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# Mappings Between Annuli of Smallest p-Harmonic Energy

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## ABSTRACT

An important topic in the calculus of variations is the study of traction-free problems, in which deformations between given domains in $\mathbb{R}^n$ are allowed to slip on the boundary, without prescribing boundary values. For annuli $\mathbb{A} = \mathbb{A}(r, R)$ and $\mathbb{A}^* = \mathbb{A}(r_*, R_*)$, we seek the traction-free minimizer of the $p$-harmonic energy among homeomorphisms in Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$. For such a mapping, the $p$-harmonic energy is defined by

$$\mathcal{E}_p[h] = \int_{\mathbb{A}} |Dh(x)|^p dx$$

Classical methods fail for traction-free problems. We will use a novel approach based on the concept of free Lagrangians, described as differential forms $L(x, h(x), Dh(x))dx$ whose integral depends only on the homotopy class of $h$. We find that the solution to the $p$-harmonic variational problem depends on the relative thickness of $\mathbb{A}$ and $\mathbb{A}^*$.

# Mappings Between Annuli of Minimal $p$-Harmonic Energy

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# Contents

# Chapter 1

# Introduction

A classical problem in the calculus of variations concerns the existence of a minimizer for a given energy integral, subject to prescribed values.

*Problem* 1.0.1. Let a bounded domain $\mathbb{X} \subset \mathbb{R}^n$ and a function $f : \partial\mathbb{X} \to \mathbb{R}$ be fixed. For a given class $\mathcal{A}$ of mappings on $\mathbb{X}$ and a stored energy function $\mathcal{E} = \mathcal{E}(x, y, M)$, does there exist a map $h^0 \in \mathcal{A}$ with $h^0|_{\partial\mathbb{X}} = f$ such that

$$\int\limits_{\mathbb{X}} \mathcal{E}(x, h(x), Dh(x))dx \geq \int\limits_{\mathbb{X}} \mathcal{E}(x, h^0(x), Dh^0(x))dx$$

for all $h \in \mathcal{A}$?

Classical methods can be used to reduce this variational problem to a problem involving differential equations. A rich theory addressing these problems exists [2, 3].

A modification of the classical problem is the so-called traction-free problem, where no values of the solution are prescribed. For a traction-free problem, mappings between given domains are allowed to slide on the boundary of the target. This area of study is compelling because new tools and nonclassical approaches are needed to answer traction-free problems. Classical methods for this problem fail. The traction-free problem for the conformal, or $n$-harmonic, energy has been solved on annuli in $\mathbb{R}^n$ [7].

We generalize these results for the $p$-harmonic energy when $p$ is greater than the dimension of the space.

**Definition 1.0.2.** Let $\mathbb{X}$ be a domain in $\mathbb{R}^n$. For $p > 1$, the $p$-harmonic energy of a mapping $h$ Sobolev class $W^{1,p}(\mathbb{X}, \mathbb{R}^n)$is given by

$$\mathcal{E}_p[h] = \int_{\mathbb{X}} |Dh(x)|^p dx \tag{1.1}$$

*Problem* 1.0.3. Does there exist a homeomorphsim $h^0$ in the Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$ such that $\mathcal{E}_p[h^0] = \inf\{\mathcal{E}_p[h]\}$, where the infimum is taken over all homeomorphsisms in $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$?

When $p = n$, we have the conformal energy. The methods of [7] are followed, but considering different powers of integrability significantly complicates the calculations. For mappings between annuli $\mathbb{A} = \mathbb{A}(r, R)$ and $\mathbb{A}^* = \mathbb{A}(r_*, R_*)$, we find the solution of the $p$-harmonic variational problem depends on the relative thickness of the annuli. We now briefly present the main results.

When $\frac{R_*}{r_*} = \frac{R}{r}$, we have a harmonic mapping $h^0(x) = \frac{r_*}{r}x$ of $\mathbb{A}$ onto $\mathbb{A}^*$. If $h$ is any homeomorphism in Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$, we can estimate $\mathcal{E}_p[h]$ using Holder's inequality and Hadamard's inequality. If $|\mathbb{X}| = \int\limits_{\mathbb{X}} dx$, then we have

$$\int\limits_{\mathbb{A}} |Dh(x)|^p dx \geq |\mathbb{A}|^{-\frac{p-n}{p}} \left( \int\limits_{\mathbb{A}} |Dh(x)|^n dx \right)^{\frac{p}{n}} \geq |\mathbb{A}|^{-\frac{p-n}{p}} \left( n^{\frac{n}{2}} \int\limits_{\mathbb{A}} J_h(x) dx \right)^{\frac{p}{n}} = \frac{n^{\frac{p}{2}} |\mathbb{A}^*|^{\frac{p}{n}}}{|\mathbb{A}|^{\frac{p-n}{n}}}$$

Equality holds throughout for the conformal mapping $h^0$, so we see it is the desired homeomorphism.

Suppose the target annulus $\mathbb{A}^*$ is conformally thinner than $\mathbb{A}$. We say we are in the contracting case when $\frac{R_*}{r_*} < \frac{R}{r}$. If $\mathbb{A}^*$ is not too thin relative to $\mathbb{A}$, we can find a $p$-harmonic energy-minimal radial homeomorphism. If $\mathbb{A}^*$ is too thin relative to $\mathbb{A}$, then there is no such homeomorphism, but we can still find a radial $p$-harmonic energy minimizer. The following two theorems state these results.

**Theorem 1.0.4.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$ with $H_+\left(\frac{R}{r}\right) < \frac{R_*}{r_*} < \frac{R}{r}$, where $H_+ : [1, \infty) \to [1, \infty)$ is an increasing function depending on $n$ and $p$, defined in (4.27). If $p > n$, then there exists a unique radial homeomorphsim $h^0(x) = H(|x|)\frac{x}{|x|}$ that maps $\mathbb{A}$ onto $\mathbb{A}^*$ such that*

$$\int\limits_{\mathbb{A}} |Dh(x)|^p \, dx \geq \int\limits_{\mathbb{A}} |Dh^0(x)|^p \, dx$$

*for every homeomorphism $h : \mathbb{A} \to \mathbb{A}^*$ of Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.*

**Theorem 1.0.5.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$ with $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$. If $p > n$, then there is no $p$-harmonic energy minimal radial homeomorphism of $\mathbb{A}$ onto $\mathbb{A}^*$, but there exists a radial* *map $h^0(x) = H(|x|)\frac{x}{|x|}$ of $\mathbb{A}$ onto $\mathbb{A}^*$, which is a limit of homeomorphsisms, such that*

$$\inf \left\{ \int_{\mathbb{A}} |Dh(x)|^p \, dx \right\} = \int_{\mathbb{A}} |Dh^0(x)|^p \, dx$$

*where the infimum is taken over the class of homeomorphisms in Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.*

Now suppose the target annulus $\mathbb{A}^*$ is conformally thicker than $\mathbb{A}$. We say we are in the expanding case when $\frac{R_*}{r_*} > \frac{R}{r}$. If $\mathbb{A}^*$ is not too thick relative to $\mathbb{A}$, there is a radial homeomorphism with minimal $p$-harmonic energy. However, in higher dimensions if $\mathbb{A}^*$ is too thick relative to $\mathbb{A}$, we have constructed homeomorphisms with smaller $p$-harmonic energy than every radial mapping of $\mathbb{A}$ onto $\mathbb{A}^*$. We have the following theorems, which are restated in more detail in Chapter 5.

**Theorem 1.0.6.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$, and suppose $p > n$. There exists a function $\mathcal{H}_+ : [1, \infty) \to [1, \infty)$, depending on $n$ and $p$ (see Theorem 5.2.4), such that if $\frac{R}{r} < \frac{R_*}{r_*} < \mathcal{H}_+\left(\frac{R}{r}\right)$, then there exists a radial homeomorphsim $h^0(x) = H(|x|)\frac{x}{|x|}$ that maps $\mathbb{A}$ onto $\mathbb{A}^*$ such that*

$$\int_{\mathbb{A}} |Dh(x)|^p \, dx \geq \int_{\mathbb{A}} |Dh^0(x)|^p \, dx$$

*for every homeomorphism $h : \mathbb{A} \to \mathbb{A}^*$ of Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.*

**Theorem 1.0.7.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$ for $n \geq 4$, and suppose $p > n$. Surprisingly, there exists a function $\mathcal{H}_-$ depending on $n$ and $p$ (see Example 5.2.1), such that whenever $\frac{R_*}{r_*} > \mathcal{H}_-\left(\frac{R}{r}\right)$, there exists a non-radial homeomorphism $h^0(x)$ that maps $\mathbb{A}$ onto $\mathbb{A}^*$ such* *that*

$$\int_{\mathbb{A}} |Dh(x)|^p \, dx \geq \int_{\mathbb{A}} |Dh^0(x)|^p \, dx$$

*for every radial homeomorphism* $h : \mathbb{A} \to \mathbb{A}^*$ *of Sobolev class* $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$*.*

In this work, Chapter 1 presents some necessary preliminary material introducing notation and describing the appropriate class of mappings under consideration. The main computational tools are then introduced in Chapter 2. These are special differential $n$-forms called free Lagrangians, whose integral will be independent of choice of mapping within the appropriate class.

Armed with the $n$-forms to integrate, we describe in Chapter 3 some inequalities we will use to estimate $\mathcal{E}_p[h]$. In Chapter 4, we construct $p$-harmonic radial mappings that are candidates for the energy minimizer. Lastly, we pull together the tools from Chapters 2-4 to give proofs of the main theorems.

## 1.1 Preliminaries

### 1.1.1 Basic notation

Throughout our paper, we will let $n > 2$ denote the dimension of the space. The sets $\mathbb{X}$ and $\mathbb{Y}$ will be bounded domains in $\mathbb{R}^n$ of finite connectivity, unless otherwise specified. The set $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$ will be the punctured Euclidean space, and $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ will be the one-point compactification of $\mathbb{R}^n$. For vectors $v, w \in \mathbb{R}^n$, we will use the inner product $\langle v, w \rangle = v^T w$, and $|v|$ will denote the Euclidean norm of $v$, given by $|v|^2 = \langle v, v \rangle$. The cross product of $n-1$ vectors $v_1, \ldots, v_{n-1}$ in $\mathbb{R}^n$ is denoted by $v_1 \times \cdots \times v_{n-1}$. In certain contexts, a point $x \in \mathbb{R}^n$ will be equated with the vector $x - 0$.

For a domain $\mathbb{X} \subset \mathbb{R}^n$, its boundary will be denoted by $\partial\mathbb{X}$. We will write the $(n-1)$-dimensional sphere of radius $t > 0$ by $\mathbb{S}_t^{n-1} = \{x \in \mathbb{R}^n : |x| = t\}$, and $\mathbb{S}^{n-1}$ will be reserved for the unit sphere. Also throughout the paper, $0 < r < R < \infty$ and $0 < r_* < R_* < \infty$ will be fixed, and $\mathbb{A}$ and $\mathbb{A}^*$ will be the annuli

$$\mathbb{A} = \{x \in \mathbb{R}^n : r < |x| < R\}, \qquad \mathbb{A}^* = \{x \in \mathbb{R}^n : r_* < |x| < R_*\}. \tag{1.2}$$

The set of all $n \times n$ real matrices will be denoted $\mathbb{R}^{n\times n}$. The transpose of a matrix $A \in \mathbb{R}^{n\times n}$ will be denoted $A^T$ and the cofactor matrix of $A$ will be denoted $A^\sharp$. For $A, B \in \mathbb{R}^{n\times n}$, we will use the inner product $\langle A|B\rangle = tr(A^T B)$. The Hilbert-Schmidt norm, also called the Frobenius norm, of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted $|A|$, and is given by $|A|^2 = \langle A|A\rangle$. The normalized Hilbert-Schmidt norm is $[A]^2 = \frac{1}{n} tr(A^T A)$. If $h : \mathbb{X} \to \mathbb{R}^n$ is a differentiable mapping, then $Dh$ will denote its Jacobi matrix. We will write the $i^{th}$ component of $h$ as $h^i$. Subscripts will denote derivatives $\frac{\partial h}{\partial x_j} = h_j$. The Jacobian determinant will be written as $J_h(x) = \det(Dh(x))$. If $\phi : \mathbb{X} \to \mathbb{R}$ is a differentiable function, its gradient will be denoted $\nabla\phi$.

If $V : \mathbb{R}^n \to \mathbb{R}^n$ is a locally integrable vector field, its divergence will be understood in the weak sense, that is, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \operatorname{div}(V)\phi = -\int_{\mathbb{R}^n} \langle V, \nabla\phi\rangle. \tag{1.3}$$

Similarly, if $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a locally integrable matrix field, then for all $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, the divergence of $M$ is defined by

$$\int_{\mathbb{R}^n} \langle \operatorname{div}(M), \phi \rangle = -\int_{\mathbb{R}^n} \langle M | D\phi \rangle. \tag{1.4}$$

We see that if $r_1, r_2, \ldots, r_n$ are the row vectors of $M$, then

$$\operatorname{div}(M) = \begin{bmatrix} \operatorname{div}(r_1) \\ \operatorname{div}(r_2) \\ \vdots \\ \operatorname{div}(r_n) \end{bmatrix}. \tag{1.5}$$

### 1.1.2 Polar Coordinates

Because of the rotational symmetry of $\mathbb{A}$ and $\mathbb{A}^*$, we find polar coordinates to be the most convenient coordinate system with which to work. In this system, we represent each point $x \in \mathbb{R}^n_0$ using a number $t > 0$, called the radial coordinate, and a point $\sigma \in \mathbb{S}^{n-1}$, called the spherical coordinate. The coordinates are given by $t = |x|$ and $\sigma = \frac{x}{|x|}$. Clearly, $x = t\sigma$.

If $x \in \mathbb{R}^n_0$ is a point, it will also be useful to use the normal and tangential vectors of $\mathbb{R}^n$ at $x$. The normal vector at $x$ is $N(x) = \frac{x}{|x|}$. The tangential vectors, denoted by $T_1(x), \ldots, T_{n-1}(x)$, form an orthonormal basis for the tangent space to $\mathbb{S}^{n-1}_t$, where $t = |x|$. We see that together, $\{N, T_1, \ldots, T_{n-1}\}$ is an orthonormal basis for $\mathbb{R}^n$ at $x$. It may only be possible to choose the vectors $T_i(x)$ to depend continuously on $x$ locally, but many important locally defined quantities we will study are actually independent of basis, and as such are well defined globally.

If $X \subset \mathbb{R}^n_0$, we can define the polar derivatives of a differentiable function $h : \mathbb{X} \to \mathbb{R}^n$. The normal derivative of $h$ at $x$ is $h_N(x) = Dh(x)N$. It is immediate from the chain rule that the normal derivative in polar coordinates is $h_N(x) = \frac{\partial h}{\partial t}(t\sigma)$. For $i = 1, 2, \ldots, n-1$, the $i^{th}$ tangential derivative of $h$ at $x$ is $h_{T_i}(x) = Dh(x)T_i$. Together, $h_N, h_{T_1}, \ldots, h_{T_{n-1}}$ are called the polar derivatives of $h$.

The Hilbert-Schimdt norm of $Dh$ is independent of basis, so we have

$$|Dh|^2 = |h_N|^2 + |h_{T_1}|^2 + |h_{T_2}|^2 + \cdots + |h_{T_{n-1}}|^2 = |h_N|^2 + (n-1)|h_T|^2, \tag{1.6}$$

where we define $|h_T|^2 = \frac{1}{n-1}\left(|h_{T_1}|^2 + |h_{T_2}|^2 + \cdots + |h_{T_{n-1}}|^2\right)$. We also note that the Jacobian can be written in terms of polar derivatives as

$$J_h = \det(Dh) = \langle h_N, h_{T_1} \times \cdots \times h_{T_{n-1}} \rangle. \tag{1.7}$$

The expression $h_{T_1} \times \cdots \times h_{T_{n-1}}$ is also independent of choice of basis.

**Proposition 1.1.1.** *If $h : \mathbb{X} \to \mathbb{Y}$ is a differentiable mapping with $J_h(x) \neq 0$ almost everywhere, then we have $(Dh^\sharp)^T \frac{x}{|x|} = h_{T_1} \times \cdots \times h_{T_{n-1}}$.*

*Proof.* The claim follows from Cramer's Rule: $Dh^\sharp(x)Dh(x) = J_h(x)I$. We begin by taking the inner product of $(Dh^\sharp)^T \frac{x}{|x|}$ with each tangential derivative of $h$. For $i = 1, \ldots, n-1$, we have

$$\left\langle (Dh^\sharp)^T \frac{x}{|x|},\, h_{T_i} \right\rangle = \langle N,\, (Dh^\sharp)Dh\, T_i \rangle = J_h(x)\langle N,\, T_i\rangle = 0. \tag{1.8}$$

Therefore, $(Dh^\sharp)^T \frac{x}{|x|}$ is orthogonal to each tangential derivative, and hence parallel to the cross product $h_{T_1} \times \cdots \times h_{T_{n-1}}$. Taking the dot product with $h_N$, we then have

$$\left\langle (Dh^\sharp)^T \frac{x}{|x|},\, h_N \right\rangle = \langle N,\, (Dh^\sharp) Dh\, N\rangle = J_h(x)\langle N,\, N\rangle = J_h(x). \tag{1.9}$$

Comparing the expressions in (1.7) and (1.9), we arrive at

$$\left\langle h_N,\, (Dh^\sharp)^T \frac{x}{|x|} \right\rangle = \left\langle h_N,\, h_{T_1} \times \cdots \times h_{T_{n-1}} \right\rangle \tag{1.10}$$

We saw that $(Dh^\sharp)^T \frac{x}{|x|}$ is parallel to $h_{T_1} \times \cdots \times h_{T_{n-1}}$, and we see from (1.7) that $h_{T_1} \times \cdots \times h_{T_{n-1}}$ is only perpendicular to $h_N$ when $J_h(x) = 0$. Therefore, $(Dh^\sharp)^T \frac{x}{|x|} - h_{T_1} \times \cdots \times h_{T_{n-1}}$ is not perpendicular to $h_N$ almost everywhere. So (1.10) implies the claim. □

When working with mappings between annuli, we will be using mappings conveniently described in polar coordinates. We say that $h : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is a radial map, or radial stretching, if there is a function $H : (0, \infty) \to (0, \infty)$, called the normal strain function of $h$, such that $h(t\sigma) = H(t)\sigma$. We will call a mapping $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ a spherical sliding, and if $h : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is given by $h(x) = H(t)\Phi(\sigma)$, it will be called a quasiradial map.

Now suppose $h : \mathbb{X} \to \mathbb{R}^n$ is a differentiable quasiradial map, where $\mathbb{X}$ is as above. We will compute the components of $h_N$ and $h_{T_i}$. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, and say $h(x) = H(|x|)\Phi\left(\frac{x}{|x|}\right)$. We see $h^i(x) = \langle h(x), e_i\rangle = H(|x|)\Phi^i\left(\frac{x}{|x|}\right)$. So

$$h^i_j(x) = \dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)\frac{x_j}{|x|} + \frac{H(|x|)}{|x|}\Phi^i_j\left(\frac{x}{|x|}\right) - \frac{H(|x|)}{|x|}\frac{x_j}{|x|}\sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|} \tag{1.11}$$

where $\dot{H}(t) = \frac{dH}{dt}$. The terms in (1.11) are the entries of $Dh$. Since $\sum_{i=1}^{n} x_i^2 = |x|^2$, we compute that the $i^{th}$ component of $h_N = Dh\, N$ is

$$\sum_{j=1}^{n} \left[ \dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)\frac{x_j}{|x|} + \frac{H(|x|)}{|x|}\Phi^i_j\left(\frac{x}{|x|}\right) - \frac{H(|x|)}{|x|}\frac{x_j}{|x|}\sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|}\right]\frac{x_j}{|x|}$$
$$=\dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)\sum_{j=1}^{n}\frac{x_j^2}{|x|^2} + \frac{H(|x|)}{|x|}\sum_{j=1}^{n}\Phi^i_j\left(\frac{x}{|x|}\right)\frac{x_j}{|x|} - \frac{H(|x|)}{|x|}\sum_{j=1}^{n}\frac{x_j^2}{|x|^2}\sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|}$$
$$=\dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right) + \frac{H(|x|)}{|x|}\left[\sum_{j=1}^{n}\Phi^i_j\left(\frac{x}{|x|}\right)\frac{x_j}{|x|} - \sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|}\right] = \dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)$$

Similarly, if $T_k = \sum_{i=1}^{n} a_i^k e_i$, then we have $\sum_{j=1}^{n} a_j^k x_j = 0$ because $\langle T_k, N\rangle = 0$. So the $i^{th}$ component of $h_{T_k} = Dh\, T_k$ is

$$\sum_{j=1}^{n} \left[ \dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)\frac{x_j}{|x|} + \frac{H(|x|)}{|x|}\Phi^i_j\left(\frac{x}{|x|}\right) - \frac{H(|x|)}{|x|}\frac{x_j}{|x|}\sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|}\right] a_i^k$$
$$= \dot{H}(|x|)\Phi^i\left(\frac{x}{|x|}\right)\sum_{j=1}^{n}\frac{x_j}{|x|}a_j^k + \frac{H(|x|)}{|x|}\sum_{j=1}^{n}\Phi^i_j\left(\frac{x}{|x|}\right)a_j^k - \frac{H(|x|)}{|x|}\sum_{j=1}^{n}\frac{x_j a_j^k}{|x}\sum_{k=1}^{n}\Phi^i_k\left(\frac{x}{|x|}\right)\frac{x_k}{|x|}$$
$$= \frac{H(|x|)}{|x|}\sum_{j=0}^{n}\Phi^i_j\left(\frac{x}{|x|}\right)a_j^k,$$

It follows that

$$h_N(x) = \dot{H}(|x|)\Phi\left(\frac{x}{|x|}\right), \qquad |h_N| = \dot{H}(|x|) \tag{1.12}$$
$$h_{T_k} = \frac{H(|x|)}{|x|}\Phi_{T_k}\left(\frac{x}{|x|}\right), \qquad |h_T| = \frac{H(|x|)}{|x|}\left[D\Phi\left(\frac{x}{|x|}\right)\right] \tag{1.13}$$

where $\left[D\Phi\left(\frac{x}{|x|}\right)\right]^2 = \frac{|\Phi_{T_1}|^2+\cdots+|\Phi_{T_{n-1}}|^2}{n-1}$ is the normalized Hilbert-Schmidt norm of the differential matrix $D\Phi : T_{\frac{x}{|x|}}\mathbb{S}^{n-1} \to T_{\Phi(\frac{x}{|x|})}\mathbb{S}^{n-1}$. Note for radial mappings, $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the identity map, for which $[D\Phi] = 1$.

