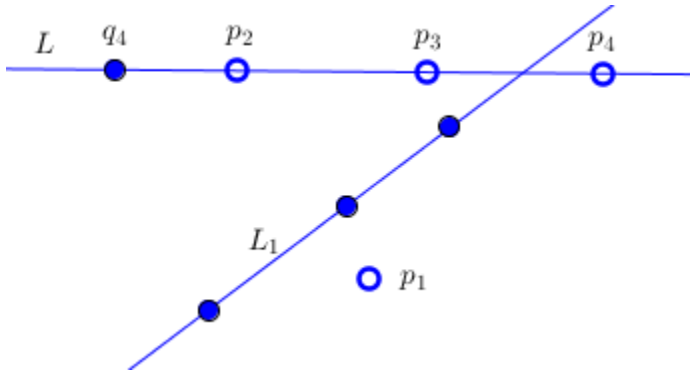
This is a contradiction, thus $m_1(S) > 3$.

Suppose $m_1(S) \geq 5$. Note that by how the points in A are constructed, it is the case that $m_1(A) \leq 4$, and since $m_1(S) \geq 5$, this means $m_1(A) = 4$, and as the p_i are in general position, the only way that $m_1(A) = 4$ is if there is a line containing $\{q_4, p_2, p_3, q_i\}$ for some $i = 1, 2, 3$. Then there is a line L containing at least five points, and it must be the previously mentioned line with p_4 on it as well. However, regardless of what point is omitted from Γ_2, L is a component of Γ_2 which means $p_4 \in \Gamma_2$, which is a contradiction. Thus $m_1(S) < 5$.

It must be the case that $m_1(S) = 4$, and now we begin our battle with this situation. As before, we will note that this breaks into cases depending on what points lie on the the line that contains four points. As the p_1, p_2, and p_3 are not collinear, we cannot have all four p_i on a line, so that removed that case instantly.

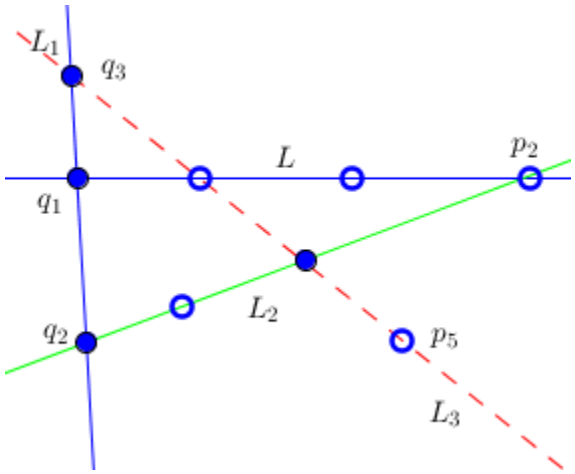
Case 1: Suppose L contains three p_i and one q_i. Suppose that $q_i = q_4$. If $p_1 \in L$, the conic $\Gamma_3 := L \cup L_1 = \Gamma_1$, which is impossible as one of the other two p_i on L will be either p_2 or p_3, and $p_2, p_3 \notin \Gamma_1$. So it must be that the p_i are p_2, p_3, and p_4. Note $|\Gamma_3 \cap \Gamma_2| \geq 5$, and that any subset of five points from $\{q_1, q_2, q_3, q_4, p_2, p_3\}$ uniquely defines Γ_3 so it must

be the case that $\Gamma_2 = \Gamma_3$, which means $p_4 \in \Gamma_2$, which is a contradiction. Thus $q_i \neq q_4$.



So L contains a $q_i \neq q_4$, say L contains q_1 (reindexing if necessary). Once again note that p_4 must be one of the points on L as otherwise we would have p_1, p_2, p_3 collinear. Let $B = \{q_2, q_3, q_4, p_i\}$ be the four points off L. If $m_1(B) = 2$, then we are done as 4.2.2 gives us a contradiction. So it must be the case that $m_1(B) \geq 3$, and as $q_4 \notin L_1$, we have $m_1(B) = 3$. Since $p_i \notin L_1$ (because $p_i \neq p_4$), we have a line, L_2,that contains $\{p_i, q_4, q_i\}$ (w.l.o.g. say q_2). Let $C := L \cup L_2$, we will show C is the desired conic. For contradiction suppose there is $p_5 \in E_\beta^+(T) \backslash C$. Note that if $L_2 \cap L \cap A = \emptyset$, C is uniquely determined by any five points of $\{q_1.q_2, q_4, p_1, p_2, p_3\}$. Also note that $|\Gamma_2 \cap C| \geq 5$, so again we can argue that $\Gamma_2 = C$, but again this means $p_4 \in \Gamma_2$, a contradiction. If instead $L_2 \cap L \cap A = \{p_2\}$ (reindex if necessary), then we consider the set $A' = S \backslash \{p_2\}$ and $S' = A' \cup \{p_5\}$. Note $|A'| = 7$, $m_1(A') = 3$, $m_2(A') = 6$, $|A' \cap C| = 6$, and $p_5 \notin A' \cup C$. Let L_3 be the line containing p_5 and q_3. If $L_3 = L_1$, i.e. $p_5 \in L_1$, then note the line L_1 and $\{p_1, p_3, p_4, q_4\}$ satisfy the assumptions of 4.2.2, giving us a contradiction. If $p_5 \notin L_1$ and $|L_3 \cap C \cap S'| \leq 1$ then $m_1(S') \leq 3$ and we can argue using 4.1.2 and 4.1.4 to get a contradiction. Finally if $|L_3 \cap C \cap S'| = 2$, then L_3 contains one point of $L_2 \cap S'$ and one point of $L \cap S'$. But now note that $m_1(S' \backslash (S' \cap L_3)) = 2$, so those four points and L_3 satisfy the assumptions of 4.2.2,

and again we get a contradiction.



Case 2: Now suppose L contains three q_i and one p_i. Actually it must be the case that $L = L_1$ and $p_4 \in L_1$, as no other p_i can be on L_1. If $m_2(S) = 6$ then the four points not on L are in general position, and thus by 4.2.2, we have a contradiction. Since $m_2(A) = 6$, $m_2(S) \leq 7$, so it must be the case that $m_2(S) = 7$. Let B be the set containing the four points not on L, and it must be that $m_1(B) = 3$ (else $m_2(S) \neq 7$). Since $m_1(B) = 3$, p_1, p_2, p_3 cannot be collinear, and $p_2, p_3 \notin \Gamma_1$, there is a line, say L_2 containing $\{p_2, p_3, q_4\}$. However it now follows that $m_1(A) = 4$ since if $m_1(A) = 3$, then we would get that $\Gamma_2 = L_2 \cup L$ which means $p_4 \in \Gamma_2$, a contradiction. So there is a line containing p_2, p_3, q_4 and one of the q_i on L (as this is the only way we can have $m_1(A) = 4$), and that line is in fact L_2. Let $C = L \cup L_2$, and note there must be a $p_5 \in E_\beta^+(T) \backslash C$, otherwise we are done. Let L_3 be the line containing p_1, p_5. If $L_3 \cap C \cap S = \emptyset$, then note $m_1(\{p_1, p_2, p_5, q_4\}) = 2$, so those four points and the line L satisfy the hypotheses of 4.2.2. If $L_3 \cap C \cap S = \{p_i\}$, then we can assume w.l.o.g. that p_i is p_2 on L_2, and now the points p_1, p_3, p_5, q_4 are in general position and none of the fall on L, so again we can apply 4.2.2 to get a contradiction. If instead $L_3 \cap C \cap S = \{q_i\}$, say q_1 on L, then note p_1, p_2, p_5, q_4 are in general position and

