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Representation Theory of Orders over Cohen-Macaulay Rings

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ABSTRACT

Orders are a certain class of noncommutative algebras over commutative rings. Originally defined by Auslander and Bridger, an R-order is an R-algebra which is a maximal Cohen-Macaulay R-module. In this thesis we consider orders, Λ, over Cohen-Macaulay local rings R possessing a canonical module, ω_R. In this case a great deal of structure is imposed on Λ.

In Chapter 3 we focus on the use of orders as noncommutative resolutions of commutative local rings. This idea was introduced by Van den Bergh [45] for R Gorenstein and we investigate the generalization to the case where R is Cohen-Macaulay. We show that if an order is totally reflexive over R and has finite global dimension, then R was already Gorenstein. Further, we investigate Gorenstein orders and give a necessary and sufficient condition for the endomorphism ring $\operatorname{End}_R(R \oplus \omega)$ to be a Gorenstein order.

The rest of the thesis focuses on various aspects of the representation theory of orders. We investigate orders which have finite global dimension on the punctured spectrum, but are not necessarily isolated singularities. In this case we are able to prove a generalization of Auslander's theorem about finite CM type [3]. We prove that if an order which satisfies $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda \leqslant n$ possesses only finitely many indecomposable n^{th} syzygies of MCM Λ-modules, then in fact $\operatorname{gldim} \Lambda_\mathfrak{p} \leqslant n + \dim R_\mathfrak{p}$ for all non-maximal primes $\mathfrak{p}$. We are then able to translate this to a condition on R by considering path algebras, since these maintain finiteness of global dimension.

Finally, we consider orders which are true isolated singularities and Iyama's higher Auslander-Reiten theory [27]. We consider the action of τ_n on n-orthogonal subcategories of $\operatorname{CM} \Lambda$ and on n-cluster tilting subcategories. For the former we are able to characterize the projective dimension of duals of modules. For the latter, we provide an obstruction to a module being τ_n-periodic, a question of great interest for the representation theory of orders of finite global dimension.

Representation Theory of Orders over Cohen-Macaulay Rings

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Chapter 1

Introduction

1.1 Commutative Rings

Here we remind the reader of some definitions and facts from commutative ring theory. Throughout, R will denote a commutative Noetherian ring of finite Krull dimension d. When necessary we will write $(R, \mathfrak{m}, k)$ to mean R is a local ring with unique maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$.

Definition 1.1.1. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring and M a finitely generated R-module. A sequence of elements $x_1 \ldots, x_n \in \mathfrak{m}$ is called an *M-regular sequence* provided x_1 is a nonzerodivisor on M and for each $i \geqslant 2$, x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$. The length of the longest M-regular sequence is independent of choice of sequence and is called the *depth* of M denoted $\operatorname{depth}_R M$. A finitely generated module M is called *maximal Cohen-Macaulay* (MCM) if $\operatorname{depth}_R M = \dim(R)$. A ring R is called *Cohen-Macaulay* (CM) if it is maximal Cohen-Macaulay as a module over itself.

In this thesis we will focus on rings which are either Cohen-Macaulay with a canonical module, ω_R, or Gorenstein. A local ring is *Gorenstein* if it has finite injective dimension as a module over itself, a non-local ring is Gorenstein if all of its localizations are Gorenstein local rings; we note that Gorenstein local rings are Cohen-Macaulay having canonical module

$\omega_R \cong R$.

1.2 Summary of Results

This thesis consists of five chapters. Following this introduction, in Chapter 2 we give some background on orders, a special class of algebras over commutative rings. In addition to motivating some of the specific orders on which we concentrate, we prove some homological results for path algebras over commutative rings. This chapter largely consists of some propositions and lemmas which are used in later chapters.

In Chapter 3, we address the notion of noncommutative crepant resolutions. Noncommutative crepant resolutions were defined by Van den Bergh in a program to settle the Bondal-Orlov conjecture that all crepant resolutions of a variety X are derived equivalent. These resolutions represent algebraic analogs of geometric resolutions of singularities. The work of Van den Bergh, Stafford-Van den Bergh, Buchweitz-Leuschke-Van den Bergh, etc. ([17, 44, 45]), deals with the study of noncommutative crepant resolutions over Gorenstein rings–here many examples and strong theorems have been found. The generalization to the Cohen-Macaulay case has proven more difficult, see [19, 20]. For various technical reasons the definition in the Gorenstein case does not extend the way one might hope. In this chapter we introduce a type of noncommutative resolution for Cohen-Macaulay rings and show that their existence actually implies that the base ring is Gorenstein. We then consider the class of Gorenstein orders, which are a noncommutative generalization of Gorenstein rings. Motivated by some examples, we find necessary and sufficient conditions for $\mathrm{End}_R(R \oplus \omega_R)$ to be a Gorenstein order. Furthermore, we are able to apply work of Iyama and Nakajima, [30], to classify exactly when a certain class of endomorphism algebras has finite global dimension.

In Chapter 4, we turn to the question of finite representation type. In the case of commutative rings we focus on finite CM type–the condition that a ring possesses only finitely many non-isomorphic indecomposable MCM modules. We focus on a classical theorem of

Auslander,[3]: a CM local ring of finite CM type must be an isolated singularity. In this chapter we introduce a homological condition on an order which controls the behavior of high syzygies. This allows us to weaken the condition of Auslander's theorem by requiring finiteness of a smaller class of modules. We then provide an example of such orders over Gorenstein local rings and show that finiteness of the class of first syzygies over such an order actually implies that R is an isolated singularity. This is a strengthening of Auslander's theorem in the case of a Gorenstein local ring. The proof adapts work of Huneke-Leuschke, [25], to a noncommutative setting where finite projective dimension is less restrictive.

Lastly, Chapter 5 focuses on higher Auslander-Reiten duality as introduced by Iyama, [27]. Iyama, with both Herschend and Oppermann, has applied this tool to the study of finite dimensional algebras which have finite global dimension, [24, 29, 31]. We focus on a key lemma of Iyama's work and the fact that for algebras of global dimension at most n, τ_n-periodic modules cannot exist. Our main results are generalization of Iyama's key lemma to the case of arbitrary finite Krull dimension and a proof of the converse. Moreover, we show that in fact projective dimension of duals of cluster tilting modules must detect the Krull dimension of the base ring. Finally, we discuss the behavior of the higher AR translation, τ_n, as a functor and give a criterion which prevents a module from being τ_n-periodic.

Chapter 2

Background on Orders

2.1 Background on Orders

Most often, R will be a Cohen-Macaulay local ring with canonical module ω_R. We begin with some definitions.

Definition 2.1.1. Suppose $(R, \mathfrak{m}, k)$ is a local ring.

- An R-algebra Λ is an R-*order* if it is a MCM R-module.
- Denote by $\operatorname{Mod}\Lambda$ the category of left Λ-modules and $\operatorname{mod}\Lambda$ the full subcategory of $\operatorname{Mod}\Lambda$ consisting of finitely generated modules. Unless specified otherwise, when we say M is a Λ-module, we always mean a finitely generated left Λ-module.
- We denote by $\operatorname{CM}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of the maximal Cohen-Macaulay R-modules.
- For a (possibly non-commutative) ring Γ, we will denote by Γ^{op} the opposite ring. If M is an abelian group with a right Γ-module structure, we will say $M \in \operatorname{mod}\Gamma^{op}$ to indicate that M is a left Γ^{op}-module.
- Suppose R is a domain. An R-algebra Λ is *birational* if $\Lambda \otimes_R K \cong M_n(K)$ where K is the fraction field of R.

- An R-algebra Λ is *symmetric* if $\operatorname{Hom}_R(\Lambda, R) \cong \Lambda$ as an Λ-Λ-bimodules.
- Λ is *non-singular* if $\operatorname{gldim}(\Lambda_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.
- We say an order Λ is an *isolated singularity* if $\operatorname{gl.dim}(\Lambda_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$ for all non-maximal prime ideals $\mathfrak{p}$ of R.
- Λ is *homologically homogeneous* if all simple Λ modules have the same projective dimension d over Λ.
- For any ring Γ, we denote by $\operatorname{Proj} \Gamma$ the full subcategory of all projective Γ-modules.
- For any module M, $\operatorname{add} M$ denotes the *additive closure* of M, i.e., the class of modules isomorphic to direct summands of finite direct sums of copies of M.

Remark 2.1.2. We note that for R equidimensional (a mild assumption which holds for e.g., a domain), an R-algebra being homologically homogeneous is equivalent to being a non-singular order, see [45, Section 3] or [19, Section 2] for details.

Given a minimal projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

We set the n^{th} *syzygy* of M to be $\Omega^n M = \ker(P_{n-1} \longrightarrow P_{n-2})$.

We will deal with syzygies a great deal in this thesis. As such, we wish to comment on their behavior over commutative rings. Since exact sequences of Λ-modules are also exact sequences of R-modules, we have the Depth Lemma. This will be utilized several times throughout.

Lemma 2.1.3 (Depth Lemma, [35, Lemma A.4])**.** *Let* $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ *be an exact sequence of* Λ*-modules.*

(1) If $\operatorname{depth}_R W < \operatorname{depth}_R V$*, then* $\operatorname{depth}_R U = \operatorname{depth}_R W + 1$

(2) $\operatorname{depth}_R U \geqslant \min\{\operatorname{depth}_R V, \operatorname{depth}_R W\}$.

(3) $\operatorname{depth}_R V \geqslant \min\{\operatorname{depth}_R U, \operatorname{depth}_R W\}$.

We will also use two convenient consequences of the Depth Lemma.

Lemma 2.1.4. *Let R be a local ring of dimension d and M an R-module. Suppose we have an exact sequence*

$$0 \longrightarrow Y_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

where the X_i are maximal Cohen-Macaulay modules, then

$$\operatorname{depth}_R Y_n = \min\{d, n + \operatorname{depth}_R M\}.$$

In particular if R is CM local, for any module M,

$$\operatorname{projdim}_R M \geqslant \dim R - \operatorname{depth}_R M.$$

If $\operatorname{projdim}_R M < \infty$, then equality holds.

Lemma 2.1.5. *Suppose Λ is an R-order over a CM local ring R of dimension at least 2. Set $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda)$. Let X be a Λ-module. We have*

$$\operatorname{depth}_R X^* \geqslant 2.$$

Proof. Note that if X is projective, the result holds since Λ is and R-order. Suppose X is not projective. Since Λ is a Noetherian ring X has a presentation

$$P_2 \longrightarrow P_1 \longrightarrow X \longrightarrow 0,$$

where P_1, P_0 are projective Λ-modules. Applying $Hom_\Lambda(-,\Lambda)$ we get an exact sequence

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0.$$

Since Λ is an R-order, $\operatorname{depth}_R P_i^* \geqslant 2$ for $i = 0, 1$. Further, $\operatorname{Tr}(-)$ is a duality, so $\operatorname{Tr} X \neq 0$. It follows from the Depth Lemma that $\operatorname{depth}_R X^* \geqslant 2$. □

We will be most interested in Cohen-Macaulay local rings having a canonical module ω_R. For more information on canonical modules, we direct the reader to [15, Section I.3] and [35, Section 11.1], but we include the relevant facts here for convenience.

Definition 2.1.6. Let $(R, \mathfrak{m}, k)$ be a CM local ring. A finitely generated module ω_R is a *canonical module* if it has finite injective dimension, is maximal Cohen-Macaulay and $\dim_k \operatorname{Ext}^d_R(k, \omega_R) = 1$.

It is known that a CM local ring possesses a canonical module if and only if it is the homomorphic image of a Gorenstein local ring; this result is due to Foxby, Reiten, and Sharp [21, 40, 42] . Canonical modules play an important role in Grothendieck local duality, and we recall a long list of their properties.

Proposition 2.1.7 ([35, Theorem 11.5])**.** *Let R be a CM local ring with a canonical module ω_R.*

(i) ω_R is unique up to isomorphism. The ring R is Gorenstein if and only if $\omega_R \cong R$.

(ii) $\operatorname{End}_R(\omega_R) \cong R$. In particular, since R is local, ω_R is an indecomposable module.

1. *Let M be a Cohen-Macaulay module of R and set $t = \operatorname{codepth}_R M := \operatorname{depth}_R R - \operatorname{depth}_R M$. Define $(-)^\vee = \operatorname{Ext}^t_R(-, \omega_R)$.*

   - *$M^\vee$ is also CM of codepth t.*
   - *$\operatorname{Ext}^i_R(M, \omega_R) = 0$ for $i \neq t$.*

- *$M^{\vee\vee}$ is naturally isomorphic to M.*

(iV) The canonical module behaves well with respect to completion, factoring out a regular sequence, and localization.

The most important fact we will use about ω_R is that it is a relatively injective object for $\mathrm{CM}(R)$. This follows from the following more general fact.

Proposition 2.1.8 ([35, Proposition 11.3]). *Let R be a CM local ring and M a finitely generated R-module. Then M is MCM if and only if $\mathrm{Ext}^i_R(M, Y) = 0$ for all $i > 0$ and Y an R-module of finite injective dimension.*

That $\mathrm{Ext}^d_R(k, \omega_R)$ is one-dimensional can be thought of as a normalizing condition. Furthermore, under mild conditions it forces ω_R to be isomorphic to an ideal of R. This will be important later in the proof of Theorem 3.3.5. To state the condition precisely we say R is *generically Gorenstein* if $R_{\mathfrak{p}}$ is a Gorenstein local ring for each minimal prime $\mathfrak{p}$ of R.

Proposition 2.1.9 ([35, Proposition 11.6]). *Let R be a CM local ring and ω a canonical module for R. If R is generically Gorenstein, then ω is isomorphic to an ideal of R and conversely. In this case, ω has constant rank 1, ω is an ideal of pure height 1 (that is every associated prime of ω has height 1) and R/ω is a Gorenstein ring of dimension* $\dim R - 1$.

Canonical modules are important tools for studying the homological behavior of local rings. We wish to study homological behavior of orders, and will thus heavily utilize the following object.

Definition 2.1.10. Let R be a Cohen-Macaulay ring with canonical module ω_R and Λ an R-order. Then the *canonical module* of Λ is $\omega_\Lambda = \mathrm{Hom}_R(\Lambda, \omega_R)$.

As with local rings, if the canonical module is projective, our ring is particularly nice. Thus, we give this condition a name.

Definition 2.1.11. Let Λ be an order over a CM local ring R with canonical module ω_R. If ω_Λ is projective as a left Λ-module, then Λ is called a *Gorenstein order*.

In fact, this condition is symmetric. In other words, if Λ is a Gorenstein R-order, so is Λ^{op}; see [32, Lemma 2.15].

Commutative Gorenstein local rings R are characterized by their canonical module being the rank one free module. It follows from this that a symmetric order over a Gorenstein local ring is a Gorenstein order. This is a key point in the theory of noncommutative crepant resolutions and will be discussed in Chapter 3.

In order to utilize this, we collect some facts about ω_Λ. Most of these follow from facts about ω_R and the tensor-hom adjunction.

Theorem 2.1.12. *Let R be a Cohen-Macaulay local ring with canonical module ω_R and Λ an R-order. Denote by ω_Λ the canonical module of Λ. Let $M \in \operatorname{CM}\Lambda$.*

- $\operatorname{Ext}^i_\Lambda(M, \omega_\Lambda) = 0$ *for $i > 0$.*
- *Denote by $(-)^\vee$ the functor $\operatorname{Hom}_\Lambda(-, \omega_\Lambda)$. Then $(-)^\vee$ is a duality on $\operatorname{CM}\Lambda$.*
- *There is a module $Y \in \operatorname{CM}\Lambda$ such that there is an exact sequence $0 \longrightarrow M \xrightarrow{\varphi} I \longrightarrow Y \longrightarrow 0$ where $I \in \operatorname{add}\omega_\Lambda$.*

Proof. This is largely an application of tensor-hom adjointness to the classical setting of canonical modules, see [35, Chapter 12]. We prove the third assertion for convenience of the reader.

By [35], the second assertion and the fact that M is also an R-module, we start with a projective cover $0 \longrightarrow Y' \longrightarrow P \longrightarrow M^\vee \longrightarrow 0$ over Λ, where Y is necessarily MCM by the Depth Lemma. We then apply $\operatorname{Hom}_R(-, \omega_R)$ and the fact that $\operatorname{Ext}^1_R(M^\vee, \omega_R) = 0$ to get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M^\vee, \omega_R) \longrightarrow \operatorname{Hom}_R(P, \omega_R) \longrightarrow \operatorname{Hom}_R(Y', \omega_R) \longrightarrow 0$$

which is isomorphic to the top row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M^\vee \otimes_\Lambda \Lambda, \omega_R) & \longrightarrow & \mathrm{Hom}_R(P \otimes_\Lambda \Lambda, \omega_R) & \longrightarrow & \mathrm{Hom}_R(Y' \otimes_\Lambda \Lambda, \omega_R) & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_\Lambda(M^\vee, \omega_\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(P, \omega_\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(Y', \omega_\Lambda) & \longrightarrow & 0 \end{array}$$

where the vertical isomorphisms are the tensor-hom adjunction, since ω_Λ is by definition $\mathrm{Hom}_R(\Lambda, \omega_R)$. Setting $Y = \mathrm{Hom}_\Lambda(Y', \omega_\Lambda)$, the bottom row is exactly the exact sequence in the assertion. □

We need several functors to study orders. Let R be a CM local ring with a canonical module ω_R and Λ an R-order. We have the following functors.

- The *canonical dual* $D_d(-) := \mathrm{Hom}_R(-, \omega_R) : \mathrm{CM}\,\Lambda \longrightarrow \mathrm{CM}\,\Lambda^{op}$.
- The *Matlis dual* $D := \mathrm{Hom}_R(-, E)$ where E is the injective hull of the residue field, k, of R. Letting f.l.R denote the full subcategory of $\mathrm{mod}\,R$ consisting of finite length R-modules, $D : \text{f.l.}R \longrightarrow \text{f.l.}R$ is a duality.
- The functor $(-)^* := \mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda$ which gives a duality $(-)^* : \mathrm{add}\,\Lambda \longrightarrow \mathrm{add}\,\Lambda^{op}$.
- The *transpose duality* $\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \longrightarrow \underline{\mathrm{mod}}\Lambda$ given by $\mathrm{Tr}\, M = \mathrm{cok}\, f_1^*$, where $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ is a minimal projective resolution of M.
- Finally, we set $(-)^\dagger = \mathrm{Hom}_R(-, R)$. When R is Gorenstein, we note that $D_d(-) = (-)^\dagger$.

Utilizing the duality $D_d : \mathrm{CM}\,\Lambda \longrightarrow \mathrm{CM}\,\Lambda$ we note that given an exact sequence

$$0 \longrightarrow N \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_i \longrightarrow M \longrightarrow 0$$

with $M, N, X_j \in \mathrm{CM}\,\Lambda$, then we get an exact sequence of Λ^{op}-modules

$$0 \longrightarrow D_d M \longrightarrow D_d X_i \longrightarrow \ldots \longrightarrow D_d X_1 \longrightarrow D_d N \longrightarrow 0.$$

Since $D_d : \mathrm{CM}\,\Lambda \longrightarrow \mathrm{CM}\,\Lambda^{op}$ is an exact duality, a second application returns the initial exact sequence. Thus we get a bijection of abelian groups

$$\mathrm{Ext}^i_{\Lambda}(M,N) \cong \mathrm{Ext}^i_{\Lambda^{op}}(D_dN, D_dM). \tag{2.1.1}$$

We also will define the stable and co-stable categories of $\mathrm{CM}\,\Lambda$, denoted by $\underline{\mathrm{CM}}\,\Lambda$ and $\overline{\mathrm{CM}}\,\Lambda$, respectively. The objects in $\underline{\mathrm{CM}}\,\Lambda$ and $\overline{\mathrm{CM}}\,\Lambda$ are the same as those in $\mathrm{CM}\,\Lambda$. We then define the homomorphism groups as follows. For $X, Y \in \mathrm{CM}\,\Lambda$, set $\mathcal{P}(X,Y)$ to be all of the Λ-homomorphisms $X \longrightarrow Y$ which factor through a projective Λ-module. Set $\mathcal{I}(X,Y)$ to be all of the homomorphisms which factor through a module in $\mathrm{add}\,\omega_\Lambda$. Then we have

$$\underline{\mathrm{Hom}}_{\Lambda}(X,Y) := \mathrm{Hom}_{\underline{\mathrm{CM}}\,\Lambda}(X,Y) = \mathrm{Hom}_{\Lambda}(X,Y)/\mathcal{P}(X,Y)$$
$$\overline{\mathrm{Hom}}_{\Lambda}(X,Y) := \mathrm{Hom}_{\overline{\mathrm{CM}}\,\overline{\mathrm{CM}}\Lambda}(X,Y) = \mathrm{Hom}_{\Lambda}(X,Y)/\mathcal{I}(X,Y).$$

Call $\underline{\mathrm{CM}}\,\Lambda$ the *stable* category of $\mathrm{CM}\,\Lambda$ and $\overline{\mathrm{CM}}\,\Lambda$ the *co-stable* category. For any subcategory $\mathcal{C}$ of $\mathrm{CM}\,\Lambda$ we denote by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ the corresponding subcategories of $\underline{\mathrm{CM}}\,\Lambda$ and $\overline{\mathrm{CM}}\,\Lambda$, respectively.