## 1.2 Theory of Homeomorphisms

We now present some definitions and remarks about the mappings to be used herein. Because our primary concern is the $p$-harmonic energy of a mapping, defined using an integral operator, we will work in the space of Sobolev mappings $W^{1,p}(\mathbb{X}, \mathbb{Y}) := \{h \in W^{1,p}(\mathbb{X}, \mathbb{R}^n) : h(x) \in \mathbb{Y} \textit{ a.e. } x \in \mathbb{X}\}$, where $p > n$. If $h \in W^{1,p}(\mathbb{X}, \mathbb{Y})$, then the derivative $Dh(x)$ is defined for almost every $x \in \mathbb{X}$. Also note that by the Meyers - Serrin theorem, smooth Sobolev mappings $C^\infty(\mathbb{X}, \mathbb{R}^n) \cap W^{1,p}(\mathbb{X}, \mathbb{R}^n)$ are dense in $W^{1,p}(\mathbb{X}, \mathbb{R}^n)$.

From the point of view of elasticity theory [1, 8], homeomorphisms and their weak limits in the Sobolev norm are the natural class of mappings to use, since they introduce no cracks or holes in $\mathbb{Y}$. These mappings are sometimes called deformations of $\mathbb{X}$ onto $\mathbb{Y}$. We briefly state some properties of these weak limits. Let $h_\nu : \mathbb{X} \to \mathbb{Y}$ be a sequence of homeomorphisms which converges weakly in $W^{1,p}(\mathbb{X}, \mathbb{Y})$ to a mapping $h : \mathbb{X} \to \overline{\mathbb{Y}}$. We know that the sequence also converges $c$-uniformly when $p \geq n$, and thus $h$ is also continuous. While the weak limit may no longer be a homeomorphsim, it does have a right inverse.

**Theorem 1.2.1.** *[6, Theorem 1.4] If $h_\nu$ is a sequence of homoeomorphsims of $\mathbb{X}$ onto $\mathbb{Y}$ which converges weakly in $W^{1,p}(\mathbb{X}, \mathbb{Y})$ to $h$, then mapping $h$ is continuous and $\mathbb{Y} \subset h(\mathbb{X}) \subset \overline{\mathbb{Y}}$. Furthermore, there exists a measurable mapping $f : \mathbb{Y} \to \mathbb{X}$ such that*

$$h \circ f = \mathrm{id} : \mathbb{Y} \to \mathbb{Y}$$

*everywhere on $\mathbb{Y}$. This right inverse mapping has bounded variation.*

On each deformation of domains we consider, we may impose two conditions without loss of generality. First, it preserves orientation within the domains. Secondly, it will preserve order of the boundary components. We will discuss the behavior of a deformation at the boundary, where the deformation need not be defined. We begin with a definition.

**Definition 1.2.2.** Let $h : \mathbb{X} \to \mathbb{Y}$ be a mapping between domains in $\mathbb{R}^n$. If $\{x_j\}$ is a sequence of points in $\mathbb{X}$ converging to a point in $\partial\mathbb{X}$ such that $\{h(x_j)\}$ also converges, then $\lim\limits_{j\to\infty} h(x_j)$ is called a cluster value of $h$. The set of all cluster values of $h$ will be denoted

$$h(\partial\mathbb{X}) = \left\{ \lim_{j\to\infty} h(x_j) : x_j \in \mathbb{X}, \lim_{j\to\infty} x_j \in \partial\mathbb{X} \right\}, \tag{1.14}$$

assuming both limits exist.

Next, it is easy to see $h(\partial\mathbb{X}) \subset \partial\mathbb{Y}$ for a homemorphsim $h$. The result is also true when we consider the weak uniform limits of homeomorphsims, see [6].

**Proposition 1.2.3.** *If $h : \mathbb{X} \to \mathbb{Y}$ is a homeomorphism, then the cluster values of $h$ lie in the boundary of $\mathbb{Y}$.*

For $y \in h(\partial\mathbb{X})$, say there exists a sequence $\{x_j\} \subset \mathbb{X}$ with $\lim\limits_{j\to\infty} x_j = x \in \partial\mathbb{X}$ and $\lim\limits_{j\to\infty}\{h(x_j)\} = y$. Since $\{h(x_j)\} \subset \mathbb{Y}$, we have that $y \in \overline{\mathbb{Y}} = \mathbb{Y} \cup \partial\mathbb{Y}$. Since $h$ is a homeomorphism, it has a continuous inverse $h^{-1}$. If $y \in \mathbb{Y}$, then $\lim\limits_{j\to\infty} h^{-1}(h(x_j)) = h^{-1}(y)$ would be in $\mathbb{X}$. But $\lim\limits_{j\to\infty} h^{-1}(h(x_j)) = \lim\limits_{j\to\infty} x_j = x \in \partial\mathbb{X}$. So $y \notin \mathbb{Y}$. Thus, $y \in \partial\mathbb{Y}$.

For $m$-connected domains $\mathbb{X}$ and $\mathbb{Y}$, let us label the boundary components of $\partial\mathbb{X}$ and $\partial\mathbb{Y}$

as

$$\partial\mathbb{X} = \mathbb{X}_0 \cup \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{m-1}$$

$$\partial\mathbb{Y} = \mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_{m-1}$$

with $\mathbb{X}_0$ and $\mathbb{Y}_0$ being the boundaries of the unbounded components of $\mathbb{R}^n - \mathbb{X}$ and $\mathbb{R}^n - \mathbb{Y}$, respectively. A deformation $h : \mathbb{X} \to \mathbb{Y}$ preserves the order of boundary components if $h(\mathbb{X}_i) \subset \mathbb{Y}_i$ for each $i = 0, \dots, m-1$. We will denote the class of all orientation-preserving, order-preserving homeomorphsims in $W^{1,p}(\mathbb{X}, \mathbb{Y})$ and their weak limits as $\mathcal{A}(\mathbb{X}, \mathbb{Y})$. These will be called admissible mappings of $\mathbb{X}$ onto $\mathbb{Y}$.

We close this section with some remarks on the specific case when $\mathbb{X} = \mathbb{A}$ and $\mathbb{Y} = \mathbb{A}^*$. We will simply use $\mathcal{A} = \mathcal{A}(\mathbb{A}, \mathbb{A}^*)$ to denote the class of admissible mappings of $\mathbb{A}$ onto $\mathbb{A}^*$. In light of Proposition 1.2.3, we see that if $h : \mathbb{A} \to \mathbb{A}^*$ is an order-preserving homeomorphsim, then $|h| : \mathbb{A} \to (r_*, R_*)$ extends continuously to the boundary as $h : \overline{\mathbb{A}} \to [r_*, R_*]$. Since $h$ is assumed to be order preserving, we must have $|h(x)| = r_*$ when $|x| = r$ and $|h(x)| = R_*$ when $|x| = R$.

## 1.3 Direct Method in the Calculus of Variations

Here, we briefly present the solution of a classical variational problem using the calculus of variations [2]. Fix a bounded domain $\mathbb{X}$ in $\mathbb{R}^n$, and let $g_0 \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$ be given. We will denote the class of $\mathscr{W}^{1,p}$ Sobolev mappings $h : \mathbb{X} \to \mathbb{R}^n$ with $h = g_0$ on $\partial\mathbb{X}$ by

$g_0 + \mathscr{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^n)$. We wish to find $h_0 \in g_0 + \mathscr{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^n)$ such that

$$\min\{\mathcal{E}_p[h] : h \in g_0 + \mathscr{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^n)\} = \mathcal{E}_p[h_0] \tag{1.15}$$

A necessary condition is that $h_0$ will satisfy the Lagrange-Euler equation. This differential equation comes from the calculus of variations. For a fixed $\phi \in C_0^\infty(\mathbb{X}, \mathbb{R}^n)$, we can consider a family of variations $h_\epsilon(x) = h_0(x) + \epsilon\phi(x)$ of $h_0$. If $h_0$ satisfies (1.15), then we see the function $\epsilon \to \mathcal{E}_p[h_\epsilon]$ has a local minimum at $\epsilon = 0$. The Lagrange-Euler equation is the differential equation given by $\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} = 0$.

**Proposition 1.3.1.** *The Lagrange-Euler equation of $\mathcal{E}_p$ is*

$$\operatorname{div}(|Dh|^{p-2}Dh) = 0 \tag{1.16}$$

*Proof.* To see this, we remark that $Dh_\epsilon = Dh_0 + \epsilon D\phi$, and simply compute

$$\frac{\mathrm{d}\mathcal{E}_p[h_\epsilon]}{\mathrm{d}\epsilon} = \int_{\mathbb{X}} \frac{\partial}{\partial \epsilon}|Dh_\epsilon|^p = \int_{\mathbb{X}} \frac{p}{2}\langle Dh_\epsilon | Dh_\epsilon\rangle^{\frac{p}{2}-1} 2\langle Dh_\epsilon | D\phi\rangle = \int_{\mathbb{X}} p|Dh_\epsilon|^{\frac{p-2}{2}}\langle Dh_\epsilon | D\phi\rangle.$$

Evaluating at $\epsilon = 0$, we get that

$$\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} = p\int_{\mathbb{X}} \langle |Dh_0|^{\frac{p-2}{2}} Dh_0 | D\phi\rangle = -p\int_{\mathbb{X}} \langle \operatorname{div}(|Dh_0|^{\frac{p-2}{2}} Dh_0), \phi\rangle. \tag{1.17}$$

Since $\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} = 0$ for all $\phi \in C_0^\infty(\mathbb{X}, \mathbb{R}^n)$, we conclude that (1.17) implies (1.16). □

The local minimizers of $\mathcal{E}_p$ are given the special name of $p$-harmonic mappings. We will see that $p$-harmonic mappings are actually the absolute minimizers of $\mathcal{E}_p$ among $g_0+\mathscr{W}_0^{1,p}(\mathbb{X},\mathbb{R}^n)$.

**Definition 1.3.2.** If $h \in W^{1,p}(\mathbb{X},\mathbb{R}^n)$ with $\operatorname{div}(|Dh|^{p-2}Dh)=0$, then $h$ is called $p$-harmonic.

**Proposition 1.3.3.** *If $h_0$ is a $p$-harmonic mapping with $h_0 = g_0$ on $\partial\mathbb{X}$, then we have $\mathcal{E}_p[h_0] = \min\{\mathcal{E}_p[h] : h \in g_0 + \mathscr{W}_0^{1,p}(\mathbb{X},\mathbb{R}^n)\}$.*

*Proof.* This is readily proved using a convexity argument. Pick $h \in g_0 + \mathscr{W}_0^{1,p}(\mathbb{X},\mathbb{R}^n)$. By Young's inequality, for $p>1$ we have $p|Dh_0|^{p-1}|Dh| \le |Dh|^p + (p-1)|Dh_0|^p$. So we see that

$$\begin{aligned} |Dh(x)|^p - |Dh_0(x)|^p &\ge p|Dh_0(x)|^{p-1}\left(|Dh(x)| - |Dh_0(x)|\right) \\ &= p|Dh_0(x)|^{p-2}\left(|Dh_0(x)||Dh(x)| - |Dh_0(x)|^2\right) \end{aligned} \tag{1.18}$$

Using the Cauchy-Schwarz inequality $|Dh||Dh_0| \ge \langle Dh|Dh_0\rangle$ and the definition of the Hilbert-Schmidt norm, integrating (1.18) yields

$$\begin{aligned} \mathcal{E}_p[h] - \mathcal{E}_p[h_0] &\ge p\int_{\mathbb{X}} |Dh_0|^{p-2}\left(\langle Dh_0|Dh\rangle - \langle Dh_0|Dh_0\rangle\right)dx \\ &= p\int_{\mathbb{X}} |Dh_0|^{p-2}\langle Dh_0|Dh - Dh_0\rangle dx \\ &= p\int_{\mathbb{X}} \langle |Dh_0|^{p-2}Dh_0|D(h-h_0)\rangle dx \end{aligned} \tag{1.19}$$

Since $h_0 = h = g_0$ on $\partial\mathbb{X}$, we have that $h_0 - h \in \mathscr{W}_0^{1,p}(\mathbb{X},\mathbb{R}^n)$. So by the definition of the divergence, we see that (1.19) is equivalent to

$$\mathcal{E}_p[h] - \mathcal{E}_p[h_0] \ge -p\int_{\mathbb{X}} \langle \operatorname{div}(|Dh_0|^{p-2}Dh_0), h - h_0\rangle. \tag{1.20}$$

Since $h_0$ satisfied (1.16), we see that $\mathcal{E}_p[h] \ge \mathcal{E}_p[h_0]$. Thus, $h_0$ is the solution to (1.15). □

These classical methods fail for the traction free problem; new methods must be used. This happens for several reasons. In the proof of Proposition 1.3.1, we considered a variation $h_\epsilon = h_0 + \epsilon\phi$. Even if $h_0$ is admissible, there is no guarantee that $h_\epsilon$ is. However, while we can be more careful about making a variation of $h_0$, bigger problems are introduced when we allow mappings to slip on the boundary. In Proposition 1.3.3, we needed that $h - h_0 \in \mathscr{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^n)$ to show that (1.19) and (1.20) were equivalent. This is not so in the traction-free case.

Finally, we remark that if $g_0\left(r\frac{x}{|x|}\right) = r_*\frac{x}{|x|}$ and $g_0\left(R\frac{x}{|x|}\right) = R_*\frac{x}{|x|}$, then the minimizer of $\mathcal{E}_p$ among $g_0 + \mathscr{W}^{1,p}(\mathbb{A}, \mathbb{R}^n)$ is attained by a map $h_0$ satisfying (1.16). This map is a reasonable candidate for the traction free minimizer in class $\mathcal{A}(\mathbb{A}, \mathbb{A}^*)$. We will investigate this more thoroughly in chapters four and five.

# Chapter 2

# Free Lagrangians

## 2.1 Differential forms

To compute integrals over domains in $\mathbb{R}^n$, differential forms provide the best framework. For an introduction to differential forms, see Chapter 10 in [11] and Chapter 4 in [4]. We will write the $n$-dimensional volume element as $dx$. The standard $n-1$-dimensional area form on $\mathbb{S}_t^{n-1}$ will be denoted $\omega$. Note that if $\omega_{n-1}$ is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, then $\int_{\mathbb{S}_t^{n-1}} d\sigma = \omega_{n-1} t^{n-1}$ for all $t > 0$.

Let $t = |x|$ for $x \in \mathbb{R}_0$. The differential of $t$ is the 1-form $dt = \sum_{i=1}^{n} \frac{x_i}{|x|} dx_i$. We obtain an $(n-1)$-form $\star dt = \sum_{i=1}^{n} \frac{(-1)^i x_i}{|x|} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ by taking the Hodge star of the differential, where the symbol $\widehat{\ }$ above a term stands for omitting that term from the wedge product. This is also the standard area form on $\mathbb{S}^{n-1}$. We see that $\star dt \wedge dt = dx$. We define another $(n-1)$-form $d\sigma$ to be a normalization of $\star dt$, namely $d\sigma = \frac{\star dt}{t^{n-1}}$. Thus, in polar coordinates, $dx = t^{n-1} d\sigma dt$. For any $t > 0$, we have $\int\limits_{\mathbb{S}_t^{n-1}} d\sigma = \omega_{n-1}$. We also remark that $d\sigma$

is a closed form. Indeed,

$$
\begin{aligned}
d(d\sigma) &= \sum_{i=1}^{n} \left( \frac{(-1)^i x_i}{|x|^n} \right)_i dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \left( \frac{n x_i^2}{|x|^{n+2}} - \frac{1}{|x|^n} \right) dx_1 \wedge \cdots \wedge \cdots \wedge dx_n \\
&= \left( \frac{n \sum\limits_{i=1}^{n} x_i^2}{|x|^{n+2}} - \frac{1}{|x|^n} \right) dx = \left( \frac{n}{|x|^n} - \frac{n}{|x|^n} \right) dx = 0.
\end{aligned}
$$

If $\mathbb{X}$ is a domain in $\mathbb{R}^n_0$ and $h : \mathbb{X} \to \mathbb{R}^n$, then the pullback via $h$ of $d\sigma$ is

$$
h^\sharp(d\sigma) = \sum_{i=1}^{n} \frac{(-1)^i h_i}{|h|^n} dh_1 \wedge \cdots \wedge \widehat{dh_i} \wedge \cdots \wedge dh_n. \tag{2.1}
$$

## 2.2 Examples of Free Lagrangians

We now introduce the concept of free Lagrangians, which will allow us to estimate integrals such as the $p$-harmonic energy. We begin with a definition.

**Definition 2.2.1.** Let $L \in \mathcal{L}^1(\mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n \times n})$ be an integrable function. The $n$-form $Ldx$ is a free Lagrangian if

$$
\int\limits_{\mathbb{X}} L(x, h(x), Dh(x))dx = \int\limits_{\mathbb{X}} L(x, g(x), Dg(x))dx
$$

whenever the deformations $h : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{X} \to \mathbb{Y}$ are homotopic.

One trivial example is an $n$-form $f(x)dx$ that does not depend on $h$. Another example is the Jacobian determinant. By the change of variable formula, we have that

$\int\limits_{\mathbb{X}} \det(Dh(x))dx = |\mathbb{Y}|$ for all orientation-preserving homeomorphsims $h : \mathbb{X} \to \mathbb{Y}$. We will now construct some examples of free Lagrangians on annuli. Throughout this section, we assume $h \in C^{\infty}(\mathbb{A}, \mathbb{A}^*)$ preserves order and orientation. Later, to consider general order-preserving orientation-preserving homeomorphisms, we will look at limits of these smooth mappings.

The first example is modelled on the Jacobian determinant $J_h(x) = \det(Dh(x))$.

**Example 2.2.2.** For any integrable function $\Phi$ on $(r_*, R_*)$, we see that

$$\int\limits_{\mathbb{A}} \Phi(|h|)J_h(x)dx = \omega_{n-1} \int\limits_{r_*}^{R_*} \Phi(s)s^{n-1}ds \tag{2.2}$$

does not depend on $h$, so $\Phi(|h|)J(h,x)dx$ is a free Lagrangian.

This formula is proved by changing variables and using polar coordinates to integrate.

$$\int\limits_{\mathbb{A}} \Phi(|h(x)|)J(h,x)dx = \int\limits_{\mathbb{A}_*} \Phi(|y|)dy = \omega_{n-1} \int\limits_{r_*}^{R_*} \Phi(s)s^{n-1}ds.$$

Our next example again uses integration by polar coordinates.

**Example 2.2.3.** For $\Phi \in \mathbb{C}^1[r_*, R_*]$, the $n$-form $\frac{\Phi'(|h|)|h|_N}{|x|^{n-1}}dx$ is a free Lagrangian. We have

$$\int\limits_{\mathbb{A}} \frac{\Phi'(|h|)|h|_N}{|x|^{n-1}}dx = \omega_{n-1}[\Phi(R_*) - \Phi(r_*)] \tag{2.3}$$

for all $h \in \mathcal{A}$.

Here, the normal derivative of a scalar function $f : \mathbb{X} \subset \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_N = \langle \nabla f, N \rangle.$$

Integrating the left-hand side of (2.3), we have

$$\begin{aligned}\int_{\mathbb{A}} \frac{\Phi'(|h|)|h|_N}{|x|^{n-1}} dx &= \int_{\mathbb{S}^{n-1}} \int_r^R \Phi'(|h|)|h|_N dt d\sigma = \int_{\mathbb{S}^{n-1}} \int_r^R [\Phi(|h|)]_N dt d\sigma \\ &= \int_{\mathbb{S}^{n-1}} \Phi(|h(t\omega)|)|_{t=r}^{t=R} d\sigma = \int_{\mathbb{S}^{n-1}} \Phi(R_*) - \Phi(r_*) d\sigma = \omega_{n-1}[\Phi(R_*) - \Phi(r_*)].\end{aligned}$$

The previous example can be thought of as a radial free Lagrangian, based on the appearance of the radial derivative. Our third example, in a fashion somewhat dual to the radial example, can be thought of as a tangential free Lagrangian.

**Example 2.2.4.** For $\Phi \in \mathbb{C}^1[r, R]$, the $n$-form $\Phi'(|x|)dt \wedge h^\sharp d\sigma$ is a free Lagrangian. We have

$$\int_{\mathbb{A}} \Phi'(|x|)dt \wedge h^\sharp d\sigma = \omega_{n-1}(\Phi(R) - \Phi(r)), \tag{2.4}$$

for all $h \in \mathcal{A}$.

The idea of the proof of (2.4) relies on the topological concept of the degree of a mapping between annuli, discussed in detail in [9].

**Definition 2.2.5.** Let $h : \mathbb{A} \to \mathbb{A}^*$ be a smooth mapping. We define

$$\deg(h) = \int\limits_{|x|=t} h^\sharp d\sigma, \tag{2.5}$$

where $r < t < R$.

To see this is well-defined, fix $r < s < t < R$. By Stokes' theorem, we have

$$\int\limits_{|x|=t} h^\sharp d\sigma - \int\limits_{|x|=s} h^\sharp d\sigma = \int\limits_{\mathbb{A}[s,t]} d(h^\sharp d\sigma) = \int\limits_{\mathbb{A}[s,t]} h^\sharp (dd\sigma) = 0.$$

To prove (2.4), we use Stokes' theorem and the closed $(n-1)$-form $d\sigma$ to compute that

$$\begin{aligned}\int\limits_{\mathbb{A}} \Phi'(|x|)dt \wedge h^\sharp d\sigma &= \int\limits_{\mathbb{A}} (d\Phi(|x|)) \wedge h^\sharp d\sigma = \int\limits_{\mathbb{A}} d\left(\Phi(|x|)h^\sharp d\sigma\right) \\ &= \int\limits_{|x|=R} \Phi(|x|)h^\sharp d\sigma - \int\limits_{|x|=r} \Phi(|x|)h^\sharp d\sigma = \Phi(R) \int\limits_{|x|=R} h^\sharp d\sigma - \Phi(r) \int\limits_{|x|=r} h^\sharp d\sigma \\ &= \deg(h)\left[\Phi(R) - \Phi(r)\right].\end{aligned}$$

To show that Example 2.2.4 is a free Lagrangian, it remains to see that (2.4) is constant up to homotopy. This follows from the following proposition.