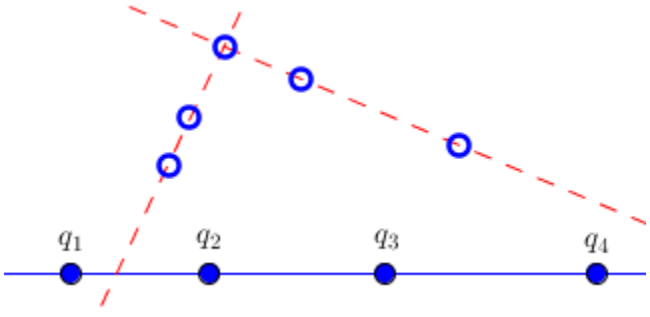
off L, so again we can use 4.2.2. A similar argument holds if q_i falls instead on L_2 or on the intersection $L \cap L_2$. If $|L_3 \cap C \cap S| = 2$ and at least one of the two points is a p_i, we can argue as we did above. If both points are q_i, one must be q_4 on L_2 and say the other is q_1 on L, however this is the same configuration that we resolved in Lemma 4.2.3, and thus this situation cannot happen either. We have have proven that there cannot exist a point p_5, and thus C is the desired conic, resolving the case when our line L contains three q_i and one p_i.

Case 3: We now move on to our last situation, that the line L contains two p_i and two q_i. As $m_1(S) = 4$, one of the q_i is q_4, and the other is one of the three q_i on L_1, w.l.o.g., say q_1, and say the other points are p_2 and p_3. Let B once again be the four points off of L, so $B = \{q_2, q_3, p_1, p_4\}$ and either $m_1(B) = 2$ or $m_1(B) = 3$ (if $m_1(B) = 4$, this means thats $\Gamma_1 = \Gamma_2$, which is impossible). If $m_1(B) = 2$, then by 4.2.2, we have a contradiction. If $m_1(B) = 3$, and we have one of the p_i on L_1 and we are back in case two as now L_1 contains three q_i and one p_i, which we have already argued. So let L_2 be the line containing three points of B and note that it must be both p_i and one of the q_i on L_1, say q_2. Let $C := L \cup L_2$, and we will show that C s the desired conic. Suppose for contradiction that there is $p_5 \in E_\beta^+(T) \backslash C$. If $p_5 \in L_1$, and if $L \cap L_2 \cap S = \emptyset$ then note we can use 4.2.2 with p_1, p_2, p_4, q_4 as they are in general position, and L_1, giving a contradiction. If $p_5 \in L_1$, and if $L \cap L_2 \cap S = \{p_i\}$, then we can use 4.2.2 again, but using the four point off of L_1 that omits $\{p_i\}$. If $p_5 \in L_1$, and if $L \cap L_2 \cap S = \{q_i\}$, then we we can apply Lemma 4.2.3 to get a contradiction. Thus $p_5 \notin L_1$, and then let L_3 be the line containing p_5 and q_3. If $L_2 \cap L_3 \cap B \neq \emptyset$ then it must that the intersection is one of the p_i, say p_1, for if the intersection is q_2, then that forces $p_5 \in L_1$. But now note that $m_1(\{q_2, q_3, p_4, p_5\}) = 2$, and all of the points are off L, so we can apply 4.2.2, and get a contradiction. If $L_2 \cap L_3 \cap B = \emptyset$,

then the same argument holds. Since p_5 can neither be on L_1 or off L_1, no such point can exist, and thus C is the desired conic that satisfies the conclusion. This resolves the third case, which finishes the $m_1(S) = 4$ case, and thus, the proof. □

Proposition 4.3.3. [14, Proposition 3.6] Let $\{q_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that all four points are collinear and $\nu(T, q_i) \geq \alpha > 2/5$. Let $\beta = \frac{2}{3}(1-\alpha)$. Then there exists a conic C (possibly reducible) such that $|E_\beta^+(T)\backslash C| \leq 1$.

Proof. Let L be the line containing the q_i, and suppose $|E_\beta^+(T)\backslash L| > 1$, (otherwise we are done), so there exist points $p_1, p_2 \in E_\beta^+(T)$ not on L, and let L_{12} be the line they lie on. We want to generate four points of $E_\beta^+(T)$ that do not lie on L such that no three are collinear. If the conic $L \cup L_{12}$ does not satisfy the conclusion then we can find two more point $p_3, p_4 \in E_\beta^+(T)$ that do not lie on our conic, and let L_{34} be the line containing these new points. If the four p_i are in general position then we are done, otherwise L_{34} contains three of the p_i, after reindexing, say it contains p_1, p_3, p_4. If the conic $L \cup L_{34}$ does not satisfy the conclusion then we can find a point $p_5 \in E_\beta^+(T)$ that is not on the new conic. If p_5 does not fall on L_{2k} for $k = 3, 4$, then take p_2, p_3, p_4, p_5 as our four points in general position. If p_5 falls on L_{2k}, say w.l.o.g. L_{23}, then we take p_1, p_2, p_4, p_5 as our four points in general position. We will reindex to the points to be p_1, p_2, p_3, p_4.



By Siu's decomposition theorem 3.1.21 we have that

$$T = a[L] + R$$

where a is the generic Lelong number of T along L. Note that $\|R\| = 1 - a$ and $\nu(R, q_i) \geq \alpha - a$. Let $\alpha' \in (\frac{2}{5}, \alpha)$ be such that $\nu(T, p_i) = \nu(R, p_i) > \frac{2}{3}(1 - \alpha') > \beta$ for $i = 1, 2, 3, 4$. Proposition 2.5 shows that there exists a current R' such that $\|R'\| = 1 - a$, R' is smooth where R has Lelong number 0, and $\nu(R', q_i) > \alpha' - a$. By 2.3.4, $R' \wedge [L]$ is a well defined measure. Now we have

$$1 - a = \int_{\mathbb{P}^2} R' \wedge [L] \geq \sum_{i=1}^{4} \nu(R' \wedge [L], q_i) \geq \sum_{i=1}^{4} \nu(R', q_i)\nu([L], q_i) > 4\alpha' - 4a$$

where the second inequality follows from 3.1.13 and the final inequality follows as $\nu([L], q_i) = 1$. So we have that $a > \frac{4\alpha' - 1}{3}$.