Finally, we will often be concerned with orders of finite global dimension and the projective dimension of modules. The following technical lemma will be used more than once.

Lemma 2.1.13. *Let A be a ring and M an A-module of finite projective dimension n. Then* $\mathrm{Ext}^n_A(M,A) \neq 0$.

Proof. Let

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M. Suppose $\mathrm{Ext}^n_A(M,A) = 0$. It follows that $\mathrm{Ext}^1_A(\Omega^{n-1}M, A) = 0$, hence the exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \Omega^{n-1}M \longrightarrow 0$$

splits and $\Omega^{n-1}M$ is projective. This is a contradiction. □

2.2 Examples of Orders

In our exposition, we will be concerned with two main examples of orders. One example, that of endomorphism rings, plays a central role in chapter 2 and has long been an object of study in the representation theory of commutative rings. We will review Auslander's *projectivization* and some facts about endomorphism rings over Cohen-Macaulay local rings. In chapter 3, we will use the notion of a path algebra over a commutative local ring. Path algebras of finite acyclic quivers over fields are examples of finite dimensional hereditary algebras. Moreover, path algebras change the global dimension of a ring in a measurable way, and we will exploit this fact to prove a generalization of Auslander's theorem [3]. We will now recall some background about these two types of orders.

2.2.1 Endomorphism Rings

Endomorphism algebras of modules are a convenient way to rephrase questions about a module M into questions about a ring $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$. One of the most useful techniques involves translating summands of the module M into projective modules. This is known as *projectivization*. We direct readers to [14, II.2] which treats the Artin algebra case. We will use the generalization of these results to commutative Notherian rings.

Proposition 2.2.1. *Let R be a commutative Noetherian ring and M an R-module. Let $\Gamma = \operatorname{End}_R(M)$ and $e_M(-) := \operatorname{Hom}_R(M, -) : \operatorname{mod} R \longrightarrow \operatorname{mod} \Gamma$ be the evaluation functor.*

- $e_M : \operatorname{Hom}_R(Z, X) \longrightarrow \operatorname{Hom}_\Gamma(e_M(Z), e_M(X))$ *is an isomorphism for $Z \in \operatorname{add} M$ and $X \in \operatorname{mod} R$.*
- *If $X \in \operatorname{add} M$ then $e_M(X) \in \operatorname{Proj} \Gamma$.*
- $e_M : \operatorname{add} M \longrightarrow \operatorname{Proj} \Gamma$ *is an equivalence of categories.*

One of the main reasons one might study endomorphism rings is their tendency to have finite global dimension, [5]. Due to this fact, endomorphism algebras have popped up in a host of places one may want a ring to have finite global dimension; in particular, they have appeared in the work of Van den Bergh[45], Leuschke[33], and Buchweitz-Leuschke-Van den Bergh[17] and others on noncommutative resolutions. These rings will be discussed in Chapter 3, where we work to extend the notion of an noncommutative crepant resolution to the case of a Cohen-Macaulay ring with canonical module.

In general, endomorphism rings are also a convenient place to find algebras which are isolated singularities. Indeed, if R is an isolated singularity, then for any MCM R-module M and non-maximal prime ideal $\mathfrak{p}$, we have that $\operatorname{End}_R(M)_\mathfrak{p} \cong \operatorname{End}_{R_\mathfrak{p}}(M_\mathfrak{p}) \cong \operatorname{End}_{R_\mathfrak{p}}(R_\mathfrak{p}^n)$ which is a matrix ring over $R_\mathfrak{p}$ and of global dimension equal to $\dim R_\mathfrak{p}$.

Finding conditions which guarantee an endomorphism ring is MCM over R seems to be quite difficult. This topic will not be addressed in this thesis, but the interested reader can turn to [34, Section M] for some details.

2.2.2 Path Algebras over Commutative Rings

The use of quivers was introduced by Gabriel in [22]. They are a foundational tool in the study of representation theory of finite dimensional algebras over fields. They are also useful for computations in homological algebra; see for example [19, Section 5.3]. As far as we can tell, very little is known about path algebras over more general commutative local rings. The following background is adapted from the treatment of path algebras over fields in [1]. Many of these results are likely well-known to experts, but lacking a good reference we include proofs.

Definition 2.2.2. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called vertices) and Q_1 (whose elements are called arrows), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $a \in Q_1$ its source $s(a) \in Q_0$ and its target $t(a) \in Q_0$ respectively.

We abbreviate $Q = (Q_0, Q_1, s, t)$ by Q. We say that Q is *finite* if Q_0 and Q_1 are finite sets.

Example 2.2.3. Consider the quiver Q

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Here, $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{a, b\}$. We have $s(a) = 1, t(a) = 2$, and $s(b) = 2, t(b) = 3$.

Definition 2.2.4. A *path* of length $l \geqslant 1$ is a sequence of l arrows $a_1a_2...a_l$ such that for $1 \leqslant i \leqslant l-1$, $t(a_i) = s(a_{i+1})$. We will also consider paths of length 0 at a vertex v. Such a path is called the *trivial path* at vertex v and is denoted e_v.

Sometimes we denote a path of length m by p and write $p = a_1...a_m$ for $a_i \in K$ or write $l(p) = m$. In 2.2.3 the only path of length 2 in Q is ab. The paths of length 1 are a and b, and the paths of length 0 are e_1, e_2, and e_3.

Definition 2.2.5. Let R be a commutative Noetherian ring and Q a quiver. The path algebra RQ of Q over R is the free module on the basis the set of all paths $a_la_{l-1}...a_1$ of length $l \geqslant 0$ in Q. The product of two basis vectors (i.e., paths) $b_k...b_1$ and $a_l...a_1$ of RQ is defined by

$$(b_k...b_1) \cdot (a_l...a_1) = b_k...b_1a_l...a_1$$

if $t(a_l) = s(b_1)$ and 0 otherwise, i.e., the product of arrows $b \cdot a$ is nonzero if and only if b leaves the vertex where a arrives. Multiplication is extended to linear combinations of basis elements R-linearly.

It is also useful to work with quotients of path algebras, and thus we record the following fact which will allow us to do computations more easily.

Proposition 2.2.6. *Let R be an algebra over a commutative local ring S . Let Q a quiver,* *and I an ideal in SQ. Then there is an isomorphism of S-algebras*

$$RQ/_{IRQ} \cong SQ/_{I} \otimes_S R.$$

Proof. We begin with the case that $I = 0$. Define a map $\Phi : SQ \times S \longrightarrow SQ$ via $\Phi(p, r) = rp$ for a path p and extending linearly. This map is clearly S-bilinear, and hence induces a map $\Phi : SQ \otimes_S R \longrightarrow RQ$. This map is onto since any basis element of RQ (i.e., a path in Q) say p, is $\Phi(p \otimes 1)$. We note that any element of $SQ \otimes R$ can be written as $\sum_{i=1}^{n} p_i \otimes s_i$ for paths p_i. Now, if

$$\Phi\left(\sum_{i=1}^{n}(p_i \otimes s_i)\right) = s_1p_1 + s_2p_2 + \cdots + s_np_n = 0,$$

it must be that $s_i = 0$ for all i, since the paths form a basis over S for SQ.

We now move to the case that I is an ideal. We note

$$S \otimes_S SQ/I \cong R \otimes_S SQ \otimes_{SQ} SQ/I \cong RQ \otimes_{SQ} SQ/I \cong RQ/IRQ,$$

where the second isomorphism follows from the $I = 0$ case. □

We remark that, in this thesis, we will focus on commutative rings which are Cohen-Macaulay. Thus, since RQ is a free R-module, RQ is maximal Cohen-Macaulay and thus an order. Finding conditions which guarantees that RQ/I is a free (or even MCM) R-module would also be an interesting question.

2.2.3 Homological behavior of Path Algebras

The main theorem of Chapter 3, Theorem 4.3.1, is homological in nature. As such, here we collect some background on the homological behavior of path algebras.

Proposition 2.2.7 ([18, Corollary IX.2.7])**.** *Let R be a local commutative ring and Λ and Γ*

be R-algebras which are free as R-modules. Suppose M is a right Λ-module and N is a left Γ-module. Then

$$\operatorname{projdim}_{\Lambda\otimes_R\Gamma} M \otimes_R N = \operatorname{projdim}_\Lambda M + \operatorname{projdim}_\Gamma N.$$

This proposition allows us to easily deduce the following fact.

Lemma 2.2.8. *Let Λ and Γ be orders over a regular local ring R and suppose* $\operatorname{gldim}\Lambda = \infty$*. Then* $\operatorname{gldim}(\Lambda \otimes_R \Gamma) = \infty$*.*

Proposition 2.2.9. *Let Q be a quiver without oriented cycles. Let R be a regular local ring of dimension d and RQ the path algebra of Q over R. Then* $\operatorname{gldim} RQ = d + 1$*.*

The proof requires a few lemmas; this is probably well known, but lacking a good reference we include a proof here for convenience. First we introduce some definitions.

Definition 2.2.10. Let R be a commutative Noetherian ring and Q an acyclic quiver.

- A *representation* of a quiver Q over R is an object $V = (V_x, \varphi_\alpha)_{x\in Q_0,\alpha\in Q_1}$ where V_x is a finitely generated R-module and φ_α is a R-linear homomorphism.
- Denote by $\operatorname{mod} RQ$ the category of representations of Q.
- Denote by $\mathcal{P}(RQ)$ the full subcategory of $\operatorname{mod} RQ$ consisting of the representations with finitely generated free R-modules at the vertices.
- For a vertex $x \in Q_0$, let $P(x) = (P(x)_y, \varphi_\alpha)$ where $P(x)_y$ is a free R-module on the basis of paths from x to y, and the φ_α are multiplication by the paths.

It is well-known that the category of representations of Q over R is equivalent to the category of finitely generated modules over RQ, see [1, Section III.2] for details. For a representation $V = (V_x, \varphi_\alpha)$ denote the corresponding module by $\widetilde{V}$, i.e., $\widetilde{V} = \bigoplus_{x\in Q_0} V_x$ as a R-module.

Lemma 2.2.11. *For any $x \in Q_0$, $\widetilde{P(x)}$ is projective in* $\operatorname{mod} RQ$.

Proof. Note that

$$RQ = \bigoplus_{p \text{ a path}} Rp.$$

We wish to show $\widetilde{P(x)} = RQe_x$, where e_x is the idempotent corresponding to x. Indeed, the equivalence assigns to $P(x)$ the sum $\bigoplus_{y \in Q_1} P(x)_y$. Multiplying on the right by e_x kills all summands of

$$RQ = \bigoplus_{p \text{ a path in } Q} Rp$$

which do not correspond to a path originating at x, and is the identity otherwise. Thus

$$RQe_x = \bigoplus_{p \text{ a path from } x \text{ to } y} Rp \cong \widetilde{P(x)}.$$

Now, since RQe_x is clearly a projective module by the above direct sum decomposition, we have that $\widetilde{P(x)}$ is projective and so $P(x)$ is a projective object in $\operatorname{mod} RQ$. □

Then Proposition 2.2.9 follows from the following lemma.

Lemma 2.2.12. *Let R be a regular local ring of dimension d, and Q an acyclic quiver.*

1. *Any $M \in \mathcal{P}(RQ)$ has a projective resolution of length 1 by direct sums of representations of the form $P(x)$.*

2. *For any projective resolution over RQ of a module $M \in \operatorname{mod} RQ$, the d^{th} syzygy of M is in $\mathcal{P}(RQ)$. If M is of finite length over R, then $\Omega^t M$ is not MCM over R for any $t < d$.*

Proof. (1) This is exactly the content of [41, Theorem 2.15].

(2) Let $M \in \operatorname{mod} RQ$. We begin by resolving M. Since kernels are computed at vertices, the depth at each vertex is maximal by the d^{th} iteration. Since the global dimension of R is d, the d^{th} syzygy of M is in $\mathcal{P}(RQ)$. We can resolve this d^{th} syzygy in one step by (1).

Hence, this gives a projective resolution of length $d+1$ over RQ. Now, suppose M has finite length over R. It follows that the R-module attached to each vertex of Q has finite length. Thus by the Depth Lemma, we must resolve at least d steps to achieve a syzygy which is in $\mathcal{P}(RQ)$. □

Our main concern with path algebras will be that their global dimension is finite provided the global dimension of R is finite. The following proposition shows that the converse also holds.

Proposition 2.2.13. *If R is a non-regular local ring and Q is an acyclic quiver, then* $\operatorname{gldim} RQ = \infty$.

Proof. Since R is commutative of infinite global dimension, there is actually a module M of infinite global dimension. Let $\overline{M}$ denote the module corresponding to a representation of RQ with a copy of M at each vertex and the zero map for each arrow. As an R-module $\widetilde{M} \cong \bigoplus_{i=1}^{n} M$ where $n = |Q_0|$. Hence, $\operatorname{projdim}_R \widetilde{M} = \infty$. But, since all indecomposable projective modules over RQ are of the form Re_x for some $x \in Q_0$, any projective RQ-module is also a projective R-module. Hence a projective resolution over RQ is also a projective resolution over R. It follows that $\widetilde{M}$ cannot have a finite length projective resolution over RQ and hence $\operatorname{gldim} RQ = \infty$. □

2.3 Background on noncommutative crepant resolutions

For a more thorough background of the origins of noncommutative (crepant) resolutions, see [34]. In algebraic geometry, a resolution of singularities is a central tool. A *resolution of singularities* for a singular algebraic variety X is a non-singular variety $\widetilde{X}$ together with a map $\pi : \widetilde{X} \longrightarrow X$ which is proper and birational. For a variety X, denote by ω_X the canonical sheaf; we come to the following definition.

Definition 2.3.1. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X. Then π is called *crepant* if $\pi^*\omega_X = \omega_{\widetilde{X}}$.

The technical details of crepant resolutions are geometric, and not immediately relevant to this thesis. Instead, we provide some informal motivation for the notion of crepancy, mostly paraphrased from [34]. Crepancy is a way of relating the method for getting a sheaf on $\widetilde{X}$ from one on X via Hom and the method for doing so via $\otimes$. In the world of Cohen-Macaulay rings, we think of an extension $R \longrightarrow S$ where S is module finite over R. The module achieved by "co-inducing" ω_R, i.e., $\mathrm{Ext}^t_R(S, \omega_R)$ where $t = \dim R - \dim S$, is a canonical module for S, yet the module "induced" from ω_R, namely $S \otimes_R \omega_R$, need not be. Crepancy is a way of remedying this situation. In other words, we are demanding $\omega_S \cong S \otimes_R \omega_R$. If we consider that Gorenstein rings have free canonical module we get the fact that for a Gorenstein variety X, a crepant resolution $\widetilde{X}$ is also Gorenstein.

In 2004, Van den Bergh defined the notion of a noncommutative crepant resolution of a ring R. This is a certain endomorphism ring of a reflexive R-module, see definition 3.2.1. The goal of this was to extend the algebraic side of algebraic geometry and investigate the Bondal-Orlov conjecture, which asserts that any two crepant resolutions of a variety X are derived equivalent.

For Gorenstein rings R, there is a natural notion of crepancy–$\mathrm{End}_R(M)$ should be a maximal Cohen-Macaulay R-module. Attempts to generalize the definition of noncommutative crepant resolution to Cohen-Macaulay rings has met with some difficulty. Chapter 3 of this thesis considers a new definition and some obstructions.

2.4 Gorenstein Projective Modules

Let Λ be a Noetherian ring. In [6], Auslander and Bridger introduce the notion of a Gorenstein projective module. It is also referred to as a totally reflexive module.

Definition 2.4.1. A Λ-module M is called *Gorenstein projective (or totally reflexive)* if the natural map $M \longrightarrow M^{**}$ is an isomorphism and $\mathrm{Ext}^i_\Lambda(M, \Lambda) = \mathrm{Ext}^i_{\Lambda^{op}}(M^*, \Lambda) = 0$ for all $i > 0$. Equivalently, M is isomorphic to $\mathrm{cok}\,\varphi$ where φ is a map in an exact sequence of projective Λ-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \stackrel{\varphi}{\longrightarrow} P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

which remains exact when dualized

$$\cdots \longrightarrow P_{-1}^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots .$$

From the second characterization, it is clear that Gorenstein projective modules are always infinite syzygies. In Chapter 4 we show if Λ is a n-canonical order, then n^{th} syzygies of maximal Cohen-Macaulay Λ-modules are Gorenstein projective, and hence are infinite syzygies.

In general, a great deal of work has been done on studying Gorenstein-projectives. They will play a role in both chapters 3 and 4 of this thesis. As such, we will mention a few important facts.

Over Gorenstein commutative rings of finite Krull dimension, Gorenstein projective modules are precisely the maximal Cohen-Macaulay modules. In [16], Buchweitz defines the singularity category of an arbitrary ring and shows that for a Gorenstein ring this category is triangle equivalent to the stable category of Gorenstein-projective modules. Additionally, a large body of work exists on studying the existence of non-trivial Gorenstein-projective modules and the number of indecomposable Gorenstein-projective modules for a ring, called the Gorenstein type.

In [16], Buchweitz works with rings, Λ, which are not necessarily commutative. He defines a notion of Gorenstein for such rings and then defines *maximal Cohen-Macaulay* Λ-modules, denoted $\mathrm{MCM}(\Lambda)$, to be what we refer to as Gorenstein-projective or totally reflexive modules. We wish to note that our definition of modules in $\mathrm{CM}\,\Lambda$ does not agree with this definition for general noncommutative rings. In particular, while Gorenstein-projectives are notoriously difficult to find over a general ring, our definition of maximal Cohen-Macaulay

Λ-modules are easy to produce–as long as R is CM local we can simply take any high enough syzygy M over Λ and by the Depth Lemma it will be in $\operatorname{CM}\Lambda$.

Chapter 3

Gorenstein and Totally Reflexive Orders as NCCRs

3.1 The Gorenstein Case

In [34], Leuschke addresses the properties we would like from a noncommutative resolution of singularites. We'd like it to be symmetric, birational and non-singular. In the case where R is Gorenstein, these conditions give a concrete description of these orders:

Theorem 3.1.1. [34, Theorem 2] *Let R be a Gorenstein normal domain of dimension d and Λ a module finite R-algebra. The following are equivalent:*

(i) Λ *is a symmetric birational R-order and has finite global dimension.*

(ii) $\Lambda \cong \mathrm{End}_R(M)$ *for some reflexive R-module M, and Λ is homologically homogenous.*

(iii) $\Lambda \cong \mathrm{End}_R(M)$ *for some reflexive R-module M, Λ is an R-order and* $\mathrm{gldim}(\Lambda) < \infty$.

Motivated by this theorem we have the following definition.

Definition 3.1.2. A *noncommutative crepant resolution of singularities* (NCCR) of a d-dimensional Gorenstein normal domain R is a ring $\Lambda = \mathrm{End}_R(M)$ for a reflexive module M such that Λ is an R-order and has finite global dimension.

To provide further motivation of this definition, we will supply a few examples. One of the first examples of such objects come from the McKay correspondence, [37]. This is a way of connecting the representation theory of finite subgroups $G \subset GL(n, k)$ for $n \geqslant 2$ and k an algebraically closed field of characteristic relatively prime to $|G|$. The main points are the following two theorems, both due to Auslander.

Theorem 3.1.3 ((Auslander))**.** *The twisted group ring $S\#G$, where $S = k[[x_1, \ldots, x_n]]$ and G is a finite group of linear automorphisms of S with order invertible in k, has finite global dimension equal to n.*

This is connected to the world of NCCRs via the following theorem.

Theorem 3.1.4. [2,4] *Let $S = k[[x_1, \ldots, x_n]]$, $n \geqslant 2$, $G \subset GL(n, k)$ a finite subgroup acting on S with $|G|$ invertible in k. Set $R = S^G$. If G contains no non-trivial pseudo-reflections then $S\#G \cong \operatorname{End}_R(S)$. In particular, $\operatorname{End}_R(S)$ has finite global dimension equal to n and is isomorphic to a finite sum of copies of S and hence is MCM over R.*

Yet another natural example of NCCRs arises from a generalization of a theorem of Auslander [5].

