July 2016

Business Calculus Students’ Reasoning about Optimization Problems: A Study of Quantitative Reasoning in an Economic Context

Thembinkosi Mkhatshwa
Syracuse University

Abstract

While the opportunity to learn mathematics via textbooks is well documented at the secondary and elementary levels, research on the opportunity to learn mathematics via textbooks at the undergraduate level has received little attention. Furthermore, research that examines the role of mathematics textbooks in students' learning of important concepts such as marginal change in applied calculus is scarce. Research on students' quantitative reasoning at the post-secondary level is lacking. This qualitative study investigated the opportunity to learn about optimization problems, marginal change, and quantitative reasoning in an economic context via a business calculus textbook and from lectures in a business calculus course. The study also investigated students' quantitative reasoning, using task based interviews conducted with 12 pairs of business calculus students, about optimization problems and marginal change in an economic context.

This study found that the textbook's presentation of optimization problems and marginal change was largely procedural with limited attention to the underlying concepts and that opportunities for students to reason about relationships between or among economic quantities such as the relationship between marginal cost and marginal revenue at a profit maximizing quantity received little attention. The presentation of optimization problems and marginal change in course lectures closely followed the presentation of these topics in the textbook. Students' interpretations of marginal change varied in different contexts and representations depending on the tasks they were given. This study provided insights into students' quantitative reasoning when analyzing multivariable situations in an economic context: students created new quantities that helped them to solve the problems in the tasks and helped them to reason about relationships among several quantities. Implications for different stakeholders including business calculus instructors and suggestions for further research are included.

Business Calculus Students' Reasoning about Optimization Problems: A Study of Quantitative Reasoning in an Economic Context

By

Thembinkosi P. Mkhatshwa

B.Sc., University of Swaziland, 2008
M.A., Marshall University, 2010
M.S., Syracuse University, 2013

DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics Education in the Graduate School of Syracuse University

Syracuse University
July 2016

Acknowledgements

I thank the Almighty God for His grace that has kept me alive and for His invaluable help while doing this research. I dedicate this dissertation to God the Father, God the Son, and God the Holy Spirit.

To my dissertation advisor, teaching mentor, and math police, Prof. Helen Doerr, thank you so much for your patience and for teaching me how to conduct high quality research. Of all the resources I had doing this research, including resources you referred me to, you were my greatest resource. There were moments when I walked into your office frustrated and without a sense of what I needed to do to make progress in this research and you gladly offered me numerous suggestions that helped me make significant progress in this dissertation. I greatly appreciate you telling me, as often as you did, the good things I was doing right in addition to pointing out areas of improvement in the course of doing this dissertation study. I sincerely appreciate your time and guidance in doing this research. Without your expertise, this dissertation study would not have been a success.

To members of my dissertation committee, Prof. Moira McDermott and Prof. Leyla Karakas, thank you for sharing your expertise and perspectives throughout the dissertation process. To my program advisor, Prof. Joanna Masingila, thank you for all you have done for me. Throughout my graduate study, I was greatly inspired by your high level of organization. To Prof. Helen Doerr and Prof. Joanna Masingila, thank you for providing a wonderful and supportive environment for me to grow professionally. You both made a significant impact on my life as a mathematics instructor and now as a researcher in undergraduate mathematics education.

To my loving and caring wife, Londiwe Shongwe, thank you for all the love, care, and support you gave me throughout my graduate study. You sacrificed a lot, including resigning from your job, to be with me and support me in every way you could during my time as a graduate student at Syracuse University, and for that I sincerely thank you. Constantly thinking about you and my son, Nathan Nkosinaye Mkhathswa, greatly motivated me to complete this dissertation study.