Define a new current:

$$S = \frac{R}{1 - a}$$

and note $\|S\| = 1$. Now we have:

$$\nu(S, p_i) > \frac{2}{3}\frac{1 - \alpha'}{1 - a} > \frac{2 - 2\alpha'}{4 - 4\alpha'} = \frac{1}{2},\ i = 1, 2, 3, 4.$$

Coman's result, 3.3.1 shows that $m_1(\{p_1, p_2, p_3, p_4\}) \geq 3$ which implies that at least three of the p_i are collinear which is a contradiction as we constructed them to be in general

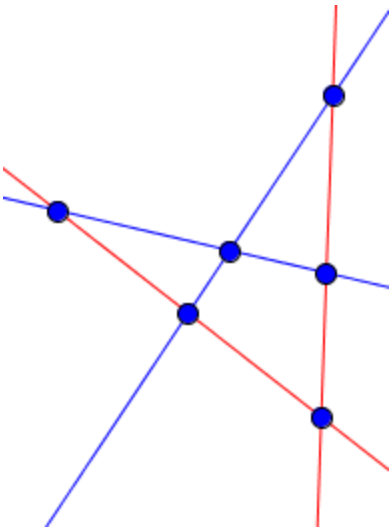
position. □

Theorem 4.1.1 now follows by combining the previous three propositions, since we only have three possibilities for the configuration of our four initial points. That is, the $\{q_i\}$ could be either all collinear, three of the four points collinear, or they are linearly independent (i.e. in general position).

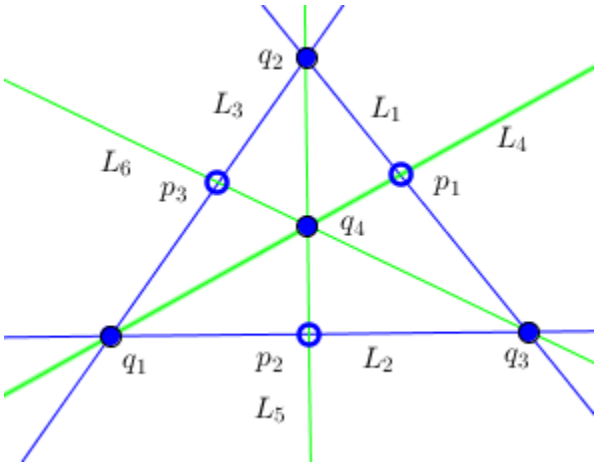
4.4 Examples

The following examples will show the necessity of allowing for $|E^+_\beta(T)\backslash C| = 1$ since we can have $E^+_\beta(T) \not\subset C$ for all conics C. Also we will see that that $\beta = \frac{2}{3}(1-\alpha)$ is sharp for this property, and that the result fails if we have less than four point with "large" Lelong number.

Example 4.4.1. Let L_i, $i = 1, 2, 3, 4$ be complex lines such that no three intersect at the same point. Define a current $T = \frac{1}{4}\sum_{i=1}^{4}[L_i]$ and let $\alpha = \frac{1}{2}$. Note that there are six points with Lelong number $\frac{1}{2}$, so we have satisfied the assumptions of the main theorem, and note that $\beta = \frac{1}{3}$. As each L_i contains exactly three points of $E^+_{1/3}(T)$, and any pair of the L_i contains exactly five of the points in $E^+_{1/3}(T)$ it follows that for any conic satisfying the result of the corollary, we have one point in $E^+_{1/3}(T)$ not on the conic.

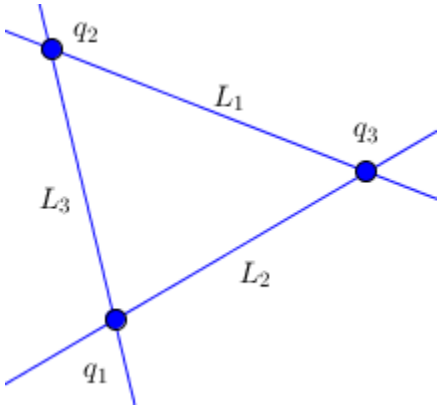


Example 4.4.2. Let L_i, $i = 1, 2, 3$, be complex lines such that they do not intersect at the same point. Let $L_1 \cap L_2 = \{q_3\}$, $L_1 \cap L_3 = \{q_2\}$, $L_3 \cap L_2 = \{q_1\}$. Let $q_4 \notin L_1 \cup L_2 \cup L_3$ and let L_4, L_5, L_6 be the lines connecting q_4 with q_1, q_2, q_3 respectively. Also $L_4 \cap L_1 = \{p_1\}$, $L_5 \cap L_2 = \{p_2\}$, $L_6 \cap L_3 = \{p_3\}$. Note that $m_1(\{p_1, p_2, p_3, q_4\}) = 2$. Finally define a current $T = \frac{1}{6}\sum_{i=1}^{6}[L_i]$. Note that $\nu(T, q_i) = \frac{1}{2}$ and $\nu(T, p_i) = \frac{1}{3}$. Let $\alpha = \frac{1}{2}$, and note that since $\beta = \frac{1}{3}$, we have that $E_\beta^+(T) = \{q_1, q_2, q_3, q_4\}$ which can clearly be contained in a conic, but $E_\beta(T) = \{q_1, q_2, q_3, q_4, p_1, p_2, p_3\}$, and $m_2(E_\beta(T)) = 5$.



Example 4.4.3. Let L_i, $i = 1, 2, 3$, be complex lines such that they do not intersect at the same point. Let $L_1 \cap L_2 = \{q_3\}$, $L_1 \cap L_3 = \{q_2\}$, $L_3 \cap L_2 = \{q_1\}$ and define a current

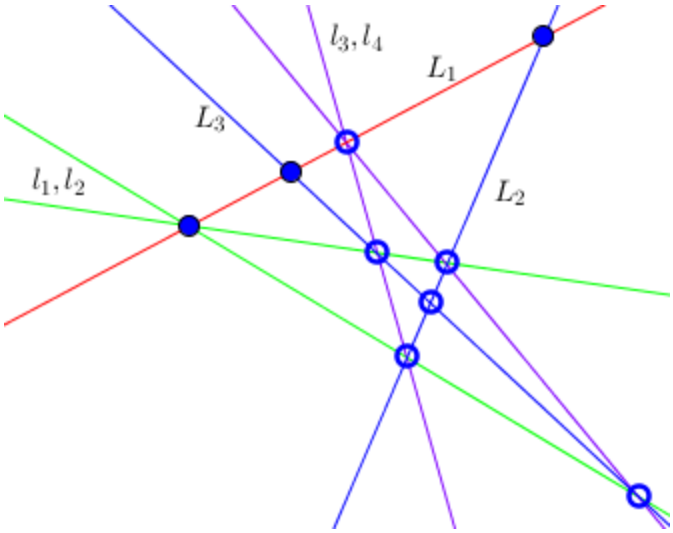
$T = \frac{1}{3}\sum_{i=1}^{3}[L_i]$. Note that $\nu(T, q_i) = \frac{2}{3}$ so if we set $\alpha = \frac{2}{3}$, then we have exactly three points with Lelong number greater than or equal to α, and $\beta = \frac{2}{9}$, thus then $E_\beta^+(T)$ contains all three lines, and $|E_\beta^+(T)\backslash C| = \infty$ for all conics C.



It is even interesting to note that the result fails in the special case where we have only three points with large Lelong number that are collinear.