**Proposition 2.2.6.** *If* $h_0 : \mathbb{A} \to \mathbb{A}^*$ *and* $h_1 : \mathbb{A} \to \mathbb{A}^*$ *are homotopic, then* $\deg(h_0) = \deg(h_1)$.

To prove this proposition, we will show that $\int\limits_{|x|=r} h_0^\sharp d\sigma = \int\limits_{|x|=r} h_1^\sharp d\sigma$. We will let $X = [0,1]\times\mathbb{S}_r^{n-1}$. If $H : [0,1]\times\mathbb{A} \to \mathbb{A}^*$ is a homotompy with $H(0,x) = h_0(x)$ and $H(1,x) = h_1(x)$, we can restrict this map to $X$. We see that the boundary of $X$ is $\{0\} \times \mathbb{S}_r^{n-1} \cup \{1\} \times \mathbb{S}_r^{n-1}$.

So by Stokes's Theorem, we have that

$$\int_{\mathbb{S}_r^{n-1}} h_1^\sharp d\sigma - \int_{\mathbb{S}_r^{n-1}} h_0^\sharp d\sigma = \int_X d(H^\sharp d\sigma) = \int_X H^\sharp (dd\sigma). \tag{2.6}$$

But recall that $d\sigma$ is a closed $(n-1)$-form, so the right-hand side of (2.6) is 0. Thus, $\int_{\mathbb{S}_r^{n-1}} h_1^\sharp d\sigma = \int_{\mathbb{S}_r^{n-1}} h_0^\sharp d\sigma$, completing the proof, and showing that (2.4) is a free Lagrangian.

To make use of the estimates in Examples 2.2-4, we will need some inequalities that relate the normal and tangential derivatives of $h$ to terms that appear in the free Lagrangians.

**Proposition 2.2.7.** *For $h \in \mathcal{A}(\mathbb{A}, \mathbb{A}^*)$, we have*

$$|h_N||h_T|^{n-1} \geq J_h(x) \tag{2.7}$$

$$|h_N| \geq |h|_N \tag{2.8}$$

$$\frac{|h_T|^{n-1}}{|h|^{n-1}} \geq |dt \wedge h^\sharp \omega| \tag{2.9}$$

*at each $x \in \mathbb{A}$. Moreover, equality holds in each instance for radial maps.*

To prove (2.7), we can write $J_h(x)$ as in (1.7) and use the estimate

$$\begin{aligned} \det(Dh) &= \langle h_N | h_{T_1} \times \cdots \times h_{T_{n-1}} \rangle \\ &\leq |h_N||h_{T_1} \times \cdots \times h_{T_{n-1}}| \\ &\leq |h_N||h_{T_1}| \ldots |h_{T_{n-1}}|. \end{aligned}$$

Since the arithmetic mean of $n-1$ values is greater than their geometric mean, we have $|h_{T_1}| \ldots |h_{T_{n-1}}| \leq |h_T|^{n-1}$, so (2.7) follows. Backwards inspection shows that equality holds for radial maps, since $h_N, h_{T_1}, \ldots, h_{T_{n-1}}$ are mutually orthogonal and $|h_{T_1}| = \cdots = |h_{T_{n-1}}| = \frac{H(t)}{t}$ for $h(x) = H(|x|)\frac{x}{|x|}$.

We obtain (2.8) simply by noting that $|h|^2 = \langle h|h\rangle$, and taking the derivative with respect to the polar coordinate $t$ of both sides. This gives $2|h||h|_N = 2\langle h_N, h\rangle$, which is equivalent to

$$|h|_N = \left\langle h_N \,\middle|\, \frac{h}{|h|} \right\rangle. \tag{2.10}$$

Applying the Cauchy-Schwarz inequality to (2.10) yields (2.8). Again for a radial map $h(x) = H(|x|)\frac{x}{|x|}$, we have that $|h(x)| = H(|x|)$ and $|h_N(x)| = \dot{H}(|x|)$, so equality holds.

To prove (2.9), we use the following proposition.

**Proposition 2.2.8.** *We can write the n-form $dt \wedge h^\sharp(d\sigma)$ as $h_S(x)dx$ where*

$$h_S(x) = \frac{1}{|h|^{n-1}} \left\langle \frac{x}{|x|}, Dh^\sharp \frac{h}{|h|} \right\rangle$$

Once we have the proposition, it suffices to show $\left|\left\langle \frac{x}{|x|}, Dh^\sharp \frac{h}{|h|} \right\rangle\right| \leq |h_T|^{n-1}$ to prove (2.9). But this follows from Proposition 1.1.1, since

$$\begin{aligned} \left|\left\langle \frac{x}{|x|}, Dh^\sharp \frac{h}{|h|} \right\rangle\right| &= \left|\left\langle (Dh^\sharp)^T \frac{x}{|x|}, \frac{h}{|h|} \right\rangle\right| \\ &\leq \left|(Dh^\sharp)^T \frac{x}{|x|}\right| \left|\frac{h}{|h|}\right| = |h_{T_1} \times \cdots \times h_{T_{n-1}}|. \end{aligned}$$

We already saw in proving (2.8) that $|h_{T_1} \times \cdots \times h_{T_{n-1}}| \leq |h_T|^{n-1}$. Equality holds when $h_{T_1} \times \cdots \times h_{T_{n-1}}$ is parallel to $\frac{h}{|h|}$, and $h_{T_1}, \ldots, h_{T_{n-1}}$ are mutually orthogonal and equal in length. This is the case for radial maps.

To prove the proposition, we begin by recalling

$$h^\sharp(d\sigma) = \sum_{i=1}^{n} (-1)^n \frac{h^i}{|h|^n} dh^1 \wedge \cdots \wedge \widehat{dh^i} \wedge \cdots \wedge dh^n.$$

We can rewrite the $(n-1)$-form $dh^1 \wedge \cdots \wedge \widehat{dh^i} \wedge \cdots \wedge dh^n$ in terms of the $(n-1)$-form basis elements $dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \ldots dx_n$, where $k = 1, \ldots, n$. Note that $dh^l = \sum_{j=1}^{n} h^l_j dx_j$. We see that when computing $dh^1 \wedge \cdots \wedge \widehat{dh^i} \wedge \cdots \wedge dh^n$, we obtain sums of terms of the form

$$h^1_{j_i} \ldots h^{i-1}_{j_{i-1}} h^{i+1}_{j_{i+1}} \ldots h^n_{j_n} dx_{j_1} \wedge \cdots \wedge dx_{j_{i-1}} \wedge dx_{j_{i+1}} \wedge \cdots \wedge dx_{j_n}.$$

If the numbers $j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n$ are not distinct, then this term is 0. Thus, we need only consider the terms where, for some $k = 1, \ldots n$, we have the set equality

$$\{j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n\} = \{1, \ldots k-1, k+1, \ldots n\}.$$

Furthermore, we have

$$dx_{\rho(1)} \wedge \cdots \wedge dx_{\rho(k-1)} \wedge dx_{\rho(k+1)} \wedge \cdots \wedge dx_{\rho(n)} = sgn(\rho) dx_1 \wedge \ldots \widehat{dx_k} \wedge \ldots dx_n$$

if $\rho \in P_k$. Hereafter, $P_k$ is the permutation group on the set $\{1, \ldots, \hat{k}, \ldots, n\}$.

From the determinant formula

$$\det \begin{bmatrix} a_1^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_m^m \end{bmatrix} = \sum_{\rho \in S^m} sgn(\rho) a^1_{\rho(1)} \dots a^m_{\rho(m)},$$

it follows that

$$dh^1 \wedge \dots \wedge \widehat{dh^1} \wedge \dots \wedge dh^n = \sum_{k=1}^{n} (-1)^{i+k} [Dh^\sharp]^i_k dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$$

We now see that

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{(-1)^i h^i}{|h|^n} dh^1 \wedge \dots \wedge \widehat{dh^i} \wedge \dots \wedge dh^n = \\
&\sum_{i=1}^{n} \frac{(-1)^i h^i}{|h|^n} \sum_{k=1}^{n} (-1)^{i+k} [Dh^\sharp]^i_k dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n = \\
&\sum_{k=1}^{n} \frac{(-1)^k}{|h|^{n-1}} \sum_{i=1}^{n} [Dh^\sharp]^i_k \frac{h^i}{|h|} dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n = \\
&\sum_{k=1}^{n} \frac{(-1)^k}{|h|^{n-1}} \left[ Dh^\sharp \frac{h}{|h|} \right]_k dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n.
\end{aligned}$$

We now wedge this term with $dt = \sum_{j=1}^{n} \frac{x_j}{|x|} dx_j$. We see that the product of the 1-form $a\ dx_j$ with the $(n-1)$-forms $b\ dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$ will be 0 if $j \neq k$. However, if $j = k$, the wedge product will be the $n$-form $(-1)^k ab\, dx$. It follows that

$$\begin{aligned} dt \wedge h^\sharp(d\sigma) &= \left(\sum_{k=1}^n \frac{x_k}{|x|} dx_k\right) \wedge \left(\sum_{i=1}^n \frac{(-1)^i h^i}{|h|^n} dh^1 \wedge \cdots \wedge \widehat{dh^i} \wedge \cdots \wedge dh^n\right) \\ &= \left(\sum_{k=1}^n \frac{x_k}{|x|} dx_k\right) \wedge \left(\sum_{k=1}^n \frac{(-1)^k}{|h|^{n-1}} \left[Dh^\sharp \frac{h}{|h|}\right]_k dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n\right) \\ &= \frac{1}{|h|^{n-1}} \sum_{k=1}^n \left[Dh^\sharp \frac{h}{|h|}\right]_k \frac{x_k}{|x|} dx = \frac{1}{|h|^{n-1}} \left\langle Dh^\sharp \frac{h}{|h|}, \frac{x}{|x|}\right\rangle dx. \end{aligned}$$

We summarize the results of this section by combining Examples 2.2.2-4 with Proposition 2.2.7 in the following lemma.

**Lemma 2.2.9.** *Let $\Phi \in C[r_*, R_*]$ be a positive function, and let $\phi \in C^1[r_*, R_*]$ and $\psi \in C^1[r, R]$ be functions with positive derivative. If $h \in \mathcal{A}$, we have the following free Lagrangian estimates:*

$$\int_{\mathbb{A}} \Phi(|h|)|h_N||h_T|^{n-1} \geq \omega_{n-1} \int_{r_*}^{R_*} \Phi(s) s^{n-1} ds \tag{2.11}$$

$$\int_{\mathbb{A}} \frac{\phi'(|h|)|h_N|}{|x|^{n-1}} dx \geq \omega_{n-1}[\Phi(R_*) - \Phi(r_*)] \tag{2.12}$$

$$\int_{\mathbb{A}} \frac{\psi'(|x|)|h_T|^{n-1}}{|h|^{n-1}} dx \geq \omega_{n-1}[\Phi(R) - \Phi(r)]. \tag{2.13}$$

*Equality holds for radial maps.*

# Chapter 3

# Algebraic Inequalities

With the free Lagrangian estimates from Lemma 2.2.9 in hand, the integration in the proofs of Theorems 1.0.3-5 poses no challenge. The hard part is to know what to integrate. This section is dedicated to establishing the inequalities we will need in Chapter 5.

## 3.1 Inequalities for the Contracting Case $\frac{R_*}{r_*} \leq \frac{R}{r}$

**Lemma 3.1.1.** *Suppose $p > n$, and let $a > 0$ and $b \geq 0$ be given. Then there exists a constant $c = c_{p,n}(a, b)$, given in (3.14), such that*

$$(X^2 + (n-1)Y^2)^{\frac{p}{2}} \geq aY^n + bXY^{n-1} - c \tag{3.1}$$

*whenever $X \geq 0$, $Y \geq 0$. Moreover, there exist constants $X^0 = X^0(a, b)$ and $Y^0 = Y^0(a, b)$, given in (3.10), such that equality holds precisely when $(X, Y) = (X^0, Y^0)$.*

*Proof.* To prove Lemma 3.1.1, we define the function $z(X,Y)$ for $X \geq 0$ and $Y \geq 0$ by

$$z(X,Y) = (X^2 + (n-1)Y^2)^{\frac{p}{2}} - aY^n - bXY^{n-1}. \tag{3.2}$$

We will find the absolute minimum of $z$ and set $-c = \min\{z(X,Y) : X \geq 0, Y \geq 0\}$. The existence of the minimum value of $z$ is easily seen. We know $z$ attains a minimum on the compact set $S = \{(X,Y) : X \geq 0, Y \geq 0, X^2 + Y^2 \leq (a+b)^{\frac{1}{p-n}}\}$. Suppose $X^2 + Y^2 = \rho^2$ for $\rho > (a+b)^{\frac{1}{p-n}}$. We estimate $z$ by

$$\begin{aligned} z &= (X^2 + (n-1)Y^2)^{\frac{p}{2}} - aY^n - bXY^{n-1} \\ &\geq \rho^p - a\rho^n - b\rho^n = \rho^p \left(1 - \frac{a+b}{\rho^{p-n}}\right) > 0 \end{aligned}$$

Since $z(X,Y) > 0$ outside of $S$, but $z\left(0, \frac{1}{2}\left(\frac{a}{(n-1)^{\frac{p}{2}}}\right)^{\frac{1}{p-n}}\right) < 0$, we conclude $z$ attains an absolute minimum. We let $(X^0, Y^0)$ be a point where $z$ attains its minimum value, so equality in (3.1) holds at $(X^0, Y^0)$.

Since $z$ is smooth, any minimum of $z$ occurs either along the line $X = 0$, or along $Y = 0$, or at a critical point of $z$.

Along $Y = 0$, we have $z(X,0) = X^p$, which is minimized when $X = 0$ by inspection. Along $X = 0$, we have $z(0,Y) = (n-1)^{\frac{p}{2}}Y^p - aY^n$, which is minimized when $Y = \left(\frac{na}{p(n-1)^{\frac{p}{2}}}\right)^{\frac{1}{p-n}}$ by elementary calculus. Writing $Y_0 = \left(\frac{na}{p(n-1)^{\frac{p}{2}}}\right)^{\frac{1}{p-n}}$, we see the minimum of $z$ possibly occurs at $(0,0)$ or $(0, Y_0)$.

Now suppose that $X_+ > 0$, $Y_+ > 0$, and $(X_+, Y_+)$ is a critical point of $z$. We compute that

$$z_X(X,Y) = p(X^2 + (n-1)Y^2)^{\frac{p-2}{2}} X - bY^{n-1} \tag{3.3}$$

$$z_Y(X,Y) = p(X^2 + (n-1)Y^2)^{\frac{p-2}{2}} (n-1)Y - naY^{n-1} - (n-1)bXY^{n-2} \tag{3.4}$$

Whenever $z_Y(X_+, Y_+) = 0$, we find that

$$p(X_+^2 + (n-1)Y_+^2)^{\frac{p}{2}} = \frac{na}{n-1} Y_+^{n-2} + bX_+ Y_+^{n-3} \tag{3.5}$$

Plugging (3.5) into (3.3) shows $z_X(X_+, Y_+) = 0$ when

$$\frac{na}{n-1} X_+ Y_+^{n-2} + bX_+^2 Y_+^{n-3} - bY_+^{n-1} = \left( \frac{na}{n-1} \frac{X_+}{Y_+} - b\left[1 - \left(\frac{X_+}{Y_+}\right)^2\right]\right) Y_+^{n-1} = 0 \tag{3.6}$$

At this point, we define a function $\phi(\xi)$ for $0 < \xi < 1$ by $\phi(\xi) = \frac{\xi}{1-\xi^2}$. We observe that (3.6) is equivalent to $\phi\left(\frac{X_+}{Y_+}\right) = \frac{(n-1)b}{na}$. We will set $V = V_n(a,b)$ to be the unique number in $[0,1)$ such that

$$\phi(V) = \frac{V}{1-V^2} = \frac{(n-1)b}{na} \tag{3.7}$$

We see this number exists because $\phi'(\xi) = \frac{1+\xi^2}{(1-\xi^2)^2} > 0$, $\phi(0) = 0$, and $\lim\limits_{\xi \to 1^-} \phi(\xi) = \infty$. Hence, $\phi$ is strictly increasing from 0 to $\infty$ on $[0,1)$. We conclude that $\frac{X_+}{Y_+} = V$. Solving

$\phi(V) = \frac{(n-1)b}{na}$ explicitly gives

$$V = \sqrt{\frac{n^2a^2}{4(n-1)^2b^2} + 1} - \frac{na}{2(n-1)b} = \frac{2(n-1)b}{\sqrt{n^2a^2 + 4(n-1)^2b^2} + na} \tag{3.8}$$

By evaluating $z_X(VY_+, Y_+) = 0$ in (3.3) and dividing both sides by $Y_+^{n-1}$, we have

$$p(V^2 + n - 1)^{\frac{p-2}{2}} VY_+^{p-n} = b \tag{3.9}$$

Combining (3.7) and (3.9), we arrive at

$$Y_+ = \left( \frac{b}{pV(V^2 + n - 1)^{\frac{p-2}{2}}} \right)^{\frac{1}{p-n}} = \left( \frac{na}{(n-1)p(1 - V^2)(V^2 + n - 1)^{\frac{p-2}{2}}} \right)^{\frac{1}{p-n}} \tag{3.10}$$

We now also have an explicit representation of $X_+ = VY_+$.

The only possible points where $z$ attains a minimum value for $X \geq 0, Y \geq 0$ are $(0, 0)$, $(0, Y_0)$, or $(X_+, Y_+)$. We now evaluate $z$ at these points. We have

$$z(0,0) = 0 \tag{3.11}$$

$$z\left(0, Y_0\right) = -\frac{p-n}{n} \left( \frac{na}{p(n-1)^{\frac{n}{2}}} \right)^{\frac{p}{p-n}} \tag{3.12}$$

$$z(X_+, Y_+) = -\frac{p-n}{n} \left( \frac{na}{(n-1)p(1 - V^2)(V^2 + n - 1)^{\frac{n-2}{2}}} \right)^{\frac{p}{p-n}}. \tag{3.13}$$

We remark that the function $\psi(\xi) = \frac{1}{(1-\xi^2)(\xi^2+n-1)^{\frac{n-2}{2}}}$ is increasing on [0, 1), readily seen from its logarithmic derivative

$$\frac{\dot{\psi}}{\psi} = \frac{2\xi}{1-\xi^2} - \frac{(n-2)\xi}{\xi^2+n-1} = \frac{(2\xi^2+2(n-1)-(n-2)+(n-2)\xi^2)t}{(1-\xi^2)(\xi^2+n-1)} = \frac{n(\xi^2+)t}{(1-\xi^2)(\xi^2+n-1)}$$

So, looking at (3.12) and (3.13), we see

$$z(X_+, Y_+) = -\frac{p-n}{n}\left(\frac{na\psi(V)}{(n-1)p}\right)^{\frac{p}{p-n}} \leq -\frac{p-n}{n}\left(\frac{na\psi(0)}{(n-1)p}\right)^{\frac{p}{p-n}} = z(0, Y_0)$$

Thus, the minimum of $z$ occurs at $(X, Y) = (X_+, Y_+)$. We can now take $X^0 = X_+$ and $Y^0 = Y_+$. Taking

$$c = \frac{p-n}{n}\left(\frac{na}{(n-1)p(1-V^2)(V^2+n-1)^{\frac{n-2}{2}}}\right)^{\frac{p}{p-n}}, \tag{3.14}$$

we establish (3.1). By backwards inspection, we see equality holds if and only if $X = X^0$ and $Y = Y^0$, proving Lemma 3.1.1 □

## 3.2 Inequalities for the Expanding Case $\frac{R}{r} < \frac{R_*}{r_*}$

The second inequality is similar, but depends on a constant, $\alpha_{n,p}$, given in Definition 3.2.1 below. This constant is defined using a function, $f(\xi)$. Hereafter, for $\xi > 1$, we will have the functions

$$f(\xi) = \frac{(\xi^2 + n - 1)^{\frac{p-2}{2}}(\xi^2 - 1)^p}{\xi^p \left(\xi^2 - \frac{n-1}{p-1}\right)^{p-1}} \tag{3.15}$$

$$g(\xi) = \frac{(\xi^2 + n - 1)^{\frac{(n-1)(p-2)}{2}}}{\xi^{n-1}(\xi^2 - 1)^{p-n}} \tag{3.16}$$

$$P(\xi) = \frac{(n-3)(p-1)}{n-1}\xi^4 - (3p - n - 4)\xi^2 + n - 1. \tag{3.17}$$

We will use these functions to define $\alpha_{n,p}$, and in the proof of Lemma 3.2.2.

We first discuss some properties of the polynomial $P(\xi)$. Note that $P(1) = \frac{-2n(p-n)}{n-1} < 0$. We see that

$$P'(\xi) = 4\frac{(n-3)(p-1)}{n-1}\xi^3 - 2(3p - n - 4)\xi$$

If $n > 3$, then $P'(\xi) < 0$ for $\xi < \sqrt{\frac{(n-1)(3p-n-4)}{2(n-3)(p-1)}}$, and $P'(\xi) > 0$ for $\xi > \sqrt{\frac{(n-1)(3p-n-4)}{2(n-3)(p-1)}}$. Looking at where $P$ is increasing and decreasing, we conclude there must be an unique number $a_{n,p} > \sqrt{\frac{(n-1)(3p-n-4)}{2(n-3)(p-1)}}$ such that $P(\xi) < 0$ for $1 < \xi < a_{n,p}$ and $P(\xi) > 0$ for $\xi > a_{n,p}$. If $n = 2$ or $n = 3$, then $P'(\xi) < 0$ for $\xi > 1$. We take $a_{2,p} = a_{3,p} = \infty$.

**Definition 3.2.1.** For $n > 3$, let $\alpha_{n,p}$ be the number such that $f(\alpha_{n,p}) = 1$. Let $\alpha_{2,p} = \alpha_{3,p} = \infty$.