Theorem 3.1.5. [26,33,39] *Let R be a CM local ring of finite representation type and let M be a representation generator of R. The ring $\operatorname{End}_R(M)$ has global dimension at most $\max\{2, d\}$ with equality in the case $d > 2$. More precisely, $\operatorname{projdim}_{\operatorname{End}_R(M)} S = 2$ for all simple $\operatorname{End}_R(M)$ modules except the one corresponding to R, which has projective dimension d.*

It should be remarked that this gives a way of constructing noncommutative crepant resolutions for 2 dimensional rings of finite type. But, if $\dim R \geqslant 3$ then for a representation generator M, $\operatorname{End}_R(M)$ is never homologically homogenous, and thus $\operatorname{End}_R(M)$ is not a noncommutative crepant resolution.

The following theorems of Van den Bergh and Van den Bergh-Stafford show how this definition mirrors the geometric case and that it influences the singularities of R.

Theorem 3.1.6. [45, Theorem 6.6.3] *Let R be a Gorenstein normal domain which is a finitely generated k-algebra with k an algebraically closed field of characteristic zero. Assume R is three-dimensional and has terminal singularities. Then R has a noncommutative crepant resolution if and only if* $\operatorname{Spec} R$ *has a commutative crepant resolution.*

Theorem 3.1.7. [44, Theorem 1.1] *Let Δ be a homologically homogeneous k-algebra with k algebraically closed and of characteristic zero, then $Z = Z(\Delta)$ has at most rational singularities.*

In particular, if a normal affine k-domain R has a noncommutative crepant resolution then it has rational singularities.

Remark 3.1.8. Note in [44] that the definition of a noncommutative crepant resolution is any homologically homogeneous ring of the form $\Delta = \operatorname{End}_R(M)$ for M reflexive and finitely generated. In the situation where R is not Gorenstein, this is stronger than Definition 3.1.2, which is why this assumption is not needed in Theorem 3.1.7. In the case where R is Gorenstein, any Λ satisfying Definition 3.1.2 is homologically homogenous and so the theorem remains true with our definition.

Remark 3.1.9. The key points here are the fact that over a Gorenstein ring of dimension d, a symmmetric order of finite global dimension is actually non-singular, and in fact even homologically homogeneous [45, Lemma 4.2]. This very strong result does not hold in the non-Gorenstein case.

Example 3.1.10. Let k be an infinite field and let R be the complete (2,1)-scroll, that is, $R = k[[x, y, z, u, v]]/I$ with I the ideal generated by the 2×2 minors of $\left(\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right)$. Then, R is a 3-dimensional CM normal domain of finite CM type [49, 16.12]. It is known $\Gamma = \operatorname{End}_R(R \oplus \omega)$ is MCM over R, and Γ is symmetric since it is an endomorphism ring over a normal domain [32, Lemma 2.10]. But, Smith and Quarles have shown $\operatorname{gldim}(\Gamma) = 4$ [43] while $\dim R = 3$. Thus Γ is a symmetric R-order of finite global dimension but it is not non-singular, thus Γ does not provide a NCCR.

The work in the Gorenstein case has largely been in producing these resolutions, and there are many results in this direction in [17, 45].

3.2 The non-Gorenstein Case

We now assume only that R is a Cohen-Macaulay local normal domain with canonical module ω_R. Following [19, 20, 32], we define

Definition 3.2.1. A *noncommutative crepant resolution* of a Cohen-Macaulay normal domain, R, is $\Lambda = \operatorname{End}_R(M)$ for a reflexive R-module M such that Λ is MCM and of finite global dimension. A *noncommutative resolution* of R is $\Lambda = \operatorname{End}_R(M)$ for a reflexive module M such that Λ is of finite global dimension.

But, by example 3.1.10, we know that this definition does not nececssarily guarantee a non-singular order.

In another example, we see that $\operatorname{End}_R(R \oplus \omega)$ does yield a noncommutative crepant resolution. This is the only other known non-Gorenstein ring of dimension 3 with finite CM type [34, Example P.4]:

Example 3.2.2. Let $R = k[[x^2, xy, xz, y^2, yz, z^2]]$. Then, R is known to have finite CM type [49, 16.10] with indecomposable MCM modules R, $\omega \cong (x^2, xy, xz)$ and $M := \operatorname{syz}_R(\omega)$. The ring $A := \operatorname{End}_R(R \oplus \omega \oplus M)$ has global dimension 3, but it is not MCM. Indeed, A has depth 2, as both $\operatorname{Hom}_R(M, R)$ and $\operatorname{Hom}_R(M, M)$ have depth 2.

In [33, Example 3.2], Leuschke points out a way to fix this example: $\Lambda = \operatorname{End}_R(R \oplus \omega)$ is in fact a noncommutative crepant resolution. This is because $R \oplus \omega \cong k[[x, y, z]]$ and so $\operatorname{End}_R(R \oplus \omega)$ is isomorphic to the twisted group ring $k[[x, y, z]]\#\mathbb{Z}_2$. This is known to have global dimension 3 and be MCM over R [49, Ch. 10].

It would be helpful to have an analog of Theorem 3.1.1 to produce examples. In order to rescue some of the results from the prior case, we strengthen the hypotheses on $\Lambda =$

$\mathrm{End}_R(M)$. Since the crepant condition (i.e., that Λ is maximal Cohen-Macaulay) can be seen as a type of symmetry condition, one might hope to impose more stringent symmetry requirements on Λ. Since MCM is equivalent to totally reflexive for Gorenstein rings, but weaker in general, we strengthen the assumptions on Λ.

Definition 3.2.3. A *strong NC resolution* of a CM normal domain R is $\Lambda = \mathrm{End}_R(M)$ for a reflexive R-module M such that Λ is totally reflexive over R and of finite global dimension.

Remark 3.2.4. This definition agrees with the original definition in the Gorenstein case since over a Gorenstein ring the totally reflexive modules are exactly the MCM modules.

This definition turns out to be quite strong. In general, the Ext-vanishing assumed on Λ along with finite global dimension force strong homological conditions on R.

Proposition 3.2.5. *Let $(R, \mathfrak{m}, k)$ be a CM local ring, and Λ a module-finite R-algebra such that $\Lambda^\dagger = \mathrm{Hom}_R(\Lambda, R)$ has finite injective dimension as a left Λ-module. Additionally suppose $\mathrm{Ext}^i_R(\Lambda, R) = 0$ for all $i > 0$. Then R is Gorenstein.*

Before the proof, we need the following lemma:

Lemma 3.2.6. *Let R be a Cohen-Macaulay normal domain and Λ an R-algebra such that $\mathrm{Ext}^n_R(\Lambda, R) = 0$ for all $n > 0$. For all $i > 0$ and all left Λ-modules B , we have that*

$$\mathrm{Ext}_R(B, R) \cong \mathrm{Ext}^i_\Lambda(B, \mathrm{Hom}_R(\Lambda, R)).$$

Proof. This follows from a well-known change of rings spectral sequence available in [18, Chapter XVI, Section 4], but we include a straight-forward proof in order to make the details clearer. We begin with an injective resolution of R over itself.

$$Q^\bullet : 0 \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \dots \longrightarrow Q^i \longrightarrow Q^{i+1} \longrightarrow \dots$$

We know that $\mathrm{Hom}_R(\Lambda, -)$ takes injective R-modules to injective Λ-modules. Thus it follows that

$$\mathrm{Hom}_R(\Lambda, Q^\bullet) : \mathrm{Hom}(\Lambda, Q^0) \longrightarrow \mathrm{Hom}(\Lambda, Q^1) \longrightarrow \dots$$

is a complex of injective Λ-modules. Further, this sequence is acyclic since $\mathrm{Ext}^i_R(\Lambda, R) = 0$, thus this is an injective resolution of $\Lambda^\dagger = \mathrm{Hom}_R(\Lambda, R)$ over Λ. Thus we have that

$$\mathrm{Ext}^i_\Lambda(B, \Lambda^\dagger) = H^i(\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_R(\Lambda, Q^\bullet))).$$

On the other hand, tensor-hom adjointness gives us, for each i, a natural isomorphism

$$\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_R(\Lambda, Q^i)) \cong \mathrm{Hom}_R(\Lambda \otimes_\Lambda B, Q^i) \cong \mathrm{Hom}_R(B, Q^i)$$

which gives an isomorphism of complexes

$$\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_R(\Lambda, Q^\bullet)) \cong \mathrm{Hom}_R(B, Q^\bullet).$$

Thus we have that

$$\mathrm{Ext}^i_\Lambda(B, \Lambda^\dagger) = H^i(\mathrm{Hom}_R(B, Q^\bullet)) = \mathrm{Ext}^i_R(B, R)$$

and the claim is established. □

And now to prove the Proposition.

Proof of Proposition 3.2.5. We must only note that $\Lambda/\mathfrak{m}\Lambda$ is a finite-dimensional vector space over k, and we then have that $\mathrm{Ext}^i_R(k, R)$ is a summand of $\mathrm{Ext}^i_R(\Lambda/m\Lambda, R)$. Since $\Lambda^\dagger$ has finite injective dimension,we have that

$$\mathrm{Ext}^i_R(\Lambda/\mathfrak{m}\Lambda, \Lambda^\dagger) = 0$$

for i sufficiently large. Then by Lemma 3.2.6 we have that $\mathrm{Ext}^i_R(\Lambda/\mathfrak{m}\Lambda, R)$ and hence $\mathrm{Ext}^i_R(k, R)$ is zero for all i sufficiently large. □

Corollary 3.2.7. *If R is a CM local normal domain possessing a strong NC resolution Λ, then R is Gorenstein.*

3.3 Gorenstein Orders

When R is Gorenstein, one of the crucial components to Theorem 3.1.1 is that a symmetric R-order Λ of finite global dimension has many desirable homological properties. We remind the reader of the definition of a Gorenstein order.

Definition 3.3.1. Let R be a d-dimensional CM local normal domain with canonical module ω_R. An R-order Λ is *a Gorenstein order* if $\omega_\Lambda := \mathrm{Hom}_R(\Lambda, \omega_R)$ is a projective left Λ-module.

If we demand our order $\Lambda := \mathrm{End}_R(M)$ be a Gorenstein order and of finite global dimension, the following implies we rescue Theorem 3.1.7.

Theorem 3.3.2. [32, Proposition 2.17] *Let Λ be an order over a Cohen-Macaulay local ring R with canonical module ω_R. The following are equivalent:*

(i) Λ *is non-singular.*

(ii) Λ *is Gorenstein and of finite global dimension.*

(iii) $\mathrm{CM}\,\Lambda = \mathrm{Proj}\,\Lambda$.

We begin by recalling the final assertion of Example 3.2.2.

Example 3.3.3. Let $R = k[[x^2, xy, xz, y^2, yz, z^2]]$. As before, we know $\Lambda = \mathrm{End}_R(R \oplus \omega)$ is a noncommutative crepant resolution and hence is a non-singular R-order. In particular, by Theorem 3.3.2, $\mathrm{End}_R(R \oplus \omega)$ is a Gorenstein R-order.

In view of this, the following question is motivated.

Question 3.3.4. *For a Cohen-Macaulay normal domain R with canonical module ω_R, when is $\operatorname{End}_R(R \oplus \omega_R)$ a Gorenstein order of finite global dimension?*

Theorem 3.3.5. *Suppose R is a henselian local ring and I is an indecomposable ideal which contains a nonzerodivisor. Then $\operatorname{End}_R(R \oplus I) \cong \operatorname{Hom}_R(\operatorname{End}_R(R \oplus I), I)$ as left $\operatorname{End}_R(R \oplus I)$-modules if and only if $I \cong I^\dagger$ as R-modules.*

Proof. ($\Rightarrow$) Let $\Lambda = \operatorname{End}_R(R \oplus I)$. We see,

$$\Lambda = \begin{pmatrix} \operatorname{Hom}_R(R,R) & \operatorname{Hom}_R(I,R) \\ \operatorname{Hom}_R(R,I) & \operatorname{Hom}_R(I,I) \end{pmatrix} \cong \begin{pmatrix} R & I^\dagger \\ I & \operatorname{End}_R(I) \end{pmatrix}.$$

The bimodule structure on $\operatorname{Hom}_R(\Lambda, I)$ is given by taking $\operatorname{Hom}_R(-, I)$ in each component and taking the transpose. Thus we have

$$\operatorname{Hom}_R(\Lambda, I) = \begin{pmatrix} I & \operatorname{End}_R(I) \\ \operatorname{Hom}_R(I^\dagger, I) & I \end{pmatrix}.$$

Since R and I are indecomposable, when we decompose as left modules, we take the column vectors. Thus if $\Lambda \cong \operatorname{Hom}_R(\Lambda, I)$ one of the following must hold

$$\begin{pmatrix} I \\ \operatorname{Hom}_R(I^\dagger, I) \end{pmatrix} \cong \begin{pmatrix} R \\ I \end{pmatrix}$$
$$\begin{pmatrix} I \\ \operatorname{Hom}_R(I^\dagger, I) \end{pmatrix} \cong \begin{pmatrix} I^\dagger \\ \operatorname{End}_R(I) \end{pmatrix}.$$

Thus, either $I \cong R$ or $I \cong I^\dagger$. In either case, the result holds.

($\Leftarrow$) We wish to show that, $\mathrm{Hom}_R(\Lambda, I)$ is a free left Λ-module. We identify

$$\Lambda = \begin{pmatrix} \mathrm{Hom}_R(R,R) & \mathrm{Hom}_R(I,R) \\ \mathrm{Hom}_R(R,I) & \mathrm{Hom}_R(I,I) \end{pmatrix} \cong \begin{pmatrix} R & I^\dagger \\ I & \mathrm{End}_R(I) \end{pmatrix}$$

as a ring, and $\Lambda = R \oplus I^\dagger \oplus I \oplus \mathrm{End}_R(I)$ as an R-module. This allows us to identify

$$\mathrm{Hom}_R(\Lambda, I) = \mathrm{Hom}_R(R,I) \oplus \mathrm{Hom}_R(I^\dagger, I) \oplus \mathrm{Hom}_R(I,I) \oplus \mathrm{Hom}_R(\mathrm{End}_R(I), I)$$

where we see that the action of Λ on $\mathrm{Hom}_R(\Lambda, I)$ is given by $(\lambda \cdot g)(\eta) = g(\eta \cdot \lambda)$ for $\lambda \in \Lambda$.

We choose an isomorphism $\varphi : I^\dagger \longrightarrow I$ and show that $f = \begin{bmatrix} 0 & \varphi & 1 & 0 \end{bmatrix}$ is a basis for the left Λ-module Λ^I. Indeed, suppose we have a map $g \in \mathrm{Hom}_R(\Lambda, I)$, i.e.,

$$g = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix} \in \mathrm{Hom}_R(R,I) \oplus \mathrm{Hom}_R(I^\dagger, I) \oplus \mathrm{Hom}_R(I,I) \oplus \mathrm{Hom}_R(\mathrm{End}_R(I), I).$$

We wish to show that there is a

$$\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \Lambda$$

so that

$$g(\eta) = \lambda \cdot f(\eta) = f(\eta \cdot \lambda) \tag{3.3.1}$$

for all $\eta \in \Lambda$ and $f = \begin{bmatrix} 0 & \varphi & 1 & 0 \end{bmatrix}$. Set

$$\eta = \begin{pmatrix} r & f \\ a & \sigma \end{pmatrix}$$

for $r \in R$, $f \in I^\dagger$, $a \in I$ and $\sigma \in \mathrm{End}_R(I)$. Computing the left hand side of (3.3.1) we get

$$g(\eta) = g_1(r) + g_2(f) + g_3(a) + g_4(\sigma).$$

The right hand side becomes

$$\begin{aligned} f(\eta \cdot \lambda) =& f \begin{pmatrix} r\lambda_1 + f\lambda_3 & r\lambda_2 + f\lambda_4 \\ a\lambda_1 + \sigma\lambda_3 & a\lambda_2 + \sigma\lambda_4 \end{pmatrix} \\ =& \varphi(r\lambda_2 + f\lambda_4) + a(\lambda_1) + \sigma(\lambda_3) \\ =& \varphi(r\lambda_2) + \varphi(f\lambda_4) + a(\lambda_1) + \sigma(\lambda_3). \end{aligned}$$

The only choice we have to make is that of $\lambda_1 \in R, \lambda_2 \in I^\dagger, \lambda_3 \in I$, and $\lambda_4 \in \mathrm{End}_R(I)$. We note that since I contains a nonzerodivisor, we have that $\mathrm{End}_R(I)$ is contained in the total quotient ring of R and thus that every R-linear morphism is also $\mathrm{End}_R(I)$-linear. It follows at once that $\mathrm{Hom}_R(\mathrm{End}_R(I), I) \cong I$. Now, since $g_4 \in \mathrm{Hom}_R(\mathrm{End}_R(I), I) \cong I$ it is just multiplication by $g_4(1) \in I$ and hence we have $g_4(\sigma) = \sigma g_4(1)$ and so we can choose $\lambda_3 = g_4(1) \in I$. The same argument works for choosing $\lambda_1 = g_3(1) \in \mathrm{End}_R(I)$ since $\mathrm{End}_R(I)$ is contained in the total quotient ring of R. Since $r \in R$ we have that $\varphi(r\lambda_2) = \varphi(\lambda_2)r$; but, φ is an isomorphism, so we can choose $\lambda_2 = \varphi^{-1}(g_1(1))$ so that $\varphi(\lambda_2) = g(1)$ and as before $\varphi(\lambda_2 r) = \varphi(\lambda_2)r = g_1(1)r = g_1(r)$. Similarly we have that $\varphi(f\lambda_4) = \lambda_4\varphi(f)$ since $\lambda_4 \in \mathrm{End}_R(I)$, but then we choose $\lambda_4 = g_2\varphi^{-1} \in \mathrm{Hom}_R(I, I)$. It follows that $\varphi(f\lambda_4) = \lambda_4\varphi(f) = g_2\varphi^{-1}\varphi(f) = g_2(f)$ and thus we have

$$\varphi(r\lambda_2) + \varphi(f\lambda_4) + a(\lambda_1) + \sigma(\lambda_3) = g_1(r) + g_2(f) + g_3(a) + g_4(\sigma).$$

This concludes the proof. □

Before our main corollary we make a note that for a normal domain, a useful invariant of a ring is the *divisor class group*, $Cl(R)$. We denote by $D(R)$ the free abelian group on all divisorial ideals of R. We then set $Cl(R)$ to be the quotient of $D(R)$ by $F(R)$, the subgroup generated by the principal ideals. If R is CM with canonical module ω and is generically Gorenstein, then by Prop 2.1.9 we know that $[\omega] \in Cl(R)$. For our purposes we need to know only that $|[\omega]| = 2$ if and only if $\omega^\dagger = \omega$.

Corollary 3.3.6. *Suppose R is a CM henselian generically Gorenstein ring with canonical module ω_R. Then $\mathrm{End}_R(R \oplus \omega_R)$ is a Gorenstein R-order if and only if $\omega_R \cong \omega_R^\dagger$. In particular if R is a CM local normal domain, then this is further equivalent to $[\omega_R]$ having order 2 in the divisor class group of R.*

Proof. $(\Leftarrow)$: this is a direct application of Theorem 3.3.5.