Example 4.4.4. Let $\{q_i\}_{i=1}^3 \cup \{p_i\}_{i=1}^6$ be points and $\{L_i\}_{i=1}^3$ be lines such that $\{q_1, q_2, q_3, p_1\} \in L_1$, $\{q_1, p_2, p_3, p_6\} \in L_2$, and $\{q_3, p_4, p_5, p_6\} \in L_3$. Also let $\{l_i\}_{i=1}^4$ be lines such that $\{q_2, p_2, p_4\} \in l_1$, $\{q_2, p_3, p_5\} \in l_2$, $\{p_1, p_2, p_5\} \in l_3$, and $\{p_1, p_3, p_4\} \in l_4$. Let $\alpha = \frac{9}{20}$, which means $\beta = \frac{11}{30}$. We will instead write them as $\alpha = \frac{81}{180}$, $\beta = \frac{66}{180}$. We now consider the current given by

$$T = \frac{46}{180}[L_1] + \frac{37}{180}\sum_{i=2}^{3}[L_i] + \frac{19}{180}\sum_{i=1}^{2}[l_i] + \frac{11}{180}\sum_{i=3}^{4}[l_i]$$



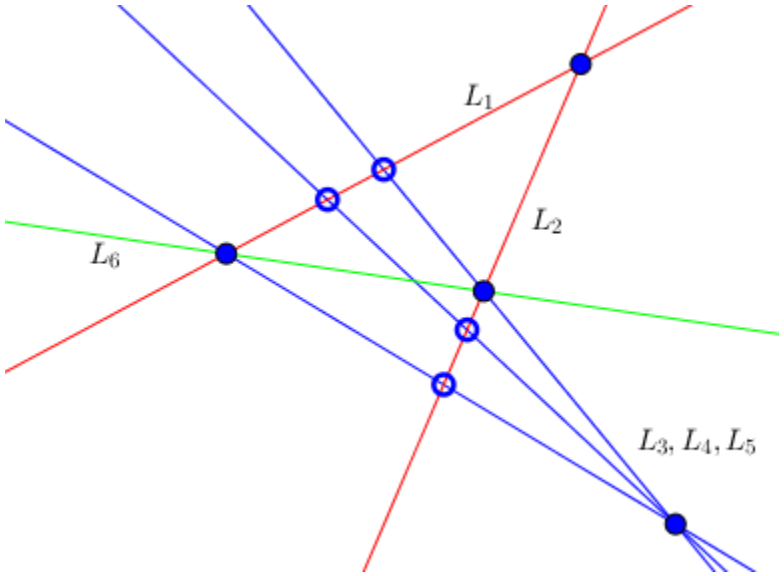
and note $\|T\| = 1$. Now calculating the Lelong numbers at each points we have:

$$\begin{aligned}
\nu(T, q_1) &= \frac{83}{180} \quad \nu(T, q_2) = \frac{84}{180}, \quad \nu(T, q_3) = \frac{83}{180} \\
\nu(T, p_1) &= \frac{68}{180} \quad \nu(T, p_2) = \frac{67}{180}, \quad \nu(T, p_3) = \frac{67}{180} \\
\nu(T, p_4) &= \frac{67}{180} \quad \nu(T, p_5) = \frac{67}{180}, \quad \nu(T, p_6) = \frac{74}{180}
\end{aligned}$$

and note $\nu(T, q_i) > \alpha$ for $i = 1, 2, 3$ and $\alpha > \nu(T, p_i) > \beta$ for $i = 1, \ldots, 6$. So we have exactly three points where T has Lelong number larger than α, and these are collinear. However there are no conics that can contain more than seven of the nine points, i.e. $|E^+_\beta(T) \backslash C| \geq 2$ for all conics C.

Example 4.4.5. Let $\{q_i\}_{i=1}^4 \cup \{p_i\}_{i=1}^4$ be points and $\{L_i\}_{i=1}^6$ be lines such that $\{q_1, q_2, p_1, p_2\} \in L_1$, $\{q_2, q_3, p_3, p_4\} \in L_2$, $\{q_2, q_4, p_2\} \in L_3$, $\{q_4, p_1, p_3\} \in L_4$, $\{q_1, q_4, p_4\} \in L_5$, and $\{q_1, q_3\} \in L_6$. Let $\alpha = \frac{9}{20}$, which means $\beta = \frac{11}{30}$. We will instead write them as $\alpha = \frac{81}{180}$, $\beta = \frac{66}{180}$. We now consider the current given by

$$T = \frac{42}{180}[L_1] + \frac{40}{180}[L_2] + \frac{27}{180}\sum_{i=3}^{4}[L_i] + \frac{28}{180}[L_5] + \frac{16}{180}[L_6]$$



and note $\|T\| = 1$. Now calculating the Lelong numbers at each points we have:

$$\begin{aligned}
\nu(T, q_1) &= \frac{86}{180} \quad \nu(T, q_2) = \frac{82}{180}, \quad \nu(T, q_3) = \frac{83}{180} \\
\nu(T, q_4) &= \frac{82}{180} \quad \nu(T, p_1) = \frac{69}{180}, \quad \nu(T, p_2) = \frac{69}{180} \\
\nu(T, p_3) &= \frac{67}{180} \quad \nu(T, p_4) = \frac{67}{180}
\end{aligned}$$

and note that any conic containing the four q_i points (i.e. any potential Γ_1) does not satisfy the conclusion of the theorem. Thus there are situations in which Γ_2 exists and is unique from any Γ_1.

As we have seen, the main result is very finely tuned. But one must wonder - can we generalize this out of $\mathbb{P}^2$? It is very limiting to be stuck in this specific space, but to prove

it, we needed to rely on the fact that bidegree $(1,1)$ is the same as bidimension $(1,1)$ in $\mathbb{P}^2$, which is not the case in higher dimensions. Furthermore, the work of Coman used to build up to my result is also confined to just $\mathbb{P}^2$. While it may seem as though this does not bode well for us, we push on regardless.

Chapter 5

Properties of Bidimension (p, p)

Currents on $\mathbb{P}^n$

5.1 Introduction

Let T be a positive closed current of bidimension (p,p) on $\mathbb{P}^n$ which has mass $\|T\| = 1$, and ω_n is the Fubini-Study form on $\mathbb{P}^n$. Our goal in this section is to gain more understanding of the geometric properties of upper level sets by attempting to generalize some of the results for bidimension $(1,1)$ currents laid out in [2] and [14] to analogous results for bidimension (p,p) currents by utilizing the tools given to us by Coman and Truong in [5]. The results contained in this chapter are from [15]. We start first by generalizing 3.3.5, which states that given a bidimension $(1,1)$ positive closed current T, $\alpha > \frac{1}{2}$, $\beta = (2-\alpha)/3$, and two points $q_1, q_2 \in \mathbb{P}^n$ such that $\nu(T, q_i) \geq \alpha$, then $E^+_\beta(T)$ can be contained in a line, with the exception of at most one point. In doing so we find that β depends on both p and α to get the following:

Theorem 5.1.1. [15, Theorem 1.1] Let T be a positive closed current of bidimension (p,p) on $\mathbb{P}^n$, $0 < p < n$, $\|T\| = 1$, $\alpha > \frac{p}{p+1}$ and $\beta = \frac{p^2+p-\alpha}{p(p+2)}$. Let q_1, q_2 be points in $\mathbb{P}^n$ such that $\nu(T, q_i) \geq \alpha$. Then either $E^+_\beta(T)$ is contained in a p-dimensional linear subspace or there exists a complex line L such that $|E^+_\beta(T) \backslash L| = p$.