To justify Definition 3.2.1, we compute that the logarithmic derivative of $f$ is

$$\begin{aligned}\frac{f'(\xi)}{f(\xi)} &= \frac{(p-2)\xi}{\xi^2+n-1} + \frac{2p\xi}{\xi^2-1} - \frac{p}{\xi} - \frac{2(p-1)\xi}{\xi^2-\frac{n-1}{p-1}} \\ &= -\frac{p(n-1)P(\xi)}{(p-1)(\xi^2+n-1)(\xi^2-1)\left(\xi^2-\frac{n-1}{p-1}\right)\xi}\end{aligned}$$

It follows from the sign of $P(\xi)$ that $f$ is increasing on $(1, a_{n,p})$ and decreasing on $(a_{n,p}, \infty)$. Since we have $f(1) = 0$ and $\lim_{\xi\to\infty} f(\xi) = 1$, we see there must be a unique number $\alpha_{n,p} \leq a_{n,p}$ with $f(\alpha_{n,p}) = 1$. So Definition 3.2.1 makes sense.

We also compute that the logarithmic derivative of $g$ is

$$\begin{aligned}\frac{g'(\xi)}{g(\xi)} &= \frac{(n-1)(p-2)\xi}{\xi^2+n-1} - \frac{2(p-n)\xi}{\xi^2-1} - \frac{n-1}{\xi} \\ &= \frac{(n-1)P(\xi)}{(\xi^2+n-1)(\xi^2-1)\xi}\end{aligned}$$

This shows that $g$ is decreasing on $(1, a_{n,p})$. We will define $g(\alpha_{2,p}) = \lim_{\xi\to\infty} g(\xi) = 0$ and $g(\alpha_{3,p}) = \lim_{\xi\to\infty} g(\xi) = 1$.

**Lemma 3.2.2.** *Let $p > n$. Let $a > 0$ and $b > 0$ be given, with $\frac{b^{p-1}}{p^{n-1}a^{p-n}} > g(\alpha_{n,p})$. Then there exists a constant $c = c(a,b)$ such that for all $X \geq 0$ and $Y \geq 0$, we have*

$$(X^2 + (n-1)Y^2)^{\frac{p}{2}} \geq aX + bXY^{n-1} - c. \tag{3.18}$$

*Moreover, there exist $X^0 = X^0(a,b)$ and $Y^0 = Y^0(a,b)$ such that equality holds if and only if $(X,Y) = (X^0, Y^0)$.*

*Proof.* To prove Lemma 3.2.2, we define the function $z(X,Y)$ for $X \geq 0$ and $Y \geq 0$ by

$$z(X,Y) = (X^2 + (n-1)Y^2)^{\frac{p}{2}} - aX - bXY^{n-1} \tag{3.19}$$

We will find the absolute minimum of $z$. We can take $(X^0, Y^0)$ to be the point where the minimum value occurs, and $-c = z(X^0, Y^0)$.

We first argue why $z$ attains a minimum value among $(X,Y)$ with $X \geq 0$ and $Y \geq 0$. Let $S = \{(X,Y) : 0 \leq X, 0 \leq Y, X^2 + Y^2 \leq \max((2a)^{\frac{1}{p-1}}, (2b)^{\frac{1}{p-n}})\}$. This is a compact set, and $z$ attains a minimum value on $S$. Note $z\left(\left(\frac{a}{2}\right)^{\frac{1}{p-1}}, 0\right) < 0$. Now suppose $(X,Y)$ is a point with $X^2 + Y^2 = \rho^2$, where $\rho > \max((2a)^{\frac{1}{p-1}}, (2b)^{\frac{1}{p-n}})$. Next, we estimate $z(X,Y)$ as

$$\begin{aligned} z &= (X^2 + (n-1)Y^2)^{\frac{p}{2}} - aX - bXY^{n-1} \\ &\geq \rho^p - a\rho - b\rho^n = \rho^p\left(1 - \frac{a}{\rho^{p-1}} - \frac{b}{\rho^{p-n}}\right) > 0 \end{aligned}$$

We conclude the minimum value of $z$ over all $(X,Y)$ with $X \geq 0$ and $Y \geq 0$ must be attained on $S$.

Any relative minimum of $z$ occurs either along the line $X = 0$, or along $Y = 0$, or at a critical point of $z$.

By inspection, we see that $z(0,Y) = (n-1)^{\frac{p}{2}} Y^p$ is minimized at $Y = 0$. By elementary calculus, we see that $z(X,0) = X^p - aX$ is minimized at $X = \left(\frac{a}{p}\right)^{\frac{1}{p-1}}$. We will set $X_0 = \left(\frac{a}{p}\right)^{\frac{1}{p-1}}$.

Now suppose that $X_+ > 0$, $Y_+ > 0$, and $(X_+, Y_+)$ is a critical point of $z$. We compute that

$$z_X(X_+, Y_+) = p(X_+^2 + (n-1)Y_+^2)^{\frac{p-2}{2}} X - a - bY_+^{n-1} = 0 \tag{3.20}$$

$$z_Y(X_+, Y_+) = (n-1)\left[p(X_+^2 + (n-1)Y_+^2)^{\frac{p-2}{2}} Y - bXY_+^{n-2}\right] = 0 \tag{3.21}$$

Writing $A = p(X_+^2 + (n-1)Y_+^2)^{\frac{p-2}{2}}$ and $B = bY_+^{n-2}$, we see that (3.20)-(3.21) is equivalent to

$$AX_+ - BY_+ = a$$

$$-BX_+ + AY_+ = 0$$

which is clearly solved by

$$X_+ = \frac{Aa}{A^2 - B^2} = \frac{\frac{A}{B}a}{\left[\left(\frac{A}{B}\right)^2 - 1\right] B} \tag{3.22}$$

$$Y_+ = \frac{Ba}{A^2 - B^2} = \frac{a}{\left[\left(\frac{A}{B}\right)^2 - 1\right] B} \tag{3.23}$$

We can now divide (3.22) by (3.23), and using the definition of $A$ and $B$, compute

$$\frac{X_+}{Y_+} = \frac{A}{B} = \frac{p\left(\left(\frac{X_+}{Y_+}\right)^2 + n - 1\right)^{\frac{p-2}{2}}}{b} Y_+^{p-n} \tag{3.24}$$

Moreover, we can plug in $\frac{X_+}{Y_+}$ for $\frac{A}{B}$ at (3.23) and using the definition of $B$, we obtain

$$Y_+^{n-1} = \frac{a}{\left[\left(\frac{X_+}{Y_+}\right)^2 - 1\right] b} \tag{3.25}$$

From (3.24) and (3.25), we obtain two different ways to write $Y_+^{(n-1)(p-n)}$ in terms of $a$, $b$, and $\frac{X_+}{Y_+}$:

$$\frac{b^{n-1}\left(\frac{X_+}{Y_+}\right)^{n-1}}{p^{n-1}\left(\left(\frac{X_+}{Y_+}\right)^2 + n - 1\right)^{\frac{(n-1)(p-2)}{2}}} = \frac{a^{p-n}}{\left(\left(\frac{X_+}{Y_+}\right)^2 - 1\right)^{p-n} b^{p-n}},$$

or equivalently,

$$g\left(\frac{X_+}{Y_+}\right) = \frac{b^{p-1}}{p^{n-1}a^{p-n}}. \tag{3.26}$$

Recall that $g$ is decreasing on $(1, a_{n,p})$, and $1 < \alpha_{n,p} < a_{n,p}$. By our assumption that $\frac{b^{p-1}}{p^{n-1}a^{p-n}} > g(\alpha_{n,p})$, there must be a unique $V$ with $1 < V < \alpha_{n,p}$ such that

$$g(V) = \frac{b^{p-1}}{p^{n-1}a^{p-n}}. \tag{3.27}$$

If $n > 3$, then $g$ is also increasing on $(a_{n,p}, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$. So when $n > 3$, we also have a unique $W > a_{n,p}$ such that

$$g(W) = \frac{b^{p-1}}{p^{n-1}a^{p-n}}. \tag{3.28}$$

We will now define constants $Y_V$, $Y_W$, $X_V$, and $X_W$ by

$$Y_V^{n-1} = \frac{a}{b(V^2-1)}, \qquad X_V = VY_V \tag{3.29}$$

$$Y_W^{n-1} = \frac{a}{b(W^2-1)}, \qquad X_W = WY_W \tag{3.30}$$

We remark that by backward analysis, $(X_V, Y_V)$ and $(X_W, Y_W)$ are the only stationary points of $z$ with $X > 0$ and $Y > 0$. Thus, the only possible points where $z$ attains a minimum value for $X \geq 0, Y \geq 0$ are $(0,0)$, $(X_0, 0)$, $(X_V, Y_V)$, or $(X_W, Y_W)$. We now evaluate $z$ at these points. We have

$$z(0,0) = 0 \tag{3.31}$$

$$z(X_0, 0) = -\frac{p-1}{p}\left(\frac{a}{p}\right)^{\frac{p}{p-1}} \tag{3.32}$$

$$z(X_V, Y_V) = -\frac{p-1}{p}\left(\frac{a}{pf(V)}\right)^{\frac{1}{p-1}} \tag{3.33}$$

$$z(X_W, Y_W) = -\frac{p-1}{p}\left(\frac{a}{pf(W)}\right)^{\frac{1}{p-1}} \tag{3.34}$$

We recall from Definition 3.2.1 that $f(\xi) < 1$ when $\xi < \alpha_{n,p}$. Thus, since $1 < V < \alpha_{n,p}$,

$$z(X_V, Y_V) = -\frac{p-1}{p}\left(\frac{a}{pf(V)}\right)^{\frac{1}{p-1}} \leq -\frac{p-1}{p}\left(\frac{a}{pf(\alpha_{n,p})}\right)^{\frac{1}{p-1}} = z(0, Y_0).$$

Similarly, if $n > 3$, then $f(\xi) \geq 1$ for $\xi > a_{n,p}$, and $W > a_{n,p} > \alpha_{n,p}$. So

$$z(X_W, Y_W) = -\frac{p-1}{p}\left(\frac{a}{pf(W)}\right)^{\frac{1}{p-1}} \geq -\frac{p-1}{p}\left(\frac{a}{pf(\alpha_{n,p})}\right)^{\frac{1}{p-1}} = z(0, Y_0).$$

Therefore, the minimum of $z$ occurs at $(X, Y) = (X_V, Y_V)$. We can now take $X^0 = X_V$ and $Y^0 = Y_V$. Taking

$$c = \frac{p-1}{p} \left( \frac{a}{pf(V)} \right)^{\frac{p}{p-1}}, \tag{3.35}$$

we establish (3.18). We see equality holds if and only if $X = X^0$ and $Y = Y^0$, proving Lemma 3.2.2. $\square$

# Chapter 4

# Radial $p$-harmonic Mappings

In this section, we will find the radial $p$-harmonic mappings from $\mathbb{A}$ to $\mathbb{A}^*$, and investigate their properties. Throughout this section, we will denote a general radial mapping from $\mathbb{A}$ to $\mathbb{A}^*$ by $h$, and $H : (r, R) \to (r_*, R_*)$ will be its strain function. We will first describe a generalized $p$-harmonic equation for radial mappings

$$\mathcal{L}(t, H, \dot{H}) = C \tag{4.1}$$

We will find some principal solutions to this equation, and then use these functions to construct all radial $p$-harmonic mappings on $\mathbb{A}$.

## 4.1 Inner Variational Equation

In Section 1.3, we saw that the $p$-harmonic mappings are the local minimizers of $\mathcal{E}_p$. Here, we compute a necessary differential equation for a radial $p$-harmonic mapping.

**Proposition 4.1.1.** *Let $h \in \mathcal{A}$ be a radial mapping with strain function $H(t) \in C^2[r, R]$. If $h$ is $p$-harmonic, then for each $t \in [r, R]$, either*

$$\dot{H}(t) = 0 \tag{4.2}$$

*or*

$$\begin{aligned} &\left(\tfrac{p-1}{n-1}\dot{H}^2(t) + H^2(t)\right) t\ddot{H}(t) = \\ &(n-1)\left(\tfrac{H(t)}{t}\right)^3 + (p-n-1)\left(\tfrac{H(t)}{t}\right)^2 \dot{H}(t) - (p-3)\tfrac{H(t)}{t}\dot{H}^2(t) - \dot{H}^3(t) \end{aligned} \tag{4.3}$$

*Proof.* We arrive at these equations by studying the inner variation equation of the type studying in [5] for dimension $n = 2$. To calculate it, we make a variation $h_\epsilon$ of $h$ as follows. Fix a test function $\phi \in C_0^\infty(r, R)$ and let $\epsilon$ be small enough so that $t \to t + \epsilon\phi(t)$ is a diffeomorphism of $(r, R)$ onto itself. Then the map $\Phi(x) = (|x| + \epsilon\phi(|x|))\frac{x}{|x|}$ is a diffeomorphism from $\mathbb{A}$ to itself. We set

$$h_\epsilon(x) = h \circ \Phi(x) = H(|x| + \epsilon\phi(|x|))\frac{x}{|x|} \tag{4.4}$$

Note that $h_\epsilon$ is radial, with strain function $H_\epsilon(t) = H(t+\epsilon\phi(t))$. The derivative of the strain function is $\dot{H}_\epsilon(t) = \dot{H}(t + \epsilon\phi(t))(1 + \epsilon\dot{\phi}(t))$.

We will now compute the inner variational equation

$$\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} = 0 \tag{4.5}$$

Recall that $p$-harmonic mappings are exactly the local minimizers of the $p$-harmonic energy.

Since $h_\epsilon = h$ when $\epsilon = 0$, we see that a $p$-harmonic mapping will satisfy (4.5).

To compute (4.5), we use (1.6) to write $|Dh|$ and (1.12)-(1.13) to write $|h_N|$ and $|h_T|$ for a radial map. We thus arrive at

$$\mathcal{E}_p[h^\epsilon] = \int_{\mathbb{A}} |Dh^\epsilon(x)|^p dx = \int_{\mathbb{A}} \left( \dot{H}_\epsilon^2(|x|) + (n-1)\frac{H_\epsilon^2(|x|)}{|x|^2} \right)^{\frac{p}{2}} dx \tag{4.6}$$

$$= \omega_{n-1} \int_r^R \left( \dot{H}^2(t+\epsilon\phi(t))(1+\epsilon\dot{\phi}(t))^2 + (n-1)\frac{H^2(t+\epsilon\phi(t))}{t^2} \right)^{\frac{p}{2}} t^{n-1} dt \tag{4.7}$$

We note $H \in C^2[r,R]$ and $\phi \in C_0^\infty(r,R)$, and thus the integrand, as well as its derivative with respect to $\epsilon$, is bounded. We compute

$$\begin{aligned} &\frac{\partial}{\partial\epsilon} \left( \dot{H}^2(t+\epsilon\phi(t))(1+\epsilon\dot{\phi}(t))^2 + (n-1)\frac{H^2(t+\epsilon\phi(t))}{t^2} \right)^{\frac{p}{2}} t^{n-1} \\ =&p \left( \dot{H}_\epsilon^2(t) + (n-1)\frac{H_\epsilon^2(t)}{t^2} \right)^{\frac{p-2}{2}} \dot{H}^2(t+\epsilon\phi(t))(1+\epsilon\dot{\phi}(t))\dot{\phi}(t) t^{n-1} \\ &+ p \left( \dot{H}_\epsilon^2(t) + (n-1)\frac{H_\epsilon^2(t)}{t^2} \right)^{\frac{p-2}{2}} \dot{H}(t+\epsilon\phi(t))\ddot{H}(t+\epsilon\phi(t))(1+\epsilon\phi(t))^2\phi(t)t^{n-1} \\ &+ (n-1)p \left( \dot{H}_\epsilon^2(t) + (n-1)\frac{H_\epsilon^2(t)}{t^2} \right)^{\frac{p-2}{2}} \frac{H(t+\epsilon\phi(t))\dot{H}(t+\epsilon\phi(t))\phi(t)}{t^2} t^{n-1} \end{aligned}$$

We may now find $\left.\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\right|_{\epsilon=0}$ by differentiating under the integral at (4.7) and setting $\epsilon = 0$. So we see

$$\left.\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \omega_{n-1} \int_r^R p|Dh|^{p-2} t^{n-1} \left( \dot{H}^2(t)\dot{\phi}(t) + \dot{H}(t)\ddot{H}(t)\phi(t) + (n-1)\frac{H(t)\dot{H}(t)\phi(t)}{t^2} \right) dt$$

Setting $\left.\frac{\mathrm{d}\mathcal{E}_p[h^\epsilon]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0$, we obtain

$$\int\limits_r^R p|Dh|^{p-2}t^{n-1}\left(\dot{H}(t)\ddot{H}(t) + (n-1)\frac{H(t)\dot{H}(t)}{t^2}\right)\phi(t)dt = -\int\limits_r^R p|Dh|^{p-2}t^{n-1}\dot{H}^2(t)\dot{\phi}(t)dt \tag{4.8}$$

We will rewrite the integrand on the left-hand side of (4.8) as

$$(n-1)p|Dh|^{p-4}\dot{H}(t)t^{n-2}\left[\left(\frac{\dot{H}^2(t)}{n-1} + \frac{H^2(t)}{t^2}\right)\ddot{H}(t)t + \dot{H}^2(t)\frac{H(t)}{t} + (n-1)\frac{H^3(t)}{t^3}\right]\phi(t)$$

Recalling that $\phi \in C_0^\infty(r, R)$, we compute the integral on the right-hand side (4.8) using integration by parts, yielding

$$\begin{aligned}
&-\int\limits_r^R p|Dh|^{p-2}t^{n-1}\dot{H}^2(t)\dot{\phi}(t)dt = \int\limits_r^R \left[p|Dh|^{p-2}\dot{H}^2(t)t^{n-1}\right]'\phi(t)dt\\
&= \int\limits_r^R (n-1)p|Dh|^{p-4}\dot{H}(t)t^{n-2}\left(\frac{p}{n-1}\dot{H}^2(t) + 2\frac{H^2(t)}{t^2}\right)\ddot{H}(t)t\phi(t)dt\\
&+ \int\limits_r^R (n-1)p|Dh|^{p-4}\dot{H}(t)t^{n-2}\left[\dot{H}^3(t) + (p-2)\dot{H}^2(t)\frac{H(t)}{t^2} - (p-n-1)\dot{H}(t)\frac{H^2(t)}{t^2}\right]\phi(t)dt
\end{aligned}$$

Thus, we obtain

$$\int_r^R |Dh|^{p-4}\dot{H}(t)t^{n-2}\left[\left(\frac{\dot{H}^2(t)}{n-1}+\frac{H^2(t)}{t^2}\right)\ddot{H}(t)t+\dot{H}^2(t)\frac{H(t)}{t^2}+(n-1)\frac{H^3(t)}{t^3}\right]\phi(t)dt$$
$$=\int_r^R |Dh|^{p-4}\dot{H}(t)t^{n-2}\left(\frac{p}{n-1}\dot{H}^2(t)+2\frac{H^2(t)}{t^2}\right)\ddot{H}(t)t\phi(t)dt$$
$$+\int_r^R |Dh|^{p-4}\dot{H}(t)t^{n-2}\left[\dot{H}^3(t)+(p-2)\dot{H}^2(t)\frac{H(t)}{t^2}-(p-n-1)\dot{H}(t)\frac{H^2(t)}{t^2}\right]\phi(t)dt,$$

or equivalently,

$$\int_r^R |Dh|^{p-4}t^{n-2}\left[(n-1)\frac{H^3}{t^3}+(p-n-1)\frac{H^2}{t^2}\dot{H}-(p-3)\frac{H}{t}\dot{H}^2-\dot{H}^3\right]\dot{H}\phi dt$$
$$=\int_r^R |Dh|^{p-4}t^{n-2}\left(\frac{p-1}{n-1}\dot{H}^2+\frac{H^2}{t^2}\right)t\ddot{H}\dot{H}\phi dt$$

Since this is true for all test functions $\phi$, the integrands must be equal. This implies

$$\left[(n-1)\frac{H^3(t)}{t^3}+(p-n)\frac{H^2(t)}{t^2}\dot{H}(t)-(p-3)\frac{H(t)}{t}\dot{H}^2(t)-\dot{H}^3(t)\right]\dot{H}(t) \tag{4.9}$$
$$=\left(\frac{p-1}{n-1}\dot{H}^2(t)+\frac{H^2(t)}{t^2}\right)t\ddot{H}(t)\dot{H}(t) \tag{4.10}$$

So the inner variational equation implies either (4.2) or (4.3). □

## 4.2 The Elasticity Function

At this point, we introduce $\eta_H(t)$, the elasticity function of $H$. After exploring some basic properties of the elasticity function, we use it to combine (4.2) and (4.3) into a nonlinear first-order differential equation $\mathcal{L}(t, H, \dot{H}) = C$.

**Definition 4.2.1.** The elasticity function of a differentiable nonzero function $H$ is

$$\eta(t) = \eta_H(t) = \frac{t\dot{H}(t)}{H(t)} \tag{4.11}$$

This function transforms nicely under scaling and dilation. Precisely, for $\lambda \in \mathbb{R}$ and $k > 0$, if $G(t) = \lambda H(kt)$ then

$$\eta_G(t) = \eta_H(kt) \tag{4.12}$$

This formula is easily verified, computing $\dot{G}(t) = k\lambda\dot{H}(kt)$, multiplying by $t$ and dividing by $G(t) = \lambda H(kt)$.

Moreover, if $H$ is invertible, a similar computation gives the elasticity function of its inverse in terms of the elasticity function.

**Proposition 4.2.2.** *If there is a nonzero differentiable function $F(t)$ such that $H(F(t)) = t$, then*

$$\eta_F(t) = \frac{1}{\eta_H(F(t))}. \tag{4.13}$$

We obtain $\dot{H}(F(t))\dot{F}(t) = 1$ by differentiating $H(F(t)) = t$. We then multiply this equation by $t = H(F(t))$ and divide by $F(t)\dot{H}(F(t))$ to obtain (4.13).

