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Geometry of Hilbert Space Frames

Stephen Sorokanich III
Syracuse University

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Stephen Sorokanich III
Candidate for a B.S. Degree in Mathematics and Physics and Renée Crown University Honors
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Capstone Project Advisor: ______________________________
Professor Leonid Kovalev

Capstone Project Reader: ______________________________
Professor Claudia Miller

Honors Director: ______________________________
Stephen Kuusisto, Director

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Abstract

In applied linear algebra, the term *frame* is used to refer to a redundant or linearly dependent coordinate system. The concept was introduced in the study of Fourier series and is pertinent in signal processing, where the reconstruction property for finite frames allows for redundant transmission of data to guard against losses due to noise. We give a brief introduction to the theory of finite frames in Section 1, including the major results that allow for the easy construction and description of frames. The subsequent sections relate to the theoretical importance of frames. As a natural extension of the definition of basis, we are lead naturally to ask the same topological questions for the space of frames as we do for the space of bases, $GL_n(\mathbb{R})$. Two particular questions are explored in Sections 2 and 3. The reconstruction property for finite frames leads to a natural generalization in the realm of measure theory. This is the scope of Section 4, culminating in the approximation theorem for "frame measures".

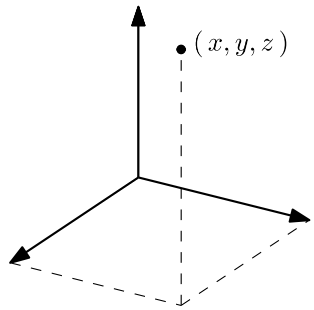
Executive Summary

The focus of this thesis, Hilbert space frames, is a bridge between pure and applied mathematics, general theory and problem solving, and the formal development of a topic versus intuitive geometric motivation. The core concept, that of a finite frame, is quite simple—yet surprising and deep results emerge from their study. Frames motivate important, and difficult, questions in the range of mathematical disciplines: analysis, topology, linear algebra, differential geometry, and abstract algebra. At the same time, for the undergraduate they offer an optimal segway into complicated topics such as operator analysis and measure theory.

Finite frames are motivated by, and have an important role in, robust data transmission and image processing, and it is immediate in their theoretical treatment how such applications emerge. As a concrete example, imagine transmitting a signal encoded as an ordered set of numbers. A noisy transmission might lead to several of these numbers being altered or even deleted. But just as we can understand a spoken sentence without necessarily hearing every word via context clues, if the transmission is in some sense "evenly spread out" over the length of the message, the signal may be read accurately enough even with the loss of some numbers. This is the crucial idea that makes frames so useful in practice. It has also inspired abstract theoretical developments of significance.

Let's make this discussion a little more concrete. A signal might be position data in three-dimensional space sent from a satellite in orbit. In this case, a transmission $\boldsymbol{T}$ will consist of three coordinates, say x, y, z.

Figure 1: A point in three dimensions, with coordinate representation

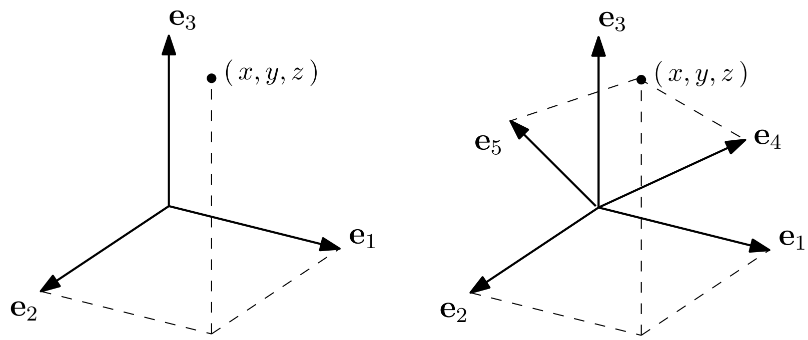


These coordinates express the distance of the satellite from some fixed reference point, along the x, y and z directions, or *axes*.

It is easy to imagine how hopeless the situation would be if our satellite were only able to transmit x and y data after some malfunction. One very simple way to avoid the threat of such a problem is with *redundancy*: although we cannot express position in space completely without 3 axes, but we are not limited to *only* 3 axes. We can simply decide to measure along more directions and describe the satellite's position with extra components. Lost data from one axis no longer matters (or doesn't matter as much), since that data can be replaced by a reading from another axis. We can make as many axes as we want until we are either confident enough against loss of data, or run out of memory to store the coordinates. Labeling these axes by $\mathbf{e}_i$'s , we get a situation like **Figure 2**.

Now this is all fine for making sure no data is lost, but the method doesn't help when it comes to manipulating it. In the case of the satellite, it isn't difficult to identify a bad transmission in some component, and to take compensating steps later on to ignore the faulty data. But when dealing with transmissions of, say, thousand-dimensional data, automation is crucial. Algorithms will transform the data, whether or not the transmission accurately reflects the truth.

Figure 2: Adding axes to prevent data loss



This is where the theory of frames begins: at the crux of this application. It is already natural to ask the motivating question of the theory. Is it possible to add axes to our data transmission in a way that faulty data along one or several axes will not affect results significantly? The way this is quantified relies on fundamental concepts in linear algebra and operator theory, but can be summarized with our example of three dimensional data. Let's call any choice of 3 axes capable of measuring any position in space a *basis*. A general position, like (x, y, z) from before, will have components with respect to each of the axes in a basis.

To be clear, there is a mathematical difference between an object's position, and the coordinates that we measure this position with. The position exists irrespective of how we choose what axes to use. We sometimes denote this pure position by a boldface symbol, like v. In order to work with this position concretely though, it is necessary to measure its coordinates. It might already be apparent that how we choose our axes will affect what coordinate representation we give a position. The position data of our satellite will certainly look different if we measure from Cape Canaveral versus Houston. In the broadest sense, this is what the branch of linear algebra is concerned with. It turns out that relating how a set of coordinates changes when changing axes is not hard.

However, as mathematicians, we would like to use the abstract boldface description of position as much as possible - it clarifies proofs and allows for greater generality. We can develop an algebra of scaling axes by their components, and adding these axes together. In this language, our coordinate representation of position, (x, y, z) will look like

$$x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3.$$

It doesn't look like such a big deal here, because we used the component data that we assumed we had. In order for the abstract boldface notation to really be useful in a unique way, we need to be able to express the components x, y, z in terms of the abstract position $\boldsymbol{v}$ instead of assuming we can measure along the axes. There is no simple mathematical formula giving the components for a general basis. For a special subset of bases though, called orthonormal bases, a classic result of linear algebra gives an explicit formula for these components in terms of the abstract position. We will often refer to this as the *reconstruction formula* with respect to an orthonormal basis, since it allows one to compute the components needed to find a position with a specific set of axes. For clarity, let's denote the reconstruction formula with respect to the axes $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ by $R(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. The reconstruction formula is the key to making redundant sets of axes less sensitive to faulty data. In particular, for a set of k axes $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, and an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, we can replace every $\boldsymbol{e}_i$ in the reconstruction formula $R(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ with the corresponding $\boldsymbol{v}_i$, and study how close this new formula is to reconstructing any given signal. In symbolic terms, we might represent this replacement by

$$R(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \rightsquigarrow R(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k).$$

It turns out, remarkably, that some redundant, non orthonormal choices of axes have a perfect reconstruction formula - the components calculated by pretending that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_k$ forms an orthonormal basis are exactly the components with respect to $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_k$! Such a set is the eponymous frame of finite frame theory. We can in fact find frames with any number of axes we want. For the satellite, this means we can take position readings with any number of components, and reconstruct the position perfectly. The guard against faulty data is now possible: the more axes we have, the less contribution each one has in the reconstruction formula. We can effectively safeguard against a bad transmission in some components by increasing the total number of axes to take components with respect to. The result of reconstruction with bad components is now practically the same signal that was sent originally, and manipulating the components will give nearly the same results as manipulating components of a basis.

The mathematical framework of this theory is developed in Section 1. It turns out that non-perfect reconstruction comes in varying degrees, and a given choice of axes can be classified by how badly it violates the reconstruction formula. These axes are just as important in theory as the frames with perfect reconstruction, since both have similar geometric characterizations. Thus, we include the non-perfectly-reconstructing choice of axes in the definition of frame.

The theory of finite frames motivates difficult questions in pure mathematics. Two of these questions, topological in nature, are the focus of Sections 2 and 3. Moreso, just as frames generalize bases, the definition of frame can be extended to include infinitely many axes. This is the focus of Section 4. The generalization requires a proper reconstruction formula with respect to infinitely many axes. The result of this generalization is surprising - the finite frames can approximate infinite frames as well as desired.

Contents

Advice to Future Honors Students

So you're writing a thesis in mathematics? There is only one piece of advice. Start early, and start typing.

While the scientist has a lab or raw data to structure their investigation, the mathematician, without these, can feel particularly lost in a vast and abstract place. At times, an argument might seem obvious until it comes to putting it to paper. This is the biggest danger, especially when writing for a deadline. Scribbled notes can contain holes that you assumed were filled, and a theory might seem a collection of disparate facts until the student finally gathers the courage to type. When typing, strive for terseness without sacrificing thorough explanation. The best mathematical writings have a refined beauty to them. They have the ability to speak simply, and paint a picture of a theory as a coherent whole. To write mathematics is an art and a balancing act, between maintaining a sufficient level of rigor, while retaining a global scope and easing the reader into a subject.

To *do* mathematics is another thing, arguably a thing that the undergraduate should limit in their prospects. To begin a thesis expecting to discover a great theorem is simply counterproductive, not to mention unreasonable. In the author's opinion, the most under appreciated aspect of mathematical discovery for undergraduates is the simple reinterpretation of existing results. It is a subtle business to reinterpret (or shall I say, re-frame), and not very glorified in the curriculum. You might hear these results referred to as "grunt work", or "machinery". They live in propositions, and might confuse the reader in their apparent obviousness. But they are *crucial*. If the mathematician has two tools, they are the formal proof, and the reframing of results. The former is what distinguishes mathematics from the other disciplines. The latter, however, is just as important, and might be called intuition. Instead of classifying finite Abelian groups, we classify finitely generated Modules over PIDs and get the result for groups as a consequence. In real analysis, we view the derivative on the real line as an approximating linear function and open up the world of multivariable calculus.

So how do you proceed writing an undergraduate thesis in mathematics? Your professor will likely give you a broad topic of investigation early on. Find a text to make your primary reference, and do as many exercises from it as possible. Then type them up. Then, perhaps six months later, type up "preliminary results" by recounting the theory from this text as best as possible, proofs included, *from memory*. This will become the first part of your thesis. All the while, you should be asking your advisor many questions about your topic. Most will be silly, but one may become another chapter of your thesis.

Inevitably, you will hit a dry spell. You will be lost trying to prove something in a topic you barely know anything about. After all, you are just an undergraduate. You won't realize it, but very likely your advisor won't know exactly where to go either, although probably for different reasons. The best and sometimes most difficult way to remove yourself from this period is simply to keep on reading, and keep on typing. Maybe read from a new book, skipping the proofs so you can "jump start" your understanding of a subject. You'll find out that some mathematicians spend their whole lives in this state of mind. Get used to dealing with your own particular response to this stage.

In the end you'll have more to show than you expected. At the same time, your results will seem completely trivial and embarrassingly obvious. From my interaction with mathematicians, I've gathered that this also happens to everyone.

1 Introduction

Finite frame theory focuses on finite collections of vectors in a Hilbert space $\mathcal{H}$ known as frames. There are several equivalent formulations of the definition of frame, and the connection between these is not immediately apparent. In this section, the basics of finite frames will be explored, along with some surprising results. These characterizations will be the foundation for subsequent sections, which both generalize much of the theory of finite frames, and answer some natural topological questions one might ask about frames. The topic has a flavor of application in many ways that motivate the theory. The results here are presented for $\mathcal{H} = \mathbb{R}^n$. Unless otherwise noted, they hold in the more general case of n dimensional Hilbert space.

Definition 1. A *finite frame* is a set of vectors $\{\boldsymbol{u}_i\} \subset \mathbb{R}^n$ with constants $0 < A \leq B < \infty$ known as frame bounds, such that for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$A\|\boldsymbol{x}\|^2 \leq \sum_i \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 \leq B\|\boldsymbol{x}\|^2. \tag{1}$$

If $A = B$, the frame is called tight. A Parseval frame is a tight frames satisfying $A = B = 1$.

A frame is restricted case of a *Bessel sequence*, which is a set $\{\boldsymbol{u}_i\}$ that only satisfies the upper bound in **(1)**,

$$\sum_i \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 \leq B\|\boldsymbol{x}\|^2.$$

It is easy to see that $\{\boldsymbol{u}_i\}$ finite implies the set is a Bessel sequence. The lower bound A furnishes the more interesting properties of frames.

Theorem 2. *$\{\boldsymbol{u}_i\}_{i=1}^k \subset \mathbb{R}^n$ is a frame if and only if its vectors span $\mathbb{R}^n$.*

Proof. ($\Longrightarrow$) By contrapositive (non-spanning implies not a frame). Suppose $\{\boldsymbol{u}_i\}$ does not span $\mathbb{R}^n$, so there exists unit vector $\boldsymbol{x} \in \text{span}\,(\boldsymbol{u}_i)^{\perp}$. It follows that

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 = 0.$$

so that the frame condition fails.

($\Longleftarrow$) Again by contrapositive (not a frame implies non-spanning). Suppose that $\{\boldsymbol{u}_i\}$ is not a frame. The upper bound is satisfied since $k < \infty$, so there must exist a sequence of unit vectors $\boldsymbol{x}_j \in \mathbb{R}^n$ with

$$\sum_{i=1}^{k} \langle \boldsymbol{x}_j, \boldsymbol{u}_i \rangle^2 \leq \frac{1}{j}.$$

Since $\{\boldsymbol{x}_j\}$ a subset of the unit sphere which is compact, and $\mathbb{R}^n$ is a Hilbert space (which is complete), there exists a convergent subsequence $\{\boldsymbol{x}_{j_k}\} \to \bar{\boldsymbol{x}} \in \mathbb{R}^n$. It must be that $\bar{\boldsymbol{x}}$ a unit vector with

$$\sum_{i=1}^{k} \langle \bar{\boldsymbol{x}}, \boldsymbol{u}_i \rangle^2 = 0,$$

which violates the lower bound condition. Thus $\{\boldsymbol{u}_i\}$ is not a frame.

□

Frames have significant application in signal processing. Their most important feature is shared with orthonormal bases, primarily the ability to reconstruct any vector using a prescribed linear combination of frame vectors. Their advantage over bases lies in their flexible size. While the number of vectors in any basis is fixed, the spanning property asserts the existence of frames with arbitrary size. The applied aspects of frames will not be explored further in this thesis, although the allusion to decomposing a signal vector into coefficients, and reconstructing it as a combination of frame vectors motivates several key theoretical results. In particular, the following discussion will introduce the analysis operator, which decomposes a vector into coefficients, and the synthesis operator, that attaches coefficients back to frame vectors. These operators are crucial in several unintuitive theorems that follow in the next section.