To my fellow graduate students, thank you for all the time we spent together and most importantly for the feedback you gave me on my work during our weekly mathematics education seminars. I am also grateful to Pastor A.O. Fadiran and Pastor O. S. Arisekola (and their families) for their spiritual support and encouragement throughout my life as a student. I am also indebted to a number of people who supported me and my family during my time as a graduate student at Syracuse University. These include, among others and in no order of importance, Mr. Bob Adkins, Mrs. Ruth Dlamini-Ndeze, Pastors Bob and Kathy Kotlarz, Pastor David Anyanwu, Prof. Alex and Mrs. Ranjana Thevaranjan, and my church family.

Table of Contents

List of Tables and Figures

Tables

Figures

Chapter 1 – Introduction

Much research over the last three decades has documented difficulties encountered by undergraduate students when reasoning about real-valued functions of a single variable when solving calculus problems that are situated in a purely mathematics context or a physics context. Much of this research (e.g., Oehrtman, Carlson, & Thompson, 2008) has focused on students' understanding of key concepts in pre-calculus and calculus respectively, namely average rates of change and instantaneous rates of change. A growing body of research (e.g., Moore & Carlson, 2012) has focused on how undergraduate students reason about quantities and relationships between quantities when analyzing mathematical situations, an idea that has come to be known as quantitative reasoning (Thompson, 2011). Other research on students' understanding of calculus has focused on students' understanding of optimization problems (Klymchuk, Zverkova, & Sauerbier, 2010; Swanagan, 2012).

Little research has specifically investigated students' understanding of real-valued functions of two or more variables and how students engage in quantitative reasoning when solving optimization problems in an economic context. There is a dearth of research on what students' reasoning about an economic context reveals about their interpretation of important concepts in business or economics such as marginal change (e.g., marginal cost). Furthermore, research on students' algebraic reasoning when solving optimization problems, especially those that are situated in a business or economics, context is lacking. In this study, algebraic reasoning refers to students' reasoning about the algebra of continuous functions and their derivatives when solving economic-based optimization problems using algebraic methods. Algebraic reasoning when solving optimization problems in an economic context includes reasoning about critical numbers (e.g., profit maximizing quantities), reasoning about extrema (e.g., maximum profit), identifying constants and variables in a given problem situation, and finding an objective function (e.g., profit function) that relates constants and variables. While research on students' opportunity to learn mathematics via textbooks is well documented at the pre-university level (e.g., Alajmi, 2012; Wijaya, van den Heuvel-Panhuzen, & Doorman, 2015), research on students' opportunity to learn mathematics via textbooks at the undergraduate level is scarce.

There are five bodies of research literature that are related to this study: opportunity to learn mathematics via textbooks, calculus learning, learning in context, students' understanding of optimization problems, and quantitative reasoning. This study sought to add to the research base on the role of context on students' learning of calculus by investigating undergraduate students' interpretation of marginal change and quantitative reasoning when solving optimization problems that are situated in the economic context of cost, revenue, and profit.

The term quantity has been defined in similar ways by several researchers (e.g., Ärlebäck, Doerr, & O'Neil, 2013; Moore & Carlson, 2012; Smith & Thompson, 2008; Thompson, 1994a; Thompson, 2011). This study used the definition of quantity proposed by Ärlebäck et al. (2013): "a quantity is the result of conceiving a quality (an attribute) of an object to have an explicit or implicit unit that enables a process of measurement" (p. 317). Using this definition, Ärlebäck and colleagues argued that a rate of change (average or instantaneous rate of change) "can be considered a quantity created by the ratio of the covariational change of two quantities" (p. 318), namely the input variable as one quantity and the output variable as the other quantity. Thompson (1994a) refers to the process of forming a new quantity from other quantities as a quantitative operation. In this study, the term quantity refers to marginal change, number of items (products) produced or sold, as well as the cost, revenue, and profit obtained by making or selling a product(s).

This study used Thompson's (1993) definition of quantitative reasoning: analyzing a situation in terms of the quantities and relationships among the quantities involved in the situation. According to Thompson, what is important in quantitative reasoning is not assigning numeric measures to quantities but rather reasoning about relationships between or among quantities. The term, reasoning quantitatively, as used in this study, refers to how students described and represented relationships between or among quantities and how they created and used new quantities to solve the problems they were given.