The lower bound on α and the fact that we allow for p points to be omitted from the line, while seemingly an arbitrary choice, comes from the conclusion of 3.3.7, which we will recap shortly for the convenience of the reader. At the end of the third section, we will investigate two examples to show that this β value is sharp for this property, and that the assumption of needing two points q_1, q_2 where the current has "large" Lelong number is necessary. We also will generalize [2, Theorem 3.12], in which Coman showed that for a bidimension $(1,1)$ current T and $\alpha \geq 1/2$, if the set $E_{1-\alpha}$ contained three collinear points on some line L, then

$E^+_\alpha(T)$ is contained on L with the exception of at most one point.

We then turn our attention to generalizing [14, Proposition 3.6] (4.3.3), which was originally proved only for bidimension $(1,1)$ currents on $\mathbb{P}^2$, by proving the following:

Theorem 5.1.2. [15, Theorem 1.2] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^n$ with $\|T\| = 1$. Let $\{q_i\}_{i=1}^4$ be four collinear points in $\mathbb{P}^n$ such that $\nu(T, q_i) \geq \alpha > 2/5$. Let $\beta = \frac{2}{3}(1-\alpha)$. Then there exists two lines L_1, L_2 such that $|E^+_\beta(T) \backslash (L_1 \cup L_2)| \leq 1$.

We close by looking at a weak generalization of [14, Theorem 1.1] (4.1.1) from $\mathbb{P}^2$ to $\mathbb{P}^n$, and making some remarks on the difficulties of attempting to make a stronger result.

We now review the tools that will be commonly used in the upcoming proofs for the convenience of the reader. To show Theorem 5.1.1, we will recall the following result from the third chapter:

Theorem 5.1.3. [5, Theorem 1.2] If T is a positive closed current of bidimension (p,p) on $\mathbb{P}^n$, $0 < p < n$, with $\|T\| = 1$, then the set $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ is either contained in a p-dimensional linear subspace of $\mathbb{P}^n$ or else it is a finite set and $|E^+_{p/(p+1)}(T, \mathbb{P}^n) \backslash L| = p$ for some line L.

We will also make specific use of the previous theorem when $p = 1$, which we are already familiar with, see Theorem 3.3.1. We will also be working on generalizing the following result to $\mathbb{P}^n$:

Theorem 5.1.4. [14, Theorem 1.1] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^2$ with $\|T\| = 1$, $\alpha > 2/5$ and $\beta = \frac{2}{3}(1-\alpha)$. Let $\{q_i\}_{i=1}^4$ be points in $\mathbb{P}^2$ such that $\nu(T, q_i) \geq \alpha$. Then there exists a conic C (possibly reducible) such that $|E^+_\beta(T) \backslash C| \leq 1$.

The proof of the above theorem utilized 3.3.3, but for bidimension $(1,1)$ currents on $\mathbb{P}^n$, the only analogous theorem to assist us is the following:

Theorem 5.1.5. [5, Theorem 3.3] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^n$ with $\|T\| = 1$. If $|E^+_{2/5}(T)| > 37$ then there exists a curve $C \subset \mathbb{P}^n$ of degree at most 2 such that $|E^+_{2/5}(T)\backslash C| \leq 1$.

5.2 Some Generalizations of Coman's Results

To start with, let us recall some basic definitions. Consider $A = \{x_1, \ldots, x_{p+1}\}$, $x_i \in \mathbb{P}^n$. By the Span (A), we mean the smallest linear subspace $V \subset \mathbb{P}^n$ that contains A. If $p \leq n$ and span(A) is a p-dimensional space, then we say $\{x_i\}_{i=1}^{p+1}$ are linearly independent. If we have $p > n+1$ points, then we say they are in general position if any $n+1$ of them are linearly independent. Assume $\|T\| = 1$ and we now prove Theorem 5.1.1.

Proof. Suppose $\{x_i\}_{i=1}^{p}$ are points in $E^+_\beta(T)\backslash\{q_1, q_2\}$, and let $A := \{q_1, q_2, x_1, \ldots, x_{p-1}\}$ and then $V_1 = \text{span}(A)$. Suppose that $\{q_1, q_2\} \cup \{x_i\}_{i=1}^{p-1}$ are linearly independent and then $V_1 := \text{span}(A)$ is a p dimensional linear subspace and $x_p \in E^+_\beta(T)\backslash V_1$, noting that if no such points exists, then there is nothing to prove. Let L_1 be the line spanned by q_1, q_2, and since the points in A are linearly independent, L_1 does not contain any other points of A or x_p. Note if $E^+_\beta(T)\backslash(A \cup \{x_p\}) = \emptyset$ then $|E^+_\beta(T)\backslash L_1| = p$, and we are done. Suppose then that $x_{p+1} \in E^+_\beta(T)\backslash(A \cup L_1)$, $x_p \neq x_{p+1}$. Let V_2 be a p-dimensional linear space containing $\{x_i\}_{i=1}^{p+1}$ (observe V_2 need not be the only such p-dimensional linear subspace containing these points). Choose α' such that $\frac{p}{p+1} < \alpha' < \alpha$ and $\nu(T, x_i) > \frac{p^2+p-\alpha'}{p(p+2)}$, and define the current R as follows:

$$R := \frac{(p+1)\alpha' - p}{(p+1)\alpha'}[V_2] + \frac{p}{(p+1)\alpha'}T.$$

Note that $\|R\| = 1$ as well as

$$\nu(R, q_i) \geq \frac{p}{(p+1)\alpha'}\nu(T, q_i) > \frac{p}{p+1} \quad i = 1, 2$$

and

$$\nu(R, x_i) > \frac{(p+1)\alpha' - p}{(p+1)\alpha'} + \frac{p}{(p+1)\alpha'}\frac{p^2 + p - \alpha'}{p(p+2)}$$

$$= \frac{(p+2)(p+1) - 1}{(p+2)(p+1)} - \frac{p}{(p+2)(p+1)\alpha'} > \frac{p}{p+1}, \quad i = 1, \ldots, p+1.$$

Thus by Coman 3.3.7, it must be the case that there is a line containing three points of $\{q_1, q_2, x_1, \ldots, x_{p+1}\}$, say L_2. We now have to break our argument into two cases, depending on if x_{p+1} is contained in V_1 or not.

