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The Elementary Theory of Normed Linear Spaces and Linear Functionals

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Abstract

In this paper, we introduce the notions of normed linear spaces and linear functionals. The organization of the paper is as follows. Section 2 gives the definition of a norm, provides examples, theorems relating to them, and spaces on which they act. Section 3 delves into linear functionals by also giving examples and ending with a standard result. Sections 4 and 5 extend the idea of normed linear spaces to complete linear spaces, with the latter ending with a fascinating theorem. The thesis concludes in Section 6, which gives a method for constructing a complete linear space.

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Preface

Background and Reflections

I began to contemplate whether or not to write a senior honors thesis about nine months before my senior year at Syracuse University. The biggest decisions I had to make at the time were choosing the topic I would focus upon, and under whom would I research it. Through the advice of fellow students and professors, I narrowed my choice to functional analysis and Professor John Lindberg Jr.

In November of 2004, I introduced myself to Professor Lindberg and asked for his permission to be my advisor. It took a few weeks before he would even accede to my request. Luckily for me he did, and to my surprise informed me that the project would begin immediately. The reason for this stemmed from his imminent research leave in the spring semester of 2006; the time at which I would graduate. The year long process of preparing a thesis commenced during the winter recess of my junior year.

The aim of our project was to closely imitate mathematical research. Because of this, Professor Lindberg settled a few guidelines; the most substantial being that I was prohibited from using any resources outside of my past and present course work. In my opinion, this rule had to be one of the most difficult aspects of writing the paper as it went against the tendencies of my inquisitive habits. It forced to be somewhat original when solving problems, as I could no longer follow an "algorithm" that so many students of mathematics, including myself, depend upon. This restriction paid great dividends in the end as it forced me to think independently, like a research mathematician.

Because of the way researchers attack problems, many of them exhibit extraordinary creativity. The most elegant mathematical proofs have taken a "romantic" approach, rather than displaying brute force computations and inductions. An excellent example of this was Israel Gelfand's new proof of a theorem by Norbert Weiner. Gelfand, in his seminal paper, introduced revolutionary techniques and tools that would pave the way for many future analysts. However, the proofs in this thesis are nowhere near as crafty (if crafty at all) as Gelfand's or any other prominent mathematician's. In fact, the purpose of writing this paper was to introduce me to such mathematical thinking and improve my skills at solving problems. At this time, I feel that I have made long strides in that direction, although still far from the purview of mathematical sharpness.

Aside from this, one of the hardest parts of the experience was developing confidence. All of the results in this thesis are basic facts in the area in which Professor Lindberg works. However, many of them are far from trivial. By the guidelines enforced by my advisor, finding proofs to such fundamental theorems was very difficult. In essence, I was given the task of developing the elementary theory of functional analysis in less than a year. The pith of this thesis was written during the summer of 2006, and many a time I doubted my abilities in mathematics simply because I couldn't get any substance down. But through the confidence of Professor Lindberg, I started to realize I could finish these proofs. Working with him helped me figure out a lot about myself, in terms of how I thought about mathematics, and ultimately, my views towards things in life. His influence has spanned far beyond my mathematical career.

A few things should be said about Professor John Lindberg. While in graduate school, he worked with Benny Gelbaum, a mathematician who did his Ph.D. thesis under Salomon Bochner. During the 1950's, Bochner was one of the world's top minds, leading the prestigious analysis group at Princeton University. Being a mathematical grandson of Bochner, Professor Lindberg inherited the same interests as him. Specifically, he studied functional analysis and Banach algebras at the University of Minnesota-Minneapolis, a place that continues to be one of the top analysis schools in the country. These areas of research are rooted in a very important subject, that being linear algebra.

Foundations and History of Thesis Topic

The theory of linear algebra is one of the best developed in the field of mathematics. Its applications are extensive—ranging from Jacobian matrices in multivariable calculus to quadratic forms in number theory. Although the more immediate uses of linear algebra deal less with complicated ideas and theorems proved, the more advanced concepts and questions are what intrigue mathematicians. In fact, we rarely see the notion of "infinite-dimensional linear spaces" used in economics or computer science, areas that have significantly been furthered by linear algebra.

In the 1920's, a Polish mathematician named Stefan Banach published a paper titled "On Operations on Abstract Sets and their Application to Integral Equations" (O'Connor) that marked the birth of "functional analysis". Simply stated, the goal of functional analysis at the time was (and generally still is) to study infinite-dimensional linear spaces, an area that eluded many mathematicians. Banach's research was broad enough to name a mathematical object after him; its name is the "Banach Space". Today, the Banach Space is so ubiquitous in research that its applications go far beyond that of functional analysis (particularly harmonic analysis, complex analysis, probability theory, and physics). It has been brought to forefront of mathematics by people such as Israel Gelfand (who was mentioned earlier) and Banach himself.

Banach Space theory is centered upon the structure known as the normed linear space. A "norm" essentially provides an abstraction of length and distance between elements in a linear space. Given certain conditions, this norm can give rise to a very interesting concept known as "completeness". More than that, the study of mappings whose domain is the normed linear space provides even more substance to this already rich theory. We commonly refer to these as linear functionals; it will be shown in this thesis how Banach Spaces and linear functionals are intimately tied (hence the name, "functional analysis").

This discussion has been written to provide a thorough yet clear introduction to normed linear spaces and linear functionals. Only an elementary knowledge in real analysis and abstract algebra is needed to comprehend this paper. Therefore, we will freely use the language seen in courses such as MAT 512 - *Introduction to Real Analysis* and MAT 534 - *Introduction to Modern Algebra*. We have assiduously tried to make definitions, lemmas, and theorems as clear and explicit as possible in order make the content easier for the reader.

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