Definition 3. For any finite sequence $\{\boldsymbol{u}_i\}_{i=1}^k$, the *analysis operator* Θ is defined by

$$\begin{aligned} \Theta : \mathbb{R}^n &\rightarrow l^2(k) \\ \boldsymbol{y} &\mapsto \langle \boldsymbol{y}, \boldsymbol{u}_n \rangle. \end{aligned}$$

The adjoint of $\Theta^* : \ell^2(k) \rightarrow \mathbb{R}^n$ is called the *synthesis operator*. Θ^* is unique, and must satisfy

$$\langle \Theta \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \Theta^* \boldsymbol{y} \rangle.$$

This holds for the operator

$$\begin{aligned} \Theta^* : \ell^2(k) &\rightarrow \mathbb{R}^n \\ \{a_n\} &\mapsto \sum_{n=1}^{k} a_n \boldsymbol{u}_n, \end{aligned}$$

since for $\boldsymbol{y} = \{y_i\}_{i=1}^k$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$,

$$\langle \Theta \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{u}_1 \rangle y_1 + \cdots + \langle \boldsymbol{x}, \boldsymbol{u}_k \rangle y_k$$

and

$$\begin{aligned} \langle \boldsymbol{x}, \Theta^* \boldsymbol{y} \rangle &= \langle \boldsymbol{x}, \sum_{n=1}^{k} y_n \boldsymbol{u}_n \rangle \\ &= \sum_{n=1}^{k} y_n \langle \boldsymbol{x}, \boldsymbol{u}_n \rangle. \end{aligned}$$

The tight frames satisfy a reconstruction property similar to the expression of a vector with respect to an orthonormal basis, and motivates the definition of the frame operator.

Definition 4. The *frame operator* $S : \mathbb{R}^n \to \mathbb{R}^n$ for frame $\{\boldsymbol{u}_i\}_{i=1}^k$ is given by

$$S(\boldsymbol{x}) = \Theta^* \Theta(\boldsymbol{x}) = \sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \boldsymbol{u}_i.$$

By **Theorem 2**, It follows that $\{\boldsymbol{u}_i\}$ a frame if and only if S is an invertible operator.

Theorem 5. *(Reconstruction Property for Tight Frames) A k-vector frame $\{\boldsymbol{u}_i\}_{i=1}^k \subset \mathbb{R}^n$ is tight with frame constant $A > 0$ if and only if*

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \boldsymbol{u}_i = A\boldsymbol{x}. \tag{2}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, I.e., $S = A \cdot Id$. If the vectors $\{\boldsymbol{u}_i\}$ are all unit vectors, $A = \frac{k}{n}$, and in general $A = \frac{1}{n} \sum_{i=1}^{k} \|\boldsymbol{u}_i\|^2$.

Proof. Suppose $\{\boldsymbol{u}_i\}_{i=1}^k$ tight, so that for all $\boldsymbol{x}$,

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 = A\|\boldsymbol{x}\|^2.$$

Note that $\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 = \left\langle \sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \boldsymbol{u}_i, \boldsymbol{x} \right\rangle = \langle S\boldsymbol{x}, \boldsymbol{x} \rangle$, and the lower frame bound implies that S is positive definite. The tight frame condition,

$$\langle S\boldsymbol{x}, \boldsymbol{x} \rangle = A\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \langle A\boldsymbol{x}, \boldsymbol{x} \rangle$$

is equivalent to

$$A \cdot Id \preceq S \preceq A \cdot Id$$

so it must be that $S = A \cdot Id$.

The value of A follows from the reconstruction of $\boldsymbol{u}_i$ by the orthonormal basis $\{\boldsymbol{e}_i\}$,

$$\sum_{i=1}^{k} \|\boldsymbol{u}_i\|^2 = \sum_{i=1}^{k}\sum_{j=1}^{n} \left|\langle \boldsymbol{e}_j, \boldsymbol{u}_i\rangle\right|^2 = \sum_{j=1}^{n}\sum_{i=1}^{k} \left|\langle \boldsymbol{e}_j, \boldsymbol{u}_i\rangle\right|^2 = \sum_{j=1}^{n} A = nA.$$

□

For any frame $\{\boldsymbol{u}_i\}_{i=1}^{k}$, and invertible operator T, it follows by **Theorem 2** that the set $\{T\boldsymbol{u}_i\}$ is also a frame. In this case we will say $\{\boldsymbol{u}_i\}$ and $\{\boldsymbol{v}_i\} = \{T\boldsymbol{u}_i\}$ are *similar frames*. In an analogous manner, for frame operator S, the frame $\{S^{-1}\boldsymbol{u}_i\}$ satisfies the property,

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle S^{-1}\boldsymbol{u}_i = S^{-1}\left(\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle \boldsymbol{u}_i\right) = S^{-1} \circ S(\boldsymbol{x}) = \boldsymbol{x}.$$

Since $S = \Theta^*\Theta$ is obviously self-adjoint, The following property also follows:

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, S^{-1}\boldsymbol{u}_i\rangle \boldsymbol{u}_i = \sum_{i=1}^{k} \langle S^{-1}\boldsymbol{x}, \boldsymbol{u}_i\rangle \boldsymbol{u}_i = S\left(S^{-1}\boldsymbol{x}\right) = \boldsymbol{x}.$$

This motivates the term *dual frame* for any frame $\{\boldsymbol{v}_i\}$ with analysis operator $\Theta_{\{\boldsymbol{v}_i\}}$, such that

$$\Theta^*_{\{\boldsymbol{u}_i\}}\Theta_{\{\boldsymbol{v}_i\}} = \Theta^*_{\{\boldsymbol{v}_i\}}\Theta_{\{\boldsymbol{u}_i\}} = \mathrm{Id}.$$

It is apparent that $\{S^{-1}\boldsymbol{u}_i\}$ is a dual frame. However, in general there exist other dual frames.

As seen above, tight frames share a useful property of orthonormal bases, and this property can be generalized to non-tight frames via the canonical dual frame. In fact the connection between frames and bases runs deeper. First, the orthogonal projection of a frame with bounds A, B is a frame for the range space. Next, any frame is the projection of a basis in a higher dimension.

Theorem 6. *(Projection Property for Finite Frames) Let $\{\boldsymbol{u}_i\}_{i=1}^k$ be a frame for $\mathbb{R}^n$ with bounds A, B and $P : \mathbb{R}^n \to \mathbb{R}^m$ be an orthogonal projection with rank$(P) = m < n$. Then $\{P\boldsymbol{u}_i\}_{i=1}^k$ is a frame for $\mathbb{R}^m$ also with bounds A, B.*

Proof. Let $H = \text{Range}(P)$. Since P is self-adjoint, and $P\big|_H = \text{Id}_H$, we have for $\boldsymbol{x} \in H$

$$\sum_{i=1}^k \langle \boldsymbol{x}, P\boldsymbol{u}_i \rangle^2 = \sum_{i=1}^k \langle P\boldsymbol{x}, \boldsymbol{u}_i \rangle^2 = \sum_{i=1}^k \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2.$$

giving the result.

□

Theorem 7. *(Dilation Property for Finite Frames) Let $\{\boldsymbol{u}_i\}_{i=1}^k$ be a frame for V, $\dim V = n$, with bounds A, B. There exists a basis $\{\boldsymbol{w}_i\}_{i=1}^k$ for $H \supset V$, $\dim H = k$ for which $\boldsymbol{u}_i = \text{Proj}_V(\boldsymbol{w}_i)$ for $1 \leq i \leq k$. In particular, if $\{\boldsymbol{u}_i\}$ is a tight frame, then $\{\boldsymbol{w}_i\}$ is an orthonormal basis for H.*

Proof. Consider the analysis operator $\Theta : \mathbb{R}^n \to \ell^2(k) \cong \mathbb{R}^k$ for $\{\boldsymbol{u}_i\}_{i=1}^k$, and let Q be the orthogonal projection onto $\Theta(V)$, and $Q^\perp$ be the projection onto $\Theta(V)^\perp$. Construct the Hilbert space $H = V \oplus \Theta(V)^\perp$, identifying V with $V \oplus \{0\}$. For standard basis $\{\boldsymbol{e}_i\} \subset \mathbb{R}^k$, define $\boldsymbol{v}_i = \boldsymbol{u}_i \oplus Q^\perp \boldsymbol{e}_i$. It is clear that $P\boldsymbol{v}_i = \boldsymbol{u}_i$ for P the orthogonal projection onto H. The claim is that $\{\boldsymbol{v}_i\}_{i=1}^k$ is in fact a basis. The following fact will be used to prove this.

Fact 1: *Two frames $\{\boldsymbol{x}_i\}^k$, $\{\boldsymbol{y}_i\}^k$ for spaces H, K are similar if and only if their analysis operators have the same range. If two Parseval frames are similar, then T must be a unitary operator.*

For the case where $\{\boldsymbol{u}_i\}^k$ a frame, we show that there exists an invertible operator T that takes $\{\boldsymbol{e}_i\}^k$ to $\{\boldsymbol{v}_i\}^k$, so that $\{\boldsymbol{v}_i\}$ must be a basis. This makes use of **Fact 1**, by first showing that Θ and $\Theta_{\{Q\boldsymbol{e}_i\}}$ have the same range. Since $\text{span}\{Q\boldsymbol{e}_i\} = \Theta(H)$, $\Theta_{\{Q\boldsymbol{e}_i\}} : \Theta(H) \to \mathbb{R}^k$, and for $\boldsymbol{y} \in \Theta(H)$,

$$\begin{aligned}\Theta_{\{Q\boldsymbol{e}_i\}}(\boldsymbol{y}) &= \sum_{i=1}^{k} \langle \boldsymbol{y}, Q\boldsymbol{e}_i \rangle \boldsymbol{e}_i \\ &= \sum_{i=1}^{k} \langle Q\boldsymbol{y}, \boldsymbol{e}_i \rangle \boldsymbol{e}_i = \sum_{i=1}^{k} \langle \boldsymbol{y}, \boldsymbol{e}_i \rangle \boldsymbol{e}_i = \boldsymbol{y}.\end{aligned}$$

It follows that Θ and $\Theta_{\{Q\boldsymbol{e}_i\}}$ have the same range, and so there exists T invertible such that $T\boldsymbol{u}_i = Q\boldsymbol{e}_i$ for $1 \le i \le k$. This gives

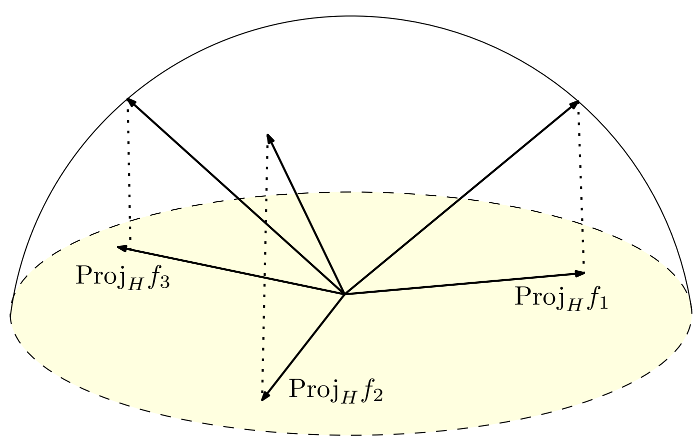
$$\boldsymbol{v}_i = \boldsymbol{u}_i \oplus Q^{\perp}\boldsymbol{e}_i = T^{-1}Q\boldsymbol{e}_i \oplus Q^{\perp}\boldsymbol{e}_i = U(Q\boldsymbol{e}_i \oplus Q^{\perp}\boldsymbol{e}_i),$$

where $U = T^{-1} \oplus \text{Id}_{Q^{\perp}}$ is invertible. This gives the result.

The case where $\{\boldsymbol{u}_i\}^k$ a tight frame follows by choosing T unitary using **Fact 1**. U will then be unitary.

□

Figure 3: The Dilation Property for a 3-vector frame in $\mathbb{R}^2$



Example 8. For $\mathbb{R}^2$, the set of n^{th} roots of unity in $\mathbb{C}$ give a tight frame for every $n > 2$.

This is seen by writing the frame operator from analysis and synthesis operator in polar coordinates. For a concrete example, take $k = 3$. Then

$$\Theta = \begin{bmatrix} \leftarrow & \boldsymbol{u}_1 & \rightarrow \\ \leftarrow & \boldsymbol{u}_2 & \rightarrow \\ \leftarrow & \boldsymbol{u}_3 & \rightarrow \end{bmatrix} = \begin{bmatrix} \cos(\frac{2\pi\cdot 0}{3}) & \sin(\frac{2\pi\cdot 0}{3}) \\ \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ \cos(\frac{2\pi(2)}{3}) & \sin(\frac{2\pi(2)}{3}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{2\pi\cdot 0}{3}} \\ e^{i\frac{2\pi}{3}} \\ e^{i\frac{4\pi}{3}} \end{bmatrix}$$

$$\Theta^* = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \cos(\frac{2\pi\cdot 0}{3}) & \cos(\frac{2\pi}{3}) & \cos(\frac{2\pi(2)}{3}) \\ \sin(\frac{2\pi\cdot 0}{3}) & \sin(\frac{2\pi}{3}) & \sin(\frac{2\pi(2)}{3}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{2\pi\cdot 0}{3}} & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \end{bmatrix}.$$

Making use of the double angle formulas $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = 1 - 2\sin^2\theta$, along with the real and imaginary parts of the identity $\sum_{k=0}^{N-1} e^{i2\pi k/N} = 0$,

$$\begin{aligned}
\Theta^*\Theta_{1,1} &= \sum_{k=0}^{2}\cos^2(\frac{2\pi k}{3}) = \sum_{k=0}^{2}\left(\cos(\frac{4\pi k}{3}) + \frac{1}{2}\right) = \frac{3}{2} \\
\Theta^*\Theta_{1,2} = \Theta^*\Theta_{2,1} &= \sum_{k=0}^{2}\cos(\frac{2\pi k}{3})\sin(\frac{2\pi k}{3}) \\
&= \frac{1}{2}\sum_{k=0}^{2}\sin(\frac{4\pi k}{3}) = 0 \\
\Theta^*\Theta_{2,2} &= \sum_{k=0}^{2}\sin^2(\frac{2\pi k}{3}) = \frac{1}{2}\sum_{k=0}^{2}\left(-\cos(\frac{2\pi k}{3}) + 1\right) = \frac{3}{2}.
\end{aligned}$$

It follows that

$$S = \Theta^*\Theta = \frac{3}{2}\cdot \mathrm{Id}$$

so the frame is tight. By an identical method, the frame operator for K-th roots of unity is $\frac{K}{2}\cdot\mathrm{Id}$. It is clear that the k-th roots of $e^{i\theta}$ and $re^{i\theta}$ are also frames for $\mathbb{R}^2$, since they are the images of k-th roots of unity under invertible linear transformations (see **Theorem 2**).