Much research has investigated how students engage in quantitative reasoning in a kinematics context (e.g., Beichner, 1994; Monk, 1992; Moore & Carlson, 2012; Smith & Thompson, 2008; Thompson, 1994a; Thompson, 2011). There is no study that has investigated students' quantitative reasoning in an economic context. The term, discrete reasoning, as used in this study, refers to the treatment of continuous quantities as if they were discrete quantities when reasoning about relationships among several quantities in an economic context. Treating the continuous quantities (number of units produced and sold, total cost, total revenue, and profit) in Task 3 (Appendix A) as if they were discrete quantities when creating the profit graph is an example of discrete reasoning.

The terms derivative and instantaneous rate of change are used interchangeably to refer to the same concept. The terms, slope and average rate of change, refer to the same concept. Similarly, the term marginal change is used to refer to marginal cost, marginal revenue, and marginal profit collectively. Marginal cost refers to the cost per additional unit produced, marginal revenue refers to the revenue generated per additional unit sold, and marginal profit refers to the profit per additional unit produced and sold. Marginal change is a discrete quantity that can be approximated using a continuous function, the derivative. Mathematically, marginal change can be calculated using average rate of change which can be approximated using instantaneous rate of change.

The Aim of this Study

The aim of this study was twofold: (i) to examine opportunities provided by a business calculus textbook (Haeussler et al., 2011) and classroom instruction for business calculus students to learn about optimization problems, marginal change, and quantitative reasoning in the economic context of cost, revenue, and profit and (ii) to examine business calculus students' (undergraduate business or economics majors) algebraic reasoning, interpretation of marginal change, and quantitative reasoning when solving optimization problems that are situated in the economic context of cost, revenue, and profit. The following research questions guided this investigation:

1. What opportunities to learn about (a) optimization problems, (b) the concept of marginal change and (c) quantitative reasoning in the context of cost, revenue, and profit do business calculus textbooks and classroom instruction provide to business calculus students?
2. How do business calculus students reason algebraically about optimization problems that are situated in the context of cost, revenue, and profit?
3. How do business calculus students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit?
4. What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit reveal about their quantitative reasoning?

Rationale for this Study

Much of the research that has investigated students' reasoning when solving optimization problems has been based on problems that lack a real-world context and require students to recall geometric properties such the Pythagorean Theorem. There is only one study (Dominguez, 2010) that examined students' reasoning about an optimization problem in an economic context. Dominguez studied how students modeled a multivariable situation involving three quantities: the profit of a historic hotel, the number of rooms booked, and the daily charge for renting a room. The focus of that study was on developing an algebraic model (profit function) for this multivariable situation. However, the study did not examine students' quantitative reasoning while modeling the multivariable situation. There is a need for more research that investigates students' ability to model multivariable optimization problems in an economic context and their quantitative reasoning, that is, their reasoning about quantities and relationships among quantities involved when analyzing multivariable situations. This study sought to address this gap.

Reasoning about optimization problems involves reasoning about rates of change, including the concept of the derivative. There is a large body of research that investigated students' understanding of rates of change in a motion context (e.g., Beichner, 1994; Bingolbali & Monaghan, 2008; Berry & Nyman, 2003; Cetin, 2009; Christensen & Thompson, 2012; Marongelle, 2004). Of the research that investigated students' understanding of rates of change in non-motion contexts, only a few (e.g., Herbert & Pierce, 2008; Wilhelm & Confrey, 2003) examined students' understanding of average rates of change in a business or economic context. This study contributes to what we know about undergraduate students' understanding of rates of change in an economic context. The scarcity of research that examines the opportunity to learn how to solve context problems provided by college mathematics textbooks, is another motivation for this study. This study examined the opportunity to learn how to solve realistic context problems (optimization and marginal change problems) that occur in business or economics provided by a business calculus textbook.