We can now rewrite (4.3) in terms of $\eta_H$. We begin by dividing both sides of (4.3) by

$\left(\frac{H(t)}{t}\right)^3$ to obtain

$$\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)\frac{\ddot{H}(t)t^2}{H(t)} = (n-1)+(p-n-1)\eta_H(t)-(p-3)\eta_H^2(t)-\eta_H^3(t) \qquad (4.14)$$

We then rewrite $\frac{\ddot{H}(t)t^2}{H(t)}$ using $\dot{\eta}_H(t)$. Differentiating $\eta_H(t) = \frac{t\dot{H}(t)}{H(t)}$ yields

$$\dot{\eta}_H(t) = \frac{tH\ddot{(}t)}{H(t)} + \frac{\dot{H}(t)}{H(t)} - \frac{t\dot{H}^2(t)}{H^2(t)}$$

Multiplying by $t$ and solving for $\frac{\ddot{H}(t)t^2}{H(t)}$, we obtain

$$\frac{\ddot{H}(t)t^2}{H(t)} = \eta_H^2(t) - \eta_H(t) + \dot{\eta}_H(t)t \qquad (4.15)$$

Plugging this into (4.14) gives

$$\begin{aligned}(n-1)+(p-n)\eta_H(t)-(p-3)\eta_H^2(t)-\eta_H^3(t) &= \left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)\left(\eta_H^2(t)-\eta_H(t)+\dot{\eta}_H(t)t\right)\\ &= \frac{p-1}{n-1}\eta_H^4(t)-\frac{p-1}{n-1}\eta_H^3(t)+\eta_H^2(t)-\eta_H(t)+\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)\dot{\eta}_H(t)t\end{aligned}$$

or equivalently

$$\begin{aligned}\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)\dot{\eta}_H(t)t &= (n-1)+(p-n+1)\eta_H(t)-(p-2)\eta_H^2(t)+\frac{p-n}{n-1}\eta_H^3(t)-\frac{p-1}{n-1}\eta_H^4(t)\\ &= (1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)\left(\eta_H^2(t)+n-1\right)\end{aligned}$$

Dividing both sides by $(1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)(\eta_H^2(t)+n-1)\,t$ yields

$$\frac{\frac{p-1}{n-1}\eta_H^2(t)+1}{(1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)(\eta_H^2(t)+n-1)}\dot{\eta}_H(t)=\frac{1}{t}. \tag{4.16}$$

We now turn to using the elasticity function $\eta_H$ to find the $p$-harmonic inner-stationary radial mapping $h(x)=H(|x|)\frac{x}{|x|}$. Multiplying both sides of (4.16) by $\eta_H$ yields

$$\frac{\frac{p-1}{n-1}\eta_H^3(t)+\eta_H(t)}{(1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)(\eta_H^2(t)+n-1)}\dot{\eta}_H(t)=\frac{\dot{H}(t)}{H(t)}. \tag{4.17}$$

If $\dot{H}(t)=0$, both sides of (4.17) are zero. If $\dot{H}(t)\neq 0$, we can divide by $\eta_H$ to recover (4.16), which is equivalent to (4.3). Thus, (4.17) is implied when $H$ satisfies either (4.2) or (4.3).

To make use of (4.17), we use the partial fraction decomposition

$$\begin{aligned}\frac{\frac{p-1}{n-1}\xi^3+\xi}{(1-\xi)\left(\frac{p-1}{n-1}\xi-1\right)(\xi^2+n-1)} = \; & \frac{1}{n}\frac{1}{1-\xi}-\left(\frac{p-1}{p^2-2p+n}\right)\frac{1}{\frac{p-1}{n-1}\xi+1}\\ & -\left(\frac{p^2-2p}{n(p^2-2p+n)}\right)\frac{\xi}{\frac{\xi^2}{n-1}+1}-\left(\frac{p^2-(n+2)p+2n}{n(p^2-2p+n)}\right)\frac{1}{\frac{\xi^2}{n-1}+1}\end{aligned} \tag{4.18}$$

We can integrate (4.17) with respect to $t$ and then multiply by $-n$ to obtain

$$\begin{aligned}\log|1-\eta_H(t)| \quad & +\alpha\log\left|\tfrac{p-1}{n-1}\eta_H(t)+1\right|+\beta\log\left(\frac{\eta_H^2(t)}{n-1}+1\right)+\gamma\tan^{-1}\left(\frac{\eta_H(t)}{\sqrt{n-1}}\right)\\ & =-n\log|H(t)|+C'\end{aligned} \tag{4.19}$$

where $C'$ is any constant of integration and hereafter,

$$\alpha=\frac{n(n-1)}{p^2-2p+n},\quad \beta=\frac{(n-1)(p-2)p}{2(p^2-2p+n)},\quad \gamma=\frac{\sqrt{n-1}(p-2)(p-n)}{p^2-2p+n} \tag{4.20}$$

We can rewrite (4.19) as

$$(1-\eta_H)\left(\frac{p-1}{n-1}\eta_H+1\right)^{\alpha}\left(\frac{\eta_H^2}{n-1}+1\right)^{\beta}\exp\left(-\gamma\tan^{-1}\left(\frac{\eta_H}{\sqrt{n-1}}\right)\right)=\frac{C}{H^n} \tag{4.21}$$

where $C$ is a constant.

Finally, multiplying both sides of (4.21) by $H^n$ gives us

$$(H-t\dot{H})\left(\frac{p-1}{n-1}t\dot{H}+H\right)^{\alpha}\left(\frac{(t\dot{H})^2}{n-1}+H^2\right)^{\beta}\exp\left(-\gamma\tan^{-1}\left(\frac{t\dot{H}}{\sqrt{n-1}H}\right)\right)=C \tag{4.22}$$

We set $\mathcal{L}$ to be first-order differential operator on the left hand side of (4.22). The generalized $p$-harmonic equation will be

$$\mathcal{L}(t,H,\dot{H})=C \tag{4.23}$$

We have that (4.23) is equivalent to (4.17). Differentiating (4.23), we recover (4.9) and (4.10). Therefore, the generalized $p$-harmonic equation is equivalent to (4.2) and (4.3). In light of Proposition 4.1.1, the strain function of a radial $p$-harmonic mapping will satisfy the generalized $p$-harmonic equation.

## 4.3 Principal Solutions

We now turn our attention to constructing $p$-harmonic radial mappings on $\mathbb{A}$. We will see these mappings will extend to the punctured space $\mathbb{R}_0$. We will build such a mapping by constructing a principal strain function which we can scale and dilate to obtain the strain function $H$ of the radial $p$-harmonic map $h$.

We saw in the previous section that the strain function of a radial $p$-harmonic mapping must satisfy the generalized $p$-harmonic equation, or equivalently, (4.21). For this to hold, $1-\eta_H$ and $\frac{p-1}{n-1}\eta_H+1$ cannot change signs. We note that $\eta_H(t)>1$ is equivalent to $\frac{\dot{H}(t)}{H(t)}>\frac{1}{t}$. Integrating this from $r$ to $R$ gives $\log\frac{R_*}{r_*}\geq\log\frac{R}{r}$. Similarly, if $0<\eta_H(t)<1$, we find $\log\frac{R_*}{r_*}\leq\log\frac{R}{r}$. Note that $\eta_H=1$ for $H=t$, and $\eta_H=-\frac{n-1}{p-1}$ for $H(t)=t^{-\frac{n-1}{p-1}}$. In these cases, we have (4.21) with $C=0$. The functions $H_0(t)=t$ and $H_\infty(t)=t^{-\frac{n-1}{p-1}}$ are our first principal solutions.

Now consider the case when $1-\eta>0$ and $\frac{p-1}{n-1}\eta+1>0$. Our goal is to construct a function $H=H_+$ such that (4.21) holds with $C=1$. The construction begins by defining a function $\Gamma_+(s)$ for $-\frac{n-1}{p-1}<s<1$ by

$$\begin{aligned}\Gamma_+(s)&=\exp\left(\int_0^s\frac{1+\frac{p-1}{n-1}\tau^2}{(1-\tau)\left(\frac{p-1}{n-1}\tau+1\right)(\tau^2+n-1)}d\tau\right)\\&=\frac{\left(\frac{p-1}{n-1}s+1\right)^{A'}\left(\frac{s^2}{n-1}+1\right)^{B'}}{(1-s)^{\frac{1}{n}}}\exp\left(D'\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right),\end{aligned}\tag{4.24}$$

where henceforth,

$$A'=\frac{p-1}{p^2-2p+n},\quad B'=\frac{p^2-(n+2)p+2n}{2n(p^2-2p+n)},\quad D'=-\frac{\sqrt{n-1}p(p-2)}{n(p^2-2p+n)}.\tag{4.25}$$

Note that

$$\frac{\dot{\Gamma}_+(s)}{\Gamma_+(s)}=\frac{1+\frac{p-1}{n-1}s^2}{(1-s)\left(\frac{p-1}{n-1}s+1\right)(s^2+n-1)}\tag{4.26}$$

We see $\Gamma_+$ is increasing, and thus invertible. We set $u_+=\Gamma_+^{-1}$. Thus, $\Gamma_+(u_+(t))=t$ and $\dot{\Gamma}_+(u_+)\dot{u}_+=1$.

We now use $\alpha$, $\beta$, and $\gamma$ from (4.20) to set $H_+$ to be

$$H_+ = (1-u_+)^{-\frac{1}{n}} \left(\frac{p-1}{n-1}u_+ + 1\right)^{-\frac{\alpha}{n}} \left(\frac{u_+^2}{n-1} + 1\right)^{-\frac{\beta}{n}} \exp\left(-\frac{\gamma}{n}\tan^{-1}\left(\frac{u_+}{\sqrt{n-1}}\right)\right) \tag{4.27}$$

We next compute the logarithmic derivative of $H_+$ using the partial fraction decomposition (4.18), and apply (4.24), to obtain

$$\frac{\dot{H}_+}{H_+} = \left[\frac{\frac{p-1}{n-1}u_+^3 + u_+}{(1-u_+)(\frac{p-1}{n-1}u_+ + 1)(u_+^2 + n - 1)}\right]\dot{u}_+ = u_+\frac{\dot{\Gamma}_+(u_+)}{\Gamma_+(u_+)}\dot{u}_+ = u_+\frac{1}{t}. \tag{4.28}$$

Multiplying both sides of (4.28) by $t$, we arrive at

$$u_+(t) = \frac{t\dot{H}_+(t)}{H_+(t)} = \eta_{H_+}(t). \tag{4.29}$$

We now observe by plugging (4.29) into (4.27) that we indeed have

$$H_+^n = \left(1-\eta_{H_+}\right)^{-1}\left(1+\frac{p-1}{n-1}\eta_{H_+}\right)^{-\alpha}\left(\frac{\eta_{H_+}}{n-1}+1\right)^{-\beta}\exp\left(-\gamma\tan^{-1}\left(\frac{\eta_{H_+}}{\sqrt{n-1}}\right)\right) \tag{4.30}$$

which clearly satisfies (4.21) with $C = 1$. The function $H_+$ is a principal solution to the generalized $p$-harmonic equation.

Here, we present some basic properties of our principal solution $H_+$. Inspecting (4.24), we see that $\Gamma_+$ is smooth, hence so are $u_+$ and $H_+$. By noting $\Gamma_+(s)$ is increasing on $\left(-\frac{n-1}{p-1}, 1\right)$, and $\Gamma_+(0) = 1$, we also know $u_+(t)$ is increasing, and that $u_+(1) = 0$. For all $0 < t < \infty$, we observe that

$$-\frac{n-1}{p-1} < u_+(t) = \eta_{H_+}(t) < 1$$

From (4.27), we see that $H_+(t) > 0$. So by (4.29), $\dot{H}_+$ and $u_+$ have the same sign. It follows that $H_+$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Thus, $H(t) \geq H(1) = 1$ for all $t$. Moreover, because $\eta_{H_+} < 1$, we have

$$\frac{d}{dt}\frac{H_+(t)}{t} = \frac{\dot{H}_+(t)}{t} - \frac{H_+(t)}{t^2} = \frac{H_+(t)}{t^2}\Big(\eta_{H_+}(t) - 1\Big) < 0. \tag{4.31}$$

It follows that $\frac{H_+(t)}{t} < \frac{H_+(1)}{1} = 1$ for $t > 1$. In particular, $H_+\left(\frac{R}{r}\right) < \frac{R}{r}$.

We now compute limits. From (4.24), we have

$$\lim_{s\to -\frac{n-1}{p-1}} \Gamma_+(s) = 0, \quad \lim_{s\to 1}\Gamma_+(s) = \infty. \tag{4.32}$$

We can use these limits for asymptotic values of $H_+$. By a continuity argument, using (4.32), we have

$$\begin{aligned}
\lim_{t\to\infty}\frac{H_+(t)}{t} &= \lim_{s\to 1}\frac{H_+(\Gamma_+(s))}{\Gamma_+(s)} \\
&= \lim_{s\to 1}\frac{(1-s)^{\frac{-1}{n}}\left(\frac{p-1}{n-1}s+1\right)^{-\frac{\alpha}{n}}\left(\frac{s^2}{n-1}+1\right)^{-\frac{\beta}{n}}\exp\left(-\frac{\gamma}{n}\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)}{(1-s)^{\frac{-1}{n}}\left(\frac{p-1}{n-1}s+1\right)^{A'}\left(\frac{s^2}{n-1}+1\right)^{B'}\exp\left(D'\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)} \\
&= \lim_{s\to 1}\left(\frac{\exp\left(\sqrt{n-1}\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)}{\left(\frac{p-1}{n-1}s+1\right)^{\frac{p+n-2}{p-2}}\left(\frac{s^2}{n-1}+1\right)^{\frac{p-1}{2}}}\right)^{\frac{p-2}{p^2-2p+n}}
\end{aligned}$$

$$\lim_{t\to 0} t^{\frac{n-1}{p-1}} H_+(t) = \lim_{s\to -\frac{n-1}{p-1}} \Gamma_+^{\frac{n-1}{p-1}}(s) H_+(\Gamma_+(s))$$

$$= \lim_{s\to -\frac{n-1}{p-1}} \frac{(1-s)^{-\frac{(n-1)}{n(p-1)}} \left(\frac{p-1}{n-1}s+1\right)^{\frac{(n-1)A'}{p-1}} \left(\frac{s^2}{n-1}+1\right)^{\frac{(n-1)B'}{(p-1)}} \exp\left(\frac{(n-1)D'}{p-1}\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)}{(1-s)^{\frac{1}{n}} \left(\frac{p-1}{n-1}s+1\right)^{\frac{\alpha}{n}} \left(\frac{s^2}{n-1}+1\right)^{\frac{\beta}{n}} \exp\left(\frac{\gamma}{n}\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)}$$

$$= \lim_{s\to -\frac{n-1}{p-1}} \left((1-s)^{\frac{p+n-2}{p-2}} \left(\frac{s^2}{n-1}+1\right)^{\frac{n-1}{2}} \exp\left(\sqrt{n-1}\tan^{-1}\left(\frac{s}{\sqrt{n-1}}\right)\right)\right)^{-\frac{p-2}{n(p-1)}}$$

Writing

$$\Theta_+ = \left(\frac{(n-1)^{\frac{p^2-p+2n-2}{2(p-2)}}}{n^{\frac{p-1}{2}}(p+n-2)^{\frac{p+n-2}{p-2}}} \exp\left(\sqrt{n-1}\tan^{-1}\left(\frac{1}{\sqrt{n-1}}\right)\right)\right)^{\frac{p-2}{p^2-2p+n}}$$

$$\Xi_+ = \left(\frac{(p+n-2)^{\frac{p+n-2}{p-2}}(p^2-2p+n)^{\frac{n-1}{2}}}{(p-1)^{\frac{n(p-1)}{p-2}} \exp\left(\sqrt{n-1}\tan^{-1}\left(\frac{\sqrt{n-1}}{p-1}\right)\right)}\right)^{-\frac{p-2}{n(p-1)}}$$

we conclude that

$$\lim_{t\to\infty} \frac{H_+(t)}{t} = \Theta_+ \tag{4.33}$$

$$\lim_{t\to 0} t^{\frac{n-1}{p-1}} H_+(t) = \Xi_+ \tag{4.34}$$

Now consider the case when $\eta > 1$. Our goal is to construct a function $H = H_-$ such that (4.21) holds for $C = -1$. We modify our construction of $H_+$, beginning by defining $\Gamma_-(s)$ for $-\frac{p-1}{n-1} < s < 1$ by

$$\begin{aligned}\Gamma_-(s) &= \exp\int_0^s \frac{\frac{p-1}{n-1}+\tau^2}{(1-\tau)\left(\frac{p-1}{n-1}+\tau\right)(1+(n-1)\tau^2)}d\tau \\ &= (1-s)^{\frac{-1}{n}}\left(1+\frac{n-1}{p-1}s\right)^{A'}(1+(n-1)s^2)^{B'}\exp\left(-D'\tan^{-1}\left(\sqrt{n-1}s\right)\right)\end{aligned} \tag{4.35}$$

Note that

$$\frac{\dot{\Gamma}_-(s)}{\Gamma_-(s)} = \frac{\frac{p-1}{n-1} + s^2}{(1-s)\left(\frac{p-1}{n-1} + s\right)(1 + (n-1)s^2)} \tag{4.36}$$

We see $\Gamma_-$ is increasing, and thus invertible. We set $u_- = \Gamma_-^{-1}$. Thus, $\Gamma_-(u_-(t)) = t$ and $\dot{\Gamma}_-(u_-)\dot{u}_- = 1$.

For for $t \neq 1$, we define our last principal solution $H_-(t)$ by

$$H_- = u_- (1-u_-)^{-\frac{1}{n}} \left(u_- + \frac{p-1}{n-1}\right)^{-\frac{\alpha}{n}} \left(\frac{1}{n-1} + u_-^2\right)^{-\frac{\beta}{n}} \exp\left(-\frac{\gamma}{n}\tan^{-1}\left(\frac{1}{\sqrt{n-1}u_-}\right)\right) \tag{4.37}$$

We can now compute its logarithmic derivative

$$\frac{\dot{H}_-}{H_-} = \frac{\frac{p-1}{n-1} + u_-^2}{u_-(1-u_-)\left(\frac{p-1}{n-1} + u_-\right)(1 + (n-1)u_-^2)}\dot{u_-} = \frac{1}{u_-}\frac{\dot{\Gamma}_-(u_-)}{\Gamma_-(u_-)}\dot{u}_- = \frac{1}{tu_-} \tag{4.38}$$

Multiplying both sides of (4.38) by $t$, we arrive at

$$\eta_{H_-}(t) = \frac{t\dot{H}_-(t)}{H_-(t)} = \frac{1}{u_-(t)}. \tag{4.39}$$

At this point, we need some remarks about $u_-$. Note that $\Gamma_-(0) = 1$, so $u_-(1) = 0$. Therefore, $\eta_{H_-}$ (and indeed, also $H_-$) is not defined by (4.37) at $t = 1$. We will set $H(1) = 0$, defining $H(t)$ for all $t > 0$. Also, since $\Gamma_-$ is increasing on $\left(-\frac{p-1}{n-1}, 1\right)$, it follows that $u(t)$ is increasing, and we have $-\frac{n-1}{p-1} < u(t) < 0$ for $0 < t < 1$, and $0 < u(t) < 1$ for $t > 1$. We see $\eta_{H_-}(t) > 1$ for $t > 1$, and $\eta_{H_-}(t) < -\frac{p-1}{n-1}$ for $0 < t < 1$.

We consider $t > 1$, where $\eta_{H_-}(t) > 1$. We momentarily restrict ourselves to this case because we are most interested when $H_-(t)$ is positive and increasing ($H_-(t) > 0$ and

$\dot{H}_-(t) > 0$). These properties for our strain function follow from the assumption that our radial map $h : \mathbb{A} \to \mathbb{A}^*$ is order preserving and orientation preserving. Plugging (4.39) into (4.37), we see

$$\begin{aligned} H_-^{-n} &= \eta_{H_-}\left(1-\tfrac{1}{\eta_{H_-}}\right)\left(\tfrac{1}{\eta_{H_-}}+\tfrac{p-1}{n-1}\right)^{\alpha}\left(\tfrac{1}{n-1}+\tfrac{1}{\eta_{H_-}^2}\right)^{\beta}\exp\left(\gamma\tan^{-1}\left(\tfrac{\eta_{H_-}}{\sqrt{n-1}}\right)\right) \\ &= \left(\eta_{H_-}-1\right)\left(1+\tfrac{p-1}{n-1}\eta_{H_-}\right)^{\alpha}\left(\tfrac{\eta_{H_-}^2}{n-1}+1\right)^{\beta}\exp\left(\gamma\tan^{-1}\left(\tfrac{\eta_{H_-}}{\sqrt{n-1}}\right)\right) \end{aligned} \tag{4.40}$$

Thus, $H_-$ satisfies (4.21) with $C = -1$.

We now present some properties of $H_-$ that follow from its construction. By (4.37), we see $H_-(t)$ and $u_-(t)$ have the same sign. So, $H_-(t) < 0$ for $t < 1$ and $H_-(t) > 0$ for $t > 1$, and $\dot{H}(t) > 0$ by (4.39). Since $H_-(1) = 0$, we have that $H_-$ is continuous on $(0, \infty)$. In fact, it is a smooth function, and we see it is increasing. Note that $\eta_{H_-}(1)$ is not defined.