Example 9. (Diagram Vectors) There are more tight frames in $\mathbb{R}^2$ than the roots of unity. For K frame vectors

$$\boldsymbol{f}_k = \begin{bmatrix} a_k \cos\theta_k \\ a_k \sin\theta_k \end{bmatrix}$$

we can write a formula for $S = \Theta^*\Theta$ in a similar fashion to **Example 8**. Since S is a symmetric matrix,

$$\Theta^*\Theta_{2,2} = \sum_{k=0}^{K} a_k^2 \sin^2\theta_k = \sum_{k=0}^{K} a_k^2 \cos^2\theta_k = \Theta^*\Theta_{1,1}.$$

I.e.,

$$\sum_{k=0}^{K} a_k^2 \left(\cos^2\theta_k - \sin^2\theta_k\right) = \sum_{k=0}^{K} a_k^2 \cos(2\theta_k) = 0.$$

If $\{\boldsymbol{f}_k\}$ is a tight frame, off-diagonal entries of S must vanish, so

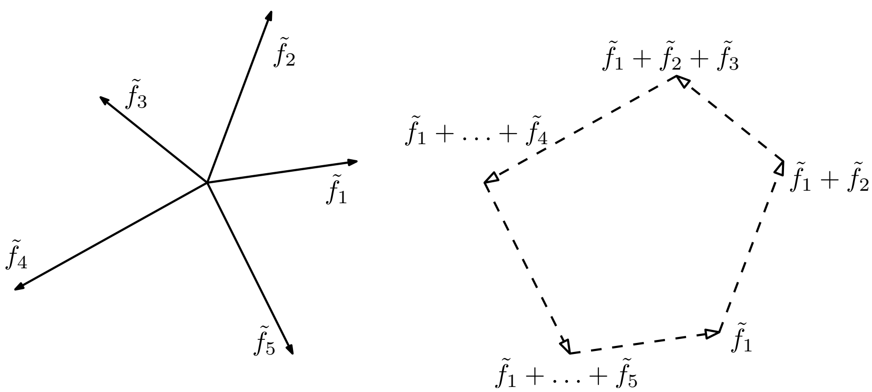
$$\Theta^*\Theta_{1,2} = \sum_{k=0}^{K} a_k^2 \cos\theta_k \sin\theta_k = \sum_{k=0}^{K} a_k^2 \sin(2\theta_k) = 0.$$

Defining the *diagram vectors* for $\{\boldsymbol{f}_k\}$ by

$$\tilde{\boldsymbol{f}}_k = \begin{bmatrix} a_k^2 \cos(2\theta_k) \\ a_k^2 \sin(2\theta_k) \end{bmatrix}$$

gives the condition $\sum_{k=0}^{K} \tilde{\boldsymbol{f}}_k = 0$, a geometrically appealing property. See [3] for a definition of diagram vectors in three dimensions.

Figure 4: The diagram vector sum-zero property



2 Moving Frames

Manifolds arise in various branches of mathematics as spaces that are "locally Euclidean". Formally, a *manifold* M is a topological space in which for every point $x \in M$ there exists a neighborhood $N(x) \subset M$ and a homeomorphism $\phi_x : N(x) \to \mathbb{R}^n$. It must be that n is the same for each ϕ_x, so that we may unambiguously refer to an n-manifold. The *tangent space* of a point $x \in M$, denoted TM_x is an n-dimensional vector space abstractly defined as an equivalence class of curves $\{\bar{\gamma} \mid \gamma(0) = x\}$ passing through x such that $\phi \circ \bar{\gamma}$ is differentiable in $\mathbb{R}^n$ for all $\bar{\gamma}$, and two curves γ_1, γ_2 are equivalent if $(\phi_x \circ \gamma_1)'(0) = (\phi_x \circ \gamma_2)'(0)$. In the cases discussed in this section, the tangent space at a point $\boldsymbol{x}$ can always be associated with those vectors $\boldsymbol{y}$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, so this warrants no further discussion.

A moving frame for an n-manifold M is a local choice of basis $(\boldsymbol{f}_i)^n(x)$ for the tangent space TM_x at every point $x \in M$, such that each $\boldsymbol{f}_i(x)$ is a vector field on M. The well-known Hairy Ball theorem states that the n-sphere S^n, where n refers to the dimension of its tangent space, does not have a nonvanishing tangent vector field for any n even. In a search for a moving frame for S^n, we are thus restricted to the case where n odd. This series of questions has in fact been resolved as well: such a frame only exists for $n = 1, 3, 7$ and makes use of the algebraic structure of $\mathbb{C}$, the quaternions in $\mathbb{R}^4$, and the octonions in $\mathbb{R}^8$. The proof that no other sphere hosts a moving frame is complicated and will not be discussed further in this document. It does, however, motivate an extension of finite Hilbert space frame theory to clarify this classic result.

In our discussion so far, the term tight frame denotes any set of vectors in space $\mathcal{H}$ that generalizes the spanning and reconstruction properties of orthonormal bases. It is an easy matter to confuse the terms moving frame and tight frame, a mistake that turns out to be lucrative. If we take moving frame to mean an ordered set of vector fields $\{f_i(x),\ 1 \leq i \leq k\}$ so that at each $x \in M$ the fields constitute a tight frame for TM_x, a new set of questions arise regarding whether moving Hilbert space frames exist on manifolds where the older definition of moving frame fails to exist. As in the paper [4], to avoid confusion with our established terminology, we will refer to a local choice of basis as a 'moving basis', so that the term 'moving frame' can be used for a general k-vector Hilbert space frame.

The question now becomes whether a moving frame for S^n exists for arbitrary n odd. In [4], the authors show that a moving tight frame for S^n does in fact exist for all odd n. The argument proceeds by selecting a convenient collection of vectors from the set $\mathcal{A}$ of tangent vectors at point $a \in S^n$. In particular, the vectors in $\mathcal{A}$ are obtained from the point $a \in S^n$, denoted

$$a = (a_1, a_2, \dots, a_{2m})$$

by swapping pairs of indices and multiplying one element of each pair by -1. As a concrete example, consider the circle S^1. For $a = (a_1, a_2)$ on the circle, $(a_2, -a_1)$ and $(-a_2, a_1)$ are both orthogonal to a. Since the tangent space is one dimensional, these two vectors constitute a frame for the tangent space of a. The tangent vector field obtained by the swap $(a_1, a_2) \to (a_2, -a_1)$, for example, is continuous by its algebraic nature. Checking orthogonality is just as simple. Using subsets of $\mathcal{A}$, the authors showed that it was possible to find a moving tight frame with $(2n-1)2^n$ vectors. The question was posed however whether this was minimal for a moving frame chosen from this set.

The goal of this section is to investigate the less restrictive question of moving *frames* for S^{2n-1}, without the tightness condition.

2.1 Exhibiting Moving Frames

First, we would like to assert the existence of a moving frame for S^{2n-1}. Using the existence of a moving tight frame, it will be shown that we can always delete a critical number of vectors from this tight frame while still maintaining the frame condition. In fact, the deleted vectors are arbitrary, so this is presumably the bluntest way of maintaining the frame bounds. It is nonetheless illustrative of the technique by which we analyze how 'close' to reconstructing a vector the frame operator is.

Theorem 10. *(Neumann series) Let $A : X \to X$ be a linear operator. If $\|cI - A\| < c$, for $c > 0$ then A is invertible.*

Proof. If $\|cI - A\| = \alpha < c$ then

$$c\|\boldsymbol{x}\| = \|(cI - A)\boldsymbol{x} + A\boldsymbol{x}\| \leq \|(cI - A)\boldsymbol{x}\| + \|A\boldsymbol{x}\| \leq \alpha\|\boldsymbol{x}\| + \|A\boldsymbol{x}\|.$$

Therefore, $\|A\boldsymbol{x}\| \geq (c - \alpha)\|\boldsymbol{x}\|$, so that A is invertible.

□

Theorem 11. *Given a unit-norm tight frame $(\boldsymbol{u}_i)_{i=1}^K$ with $K/n > 1$, the set $(\boldsymbol{u}_i)_{i=1}^K \setminus \{\boldsymbol{u}_j\}$ after removing any vector $\boldsymbol{u}_j$ is a $K-1$ frame.*

Proof. The frame operator is $T = \sum_{i=1}^{K} \boldsymbol{u}_i \otimes \boldsymbol{u}_i = \frac{K}{n} I$, with norm $\|T\| = \frac{K}{n}$. Let $\tilde{T}$ be the operator

$$\tilde{T} = \sum_{i=1}^{K-1} \boldsymbol{u}_i \otimes \boldsymbol{u}_i.$$

If $\tilde{T}$ is invertible, it follows that $(\boldsymbol{u}_i)_{i=1}^{K-1}$ is a frame. Since $\boldsymbol{u}_K$ is a unit vector, $\boldsymbol{u}_K \otimes \boldsymbol{u}_K$ is an orthogonal projection with norm 1. Thus,

$$\|\tilde{T} - \frac{K}{n} I\| = \|T - \boldsymbol{u}_K \otimes \boldsymbol{u}_K - \frac{K}{n} I\| \leq \|T - \frac{K}{n} I\| + \|\boldsymbol{u}_K \otimes \boldsymbol{u}_K\| = 1 < \frac{K}{n}.$$

Theorem 10 now implies that $\tilde{T}$ is invertible, giving the result.

□

Corollary 12. *Given a UNTF $(\boldsymbol{u}_i)_{i=1}^{K}$ with $K/n > m \in \mathbb{N}$, there exists a frame with $K - m$ vectors by removing m vectors from $(\boldsymbol{u}_i)_{i=1}^{K}$.*

Proof. Use **Theorem 11** on the sum $\sum_{i=1}^{m} \boldsymbol{u}_i \otimes \boldsymbol{u}_i$,

$$\|\tilde{T} - \frac{K}{n} I\| \leq \|T - \frac{K}{n} I\| + \sum_{i=1}^{m} \|\boldsymbol{u}_i \otimes \boldsymbol{u}_i\| = m < \frac{K}{n}.$$

□

The above result relies on keeping the modified frame operator $\tilde{T}$ appropriately close to the tight frame operator $\frac{K}{n} I$, and follows from a general theorem for invertible operators. A natural question to ask is whether this bound on number of dropped vectors is strict, and if it any simple cases allow us to push this number further.

Theorem 13. *Let $(\boldsymbol{u}_i)_{i=1}^{K}$ be a moving UNTF with frame constant $\frac{K}{n}$. If for some subset $\{\boldsymbol{u}_{k_i}\}_{i=1}^{K/n}$, the dimension of* $\text{span}(\{\boldsymbol{u}_{k_j}\})$ *is > 1, then $(\boldsymbol{u}_i)_{i=1}^{K-m} \backslash \{\boldsymbol{u}_k\}$ is a frame.*

Proof. The norm $\sum_{i=1}^{K/n} \|\boldsymbol{u}_{k_i} \otimes \boldsymbol{u}_{k_i}\|$ is bounded above by $\frac{K}{n}$ as in the previous result. In fact, the inequality is strict, for unit vectors $\boldsymbol{u}, \boldsymbol{x}$

$$|\boldsymbol{u} \otimes \boldsymbol{u}(\boldsymbol{x})| = |\langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}| = |\langle \boldsymbol{u}, \boldsymbol{x} \rangle| \leq 1$$

and the upper bound is attained exactly when $\boldsymbol{x} \in \operatorname{span}(\boldsymbol{u})$. Since by assumption, there exist some $\boldsymbol{u}_{k_i} \notin \operatorname{span}(\boldsymbol{u}_{k_j})$, the inequality above must be strict for one of $|\boldsymbol{u}_{k_i} \otimes \boldsymbol{u}_{k_i}(\boldsymbol{x})|$, $|\boldsymbol{u}_{k_j} \otimes \boldsymbol{u}_{k_j}(\boldsymbol{x})|$, which bounds the norm strictly and gives the result.

□

2.2 A Criterion on the Standard Basis

Lemma 14. *Let T be an operator. If*

$$\left|T\boldsymbol{e}_i - \boldsymbol{e}_i\right| < \frac{1}{\sqrt{n}}.$$

for every standard basis vector $\boldsymbol{e}_i$, then T is invertible. I.e., if $T = \sum_{i=1}^{k} \boldsymbol{u}_i \otimes \boldsymbol{u}_i$, then it is a frame operator.

Proof. T is noninvertible if and only if $\operatorname{Range}(T)$ is contained in a hyperplane of $\mathbb{R}^n$. The proof follows from finding the hyperplane H that is 'closest' to the set $\{\boldsymbol{e}_i\}^n$. As indicated, the word closest must be clarified. For hyperplane H let $d_H(\boldsymbol{e}_i) = \left|\operatorname{Proj}_{H^\perp}(\boldsymbol{e}_i)\right|$. For the problem at hand, a closest hyperplane to $\{\boldsymbol{e}_i\}$ will minimize

$$D\left(H\right) = d_H(\boldsymbol{e}_1) + d_H(\boldsymbol{e}_2) + \cdots + d_H(\boldsymbol{e}_n).$$

If $\left|T\boldsymbol{e}_i - \boldsymbol{e}_i\right| < \frac{1}{n}D(H)$ for $0 \leq i \leq n$, then $T\boldsymbol{e}_i$ cannot send the standard basis to a hyperplane.

H can be found by Lagrange multipliers. H will consist of vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ that satisfy

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0,$$

i.e., $\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle = 0$ for normal vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$. It is easy to express $d_H(\boldsymbol{e}_i)$, since

$$\mathrm{Proj}_{H^\perp}(\boldsymbol{e}_i) = \frac{1}{\|\boldsymbol{\alpha}\|^2} \langle \boldsymbol{\alpha}, \boldsymbol{e}_i \rangle \boldsymbol{\alpha}.$$

We can restrict $\|\boldsymbol{\alpha}\|^2 = 1$, so that $d_H(\boldsymbol{e}_i) = |\alpha_i| = \sqrt{\alpha_i^2}$. This gives the bounded minimization problem

$$\begin{array}{ll} \text{minimize} & f(\alpha_1, \ldots, \alpha_n) = \sqrt{\alpha_1^2} + \sqrt{\alpha_2^2} + \cdots + \sqrt{\alpha_n^2} \\ \text{subject to} & \mathrm{g}(\mathrm{ff}_1, \ldots, \mathrm{ff}_\mathrm{n}) = \mathrm{sqrt}\left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2\right) - 1 = 0. \end{array}$$

The Lagrangian is given by

$$\Lambda = \left(\sqrt{\alpha_1^2} + \sqrt{\alpha_2^2} + \cdots + \sqrt{\alpha_n^2}\right) + \lambda\left(\mathrm{sqrt}\left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2\right) - 1\right)$$

so a minimum will solve the system,

$$\begin{aligned} \frac{\partial \Lambda}{\partial \alpha_i} &= \frac{\alpha_i}{\sqrt{\alpha_i^2}} + \lambda\left(\frac{\alpha_i}{\mathrm{sqrt}(\alpha_1^2 + \cdots + \alpha_n^2)}\right) = 0 \\ \frac{\partial \Lambda}{\partial \lambda} &= \mathrm{sqrt}\left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2\right) - 1 = 0. \end{aligned}$$

The second equation is simply $\|\boldsymbol{\alpha}\|^2 = 1$. Thus, the equations $\frac{\partial \Lambda}{\partial \alpha_i} = 0$ become

$$\lambda \alpha_i = \pm 1,$$

i.e., $\alpha_i = \frac{\pm 1}{\lambda}$, so that $\|\boldsymbol{\alpha}\|^2 = \frac{1}{\lambda^2} n = 1$ gives $\lambda = \frac{1}{\sqrt{n}}$. It follows that $d_H(\boldsymbol{e}_i) = \frac{1}{\sqrt{n}}$ for $0 \leq i \leq n$, giving the result.