There are several reasons that warrant the investigation carried out in this study. First, students' learning of mathematics has been shown to be largely dependent on mathematics textbooks as an opportunity to learn (e.g., Reys et al., 2004; Begle, 1973). Second, a large number of students enroll in business calculus nationwide every year. According to Gordon (2008), this number is more than 300,000 students. Third, given the importance of quantitative skills in business or economics (e.g., Butler, Finegan, & Siegfried, 1998; Von Allmen, 1996), there is a need to investigate students' quantitative reasoning in a business or economic context. Fourth, understanding optimization problems and marginal change in a business or economic context is vital in fields such as marketing, managerial accounting, supply chain management, finance, and economics.

Chapter 2 – Related Literature

Five bodies of research literature are related to this study. These bodies of research are: (1) opportunity to learn via mathematics textbooks, (2) calculus learning, (3) learning in context, (4) students' understanding of optimization problems, and (5) quantitative reasoning. This chapter is organized as follows: it begins with a description of the theoretical frameworks guiding the study followed by a review of research literature in each of the aforementioned bodies of research. The chapter concludes with a discussion of knowledge gaps in the research literature.

Theoretical Frameworks

This study draws on two theories, namely the theory of realistic mathematics education (RME) and the theory of quantitative reasoning.

The theory of realistic mathematics education. Realistic mathematics education is a theory of teaching and learning in mathematics education that originated in the Netherlands in the early 1970s. This theory has been used as a theoretical perspective for research in several nations of the world including the United States, England, South Africa, and Japan to name but a few (Zulkardi, 1999). Although Hans Freudenthal is widely recognized as the founder of RME, other researchers (e.g., Gravemeijer, 1994; Treffers, 1987) have significantly contributed to the development of RME. As a theory of learning, RME emphasizes that students should be asked to solve realistic contextual problems which are not only realistic in the sense of being connected to a real-world context but also that the context of these problems should be experientially real to the students. That is, the students should be asked to solve "problem situations which they can imagine" (van den Heuvel-Panhuizen, 2000, p. 4).

The idea of vertical and horizontal mathematising is an important feature of RME (Barnes, 2004; Freudenthal, 1991; Gravemeijer, 1994; Treffers, 1987). Treffers (1987) explained vertical and horizontal mathematising as follows:

> We distinguish horizontal and vertical mathematisation in order to account for the difference between transforming a problem field into a mathematical problem on the one hand, and processing within the mathematical system on the other hand. In the horizontal component the way towards mathematics is paved via model formation, schematizing, and symbolising. The vertical sketch [mathematising] is concerned with mathematical processing and level raising in the structuring of the problem field under consideration. We admit that this distinction between horizontal and vertical components is a bit artificial given the fact that they may be strongly related. (p. 247).

According to Freudenthal (1991), horizontal mathematising "leads from the world of life to the world of symbols" (p. 41) while vertical mathematising involves the manipulation of symbols. I illustrate my understanding of horizontal and vertical mathematising using the following example: Given an experientially real optimization problem (in text form) and asked to determine the maximum profit that a particular company makes in a given time period, horizontal mathematising involves the formulation of an algebraic model (objective function) based on the information given in the problem situation. Vertical mathematising involves finding the derivative of the model, finding a zero (critical number) for the derivative of the model, evaluating the model at the critical number to find the relative maximum, and comparing the relative maximum with the values of the model at the end points of the domain to determine the absolute maximum which is the maximum profit.

Influence of RME on this study. The design of my study was influenced by RME in three phases, namely task design, data collection, and data analysis. Using RME as a theoretical framework for my study, I designed mathematical tasks with the exception of Task 1 in Appendix A that are realistic to the students. Some of the tasks, especially Tasks 1 which I used to collect data that provided answers to the first research question, were designed in such a way that they afforded my study participants (students) the opportunity to engage in both horizontal and vertical mathematising. RME also influenced the recruitment of participants for my study: only business calculus students for whom the real-world context of cost, revenue, and profit as used in the tasks may be experientially real were recruited to participate in the study. During data analysis, I used the RME ideas of vertical and horizontal mathematising to create a data codebook. Part of my coding of data focused on the nature of horizontal and vertical mathematising demonstrated by the students as they reasoned about the tasks.