$(\Rightarrow)$: We must address the fact that Λ being a Gorenstein order may not imply that $\mathrm{Hom}_R(\Lambda, \omega) \cong \Lambda$ but only that $\mathrm{Hom}_R(\Lambda, \omega)$ is a summand of a finite sum of copies of Λ as a Λ-module. We note that as R-modules

$$\Lambda \cong R^2 \oplus \omega \oplus \omega^\dagger$$
$$\mathrm{Hom}_R(\Lambda, \omega) \cong \omega^2 \oplus R \oplus \mathrm{Hom}_R(\omega^\dagger, \omega).$$

Since the functor $\mathrm{Hom}_R(R \oplus \omega, -) : \mathrm{add}(R \oplus \omega) \longrightarrow \mathrm{proj}\Lambda$ is an equivalence, we have that the indecomposable projective Λ-modules are $P_1 = \mathrm{Hom}_R(R \oplus \omega, R)$ and $P_2 = \mathrm{Hom}_R(R \oplus \omega, \omega)$. As R-modules, we have

$$P_1 \cong R \oplus \omega^\dagger$$
$$P_2 \cong R \oplus \omega.$$

By considering ranks, we see $\mathrm{Hom}_R(\Lambda, \omega) \cong P_i \oplus P_j$ for $i, j \in \{1, 2\}$. In the case that $\mathrm{Hom}_\Lambda(\Lambda, \omega)$ is P_1^2 or $P_1 \oplus P_2$ it is clear that either R is Gorenstein or $\omega^\dagger \cong \omega$ since R, ω, and $\omega^\dagger$ are all rank one R-modules at the associated primes; in either case, the result holds. Thus, we must only deal with the case

$$\mathrm{Hom}_\Lambda(\Lambda, \omega) \cong P_2^2$$

so that we deduce $\mathrm{Hom}_R(\omega^\dagger, \omega) \cong R$. But, as R is CM, we have that R satisfies Serre's condition (S_1) and we know $\omega^\dagger$ satisfies (S_2) as it is the dual of a finitely generated R-module. Thus, we know $\omega^\dagger$ is ω-reflexive by [23, Lemma 1.5], and we see

$$\omega^\dagger \cong \mathrm{Hom}_R(\mathrm{Hom}_R(\omega^\dagger, \omega), \omega) \cong \mathrm{Hom}_R(R, \omega) \cong \omega. \qquad \square$$

3.3.1 Examples

Since we now have a criterion on the canonical module, we will consider some invariant subrings under actions by cyclic groups, where the order of ω_R is easily computed. We start with a theorem of Weston [48].

Hypotheses 3.3.7. Let $S = k[x_1, \ldots, x_n]$ and G a finite subgroup of $GL_n(k)$ with generators $g_1, \ldots, g_t$ which acts linearly on the variables. For each $j = 1, \ldots, t$ let ζ_j be a primitive $|g_j|^{th}$ root of unity in k. Then for each j there exists a basis for $kx_1 \oplus \cdots \oplus kx_n$ so that

$$g_j = \begin{bmatrix} \zeta_j^{a_{1j}} & 0 & \ldots & 0 \\ 0 & \zeta_j^{a_{2j}} & \ldots & 0 \\ 0 & \ldots & \zeta_j^{a_{3j}} & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \zeta_j^{a_{nj}} \end{bmatrix}$$

for integers a_{ij} with $1 \leqslant a_{ij} \leqslant |g_j|$ for $j = 1, \ldots, t$. Set $d_{ij} = \gcd(a_{1j} \ldots, \widehat{a_{ij}}, \ldots a_{nj}, |g_j|)$ and m_j the least integer so that $m_j \sum_{i=1}^n d_{ij} a_{ij} = 0 \bmod |g_j|$. Let $R = S^G$. When G is cyclic (i.e., $t = 1$), we will suppress the use of j, as it is not needed.

Theorem 3.3.8. [47, Theorem 2.2] *With Hypotheses 3.3.7 the class* $[\omega]$ *in* $Cl(R)$ *of the canonical* R *module* ω *has order* $m := \mathrm{lcm}(m_1, \ldots, m_t)$

Corollary 3.3.9. *Let* $S = k[x_1, \ldots, x_n]$ *(or,* $k[[x_1, \ldots, x_n]]$ *) and* $G \subset GL(n, k)$ *be a finite subgroup acting linearly on the variables and set* $R = S^G$*. If* $\mathrm{End}_R(R \oplus \omega)$ *is a Gorenstein* R*-order, then* G *is of even order.*

Proof. First we note that by [48, Theorem 3.1], it suffices to treat the polynomial ring case. We adopt the notation of Hypotheses 3.3.7. By Theorem 3.3.8 the order of ω is $\mathrm{lcm}(m_1, \ldots, m_t)$. Now, Corollary 3.3.6 says that if $\mathrm{End}_R(R \oplus \omega)$ is a Gorenstein algebra, then we have $|[\omega]| = 2$, since R is normal. This means at least one $m_i = 2$, call it m_1. Then we have that $m_1 \sum_{j=1}^{n} a_{1j} d_{1j} = l|g_1|$. But as 2 is prime, it must divide l or $|g_1|$. It cannot divide l as then a smaller integer would be chosen instead of m_1. Thus it must be that $|g_1|$ is even, and hence G must be of even order. □

For the remainder of this section we will focus on Veronese subrings, so we introduce some notation. Let $k[[x_1, \ldots, x_n]]^{(a)}$ denote the a^{th} Veronese subring of $k[[x_1, \ldots, x_n]]$. I.e., it is the subring generated by all monomials of degree a in $x_1, \ldots, x_n$. Alternatively, fix a primitive a^{th} root of unity $\zeta \in k$. Then $k[[x_1, \ldots, x_n]]^{(a)}$ is the ring of invariants of the group action of the cyclic group $C_a = \langle \sigma \rangle$ where σ has order a and $\sigma \cdot x_i = \zeta x_i$.

Remark 3.3.10. Note that the converse to this is not true, since it is possible to have $|G|$ even, but the order of the canonical module not be 2. For example, let $R = k[[x_1, x_2, x_3, x_4]]^{(2)} = k[[x_i x_j]]_{1 \leqslant i \leqslant j \leqslant 4}$. Here G is cyclic of order 2, $G \subset \mathrm{SL}_2(k)$, and G contains no pseudo-reflections, hence R is Gorenstein [46, Theorem 1]. It follows that the order of $[\omega]$ is 1.

This gives us the ability to produce ample examples of Gorenstein algebras over the power series ring in n variables.

3.4 Steady NCCRs and global dimension

In this section we will suppose we are working within the conditions of 3.3.7. We start with the following definitions, from [30]:

Definition 3.4.1. Let R be a d-dimensional CM local normal domain with canonical module ω_R.

- A module M is *steady* if it is a generator and $\mathrm{End}_R(M) \in \mathrm{add}_R M$.
- If M is steady and $\mathrm{End}_R(M)$ is a noncommutative crepant resolution of R then we say $\mathrm{End}_R(M)$ is a *steady NCCR*.
- If M is a direct sum of reflexive modules of rank one, then we call M *splitting*.
- We say $\mathrm{End}_R(M)$ is a *splitting NCCR* if it is an NCCR and M is splitting.
- If $M = M_1 \oplus \cdots \oplus M_n$ is a decomposition of M into indecomposables, we say M is *basic* if the M_i are mutually nonisomorphic.

Remark 3.4.2. Let R be a d-dimensional CM local normal domain with canonical module ω_R. We see that if the conditions of Corollary 3.3.6 are satisfied, then $\mathrm{End}_R(R \oplus \omega) \cong R \oplus R \oplus \omega \oplus \omega \in \mathrm{add}_R(R \oplus \omega)$ so that $R \oplus \omega$ is a steady splitting module. If R is not Gorenstein, then $R \oplus \omega$ is basic.

Theorem 3.4.3. *Let $R = S^G$ be a subring of $S = k[[x_1, \ldots, x_n]]$ for k an algebraically closed field of characteristic zero with G a finite abelian subgroup of $GL_n(k)$ and such that $[\omega]$ has order 2 in $Cl(R)$. Then* $\mathrm{End}_R(R \oplus \omega)$ *has finite global dimension if and only if R is isomorphic to a ring of the form $T[[x_1, x_2, \ldots, x_{n-j}]]$ for j odd and $3 \leqslant j \leqslant n$. where $T = k[[x_1, \ldots, x_j]]^{(2)}$.*

Before the proof, we need the following result of Iyama and Nakajima:

Lemma 3.4.4. [30, Theorem 3.1] *Let R be a d-dimensional CM local normal domain. Then the following are equivalent:*

- *R is a quotient singularity associated with a finite abelian group $G \subset Gl_d(k)$ (i.e., $R = S^G$ where $S = k[[x_1, \dots, x_d]]$.)*
- *R has a unique basic module giving a splitting NCCR.*
- *R has a steady splitting NCCR.*
- *There exists a finite subgroup G of $Cl(R)$ such that $\bigoplus_{X \in G} X$ gives an NCCR of R.*
- *$Cl(R)$ is a finite group and $\bigoplus_{X \in Cl(R)} X$ gives an NCCR of R.*

In this case, S is the unique basic splitting R-module giving an NCCR.

Furthermore, if R is a completion of a toric variety all of the above conditions are equivalent to the following:

- *$Cl(R)$ is a finite group.*

Proof of Theorem 3.4.3. ($\Rightarrow$): Suppose that $\mathrm{End}_R(R \oplus \omega)$ has finite global dimension. Since $|[\omega]| = 2$ we know that $\mathrm{End}_R(R \oplus \omega)$ is a Gorenstein R-order by Corollary 3.3.6. Thus, by 3.3.2 the global dimension of $\mathrm{End}_R(R \oplus \omega)$ is d. Then $\mathrm{End}_R(R \oplus \omega)$ is a noncommutative crepant resolution of R. Indeed, $\mathrm{End}_R(R \oplus \omega)$ is MCM since $\omega \cong \omega^\dagger$ and thus $\mathrm{End}_R(R \oplus \omega) \cong R^2 \oplus \omega^2$. This NCCR is steady and splitting and $R \oplus \omega$ is basic by Remark 3.4.2. Thus $R \oplus \omega \cong S$ by Lemma 3.4.4. It follows from Galois theory that $|G| = \mathrm{rank}_R S = 2$. We see immediately that $n \geqslant 3$ since otherwise R would be Gorenstein and $|[\omega]| = 1$. Then $G = \langle \sigma \rangle$ where $\sigma^2 = 1$. The minimal polynomial of σ is $(t-1)(t+1)$ and so σ is diagonalizable with eigenvalues $\lambda = \pm 1$. Thus there is a basis for $V = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$ where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Notice that the number of negative entries is exactly the quantity j from the theorem. We claim there must be at least 3 entries which are negative, and that the number of negative entries must be odd. Indeed, if $\sigma = I_n$, then $|G| = 1$ and hence there must be at least one negative entry. Now, if there is exactly one then $a_1 = a_2 = \cdots = a_{n-1} = 2$ and $a_n = 1$ in the notation of Hypotheses 3.3.7. It would follow that $d_1 = d_2 = d_3 = \cdots = d_{n-1} = 1$ and $d_n = 2$, hence $\sum_{i=1}^n a_i d_i = 2n$ and $|[\omega]| = 1$, by Theorem 3.3.8. Now, let l be the largest index so that the l^{th} row contains a positive entry. We have that $l \leqslant n-2$. Then it must be that $d_i = 1$ for all $i = 1, \ldots, n$. Thus $a_i d_i = 2$ in the case that $i \leqslant l$ and $a_i d_i = 1$ in the case that $i > l$. Since we need that $\sum_{i=1}^n a_i d_i$ is odd, it must be that $n - l$ is odd, hence there are an odd number of negative entries in σ. It follows at once that R is of the form indicated.

($\Leftarrow$): Suppose R is of the indicated form. As above, we then have $d_i = 1$ for $i = 1, \ldots, n$. Then by [48, Example 2.3] $\omega \cong x_1 x_2 \ldots x_n S \cap R \cong (x_j, \ldots, x_n)$ and hence $R \oplus \omega \cong k[[x_1, \ldots, x_n]]$ as R-modules and thus, $\mathrm{End}_R(R \oplus \omega) \cong \mathrm{End}_R(S)$ which is known to have finite global dimension, see [30, Example 2.3].

□

Example 3.4.5. It should be noted that the condition $|[\omega]| = 2$ (in particular, that R is not Gorenstein) is needed. If we do not require this, the theorem is false. Let $R = k[[x, y]]^{\mathbb{Z}_2}$ where the group acts via $x \mapsto y$ and $y \mapsto x$. Then $R \cong k[[xy, x + y]]$ and hence is a regular local ring. Thus $\mathrm{End}_R(R \oplus \omega) = \mathrm{End}_R(R^2)$ has finite global dimension as it is Morita equivalent to R. Similar examples exist for larger n.

Chapter 4

Auslander's Theorem and Path Algebras

In this chapter we examine orders which exhibit some behavior seen in commutative rings. Specifically, we note that by the Auslander-Buchsbaum formula [7], maximal Cohen-Macaulay modules over commutative rings are either projective or have infinite projective dimension. We consider non-commutative rings where a similar result holds for high syzygies and prove that finite projective dimension of the canonical module, ω_Λ is sufficient to guarantee this condition. By moving to syzygies of MCM modules we are able to strengthen a theorem of Auslander which states that a CM local ring of finite CM type must be an isolated singularity[3], see Theorem 4.3.1. It turns out, path algebras are a natural example of orders whose canonical modules have finite projective dimension and thus provide a good setting to apply our theorem. We are able to deduce that if for a complete Gorenstein domain R, the path algebra RQ has finitely many nonisomorphic syzygies of MCM modules, then in fact R must be an isolated singularity. Finally, we investigate the ascent of finite syzygy type to and from the Henselization of an order.

We begin by studying a condition which produces a Auslander-Buchsbaum type formula in orders.

4.1 Projective Dimension and the Canonical module

Recall that a great deal of work has been done to study Gorenstein orders. These are natural candidates in the CM local ring case for noncommutative crepant resolutions. One of the reasons that Gorenstein orders are exceptionally useful is that they satisfy an Auslander-Buchsbaum formula. Throughout R is a d-dimensional Cohen-Macaulay local ring with canonical module ω and Λ will be an R-order. We remind the reader that $\omega_\Lambda := \operatorname{Hom}_R(\Lambda, \omega)$ is the canonical module of Λ. Recall that this is both a Λ- and Λ^{op}-module.

Lemma 4.1.1. [32, Lemma 2.16] *Let Λ be a Gorenstein R-order. Then for any $X \in \operatorname{mod} \Lambda$ with $\operatorname{projdim}_\Lambda X < \infty$ we have*

$$\operatorname{projdim}_\Lambda X + \operatorname{depth}_R X = \dim R.$$

We begin this section by generalizing this result.

Theorem 4.1.2. *Let Λ be an R-order with $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$. For any $X \in \operatorname{mod} \Lambda$ with $\operatorname{projdim}_\Lambda X < \infty$ we have*

$$\dim R \leqslant \operatorname{projdim}_\Lambda X + \operatorname{depth}_R X \leqslant \dim R + n.$$

Moreover, if $\operatorname{gldim} \Lambda = n + d$, then $\operatorname{projdim}_\Lambda M \leqslant \operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$ for all $M \in \operatorname{CM} \Lambda$.

With this motivation, we give a name to this condition.

Definition 4.1.3. Let R be a CM local ring with canonical module ω. Let Λ be an R-order. We call Λ *n-canonical* if $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$.

Now we move on to prove Theorem 4.1.2, which follows from the strong condition of Ext-vanishing imposed by being n-canonical.

Lemma 4.1.4. *Suppose Λ is an n-canonical order over a CM local ring R with canonical module ω. If $M \in \operatorname{CM}\Lambda$ then $\operatorname{Ext}^i_\Lambda(M,\Lambda) = 0$ for $i > n$. In particular, if $X \in \Omega^n \operatorname{CM}\Lambda$, then $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i > 0$.*

Proof. Begin by taking a projective resolution of ω_Λ as a left module over Λ^{op}

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow \omega_\Lambda \longrightarrow 0$$

and apply $D_d(-)$ to get a resolution of left Λ-modules,

$$0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow 0$$

with $I_j \in \operatorname{add}\omega_\Lambda$. Decompose this exact sequence into short exact sequences

$$0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow Y_1 \longrightarrow 0$$

$$0 \longrightarrow Y_i \longrightarrow I_i \longrightarrow Y_{i+1} \longrightarrow 0 \text{ for } i = 1, \ldots, n-2$$

$$0 \longrightarrow Y_{n-1} \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow 0.$$

Since $I_k \in \operatorname{add}\omega_\Lambda$ for all k, we have $\operatorname{Ext}^i_\Lambda(M, I_k) = 0$ for all $i > 0$, $M \in \operatorname{CM}\Lambda$, $k = 1, \ldots, n$. Then, we see

$$\operatorname{Ext}^{n+j}_\Lambda(M,\Lambda) \cong \operatorname{Ext}^{n+j-1}_\Lambda(M, Y_1) \cong \ldots \cong \operatorname{Ext}^1_\Lambda j(M, I_n) = 0$$

for any $M \in \operatorname{CM}\Lambda$, $j > 0$. This concludes the proof of the first statement. For the second statement, the usual dimension shift in the first argument of $\operatorname{Ext}^{n+1}_\Lambda(M,\Lambda)$ gives

$$\operatorname{Ext}^j_\Lambda(\Omega^n M, \Lambda) \cong \operatorname{Ext}^{n+j}_\Lambda(M,\Lambda) = 0$$

for $j > 0$. □

With this in hand, we are able to prove our generalize Auslander-Buchsbaum theorem.

Proof of Theorem 4.1.2. We start by showing that if $X \in \operatorname{CM}\Lambda$ satisfies $\operatorname{projdim}_\Lambda X < \infty$, then $\operatorname{projdim}_\Lambda X \leqslant n$. By Lemma 4.1.4, if $X \in \operatorname{CM}\Lambda$, then $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for $i > n$. Since $\operatorname{Ext}^r_\Lambda(X, \Lambda) \neq 0$ for $r = \operatorname{projdim}_\Lambda X$ by Lemma 2.1.13, we must have either $\operatorname{projdim}_\Lambda X \leqslant n$ or $\operatorname{projdim}_\Lambda X = \infty$. Applying the Depth Lemma and the fact that Λ is MCM over R we know that given any module $X \in \operatorname{mod}\Lambda$ with $\operatorname{depth}_R X = t$, the $(d-t)^{th}$ syzygy must be in $\operatorname{CM}\Lambda$ so that

$$\operatorname{projdim}_\Lambda X \leqslant d - t + n = \dim R - \operatorname{depth}_R X + n.$$

The right side inequality follows at once from this. To prove the left inequality we simply note that projective modules must be in $\operatorname{CM}\Lambda$. By the Depth Lemma again, if $\operatorname{depth}_R X = t$, then the first syzygy which could be projective is the $(d-t)^{th}$, as each syzygy can go up in depth by at most 1. Thus

$$\operatorname{projdim}_\Lambda X \geqslant d - \operatorname{depth}_R X.$$

This concludes the proof. □

Remark 4.1.5. Note that the second inequality of Theorem 4.1.2 surely can not be strengthened to an equality. Indeed, suppose $M \in \operatorname{CM}\Lambda$ has $\operatorname{projdim}_\Lambda M = n$. Then of course $\Omega M \in \operatorname{CM}\Lambda$ has $\operatorname{projdim}_\Lambda \Omega M = n - 1$, but $\operatorname{depth}_R \Omega M = \operatorname{depth}_R M$.

It is clear by the work above that any order Λ with $\operatorname{gldim}\Lambda < \infty$ is an n-canonical order for some n. The first question we address is whether these can exist with infinite global dimension. We will show that in fact they do, and that the construction is natural. We begin by identifying the canonical module of a tensor product of orders.

Lemma 4.1.6. *Let Λ_1 and Λ_2 be algebras over a Gorenstein local ring R such that Λ_1 and Λ_2 are free R-modules. Then $\Lambda_1 \otimes_R \Lambda_2$ is an R-order and $\omega_{\Lambda_1 \otimes \Lambda_2} \cong \omega_{\Lambda_1} \otimes_R \omega_{\Lambda_2}$ as both $\Lambda_1 \otimes_R \Lambda_2$- and $(\Lambda_1 \otimes_R \Lambda_2)^{op}$-modules.*

Proof. It is easy to check that since Λ_1 and Λ_2 are finitely generated free R-modules we have an R-isomorphism

$$\begin{aligned}\omega_{\Lambda_1} \otimes \omega_{\Lambda_2} =& \operatorname{Hom}_R(\Lambda_1, R) \otimes_R \operatorname{Hom}_R(\Lambda_2, R)\\ \cong& \operatorname{Hom}_R(\Lambda_1 \otimes_R \Lambda_2, R)\\ =&\omega_{\Lambda_1 \otimes \Lambda_2}.\end{aligned}$$

The isomorphism $\Phi : \operatorname{Hom}_R(\Lambda_1, R) \otimes_R \operatorname{Hom}_R(\Lambda_2, R) \longrightarrow \operatorname{Hom}_R(\Lambda_1 \otimes_R \Lambda_2, R)$ is given by $[\Phi(f \otimes g)](a \otimes b) = f(a)g(b)$. We must only show that Φ is a morphism of $\Lambda_1 \otimes_R \Lambda_2$-modules.