□

3 Connectedness Theorems for Frames

3.1 Connectedness of $\mathcal{F}_n^k$

This section is concerned with topological properties of different classes of frames, primarily connectedness and path-connectedness of finite frames, finite unit-norm frames, and tight frames. These are properties of the set of frames as a whole, an element of which is the ordered set $\{\boldsymbol{u}_i\}_{i=1}^k$ that constitutes a frame with k vectors over a given Hilbert space $\mathcal{H}$. Recall that a set S is called disconnected if there exist nonempty sets $A, B \subset S$ such that $\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$ and $A \cup B = S$. If a set is not disconnected, then it is connected. A path connected set is one for which given any two points a, b in the set, there exists a continuous function $\gamma : [0,1] \to S$ with $\gamma(0) = a$ and $\gamma(1) = b$.

Connectedness and path-connectedness are basic topological properties which, like openness, describe how well-behaved a set is, and lend geometric intuition to the understanding of a general space, beyond the confines of the low-dimensional Euclidean spaces. An important result that is used in our discussion of frames relates to the path-connectedness of $GL_n(V)$, the invertible linear operators on vector space $V = \mathbb{R}^n$.

Since the determinant is a continuous function from $M_{n\times n}(\mathbb{R})$ to $\mathbb{R}[x]$, the preimage of $\mathbb{R}\backslash\{0\}$ (a disconnected set) must be disconnected. It is natural to ask whether the preimages of $\mathbb{R}^+$ and $\mathbb{R}^-$ are connected. That is, whether the matrices with determinant 0 are the only obstructions to connectedness. It turns out that they are.

Theorem 15. *The sets* $GL^{+}(\mathbb{R}^n) = \{T \in GL(\mathbb{R}^n) \big| \det T > 0\}$, $GL^{-}(\mathbb{R}^n) = \{T \in GL(\mathbb{R}^n) \big| \det T < 0\}$ *are path connected.*

Proof. See [11].

□

Theorem 15 raises an important question with connection to finite frames. Since an element of $GL(\mathbb{R}^n)$ is a collection of column vectors that forms a basis, the previous proof describes a path through $GL(\mathbb{R}^n)$ as a path through the bases for $\mathbb{R}^n$. In $\mathbb{R}^n$, a basis does not contain enough vectors to avoid all vectors entering a hyperplane when continuously moving from one basis to another. It seems plausible that including more vectors is one way of getting around this problem. Instead of moving a basis continuously to another basis, the question becomes whether it is possible to continuously move one k-vector frame to another. With this, we introduce a notation for the topological space of frames.

Definition 16. $\mathcal{F}_n^k$ = the set of all k-vector frames on $\mathbb{R}^n$, with topology induced from $\mathbb{R}^n$.

A matrix $M_{m\times n}$ with $m < n$ is said to be full-rank if $\mathrm{Range}(\mathrm{M_{m\times n}})$ has dimension m. If $M_{m\times n}$ has columns representing vectors in a frame $(\boldsymbol{u}_i)_{i=1}^n$, $\boldsymbol{u}_i \in \mathbb{R}^m$ by the dilation property it is natural to consider the related column matrix $N_{n\times n}$ corresponding to some basis $(\boldsymbol{v}_i)^n$ of $\mathbb{R}^n$, in which projecting each $\boldsymbol{v}_i$ onto its first m coordinates gives $\boldsymbol{u}_i$. We will call the matrix $N_{n\times n}$ a dilation of $M_{m\times n}$. It is clear that a given frame matrix $M_{m\times n}$ might have several dilations.

Figure 5: The dilation of frame matrix M

$$M = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}, \quad N = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_n \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & & \boldsymbol{v}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

Lemma 17. *A real matrix $M_{m\times n}$, $m < n$ is full-rank if and only if can be dilated to a matrix with $det > 0$.*

Proof. ($\Longrightarrow$) Let $m \le n$, and matrix

$$M = \begin{pmatrix} \uparrow & & \uparrow \\ \boldsymbol{m}_1 & \dots & \boldsymbol{m}_n \\ \downarrow & & \downarrow \end{pmatrix}, \boldsymbol{m}_i \in \mathbb{R}^m$$

be full rank, so that $(\boldsymbol{m}_i)_{i=1}^n$ is a spanning set (frame) for $\mathbb{R}^m$. By the dilation property for frames (**Theorem 7**), it is possible to dilate each $\boldsymbol{m}_i$ to $\boldsymbol{m}_i \oplus \boldsymbol{v}_i$, $\boldsymbol{v}_i \in \mathbb{R}^{n-m}$ so that $(\boldsymbol{m}_i \oplus \boldsymbol{v}_i)_{i=1}^n$ is a basis for $\mathbb{R}^n$. If $\boldsymbol{m}_i' = \boldsymbol{m}_i \oplus \boldsymbol{v}_i$, the matrix

$$M' = \begin{pmatrix} \uparrow & & \uparrow \\ \boldsymbol{m}_1' & \dots & \boldsymbol{m}_n' \\ \downarrow & & \downarrow \end{pmatrix}, \boldsymbol{m}_i' \in \mathbb{R}^n$$

has $\det M' \neq 0$. If $\det M' < 0$, the matrix M'' obtained by multiplying the last row of M' by -1 has full span, and by the properties of determinants $\det M'' = -\det M' > 0$.

($\Longleftarrow$) Suppose $M_{m\times n}$ has dilation N with $\det N > 0$. The columns of N form a basis for $\mathbb{R}^n$, say $(\boldsymbol{u}_i)^n$. Since the orthogonal projection of a frame is also a frame for the projected subspace, it follows that $(P\boldsymbol{u}_i)^n$ is a frame for $\mathbb{R}^m$, where $P : \mathbb{R}^n \to \mathbb{R}^m$ projects onto the first m coordinates. (see Proposition 5.1 in [6]). Therefore $(P\boldsymbol{u}_i)^n$ has full rank, and its frame matrix is precisely M.

□

Fact 18. *The continuous image of a path-connected space is path connected.*

Theorem 19. *$\mathcal{F}_n^k$ is path-connected.*

Proof. Using **Lemma** 17, we can dilate the frame matrix of any $(\boldsymbol{u}_i)^k \in \mathcal{F}_n^k$ to a member of GL_k^+. For any $(\boldsymbol{u}_i)^k$,$(\boldsymbol{v}_i)^k \in \mathcal{F}_n^k$, choose corresponding dilations $U, V \in GL_k^+$. By the path-connectivity of GL_k^+, there exists a path $f : [0,1] \to GL_k^+$ such that $f(0) = U$ and $f(1) = V$. The projection operator

$$P = \begin{pmatrix} I^n & 0 \\ 0 & 0 \end{pmatrix}$$

is continuous, so it follows by **Fact 18** that $P \circ f(t) \equiv q(t)$ is a path. Moreso, the backward direction of **Lemma 17** ensures that the $P \circ f(t)$ is isomorphic to $\mathcal{F}_n^k$ for all t. This gives the desired path, since $q(0) = (\boldsymbol{u}_i)^k$ and $q(1) = (\boldsymbol{v}_i)^k$.

□

The path connectedness of $\mathcal{F}_n^k$ is an immediate consequence of the path-connectedness of GL_+, and the dilation property for frames. We run into more difficulty when considering the path-connectedness of finite unit-norm frames (FUNF), since the path found in **Theorem** 19,

$$f(t) = \left(\begin{pmatrix} \boldsymbol{u}_1(t) \\ \boldsymbol{q}_1(t) \end{pmatrix} \ \cdots \ \begin{pmatrix} \boldsymbol{u}_k(t) \\ \boldsymbol{q}_k(t) \end{pmatrix} \right)$$

which gives $u_i \in \mathbb{R}^n$ as the *ith* frame vector in the path $P \circ f(t)$, might have $u_i = 0$ for some t, making the path $\boldsymbol{u}_i(t)$ non-normalizable. If $\boldsymbol{u}_i(t)$ nonzero for all $t \in [0,1]$, then showing the path-connectedness of FUNF would be a matter of normalizing each of the $\boldsymbol{u}_i(t)$ via multiplication by some$\lambda_i(t)$, making the new path

$$\Lambda(t)Pf(t)$$

with

$$\Lambda(t) = \begin{pmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_k(t) \end{pmatrix}.$$

Since $Pf(t)$ must be continuous with respect to the norm of each of its columns and by the continuity of matrix multiplication, the normalization constants $\lambda_i(t)$ must be continuous. It remains to be shown that such a nonzero path can always be found.

3.2 Connectedness of FUNF

As stated in the previous section, the path-connectedness of $\mathcal{F}_n^k$ does not ensure the path-connectedness of $FUNF$, since $\gamma : [0,1] \to \mathcal{F}_n^k$ might pass through a frame $(\boldsymbol{u}_i)^k$ in which $\boldsymbol{u}_j = 0$ for some j. This is in fact the only problem, since it was reasoned that normalization is a continuous process and preserves the path.

We are therefore concerned with the set of frames with vanishing vectors, or frame matrices with vanishing components in their column vectors. More precisely, the following two subsets of GL_+ will be considered:

Definition 20. Writing $M \in GL_+(k)$ as

$$M = \begin{pmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_i & \dots & \boldsymbol{u}_k \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_i & \dots & \boldsymbol{v}_k \end{pmatrix}$$

with column vectors $\boldsymbol{u}_i \in \mathbb{R}^n$ and $\boldsymbol{v}_i \in \mathbb{R}^{k-n}$, denote the sets:

$$\begin{aligned} T_0 &\equiv \{M \in GL_+^k \mid \exists i, \boldsymbol{u}_i = 0\} \\ T_1 &\equiv \{M \in GL_+^k \mid \boldsymbol{u}_i \neq 0 \, \forall i\} \end{aligned}.$$

Lemma 21. *T_1 is open in GL_+, T_0 is closed.*

Proof. The multiplication map $m_i : GL_+ \to \mathbb{R}^k$ defined by $m_i(M) = M\boldsymbol{e}_i$, and the projection map $P = \begin{pmatrix} I^n & 0 \\ 0 & 0 \end{pmatrix}$ are both continuous maps, so the map

$$\begin{aligned} \Omega_i : GL_+ &\to \mathbb{R}^n \\ M &\mapsto PM\boldsymbol{e}_i \end{aligned}$$

is continuous. Since $\boldsymbol{u}_i = 0$ if and only if $\Omega_i(M) = 0$, and $\{\boldsymbol{0}\} \in \mathbb{R}^n$ is closed, it follows that T_0 is closed. Since $T_1 = T_0^c$, it must be open.

□

Fact 22. *A proper subspace of $\mathbb{R}^n$ is closed with empty interior.*

Proof. Let $W \subset \mathbb{R}^n$ be proper, so that $W^\perp \neq \emptyset$. For any $\boldsymbol{v} \notin W$, and projection $P_W(\boldsymbol{v})$ is the closest element of W to $\boldsymbol{v}$. If $\epsilon = d(\boldsymbol{v}, P_W(\boldsymbol{v}))$ then it follows that

$$N_\epsilon(\boldsymbol{v}) \subset W^c$$

so that W must be closed. To show W has empty interior, take any open ball centered around $\boldsymbol{w} \in W$, $N_\epsilon(\boldsymbol{w})$. Since $W^\perp$ is a subspace, $\boldsymbol{v} \in W^\perp$ implies $c\boldsymbol{v} \in W^\perp$ for all $c \in \mathbb{R}$. In particular, we can find a $\boldsymbol{v} \in W^\perp$ with arbitrarily small norm, $|\boldsymbol{v}| < \epsilon$. The vector $\boldsymbol{w} + \boldsymbol{v} \in W^c$ satisfies

$$d(\boldsymbol{w} + \boldsymbol{v}, \boldsymbol{w}) = |\boldsymbol{w} + \boldsymbol{v} - \boldsymbol{w}| = |\boldsymbol{v}| < \epsilon$$

so that $\boldsymbol{w} + \boldsymbol{v} \in N_\epsilon(w)$, showing that W has empty interior.

□

Proposition 23. *The union of finitely many proper subspaces is closed with empty interior.*

Proof. Let $W_i \subset \mathbb{R}^n$ be a proper subspace for all $i \leq N$. As the union of finitely many closed sets, $\bigcup_{i=1}^N W_i$ is closed. Since each W_i has empty interior, $W_i^\complement$ is a dense open subset of $\mathbb{R}^n$. Baire's Theorem (see Exercise 3.25 in [8]) implies that

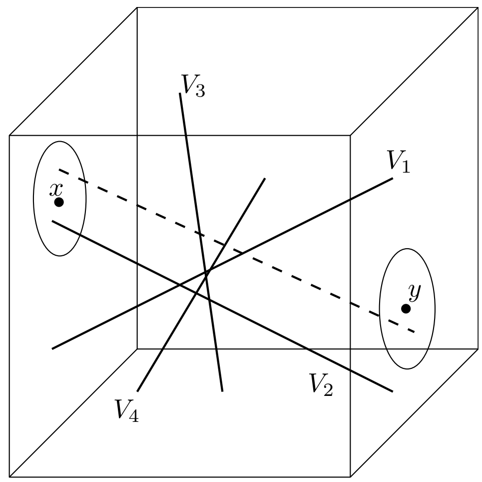
$$G = \left(\bigcup^N W_i\right)^\complement = \bigcap^N W_i^\complement$$

is dense in $\mathbb{R}^n$. Since N is finite, G is also open, giving the result.

□

Theorem 24. *Let W be an n-dimensional vector space, $V_i \subset W$ a subspace with codimension ≥ 2 for all $1 \leq i \leq N$. For any $\boldsymbol{x}, \boldsymbol{y} \in W \backslash V$ and $\epsilon > 0$, it is possible to choose $\boldsymbol{x}' \in N_\epsilon(\boldsymbol{x})$, $\boldsymbol{y}' \in N_\epsilon(\boldsymbol{y})$ so that the line segment connecting them does not intersect any V_i.*

Figure 6: A connecting linear path for $\dim W = 3$, $\dim V_i = 1$



Proof. For clarity, assume $N = 1$. The case $N > 1$ will be described subsequently. The worst case scenario occurs when the codimension of V is precisely 2. Any subspace with larger codimension can be embedded into a codimension 2 subspace, so it is acceptable to consider only this case. Consider the line segment

$$\gamma(t) = t\boldsymbol{x} + (1-t)\boldsymbol{y}, \quad 0 \le t \le 1.$$

If $\gamma(t)$ does not intersect intersect V at any point t then we are done. So suppose there exists some t with $\gamma(t) = \boldsymbol{v} \in V$. Shift everything by $-\boldsymbol{v}$, so that the intersection is at the origin and the line segment $\gamma(t)$ is a subset of linear subspace $L = \{t\boldsymbol{x} + (1-t)\boldsymbol{y} \,\big|\, t \in \mathbb{R}\}$. This will be the only point of intersection with V, since $L \cap V$ a proper subspace of L. Let $L^\perp$ be the orthogonal complement of L. This is a subspace with codimension 1, since $\dim(L) = 1$, and $(\mathrm{Proj}_{L^\perp} V) \cap L = \mathbf{0}$ since $\mathrm{Proj}_{L^\perp} V \subset L^\perp$.