We now compute limits. From (4.35), we have

$$\lim_{s\to -\frac{p-1}{n-1}} \Gamma_-(s) = 0, \qquad \lim_{s\to 1} \Gamma_-(s) = \infty. \tag{4.41}$$

We can use these limits for asymptotic values of $H_-$. By a continuity argument, using (4.41), we have

$$\begin{aligned} \lim_{t\to\infty}\frac{H_-(t)}{t} &= \lim_{s\to 1}\frac{H_-(\Gamma_-(s))}{\Gamma_-(s)} \\ &= \lim_{s\to 1}\frac{s\,(1-s)^{-\frac{1}{n}}\left(s+\frac{p-1}{n-1}\right)^{-\frac{\alpha}{n}}(1+(n-1)s^2)^{-\frac{\beta}{n}}\exp\left(-\frac{\gamma}{n}\tan^{-1}\left(\frac{1}{\sqrt{n-1}s}\right)\right)}{(1-s)^{\frac{-1}{n}}\left(1+\frac{n-1}{p-1}s\right)^{A'}(1+(n-1)s^2)^{B'}\exp\left(-D'\tan^{-1}\left(\sqrt{n-1}s\right)\right)} \\ &= \lim_{s\to 1}\frac{\left(\frac{p-1}{n-1}\right)^{\frac{p-1}{p^2-2p+n}}s\exp\left[\frac{\sqrt{n-1}(p-2)}{p^2-2p+n}\left(\tan^{-1}\left(\frac{1}{\sqrt{n-1}s}\right)-\frac{p\pi}{2n}\right)\right]}{\left(\frac{p-1}{n-1}+s\right)^{\frac{p+n-2}{p^2-2p+n}}(1+(n-1)s^2)^{\frac{(p-1)(p-2)}{2(p^2-2p+n)}}} \end{aligned}$$

$$\lim_{t\to 0} t^{\frac{n-1}{p-1}} H_-(t) = \lim_{s\to -\frac{p-1}{n-1}} \Gamma_-^{\frac{n-1}{p-1}}(s) H_-(\Gamma_-(s))$$

$$= \lim_{s\to -\frac{p-1}{n-1}} \frac{\left[(1-s)^{-\frac{1}{n}}\left(1+\frac{n-1}{p-1}s\right)^{A'}(1+(n-1)s^2)^{B'}\exp\left(-D'\tan^{-1}\left(\sqrt{n-1}s\right)\right)\right]^{\frac{n-1}{p-1}}}{s^{-1}(1-s)^{\frac{1}{n}}\left(s+\frac{p-1}{n-1}\right)^{\frac{\alpha}{n}}(1+(n-1)s^2)^{\frac{\beta}{n}}\exp\left(\frac{\gamma}{n}\tan^{-1}\left(\frac{1}{\sqrt{n-1}s}\right)\right)}$$

$$= \lim_{s\to -\frac{p-1}{n-1}} \frac{s\exp\left[\frac{\sqrt{n-1}(p-2)}{n(p^2-2p+n)}\left(\frac{(n-1)p}{p-1}\tan^{-1}(\sqrt{n-1}s)-(p-n)\tan^{-1}\left(\frac{1}{\sqrt{n-1}s}\right)\right)\right]}{\left(\frac{p-1}{n-1}\right)^{\frac{n-1}{p^2-2p+n}}(1-s)^{\frac{p+n-2}{n(p-1)}}(1+(n-1)s^2)^{\frac{(n-1)(p-2)}{2n(p-1)}}}$$

Writing

$$\Theta_- = \frac{\left(\frac{p-1}{n-1}\right)^{\frac{p-1}{p^2-2p+n}}\exp\left[\frac{\sqrt{n-1}(p-2)}{p^2-2p+n}\left(\tan^{-1}\left(\frac{1}{\sqrt{n-1}}\right)-\frac{p\pi}{2n}\right)\right]}{\left(\frac{p+n-2}{n-1}\right)^{\frac{p+n-2}{p^2-2p+n}}(n)^{\frac{(p-1)(p-2)}{2(p^2-2p+n)}}}$$

$$\Xi_- = -\frac{\left(\frac{p-1}{n-1}\right)^{\frac{p^2-2p+1}{p^2-2p+n}}\exp\left[-\frac{\sqrt{n-1}(p-2)}{n(p^2-2p+n)}\left(\frac{(n-1)p}{p-1}\tan^{-1}\left(\frac{p-1}{\sqrt{n-1}}\right)-(p-n)\tan^{-1}\left(\frac{\sqrt{n-1}}{p-1}\right)\right)\right]}{\left(\frac{p+n-2}{n-1}\right)^{\frac{p+n-2}{n(p-1)}}\left(\frac{p^2-2p+n}{n-1}\right)^{\frac{(n-1)(p-2)}{2n(p-1)}}}$$

we conclude that $\lim_{t\to\infty}\frac{H_-(t)}{t} = \Theta_-$ and $\lim_{t\to 0} t^{\frac{n-1}{p-1}} H_-(t) = \Xi_-$.

Because $\eta_{H_-}(t) > 1$ for $t > 1$, we have

$$\frac{d}{dt}\frac{H_-(t)}{t} = \frac{\dot{H}_-(t)}{t} - \frac{H_-(t)}{t^2} = \frac{H_-(t)}{t^2}\Big(\eta_{H_-}(t) - 1\Big) > 0 \tag{4.42}$$

for all $t > 1$. It follows that $\frac{H_-(t)}{t}$ is increasing on $(1,\infty)$. In particular, $\frac{t}{H_-(t)} > \frac{1}{\Theta_-}$ for $t > 1$.

Finally, we remark that when $t < 1$, we have seen that $u_-(t)$, $H_-(t)$, and $\eta_{H_-}(t)$ are all negative. By plugging (4.39) into (4.37) and taking the logarithm, we obtain

$$\begin{aligned} -\tfrac{1}{n}\log\Big(1-\eta_{H_-}\Big) - \tfrac{\alpha}{n}\log\Big|1+\tfrac{p-1}{n-1}\eta_{H_-}\Big| - \tfrac{\beta}{n}\log\Big(\eta_{H_-}^2+n-1\Big) - \tfrac{D}{A}\tan^{-1}\left(\tfrac{\eta_{H_-}}{\sqrt{n-1}}\right) \\ = \log|H| \end{aligned} \tag{4.43}$$

which satisfies (4.19) with $C' = 0$.

## 4.4 Generating all radial $p$-harmonics

We saw that the radial $p$-harmonic map $h : \mathbb{A}(r, R) \to \mathbb{A}^*(r_*, R_*)$ is the solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(|Dh|^{p-2}Dh) = 0 & \\ h(x) = r_* \frac{x}{|x|} & |x| = r \\ h(x) = R_* \frac{x}{|x|} & |x| = R \end{cases}$$

In this section, we will find a strain function $H : (r, R) \to \mathbb{R}$ which, given $a, b \in \mathbb{R}$, satisfies

$$\mathcal{L}(t, H(t), \dot{H}(t)) = C \tag{4.44}$$

$$H(r) = a \tag{4.45}$$

$$H(R) = b \tag{4.46}$$

Here, $C$ is some constant. The system would be overdetermined if the constant was fixed. We can build solutions to (4.44)-(4.46) using the principal solutions $H_0$, $H_\infty$, $H_+$, and $H_-$. We saw these four functions satisfy (4.44) when $C = 0$ or $C = \pm 1$. Note only in certain cases will this function $H$ be the strain function of a radial orientation-preserving, order-preserving homeomorphism $h : \mathbb{A} \to \mathbb{A}^*$.

We recall that (4.44) can be written in terms of the elasticity function as (4.21). Since the elasticity function transforms nicely under scaling and dilation, we see $H(t) = \lambda H_i(kt)$ satisfies (4.21) whenever $H_i$ is a principal solution. Thus, given $a$ and $b$, we search for the correct scaling factor $\lambda$ and dilating factor $k$ of a principal solution $H_i$ so that $H(t) = \lambda H_i(kt)$ satisfies (4.45)-(4.46). These factors, as well as the choice of principal solution, will depend on the ratio of $b$ to $a$. We now consider several cases.

First, suppose that $a = 0$. Then the function $H(t) = \frac{b}{H_-(\frac{R}{r})} H_-(\frac{t}{r})$ satisfies (4.44)-(4.46). For all other cases, we will suppose $a \neq 0$.

We can eliminate one constant from $\lambda H(kt)$ just by dividing. If $\lambda H(kt)$ satisfies (4.44)-(4.46) for $a \neq 0$, then $\frac{H(kR)}{H(kr)} = \frac{b}{a}$. We now define two functions $Q_+(t) = \frac{H_+(tR)}{H_+(tr)}$ and $Q_-(t) = \frac{H_-(tR)}{H_-(tr)}$. We take a moment to establish some properties of these functions.

We first consider $Q_+(t)$. We see that it is defined for $0 < t < \infty$, and $Q_+(t) > 0$ for all $t$. We compute the logarithmic derivative to be

$$\frac{\dot{Q}_+(t)}{Q_+(t)} = \frac{R\dot{H}_+(tR)}{RH_+(tR)} - \frac{r\dot{H}_+(tr)}{rH_+(tr)} = \frac{1}{t}\left(\eta_{H_+}(tR) - \eta_{H_+}(tr)\right).$$

Since $R > r$ and $\eta_{H_+} = u_+$ is increasing, it follows that $\dot{Q}_+ > 0$, so $Q_+$ is strictly increasing. We can now use the asymptotic limits of $H_+$ to show that

$$\lim_{t\to 0} Q_+(t) = \lim_{t\to 0} \frac{H_+(tR)}{H_+(tr)} = \lim_{t\to 0} \frac{r^{\frac{n-1}{p-1}}(tR)^{\frac{n-1}{p-1}} H_+(tR)}{R^{\frac{n-1}{p-1}}(tr)^{\frac{n-1}{p-1}} H_+(tr)} = \frac{r^{\frac{n-1}{p-1}}}{R^{\frac{n-1}{p-1}}} \tag{4.47}$$

$$\lim_{t\to \infty} Q_+(t) = \lim_{t\to \infty} \frac{H_+(tR)}{H_+(tr)} = \lim_{t\to \infty} \frac{R\frac{H_+(tR)}{tR}}{r\frac{H_+(tr)}{tr}} = \frac{R}{r} \tag{4.48}$$

We now consider $Q_-(t)$. Since $H_-(1) = 0$, we see that $Q_-(t)$ is undefined for $t = \frac{1}{r}$, and $Q_-(\frac{1}{R}) = 0$. We will show that $Q_-$ is decreasing on $\left(0, \frac{1}{r}\right)$ and on $\left(\frac{1}{r}, \infty\right)$. Moreover, we have the limits

$$\lim_{t\to 0^+} Q_-(t) = \left(\frac{r}{R}\right)^{\frac{n-1}{p-1}} \tag{4.49}$$

$$\lim_{t\to \frac{1}{r}^-} Q_-(t) = -\infty \tag{4.50}$$

$$\lim_{t\to \frac{1}{r}^+} Q_-(t) = \infty \tag{4.51}$$

$$\lim_{t\to \infty} Q_-(t) = \frac{R}{r} \tag{4.52}$$

If $t > \frac{1}{r}$, then $H_-(tR) > H_-(tr) > 0$. Thus, $Q_-(t)$ is positive on $\left(\frac{1}{r}, \infty\right)$. If $0 < t < \frac{1}{R}$, then $H_-(tr) < H_-(tR) < 0$. Thus, $Q(t)$ is also positive on $\left(0, \frac{1}{R}\right)$. But if $\frac{1}{R} < t < \frac{1}{r}$, we have $H_-(tr) < 0 < H_-(tR)$, so $Q_-(t)$ is negative on $\left(\frac{1}{R}, \frac{1}{r}\right)$. Considering the sign of $Q_-$ near $\frac{1}{r}$ gives (4.50) and (4.51).

The other limits use the asymptotic limits of $H_-$, establishing

$$\lim_{t\to 0} Q_-(t) = \lim_{t\to 0} \frac{H_-(tR)}{H_-(tr)} = \lim_{t\to 0} \frac{r^{\frac{n-1}{p-1}}(tR)^{\frac{n-1}{p-1}} H_-(tR)}{R^{\frac{n-1}{p-1}}(tr)^{\frac{n-1}{p-1}} H_-(tr)} = \frac{r^{\frac{n-1}{p-1}}}{R^{\frac{n-1}{p-1}}},$$

$$\lim_{t\to \infty} Q_-(t) = \lim_{t\to \infty} \frac{H_-(tR)}{H_-(tr)} = \lim_{t\to \infty} \frac{R\frac{H_-(tR)}{tR}}{r\frac{H_-(tr)}{tr}} = \frac{R}{r}.$$

We will now show $Q_-(t)$ is decreasing on $\left(0, \frac{1}{r}\right)$ and on $\left(\frac{1}{r}, \infty\right)$. The logarithmic derivative of $Q_-(t)$ at $t \neq \frac{1}{R}, \frac{1}{r}$ is

$$\frac{\dot{Q}_-(t)}{Q_-(t)} = \frac{R\dot{H}_-(tR)}{RH_-(tR)} - \frac{r\dot{H}_-(tr)}{rH_-(tr)} = \frac{1}{t}\left(\eta_{H_-}(tR) - \eta_{H_-}(tr)\right).$$

We recall that $\eta_{H_-}(t) = \frac{1}{u_-(t)}$ is decreasing and negative on $(0, 1)$, while it is decreasing and positive on $(1, \infty)$. This implies that $\eta_{H_-}(tR) - \eta_{H_-}(tr) < 0$ when $0 < t < \frac{1}{R}$ or when $t > \frac{1}{r}$. Since $Q_-(t) > 0$ for these values of $t$, we conclude $Q_-$ is decreasing on $(0, \frac{1}{R})$ and $(\frac{1}{r}, \infty)$. Moreover, if $\frac{1}{R} < t < \frac{1}{r}$, we have that $\eta_{H_-}(tR) > 0 > \eta_{H_-}(tr)$, so $\eta_{H_-}(tR) - \eta_{H_-}(tr) > 0$. But $Q_-(t) < 0$ for $\frac{1}{R} < t < \frac{1}{r}$, implying that $Q_-$ is decreasing on $(\frac{1}{R}, \frac{1}{r})$. To conclude $Q_-$ is decreasing on $(0, \frac{1}{r})$, we remark that $Q_-(\frac{1}{R}) = 0$, $Q_-(t) > 0$ for $t < \frac{1}{R}$, and $Q_-(t) < 0$ for $\frac{1}{R} < t < \frac{1}{r}$.

We now return to considering cases where $a \neq 0$. First, suppose $\frac{b}{a} < \left(\frac{r}{R}\right)^{\frac{n-1}{p-1}}$. We have that $Q_-(t)$ is strictly decreasing on $\left(0, \frac{1}{r}\right)$. Using limits (4.49) and (4.50), we see there exists a unique $k$ with $0 < k < \frac{1}{r}$ such that $Q_-(k) = \frac{b}{a}$. We then see that $H(t) = \frac{a}{H_-(kr)} H_-(kt)$ satisfies (4.46)-(4.46).

If $\frac{b}{a} = \left(\frac{r}{R}\right)^{\frac{n-1}{p-1}}$, then we see that $H(t) = aH_\infty(\frac{t}{r}) = a\left(\frac{r}{t}\right)^{\frac{n-1}{p-1}}$ satisfies (4.44)-(4.46).

Suppose $\left(\frac{r}{R}\right)^{\frac{n-1}{p-1}} < \frac{b}{a} < \frac{R}{r}$. We have that $Q_+(t)$ is strictly increasing on $(0, \infty)$. Using limits (4.47) and (4.48), we conclude there exists a unique $k$ such that $Q_+(k) = \frac{b}{a}$. We then see that $H(t) = \frac{a}{H_+(kr)} H_+(kt)$ satisfies (4.44)-(4.46).

If $\frac{b}{a} = \frac{R}{r}$, then we see that $H(t) = aH_0(\frac{t}{r}) = \frac{a}{r}t$ satisfies (4.44)-(4.46).

Now assume $\frac{R}{r} < \frac{b}{a}$. We have that $Q_-(t)$ is strictly decreasing on $(\frac{1}{r}, \infty)$ Using limits (4.51) and (4.52), we conclude there exists a unique $k$ with $\frac{1}{r} < k$ such that $Q_-(k) = \frac{b}{a}$. We then see that $H(t) = \frac{a}{H_-(kr)} H_-(kt)$ satisfies (4.44)-(4.46).

Thus, in all cases, we have found a strain function $H(t)$ satisfying (4.44)-(4.46). So the map $h(x) = H(|x|)\frac{x}{|x|}$ is a $p$-harmonic mapping of $\mathbb{A}$ into $\mathbb{R}^n$ such that $|h(x)| = a$ for $|x| = r$ and $|h(x)| = b$ for $|x| = R$.

## 4.5 Existence of Radial $p$-harmonic homeomorphisms

We are interested in finding radial maps that are $p$-harmonic order-preserving homeomorphisms of $\mathbb{A}$ onto $\mathbb{A}^*$. Thus, we consider solutions of (4.44)-(4.46) when $a = r_*$ and $b = R_*$, with $\frac{b}{a} > 1$.

When $\frac{R_*}{r_*} > \frac{R}{r}$, we saw that there exists $k > \frac{1}{r}$ such that $H(r) = R_*$ and $H(R) = R_*$ for $H(t) = \frac{r_*}{H_-(kr)} H_-(kt)$. Because $H_-$ is strictly increasing, we have that $h(x) = H(|x|)\frac{x}{|x|}$ is the desired homeomorphism.

If $\frac{R_*}{r_*} = \frac{R}{r}$, then the conformal map $h(x) = \frac{r_*}{r} x$ is an order-preserving $p$-harmonic homeomorphsim.

Suppose $\frac{R_*}{r_*} < \frac{R}{r}$. We defined a constant $k$ by $Q_+(k) = \frac{R_*}{r_*}$, where $Q_+(t) = \frac{H_+(tR)}{H_+(tr)}$ was a strictly increasing function. Taking $H(t) = \frac{r_*}{H_+(kr)} H_+(kt)$ gave $H(r) = r_*$ and $H(R) = R_*$. Since $H_+$ is increasing on $(1, \infty)$, we have that $h(x) = H(|x|)\frac{x}{|x|}$ is the desired homeomorphism if $kr \geq 1$, or equivalently,

$$\frac{R_*}{r_*} = Q_+(k) \geq Q_+\left(\frac{1}{r}\right) = \frac{H_+(\frac{R}{r})}{H_+(\frac{r}{r})} = H_+\left(\frac{R}{r}\right).$$

Recall that $H_+$ is increasing on $(1, \infty)$ but decreasing on $(0, 1)$. If $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$, then $kr < 1$. So $h(x) = H(|x|)\frac{x}{|x|}$ is not a homeomorphism, since $H(t)$ is decreasing on $\left(r, \frac{1}{k}\right)$ but increasing on $\left(\frac{1}{k}, R\right)$. The condition $H_+\left(\frac{R}{r}\right) \leq \frac{R_*}{r_*} < \frac{R}{r}$ says $\frac{R_*}{r_*}$ is not too small, relative to $\frac{R}{r}$. Note in the case that $p = n = 2$, the term $H_+(\frac{R}{r}) = \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$ is the classical Nitsche bound for the existence of harmonic homeomorphsims between annuli [10]. What we have seen is that there is no radial $p$-harmonic homeomorphism from $\mathbb{A}$ onto $\mathbb{A}^*$ if $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$.

We now construct an important map that is the limit of homeomorphsims when $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$. We recall that $H_+(t)$ is increasing on $(1, \infty)$, so $H_+(\frac{R}{t})$ is decreasing on $(0, R)$. Moreover, since $H_+(1) = 1$, we have

$$H_+\left(\frac{R}{R}\right) < \frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right).$$

Since $H_+$ is continuous, we conclude that there is some number $\rho$ with $r < \rho < R$ such that $H_+\left(\frac{R}{\rho}\right) = \frac{R_*}{r_*}$. We then set

$$H^0(t) = \begin{cases} r_* H_+\left(\frac{t}{\rho}\right) & \rho \le t \le R \\ r_* & r \le t \le \rho \end{cases} \tag{4.53}$$

Since $\dot{H}_+(1) = 0$, we see that $H^0 \in C^1(r, R)$. In fact, $H^0$ is a limit of homeomorphisms. Moreover, $H^0$ is increasing, and it satisfies (4.44)-(4.46). The radial map $H^0(|x|)\frac{x}{|x|} : \mathbb{A} \to \mathbb{A}^*$ is now an admissible map.

# Chapter 5

# Radial $p$-harmonics and Minimal Energy

In this section, we prove the main results. We will require the free Lagrangian estimates from Lemma 2.2.9, and the algebraic inequalities given in Lemma 3.1.1 and Lemma 3.2.2. Throughout this section, we will let $h \in \mathcal{A}(\mathbb{A}, \mathbb{A}^*)$ be arbitrary, and we will think of $t = |x|$ and $s = |h(x)|$, where $r < t < R$ and $r_* < s < R_*$. We will consider the problem in two subsections.

## 5.1 The contracting case $\frac{R_*}{r_*} \leq \frac{R}{r}$

We begin with the proof of Theorem 1.0.4, restated below.

**Theorem 5.1.1.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$ with $H_+\left(\frac{R}{r}\right) < \frac{R_*}{r_*} < \frac{R}{r}$, where $H_+$ refers to the principal solution to the generalized p-harmonic equation. Then there exists a radial* *homeomorphsim* $h^0(x) = H(|x|)\frac{x}{|x|}$ *that maps* $\mathbb{A}$ *onto* $\mathbb{A}^*$ *such that*

$$\int_{\mathbb{A}} |Dh(x)|^p \, dx \geq \int_{\mathbb{A}} |Dh^0(x)|^p \, dx$$

*for every homeomorphism* $h : \mathbb{A} \to \mathbb{A}^*$ *of Sobolev class* $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.

*Proof.* The existence of such a radial homeomorphism has already been shown. Let $h$ : $\mathbb{A} \to \mathbb{A}^*$ be any admissible homeomorphism. We apply Lemma 3.1.1, with carefully chosen functions $a = a(|x|, |h|)$ and $b = b(|h|)$ as our constants. Since $a$ and $b$ depend on $|x|$ and $|h|$, so does the constant $c = c(a, b)$ from the lemma. Letting $X = |h_N(x)|$ and $Y = |h_T(x)|$, we now have the pointwise inequality

$$(|h_N|^2 + (n-1)|h_T|^2)^{\frac{p}{2}} \geq a(|x|, |h|)|h_T|^n + b(|h|)|h_N||h_T|^{n-1} - c(|x|, |h|) \tag{5.1}$$

The functions $a$ and $b$ will be chosen so that equality holds for the radial $p$-harmonic homeomorphsim $h^0 : \mathbb{A} \to \mathbb{A}^*$.

Observe that $b(|h|)|h_N||h_T|^{n-1}dx$ is a free Lagrangian. We will also choose an appropriate function $A = A(|x|, |h|)$ so that $a(|x||h|)A(|x|, |h|)|h_T|^{n-1}dx$ is a free Lagrangian. From Young's inequality, we have

$$aA|h_T|^{n-1} \leq \frac{n-1}{n} a|h_T|^n + \frac{1}{n} aA^n \tag{5.2}$$

with equality when $A(|x|,|h|)=|h_T(x)|$. Combining (5.1) and (5.2) yields

$$(|h_N|^2+(n-1)|h_T|^2)^{\frac{p}{2}} \geq \frac{n}{n-1}a(|x|,|h|)A(|x|,|h|)|h_T|^{n-1}+b(|h|)|h_N||h_T|^{n-1} \tag{5.3}$$

$$-\left(\frac{1}{n-1}a(|x|,|h|)A^n(|x|,|h|)+c(|x|,|h|)\right) \tag{5.4}$$

We will write

$$\frac{1}{n-1}a(|x|,|h|)A^n(|x|,|h|)+c(|x|,|h|)=B(|x|,|h|) \tag{5.5}$$

If $H(|x|)=|h^0(x)|$, then we will show, for all $r_*\leq s\leq R_*$, that

$$B(|x|,s)\leq B(|x|,H(|x|)) \tag{5.6}$$

Thus, we may estimate the term in (5.4) using a term independent of $h$. Equality will hold when $h=h^0$. Integrating all this and using (1.6), we obtain

$$\int\limits_{\mathbb{A}}|Dh|^p dx \geq \int\limits_{\mathbb{A}} \frac{n}{n-1}a(|x|,|h|)A(|x|,|h|)|h_T|^{n-1}dx+\int\limits_{\mathbb{A}} b(|h|)|h_N||h_T|^{n-1}dx \tag{5.7}$$

$$-\int\limits_{\mathbb{A}} B(|x|,H(|x|))dx \tag{5.8}$$

Equality holds throughout for $h^0$, so this establishes $\mathcal{E}_p[h]\geq\mathcal{E}_p[h^0]$.