Let $f \otimes g \in \operatorname{Hom}_R(\Lambda_1, R) \otimes_R \operatorname{Hom}_R(\Lambda_2, R)$. We compute, for any $\lambda_1 \otimes \lambda_2 \in \Lambda_1 \otimes \Lambda_2$,

$$\begin{aligned}[(\lambda_1 \otimes \lambda_2) \cdot \Phi(f \otimes g)](\eta_1 \otimes \eta_2) =& \Phi[f \otimes g]((\eta_1 \otimes \eta_2) \cdot (\lambda_1 \otimes \lambda_2))\\ =& f(\eta_1\lambda_1)g(\eta_2\lambda_2).\end{aligned}$$

On the other hand, $\Phi(f \otimes g)$ is the composition of $f \otimes g$ followed by multiplication $\mu : R \otimes R \longrightarrow R$. We see then

$$\begin{aligned}[\Phi((\lambda_1 \otimes \lambda_2) \cdot f \otimes g))](\eta_1 \otimes \eta_2) =& \mu \circ [(\lambda_1 \otimes \lambda_2) \cdot f \otimes g]((\eta_1 \otimes \eta_2))\\ =& \mu \circ f \otimes g((\eta_1 \otimes \eta_2)(\lambda_1 \otimes \lambda_2))\\ =& \mu \circ f \otimes g(\eta_1\lambda_1 \otimes \eta_1\lambda_1)\\ =& f(\eta_1\lambda_1)g(\eta_1\lambda_1).\end{aligned}$$

Checking that it is a $(\Lambda_1 \otimes_R \Lambda_2)^{op} = \Lambda_1^{op} \otimes_R \Lambda_2^{op}$-module morphism is similar. □

Since we are now able to find the canonical module of orders which are free over R, we get the following examples for R a regular local ring. Recall that over a regular local ring the MCM modules are exactly the free modules.

Theorem 4.1.7. *Let $(R, \mathfrak{m}, k)$ be a regular local ring. Suppose $\Lambda_1,\ \Lambda_2$ are n_1-canonical and n_2-canonical R-orders,respectively. Then $\Lambda_1 \otimes \Lambda_2$ is an $n_1 + n_2$-canonical R-order.*

Proof. Since Λ_1 and Λ_2 are MCM over R, and R is a regular local ring, then in fact they are free. Then, noting that $(\Lambda_1 \otimes_R \Lambda_2)^{op} = \Lambda_1^{op} \otimes_R \Lambda_2^{op}$, this follows from Lemma 4.1.6 and Lemma 2.2.8; indeed, we see

$$\begin{aligned}\operatorname{projdim}_{\Lambda_1^{op}\otimes\Lambda_2^{op}}(\omega_{\Lambda_1\otimes\Lambda_2}) &= \operatorname{projdim}_{\Lambda_1^{op}\otimes\Lambda_2^{op}} \omega_{\Lambda_1} \otimes \omega_{\Lambda_2} \\ &= \operatorname{projdim}_{\Lambda_1^{op}} \omega_{\Lambda_1} + \operatorname{projdim}_{\Lambda_2^{op}} \omega_{\Lambda_2}.\end{aligned}$$

□

Now, in order to produce an order of infinite global dimension where $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$, we can take a Gorenstein order of infinite global dimension as one factor, and an order of finite global dimension as the other. The following example can be considered whenever we have a commutative Gorenstein local ring R.

In order to reduce to the complete case, we need a lemma.

Lemma 4.1.8. *Suppose R is a CM local ring with a canonical module ω_R and that $R \hookrightarrow S$ is a faithfully flat (commutative) ring extension such that* $\dim S = \dim R$ *and S has a canonical module $\omega_S = \omega_R \otimes_R S$ (e.g., if $S = \widehat{R}$). We have that Λ is an n-canonical R-order if and only if $\Lambda \otimes_R S$ is an n-canonical S-order.*

Proof. We really only need to prove two facts. First we note that since S is faithfully flat

$$\operatorname{Hom}_R(M, N) \otimes_R S \cong \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S).$$

It follows at once that $\omega_{\Lambda\otimes_R S} \cong \omega_\Lambda \otimes_R S$ over S. Verifying that this is a $\Lambda\otimes_R S$-isomorphism is straightforward. Next, since exactness of Λ-module sequences can be checked as R-modules, S is faithfully flat over R, and $- \otimes_R S$ takes projective Λ-modules to projective $\Lambda \otimes_R S$-modules, we see that

$$\operatorname{projdim}_\Lambda \omega_\Lambda = \operatorname{projdim}_{\Lambda\otimes_R S} \omega_{\Lambda\otimes_R S}.$$

The lemma follows at once from these two observations. □

Theorem 4.1.9. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Gorenstein local domain. Suppose Q is an acyclic quiver. Then $\Lambda = RQ$ is a 1-canonical R-order. If R is not regular, then* $\operatorname{gldim} \Lambda = \infty$.

Proof. We reduce to the case where R is complete. Let $\widehat{R}$ denote the completion of R with respect to the maximal ideal. By Lemma 4.1.8, we see that RQ is 1-canonical if and only if $RQ \otimes_R \widehat{R}$ is one canonical. But, by Proposition 2.2.6, we know that $RQ \otimes \widehat{R} \cong \widehat{R}Q$. Thus we see RQ is 1-canonical if and only if $\widehat{R}Q$ is 1-canonical. Thus we may assume R is complete.

Now, by Cohen's Structure Theorem for complete local rings, [36]Theorem 8.24, R is an order over some d-dimensional regular local ring S. Since R is a Gorenstein local ring and an order over S, we have $R \cong \omega_R \cong \operatorname{Hom}_S(R, S)$ and $\operatorname{projdim}_S \omega_R = 0$ since R is MCM over S and hence free; i.e., R is a 0-canonical S-order. Further, by Proposition 2.2.9, we know that $\operatorname{gldim} SQ = d + 1$ and hence by Theorem 4.1.2, $\operatorname{projdim}_{SQ^{op}} \omega_{SQ} = 1$; i.e., SQ is a 1-canonical S-order. Now, by Proposition 2.2.6 and Theorem 4.1.7, $\Lambda := RQ \cong R \otimes_S SQ$ is a 1-canonical S-order.

All that is left is to establish that Λ is in fact an R-order (indeed, it is R-free) and that $\operatorname{Hom}_S(\Lambda, S) \cong \operatorname{Hom}_R(\Lambda, R)$, i.e. that the canonical module of Λ as an R-order agrees with that as an S-order. For the final assertion, we see

$$\operatorname{Hom}_R(\Lambda, R) \cong \operatorname{Hom}_R(\Lambda, \operatorname{Hom}_S(R, S)) \cong \operatorname{Hom}_S(\Lambda \otimes_R R, S) \cong \operatorname{Hom}_R(\Lambda, S).$$

It is straight-forward to verify this is also an isomorphism of Λ-modules. Lastly, by Lemma 2.2.8 we know that if R is not regular, we have $\operatorname{gldim} \Lambda = \infty$. □

One might ask if we can extend the previous theorem to include RQ/I for some admissable ideal I. One problem is that, in general, RQ/I may not be free (or even MCM) over R and so some of our isomorphisms fail to hold. If I is monomial and admissable and generated by paths this is satisfied and the theorem generalizes accordingly.

Precisely, if Q and I are a quiver and an admissable monomial ideal so that $\operatorname{gldim}(SQ/I) = n+d$, then RQ/IRQ is an n-canonical order with infinite global dimension. Note that Theorem 4.1.9 in addition to Theorem 4.1.7 allow us to produce n-canonical orders of infinite global dimension for arbitrary n.

Here we have chosen only to present the simpler version of Theorem 4.1.9 since the proof is much nicer, and in Corollary 4.4.5 it still exhibits the usefulness of Theorem 4.3.1.

4.2 n-Isolated Singularities

Our main theorem of this chapter is that if an order Λ is n-canonical and has only finitely many nonsiomorphic indecomposable modules in $\Omega^n \operatorname{CM} \Lambda$, then Λ has finite global dimension on the punctured spectrum of R. Here we give such orders a name and investigate some of their properties. These are direct generalizations of isolated singularities and non-singular orders which have been investigated by various authors; see e.g., [27, 29, 32].

Definition 4.2.1. Let Λ be an order over a CM ring R. We call Λ an *n-isolated singularity* if

$$\operatorname{gldim} \Lambda_{\mathfrak{p}} \leqslant n + \dim R_{\mathfrak{p}}$$

for all non-maximal prime ideals $\mathfrak{p} \in \operatorname{Spec} R$. We say Λ is *n-nonsingular* if $\operatorname{gldim} \Lambda_{\mathfrak{p}} = n + \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

The following category is of central importance here, so we endow it with its own notation.

Notation 4.2.2. Denote by $\mathcal{S}$ the additive closure of the full subcategory of n^{th} syzygies of maximal Cohen-Macaulay Λ-modules, i.e., $\mathcal{S} = \operatorname{add}\{\Omega^n \operatorname{CM}(\Lambda)\}$

The rest of this Chapter focuses on the class of modules $\mathcal{S}$. Generally, over n-canonical orders, modules in $\mathcal{S}$ exhibit similar behavior to MCM modules over Gorenstein local rings. We begin by noting that Λ is an n-isolated singularity if and only if modules in $\mathcal{S}$ are projective on the punctured spectrum. To do this, we need the following lemma.

Lemma 4.2.3. *Let Λ be an n-isolated singularity over a CM local ring R. Then if $M \in$ CM(Λ) we have*

$$\operatorname{projdim} M_{\mathfrak{p}} \leqslant n$$

for all non-maximal primes $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. Let $M \in \operatorname{CM}\Lambda$. It follows that $M_{\mathfrak{p}} \in \operatorname{CM}\Lambda_{\mathfrak{p}}$. Pick a maximal $M_{\mathfrak{p}}$-regular sequence $x_1, \ldots, x_t \in \mathfrak{p}R_{\mathfrak{p}}$. We have an exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{x_1} M_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}/x_1 M_{\mathfrak{p}} \longrightarrow 0$$

which induces an exact sequence

$$\operatorname{Ext}^i_{\Lambda_{\mathfrak{p}}}(M, -) \xrightarrow{x_1} \operatorname{Ext}^i_{\Lambda_{\mathfrak{p}}}(M, -) \longrightarrow \operatorname{Ext}^{i+1}_{\Lambda_{\mathfrak{p}}}(M/x_1 M, -) \longrightarrow \operatorname{Ext}^{i+1}_{\Lambda_{\mathfrak{p}}}(M, -).$$

It follows from this that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}/x_1 M_{\mathfrak{p}} = \operatorname{projdim}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} + 1$, and by induction we get $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}}/(x_1, \ldots, x_t) M_{\mathfrak{p}} = \operatorname{projdim}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim(R_{\mathfrak{p}})$. Since $\operatorname{gldim}\Lambda_{\mathfrak{p}} = n + \dim(R_{\mathfrak{p}})$, it must be that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}} M_{\mathfrak{p}} \leqslant n$. □

From this we get the following useful characterization of n-isolated singularities.

Corollary 4.2.4. *Let Λ be an order over a CM local ring R. Then Λ is an n-isolated singularity if and only if for all $X \in \mathcal{S}$, $X_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}$-module for all non-maximal primes $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. ($\Rightarrow$): This follows at once from the previous lemma.

($\Leftarrow$): This is clear by the Depth Lemma. Since any d^{th} syzygy of a Λ-module is in $\operatorname{CM}\Lambda$, we have n^{th} syzygies of modules in $\operatorname{CM}\Lambda$ are projective on the punctured spectrum. It follows that $(n+d)^{th}$ syzygies of arbitrary (finitely generated) Λ-modules are projective on the punctured spectrum. This implies Λ is an n-isolated singularity. □

The following lemma will be useful later, as it detects n-isolated singularities.

Lemma 4.2.5. *Let R be a CM local ring with canonical module ω. Let Λ be an R-order. Then Λ is an n-isolated singularity if and only if $\ell_R(\operatorname{Ext}^1_\Lambda(N, M)) < \infty$ for all $M, N \in \mathcal{S}$, where $\ell_R(-)$ denotes the length of an R-module.*

Proof. ($\Rightarrow$): Recall that an R-module M has finite length if and only if $\ell_R(M) < \infty$ for all non-maximal $\mathfrak{p} \in \operatorname{Spec} R$. Since $N \in \mathcal{S}$, by Corollary 4.2.4 we know $N_\mathfrak{p}$ is projective on the punctured spectrum. It follows that $\operatorname{Ext}_\Lambda(N, M)_\mathfrak{p} = \operatorname{Ext}_{\Lambda_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all non-maximal $\mathfrak{p} \in \operatorname{Spec} R$.

($\Leftarrow$): Suppose $\ell(\operatorname{Ext}^1_\Lambda(N, M)) < \infty$ for all $M, N \in \mathcal{S}$. Suppose there is a prime ideal $\mathfrak{p}$ for which $\operatorname{gldim} \Lambda_\mathfrak{p} > n + \dim(R_\mathfrak{p})$. Then there must exist a maximal Cohen-Macaulay module M so that $\operatorname{projdim} X \geqslant 1$ for $X = \Omega^n M$. Consider the short exact sequence

$$0 \longrightarrow \Omega X \longrightarrow F \longrightarrow X \longrightarrow 0,$$

where F is a free module since $X, \Omega X \in \mathcal{S}$, by assumption $\operatorname{Ext}^1_\Lambda(X, \Omega X) = 0$. This means the above sequence splits, and hence X is projective, a contradiction. Thus it must be that $\operatorname{gldim} \Lambda_\mathfrak{p} \leqslant n + \dim(R_\mathfrak{p})$. □

The next proposition illustrates that n^{th} syzygies (of MCM modules) over an n-isolated singularity behave like MCM modules over an isolated singularity. This is shown for the $n = 0$ case in [27, Theorem 1.3.1]; the proof is largely the same except the $d = 2$ case of part (1).

Proposition 4.2.6. *Let Λ be an n-isolated singularity over a d-dimensional CM local ring R. For $X \in \mathcal{S}$:*

1. $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$ *for* $i = 1, \ldots, d$.
2. $\operatorname{Ext}^i_\Lambda(X, Y)$, $\operatorname{Tor}_i^\Lambda(Z, X)$, *and* $\underline{\operatorname{Hom}}_\Lambda(X, Y)$ *are all finite length over* R *for any* $Y \in \operatorname{mod} \Lambda$ *and* $Z \in \operatorname{mod} \Lambda^{op}$.

Proof. For assertion (1), we note that if $d = 0$ there is nothing to show. In the case where $d = 1$, the fact that X is projective on the punctured spectrum implies that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{op}, \Lambda)$ has finite length since $\operatorname{Tr} X^{op}_{\mathfrak{p}} = 0$ for any non-maximal prime ideal $\mathfrak{p}$. Then the well-known exact sequence (see, e.g., [35, Proposition 12.8])

$$0 \longrightarrow \operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{op}, \Lambda) \longrightarrow X \longrightarrow X^{**} \longrightarrow \operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{op}, \Lambda) \longrightarrow 0$$

shows that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{op}, \Lambda)$ embeds in X. But, $\operatorname{depth} X \geqslant 1$ since $d \geqslant 1$ so that X is torsion-free. Thus, $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$.

Now suppose $d \geqslant 2$. We still have that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$ by the above case. Thus, we have an exact sequence

$$0 \longrightarrow X \longrightarrow X^{**} \longrightarrow \operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{op}, \Lambda) \longrightarrow 0.$$

By virtue of being a dual module, X^{**} has depth over R at least 2, by Lemma 2.1.5. Since $\operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{op}, \Lambda)$ is of finite length, hence depth zero, the Depth Lemma implies $\operatorname{depth}_R X = 1$. This is a contradiction since $d \geqslant 2$ and $X \in \operatorname{CM} \Lambda$. Thus, we must have $\operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$. It now follows from the above exact sequence that $X \cong X^{**}$.

Finally, suppose $\operatorname{Tr} X^{op} \in {}^{\perp_{k-1}}\Lambda$ for some $3 \leqslant k \leqslant d$. We begin with a projective resolution

$$... \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \operatorname{Tr} X^{op} \longrightarrow 0.$$

Dualizing the above exact sequence, and utilizing the fact that $X \cong X^{**}$, we get an exact sequence

$$0 \longrightarrow X \longrightarrow P_2^* \longrightarrow P_3^* \longrightarrow \ldots \longrightarrow P_{k-1}^* \longrightarrow (\Omega^k X)^* \longrightarrow \operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{op}, \Lambda) \longrightarrow 0,$$

where $\operatorname{depth}_R(\Omega^k X)^* \geqslant 2$, again by Lemma 2.1.5 since it is a dual module. Now, if $\operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{op}, \Lambda) \neq 0$ the Depth Lemma implies $\operatorname{depth}_R X \leqslant d - 1$, since $\operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{op}, \Lambda)$

has finite length and $P_i^* \in \operatorname{CM}\Lambda$ for all i. This is impossible since $X \in \operatorname{CM}\Lambda$. Thus, it must be that $\operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$. Thus part (1) is proved by induction. □

The next result shows that n-nonsingular orders have trivial category of n^{th} syzygies. This is the analog of [32, Prop 2.17], and the proof is largely the same.

Proposition 4.2.7. *Let Λ be an order over a CM ring R with canonical module ω_R. The following are equivalent:*

1. *Λ is n-nonsingular.*
2. *$\operatorname{gldim}\Lambda_{\mathfrak{m}} \leqslant n + d$ for all maximal ideals $\mathfrak{m} \in \operatorname{Spec} R$.*
3. *$\operatorname{CM}\Lambda \subset \operatorname{projdim}_{\leqslant n}\Lambda$.*
4. *$\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda \leqslant n$ and $\operatorname{gldim}\Lambda < \infty$.*

Proof. The first 3 implications are the same argument as [32], but we include them for the convenience of the reader. (1) ⇒ (2) This is immediate.

(2) ⇒ (3) This proof is nearly identical to the proof of Lemma 4.2.3.

(3) ⇒ (4) Since $\omega_\Lambda \in \operatorname{CM}\Lambda$, we know it has projective dimension at most n by (3). Further, since each d^{th} syzygy is MCM by the depth lemma, we also have $\operatorname{gldim}\Lambda < \infty$.

(4) ⇒ (1) Fix a non-maximal prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ and let X be in $\operatorname{CM}(\Lambda_{\mathfrak{p}})$. We will show that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}} X \leqslant n$ and the depth lemma will conclude the proof as in the previous step. Since localization can only reduce projective dimension, we have that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}^{op}} \omega_{\Lambda_{\mathfrak{p}}} \leqslant n$ and $\operatorname{gldim}\Lambda_{\mathfrak{p}} < \infty$. The result then follows from Theorem 4.1.2 □

Remark 4.2.8. One might ask if we can strengthen condition (3) to be a set equality. If $n \geqslant 1$, the answer is no: consider a regular sequence $\underline{x} = x_1, \ldots, x_d$ on Λ, and take the Koszul complex over Λ on $\underline{x}$. Then this is exact, and has length d. Then $\Omega^{d-1}(\Lambda/\underline{x}\Lambda)$ has depth $d-1$ by the Depth Lemma, but the end of the Koszul complex gives a length one resolution. Thus $\Omega^{d-1}(\Lambda/\underline{x}\Lambda) \in \operatorname{projdim}_{\leqslant n}\Lambda$ but is not in $\operatorname{CM}\Lambda$. We finally note that (3) is equivalent to $\mathcal{C} \subset \operatorname{Proj}\Lambda$.

4.3 Gorenstein Projectives and Auslander's Theorem

The goal of this section is to show the following variation of Auslander's Theorem, [3, Corollary A.2].

Theorem 4.3.1. *Let R be a CM local ring R with canonical module, ω_R and let Λ be an R-order. Suppose* $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$. *If Λ has only finitely many nonisomoprhic indecomposable modules in $\mathcal{S}$, then Λ is an n-isolated singularity.*

The proof of this will rely on the notion of *Gorenstein projective* modules, introduced in Chapter 2.

Our interest in Gorenstein projectives is motivated by the following fact. We let $G\operatorname{Proj}\Lambda$ denote the subcategory of all Gorenstein projective modules, and $\underline{G\operatorname{Proj}}\Lambda$ the corresponding stable category.

Proposition 4.3.2. *Let R be a CM local ring with canonical module ω_R. Suppose Λ is an R-order with* $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$, *where $n \geqslant 2$. Let M be a non-projective Λ-module; then* $M \in G\operatorname{Proj}$ *if and only if $M \in \mathcal{S}$.*

Before the proof we note that the only reason we require M to be non-projective is that it is not necessarily true that $\operatorname{Proj}\Lambda \subset \mathcal{S}$, but certainly all projectives are Gorenstein projective.