Now $\mathrm{Proj}_{L^\perp} V$ must be a proper subspace of $L^\perp$, since V was assumed to have codimension 2. Using **Fact 22** 22, it follows that $\mathrm{Proj}_{L^\perp} V$ has empty interior, so that for any

$\epsilon > 0$ there exists $\boldsymbol{\ell} \in L^{\perp}$, $\boldsymbol{\ell} \in N_{\epsilon}(\mathbf{0})$ with $\boldsymbol{\ell} \notin \text{Proj}_{L^{\perp}} V$. Shifting L by $\boldsymbol{\ell}$,

$$L \to L + \boldsymbol{\ell} = L'$$

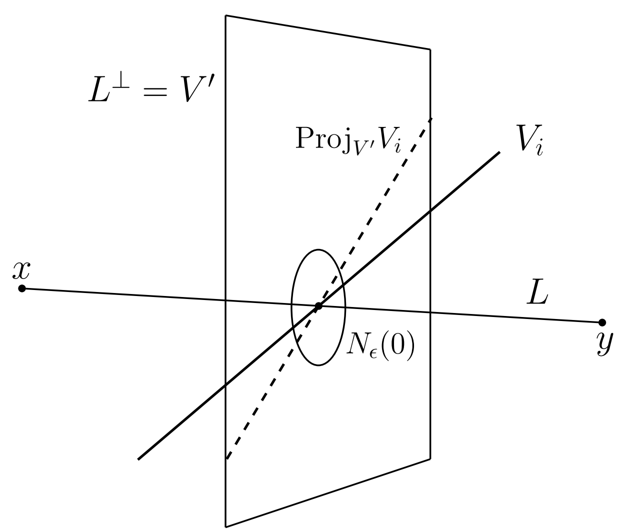
gives $L' \cap L^{\perp} = \boldsymbol{\ell}$. It must be that $L' \cap V = \emptyset$, since $\text{Proj}_{L^{\perp}} L' = \boldsymbol{\ell}$ and $\text{Proj}_{L^{\perp}}^{-1} \boldsymbol{\ell}$ does not intersect V. Choose ϵ small enough so that $N_{\epsilon}(\boldsymbol{x}) \cap L_{\perp} = \emptyset$ and $N_{\epsilon}(\boldsymbol{y}) \cap L_{\perp} = \emptyset$, and set $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\ell} \in N_{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{\ell} \in N_{\epsilon}(\boldsymbol{y})$ for $\boldsymbol{\ell}$ as above. This gives the result for a single subspace. The result for $N > 1$ follows since the union

$$\bigcup \text{Proj}_{L^{\perp}} V_i$$

is closed with empty interior by **Proposition** 23, and $\boldsymbol{\ell}$ can be found in a similar manner.

□

Figure 7: The construction of L, V', and $N_{\epsilon}(\mathbf{0})$ for $\dim(W) = 3$ and $\dim V_i = 1$.



Lemma 25. *(Paving Lemma) For any open set S, and path $\gamma : [0,1] \to S$, there exists a subdivision $0 = x_0 < x_1 < \ldots, < x_n = 1$ such that $\gamma\big|_{[k,k+1]} \subset B_k$ (an open ball) for all k. In*

particular,

$$\bigcup_{i=0}^{n} B_\epsilon(\gamma(x_i)) \subset S$$

and $\gamma([x_i, x_{i+1}]) \subseteq B_\epsilon(\gamma(x_i))$ *for all* $i \in \{0, 1, \ldots, n\}$.

Theorem 26. *FUNF is path connected*

Proof. For any $A, B \in FUNF$, take the path $\gamma : [0,1] \to \mathcal{F}_n^k$ that connects A and B in the finite frames. This path was found by dilating the frame matrix to GL_+. As stated, γ may include elements of T_0, which cannot give a normalized frame. However, the path γ can be modified to stay within T_1. Note that T_0 is the finite union of subspaces each with codimension ≥ 2, since for

$$M = \begin{pmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_i & \ldots & \boldsymbol{u}_k \\ \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_i & \ldots & \boldsymbol{v}_k \end{pmatrix}$$

$\boldsymbol{u}_i$ is assumed to have dimension ≥ 2, and T_0 is the intersection $T_0^i \cap GL_+$ where T_0^i is the subspace

$$T_0^i = \begin{pmatrix} \boldsymbol{u}_1 & \ldots & 0 & \ldots & \boldsymbol{u}_k \\ \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_i & \ldots & \boldsymbol{v}_k \end{pmatrix} \subset M_{n \times n}.$$

Since GL_+ is open , the paving lemma gives subdivision $\{x_i\}_{i=0}^n$ of $[0,1]$ and $B_{\epsilon_i}(\gamma(x_i)) \subseteq GL_+$. We are concerned with the set of $j \in 0, \ldots, n$ for which

$$B_{\epsilon_j}(\gamma(x_j)) \cap T_0 \neq \emptyset.$$

Without loss of generality, it can be assumed that $\gamma(x_j), \gamma(x_{j+1}) \notin T_0$. If $\gamma(x_j) \in T_0$, then since T_0 is closed with empty interior (**Lemma** 21) there exists some $\boldsymbol{u}_j \in T_1$

such that $\boldsymbol{u}_j$ lies in neighborhood $N_\delta(\gamma(x_j)) \subset B_\epsilon(\gamma(x_j))$. The path γ can be diverted through $\boldsymbol{u}_j$ without leaving GL_+. Relabeling $x_j \to u_j$ when necessary then gives the desired subdivision.

The path $p_i : [0,1] \to GL_+$ through $B_\epsilon(\gamma(x_i))$ given by

$$p_i(t) = t\gamma(x_i) + (1-t)\gamma(x_{i+1}).$$

is a line segment through $M_{n\times n}(\mathbb{R})$. Suppose $p_i([0,1]) \cap T_0$ nonempty. Since T_1 is open, there exists $\epsilon > 0$ such that

$$N_\epsilon(\gamma(x_i)),\ N_\epsilon(\gamma(x_{i+1})) \subset T_1.$$

Theorem 24 now gives new points $x_i' \in N_\epsilon(\gamma(x_i))$ and $x_{i+1}' \in N_\epsilon(\gamma(x_{i+1}))$ such that the path

$$p_i' = tx_i' + (1-t)x_{i+1}'$$

does not intersect T_0. Since $\gamma(x_i)$ and x_i' can be connected by a path as members of $N_\epsilon(\gamma(x_i)) \subset T_1$ (the same holds for $\gamma(x_{i+1})$ and x_{i+1}'), the new path formed from them makes the path through $B_\epsilon(\gamma(x_i))$ avoid T_0. Repeating this finitely many times for all $B_{\epsilon_i}(\gamma(x_i))$ completes the theorem.

□

4 Frame Measures

The aim of this chapter is to generalize the concept of frame to infinite collections of vectors $\{\boldsymbol{u}_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^n$. While a Bessel sequence only becomes interesting for infinite sequences, this still restricts discussion to countable sets. One apparent way to generalize to uncountable sets is to reinterpret condition (1) in a measure-theoretic framework. The summation in the frame condition is then replaced by integration with respect to a measure $\mu : \mathbb{R}^n \to \mathbb{R}$. The properties of a collection of vectors are now studied as properties of a measure on $\mathbb{R}^n$. Finite frames follow immediately as a special case of 'frame measures'. It will be shown that this reinterpretation admits similar notions of dual frame and the geometric interpretation for $\mathbb{R}^2$ frame measures in **Example 9**.

The definition of frame measure is most naturally defined as a Borel measure, which is the smallest measure defined on all open sets of a topological space X.

Definition 27. (*Borel Measure*). Let X be a locally-compact Hausdorff topological space, and $\mathfrak{B}(X)$ be the smallest σ- algebra that contains the open sets of X. $\mathfrak{B}(X)$ is the Borel σ-algebra on X. Any measure μ defined on $\mathfrak{B}(X)$ is called a Borel measure.

The definition of Radon measure follows from the Borel measure.

Definition 28. A *Radon measure* on a locally-compact Hausdorff space is a Borel measure satisfying the following properties:

1. $\mu(K) < \infty$ for every compact K.
2. $\mu(E) = \sup_{K \subset E} \mu(K)$ for every Borel set E, and compact K.

With these we define a frame measure μ.

Definition 29. A *frame measure* is a measure space (X, μ) with *square integrability*, $\int_X \boldsymbol{r}^2 \, d\mu(\boldsymbol{r}) < \infty$, and constants $0 < A \leq B < \infty$ so that for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$B\|\boldsymbol{x}\|^2 \leq \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle^2 d\mu(\boldsymbol{r}) \leq A\|\boldsymbol{x}\|^2. \tag{3}$$

The measure is called a *tight frame measure* if $A = B$.

Note the similarity to formula (1). The first connection with finite frames follows from this definition. In particular, it is possible to recover the definition of finite frames from the notion of a frame measure, indicating that (3) is an appropriate generalization.

Example 30. The Dirac measure (δ_{u_i}) which gives mass c_i to each of the vectors $\{\boldsymbol{u}_i\}_{i=1}^k$,

$$\delta_{\{u_i\}}(E) = \begin{cases} 0, & \{\boldsymbol{u}_i\} \cap E = \emptyset \\ \sum_{u_i \in E} c_i, & \{\boldsymbol{u}_i\} \cap E \neq \emptyset \end{cases}$$

is identical to a finite frame when $c_i = 1$ for all i, since the finite frame operator as an integral over the measure becomes the finite sum

$$\sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle^2 = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle^2 \, d\mu(\boldsymbol{r}).$$

For a general description of the Dirac measure, see [7].

It is necessary to introduce the support of a measure, which we define using the usual topology on $\mathbb{R}^n$.

Definition 31. The *support* of a Borel measure μ is the set

$$\text{supp}(\mu) \equiv \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \in N_{\boldsymbol{x}} \implies \mu(N_{\boldsymbol{x}}) > 0\}$$

where $N_{\boldsymbol{x}}$ is any neighborhood of $\boldsymbol{x}$.

With this it is now possible to reproduce the spanning property of finite frames with frame measures.

Theorem 32. *(Spanning property for frame measures)*
(i) *If* $\text{supp}(\mu)$ *not contained in any hyperplane of* $\mathbb{R}^n$ *and* $\int \|\boldsymbol{r}\|^2 d\mu < \infty$*, then* $(\mathbb{R}^N, \chi, \mu)$ *a frame measure.*
(ii) *If* $(\mathbb{R}^n, \chi, \mu)$ *a frame measure, then* $\text{supp}(\mu)$ *not contained in any hyperplane of* $\mathbb{R}^n$.

Proof. **(i)** By the Cauchy-Schwarz inequality, $\langle \boldsymbol{x}, \boldsymbol{r}\rangle^2 \leq \|\boldsymbol{x}\|^2 \|\boldsymbol{r}\|^2$, so the measure inequality

$$\int \langle \boldsymbol{x}, \boldsymbol{r}\rangle^2 d\mu \leq \|\boldsymbol{x}\|^2 \int \|\boldsymbol{r}\|^2 d\mu \leq \|\boldsymbol{x}\|^2 A,$$

gives the upper frame bound. So we are only concerned with the lower bound. By contrapositive, supposing μ not a frame measure, then the lower bound fails, and we can find a sequence of vectors $\boldsymbol{y}_m$, with $\|\boldsymbol{y}_m\| = 1$ such that

$$\int \langle \boldsymbol{y}_m, \boldsymbol{r}\rangle^2 d\mu < \frac{1}{m}.$$

$\{\boldsymbol{y}_m\}$ must have a convergent subsequence , $\{\boldsymbol{y}_{m_j}\} \to \boldsymbol{y}$. Since the sequence of functions $\langle \boldsymbol{y}_m, \cdot\rangle^2$ converges pointwise to $\langle \boldsymbol{y}, \cdot\rangle^2$, and are pointwise bounded above by $g = A\|\boldsymbol{y}\|$ (a measurable (constant) function), by the dominated convergence theorem it follows that

$$0 = \lim_{m\to\infty} \int \langle \boldsymbol{y}_m, \boldsymbol{r}\rangle^2 d\mu = \int \langle \boldsymbol{y}, \boldsymbol{r}\rangle^2 d\mu.$$

Since $\langle \boldsymbol{y}, \cdot\rangle^2$ positive, and the μ nontrivial, it follows that $\mathrm{supp}(\mu)$ cannot span $\mathbb{R}^n$.

(ii) By Contrapositive. Suppose $(\mathbb{R}^N, \chi, \mu)$ has support completely contained in hyperplane $H \subset \mathbb{R}^n$. Then there exists nonzero vector $\boldsymbol{x} \in H^\perp$. Since $\boldsymbol{x}$ is orthogonal to all $\boldsymbol{r} \in \mathrm{supp}(\mu)$, the integral

$$\int \langle \boldsymbol{x}, \boldsymbol{r}\rangle^2 d\mu(\boldsymbol{r}) = 0,$$

so the lower frame bound condition fails. **(ii)** follows.

□

Theorem 33. *If (μ) a frame measure, then the frame bound B satisfies*

$$B \le \int \|\boldsymbol{r}\|^2 d\mu$$

with equality holding when μ is a tight frame measure.

Proof. Using the reconstruction property on the standard orthonormal basis $\{\boldsymbol{e}_i\}^N \subset \mathbb{R}^N$, and by Holder's inequality,

$$B = \frac{1}{N}\sum_{n=1}^{N} B\|\boldsymbol{e}_n\| \le \frac{1}{N}\sum_{n=1}^{N} \| \int \langle \boldsymbol{e}_n, \boldsymbol{r}\rangle^2 d\mu\| = \frac{1}{N}\sum_{n=1}^{N} \int \|\boldsymbol{r}\|^2 d\mu = \int \|\boldsymbol{r}\|^2 d\mu.$$

This gives the equality when μ is tight.

Together, **Theorems 33** and **34** provide a correlate of the spanning property for finite frames. Note however that the $\int \boldsymbol{r}^2 \, d\mu < \infty$ property is trivial in the finite case, and so the extra assumption of square integrability does not manifest itself.

□

Theorem 34. *(Reconstruction property for tight frame measures) For (μ) is a tight frame measure, for all $\boldsymbol{x} \in \mathbb{R}^n$,*

$$\boldsymbol{x} = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r} \, d\mu(\boldsymbol{r})$$

Proof. The operator $T : \mathbb{R}^n \to \mathbb{R}$, given by

$$T(\boldsymbol{x}) = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r} \, d\mu$$

is positive semidefinite, since $\langle \boldsymbol{x}, T\boldsymbol{x} \rangle = A\|\boldsymbol{x}\| \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$, and also self-adjoint, since

$$\langle \boldsymbol{y}, T\boldsymbol{x} \rangle = \langle \boldsymbol{y}, \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r} d\mu \rangle = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \langle \boldsymbol{y}, \boldsymbol{r} \rangle d\mu = \langle \boldsymbol{x}, \int \langle \boldsymbol{y}, \boldsymbol{r} \rangle \boldsymbol{r} d\mu \rangle = \langle \boldsymbol{x}, T\boldsymbol{y} \rangle = \langle T\boldsymbol{y}, \boldsymbol{x} \rangle.$$

Now since $\langle \boldsymbol{x}, T\boldsymbol{x} \rangle = A\|\boldsymbol{x}\|^2$ for all $\boldsymbol{x}$, it must be that $\langle T\boldsymbol{x} - A\boldsymbol{x}, \boldsymbol{x} \rangle = 0 \, \forall \boldsymbol{x}$, and so $T = A \cdot \mathrm{Id}$ since $\langle \boldsymbol{y}, T\boldsymbol{x} \rangle$ defines an inner product.