To choose $a$ and $b$, we must first define a function $\eta(s)$ for $r_*\leq s\leq R_*$. Recall that when $H_+\left(\frac{R}{r}\right)\leq\frac{R_*}{r_*}<\frac{R}{r}$, we have an increasing function $H(t)$ whose elasticity function $\eta_H$

satisfies

$$(1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)^{\alpha}(\eta_H^2(t)+n-1)^{\beta}\exp\left(\gamma\tan^{-1}\left(\frac{\eta_H(t)}{\sqrt{n-1}}\right)\right)=\frac{C}{H^n(t)} \quad (5.9)$$

where $C>0$ is a constant. Here, $0\leq\eta_H(t)<1$. Since $H$ is nonconstant, we see from (4.16) that $\dot{\eta}_H(t)$ satisfies

$$\dot{\eta}_H(t)=\frac{(1-\eta_H(t))\left(\frac{p-1}{n-1}\eta_H(t)+1\right)(\eta_H^2(t)+n-1)}{\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)t} \quad (5.10)$$

For $r_*\leq s\leq R_*$, let $F(s)$ be the function with $F(H(t))=t$. We now set $\eta(s)=\eta_H(F(s))$. From Proposition 4.2.2, we have $\eta(s)=\frac{1}{\eta_F(s)}$. Using this fact and (5.10), we obtain

$$\dot{\eta}(s)=\frac{(1-\eta(s))\left(\frac{p-1}{n-1}\eta(s)+1\right)(\eta^2(s)+n-1)}{\left(\frac{p-1}{n-1}\eta^2(s)+1\right)\eta(s)s} \quad (5.11)$$

We remark that we can use (5.9) to define $\eta(s)$ implicitly for $r_*\leq s\leq R_*$ by the formula

$$(1-\eta(s))\left(\frac{p-1}{n-1}\eta(s)+1\right)^{\alpha}(\eta(s)+n-1)^{\beta}\exp\left(\gamma\tan^{-1}\left(\frac{\eta(s)}{\sqrt{n-1}}\right)\right)=\frac{C}{s^n} \quad (5.12)$$

Since $H$ is the unique function satisfying (5.9), these definitions are equivalent. Differentiating (5.12), we recover (5.11).

We are now ready to define $a$ and $b$. We set

$$a(t,s) = \frac{(n-1)p}{n}(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))\left(\frac{H(t)}{t}\right)^{p-n}\left(\frac{H(t)}{s}\right)^{n} \tag{5.13}$$

$$b(s) = np(\eta^2(s)+n-1)^{\frac{p-2}{2}}\eta(s)\left(\frac{s}{F(s)}\right)^{p-n} \tag{5.14}$$

We the have

$$c(t,s) = \frac{p-n}{n}\left[\frac{(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))H^n(t)}{(V^2+n-1)^{\frac{n-2}{2}}(1-V^2)s^n}\right]^{\frac{p}{p-n}}\left(\frac{H(t)}{t}\right)^{p} \tag{5.15}$$

where $V = V(t,s)$ is the function defined at (3.7) by

$$\frac{V}{1-V^2} = \frac{(n-1)b}{na} = \frac{\eta(s)}{1-\eta_H^2(t)}\left[\frac{(\eta^2(s)+n-1)^{\frac{p-2}{2}}\frac{s^p}{F^{p-n}(s)}}{(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}\frac{H^p(t)}{t^{p-n}}}\right] \tag{5.16}$$

Note that $V(t,H(t)) = \eta_H(t)$. We record here that the derivatives of $V(t,s)$ found by logarithmic differentiation are

$$\frac{1+V^2(t,s)}{V(t,s)[1-V^2(t,s)]}V_s(t,s) = \frac{\eta^2(s)+n-1}{s\eta^2(s)} \tag{5.17}$$

$$\frac{1+V^2(t,s)}{V(t,s)[1-V^2(t,s)]}V_t(t,s) = \frac{\frac{p-n}{n-1}(\eta_H^2(t)+n-1)(1-\eta_H(t))}{\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)(\eta_H(t)+1)t} \tag{5.18}$$

Recall equality holds in (3.1) for $Y^0 = \left(\frac{na}{(n-1)p(V^2+n-1)^{\frac{p-2}{2}}(1-V^2)}\right)^{\frac{1}{p-n}}$ and $X^0 = VY^0$.

Therefore, equality holds in (5.2) when

$$|h_T| = \frac{H(|x|)}{|x|}\left(\frac{(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(|x|))H^n(|x|)}{(V^2(|x|,|h|)+n-1)^{\frac{p-2}{2}}(1-V^2(|x|,|x|))|h|^n}\right)^{\frac{1}{p-n}} \tag{5.19}$$

$$|h_N| = \frac{V(|x|,|h|)H(|x|)}{|x|}\left(\frac{(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(|x|))H^n(|x|)}{(V^2(|x|,|h|)+n-1)^{\frac{p-2}{2}}(1-V^2(|x|,|x|))|h|^n}\right)^{\frac{1}{p-n}} \tag{5.20}$$

In particular, when $h = h^0$, we have equality in (5.2).

Next, we set

$$A(t,s) = \frac{s}{t} \tag{5.21}$$

Equality in (5.3)-(5.4) holds when $A(|x|,|h|) = \frac{|h|}{|x|} = |h_T|$. This occurs for any radial map, or a radial map composed with a rotation. We also see that

$$a(t,s)A(t,s) = (\eta_H^2(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))\left(\frac{H^p(t)}{t^{p-n+1}}\right)\frac{1}{s^{n-1}} \tag{5.22}$$

Thus, $a(|x|,|h|)A(|x|,|h|)|h_T|^{n-1}dx$ is a free Lagrangian.

All that remains is to prove (5.6). We investigate the derivative of $B(t,s)$ with respect to $s$ to maximize $B(t,s)$ for each $r < t < R$. We have

$$B(t,s) = \frac{p}{n}(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))\left(\frac{H(t)}{t}\right)^p \tag{5.23}$$

$$+\frac{p-n}{n}\left[\frac{(\eta_H(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))H^n(t)}{(V^2(t,s)+n-1)^{\frac{n-2}{2}}(1-V^2(t,s))s^n}\right]^{\frac{p}{p-n}}\left(\frac{H(t)}{t}\right)^p \tag{5.24}$$

Differentiating $B(t,s)$ with respect to $s$ and using (5.17), we get

$$B_s(t,s) = -\frac{pH^p(t)}{nt^p}\left[\frac{(\eta_H(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))H^n(t)}{(V^2(t,s)+n-1)^{\frac{n-2}{2}}(1-V^2(t,s))s^n}\right]^{\frac{p}{p-n}}\left[\frac{(n-2)VV_s}{V^2+n-1}-\frac{2VV_s}{1-V^2}+\frac{n}{s}\right] \tag{5.25}$$

$$= \frac{pH^p(t)}{t^p s}\left[\frac{(\eta_H(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))H^n(t)}{(V^2(t,s)+n-1)^{\frac{n-2}{2}}(1-V^2(t,s))s^n}\right]^{\frac{p}{p-n}}\left[\frac{V^2[\eta^2(s)+n-1]}{\eta(s)[V^2+n-1]}-1\right] \tag{5.26}$$

We know see that $B_s(t,s)$ and $\frac{V^2(t,s)}{V^2(t,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)}-1$ have the same sign. Since $V(t,H(t)) = \eta(H(t)) = \eta_H(t)$, we have for fixed $r < t < R$ that $B(t,s)$ has a critical point at $s = H(t)$. Next, differentiating $\frac{V^2(t,s)}{V^2(t,s)+n-1}$ with respect to $t$ and using (5.18) yields

$$\left(\frac{V^2(t,s)}{V^2(t,s)+n-1}\right)_t = \frac{2(n-1)V(t,s)}{[V^2(t,s)+n-1]^2}V_t(t,s) \tag{5.27}$$

$$= \frac{2(p-n)[\eta_H^2(t)+n-1][1-\eta_H(t)][1-V^2(t,s)]V^2(t,s)}{t\left(\frac{p-1}{n-1}\eta_H^2(t)+1\right)(\eta_H(t)+1)[V^2(t,s)+n-1]^2[V^2(t,s)+1]} \tag{5.28}$$

Thus, we see that $\frac{V^2(t,s)}{V^2(t,s)+n-1}$ is increasing in $t$.

Now pick $r_* \leq s \leq H(t)$. We can find $r \leq \tau \leq t$ with $s = H(\tau)$ since $H$ is increasing and maps onto $[r_*, R_*]$. Since $\eta(s) = \eta_H(F(s))$, we have by (5.16) that $V(\tau,s) = \eta(s)$. Since $\tau \leq t$ and $\frac{V^2(t,s)}{V^2(t,s)+n-1}$ is increasing in $t$, we have that

$$\frac{V^2(t,s)}{V^2(t,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)}-1 \geq \frac{V^2(\tau,s)}{V^2(\tau,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)}-1 = 0.$$

Because $\frac{V^2(t,s)}{V^2(t,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)}-1$ and $B_s(t,s)$ have the same sign, we see that $B_s(t,s) \geq 0$ for $r_* \leq s \leq H(t)$.

Similarly, suppose $H(t) \leq s \leq R_*$. We can find $t \leq \tau \leq R$ with $s = H(\tau)$ since $H$ is increasing and maps onto $[r_*, R_*]$. Since $t \leq \tau$ and $\frac{V^2}{1-V^2}$ is increasing in $t$, we have that

$$\frac{V^2(t,s)}{V^2(t,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)} - 1 \leq \frac{V^2(\tau,s)}{V^2(\tau,s)+n-1}\frac{\eta^2(s)+n-1}{\eta(s)} - 1 = 0.$$

We see $B_s(t,s) \leq 0$ for $H(t) \leq s \leq R_*$. Therefore, $B(t, H(t)) = \max\{B(t,s) : r_* \leq s \leq R_*\}$. This establishes that $h^0$ is in fact a minimizer.

We remark that by backwards inspection, equality can only occur for maps $h : \mathbb{A} \to \mathbb{A}$ with $|h(x)| = H(|x|)$ and $|h_T(x)| = \frac{H(|x|)}{|x|}$. The only such map, up to rotation, is the radial $p$-harmonic homeomorphism $h^0$. □

Recall that when $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$, there is no radial $p$-harmonic homeomorphsim between $\mathbb{A}$ and $\mathbb{A}^*$. Theorem 1.0.5, restated as Theorem 5.1.2, addresses this case, and the proof is similar to the one above.

**Theorem 5.1.2.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$ with $\frac{R_*}{r_*} < H_+\left(\frac{R}{r}\right)$. The map $h^0(x) = H(|x|)\frac{x}{|x|}$, where $H = H^0$ is defined in (4.53), is the limit of homeomorphisms in $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$, and we have*

$$\inf\left\{\int_{\mathbb{A}} |Dh(x)|^p dx\right\} = \int_{\mathbb{A}} |Dh^0(x)|^p dx$$

*where the infimum is taken over homeomorphisms in Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.*

*Proof.* To prove this theorem, we recall there exists $r < \rho < R$ such that $H_+\left(\frac{R}{\rho}\right) = \frac{R_*}{r_*}$, and break up the domain $\mathbb{A} = \mathbb{A}(r,\rho] \cup \mathbb{A}(\rho, R)$. We will then choose functions $A$, $a$, and $b$ and proceed as in the proof of Theorem 5.1.1. Applying Lemma 3.1.1 and Young's inequality with $X = |h_N(x)|$ and $Y = |h_T(x)|$, we obtain

$$\int_{\mathbb{A}} |Dh|^p dx \geq \int_{\mathbb{A}(r,\rho]} \left(a|h_T|^n + b|h_N||h_T|^{n-1} - c\right) dx + \int_{\mathbb{A}(\rho,R)} \left(a|h_T|^n + b|h_N||h_T|^{n-1} - c\right) dx \tag{5.29}$$

$$\geq \int_{\mathbb{A}(r,\rho]} \frac{n}{n-1} aA|h_T|^{n-1} dx + \int_{\mathbb{A}(r,\rho]} b|h_N||h_T|^{n-1} dx - \int_{\mathbb{A}(r,\rho]} \left[\frac{1}{n-1} aA^n + c\right] dx + \tag{5.30}$$

$$\int_{\mathbb{A}(\rho,R)} \frac{n}{n-1} aA|h_T|^{n-1} dx + \int_{\mathbb{A}(\rho,R)} b|h_N||h_T|^{n-1} dx - \int_{\mathbb{A}(\rho,R)} \left[\frac{1}{n-1} aA^n + c\right] dx \tag{5.31}$$

We begin with (5.31). We choose $a$, $b$, and $A$ as

$$a(t,s) = \frac{(n-1)^{\frac{p}{2}} p}{n} \left(\frac{r_*}{t}\right)^{p-n} \tag{5.32}$$

$$b(s) = 0 \tag{5.33}$$

$$A(t,s) = \frac{r_*}{t} \left(\frac{r_*}{s}\right)^{n-1} \tag{5.34}$$

In the context of Lemma 3.1.1, we have $V = V(a,b) = 0$ and $c = c(a,b) = \frac{(p-n)(n-1)^{\frac{p}{2}}}{n} \left(\frac{r_*}{t}\right)^p$.

Thus, (5.31) reads

$$\int_{\mathbb{A}(r,\rho]} \frac{(n-1)^{\frac{p-2}{2}} p r_*^p}{|x|^{p-n+1}} \frac{|h_T|^{n-1}}{|h|^{n-1}} dx - \int_{\mathbb{A}(r,\rho]} \frac{(n-1)^{\frac{p}{2}}}{n} \left(\frac{r_*}{|x|}\right)^p \left[\frac{p r_*^{n(n-1)}}{(n-1)|h|^{n(n-1)}} + p - n\right] dx \tag{5.35}$$

$$\geq \int_{\mathbb{A}(r,\rho]} \frac{(n-1)^{\frac{p-2}{2}} p r_*^p}{|x|^{p-n+1}} \frac{|h_T|^{n-1}}{|h|^{n-1}} dx - \int_{\mathbb{A}(r,\rho]} (n-1)^{\frac{p-2}{2}} [p-n+1] \left(\frac{r_*}{|x|}\right)^p dx \tag{5.36}$$

The inequality holds for all maps $h : \mathbb{A} \to \mathbb{A}^*$ since $|h| \geq r_*$. Equality holds throughout when $|h_N| = 0$, $|h_T| = \frac{r_*}{|x|} = \frac{r_*}{|x|}\left(\frac{r_*}{|h|}\right)^{n-1}$ and $|h| = r_*$. This happens on $\mathbb{A}(r, \rho]$ precisely for the map $h^0$.

We now turn our attention to the annulus $\mathbb{A}(r, R)$. Recall there must be some $C > 0$ such that

$$(1 - \eta_{H^0}(t))\left(\frac{p-1}{n-1}\eta_{H^0}(t) + 1\right)^{\alpha}\left(\frac{\eta_{H^0}(t)}{n-1} + 1\right)^{\beta}\exp\left(\gamma\tan^{-1}\left(\frac{\eta_{H^0}(t)}{\sqrt{n-1}}\right)\right) = \frac{C}{(H^0(t))^n} \tag{5.37}$$

We will again denote the right inverse of $H^0$ by $F : [r_*, R_*] \to [\rho, R]$. We define $\eta(s)$ implicitly for $r_* \leq s \leq R_*$ by the formula

$$(1 - \eta(s))\left(\frac{p-1}{n-1}\eta(s) + 1\right)^{\alpha}(\eta(s) + n - 1)^{\beta}\exp\left(\gamma\tan^{-1}\left(\frac{\eta(s)}{\sqrt{n-1}}\right)\right) = \frac{C}{s^n} \tag{5.38}$$

From (5.37), we have $\eta(H^0(t)) = \eta_{H^0}(t)$. Differentiating (5.38), we obtain

$$\dot{\eta}(s) = \frac{(1 - \eta(s))\left(\frac{p-1}{n-1}\eta(s) + 1\right)(\eta^2(s) + n - 1)}{\left(\frac{p-1}{n-1}\eta^2(s) + 1\right)\eta(s)s} \tag{5.39}$$

We are now ready to define $A$, $a$, and $b$ for $\rho \leq t < R$ and $r_* < s < R_*$. We set

$$a(t, s) = \frac{(n-1)p}{n}(\eta_H^2(t) + n - 1)^{\frac{p-2}{2}}(1 - \eta_H^2(t))\left(\frac{H(t)}{t}\right)^{p-n}\left(\frac{H(t)}{s}\right)^{n} \tag{5.40}$$

$$b(s) = np(\eta^2(s) + n - 1)^{\frac{p-2}{2}}\eta(s)\left(\frac{s}{F(s)}\right)^{p-n} \tag{5.41}$$

$$A(t, s) = \frac{s}{t} \tag{5.42}$$

We the have

$$c(t,s)=\frac{p-n}{n}\left[\frac{(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(t))H^n(t)}{(V^2+n-1)^{\frac{n-2}{2}}(1-V^2)s^n}\right]^{\frac{p}{p-n}}\left(\frac{H(t)}{t}\right)^p \tag{5.43}$$

where $V=V(t,s)$ is the function defined at (3.7) by

$$\frac{V}{1-V^2}=\frac{(n-1)b}{na}=\frac{\eta(s)}{1-\eta_H^2(t)}\left[\frac{(\eta^2(s)+n-1)^{\frac{p-2}{2}}\frac{s^p}{F^{p-n}(s)}}{(\eta_H^2(t)+n-1)^{\frac{p-2}{2}}\frac{H^p(t)}{t^{p-n}}}\right] \tag{5.44}$$

Note that these are the same functions chosen in the proof of Theorem 5.1. The integrals in (5.32) are now

$$\int\limits_{\mathbb{A}(\rho,R)} p(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(|x|))\frac{H^p(|x|)}{|x|^{p-n+1}}\frac{|h_T|^{n-1}}{|h|^{n-1}}dx+ \tag{5.45}$$

$$\int\limits_{\mathbb{A}(\rho,R)} np(\eta^2(|h|)+n-1)^{\frac{p-2}{2}}\eta(|h|)\left(\frac{|h|}{F(|h|)}\right)^{p-n}|h_N||h_T|^{n-1}dx- \tag{5.46}$$

$$\int\limits_{\mathbb{A}(\rho,R)} \frac{p}{n}(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(|x|))\left(\frac{H(|x|)}{|x|}\right)^p dx+ \tag{5.47}$$

$$\int\limits_{\mathbb{A}(\rho,R)} \frac{p-n}{n}\left[\frac{(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}(1-\eta_H^2(|x|))H^n(t)}{(V^2+n-1)^{\frac{n-2}{2}}(1-V^2)|h|^n}\right]^{\frac{p}{p-n}}\left(\frac{H(|x|)}{|x|}\right)^p dx \tag{5.48}$$

The values of the first three integrals are independent of $h$. The integrand of the fourth is again maximized at $|h|=H(|x|)$, so we can estimate it from below using an integral independent of $h$, as in the proof of Theorem 5.1. Thus, for every admissible $h : \mathbb{A} \to \mathbb{A}^*$,

$$\int\limits_{\mathbb{A}} |Dh(x)|^p dx = \int\limits_{\mathbb{A}(r,\rho]} |Dh(x)|^p dx + \int\limits_{\mathbb{A}(\rho,R)} |Dh(x)|^p dx \tag{5.49}$$

$$\geq \int\limits_{\mathbb{A}(r,\rho]} |Dh^0(x)|^p dx + \int\limits_{\mathbb{A}(\rho,R)} |Dh^0(x)|^p dx = \int\limits_{\mathbb{A}} |Dh^0(x)|^p dx \tag{5.50}$$

Equality holds throughout for $h^0$, so it is the minimizer. □

## 5.2 The Expanding Case $\frac{R_*}{r_*} > \frac{R}{r}$

In this case, we saw that there exists a unique $p$-harmonic order preserving radial homeomorphism $h^0 : \mathbb{A} \to \mathbb{A}^*$ of the form $h^0(x) = \lambda H_-(k|x|)\frac{x}{|x|}$ where $\lambda > 0$ and $kr > 1$. We write $H(t) = \lambda H_-(kt)$. By classical methods, this homeomorphism is the minimizer among all radial mappings of $\mathbb{A}$ into $\mathbb{A}^*$. Theorem 1.0.7 states the surprising fact that the radial minimizer is not always the traction-free minimizer of $\mathcal{E}_p$. This is proved by the following example.

**Example 5.2.1.** Suppose $n \geq 4$ and $\frac{R_*}{r_*} > \frac{H_-(\delta)}{H_-(\delta\frac{r}{R})}$, where $\delta = \Gamma_-\left(\sqrt{\frac{n-3}{(n-1)(p-n+1)}}\right)$, and let $h^0$ be the minimizer among radial maps. Then there exists a map $h^1 \in \mathcal{A}(\mathbb{A}, \mathbb{A}^*)$ with $\mathcal{E}_p[h^1] < \mathcal{E}_p[h^0]$.