Proof. Since Gorenstein projectives occur as syzygies in complete resolutions, it is clear that, excluding projective modules, $G\operatorname{Proj}\Lambda \subset \operatorname{add}\Omega^n(\operatorname{CM}\Lambda)$. We show the reverse inclusion. Let $M = \Omega^n X$ for a maximal Cohen-Macaulay module X, and suppose M is nonprojective. By Lemma 4.1.4 we have that $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i > 0$. Then, by dualizing a projective resolution of M, we get an exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots .$$

Since M necessarily has infinite projective dimension by Lemma 3.4.3, we see M^* is an arbitrarily high syzygy. By Lemma 4.1.4 again we have $\mathrm{Ext}^i_\Lambda(M^*, \Lambda) = 0$ for $i > 0$. All that remains to show is that M is reflexive. Note that $\operatorname{Tr} M^{op}$ fits into the above exact sequence as follows

$$0 \longrightarrow \operatorname{Tr} M^{op} \longrightarrow P_2^* \longrightarrow P_3^* \longrightarrow \cdots .$$

Thus, $\operatorname{Tr} M^{op}$ is also an arbitrarily high syzygy and satisfies the same Ext vanishing as M. Thus the exact sequence from before

$$0 \longrightarrow \mathrm{Ext}^1_\Lambda(\operatorname{Tr} M^{op}, \Lambda) \longrightarrow M \longrightarrow M^{**} \longrightarrow \mathrm{Ext}^2_\Lambda(\operatorname{Tr} M^{op}, \Lambda) \longrightarrow 0$$

implies that $M \cong M^{**}$. □

The key use of Gorenstein projectives is that they are closed under extensions. This has been shown in various places, see e.g., [13, Proposition 5.1].

Corollary 4.3.3. *Let Λ be a n-canonical order over a CM local ring R with canonical module ω. Then $\mathcal{S}$ is closed under extensions.*

4.4 Main Theorem

We now return to proving the following main theorem of this chapter.

Theorem 4.4.1. *Let R be a CM local ring with canonical module ω. Let Λ be an n-canonical R-order. If Λ has only finitely many nonisomorphic indecomposable modules in $\mathcal{S}$, then Λ is an n-isolated singularity.*

The proof of this involves several lemmas. It follows closely Huneke and Leuschke's proof of Auslander's theorem, [25]. We start with Miyata's Theorem.

Lemma 4.4.2. [38, Theorem 2] *Let Λ be a module finite algebra over a commutative Noetherian ring R. Suppose we have an exact sequence of finitely generated Λ-modules*

$$M \longrightarrow X \longrightarrow N \longrightarrow 0$$

and that $X \cong M \oplus N$. Then the sequence is a split short exact sequence.

From this we are able to deduce the following lemma about $\operatorname{Ext}^1_\Lambda(N, M)$. This is originally due to Huneke-Leuschke, [25, Theorem 1]. Here we include a simpler proof contained in [35, Lemma 7.10]. Note that the $r = 0$ case of this lemma is exactly Miyata's theorem, Lemma 4.4.2.

Lemma 4.4.3. *Let $(R, \mathfrak{m})$ be a CM local ring and Λ an R-order. Fix $r \in \mathfrak{m}$. Suppose we have an exact sequence of Λ-modules*

$$\alpha : \ 0 \longrightarrow M \longrightarrow X_\alpha \longrightarrow N \longrightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccccccc}
\alpha : \ 0 & \longrightarrow & M & \xrightarrow{i} & X_\alpha & \longrightarrow & N & \longrightarrow & 0 \\
& & {\scriptstyle r}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\| & & \\
r\alpha : \ 0 & \longrightarrow & M & \xrightarrow[j]{} & X_{r\alpha} & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

If $X_\alpha \cong X_{r\alpha}$, then $\alpha \in r \operatorname{Ext}^1_\Lambda(N, M)$.

Proof. The commutative diagram in the lemma is a pushout diagram, so we have an exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \oplus X_\alpha \xrightarrow{g} X_{r\alpha} \longrightarrow 0,$$

where $f = \begin{bmatrix} r \\ -i \end{bmatrix}$ and $g = \begin{bmatrix} j & f \end{bmatrix}$. Since $M \oplus X_\alpha \cong M \oplus X_{r\alpha}$, Lemma 4.4.2 implies this exact sequence is split. From this we get that the induced map on Ext,

$$\begin{bmatrix} r \\ -i_* \end{bmatrix} : \operatorname{Ext}^1_\Lambda(N, M) \longrightarrow \operatorname{Ext}^1_\Lambda(N, M) \oplus \operatorname{Ext}^1_\Lambda(N, X_\alpha),$$

is a split monomorphism. Let $h : \operatorname{Ext}^1_\Lambda(N, M) \oplus \operatorname{Ext}^1_\Lambda(N, X_\alpha) \longrightarrow \operatorname{Ext}^1_\Lambda(N, M)$ be a splitting, i.e., a left inverse of this split monomorphism. Now apply $\operatorname{Hom}_\Lambda(N, -)$ to the exact sequence

$$\alpha :\ 0 \longrightarrow M \longrightarrow X_\alpha \longrightarrow N \longrightarrow 0;$$

this yields an exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_\Lambda(N, N) \xrightarrow{\delta} \operatorname{Ext}^1_\Lambda(N, M) \xrightarrow{i_*} \operatorname{Ext}^1_\Lambda(N, X_\alpha) \longrightarrow \cdots .$$

The connecting map δ takes 1_N to α, hence $i_*(\alpha) = 0$, as the sequence is exact. From this we get that

$$\begin{bmatrix} r \\ -i_* \end{bmatrix} (\alpha) = (r\alpha, 0).$$

It follows that

$$\alpha = h \circ [r \ \ -i_*]^T(\alpha) = h(r\alpha, 0) = rh(\alpha, 0) \ \in r \operatorname{Ext}^1_\Lambda(N, M). \qquad \square$$

Now, we are able to prove the following lemma from which the main theorem follows. This proof in the commutative case is again due to Huneke-Leuschke [25]; the generalization to the case of orders is straightforward, but included for convenience.

Lemma 4.4.4. *Suppose Λ is an order over a local ring $(R, \mathfrak{m}, k)$. Let*

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0$$

be an exact sequence of Λ-modules. If there are only finitely many choices for X up to isomorphism, then $\operatorname{Ext}^1_\Lambda(N, M)$ is a finite length R-module.

Proof. Let $\alpha \in \operatorname{Ext}^1_\Lambda(N, M)$ and $r \in \mathfrak{m}$. It is well known that an R-module M has finite length if and only if for all $r \in \mathfrak{m}$ and $x \in M$ there is an integer n so that $r^n x = 0$. Thus, we must only show that $r^n\alpha = 0$ for $n \gg 0$. For any integer n we consider a representative

$$r^n\alpha : \quad 0 \longrightarrow M \longrightarrow X_n \longrightarrow N \longrightarrow 0.$$

Since only finitely many X_n can exist up to isomorphism there is an infinite sequence $n_1 < n_2 < n_3 < \ldots$ such that $X_{n_i} \cong X_{n_j}$ for all pairs i, j. Set $\beta = r^{n_1}\alpha$, and let $i > 1$. Then $r^{n_i}\beta = r^{n_i - n_1}\alpha$. We show $\beta = 0$. We have, for each i, a commutative diagram

$$\begin{array}{rcccccccc}
\beta : & 0 & \longrightarrow & M & \longrightarrow & X_{n_1} & \longrightarrow & N & \longrightarrow 0 \\
 & & & {\scriptstyle r^{n_i - n_1}}\downarrow & & \downarrow & & \| & \\
r^{n_i - n_1}\beta : & 0 & \longrightarrow & M & \longrightarrow & X_{n_i} & \longrightarrow & N & \longrightarrow 0.
\end{array}$$

By Lemma 4.4.3, since $X_{n_1} \cong X_{n_i}$, we have $\beta \in r^{n_i - n_1}\operatorname{Ext}^1_\Lambda(N, M)$ for every i. Since the sequence of n_i is infinite and strictly increasing, this means $\beta \in \mathfrak{m}^t \operatorname{Ext}^1_\Lambda(N, M)$ for all t. Finally, the Krull Intersection Theorem [36, Theorem 8.10] implies $\beta = 0$. □

Finally, we provide the proof of the main theorem following this lemma.

Proof of Theorem 4.3.1. Let $M, N \in \mathcal{S}$. By Lemma 4.2.5 we must only show that $\ell(\operatorname{Ext}^1_\Lambda(N, M)) < \infty$. Consider any sequence $\alpha \in \operatorname{Ext}^1_\Lambda(N, M)$,

$$\alpha : \ 0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0.$$

By the work above, specifically Corollary 4.3.3, we know $X \in \mathcal{S}$. Now since M and N are finitely generated and there are only finitely many indecomposable modules in $\mathcal{S}$, there are only finitely many possibilities for X. Namely, X must be one of the finitely many modules in $\mathcal{S}$ generated by at most $\mu_\Lambda(M) + \mu_\Lambda(N)$ where $\mu_\Lambda(Y)$ denotes the minimum number of generators of Y over Λ. Thus, $\ell(\operatorname{Ext}^1_\Lambda(N, M)) < \infty$ by Lemma 4.4.4. $\square$

In view of Theorem 4.1.9, we arrive at the following strengthening of Auslander's theorem in the case where R is a suitable Gorenstein local ring.

Corollary 4.4.5. *Let R be a Gorenstein local domain, and let Q be an acyclic quiver. If there exist only finitely many nonisomorphic indecomposable modules in $\Omega \operatorname{CM}(RQ)$, then R is an isolated singularity, i.e.,*

$$\operatorname{gldim} R_\mathfrak{p} = \dim(R_\mathfrak{p})$$

for all non-maximal primes ideals $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. We only need to notice that by Theorem 4.1.9 RQ is a 1-canonical order. Thus by Theorem 4.3.1 if there are only finitely many indecomposable modules in $\Omega^n \operatorname{CM}(RQ)$ we must have that RQ is a 1-isolated singularity. But, by Proposition 2.2.13, this is only possible if $\operatorname{gldim} R_\mathfrak{p} < \infty$ for all non-maximal primes $\mathfrak{p}$. Since R is commutative, this is only possible if $\operatorname{gldim} R_\mathfrak{p} = \dim(R_\mathfrak{p})$. $\square$

We give a quick example which demonstrates that $\Omega \operatorname{CM} \Lambda$ has the propensity to be much smaller than $\operatorname{CM} \Lambda$.

Example 4.4.6. Let R be a regular local ring and Q be the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

It is known that RQ has infinitely many indecomposable modules in $\operatorname{CM} \Lambda$. But, since $\operatorname{gldim} RQ = \dim R + 1$, we know $\Omega \operatorname{CM} RQ = \operatorname{Proj} RQ$ which definitely has finitely many indecomposables.

Chapter 5

Higher AR Theory in positive dimension

Auslander-Reiten (AR) theory was developed by Maurice Auslander and Idun Reiten over the course of several papers, including [8–12] . It has been used in the representation theory of hereditary finite dimensional algebras over fields, [14] and the representation theory of commutative Noetherian rings, [35, 49]. In [27], Osamu Iyama generalizes this to algebras of higher global dimensions and orders over commutative Noetherian rings. In this chapter we consider results of Iyama on cluster tilting subcategories for finite dimensional algebras, [29]. Specifically, we examine orders of finite global dimension $n + \dim R$, and use Iyama's higher Auslander-Reiten theory to examine the category of maximal Cohen-Macaulay modules.

This chapter contains two main results. In the case of algebras over Artinian rings, sufficient Ext vanishing implies vanishing of dual modules, [29, Lemma 2.16]. Our first main result, Theorem 5.2.4 proves that this in fact characterizes the Krull dimension 0 case. Moreover, we extend the result to arbitrary finite Krull dimension and produce a condition on our order Λ which detects the Krull dimension of the base ring R. In [29], Iyama characterizes cluster tilting subcategories over finite dimensional algebras of finite global dimension. In particular, he proves that if Γ is a finite dimensional algebra of global dimension n, then there are no τ_n-periodic modules in any n-cluster tilting subcategory of $\operatorname{mod} \Gamma$. The proof of this does not extend to higher Krull dimensions. We investigate the possible existence of τ_n-periodic modules, and provide a sufficient condition, Theorem 5.3.7, which prevents a module from being τ_n-periodic.

In this chapter, R is a d-dimensional Cohen-Macaulay local ring with canonical module ω. We will let Λ be an R-order which is an isolated singularity.

5.1 Background on Higher AR Theory

Higher AR Theory was introduced by Iyama in [27] in order to study algebras which have finite global dimension, but which are not hereditary (i.e., algebras Γ such that $\operatorname{gldim}\Gamma = n > 1$). A first resultof Iyama's work is a higher (homological) dimensional analog of the Auslander Correspondence, [26, Theorem 0.2]. Since, what were originally called "maximal-$(n-1)$ orthogonal" subcategories, have been renamed n-cluster tilting subcategories, and a great deal of work has been done in finding and studying them. We begin by introducing higher AR theory and some of the tools involved. The following proposition is a special case of Proposition 4.2.6 where $n = 0$.

Proposition 5.1.1. [27, Prop 1.3.1] *Suppose Λ is an R-order which has at most an isolated singularity.*

1. *X is in $\operatorname{CM}\Lambda$ if and only if $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$ for $i = 1, \ldots, d-1$.*

2. *$\underline{\operatorname{Hom}}_\Lambda(X, Y), \operatorname{Ext}^i_\Lambda(X, Y)$ and $\operatorname{Tor}^\Lambda_i(Z, X)$ are all finite length R-modules for any $X \in \operatorname{CM}\Lambda$, $Y \in \operatorname{mod}\Lambda$ and $Z \in \operatorname{mod}\Lambda^{op}$*

One of the main reasons to ask that Λ be an isolated singularity is that being in $\operatorname{CM}\Lambda$ is equivalent to a module being a d^{th} syzygy, which mirrors the commutative case, [35, Corollary A.15]. The proof is very similar to the commutative case. The fact that Λ is an isolated singularity replaces the assumption that R is Gorenstein on the punctured spectrum.

Proposition 5.1.2. *Let R be a CM local ring of dimension d. Let Λ be an R-order which has at most an isolated singularity. If $M \in \operatorname{CM}\Lambda$, then there exists a non-zero Λ-module X so that $M \cong \Omega^d X$.*

Proof. We define Serre's condition (S_n). Say an R-module (or a Λ-module) M satisfies (S_n) if

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant \min\{n, \dim R_\mathfrak{p}\} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R.$$

We show that if $M \in \operatorname{mod}\Lambda$ satisfies (S_k) for some $k \geqslant 1$, then there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} F \longrightarrow N \longrightarrow 0$$

where F is a finitely generated free Λ-module and N satifies (S_{k-1}). This will finish the proof since $M \in \operatorname{CM}\Lambda$ satifies (S_n) for all $n \geqslant 0$.

Begin with an exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow M^* \longrightarrow 0 \tag{5.1.1}$$

where G is a finitely generated free Λ-module. Setting $F = G^*$ we have an exact sequence

$$0 \longrightarrow M^{**} \xrightarrow{\beta} F \longrightarrow K^* \longrightarrow \operatorname{Ext}^1_\Lambda(M^*, \Lambda) \longrightarrow 0. \tag{5.1.2}$$

Now set $\sigma : M \longrightarrow M^{**}$ to be the natural morphism, $\alpha = \beta\sigma$ and $N = \operatorname{cok}\alpha$. Then we have a sequence

$$0 \longrightarrow M \xrightarrow{\alpha} F \longrightarrow N \longrightarrow 0 \tag{5.1.3}$$

as desired. We must only verify that it is exact and that N satisfies (S_{k-1}).

To prove that the sequence

$$0 \longrightarrow M \xrightarrow{\alpha} F \longrightarrow N \longrightarrow 0 \tag{5.1.4}$$

is exact, we must only show that α is injective. Suppose $L = \ker \alpha \neq 0$. Choose an associated prime $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{p}} L$. Then $\operatorname{depth}_{R_\mathfrak{p}} L_\mathfrak{p} = 0$.

We claim that $\operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p} \geqslant 1$. Since $k \geqslant 1$ and M satisfies (S_k), this will imply $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant 1$. Since Λ is an isolated singularity, $\Lambda_\mathfrak{q}$ is nonsingular for all minimal primes $\mathfrak{q}$. Further, $M_\mathfrak{q} \in \mathrm{CM}\,\Lambda_\mathfrak{q}$ for all minimal primes $\mathfrak{q}$, since $R_\mathfrak{q}$ is zero-dimensional. Thus $M_\mathfrak{q}$ is $\Lambda_\mathfrak{q}$-projective by [32, Prop 2.17]. Then $\sigma_\mathfrak{q}$ is an isomorphism for all minimal primes $\mathfrak{q}$. Thus, $L_\mathfrak{q} = 0$ for all mininal primes $\mathfrak{q}$. Since $L \neq 0$ there must be at least one non-maximal prime ideal, hence $\dim R_\mathfrak{p} \geqslant 1$. It follows that $\operatorname{depth}_R M_\mathfrak{p} \geqslant 1$ since M satifies (S_k), which contradicts that $\operatorname{depth}_R L_\mathfrak{p} = 0$, thus it could not be that $L \neq 0$.

We now show that N satisfies (S_{k-1}). Let $\mathfrak{p}$ be a prime of height h. If $h \leqslant k-1$ we must show that $N_\mathfrak{p} \in \mathrm{CM}\,\Lambda$. Since $\Lambda_\mathfrak{p}$ is non-singular, and $M_\mathfrak{p} \in \mathrm{CM}\,\Lambda_\mathfrak{p}$ the canonical map $\sigma_\mathfrak{p}$ is an isomorphism. Also, $M_\mathfrak{p}^* \in \mathrm{CM}\,\Lambda_\mathfrak{p}$ so $\operatorname{Ext}^1_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}^*, \Lambda_\mathfrak{p}) = 0$. Together with sequence (5.1.2) this implies that $N_\mathfrak{p} \cong K_\mathfrak{p}^*$. The Depth Lemma and the sequence (5.1.1) imply that $K_\mathfrak{p} \in \mathrm{CM}\,\Lambda$, and therefore $K_\mathfrak{p}^* \in \mathrm{CM}\,\Lambda$.

To finish the proof, suppose now that $h \geqslant k$. We need to show that $\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \geqslant k-1$. Suppose the contrary, that $\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} < k-1$. Since F is free and hence $\operatorname{depth}_{R_p} F_\mathfrak{p} \geqslant k-1$, the Depth Lemma and sequence (5.1.4) imply that

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 1 + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} < k.$$

This is a contradiction to the fact that M satifies (S_k). □

Our goal is to study $\mathrm{CM}\,\Lambda$ by taking n^{th} syzygies and using Auslander-Reiten theory. Here we will outline the basic tools we will use, beginning with the n-AR translation. Recall that $D_d := \operatorname{Hom}_R(-, \omega_R) : \mathrm{CM}\,\Lambda \longrightarrow \mathrm{CM}\,\Lambda$ is a duality.

Definition 5.1.3. Assume Λ is an isolated singularity. We let

$$\tau_1 = D_d \Omega^d \operatorname{Tr}$$
$$\tau_1^- = \Omega^d \operatorname{Tr} D_d$$

be the usual AR translations [35, Chapter 12]. For $n > 1$ we define the n-AR translation

$$\tau_n = \tau_1 \Omega^{n-1}$$
$$\tau_n^- = \tau_1^- \Omega^{n-1}$$

It is well known that for a hereditary k-algebra, the functor $\tau_1 : \underline{\mathrm{CM}}\,\Lambda \longrightarrow \overline{\mathrm{CM}}\Lambda$ is an equivalence. The main problem which arises in higher global dimensions is that τ_n : $\underline{\mathrm{CM}\Lambda} \longrightarrow \overline{\mathrm{CM}}\Lambda$ is no longer an equivalence, [27]. Instead, we restrict the domain and codomain of this functor.

Notation 5.1.4. For $X, Y \in \operatorname{mod}\Lambda$ write $X \perp_n Y$ if $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for $i = 1, \ldots, n$. For full subcategories $\mathcal{C}$ and $\mathcal{D}$ of $\operatorname{mod}\Lambda$, write $\mathcal{C} \perp_n \mathcal{D}$ if $X \perp_n Y$ for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Put $\mathcal{C}^{\perp_n} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0\ (0 < i \leqslant n)\}$ and ${}^{\perp_n}\mathcal{C} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0\ (0 < i \leqslant n)\}$. Set

$$\mathcal{X}_n = {}^{\perp_{n-1}}\Lambda \cap \mathrm{CM}\,\Lambda, \qquad \mathcal{Y}_n = \omega_\Lambda^{\perp_{n-1}} \cap \mathrm{CM}\,\Lambda.$$

We note that $D_d : \mathcal{X}_n \longrightarrow \mathcal{Y}_n$.