□

Example 35. The Lebesgue measure on the circle (S^1, χ, λ) is a tight frame measure with frame constant $A = \frac{1}{2}$. For any $\boldsymbol{x} = (a, b) \in \mathbb{R}^2$, and $f = (\cos\theta, \sin\theta)$

$$\begin{aligned} \int \langle \boldsymbol{x}, f \rangle f \, d\lambda &= \int_0^{2\pi} \begin{pmatrix} (a\cos(\theta) + b\sin(\theta))\cos(\theta) \\ (a\cos(\theta) + b\sin(\theta))\sin(\theta) \end{pmatrix} d\theta \\ &= \int_0^{2\pi} \begin{pmatrix} a\cos^2(\theta) \\ b\sin^2(\theta) \end{pmatrix} d\theta \\ &= \left(\frac{a}{2}, \frac{b}{2} \right) = \frac{1}{2}(a, b) \end{aligned}$$

4.1 Analysis and Synthesis Operators

The operator T in **Theorem 35** is the analog of the frame operator S for finite frames. This is not coincidental. In fact, we are able to make a measure theoretic generalization of the analysis/synthesis operators Θ, Θ^*. With the proper definitions, this will lead to a generalization of dual frame, which given frame measure μ will be be a new frame measure η related to μ that satisfies the reconstruction property.

Definition 36. The *analysis operator* $\Theta : \mathbb{R}^n \to L^2$ takes $\boldsymbol{x} \in \mathbb{R}^n \to \Theta(\boldsymbol{x})$, and is defined by

$$\Theta(\boldsymbol{x}) = \langle \boldsymbol{x}, \cdot \rangle.$$

The *synthesis operator* is the adjoint of the analysis operator, $\Theta^* : L^2 \to \mathbb{R}^n$.

Note that $\Theta(\boldsymbol{x})$ is always L^2 integrable by definition of frame measure.

Theorem 37. $\Theta^*\Theta(x) = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r} \, d\mu$. *In particular,* $\Theta^*(\boldsymbol{g}) = \int \boldsymbol{g}(\boldsymbol{r})\boldsymbol{r} \, d\mu(\boldsymbol{r})$.

Proof. First we determine the form of Θ^*, the unique operator satisfying

$$\langle \Theta(\boldsymbol{x}), \boldsymbol{v} \rangle = \langle \boldsymbol{x}, \Theta^*(\boldsymbol{v}) \rangle \; \forall \boldsymbol{x} \in \mathbb{R}^n, \, \boldsymbol{v} \in L^2().$$

Since

$$\begin{aligned} \langle \Theta(\boldsymbol{x}), \boldsymbol{v} \rangle &= \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{v}(\boldsymbol{r}) \, d\mu(\boldsymbol{r}) \\ &= \left\langle \boldsymbol{x}, \int \boldsymbol{r}\boldsymbol{v}(\boldsymbol{r}) \, d\mu(\boldsymbol{r}) \right\rangle \end{aligned}$$

the last line must be $\langle \boldsymbol{x}, \Theta^*(\boldsymbol{v}) \rangle$, so that $\Theta^*(\boldsymbol{v}) = \int \boldsymbol{v}(\boldsymbol{r})\boldsymbol{r}\, d\mu(\boldsymbol{r})$.

□

Now by our definition of tight frame measure, μ is A-tight for $\mathbb{R}^n$ if and only if the analysis operator satisfies

$$\int \|\Theta(\boldsymbol{x})\|^2\, d\mu(\boldsymbol{r}) = A\|\boldsymbol{x}\|^2$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. Likewise, the reconstruction property for tight frames can be rephrased as the requirement $T = \Theta^*\Theta = A \cdot \text{Id}$. The self-adjoint property of T follows immediately from the decomposition.

Drawing from the theme of signal analysis, the dual frame allowed for perfect reconstruction of any signal vector using the analysis operator of frame $\{\boldsymbol{u}_i\}$ and the synthesis operator of its dual $\{S^{-1}\boldsymbol{u}_i\}$, or vice versa. In the context of frame measures, the dual measure η will have to satisfy $\boldsymbol{x} = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle\, d\eta(\boldsymbol{r})$, but if there is no relation between η and the original frame measure μ, nothing new gained from beyond an arbitrary tight frame measure. We require a method of obtaining new measures from old in order to make sense of a dual frame measure.

Definition 38. Let (X, β, μ) be a measure space. A function $\phi : X \to Y$ from (X, β) to measure space (Y, C) is *measurable* if the preimage of a measurable set is measurable.

Definition 39. Let (X, β, μ) be a measure space, and $\phi : X \to Y$ a measurable function from (X, β) to measurable space (Y, C). The *pushforward measure* $\phi_*\mu : C \to [0, \infty]$ is defined by the formula $\phi_*\mu(E) = \mu(\phi^{-1}(E))$.

Fact 40. *$\phi_*\mu$ is a measure on C, and for $f : Y \to \mathbb{R}$ measurable, $\int_Y f\, d\phi_*\mu = \int_X (f \circ \phi)\, d\mu$.*

Lemma 41. *The spherical projection $P : \boldsymbol{x} \mapsto \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ and the inverse frame operator T^{-1} are measurable functions.*

Proof. This follows since P, T^{-1} are both uniformly continuous, and μ is a Borel measure by definition.

□

Theorem 42. *The pushforward $S_*^{-1}\mu$ of frame measure μ by invertible linear operator S is itself a frame measure*

Proof. By **Theorem** 33, since $\int \|\boldsymbol{r}\|^2 d\mu < \infty$.
(i) $\mathrm{supp}(S_*^{-1}\mu)$ not contained in any hyperplane. The nonzero-measure sets of $S_*^{-1}\mu$ are those sets E such that

$$\mu((S^{-1})^{-1}(E)) = \mu(S(E)) \neq 0.$$

Now S cannot send sets with nonzero support to a hyperplane, since it is invertible and $\mathrm{supp}(\mu)$ not contained in any hyperplane. Therefore $\mu(S(E))$ nonzero for a set not contained in any hyperplane.
(ii) Using the inequality $\|S\boldsymbol{r}\| \leq \|S\|\|\boldsymbol{r}\|$, where $\|S\|$ denotes the operator norm of S,

$$\int \|\boldsymbol{r}\|^2 S_*^{-1} d\mu = \int \|S^{-1}(\boldsymbol{r})\|^2 d\mu \leq \int \|S^{-1}\|^2 \|\boldsymbol{r}\|^2 d\mu = \|S^{-1}\|^2 \int \|\boldsymbol{r}\|^2 d\mu.$$

Now since S is an invertible operator on a finite dimensional space, $0 < \|S\| < \infty$, so $\|S^{-1}\| = 1/\|S\| < \infty$. By the assumption that $\int \|\boldsymbol{r}\|^2 d\mu < \infty$, it follows that $\int \|\boldsymbol{r}\|^2 S_*^{-1} d\mu < \infty$.

□

Theorem 43. *The pushforward of a frame measure μ under any invertible linear transformation T is a Borel measure. In particular, the pushforward of a frame measure by an invertible operator is a frame measure.*

Proof. This is equivalent to showing that T^{-1} is Borel measurable, since

$$\int_Y f\, dT_*\mu = \int_X (f \circ T)\, d\mu$$

and T^{-1} Borel measurable implies $T(E)$ a Borel (open) set for all $E \subset Y$. (Or since any invertible linear map is a homeomorphism).

□

The most appropriate way to generalize the concept of dual frame is through the relation

$$\Theta^*_{\{\boldsymbol{u}_i\}}\Theta_{\{S\boldsymbol{u}_i\}} = \Theta^*_{\{S\boldsymbol{u}_i\}}\Theta_{\{\boldsymbol{u}_i\}} = \mathrm{Id}. \tag{4}$$

That is, the correct combination of analysis and synthesis operators allows for reconstruction. For the finite frame $\{\boldsymbol{u}_i\}$, $\Theta_{\{\boldsymbol{u}_i\}} : \mathbb{R}^n \to \ell^2(k) \cong \mathbb{R}^k$ gives the coefficients with respect to the frame vectors as a finite list. The analogous operator for frame measures gives an infinite list of coefficients, in the form of an integrable linear functional $\Theta : \mathbb{R}^n \to L^2$. The synthesis operator $\Theta^*_{\{\boldsymbol{u}_i\}} : \ell^2(k) \to \mathbb{R}^n$ takes a list of coefficients to a finite linear combination of frame vectors, and similarly, the synthesis operator for frame measures takes an infinite list of coefficients to an infinite linear combination of frame vectors, $\Theta^* : L^2 \to \mathbb{R}^n$ by integrating with respect to a measure.

For T a linear operator, denote the linear functional $\Theta_T(\boldsymbol{x}) = \langle T\boldsymbol{x}, \cdot\rangle$, and the map Θ^*_T from $L^2 \to \mathbb{R}^n$ given by $\Theta^*_T(g) = \int g(\boldsymbol{r})T\boldsymbol{r}\, d\mu$. The following theorem describes how the canonical dual operators are defined, and how the operator T in Θ_T,Θ_T^* can be absorbed into the measure.

Theorem 44. *The pushforward of frame measure μ under the inverse frame operator T^{-1} satisfies the reconstruction property*

$$\boldsymbol{x} = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle T\boldsymbol{r}\, dT_*^{-1}\mu = \int \langle \boldsymbol{x}, T\boldsymbol{r} \rangle \boldsymbol{r}\, dT_*^{-1}\mu.$$

Proof. Since T is invertible, measurable and self-adjoint,

$$\begin{aligned} \boldsymbol{x} = TT^{-1}(\boldsymbol{x}) &= \int \langle T^{-1}(\boldsymbol{x}), \boldsymbol{r} \rangle \boldsymbol{r}\, d\mu(\boldsymbol{r}) \\ &= \int \langle \boldsymbol{x}, T^{-1}\boldsymbol{r} \rangle \boldsymbol{r}\, d\mu(\boldsymbol{r}) \\ &= \int \langle \boldsymbol{x}, T^{-1}\boldsymbol{r} \rangle \boldsymbol{r}\, d(TT^{-1})_*\mu \\ &= \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle T\boldsymbol{r}\, dT_*^{-1}\mu. \end{aligned}$$

The second line gives $\boldsymbol{x} = \Theta^*\Theta_{T^{-1}}$, and the fourth line gives $\boldsymbol{x} = \Theta_T^*\big|_{dT_*^{-1}}\Theta$, where $\Theta_T^*\big|_{dT_*^{-1}}(v) = \int v(\boldsymbol{r}) T\boldsymbol{r}\, dT_*^{-1}\mu(\boldsymbol{r})$. Reversing the order of T, T^{-1} gives

$$\begin{aligned} \boldsymbol{x} = T^{-1}T\boldsymbol{x} &= \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle T^{-1}\boldsymbol{r}\, d\mu \\ &= \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle T^{-1}\boldsymbol{r}\, d(TT^{-1})_*\mu \\ &= \int \langle \boldsymbol{x}, T\boldsymbol{r} \rangle \boldsymbol{r}\, dT_*^{-1}\mu, \end{aligned}$$

giving $\boldsymbol{x} = \Theta^*\big|_{dT_*^{-1}}\Theta_T$.

□

Example 45. (Finite Dual Frame) Recall from **Example 30** the Dirac measure (δ_{u_i}) which gives mass 1 to each of the vectors $\{\boldsymbol{u}_i\}_{i=1}^k$. In this case the frame operator is the finite frame operator S, and the pushforward measure under S^{-1} gives weight 1 to each of the vectors $\{S^{-1}\boldsymbol{u}_i\}_{i=1}^k$. We then have

$$\int \langle \boldsymbol{x}, S\boldsymbol{r}\rangle \boldsymbol{r}\, dS_*^{-1}\mu = \sum_{i=1}^k \langle \boldsymbol{x}, SS^{-1}\boldsymbol{u}_i\rangle S^{-1}\boldsymbol{u}_i = \sum_{i=1}^k \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle S^{-1}\boldsymbol{u}_i = \boldsymbol{x}$$

as well as

$$\int \langle \boldsymbol{x}, \boldsymbol{r}\rangle S\boldsymbol{r}\, dS_*^{-1}\mu = \sum_{i=1}^k \langle \boldsymbol{x}, S^{-1}\boldsymbol{u}_i\rangle SS^{-1}\boldsymbol{u}_i = \sum_{i=1}^k \langle \boldsymbol{x}, S^{-1}\boldsymbol{u}_i\rangle \boldsymbol{u}_i = \boldsymbol{x}.$$

4.2 Examples

Example 46. (Gaussian measure) Consider the usual n-dimensional Lebesgue measure λ^3. The Gaussian measure γ^3 is defined by

$$\gamma^3(A) = \int_A \exp(-\|\boldsymbol{x}\|^2) d\lambda^3(\boldsymbol{x}).$$

γ^n is a tight frame measure, since for $\boldsymbol{x} = (x_1, x_2, x_3)$

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r} \, d\lambda^n(\boldsymbol{r}) &= \int (x_1 r_1 + x_2 r_2 + x_3 r_3) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} e^{-(r_1^2 + r_2^2 + r_3^2)} \, dr_1 dr_2 dr_3 \\ &= \int \begin{pmatrix} x_1 r_1^2 + x_2 r_1 r_2 + x_3 r_1 r_3 \\ x_1 r_1 r_2 + x_2 r_2^2 + x_3 r_2 r_3 \\ x_1 r_1 r_3 + x_2 r_2 r_3 + x_3 r_3^2 \end{pmatrix} e^{-(r_1^2 + r_2^2 + r_3^2)} \, dr_1 dr_2 dr_3\end{aligned}$$

The integral is symmetric with respect to the three coordinates, so we can determine its matrix by its value for $\boldsymbol{e}_1$.

$$\int_{\mathbb{R}} r_1^2 e^{-(r_1^2 + r_2^2 + r_3^2)} \, dr_1 dr_2 dr_3 = \pi \int r_1^2 e^{-r_1^2} \, dr_1 = \pi \frac{2}{2^2} \frac{\sqrt{\pi}}{2} = \frac{\pi^{3/2}}{4}$$

And

$$\int_{\mathbb{R}} r_1 r_2 e^{-(r_1^2 + r_2^2 + r_3^2)} \, dr_1 dr_2 dr_3 = \sqrt{\pi} \int_{\mathbb{R}} r_1 r_2 e^{-(r_1^2 + r_2^2)} dr_1 dr_2 = 0$$

Since the off-diagonal terms are odd functions. So the matrix for T is given by

$$T = \frac{\pi^{3/2}}{4} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

that is, the Gaussian measure is tight, with frame constant $\frac{\pi^{3/2}}{4}$.