*Proof.* Writing $h^0(x) = H(|x|)\frac{x}{|x|}$, we first show the assumption that $\frac{R_*}{r_*} > \frac{H_-(\delta)}{H_-(\delta\frac{r}{R})}$ implies $\eta_H(t) > \sqrt{\frac{(n-1)(p-n+1)}{n-3}}$. Indeed, recall that we defined a constant $k > \frac{1}{r}$ in Chapter 4 by $Q_-(k) = \frac{R_*}{r_*}$, where $Q_-(t) = \frac{H_-(tR)}{H_-(tr)}$ is a decreasing function. Our assumption $\frac{R_*}{r_*} > \frac{H_-(\delta)}{H_-(\delta\frac{r}{R})}$ simply means $Q_-(k) > Q_-(\frac{\delta}{R})$.. Because $Q_-(t)$ is decreasing, we have $kR < \delta$. We also recall that the elasticity function $\eta_{_{H_-}}(t) = \frac{1}{u_-(t)}$ is decreasing. Thus, since $H(t) = \lambda H_-(kt)$ for some $\lambda > 0$, we have for $r < t < R$

$$\eta_H(t) = \eta_{_{H_-}}(kt) > \eta_{_{H_-}}(kR) > \eta_{_{H_-}}(\delta) = \frac{1}{u(\delta)} = \sqrt{\frac{(n-1)(p-n+1)}{n-3}}. \tag{5.51}$$

Next, we construct $h^1 : \mathbb{A} \to \mathbb{A}^*$, and then prove that $\mathcal{E}_p[h^1] < \mathcal{E}_p[h^0]$. Let $\Pi : \mathbb{S}^{n-1} \to \widehat{\mathbb{R}}^{n-1}$ be the stereographic projection of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ through the south pole onto $\widehat{\mathbb{R}}^{n-1}$, and let $f^\tau : \widehat{\mathbb{R}}^{n-1} \to \widehat{\mathbb{R}}^{n-1}$ be the dilation given by $f^\tau(x) = \tau x$. The map $\Phi^\tau = \Pi^{-1} \circ f^\tau \circ \Pi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is called a spherical homothety, and

$$\fint_{\mathbb{S}^{n-1}} [D\Phi^\tau]^{n-1} d\sigma = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} [D\Phi^\tau]^{n-1} d\sigma = 1 \tag{5.52}$$

For $\alpha^2 > \frac{(n-1)(p-n+1)}{n-3}$, take $1 < \tau < \sqrt{\frac{n-3}{(n-1)(p-n+1)}}\alpha$. It is a fact that

$$[D\Phi^\tau]^{n-1} \le \tau^{n-1} < \left(\frac{n-3}{(n-1)(p-n+1)}\alpha^2\right)^{\frac{n-1}{2}} \tag{5.53}$$

Define $F(s) = \left(\alpha^2 + (n-1)s^{\frac{2}{n-1}}\right)^{\frac{p}{2}}$ for $s > 0$. Then differentiation yields

$$\dot{F}(s) = p\left(\alpha^2 + (n-1)s^{\frac{2}{n-1}}\right)^{\frac{p-2}{2}} s^{\frac{3-n}{n-1}}$$
$$\ddot{F}(s) = p\left(\alpha^2 + (n-1)s^{\frac{2}{n-1}}\right)^{\frac{p-4}{2}} s^{\frac{4-2n}{n-1}} \left((p-n+1)s^{\frac{2}{n-1}} - \frac{n-3}{n-1}\alpha^2\right)$$

So when $0 < s < \left(\frac{n-3}{(n-1)(p-n+1)}\alpha^2\right)^{\frac{n-1}{2}}$ we see that $F$ is concave. Hence using (5.53), we can apply Jensen's inequality, showing

$$\fint_{\mathbb{S}^{n-1}} (\alpha^2 + (n-1)[D\Phi^\tau]^2)^{\frac{p}{2}} = \fint_{\mathbb{S}^{n-1}} F\left([D\Phi^\tau]^{n-1}\right) \tag{5.54}$$

$$< F\left(\fint_{\mathbb{S}^{n-1}} [D\Phi^\tau]^{n-1}\right) = F(1) = (\alpha^2 + n - 1)^{\frac{p}{2}} \tag{5.55}$$

We now define a quasiradial map $h^1(x) = H(|x|)\Phi^\tau\left(\frac{x}{|x|}\right)$, and show that $\mathcal{E}_p[h^1] < \mathcal{E}_p[h^0]$. We integrate in polar coordinates. Using inequality (5.55) with $\alpha = \eta_H(t) > \sqrt{\frac{(n-1)(p-n+1)}{n-3}}$, we have

$$\mathcal{E}_p[h^1] = \int_{\mathbb{A}} \left(\dot{H}^2(|x|) + (n-1)\frac{H^2(|x|)}{|x|^2}[D\Phi^\tau]^2\right)^{\frac{p}{2}} dx = \int_{\mathbb{A}} \frac{H^p(|x|)}{|x|^p}\left(\eta_H^2(|x|) + (n-1)[D\Phi^\tau]^2\right)^{\frac{p}{2}} dx$$

$$= \int_r^R \int_{\mathbb{S}^{n-1}} \frac{H^p(t)}{t^p}\left(\eta_H^2(t) + (n-1)[D\Phi^\tau]^2\right)^{\frac{p}{2}} t^{n-1} d\sigma dt$$

$$= \omega_{n-1}\int_r^R \frac{H^p(t)}{t^{p-n+1}} \fint_{\mathbb{S}^{n-1}} \left(\eta_H^2(t) + (n-1)[D\Phi^\tau]^2\right)^{\frac{p}{2}} d\sigma dt$$

$$< \omega_{n-1}\int_r^R \frac{H^p(t)}{t^{p-n+1}}\left(\eta_H^2(t) + n - 1\right)^{\frac{p}{2}} dt = \omega_{n-1}\int_r^R \left(\dot{H}^2(t) + (n-1)\frac{H^2(t)}{t^2}\right)^{\frac{p}{2}} t^{n-1} dt$$

$$= \int_{\mathbb{A}} \left(\dot{H}^2(|x|) + (n-1)\frac{H^2(|x|)}{|x|^2}\right)^{\frac{p}{2}} dx = \mathcal{E}_p[h^0]$$

□

This example shows that if $\frac{R_*}{r_*}$ is too big, relative to $\frac{R}{r}$, then radial symmetry is lost in the minimizer of $\mathcal{E}_p$. However, we can show that the radial map $h^0$ is the minimizer of $\mathcal{E}_p$, provided $\frac{R_*}{r_*}$ is not too large.

**Lemma 5.2.2.** *Let* $Q(\xi) = \frac{(n-3)(p-1)}{n-1}\xi^3 + (p-3)\xi^2 - (2p-n-1)\xi - (n-1)$*. There exists a number* $b_{n,p} > 1$ *such that* $Q(b_{n,p}) = 0$ *and* $Q(\xi) < 0$ *for* $1 < \xi < b_{n,p}$*. If* $n > 3$*, then* $b_{n,p} < \sqrt{\frac{n-1}{n-3}}$.

The proof of the lemma is elementary. First, note $Q(1) = \frac{-2(p+n-2)}{n-1} < 1$. We now consider cases. For $n = 2$, we see that

$$Q(\xi) = -(p-1)\xi^3 + (p-3)\xi^2 - (2p-3)\xi - 1 \tag{5.56}$$

$$= (p-3)\xi^2(1-\xi) - 2\xi^3 - (2p-3)\xi - 1 \tag{5.57}$$

If $3 > p > 2$, then (5.56) shows that $Q(\xi) < 0$ for all $\xi > 0$. If $p > 3$, then (5.57) shows that $Q(\xi) < 0$ for all $\xi > 1$. Thus, we can take $b_{2,p} = \infty$.

For $n = 3$, we have

$$\begin{aligned} Q(\xi) &= (p-3)\xi^2 - (2p-4)\xi - 2 \\ &= (p-3)\left(\xi - \frac{p-2-\sqrt{p^2-2p-2}}{p-3}\right)\left(\xi - \frac{p-2+\sqrt{p^2-2p-2}}{p-3}\right) \end{aligned}$$

We observe that $\frac{p-2-\sqrt{p^2-2p-2}}{p-3} < 1 < \frac{p-2+\sqrt{p^2-2p-2}}{p-3}$ since $Q(1) < 0$. We take $b_{3,p} = \frac{p-2+\sqrt{p^2-2p-2}}{p-3}$.

Now suppose $n > 3$. The we have $Q\left(\sqrt{\frac{n-1}{n-3}}\right) = (p-n)\sqrt{\frac{n-1}{n-3}}\left[\sqrt{\frac{n-1}{n-3}} - 1\right] > 0$. Since $Q(1) < 0$, there exists a number $1 < b < \sqrt{\frac{n-1}{n-3}}$ with $Q(b) = 0$. By the fundamental theorem of algebra, there are no more than three such numbers. Take $b_{n,p}$ to be the smallest number in $\left(1, \sqrt{\frac{n-1}{n-3}}\right)$ with $Q(b_{n,p}) = 0$.

Since $g(\xi)$ is decreasing on $(1, \alpha_{n,p})$, there is a number $1 < \alpha'_{n,p} < \alpha_{n,p}$ so that $g(\alpha'_{n,p}) =$

$\left(\frac{R}{r}\right)^{(n-1)(p-n)} g(\alpha_{n,p})$.

**Definition 5.2.3.** Set $\alpha_0 = \min\{\alpha'_{n,p}, b_{p,n}\}$. Let $\delta_0 = \Gamma_-\left(\frac{1}{\alpha_0}\right)$. This constant depend on $n$, $p$, and $\frac{R}{r}$.

We are now ready to restate Theorem 1.0.5.

**Theorem 5.2.4.** *Let $\mathbb{A}$ and $\mathbb{A}^*$ be annuli in $\mathbb{R}^n$. Let $H_-$ be the principal solution of the generalized p-harmonic equation and $\alpha_0$ be the constant defined in Definition 5.2.3. If $\frac{R}{r} < \frac{R_*}{r_*} < \frac{H_-\left(\delta_0 \frac{R}{r}\right)}{H_-(\delta_0)}$, then there exists a radial p-harmonic homeomorphism $h^0(x) = H(|x|)\frac{x}{|x|}$ with*

$$\int_{\mathbb{A}} |Dh(x)|^p dx \geq \int_{\mathbb{A}} |Dh^0|^p dx$$

*for every homeomorphism $h : \mathbb{A} \to \mathbb{A}^*$ of Sobolev class $W^{1,p}(\mathbb{A}, \mathbb{A}^*)$.*

*Proof.* We begin by showing that $\frac{R}{r} < \frac{R_*}{r_*} < \frac{H_-(\delta_0 \frac{R}{r})}{H_-(\delta_0)}$ is equivalent to $1 < \eta_H(t) < \alpha_0$ for all $r < t < R$. We recall that $k$ is defined by $Q_-(k) = \frac{R_*}{r_*}$, where $Q_-(t) = \frac{H_-(tR)}{H_-(tr)}$ is a decreasing function on $\left(\frac{1}{r}, \infty\right)$. Moreover, for $r < t < R$, we have $\eta_H(t) = \eta_{H_-}(kt) = \frac{1}{u_-(kt)}$, which is decreasing. Thus,

$$\eta_H(t) = \eta_{H_-}(kt) < \alpha_0,\ \forall t \in (r, R) \Leftrightarrow \eta_{H_-}(kR) = \frac{1}{u(kR)} < \alpha_0$$
$$\Leftrightarrow u(kR) < \frac{1}{\alpha_0} \Leftrightarrow kR < \delta_0 \Leftrightarrow \frac{R_*}{r_*} < Q_-\left(\frac{\delta_0}{R}\right) = \frac{H_-(\delta_0 \frac{R}{r})}{H_-(\delta_0)}$$

Therefore, $1 < \eta_H(t) < \alpha_0$ is equivalent to $\frac{R}{r} < \frac{R_*}{r_*} < \frac{H_-(\delta_0 \frac{R}{r})}{H_-(\delta_0)}$.

Recall that when $\frac{R}{r} < \frac{R_*}{r_*}$, we have an increasing function $H(t)$ satisfying

$$(1-\eta_H)\left(\frac{p-1}{n-1}\eta_H+1\right)^A(\eta_H^2+n-1)^B\exp\left(D\tan^{-1}\left(\frac{\eta_H}{\sqrt{n-1}}\right)\right)=\frac{C}{H^n}, \tag{5.58}$$

where $C<0$ is a constant. For $r_* < s < R_*$, let $F(s)$ be the function with $F(H(t)) = t$, and let $\eta(s) = \eta_H(F(s))$. From (5.58), we see that $\eta(s)$ satisfies

$$(1-\eta)\left(\frac{p-1}{n-1}\eta+1\right)^A(\eta^2+n-1)^B\exp\left(D\tan^{-1}\left(\frac{\eta}{\sqrt{n-1}}\right)\right)=\frac{C}{s^n}. \tag{5.59}$$

This could be taken as the definition of $\eta(s)$.

We can define functions $a(s,t)$ and $b(s)$ to be

$$a(s,t) = \frac{p\left(\eta^2(s)+n-1\right)^{\frac{p-2}{2}}\left(\eta^2(s)-1\right)}{\eta(s)}\frac{s^{p-1}}{F^{p-1}(s)}\left(\frac{F(s)}{t}\right)^{n-1} \tag{5.60}$$

$$b(s) = \frac{p(\eta^2(s)+n-1)^{\frac{p-2}{2}}}{\eta(s)}\frac{s^{p-n}}{F^{p-n}(s)} \tag{5.61}$$

Recall the function $f(\xi)$ from Definition 3.2.1. We remark that $\alpha_{2,p}=\infty$. Thus, if $n=2$, we have $g(\alpha_{2,p})=0$, so $\frac{b^{p-1}}{p^{n-1}a^{p-n}} > g(\alpha_{2,p})$ automatically if $n=2$. Suppose that $n\geq 3$. Note by (5.60)-(5.61), we have that

$$\frac{b^{p-1}}{p^{n-1}a^{p-n}} = g(\eta(s))\left(\frac{t}{F(s)}\right)^{(n-1)(p-n)} \geq g(\eta(s))\left(\frac{r}{R}\right)^{(n-1)(p-n)} \tag{5.62}$$

Recalling our assumption that $1<\eta_H(t)=\eta(H(t))<\alpha_0$, and that $g$ is decreasing on $(1,\alpha_0)$, the definition of $\alpha'_{n,p}$ shows

$$\frac{b^{p-1}}{p^{n-1}a^{p-n}} \geq g(\eta(s)) \left(\frac{r}{R}\right)^{(n-1)(p-n)} \geq g(\alpha_{n,p}). \tag{5.63}$$

We may now invoke Lemma 3.2.2. Letting $X = |h_N(x)|$ and $Y = |h_T(x)|$ in Lemma 3.2.2 and using (1.6), we have

$$|Dh|^p \geq a|h_N| + b|h_N||h_T|^{n-1} - c. \tag{5.64}$$

Equality holds when $X = X^0(a, b)$ and $Y = Y^0(a, b)$. By our choice of $a$ and $b$, we have that

$$Y^0 = \left(\frac{\eta^2(s) - 1}{V^2 - 1}\right)^{\frac{1}{n-1}} \frac{s}{t} \qquad\qquad X^0 = \left(\frac{\eta^2(s) - 1}{V^2 - 1}\right)^{\frac{1}{n-1}} \frac{Vs}{t} \tag{5.65}$$

where $V = V(t, s)$ is defined by

$$g(V) = g(\eta(s)) \left(\frac{t}{F(s)}\right)^{(n-1)(p-n)}. \tag{5.66}$$

We see when $t = |x|$ and $s = H(|x|)$ that $V = \eta_H(x)$, so $Y^0 = \frac{H(|x|)}{|x|}$ and $X^0 = \dot{H}(|x|)$.

Integrating (5.64) with $t = |x|$ and $s = |h(x)|$, we see we have

$$\int_{\mathbb{A}} |Dh|^p \geq \int_{\mathbb{A}} \frac{p\left(\eta^2(|h|)+n-1\right)^{\frac{p-2}{2}}\left(\eta^2(|h|)-1\right)}{\eta(|h|)} \frac{|h|^{p-1}}{F^{p-n}(|h|)|x|^{n-1}} |h_N| dx \tag{5.67}$$

$$+ \int_{\mathbb{A}} \frac{p(\eta^2(|h|)+n-1)^{\frac{p-2}{2}}}{\eta(|h|)} \frac{|h|^{p-n}}{F^{p-n}(|h|)} |h_N||h_T|^{n-1} dx \tag{5.68}$$

$$- \int_{\mathbb{A}} (p-1) \left( \frac{p\left(\eta^2(|h|)+n-1\right)^{\frac{p-2}{2}}\left(\eta^2(|h|)-1\right)}{\eta(|h|)} \frac{|h|^{p-1}}{F^{p-n}(|h|)|x|^{n-1}} \right)^{\frac{p}{p-1}} [f(V)]^{-\frac{1}{p-1}} dx \tag{5.69}$$

The integrals on the right-hand side of (5.67) and in (5.68) are free Lagrangians. To finish showing that $\mathcal{E}_p[h] \geq \mathcal{E}_p[h^0]$, we only need study the integral in (5.69).

We will now let

$$c(t,s) = \left( \frac{p\left(\eta^2(s)+n-1\right)^{\frac{p-2}{2}}\left(\eta^2(s)-1\right)s^{p-1}}{\eta(s)F^{p-n}(s)t^{n-1}} \right)^{\frac{p}{p-1}} [f(V)]^{-\frac{1}{p-1}} \tag{5.70}$$

where $V$ is defined by (5.66). We claim that

$$\max\{c(t,s) : r_* \leq s \leq R_*\} = c(t, H(t)). \tag{5.71}$$

To prove our claim, we will show that

$$c_s(t,s) \geq 0 \qquad s \leq H(t) \tag{5.72}$$

$$c_s(t,s) \leq 0 \qquad s \geq H(t) \tag{5.73}$$

We will need to find the sign of $c_s$.

Taking the logarithmic derivative of (5.70) with respect to $s$, we come to

$$\frac{(p-1)c_s(t,s)}{c(t,s)} = p\left[\frac{(p-2)\eta(s)\dot{\eta}(s)}{\eta^2(s)+n-1} + \frac{2\eta(s)\dot{\eta}(s)}{\eta^2(s)-1} - \frac{\dot{\eta}(s)}{\eta(s)} + \frac{p-1}{s} - \frac{(p-n)\dot{F}(s)}{F(s)}\right] - \frac{f'(V)V_s}{f(V)} \tag{5.74}$$

$$= p\left[\frac{(p-1)\eta^4(s)-(p-n-2)\eta^2(s)+n-1}{\eta(s)(\eta^2(s)-1)(\eta^2(s)+n-1)}\dot{\eta}(s) + \frac{(p-1)\eta(s)-(p-n)}{s\eta(s)}\right] \tag{5.75}$$

$$- \frac{(n-1)pP(V)V_s}{(p-1)V(V^2+n-1)(V^2-1)\left(V^2-\frac{n-1}{p-1}\right)} \tag{5.76}$$

We now take the logarithmic derivative of (5.59), yielding

$$\frac{\frac{p-1}{n-1}\eta^3(s)+\eta(s)}{(1-\eta(s))\left(\frac{p-1}{n-1}\eta(s)+1\right)(\eta^2(s)+n-1)}\dot{\eta}(s) = \frac{1}{s}. \tag{5.77}$$

Using this relationship, (5.75) simplifies to give

$$\frac{(p-1)\eta^4-(p-n-2)\eta^2+n-1}{\eta(\eta^2-1)(\eta^2+n-1)}\dot{\eta} + \frac{(p-1)\eta-(p-n)}{s\eta} = \frac{Q(\eta(s))}{s\eta^2(\eta+1)\left(\frac{p-1}{n-1}\eta^2+1\right)} \tag{5.78}$$

To compute (5.76), we begin with the logarithmic derivative of (5.66),

$$\frac{(n-1)P(V)V_s}{V(V^2+n-1)(V^2-1)} = \frac{(n-1)P(\eta(s))\dot{\eta}(s)}{\eta(s)(\eta^2(s)+n-1)(\eta^2(s)-1)} - \frac{(n-1)(p-n)}{s\eta(s)} \tag{5.79}$$

$$= \frac{Q(\eta(s))(\frac{p-1}{n-1}\eta^2(s)-1)}{s\eta^2(\eta+1)\left(\frac{p-1}{n-1}\eta^2+1\right)} \tag{5.80}$$

From here, it is clear that (5.76) simplifies as

$$\frac{(n-1)pP(V)V_s}{(p-1)V(V^2+n-1)(V^2-1)\left(V^2-\frac{n-1}{p-1}\right)}=\frac{(n-1)pQ(\eta(s))(\frac{p-1}{n-1}\eta^2(s)-1)}{(p-1)s\eta^2(\eta+1)\left(\frac{p-1}{n-1}\eta^2+1\right)\left(V^2-\frac{n-1}{p-1}\right)}. \tag{5.81}$$

Plugging (5.81) and (5.78) into (5.74)- (5.76), we find

$$\frac{c_s(t,s)}{c(t,s)}=p\frac{Q(\eta(s))}{s\eta^2(s)(\eta(s)+1)\left(\frac{p-1}{n-1}\eta^2(s)+1\right)}\left(1-\frac{\frac{p-1}{n-1}\eta^2(s)-1}{\frac{p-1}{n-1}V^2-1}\right) \tag{5.82}$$

We now recall that $g(V)=g(\eta(s))\left(\frac{t}{F(s)}\right)^{(n-1)(p-1)}$ and that $g$ is decreasing on $(1,a_{n,p})$. Note $s\le H(t)$ is equivalent to $\frac{t}{F(s)}\ge 1$. Since $\eta(s)\le\alpha_0<a_{n,p}$ we have $g(\eta(s))\le g(V)$ for $s\le H(t)$. This implies that $V<\eta(s)$, so $\frac{\frac{p-1}{n-1}\eta^2(s)-1}{\frac{p-1}{n-1}V^2-1}\ge 1$. Recalling that $Q(\eta(s))<0$ since $\eta(s)\le b_{n,p}$, we see by (5.82) that $c_s(t,s)\ge 0$. Similarly, if $s\ge H(t)$, we will have $V>\eta(s)$, so $c_s(t,s)\le 0$.

This establishes (5.73) and (5.72), proving the claim. Therefore, we have that

$$\int_{\mathbb{A}}\left(\frac{p\left(\eta^2(|h|)+n-1\right)^{\frac{p-2}{2}}\left(\eta^2(|h|)-1\right)}{\eta(|h|)}\frac{|h|^{p-1}}{F^{p-n}(|h|)|x|^{n-1}}\right)^{\frac{p}{p-1}}[f(V)]^{-\frac{1}{p-1}}$$
$$\le\int_{\mathbb{A}}(\eta_H^2(|x|)+n-1)^{\frac{p-2}{2}}((p-1)\eta_H^2(|x|)-(n-1))\left(\frac{H(|x|)}{|x|}\right)^p,$$

with equality holding when $|h(x)|=H(|x|)$. Using this estimate in (5.69) finishes the proof. □

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