Case 1: Suppose that $x_{p+1} \notin V_1$. Note that L_2 cannot contain 3 points of the set A as those points are linearly independent. Thus it must be the case that L_2 contains both x_p, x_{p+1} as otherwise if L_2 only contains one of them, the other two points would be from the set A, which means either x_p or x_{p+1} would be in the span of A, which is a contradiction. So L_2 contains x_p, x_{p+1} and some $y \in A$. Now note we can find a new point $x_{p+2} \in E^+_\beta(T)\backslash(A \cup L_2)$, as otherwise $|E^+_\beta(T)\backslash L_2| = p$ and we would be done. Let $B := \{x_i\}_{i=1}^{p+2}$, and let U_i be a p-dimensional linear space containing $B\backslash\{x_i\}$. Choose α' such that $\frac{p}{p+1} < \alpha' < \alpha$ and $\nu(T, x_i) > \frac{p^2+p-\alpha'}{p(p+2)}$ for $i = 1, \ldots, p+2$. We now consider a new current S given by:

$$S := \frac{(p+1)\alpha' - p}{(p+1)(p+2)\alpha'} \sum_{i=1}^{p+2} [U_i] + \frac{p}{(p+1)\alpha'} T$$

note that $\|S\| = 1$ as well as

$$\nu(S, q_i) > \frac{p}{p+1} \quad i = 1, 2$$

and

$$\nu(S, x_i) > \frac{(p+1)\alpha' - p}{(p+1)(p+2)\alpha'}(p+1) + \frac{p}{(p+1)\alpha'} \frac{p^2 + p - \alpha'}{p(p+2)}$$

$$= \frac{(p+1)^2\alpha' - (p+1)p + p^2 + p - \alpha'}{(p+1)(p+2)\alpha'} = \frac{p}{p+1}, \quad i = 1, \ldots, p+2.$$

Thus by 3.3.7, there exists a complex line L_3 that will contain four points of $A \cup \{x_p, x_{p+1}, x_{p+2}\}$. As the points in A are in general position, L_3 can only contain at most two points of A. If L_3 contains two points of A, then L_3 will also contain at least one of x_p or x_{p+1} which means that at least one of x_p, x_{p+1} lies in V_1, which is impossible. So L_3 can only contain one point of A which means L_3 contains x_p, x_{p+1}, x_{p+2}. But now that means $L_3 = L_2$, and this is a contradiction as $x_{p+2} \notin L_2$. So no such point x_{p+2} can exist, and L_2 is the line that satisfies the conclusion of the theorem.

Case 2: Suppose that $x_{p+1} \in V_1$. As L_2 must contain three points of $A \cup \{x_p, x_{p+1}\}$, it must be the case that L_2 contains x_{p+1}, x_j, x_k (note that x_j, x_k may actually be the q_i, but that is irrelevant) as the points in A are in general position and $x_p \notin \text{span}(A)$. After reindexing, say that L_2 contains x_1, x_2, x_{p+1}. Arguing as we did in the first case, we can find a new point $x_{p+2} \in E_\beta^+(T) \backslash L_2$ and a line L_3 containing four points of $A \cup \{x_p, x_{p+1}, x_{p+2}\}$.

Note $L_3 \subset V_1$. As the points of A are in general position, and since $x_{p+2} \notin L_2$, it must be the case that L_3 contains x_{p+1}, x_{p+2} and (after reindexing) x_3, x_4. But now observe that L_2 is the line that spans x_1, x_{p+1}, L_3 is the line that spans x_3, x_{p+1}, and $L_2 \cap L_3 \neq \emptyset$, so we have a 2-dimensional linear space containing x_1, x_2, x_3 and x_4, which is a contradiction as they are linearly independent. Thus we cannot have such a point x_{p+2}, and L_2 is the desired line that satisfies the conclusion of our theorem.

□

Remark: When $p = 1$, we have that $\alpha > 1/2$ and that $\beta = (2 - \alpha)/3$, which is exactly the version proved by Coman, 3.3.5.

Consider now a positive closed bidimension (p, p) current T on $\mathbb{P}^n$. We now show that if T has a "small" Lelong number at a sufficient number of points in a p-dimensional linear subspace V, then either $E^+_\alpha(T) \subset V$ or that the line L satisfying the conclusion of Theorem 5.1.1 is contained in V.

Theorem 5.2.1. [15, Theorem 3.1] Let T be a positive closed current of bidimension (p, p) on $\mathbb{P}^n$ with $\|T\| = 1$, $\alpha \geq \frac{p}{p+1}$ and suppose there are points $x_1, \ldots, x_{p+2} \in E_{1-\frac{\alpha}{p}}$ such that $\{x_i\}_{i=1}^{p+2}$ span a p-dimensional linear subspace V. Then either $E^+_\alpha(T)$ is contained in V or there exists a complex line $L \subset V$ such that $|E^+_\alpha(T) \backslash L| = p$.

Proof. Let $\{x_i\}$ and V be as stated above, and suppose there exists a point $q_1 \in E^+_\alpha(T) \backslash V$, noting if no such point exists, we are already done. Choose $\alpha' > \alpha$ such that $\nu(T, q_1) > \alpha'$, and thus $\nu(T, x_i) \geq 1 - \frac{\alpha}{p} > 1 - \frac{\alpha'}{p}$. We consider now the current R given by

$$R = \frac{(p+1)\alpha' - p}{(p+1)\alpha'}[V] + \frac{p}{(p+1)\alpha'}T$$

and observe that $\|R\| = 1$, $\nu(R, q_1) > \frac{p}{p+1}$ and

$$\nu(R, x_i) > \frac{(p+1)\alpha' - p}{(p+1)\alpha'} + \frac{p - \alpha'}{(p+1)\alpha'} = \frac{p}{p+1}.$$

Thus by 3.3.7, either $\{q_1, x_1, \ldots, x_{p+2}\}$ are in a p-dimensional linear subspace or there is a line L such that $|E^+_{\frac{p}{p+1}}(R) \backslash L| = p$. Since $\{x_i\}_{i=1}^{p+2}$ uniquely define V, and $q_1 \notin V$, it must be the case that there is a line L containing exactly 3 of the $p+3$ points. As L must contain two points in V, it must be the case that $L \subset V$ and thus $q_1 \notin L$, and say after reindexing that L contains the points x_1, x_2, and x_3.

We now show via contradiction that L is the line satisfying the conclusion of the theorem. Suppose there is $q_2 \in E^+_\alpha(T) \backslash L$, $q_2 \neq q_1, x_i$. Choose $\alpha' > \alpha$ such that $\nu(T, q_1) > \alpha'$ and $\nu(T, q_2) > \alpha'$. Using the current R as above we get $\nu(R, q_i) > \frac{p}{p+1}$ and $\nu(R, x_i) > \frac{p}{p+1}$. Again we apply 3.3.7 and since we cannot contain the $p+4$ points in V, we get that there is a line L_1 that must contain 4 of the $p+4$ points. As two of those points must be in V, $L_1 \subset V$ and thus L_1 must contain at least three of the x_i points. As the x_i span V, the only three collinear points in V are x_1, x_2, x_3, thus it is the case that $L_1 = L$, but then either $q_2 \in L$ which cannot happen, or L contains 4 of the x_i points, which contradicts the fact that they span V. Thus no such q_2 can exist, and L is the line that satisfies the conclusion.

□

Remark: When $p = 1$, the previous theorem is exactly Theorem 3.12 from [2].

We close this section by looking at some examples that will demonstrate the necessity of the assumptions of Theorem 5.1.1.