Lemma 5.1.5. [27, Theorem 1.4.1] *$\tau_n : \underline{\mathcal{X}}_n \longrightarrow \overline{\mathcal{Y}}_n$ and $\tau_n^- : \overline{\mathcal{Y}}_n \longrightarrow \underline{\mathcal{X}}_n$ are mutually quasi-inverse equivalences. Moreover, τ_n induces a bijection from indecomposable non-projectives in $\mathcal{X}_n$ to indecomposable non-injectives in $\mathcal{Y}_n$.*

One of the main tools in the use of AR theory is *AR Duality*. We have a generalization as follows:

Theorem 5.1.6 (n-AR Duality, [27, Theorem 1.5])**.** *Let Λ be as above. For any $0 < i < n$ there exist functorial isomorphisms below for any $X \in \mathcal{X}_n$, $Y \in \mathcal{Y}_n$, and $Z \in \operatorname{CM}\Lambda$.*

$$\operatorname{Ext}^{n-i}_\Lambda(X,Z) \cong D\operatorname{Ext}^i_\Lambda(Z,\tau_n X) \qquad \underline{\operatorname{Hom}}_\Lambda(X,Z) \cong D\operatorname{Ext}^n_\Lambda(Z,\tau_n X)$$

$$\operatorname{Ext}^{n-i}_\Lambda(Z,Y) \cong D\operatorname{Ext}^i_\Lambda(\tau_n^- Y,Z) \qquad \overline{\operatorname{Hom}}_\Lambda(Z,Y) \cong D\operatorname{Ext}^n_\Lambda(\tau_n^- Y,Z)$$

This is stated in [27] for R a Gorenstein ring. We include a proof in the Cohen-Macaulay case for the convenience of the reader. For the proof we will need the following:

Notation 5.1.7. For canonical module, ω, of R, we have an injective resolution over R,

$$0 \longrightarrow \omega \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_d \longrightarrow 0.$$

Here I_d is $E_R(k)$, the injective hull of the residue field of R.

Lemma 5.1.8. [27, Proposition 1.1.3] *Let Λ be a Noetherian ring and $n \geqslant 1$. For any i ($0 < i < n$), $X \in \mathcal{X}_n$ and $Y \in \operatorname{mod}\Lambda$ there exist functorial isomorphisms*

$$\operatorname{Tor}^\Lambda_{n-i}(\operatorname{Tr}\Omega^{n-1}X,Y) \cong \operatorname{Ext}^i_\Lambda(X,Y) \qquad \operatorname{Tor}^\Lambda_n(\operatorname{Tr}\Omega^{n-1}X,Y) \cong \underline{\operatorname{Hom}}_\Lambda(X,Y)$$

Proof of Theorem 5.1.6. We prove the first isomorphism as the rest are similar. For any injective module I, we have a functorial isomorphism $\operatorname{Hom}_R(\operatorname{Tor}^\Lambda_i(W,Z),I) \cong \operatorname{Ext}^i_\Lambda(W,\operatorname{Hom}_R(Z,I))$ via [18, VI, Prop. 5.1]. Begin with the injective resolution,

$$0 \longrightarrow \omega \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_d \longrightarrow 0.$$

Then, applying $\operatorname{Hom}_R(Z,-)$ and the fact that Z is MCM over R, hence $\operatorname{Ext}^i_R(Z,\omega) = 0$ for $i > 0$, we obtain an exact sequence

$$0 \longrightarrow D_d Z \longrightarrow \operatorname{Hom}_R(Z,I_0) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(Z,I_d) \longrightarrow 0.$$

By the remark above, we know $\mathrm{Hom}_R(\mathrm{Tor}_i^\Lambda(W,Z),I_j) \cong \mathrm{Ext}_\Lambda^i(W,\mathrm{Hom}_R(Z,I_j))$. Further, since $\mathrm{Tor}_i^\Lambda(W,Z)$ has finite length for $i>0$ by Proposition 5.1.1 and [36, Theorem 18.4] we see $0=\mathrm{Hom}_R(\mathrm{Tor}_i^\Lambda(W,Z),I_j)=\mathrm{Ext}_\Lambda^i(W,\mathrm{Hom}_R(Z,I_j))$ for $i>0, j<d$. It follows easily that $\mathrm{Hom}_R(\mathrm{Tor}_i^\Lambda(W,Z),I_d)=\mathrm{Ext}_\Lambda^i(W,\mathrm{Hom}_R(Z,I_d))=\mathrm{Ext}_\Lambda^{i+d}(W,D_dZ)=\mathrm{Ext}_\Lambda^i(\Omega^dW,D_dZ)=\mathrm{Ext}_\Lambda^i(Z,D_d\Omega^dW)$. The result follows by setting $W=\mathrm{Tr}\,\Omega^{n-1}X$. □

Lemma 5.1.9. *For $X\in\mathrm{CM}(\Lambda)$, $\tau_n(X)=0$ if and only if $\mathrm{projdim}_\Lambda X<n$.*

Proof. We must show the "only if". We suppose $\tau_n(X)=D_d\Omega^d\,\mathrm{Tr}\,\Omega^{n-1}(X)=0$. Since D_d is a duality on $\mathrm{CM}(\Lambda)$, we have $\Omega^d\,\mathrm{Tr}\,\Omega^{n-1}(X)=0$. Thus, it must be that we have a projective resolution

$$0\longrightarrow P_m\longrightarrow P_{m-1}\longrightarrow\cdots\longrightarrow P_0\longrightarrow \mathrm{Tr}\,\Omega^{n-1}(X)\longrightarrow 0$$

where $m\leqslant d$. By the definition of Tr we get an exact sequence

$$0\longrightarrow\Omega^{n-1}(X)\longrightarrow P_2^*\longrightarrow P_3^*\longrightarrow\ldots\longrightarrow P_m^*\longrightarrow\mathrm{Ext}_\Lambda^m(\mathrm{Tr}\,\Omega^{n-1}(X),\Lambda)\longrightarrow 0$$

but $\mathrm{Ext}_\Lambda^m(\mathrm{Tr}\,\Omega^{n-1}(X),\Lambda)=0$ by 5.1.1 and hence the sequence above splits and $\Omega^{n-1}(X)$ is projective. □

5.2 Ext vanishing and Dual modules

In this section, we prove that [29, Lemma 2.3 (b)], a key ingredient in Iyama's study of so-called n-representation type characterizes the case when $d=0$. Moreover, we extend the result to arbitrary finite $d\geqslant 0$ and are able to produce a condition on our order Λ which detects the Krull dimension of the base ring R. In his paper, Iyama begins with a finite dimensional algebra Γ which has global dimension n. To consider the natural generalization of these objects in the setting of orders, we will strengthen our assumptions on Λ. Let R be a CM local ring of dimension d. Suppose Λ is an order over R such that Λ is an isolated singularity and $\mathrm{gldim}\,\Lambda\leqslant n+d$. The new assumption of finite global dimension allows us to use a different characterization of τ_n. Note that if $X \in \mathrm{CM}(\Lambda)$ and $\operatorname{projdim}_\Lambda X = n$ (the situation where $\tau_n(X) \neq 0$, by Proposition 5.1.9), then we have a projective resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

It follows that $\operatorname{Tr}\Omega^{n-1}X = \operatorname{cok}(P_{n-1}^* \longrightarrow P_n^*) = \operatorname{Ext}^n_\Lambda(X,\Lambda)$. Thus we have that

$$\tau_n(-) = D_d\Omega^d \operatorname{Ext}_\Lambda(-,\Lambda) : \underline{\mathcal{X}_n} \longrightarrow \overline{\mathcal{Y}_n}$$
$$\tau_n^-(-) = \Omega^d \operatorname{Ext}_\Lambda(D_d\Lambda,-) : \overline{\mathcal{Y}_n} \longrightarrow \underline{\mathcal{X}_n}.$$

Also with our new assumptions on the global dimension, we can say more about the projective dimension of modules $X \in \mathcal{X}_n$. Since Λ is an isolated singularity any $X \in \mathrm{CM}\,\Lambda$ is a d^{th} syzygy by Proposition 5.1.2.Together with the fact that $\operatorname{gldim}\Lambda \leqslant n+d$ we get that $\operatorname{projdim}_\Lambda X \leqslant n$ for any $X \in \mathrm{CM}\,\Lambda$. By Lemma 2.1.13 we see that $\operatorname{projdim}_\Lambda X \in \{0,n\}$ for any $X \in \mathcal{X}_n$.

The following result is a key part of Iyama's n-cluster tilting theory [29] and the work of Hirschend-Iyama-Oppermann [24] and Iyama-Oppermann [31] on n-representation type. Our goal is to prove a converse to this and generalize it to higher Krull dimension.

Proposition 5.2.1. [29, Lemma 2.3(b)] *Let Γ be a finite dimensional algebra with $\operatorname{gldim}\Gamma \leqslant n$, where $n > 1$. If X has no projective summands and $\operatorname{Ext}^i_\Gamma(X,\Gamma) = 0$ for $i = 1\ldots,n-1$, then $\operatorname{Hom}_\Gamma(X,\Gamma) = 0$.*

We wish to show that this actually characterizes the $d = 0$ case. We start by investigating this condition for orders where $d \geqslant 1$, noting that if $n \leqslant d-1$, Prop 5.2.1 does not hold. This is due to the behavior of depth over R with respect to exact sequences.

Due to Lemma 2.1.4, Λ-modules which have finite length over R cannot have projective resolutions (even over Λ) of length less than d. From this observation we have the following lemma.

Lemma 5.2.2. *Let $d \geqslant n+1 \geqslant 2$. Suppose Λ is an R-order which is an isolated singularity with* $\operatorname{gldim} \Lambda \leqslant n+d$ *and $X \in \mathcal{X}_n$. Then $X^* \neq 0$.*

Proof. Consider a projective resolution over Λ of X,

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

By our hypotheses on X, applying $(-)^*$ gives an exact sequence

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots \longrightarrow P_n^* \longrightarrow \operatorname{Ext}^n_\Lambda(X, \Lambda) \longrightarrow 0. \tag{5.2.1}$$

Since Λ is an isolated singularity, $\operatorname{Ext}^n_\Lambda(X,\Lambda)$ has finite length (hence depth 0) as an R-module. If $X^* = 0$, then $\operatorname{projdim} \operatorname{Ext}^n_\Lambda(X,\Lambda) < d$ which is impossible by the Depth Lemma. □

We wish to study the behavior of dual modules of modules in $\mathcal{X}_n = {}^{\perp_{n-1}}\Lambda \cap \operatorname{CM} \Lambda$. In view of 5.2.1, we may begin by studying the behavior of $\operatorname{Ext}^n_\Lambda(X,\Lambda)$:

Lemma 5.2.3. *Suppose Λ is an isolated singularity and has* $\operatorname{gldim} \Lambda \leqslant n+d$*. Then for any $X \in \mathcal{X}_n$ which is not projective,*

$$\operatorname{projdim}_\Lambda \operatorname{Ext}^n_\Lambda(X,\Lambda) = n+d.$$

For any $Y \in \mathcal{Y}_n$ which is not injective,

$$\operatorname{projdim}_\Lambda \operatorname{Ext}^n_\Lambda(D_d Y,\Lambda) = n+d.$$

Proof. Let $X \in \mathcal{X}_n$ not projective. Then we know that $\tau_n(X) = D_d \Omega^d \operatorname{Ext}^{\mathrm{n}}_\Lambda(X,\Lambda) \in \mathcal{Y}_n$, hence for all $i = 1, \ldots, n-1$

$$0 = \operatorname{Ext}^i_\Lambda(D_d\Lambda, D_d\Omega^d \operatorname{Ext}^n_\Lambda(X,\Lambda)) \tag{5.2.2}$$
$$= \operatorname{Ext}^i_\Lambda(\Omega^d \operatorname{Ext}^{\mathrm{n}}_\Lambda(X,\Lambda),\Lambda) \tag{5.2.3}$$
$$= \operatorname{Ext}^{i+d}_\Lambda(\operatorname{Ext}^{\mathrm{n}}_\Lambda(X,\Lambda),\Lambda) \tag{5.2.4}$$

Hence $\operatorname{Ext}^i_\Lambda(\operatorname{Ext}^n_\Lambda(X,\Lambda),\Lambda) = 0$ for $i = d+1, \ldots, d+n-1$. Since $\operatorname{Ext}^n_\Lambda(X,\Lambda)$ has finite length, any projective resolution must have length at least d by Lemma 2.1.4. Since $\operatorname{Ext}^k_\Lambda(Y,\Lambda) \neq 0$ for $\operatorname{projdim} Y = k$, it must be that $\operatorname{projdim}_\Lambda \operatorname{Ext}^n_\Lambda(X,\Lambda) = d$ or $n+d$. But, by 5.1.1, $\operatorname{Ext}^d_\Lambda(\operatorname{Ext}^n_\Lambda(X,\Lambda),\Lambda) = 0$ since $\operatorname{Ext}^n_\Lambda(X,\Lambda) = \operatorname{Tr}\Omega^{n-1}X$ for $\operatorname{gldim}\Lambda \leqslant n+d$. Thus, it must be that $\operatorname{projdim}_\Lambda \operatorname{Ext}^{\mathrm{n}}_\Lambda(X,\Lambda) = n+d$.

The proof of the other assertion is dual to this. □

From this we get the following characterization of projective dimension of dual modules for orders which have finite global dimension, the main theorem of this section.

Theorem 5.2.4. *Let Λ be an R-order which is an isolated singularity with* $\operatorname{gldim}\Lambda = n+d$, *where $n > 0$. The following are equivalent:*

1. $d = k+1$
2. *For all non-projective indecomposables $X \in \mathcal{X}_n$,* $\operatorname{projdim}_\Lambda \operatorname{Hom}_\Lambda(X,\Lambda) = k$;
3. *For all non-injective indecomposables $Y \in \mathcal{Y}_n$,* $\operatorname{projdim}_\Lambda \operatorname{Hom}_\Lambda(D_d\Lambda, Y) = k$;

If $\Lambda \in \mathcal{Y}_n$ (e.g., if Λ has an n-cluster tilting subcategory, see definition 5.3.1), then this is further equivalent to:

4. $\operatorname{projdim}_\Lambda(\operatorname{Hom}_\Lambda(\tau_n^-(\Lambda),\Lambda) = k$;
5. $\operatorname{projdim}_\Lambda(\operatorname{Hom}_\Lambda(D_d\Lambda, \tau_n(D_d\Lambda)) = k$;

In particular all dual modules of non-projective $X \in \mathcal{X}_n$ have the same projective dimension.

Proof. ((1)⇒(3)) Let $X \in \mathcal{X}_n$ with no projective summands and suppose $d = k+1$; it follows that $\operatorname{gldim} \Lambda = n + k + 1$ so that the sequence

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow \cdots \longrightarrow P_n^* \longrightarrow \operatorname{Ext}^n_\Lambda(X, \Lambda) \longrightarrow 0$$

indicates that $\operatorname{projdim}_\Lambda X^* \leqslant k$. Now if $\operatorname{projdim} X^* < k$, we have via the exact sequence above that $\operatorname{projdim} \operatorname{Ext}^n_\Lambda(X, \Lambda) < d + n$. This is impossible via Lemma 5.2.3, thus it must be $\operatorname{projdim} X^* = k$.

((3)⇒(1)) We know by (1) ⇒ (3) that for $X \in \mathcal{X}_n$, we have $\operatorname{projdim}_\Lambda X^* = d - 1$. Thus if all $X \in \mathcal{X}_n$ have $\operatorname{projdim}_\Lambda X^* = k$, it is clear $d = k + 1$.

The proof of (1) ⇔ (2) is dual to this.

In the case that $\Lambda \in \mathcal{Y}_n$, the assertions (2) ⇔ (4) and (3) ⇔ (5) are clear since $\tau_n(\Lambda) \in \mathcal{X}_n$ and $\tau_n(D_d\Lambda) \in \mathcal{Y}_n$. □

Remark 5.2.5.

1. Note that, in the case $n = 0$ in the above theorem, Λ is non-singular. From Theorem 3.3.2 it follows that $\operatorname{CM} \Lambda = \operatorname{Proj} \Lambda$. This implies that $\tau_n^-(\Lambda)$ and $\tau_n(D_d\Lambda)$ are both projective. In this case, the argument fails; k (which is necessarily 0) and d need not be related.

2. The theorem remains true with the convention that $\operatorname{projdim}_\Lambda X = -1$ if and only if $X = 0$. In other words, this theorem shows that Lemma 5.2.1 completely characterizes the case $d = 0$.

5.3 n-Cluster Tilting for Orders

Cluster tilting subcategories were introduced by Iyama in [27]. They play a pivotal role in the study of algebras which are not hereditary. In another article, [28], Iyama explains their introduction by noting that for a k-algebra Γ the pair (τ_n, τ_n^-) is an adjoint pair, but in general these are not equivalences $\underline{\text{mod}}\Gamma \leftrightarrow \overline{\text{mod}}\Gamma$. Cluster tilting subcategories are restrictions of the domain and codomain of these functors. They provide the natural setting for which a higher version of Auslander Reiten duality and higher almost split sequences exist.

Iyama has many results on cluster tilting for algebras. In particular, he shows that if an algebra of global dimension at most n possesses a cluster tilting object, then it cannot have τ_n-periodic modules. This is a key fact in the definition of n-representation finite and infinite algebras [24, 31]. In the case of orders, it is not clear if τ_n-periodic modules can exist, even with an assumption on the global dimension. The main result of this section is a criterion which indicates a module must not be τ_n-periodic.

For what follows, we will again assume Λ is an order with an isolated singularity and $\text{gl.dim}\Lambda = n + d$. We begin with the definitions in which we are interested.

Definition 5.3.1. A subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called *covariantly finite* if any object $X \in \mathcal{A}$ has a *left $\mathcal{C}$-approximation*, i.e., there is an object $C \in \mathcal{C}$ and a map $X \longrightarrow C$ such that the sequence of functors

$$(-, C) \longrightarrow (-, X) \longrightarrow 0$$

is exact on $\mathcal{C}$, where $(A, B) := \text{Hom}_{\mathcal{A}}(A, B)$ for any $A, B \in \mathcal{A}$. *Contravariantly finite* and *right $\mathcal{C}$-approximation* are defined dually. A subcategory $\mathcal{C}$ is *functorially finite* if it is both covariantly and contravariantly finite.

Finally, a subcategory $\mathcal{C} \subset \text{CM}\Lambda$ is called an *n-cluster tilting subcategory* if $\mathcal{C}$ is a functorially finite subcategory and

$$\begin{aligned}\mathcal{C} =& \{X \in \text{CM}\Lambda \mid \text{Ext}^i_\Lambda(X, \mathcal{C}) = 0,\ i = 1, \ldots, n-1\} \\ =& \{X \in \text{CM}\Lambda \mid \text{Ext}^i_\Lambda(\mathcal{C}, X) = 0,\ i = 1, \ldots, n-1\}\end{aligned}$$

We call $M \in \text{CM}(\Lambda)$ an *n-cluster tilting object* if $\text{add}M$ is an n-cluster tilting subcategory.

Notation 5.3.2. For an order Λ over a Cohen-Macaulay local ring R with canonical module ω_R which contains an n-cluster tilting subcategory $\mathcal{C}$ define $\mathcal{M} = \operatorname{add}\{\tau_n^l(\omega_\Lambda) \mid l \geqslant 0\}$.

Theorem 5.3.3. [27, Theorem 2.3] *Let $\mathcal{C}$ be an n-cluster tilting subcategory of* $\mathrm{CM}(\Lambda)$*. Then:*

- $\tau_n(X) \in \mathcal{C}$ *and* $\tau_n^-(X) \in \mathcal{C}$ *for any* $X \in \mathcal{C}$*.*
- $\tau_n : \underline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ *and* $\tau_n^- : \overline{\mathcal{C}} \longrightarrow \underline{\mathcal{C}}$ *are mutually quasiinverse equivalences.*
- τ_n *gives a bijection from indecomposable non-projective objects in* $\mathcal{C}$ *to indecomposable non-injective objects in* $\mathcal{C}$*.*

We start with the generalization of the foundation of n-cluster tilting and n-Auslander Reiten theory. The following is a generalization of [29, Proposition 1.3], which is foundational in Iyama's theory. The proof is the same as in [29].