The next example shows the explicit construction of the analysis and synthesis operators for a given frame measure.

Example 47. (Analysis/Synthesis operators for S^1 measure). Recall from before the Lebesgue measure on the circle, which is a tight. We use **Theorem 20** to find the analysis and synthesis operators, and show that $\Theta^*\Theta = \frac{1}{2}\mathrm{Id}$. $\Theta_{\boldsymbol{x}}(\boldsymbol{r})$ is simply $\boldsymbol{x}^T$, while

$$\Theta^*(\boldsymbol{g}) \;=\; \int_{S^1} \boldsymbol{g}(\boldsymbol{r})\boldsymbol{r}\, d\theta.$$

$\boldsymbol{g}$ is the vector transpose, $\boldsymbol{g} = [g_1, g_2]$, so $\boldsymbol{g}(\boldsymbol{r}) = (g_1 r_1 + g_2 r_2)$. The integral becomes

$$\begin{aligned}
\Theta^*(\boldsymbol{g}) &= \int_0^{2\pi} (g_1 \cos\theta + g_2 \sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} g_1 \cos^2\theta + g_2 \cos\theta \sin\theta \\ g_1 \cos\theta \sin\theta + g_2 \sin^2\theta \end{pmatrix} d\theta \\
&= \frac{1}{2} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.
\end{aligned}$$

Now $\Theta^*\Theta(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the identity discussed previously.

Next we characterize the tight frame measures on $\mathbb{R}^2$ expressible in terms of the standard Lebesgue measure λ^2.

Example 48. Suppose $\hat{\mu}$ a frame measure on $\mathbb{R}^2$ such that

$$\hat{\mu}(E) = \int_E f(\boldsymbol{r})\, d\lambda(\boldsymbol{r}).$$

for some $f : \mathbb{R}^n \to \mathbb{R}$. The frame operator can be given for $\boldsymbol{x} = (x_1, x_2)$ by

$$\begin{aligned}
\Theta^*\Theta(x) &= \int \begin{pmatrix} x_1 r_1^2 + x_2 r_2 r_1 \\ x_1 r_1 r_2 + x_2 r_2^2 \end{pmatrix} f(\boldsymbol{r})\, dr_1 dr_2 \\
&= \begin{pmatrix} \int f(\boldsymbol{r}) r_1^2 & \int f(\boldsymbol{r}) r_1 r_2 \\ \int f(\boldsymbol{r}) r_1 r_2 & \int f(\boldsymbol{r}) r_2^2 \end{pmatrix}.
\end{aligned}$$

This is similar to the condition introduced for finite frames in $\mathbb{R}^2$. We can express the frame matrix in terms of $\boldsymbol{r} = r(\cos\theta, \sin\theta)$ so the frame operator becomes

$$\begin{aligned}
\Theta^*\Theta &= \begin{pmatrix} \int f(\boldsymbol{r}) r^2 \cos^2\theta & \int f(\boldsymbol{r}) r^2 \cos\theta \sin\theta \\ \int f(\boldsymbol{r}) r^2 \cos\theta \sin\theta & \int f(\boldsymbol{r}) r^2 \sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \int f(\boldsymbol{r}) r^2 \cos^2\theta & \int f(\boldsymbol{r}) r^2 \sin 2\theta \\ \int f(\boldsymbol{r}) r^2 \sin 2\theta & \int f(\boldsymbol{r}) r^2 \sin^2\theta \end{pmatrix}
\end{aligned}$$

So that we have tightness if and only if

$$\int f(\boldsymbol{r}) \begin{pmatrix} r^2 \cos 2\theta \\ r^2 \sin 2\theta \end{pmatrix} d\lambda = \mathbf{0}.$$

This is the direct generalization of the diagram vectors from **Example 9**.

4.2.1 The Frame Inertia Property

We have seen in the theory of finite frames that tight frames are in some sense "evenly dispersed" (Example 9). The authors of [2] push this idea further by introducing a repulsive mathematical force between vectors in a frame, showing that a configuration is in equilibrium under this force precisely when a frame is tight. In analogy to the equilibrium configuration of finitely many electrons on a spherical shell, a tight frame has “optimally spaced vectors”.

It remains unclear if this characterization can be applied to frame measures. However, drawing a physical analogy with distributions of charge, we are able to characterize the tight frame measures as somehow “evenly distributed” around the origin. To do this, we introduce the inertia tensor of a distribution of mass.

Definition 49. The inertia tensor of a finite set of vectors $\{\boldsymbol{r}_k\}_N$ with masses $\{m_k\}_N$ is the operator defined by

$$\boldsymbol{I} = \sum_{k=1}^{N} m_k \left(\langle \boldsymbol{r}_k, \boldsymbol{r}_k \rangle \boldsymbol{E} - \boldsymbol{r}_k \otimes \boldsymbol{r}_k\right) \tag{5}$$

where $\boldsymbol{E} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_n \otimes \boldsymbol{e}_n$, the identity tensor. For a general measure, the inertia tensor is defined by

$$\boldsymbol{I} = \int \left(\langle \boldsymbol{r}, \boldsymbol{r} \rangle \boldsymbol{E} - \boldsymbol{r} \otimes \boldsymbol{r}\right) d\mu(\boldsymbol{r}). \tag{6}$$

Theorem 50. *μ is a tight frame measure on $\mathbb{R}^N$ if and only if its inertia tensor is $M\boldsymbol{E}$ for $M \in \mathbb{R}$.*

Proof. If μ is a tight frame measure with frame constant A, then by the reconstruction property,

$$A\boldsymbol{E} = \int (\boldsymbol{r} \otimes \boldsymbol{r})\, d\mu(\boldsymbol{r})$$

so that $\boldsymbol{I} = M\boldsymbol{E}$ for $M =$. Next, suppose $\boldsymbol{I} = M\boldsymbol{E}$, for $M \in \mathbb{R}$. Then since $\int r^2 \boldsymbol{E}$ is always a multiple of identity, we must have $\int \boldsymbol{r} \otimes \boldsymbol{r}\, d\mu$ a multiple of identity as well. By the reconstruction property, it follows that the frame operator is a multiple of identity, so that μ is tight.

□

Corollary 51. *The inertia tensor for a finite tight k-vector frame in* $\mathbb{R}^N$ *is* $(k - \frac{k}{N})\boldsymbol{E}$.

The usefulness of this observation is not necessarily mathematical, but provides a nice physical interpretation of tight frame measures. The tensor can be written in matrix form, where the i, j off-diagonal entry is given by

$$\int x_i x_j \, d\mu$$

for $(x_1, x_2, \ldots, x_n) = \boldsymbol{x} \in \mathbb{R}^n$, and diagonal entry i, i is given by

$$\int \left(\sum_{j \neq i} x_j^2 \right) \, d\mu.$$

In physics, the inertia matrix of an object rotating about a given point relates the angular velocity $\boldsymbol{\omega}$ to the angular momentum $\boldsymbol{L}$ via

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}.$$

In other words, for an object spinning around a fixed point about an axis parallel to $\boldsymbol{\omega}$, the angular momentum points in the direction of $\boldsymbol{I}\boldsymbol{\omega}$. An example of an object with inertia matrix given by a multiple of identity is a cube rotating about its center, or a sphere about its center. The inertia matrix does not specify the distribution of mass exactly, but it is easy to use this characterization to identify non-tight frame measures based on symmetry and geometric intuition.

4.3 Convergence of Measures

The approximation of finite frames is either ambiguous, or trivial. If we wish to approximate the frame operator $S_{\{u_i\}}$, the result follows by the fact that every positive definite operator S is the frame operator for some $\{v_i\}$, a proof of which can be found in [6], and follows by diagonalizing S. Since the positive definite operators are dense in $\mathcal{L}(\mathbb{R}^n)$, the approximation of any operator by frame operators follows.

On the other hand, there are several modes of convergence for a sequence of measures $\{\mu_i\} \to \mu$. It can be shown that the limit μ must always be a measure. It is then natural to ask whether an arbitrary frame measure μ is the limit $\lim_{n\to\infty}\{\mu_n\}$, where the μ_n come from a much simpler set of frame measures. A natural candidate for this set are the discrete frame measures from **Example 13** , which include the finite frames. The major result of this section is the approximation of an arbitrary frame measure by discrete frame measures.

The mode of convergence used will be the *weak convergence* of sequences of frame measures. We are interested in retaining the frame bounds for the original measure μ across all μ_i in an approximating sequence. This will ultimately result in a statement of the density of discrete frame measures in the space of frame measures, much like density of frames in $\mathcal{L}(\mathbb{R}^n)$.

Definition 52. Given a measure μ on X, we say the sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ *converges weakly* to μ if for every $f \in C_b$ (continuous bounded functions $f : X \to \mathbb{R}$)

$$\int f\, d\mu_n \to \int f\, d\mu.$$

Before the equivalence statement, we describe what a weakly converging sequence of finite frame measures $\{\mu_i\} \to \mu$ must look like. The following lemmas provide a description that gives a natural choice for an approximating sequence. The following lemma says that we must choose our discrete frame vectors of a μ_i close enough to the support of μ if we want weak convergence.

Lemma 53. *If the discrete frame measures $\{\boldsymbol{u}_i\}_{n\in\mathbb{N}}$ converge weakly to μ, then for every open set U with* $\operatorname{supp}(\mu) \subset U$,

$$\sum_{\boldsymbol{u}_n \notin U} c_n \to 0.$$

Proof. There exists a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ so that $f(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \operatorname{supp}(\mu)$, and $f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in U^c$. Integrating f,

$$\int f \, d\mu = \mu(\mathbb{R}^k)$$

while for each $n \in \mathbb{N}$, integrating f with respect to $\{\boldsymbol{u}_i\}_n$ gives

$$\sum_i c_i \cdot f(\boldsymbol{u}_i) = \sum_{\boldsymbol{u}_j \in U} c_j.$$

Now since $\{\boldsymbol{u}_i\}_n \to \mu$, we have $\sum c_i = \mu(\mathbb{R}^k)$, but for each $\{\boldsymbol{u}_i\}_n$ this is the same as

$$\sum_{\boldsymbol{u}_j \in U} c_j \to \mu(\mathbb{R}^k).$$

Since the weights c_j are positive, it must be that $\sum_{\boldsymbol{u}_n \notin U} c_n \to 0$ for n large enough.

□

This characterization still allows for the possibility that the approximating frames

$\{\boldsymbol{u}_i\}_n$ contain vectors not in $\text{supp}(\mu)$, as long as the weights c_i of these non-support vectors all vanish as $i \to \infty$. Of course, an easy guess for a converging sequence $\{\boldsymbol{u}_i\}_n \to \mu$ might ignore vectors outside $\text{supp}(\mu)$ altogether.

The next lemma says that the vectors in a sequence of discrete frame measures $\{\boldsymbol{u}_i\}_n \to \mu$ must in some way sample $\text{supp}(\mu)$ uniformly.

Lemma 54. *If the discrete frame measures $\{\boldsymbol{u}_i\}_{n\in\mathbb{N}}$ converge weakly to μ, then given $\epsilon > 0$, and $\boldsymbol{x} \in supp(\mu)$, $\exists N \in \mathbb{N}$ such that*

$$\boldsymbol{u}_k \in N_\epsilon(\boldsymbol{x})$$

for some $\boldsymbol{u}_k \in \{\boldsymbol{u}_i\}_n$, and all $n > N$.

Proof. There exists a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ so that for fixed $\epsilon' < \epsilon$,

$$f(\boldsymbol{r}) = \begin{cases} 1, & \boldsymbol{r} \in \overline{N_{\epsilon'}(\boldsymbol{x})} \\ 0 & \boldsymbol{r} \in \mathbb{R}^k \backslash N_\epsilon(\boldsymbol{x}) \end{cases}.$$

If for infinitely many $n \in \mathbb{N}$, we can't find $\boldsymbol{u}_k \in \{\boldsymbol{u}_i\}_n$ so that $\boldsymbol{u}_k \in N_\epsilon(\boldsymbol{x})$, then since

$$\int f \, d\mu = \mu(N_\epsilon(\boldsymbol{x}))$$

but $\sum c_i \cdot f(\boldsymbol{u}_i) = 0$ for infinitely many $\{\boldsymbol{u}_i\}_n$, it is impossible that the discrete measures converge to μ.

□

Since $\mathbb{R}^k$ is separable, **Lemma 54** poses no restrictions to approximating some μ with unbounded support. However, as a simplifying case, the following statements will be considered for $\text{supp}(\mu)$ compact. In this setting, **Lemma 54** says that given a finite open cover of $\text{supp}(\mu)$ by balls of radius $\frac{1}{n}$, we can find N so that $n' > N$ implies $\mu_{n'}$ contains a $\boldsymbol{u}_k$ in each $\frac{1}{n}$-neighborhood. For convenience, assuming the existence of $\{\mu_n\}$, let's always take a subsequence $\{\mu_m\}$ so that μ_m satisfies this property for balls of radius $\frac{1}{m}$.

So far, the necessary conditions on the weights c_i have only required that they vanish for vectors outside $\text{supp}\mu$. If we wish to show by construction that an approximation by discrete measures exists, we are left in the dark as to how to choose the $c_i's$ to pair with the $\boldsymbol{u}_i \in \text{supp}\mu$. Indeed, an intuitive choice of $\{c_i\}$ will prove to be a viable one. Assuming $\text{supp}(\mu)$ compact, and the μ_i satisfying the open-cover property, let c_i be the mass of the neighborhood containing $\boldsymbol{u}_i$,

$$c_i = \mu(N_{\frac{1}{n}}(\boldsymbol{u}_i)).$$

$N_{\frac{1}{n}}(\boldsymbol{u}_i)$ is the open ball containing $\boldsymbol{u}_i$. Since weak convergence is tested against $f \in C_b(\mathbb{R}^k)$, this choice of $\{c_i\}_n$ will ensure weak convergence to $\int f\,d\mu$ for all f. This is the basic approximation that we will use in the approximation theorem. In the following discussion, each μ_i will be modified so that the frame bounds of μ are retained.

As will be shown, the uniformity condition in **Lemma 53** ensures that for frame bounds $B \neq A$, μ_n large enough will ensure these bounds are preserved. First, a lemma describing the convergence of the frame operator $T_{\mu_i} \to T_\mu$.

Lemma 55. *If the sequence of frame measures $\{\mu_i\}$ converges weakly to frame measure μ (irrespective of frame bounds), then*

$$T_{\mu_i} \to T_\mu.$$

Proof. This follows immediately from the definition of weak convergence, since $\Theta_{\boldsymbol{x}}(\boldsymbol{r})\boldsymbol{r}$ a continuous function, and

$$T_{\mu_i}(\boldsymbol{x}) = \int \Theta_{\boldsymbol{x}}(\boldsymbol{r})\boldsymbol{r}\,d\mu_i$$

□

Theorem 56. *(Approximation of non-tight frame measures) Given frame measure μ, supp(μ) compact, and frame bounds $B \neq A$, there exists a sequence of discrete measures*

$$\mu_i : \mathbb{R}^n \to \mathbb{R}$$

converging weakly to μ. Moreso, every μ_i a frame measure with bounds A, B.