Example: We will first show that we need two points with Lelong number larger than $\frac{p}{p+1}$. To see this, let $A := \{q, x_1, x_2, \dots x_{p+1}\}$ be linearly independent points in $\mathbb{P}^n$, and let $V_j := \text{span}(A \backslash \{x_j\})$. Further, let $L_j := \cap_{i=1, i\neq j}^{p+1} V_i$, so L_j is the line spanning q and x_j. Consider the following current:

$$T = \frac{1}{p+1} \sum_{i=1}^{p+1} [V_i]$$

and note $\nu(T, q) = 1$, and that q is the only point where T has Lelong number strictly larger than $\frac{p}{p+1}$. Also note along any line L_j, T has Lelong number $\frac{p}{p+1}$ and given any V_i, there is some line L_k not contained in V_i. Since $\beta < \frac{p}{p+1}$, $E_\beta^+(T)$ is not contained in a p-dimensional linear subspace, and no matter what line L we consider, $|E_\beta^+(T) \backslash L| = \infty$.

Example: We will now show that the value β is sharp for this property. Let $p = 1$, thus $\beta = \frac{2-\alpha}{3}$. Let L_i, $i = 1, 2, 3, 4$ be complex lines and q_1, q_2, p_1, p_2 be points such that $L_1 \cap L_3 \cap L_4 = \{q_1\}$, $L_1 \cap L_2 = \{q_2\}$, $L_2 \cap L_3 = \{p_1\}$, and $L_2 \cap L_4 = \{p_2\}$. We consider the following current

$$T = \frac{7}{15}[L_1] + \frac{6}{15}[L_2] + \frac{1}{15} \sum_{i=3}^{4} [L_i]$$

and note $\|T\| = 1$. Now calculating the Lelong numbers at each of the four previously mentioned points we have:

$$\nu(T, q_1) = \frac{9}{15}, \quad \nu(T, q_2) = \frac{13}{15}, \quad \nu(T, p_1) = \frac{7}{15}, \quad \nu(T, p_2) = \frac{7}{15}.$$

Let $\alpha = \frac{9}{15} = \frac{3}{5}$, noting that $\nu(T, q_i) \geq \alpha$. Further, $\beta = \frac{7}{15}$, and we observe that $E_\beta(T) =$

$L_1 \cup \{p_1, p_2\}$, and thus $|E_\beta(T)\backslash L_1| = 2$ and $|E_\beta(T)\backslash L| = \infty$ for any line $L \neq L_1$. Finally we observe that $E^+_\beta(T) = \{q_1, q_2\}$, and $|E^+_\beta(T)\backslash L_1| = 0$, showing that the parameter β is the best it can be.

We now look at one last interesting example. In the statement of 3.3.7, Coman mentions that if $E^+_\alpha(T)$ is not contained in a p-dimensional linear subspace, then the upper level set must be finite. This however is no longer true in Theorem 5.1.1 for $E^+_\beta(T)$, as we see below.

Example: Suppose that A_i, $i = 1, \ldots, p+2$ are p-dimensional linear subspaces of $\mathbb{P}^n$, $n > p$, such that $L = \bigcap_{i=1}^p A_i$. Let V be the $(p-1)$-dimensional linear space given by $A_{p+1} \cap A_{p+2}$. Let $\{q_i\} = L \cap A_{p+i}$, $i = 1, 2$, and $\{x_i\} = (\bigcap_{j\neq i, j=1}^p A_j) \cap V$, $i = 1, \ldots, p$. Finally we suppose the A_i are arranged so that $\{q_1, q_2, x_1, \ldots, x_p\}$ cannot be contained in a p-dimensional linear subspace. Consider now the following current:

$$T = \frac{1}{p+1}\sum_{i=1}^{p}[A_i] + \frac{1}{2(p+1)}([A_{p+1}] + [A_{p+2}]).$$

It is clear that $\|T\| = 1$, and that $\nu(T, q_i) > \frac{p}{p+1}$. We note then for any α such that $\nu(T, q_i) \geq \alpha > \frac{p}{p+1}$, we have $\beta < \frac{p}{p+1}$. We now observe that $\nu(T, x_i) = \frac{p}{p+1}$, and give any point $y \in L$, $\nu(T, y) = \frac{p}{p+1}$. Thus we have that $E^+_\beta(T)$ is not contained in a p-dimensional linear subspace, $|E^+_\beta(T)\backslash L| = p$, however $E^+_\beta(T)$ is not finite.

5.3 Generalizations of Results from $\mathbb{P}^2$ to $\mathbb{P}^n$

Our goal in this section is to attempt to generalize 4.1.1 from $\mathbb{P}^2$ to $\mathbb{P}^n$. In the original proof, we relied heavily on 3.3.3, which is only valid in $\mathbb{P}^2$. Later attempts to generalize 3.3.3 have

yielded good results for all bidimensions except bidimension $(1,1)$, which is sadly the case we would need (see [5] for more details). If we have the situation in which our current has a "large" Lelong number at four points that are all on a line, then we can avoid the necessity of using 3.3.3, and can generalize it to $\mathbb{P}^n$ with ease. We now prove Theorem 5.1.2.

Proof. Let L_1 be the line containing q_1, q_2, q_3, and q_4. By Siu's decomposition theorem 3.1.21 we have that

$$T = a[L_1] + R,$$

where a is the generic Lelong number of T along L_1. Note that $\|R\| = 1 - a$ and $\nu(R, q_i) \geq \alpha - a$. By [17] there is a bidegree $(1,1)$ current S such that $\|S\| = \|R\|$ and $\nu(S, x) = \nu(R, x)$ at all x. Proposition 2.5 shows that there exists a current S' such that $\|S'\| = 1 - a$, S' is smooth where S has Lelong number 0, and $\nu(S', q_i) > \alpha - a - \epsilon$. By 2.3.4, $S' \wedge [L_1]$ is a well defined measure. Now we have

$$1 - a = \int_{\mathbb{P}^n} S' \wedge [L_1] \geq \sum_{i=1}^{4} \nu(S' \wedge [L_1], q_i)$$

$$\geq \sum_{i=1}^{4} \nu(S', q_i)\nu([L_1], q_i) > 4\alpha - 4a - 4\epsilon,$$

where the second inequality follows from 3.1.13 and the final inequality follows as $\nu([L_1], q_i) = 1$. So we have that $a \geq \frac{4\alpha - 1}{3}$ as $\epsilon \searrow 0$.

Define a new current:

$$R' = \frac{R}{1-a}$$

and note R' is a bidimension $(1,1)$ current with $\|R'\| = 1$. Now we have for $x \in E_\beta^+(T)\backslash L_1$:

$$\nu(R', x) > \frac{2}{3}\frac{1-\alpha}{1-a} \geq \frac{2-2\alpha}{4-4\alpha} = \frac{1}{2}.$$

Theorem 3.3.1 shows that there is a point y and a line L_2 such that $E_\beta^+(T)\backslash L_1 \subset L_2 \cup \{y\}$, and the theorem is proven. □

Remark: For $n \geq 3$, the two lines need not intersect.

Using Theorem 5.1.2 combined with Theorem 5.1.5, we can get the following partial result:

Theorem 5.3.1. [15, Theorem 4.1] Let T be a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^n$ with $\|T\| = 1$, $\alpha > 2/5$ and $\beta = \frac{2}{3}(1-\alpha)$. Assume we have one of the following:

i) $\alpha < 1/2$ and $E_\alpha^+(T)$ contains 4 collinear points, or

ii) $\alpha < 1/2$, $|E_\alpha^+(T)| > 37$, or

iii) $\alpha \geq 1/2$ and $|E_\alpha^+(T)| > 4$

Then there is a curve C in $\mathbb{P}^n$ of degree at most 2 such that $|E_\beta^+(T)\backslash C| \leq 1$.