Proposition 5.3.4. *Let Λ be a R-order with M an n-cluster tilting module in* $\mathrm{CM}\,\Lambda$*.*

(1) Exactly one of the following happens for each indecomposable X in $\operatorname{add} M$*:*

  (a) $\tau_n^l(X) \cong X$ *for some* $l \geqslant 0$*.*

  (b) $X \cong \tau_n^l(I)$ *for some injective module I and* $X \cong \tau_n^m(P)$ *for some projective module P.*

(2) A bijection from indecomposable injective Λ-modules to indecomposable projective Λ-modules is given by $I \mapsto \tau_n^{l_I}(I)$ *where l_I is the maximal integer so* $\tau_n^l(I) \neq 0$*.*

Note that part (1) says that if $X \in \operatorname{add} M$ is not τ_n-periodic, then $X \in \operatorname{add}\{\tau_n^l(\omega_\Lambda) \mid l \in \mathbb{Z}\}$.

We wish to address the existence of τ_n-periodic modules, which do not exist in the $d = 0$ case, see [29, Proposition 1.3]. Our main result is Theorem 5.3.7. Before it is presented we define the extinguishing time of Λ.

Definition 5.3.5. Let Λ be an order over a CM local ring R of dimension d with a canonical module. Suppose Λ has global dimension at most $n+d$ and possesses an n-cluster tilting module M. Then the *extinguishing time* of Λ is $T := \min\{s \mid \ \tau_n^s(\omega_\Lambda) = 0\}$.

Remark 5.3.6. In the case of algebras, finite global dimension and existence of a cluster tilting module M is enough to yield that $\tau_n^l(\omega_\Lambda) = 0$ for some l implies that $\tau_n^l(X) = 0$ for any $X \in \operatorname{add} M$. Thus finite extinguishing time is equivalent to τ_n-finite. This is not clear in the case of orders. It is still true, by Proposition 5.3.4 (1)(b), that $\tau_n^T(X) = 0$ for any $X \in \mathcal{M}$. In particular, if $\operatorname{gldim} \Lambda \leqslant n+d$ and Λ possesses an n-cluster tilting module M, then the extinguishing time of Λ is finite.

Given a finite dimensional algebra Γ of global dimension at most n with an n-cluster tilting object, Iyama has shown that Theorem 5.3.4 (1b) cannot occur. In particular the unique n-cluster tilting object is given by $\bigoplus_{l\in\mathbb{Z}} \tau_n^l(\omega_\Lambda)$ which is a finitely generated Γ-module. The necessary step of taking a d^{th} syzygy in the middle of τ_n renders Iyama's proof non-applicable to our case. It is not clear if τ_n-periodic modules can exist for orders of finite global dimension. Their (non)existence is crucial to a theory of higher-representation type. The following theorem is a first step in investigating such objects.

Theorem 5.3.7. *Suppose Λ is an R-order of global dimension at most $n+d$. Suppose Λ has a n-cluster tilting module $M \in \operatorname{CM}\Lambda$. If $X \in \operatorname{add} M$ is τ_n-periodic, then there exists an integer $0 \leqslant s \leqslant T$ so that $\operatorname{Ext}^n_\Lambda(\tau_n^-(\Lambda), \tau_n^s(X)) \neq 0$. In other words, if $\operatorname{Ext}^n_\Lambda(\tau_n^-(\Lambda), \tau_n^l(X)) = 0$ for all $0 \leqslant l \leqslant T$, then $X \in \operatorname{add}\{\tau_n^l(\omega_\Lambda) \mid l \in \mathbb{Z}\}$.*

For the proof we will use a few lemmas.

Lemma 5.3.8. *Let Λ be an order with $\operatorname{gldim}\Lambda \leqslant n+d$. Given an exact sequence*

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$$

with $\operatorname{Ext}^{n-1}_\Lambda(X,\Lambda)=0$, $Y,Z\in\operatorname{CM}\Lambda$, *we have an exact sequence*

$$0\longrightarrow\tau_n(X)\longrightarrow\tau_n(Z)\oplus I\longrightarrow\tau_n(Y)\longrightarrow 0$$

where $I\in\operatorname{add}\omega_\Lambda$.

Proof. Since $\operatorname{gldim}\Lambda\leqslant n+d$, we know that $\tau_n(-)=D_d\Omega^d\operatorname{Ext}^n_\Lambda(-,\Lambda)$. From the long exact sequence of $\operatorname{Ext}_\Lambda(-,\Lambda)$ and the fact that $\operatorname{Ext}^{n-1}_\Lambda(X,\Lambda)=0$ we have an exact sequence

$$0\longrightarrow\operatorname{Ext}^n_\Lambda(Y,\Lambda)\longrightarrow\operatorname{Ext}^n_\Lambda(Z,\Lambda)\longrightarrow\operatorname{Ext}^n_\Lambda(X,\Lambda)\longrightarrow 0$$

since $Y\in\operatorname{CM}\Lambda$ and hence has projective dimension at most n by Proposition 5.1.2. The Horseshoe Lemma yields an exact sequence

$$0\longrightarrow\Omega^d\operatorname{Ext}^n_\Lambda(Y,\Lambda)\longrightarrow\Omega^d\operatorname{Ext}^n_\Lambda(Z,\Lambda)\oplus P\longrightarrow\Omega^d\operatorname{Ext}^n_\Lambda(X,\Lambda)\longrightarrow 0$$

for some projective module P. Then since each term is in $\operatorname{CM}\Lambda$ we have an exact sequence

$$0\longrightarrow D_d\Omega^d\operatorname{Ext}^n_\Lambda(X,\Lambda)\longrightarrow D_d\Omega^d\operatorname{Ext}^n_\Lambda(Z,\Lambda)\oplus D_dP\longrightarrow D_d\Omega^d\operatorname{Ext}^n_\Lambda(Y,\Lambda)\longrightarrow 0.$$

This is exactly the desired sequence. □

We can use this lemma to examine the behavior of τ_n on long exact sequences as well.

Lemma 5.3.9. *Let* Λ *be as in Lemma 5.3.8. Suppose there is an exact sequence*

$$0\longrightarrow X\longrightarrow I_0\longrightarrow I_1\longrightarrow\longrightarrow\cdots\longrightarrow I_k\longrightarrow 0$$

for some $1\leqslant k\leqslant n$ *with* $\operatorname{Ext}^j_\Lambda(I_m,\Lambda)=0$ *for* $0\leqslant m\leqslant k$ *and* $1\leqslant j\leqslant n-1$. *Set* $Y_m=\ker(I_{m-1}\longrightarrow I_m)$ *for* $1\leqslant m\leqslant k$. *If* $\operatorname{Ext}^{n-1}_\Lambda(Y_i,\Lambda)=0$ *for* $i=0,\dots,k-1$ *then we* *have an exact sequence*

$$0 \longrightarrow \tau_n(X) \longrightarrow \tau_n(I_0) \oplus I_{0,1} \longrightarrow \cdots \longrightarrow \tau_n(I_{k-1}) \oplus I_{k-1,1} \longrightarrow \tau_n(I_k) \longrightarrow 0$$

where $I_{i,j} \in \operatorname{add} \omega_\Lambda$.

Proof. We first break the long exact sequence into short exact sequences. We have

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow Y_1 \longrightarrow 0 \tag{5.3.1}$$

$$0 \longrightarrow Y_l \longrightarrow I_l \longrightarrow Y_{l+1} \longrightarrow 0 \qquad l = 1, \ldots, k-1. \tag{5.3.2}$$

By assumption, the left hand terms of all exact sequences 5.3.1 and 5.3.2 satisfy $\operatorname{Ext}_\Lambda^{n-1}(-, \Lambda) = 0$. Now, Lemma 5.3.8 yields exact sequences

$$0 \longrightarrow \tau_n(X) \longrightarrow \tau_n(I_0) \oplus I_{0,1} \longrightarrow \tau_n(Y_1) \longrightarrow 0$$

$$0 \longrightarrow \tau_n(Y_i) \longrightarrow \tau_n(I_i) \oplus I_{i,0} \longrightarrow \tau_n(Y_{i+1}) \longrightarrow 0 \qquad i = 1, \ldots, k-1.$$

These can then clearly be pieced together to give the long exact sequence desired. □

With this we are able to prove our main result on the existence of τ_n-periodic modules.

Proof of Theorem 5.3.7. Note that since $X \in \operatorname{add} M$, we also have $D_d X \in \operatorname{add} M$. Since X is assumed to be τ_n-periodic it cannot be projective. It is easy to see that by definition

$$D_d \tau_n^-(-) = \tau_n D_d(-).$$

From this it follows that if X is τ_n-periodic, then $D_d X$ is τ_n-periodic (over Λ^{op}); in particular, $\tau_n(D_d X) \neq 0$. By Lemma 5.1.9 since $\tau_n(D_d X) \neq 0$, we have $\operatorname{projdim}_{\Lambda^{op}} D_d X = n$. We begin with a projective resolution of $D_d X$ over Λ^{op}

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow D_d X \longrightarrow 0.$$

Since $X \in \operatorname{add} M$, we know $\operatorname{Ext}^i_{\Lambda^{op}}(D_d X, \Lambda) = \operatorname{Ext}^i_\Lambda(D_d\Lambda, X) = 0$ for $i = 1, \dots, n-1$ by (2.1.1) and the definition of n-cluster tilting. Thus we have an exact sequence

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

where $I_j \in \operatorname{add} \omega_\Lambda$. In order to prove the Theorem, we wish to employ Lemma 5.3.9 repeatedly. We want to, for any $l \geqslant 0$, achieve an exact sequence

$$0 \longrightarrow \tau_n^l(X) \longrightarrow J_0^{(l)} \longrightarrow J_1^{(l)} \longrightarrow \dots \longrightarrow J_{n-1}^{(l)} \longrightarrow \tau_n^l(I_n) \longrightarrow 0 \tag{5.3.3}$$

with $J_j^{(l)} \in \mathcal{M}$ for all j, $0 \leqslant l \leqslant T$. From Lemma 5.3.9 it is clear we must only establish the following claim

Claim: $\operatorname{Ext}^{n-1}_\Lambda(\tau_n^l(Y_k), \Lambda) = 0$ for $l \geqslant 0$ and $k = 1, \dots, n-1$.

With this claim we have the long exact sequence (5.3.3) desired. Indeed, after an application of $\tau_n(-)$, we have short exact sequences

$$0 \longrightarrow \tau_n(X) \longrightarrow J_1^{(1)} \longrightarrow \tau_n(Y_1) \longrightarrow 0$$
$$0 \longrightarrow \tau_n(Y_i) \longrightarrow J_i^{(1)} \longrightarrow \tau_n(Y_{i+1}) \longrightarrow 0 \qquad i = 1, \dots, k-1.$$

By the claim we can apply $\tau_n(-)$ again (and so on), yielding, for any l, exact sequences

$$0 \longrightarrow \tau_n^l(X) \longrightarrow J_1^{(l)} \longrightarrow \tau_n^l(Y_1) \longrightarrow 0$$
$$0 \longrightarrow \tau_n^l(Y_i) \longrightarrow J_i^{(l)} \longrightarrow \tau_n^l(Y_{i+1}) \longrightarrow 0 \qquad i = 1, \dots, k-1$$

where $J_i^{(l)} \in \mathcal{M}$. These can clearly be pieced together to form the sequence (5.3.3). Then, since $\tau_n^T(I_n) = 0$, we have an exact sequence

$$0 \longrightarrow \tau_n^T(X) \longrightarrow J_0^{(T)} \longrightarrow J_1^{(T)} \longrightarrow \dots \longrightarrow J_{n-1}^{(T)} \longrightarrow 0.$$

Now we can apply Lemma 5.3.9 again since $n-1<n$. Since $\tau_n^T(J_{n-1}^{(T)})=0$ by remark 5.3.6, we eventually get

$$0 \longrightarrow \tau_n(X) \longrightarrow J_1^{(2T)} \longrightarrow \cdots \longrightarrow J_{n-3}^{(2T)} \longrightarrow J_{n-2}^{(2T)} \longrightarrow 0.$$

We can repeat this until we arrive at the exact sequence $0 \longrightarrow \tau_n^R(X) \longrightarrow J_N \longrightarrow 0$ for $J_N \in \mathcal{M}$. Since X is indecomposable, this indicates $X \in \mathcal{M}$. We must now only prove the claim.

Proof of Claim. The case for $I_n = Y_n$ is handled first. By n-AR Duality, Theorem 5.1.6, we see that $\operatorname{Ext}^1_\Lambda(\tau_n^l(Y_1),\Lambda) = D\operatorname{Ext}^{n-1}_\Lambda(\tau_n^-(\Lambda),\tau_n^l(Y_1))$ for all l. Then, applying $\operatorname{Hom}_\Lambda(\tau_n^-(\Lambda),-)$ to the exact sequence

$$0 \longrightarrow \tau_n^l(X) \longrightarrow J \longrightarrow \tau_n^l(Y_1) \longrightarrow 0,$$

which we get from the long exact sequence in $\operatorname{Ext}_\Lambda(\tau_n^-(\Lambda),-)$ and the fact that $J \in \operatorname{add} M$, we get an exact sequence

$$0 \longrightarrow \operatorname{Ext}^{n-1}_\Lambda(\tau_n^-(\Lambda),\tau_n^k(Y_1)) \longrightarrow \operatorname{Ext}^n_\Lambda(\tau_n^-(\Lambda),\tau_n^k(X)).$$

This part of the claim follows immediately from our assumption on X.

The rest of the claim is proved by induction on l. For $l=0$, we simply have the exact sequence

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0.$$

Then, for any $k=1,\dots,n-1$, we have

$$\operatorname{Ext}^{n-1}_\Lambda(Y_k,\Lambda) \cong \operatorname{Ext}^{n-2}_\Lambda(Y_{k-1},\Lambda) \cong \ldots \cong \operatorname{Ext}^{n-k}_\Lambda(X,\Lambda) = 0.$$

Thus, the base case is established. Now suppose the claim holds for some $l-1>0$. Since the first part is established for all l, we can use Lemma 5.3.9 to produce an exact sequence

$$0 \longrightarrow \tau_n^l(X) \longrightarrow J_0^{(l)} \longrightarrow \cdots \longrightarrow J_{n-1}^{(l)} \longrightarrow \tau_n^l(I_n) \longrightarrow 0.$$

It is clear from the proof of Lemma 5.3.9 that $\tau_n^l(Y_k) = \ker(J_k^{(l)} \longrightarrow J_{k+1}^{(l)})$. Then since $J_j^{(l)} \in \mathcal{M}$ for all j, we can again dimension shift to get

$$\mathrm{Ext}_\Lambda^{n-1}(\tau_n^l(Y_k), \Lambda) \cong \mathrm{Ext}_\Lambda^{n-2}(\tau_n^l(Y_{k-1}), \Lambda) \cong \ldots \cong \mathrm{Ext}_\Lambda^{n-k}(\tau_n^l(X), \Lambda) = 0$$

for all $k = 1, \ldots, n-1$. This establishes the claim, and thus the Theorem. $\square$.

Bibliography

[1] Ibrahim Assem, Andrzej Skowroński, and Sonia Trepode, *The representation dimension of a selfinjective algebra of euclidean type*, J. Algebra **459** (2016), 157–188. MR3503970

[2] Maurice Auslander, *On the purity of the branch locus*, Amer. J. Math. **84** (1962), 116–125. MR0137733

[3] ——, *Isolated singularities and existence of almost split sequences*, Representation theory, II (Ottawa, Ont., 1984), 1986, pp. 194–242. MR842486

[4] ——, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531. MR816307

[5] ——, *Representation dimension of artin algebras,* Selected works of Maurice Auslander. Part 1, American Mathematical Society, Providence, RI, 1999. Edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Øyvind Solberg. MR1674397

[6] Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR0269685

[7] Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405. MR0086822

[8] Maurice Auslander and Idun Reiten, *Representation theory of Artin algebras. III. Almost split sequences*, Comm. Algebra **3** (1975), 239–294. MR0379599

[9] ——, *Stable equivalence of dualizing R-varieties. V. Artin algebras stably equivalent to hereditary algebras*, Advances in Math. **17** (1975), no. 2, 167–195. MR0404243

[10] ——, *Representation theory of Artin algebras. IV. Invariants given by almost split sequences*, Comm. Algebra **5** (1977), no. 5, 443–518. MR0439881

[11] ——, *Representation theory of Artin algebras. V. Methods for computing almost split sequences and irreducible morphisms*, Comm. Algebra **5** (1977), no. 5, 519–554. MR0439882

[12] ———, *Representation theory of Artin algebras. VI. A functorial approach to almost split sequences*, Comm. Algebra **6** (1978), no. 3, 257–300. MR0472919

[13] ———, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1991), no. 1, 111–152. MR1097029

[14] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original. MR1476671

[15] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956

[16] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings*, Unpublished Manuscript, 1987.

[17] Ragnar-Olaf Buchweitz, Graham J. Leuschke, and Michel Van den Bergh, *Non-commutative desingularization of determinantal varieties I*, Invent. Math. **182** (2010), no. 1, 47–115. MR2672281

[18] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original. MR1731415

[19] Hailong Dao, Eleonore Faber, and Colin Ingalls, *Noncommutative (crepant) desingularizations and the global spectrum of commutative rings*, Algebr. Represent. Theory **18** (2015), no. 3, 633–664. MR3357942

[20] Hailong Dao, Osamu Iyama, Ryo Takahashi, and Charles Vial, *Non-commutative resolutions and Grothendieck groups*, J. Noncommut. Geom. **9** (2015), no. 1, 21–34. MR3337953

[21] Hans-Bjø rn Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973). MR0327752

[22] Peter Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. 6 (1972), 309. MR0332887

[23] Robin Hartshorne, *Generalized divisors and biliaison*, Illinois J. Math. **51** (2007), no. 1, 83–98 (electronic). MR2346188

[24] Martin Herschend, Osamu Iyama, and Steffen Oppermann, *n-representation infinite algebras*, Adv. Math. **252** (2014), 292–342. MR3144232

[25] Craig Huneke and Graham J. Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391–404. MR1933863

[26] Osamu Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82. MR2298820

[27] ______, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50. MR2298819

[28] ______, *Auslander-Reiten theory revisited*, Trends in representation theory of algebras and related topics, 2008, pp. 349–397. MR2484730

[29] ______, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61. MR2735750

[30] Osamu Iyama and Yusuke Nakajima, *On steady non-commutative crepant resolutions*, arXiv preprint arXiv:1509.09031 (2015).

[31] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614. MR2833569

[32] Osamu Iyama and Michael Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math. **197** (2014), no. 3, 521–586. MR3251829

[33] Graham J. Leuschke, *Endomorphism rings of finite global dimension*, Canad. J. Math. **59** (2007), no. 2, 332–342. MR2310620

[34] ______, *Non-commutative crepant resolutions: scenes from categorical geometry*, Progress in commutative algebra 1, 2012, pp. 293–361. MR2932589

[35] Graham J. Leuschke and Roger Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012. MR2919145

[36] Hideyuki Matsumura, *Commutative ring theory*, Second, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461

[37] John McKay, *Graphs, singularities, and finite groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), 1980, pp. 183–186. MR604577

[38] Takehiko Miyata, *Note on direct summands of modules*, J. Math. Kyoto Univ. **7** (1967), 65–69. MR0214585

[39] Christopher L. Quarles, *Krull-Schmidt rings & noncommutative resolutions of singularities*, Master's thesis, University of Washingon (2005).

[40] Idun Reiten, *The converse to a theorem of Sharp on Gorenstein modules*, Proc. Amer. Math. Soc. **32** (1972), 417–420. MR0296067

[41] Ralf Schiffler, *Quiver representations*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2014. MR3308668

[42] R. Y. Sharp, *On Gorenstein modules over a complete Cohen-Macaulay local ring*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 425–434. MR0289504

[43] S. Paul Smith and Christopher Quarles, *On an example of Leuschke*, Private communication (2005).

[44] J. T. Stafford and M. Van den Bergh, *Noncommutative resolutions and rational singularities*, Michigan Math. J. **57** (2008), 659–674. Special volume in honor of Melvin Hochster. MR2492474

[45] Michel van den Bergh, *Non-commutative crepant resolutions*, The legacy of Niels Henrik Abel, 2004, pp. 749–770. MR2077594

[46] Keiichi Watanabe, *Certain invariant subrings are Gorenstein. I, II*, Osaka J. Math. **11** (1974), 1–8; ibid. 11 (1974), 379–388. MR0354646

[47] Dana Weston, *Divisorial properties of the canonical module for invariant subrings*, Comm. Algebra **19** (1991), no. 9, 2641–2666. MR1125195

[48] ______, *Stability of the divisor class group upon completion*, Illinois J. Math. **51** (2007), no. 1, 313–323. MR2346200

[49] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990. MR1079937

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