Proof. The proof is by construction. For $\mu_n = \{c_k, \boldsymbol{u}_k\}$ Choose $\{\boldsymbol{u}_k\}$ so that the vectors satisfy the uniformity condition of **Lemma 34** for an open cover by balls of radius $\frac{1}{n}$. Let the weights $\{c_k\}$ be the μ-measures of those open balls. It was reasoned before that such a choice of μ_n will converge weakly to μ. It must be shown that for n large enough,

$$B\|\boldsymbol{x}\|^2 \leq \sum_i \langle \boldsymbol{x}, \sqrt{c_i}\boldsymbol{u}_i \rangle^2 \leq A\|\boldsymbol{x}\|^2.$$

Consider the case for the upper bound A (the case for B can be handled similarly). Let T_n be the frame operator for $\{\boldsymbol{u}_i\}_n$. By the continuity of eigenvalues, there is an N such that $n > N$ implies the eigenvalues of T_μ are within ϵ of the eigenvalues of T_{μ_n}. Since the frame bounds are determined by $B = \lambda_{min}$ and $A = \lambda_{max}$ of T_μ, this ensures all eigenvalues of T_{μ_n} are less than A except for possibly the largest one. Thus it is possible that our sequence $\{\mu_n\}$ has no subsequence $\{\mu_{n_i}\}$ with $\lambda_{max}(\mu_{n_i}) < A$. The rest of the proof will continue after the next results.

□

4.4 Perturbation of Frame Operators

The construction given approximates frame measure μ weakly by finite frames μ_i.

In fact the method is general for approximating any measure by discrete measures. We have that the limit of frame operators converges to the correct operator $T_{\mu_i} \to T_{\mu}$. The only problem being that the eigenvalues $\{\lambda_k\}_{\mu_i}$ of T_{μ_i} may approach $\{\lambda_k\}_{\mu}$ of T in a way where $(\lambda_i)_{max} > \lambda_{max}$ and/or $(\lambda_i)_{min} < \lambda_{min}$ for infinitely many $i \in \mathbb{N}$. The following theorems show that the μ_i can be perturbed

$$\mu_i \to \mu_i'$$

so that $\lim_{i\to\infty} \mu_i' \to \mu$ and each μ_i has frame bounds A, B.

Theorem 57. *Let* $\Omega : X \to X$ *.*

1. Ω has spectrum $\{\lambda_k\}$ with eigenvalues $\{v_k\}$ if and only if the operator $\Omega + \epsilon I$ has spectrum $\{\epsilon + \lambda_k\}$ with eigenvalues $\{v_k\}$.
2. Ω has spectrum $\{\lambda_k\}$ with eigenvalues $\{v_k\}$ if and only if $\epsilon\Omega$ has spectrum $\{\epsilon\lambda_k\}$ with eigenvalues $\{v_k\}$.

Proof. Since $(\Omega + \epsilon I)v_k = \Omega v_k + \epsilon v_k = (\lambda_k + \epsilon)v_k$. The second equality follows just as directly since $\epsilon\Omega v_k = \epsilon\Omega\lambda_k v_k$.

□

Let $\sigma(T) = \{\lambda_k\}$. In considering the frame operator T, the above result means $\text{diam}(\sigma(T))$ can be scaled and shifted by multiplication by $\epsilon \in \mathbb{R}$, or addition of the simple matrix ϵI. Moreso, the next theorem says that these operations on the frame operator T_i have a simple interpretation as modifications to the frame μ_i, using the following definition.

Definition 58. Denote by $M_A^B(\mathbb{R}^n)$ the space of all frame measures on $(\mathfrak{B}, \mathbb{R}^n)$ with frame bounds A and B. For two measures $\mu_1, \mu_2 \in M_A^B(\mathbb{R}^n)$, their linear combination $c_1\mu_1 + c_2\mu_2$ is the Borel measure defined on open sets by

$$(c_1\mu_1 + c_2\mu_2)(E) = c_1\mu_1(E) + c_2\mu_2(E).$$

Theorem 59. *Let μ_i have frame operator T_{μ_i}*

1. $T_{\mu_i} + \epsilon I$ is the frame operator for the frame measure $\mu_i + \{\boldsymbol{u}_i\}$, where $\{\boldsymbol{u}_i\}$ is an orthogonal basis with $\|\boldsymbol{u}_i\| = \sqrt{\epsilon}$.

2. ϵT_{μ_i} is the frame operator for the frame measure $\epsilon\mu_i$.

Proof. The frame operator for $\mu_i + \{\boldsymbol{u}_i\}$ is

$$T_{\mu_i+\{\boldsymbol{u}_i\}}(\boldsymbol{x}) = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r}\, d(\mu_i + \{\boldsymbol{u}_i\}) = \int \langle \boldsymbol{x}, \boldsymbol{r} \rangle \boldsymbol{r}\, d\mu_i + \sum_i \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \boldsymbol{u}_i = T_{\mu_i}\boldsymbol{x} + \epsilon \boldsymbol{x}.$$

The frame operator for $\epsilon\mu_i$ is obviously ϵT_{μ_i} by linearity of the integral. These are frame measures.

□

So for $\sigma(T_{\mu_i})$, we can shift and scale its spectrum by simply appending an orthogonal basis, or scaling the measure. Using this procedure, the proof of weak approximation of frame measures can be completed.

Proof. **(Theorem 56 Continued)** Let the finite frames $\mu_i \to \mu$ weakly. There exist perturbations μ_i' of μ_i so that $\mu_i' \to \mu$ and each μ_i' has frame bounds A, B.

From before we are concerned only with $(\lambda_i)_{max}$ and $(\lambda_i)_{min}$. There exists subsequence n_i so that $1 - \frac{1}{i} < \frac{\text{diam}(\sigma(T_{\mu_{n_i}}))}{\text{diam}(\sigma(T_\mu))} < 1 + \frac{1}{i}$. Picking $\epsilon_i = (1 + \frac{1}{i})^{-1}$, it follows that

$$\text{diam}(\sigma(\epsilon_i T_{n_i})) < \frac{\text{diam}(\sigma(T_\mu))}{\text{diam}(\sigma(T_{\mu_{n_i}}))}\text{diam}(\sigma(T_{\mu_{n_i}})) = \text{diam}(\sigma(T_\mu)).$$

By Theorem 9, $\epsilon_i \mu_{n_i}$ are frame measures with corresponding operators $\epsilon_i T_{n_i}$. Since $\epsilon_i \to 1$, it follows that $\epsilon_i \mu_{n_i} \to \mu$.

Now consider subsequence $\{\eta_j\} = \{n_{i_j}\} \subset \{n_i\}$ so that

$$\lambda_{min} - \frac{1}{j} < \lambda_{min}(\epsilon_j T_{\mu_{\eta_j}}) < \lambda_{min} + \frac{1}{j}$$

. Picking $\xi_j = \frac{1}{j}$ we have that

$$\lambda_{min} < \lambda_{min}\left(\xi_j I + \epsilon_j T_{\mu_{\eta_j}}\right).$$

Since $\xi_j I \to \mathbf{0}$, it follows that $\xi_j I + \epsilon_j T_{\mu_{\eta_j}} \to T_\mu$.

For the subsequence η_j, define

$$T_{\mu_j'} = \xi_j I + \epsilon_j T_{\mu_{\eta_j}}.$$

Where μ_j' is the frame measure from Theorem 9. The perturbed frames μ_i' now have shifted and scaled eigenvalues so that diam $\left(\sigma(T_{\mu_j'})\right) <$ diam $(\sigma(T_\mu))$, and $\lambda_{min}(T_{\mu_j'}) > \lambda_{min}(T_\mu)$. The perturbed sequence μ_j' is the weakly converging sequence we have been seeking.

□

We have shown that an approximation $\{\mu_i\}$ exists for non-tight frame measure μ. The method employed does not work for tight frames however, since perturbing the spectrum of T_{μ_n} to a single value is impossible by appending orthonormal bases. In other words, it is possible to contract and shift the spectrum of T_{μ_n} under the current method, but its span will remain finite in length.

Theorem 60. *(Approximation of Tight Frame Measures) Given tight frame measure μ, supp(μ) compact, and frame constant A, there exists a sequence of discrete measures*

$$\mu_i : \mathbb{R}^n \to \mathbb{R}$$

converging weakly to μ. Moreso, every μ_i a tight frame measure with frame constant A.

Proof. Beginning with the same $\{\mu_i\}$ from the approximation of non-tight frame measures, we again modify μ_i so that it is tight with constant A. For $\epsilon > 0$, from before, for N large enough, the frame condition gives

$$(A-\epsilon)\|\boldsymbol{x}\|^2 \leq \langle T_{\mu_n}\boldsymbol{x}, \boldsymbol{x}\rangle \leq A\|\boldsymbol{x}\|^2$$

or

$$A\|\boldsymbol{x}\|^2 \leq \langle T_{\mu_n}\boldsymbol{x}, \boldsymbol{x}\rangle \leq (A+\epsilon)\|\boldsymbol{x}\|^2.$$

Without loss of generality, assume the first inequality, so that $-A\|\boldsymbol{x}\|^2 \leq \langle -T_{\mu_n}\boldsymbol{x}, \boldsymbol{x}\rangle \leq (\epsilon - A)\|\boldsymbol{x}\|^2$ gives for all $\boldsymbol{x}$,

$$0 \leq \langle (AI - T_{\mu_n})\boldsymbol{x}, \boldsymbol{x}\rangle \leq \epsilon\|\boldsymbol{x}\|^2. \tag{7}$$

Thus $AI - T_{\mu_n}$ is a positive semidefininte operator and therefore has rank 1 tensor product decomposition by the spectral theorem [6]

$$AI - T_{\mu_n} = \sum_{k=1}^{n} \lambda_k \boldsymbol{r}_k \otimes \boldsymbol{r}_k.$$

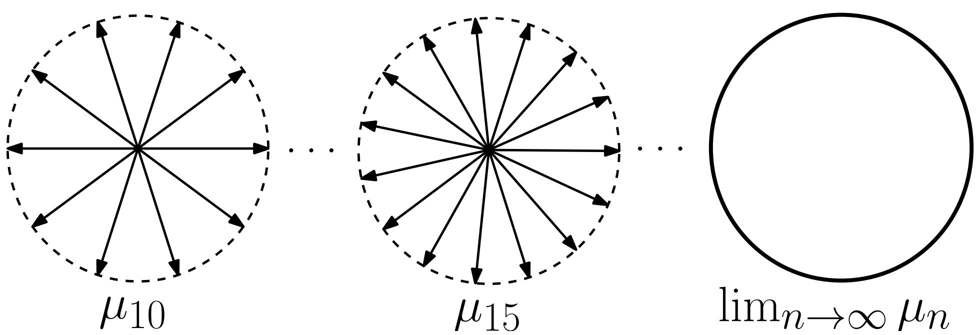
Adding this operator to T_{μ_n} is the same as appending the discrete frame $\{\boldsymbol{r}_k\}_{k=1}^n$ with weights $\{\sqrt{\lambda_k}\}_{k=1}^n$ to the frame μ_i (all eigenvalues λ_k are positive so this is a measure). The resulting frame operator is $\tilde{T}_{\mu_n} = T_{\mu_n} + \sum_{k=1}^n \lambda_k \boldsymbol{r}_k \otimes \boldsymbol{r}_k$, so

$$AI - \tilde{T}_{\mu_n} = \sum_{k=1}^{n} \lambda_k \boldsymbol{r}_k \otimes \boldsymbol{r}_k - \sum_{k=1}^{n} \lambda_k \boldsymbol{r}_k \otimes \boldsymbol{r}_k = 0$$

thus $\tilde{T}_{\mu_n}$ is the operator for a tight frame. Formula **(5)** says $\lambda_k \to 0$ as $\mu_n \to \mu$, so the new frame measures still converge to μ.

□

Figure 8: The approximation of the circle measure (**Example 18**) by tight frame measures.



5 Afterword and Looking Ahead

The developments of the previous sections are interesting on their own, but should primarily be seen as recasting frame theory in a new light. Through the specific problems

encountered, we see the primary importance of the frame bounds in dictating important theoretical questions. Such a question might proceed as "Can some mathematical property be preserved in a given subset of frames *while also preserving the frame bounds*?" We saw that these properties could naturally be topological ones in Sections 2 and 3, but very easily could be algebraic or analytic. For example, we could try to refine the path-connectedness of $FUNF$ result (**Theorem 26**) to ask about paths of *smallest length* connecting arbitrary unit-norm frames, becoming a problem in analysis. In the cases before, the subsets of the frames were $FUNF \subset \mathcal{F}_n^k$, or sets of frames in $\mathcal{F}_n^k$ with like bounds A, B. We could ask the same connectedness question for $FUNTF \subset FUNF$. This is in fact a very complicated problem. The path-connectedness of $FUNTF$, also known as the *Frame Homotopy Problem,* was shown in 2013 using rather deep results from abstract algebra [1]. Over the course of this investigation, other new questions arose which have no immediate answer. The biggest comes from Section 2. Can we delete more vectors from a moving tight frame than the number described? The most ambitious question is whether there exists a moving FUNF with smallest redundancy, i.e., a moving frame for S^{2n-1} with $2n$ vectors? The preliminary lemmas used to prove **Theorem 11** are general and apply to any operator. Is there a subclass of moving frames for which we can make stronger conclusions about the invertibility of their frame operators when deleting larger numbers of vectors from the fields? Another question from Section 2 regards preserving frame bounds at every frame in a moving frame. I.e., is it possible to make a moving frame $(\boldsymbol{f}_i)_{i=1}^k$ so that the frame $(\boldsymbol{f}_i(\boldsymbol{x}))_{i=1}^k$ has the same bounds A, B for every $\boldsymbol{x} \in S^{2n+1}$?
Regarding Section 4, the biggest remaining question is motivated by the correlation we've seen with the theory of finite frames. In particular, we were not able to conclude an analogous Dilation Property (**Theorem 7**) for frame measures. The infinite-dimensional Hilbert space obtained by making every $\boldsymbol{v} \in \text{supp}(\mu)$ a formal basis element does not give the same kind of geometric insight as the finite dimensional dilation property. Does

a more revealing dilation property exist for frame measures? As stated before, the concept of frame force for finite frames gives a deep geometric insight into tight frames. Beyond the characterization of tight frame measures with the inertia property, does there exist a more revealing notion of a force for frame measures?

The result in [1] on the connectedness of $FUNTF$ is complicated beyond the level of this thesis, and the question arises whether any simpler proof exists. As is often the case in mathematics, the push to find a simpler proof motivated many of the results in Section 3. While no simpler proof was discovered, the investigation generated new questions and results of independent interest.

The significance of finite frames with respect to signal processing was expounded upon in the Executive Summary section. The results of this investigation have significance primarily in their theoretical application. In part, the importance of these results is evident in their generality. Any time we deal with a spanning set, full rank operator, or nonvanishing vector field, we are in essence dealing with frames and moving frames. At the same time, as was seen in the section on connectedness theorems, a discourse on frames is inevitably a discourse on positive definite and semidefinite operators—which have a tremendous scope of application in pure fields such as geometry, as well as uses in electrical engineering and quantum mechanics.

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