Proof. (i) This follows immediately from Theorem 5.1.2.

(ii) By Theorem 5.1.5, we know that there is a curve C_1 such that $|E_\alpha^+(T)\backslash C_1| \leq 1$. If C_1 omits a point of $E_\beta^+(T)$, call it p_1, then we can find a point $p_2 \in E_\beta^+(T)\backslash C_1$, otherwise

we are done. If C_1 omits no points of $E^+_\alpha(T)$, then again we can find $p_1, p_2 \in E^+_\beta(T)\backslash C_1$, otherwise we are done. In either case, note that C_1 must contain at least 38 points of $E^+_\beta(T)$. We consider the cases of if C_1 is an irreducible degree 2 curve, or not.

Case 1: First suppose C_1 is an irreducible degree 2 curve. Let α' be such that $\alpha > \alpha' > 2/5$, and $\nu(T, p_i) > \frac{2}{3}(1-\alpha') > \beta$, and let L_{12} be the line spanned by p_1, p_2. Define a current R as follows:

$$R = \frac{5\alpha' - 2}{5\alpha'}[L_{12}] + \frac{2}{5\alpha'}T$$

and note $\|R\| = 1$. We have the following inequalities:

$$\nu(R, q) > \frac{2}{5\alpha'}\alpha > \frac{2}{5},\ q \in E^+_\alpha(T)$$

and

$$\nu(R, p_i) > \frac{5\alpha' - 2}{5\alpha'} + \frac{4 - 4\alpha'}{15\alpha'} > \frac{2}{5},\ i = 1, 2.$$

So by the Theorem 5.1.5, we can find a conic C_2 such that $|E^+_\alpha(R)\backslash C_2| \leq 1$. Since C_1 is irreducible, it is a plane conic by [10, Proposition 0]. If C_2 is irreducible as well, then since $|C_1 \cap C_2| > 4$, Bezout's theorem show that $C_1 = C_2$, which is impossible as this means one of the p_i are now on C_1. If C_2 is reducible, then it decomposes into two lines, and can only intersect C_1 at most four times, which is a contradiction.

Case 2: If C_1 is a reducible conic then $C_1 = L_1 \cup L_2$, for some pair of complex lines. But note then that for one of the lines, say w.l.o.g. L_1, $|L_1 \cap E^+_\alpha(T)| > 4$, and we have at least

four collinear points in $E^+_\alpha(T)$, so we are back to situation (i).

(iii) By 3.3.1 there is a line L such that $|E^+_{1/2}(T)\backslash L| \leq 1$, so L contains at least four points of $E^+_\alpha(T)$, and we are done by Theorem 5.1.2.

□

Closing Remarks: The span of two non-concurrent lines has dimension 3, which is the reason that the bidimension $(1,1)$ case did not generalize into $\mathbb{P}^n$ (see [5] Theorem 1.3, Proposition 3.2, and the remarks following Proposition 3.2 for more details). This still leaves the following open question: Suppose now that T is a positive closed bidimension $(1,1)$ current on $\mathbb{P}^n$. If we allow for a pair of non-concurrent lines, does there exist a degree 2 curve C such that $|E^+_{2/5}(T)\backslash C| \leq 1$? Perhaps one day we will know, but that day is not today.

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AMS Special Section, *SUNY Buffalo*, Buffalo, NY, *Geometric Properties of Upper Level Sets of Lelong Numbers of Currents on* $\mathbb{P}^n$.

MGO 42^{nd} **Annual Conference**, *Syracuse University*, Syracuse, NY, *Algebraic Geometric Properties of Upper Level Sets of Lelong Numbers of Closed Positive Bidimension* (p,p) *Currents on* $n-$*dimensional Complex Projective Space... and other fun math related things... .*

2016 **Analysis Seminar and MGO Colloquium**, *Syracuse University*, Syracuse, NY, *A Property of Upper Level Sets of Lelong Numbers of Currents on* $\mathbb{P}^2$.

MGO 41^{st} **Annual Conference**, *Syracuse University*, Syracuse, NY, *Not Quite Harmonic....*

MGO Colloquium, *Syracuse University*, Syracuse, NY, *Liouville's Theorem and Other Fun Things.*

Other Conferences and Workshops Attended

2017 **Northeast Analysis Network Conference**, *Syracuse University*, Syracuse, NY.

Midwest Several Complex Variables, *Brown University*, Providence, RI.

2017 Global Research Symposium: Geometry, Analysis, and Probability, *Korea Institute for Advance Study*, Seoul, South Korea.

2016 **Harmonic analysis, complex analysis, spectral theory and all that**, *IMPAN*, Bedlewo, Poland.

Midwest Several Complex Variables, *University of Toledo*, Toledo, OH.

Winter School in Sanya for Complex Analysis and Geometry, Sanya, China.

Classes Taught

- The following are the courses for which I was the primary instructor who prepared and presented the lectures as well as the quizzes, exams, and homework.

MAT121: Probability and Statistics I: Summer 2012

MAT183: Elements of Modern Mathematics: Summer 2014

MAT222: Probability and Statistics II: Spring 2012, Spring 2013

MAT295: Calculus I: Fall 2013, Spring 2014, Spring 2016, Fall 2016, Fall 2017

MAT296: Calculus II: Fall 2014, Spring 2017

MAT397: Calculus III: Fall 2015, Summer 2017, Spring 2018

Other.

Grader for MAT601/602 Real Analysis I/II: Fall 2015-Present

Service

Mathematics Graduate Organization (MGO), Syracuse University

- The MGO is the official department organization that is responsible for the ongoings that involve all of the math, applied math, and math ed. graduate students. Of particular interest is the annual conference, which is organized entirely by the graduate students, who reach out to mathematics professors and students from other universities to attend and give presentations.

2014-2015 **MGO President**.

2015-2016 **MGO Vice-President**.

- Duties included organizing the 40th and 41st Annual MGO Conference held in April 2015 and April 2016 at Syracuse University, working with faculty to help better the graduate program, organizing the annual department picnic. During my time as president, I was able to register Syracuse University's math department as a graduate student chapter of the AMS.

Graduate Student Organization (GSO), Syracuse University

2013-2015 **GSO Senator, Mathematics department representative**.

- Duties included representing the graduate students of the mathematics department. This involved requesting funding for events and voting on behalf of the department on policy changes that would impact the graduate students.

Other

Summer 2016 **Syracuse University Project Advance (SUPA)**.

- Duties included revising the exams and curriculum of the calculus courses taught at local high schools. The SUPA version of the calculus courses offered to senior high school students would fulfill the Calculus 1 requirement at Syracuse University, as well as count for credit at various other universities.

Professional Memberships.

- American Mathematical Society

Professional References

- Dan Coman
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  dcoman@syr.edu

- Eugene Poletsky
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  Syracuse University, Syracuse, NY.
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- Tadeusz Iwaniec
  Department of Mathematics
  Syracuse University, Syracuse, NY.
  tiwaniec@syr.edu

- Duane Graysay (Teaching)
  Department of Mathematics
  Syracuse University, Syracuse, NY.
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