RME does not specifically address the type of mathematising that takes place when moving from the world of symbols back to the world of life. This could, for example, include interpreting the relative maximum of the model as the maximum profit realized by the company for the time period under consideration. It could also mean starting with a mathematical model and have students imagine a realistic situation that could be modeled using the model. I think of this as reverse horizontal mathematising which in a language similar to that used by Freudenthal (1991) would mean moving from the world of symbols back to the world of life. According to van den Heuvel-Panhuizen (2010), critics of RME argue that the adoption of RME has come with a de-emphasis on algorithmic calculations in favor of context problems in mathematics instruction. These critics argue that "mathematics should not be taught in context" (p. 1) which implies that mathematics should not be learned in context either. This means that from the critics' point of view, real-world contexts do not play a significant role in students' learning of mathematics.

The theory of quantitative reasoning. The theory of quantitative reasoning is an evolving theory in mathematics education whose origin can be traced to the early work of Thompson (1990). Quantitative reasoning, as defined by Thompson (1993), is the analysis of a situation in terms of the quantities and relationships among the quantities involved in the situation. According to Thompson, what is important in quantitative reasoning is not assigning numeric measures to quantities but rather reasoning about relationships between or among quantities. In economics, for instance, one may analyze how the total profit of a particular company changes with increased production and sales of the company's product by analyzing how the total cost and total revenue of the company are changing qualitatively, that is, without mentioning numerical values of total cost, total revenue, and total profit. For example, saying that total cost decreases steadily with increased production of the company's product while total revenue and total profit increases rapidly with increased production and sales of the company's product is an example of reasoning quantitatively that does not focus on numerical values of total cost, total revenue, and total profit but rather on the relationships among the quantities: total cost, total revenue, and total profit.

Thompson (2011) describes three tenets that are central to the theory of quantitative reasoning. These tenets are: a quantity, quantification, and quantitative operations. According to Thompson (1990), "a quantity is a quality of something that one has conceived as admitting some measurement process" (p. 5). Thompson added that "part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit" (p. 5). In his recent work, Thompson (2011) clearly stated that a quantity is a mental construction. Thompson (1993) distinguished between a quantity and a numerical value: a quantity has a unit of measurement and a numerical value does not. Thompson (1993) added that “quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them” (pp. 165-166). For example, we can think of Company A’s profit for a given trading period, Company B’s profit for the same trading period, and the amount by which Company A’s profit is bigger (or smaller) than Company B’s profit, without having to know the actual profit values.

The second tenet of the theory of quantitative reasoning is quantification. According to Thompson (2011), quantification “is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute’s measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit” (p. 37). In a sense, the process of quantification entails assigning numerical values to the attributes of an object. Thompson (2011) asserted that the process of quantification may vary from as little as taking a few minutes to as long as taking generations.

The third tenet of the theory of quantitative reasoning is a quantitative operation. A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). Thompson (1993) stated that “comparing two quantities with the intent to find the excess of one against the other” (p. 166) is a specific example of a quantitative operation formed by comparing two quantities additively. In economics, for example, comparing (by way of finding the difference) total revenue and total cost with the intent to find the excess (profit or loss) of total revenue against total cost is a quantitative operation known as a quantitative difference. Thompson (1993) distinguished between a numerical difference and a quantitative difference. According to Thompson, a numerical difference is “the result of subtracting” (p. 166). A quantitative difference, however, “is not always evaluated by subtraction, and subtraction may be used to evaluate quantities other than a quantitative difference" (p. 166). For example, saying that Company A's profit for a given trading period was by far more than Company B's profit for the same trading period is an example of a quantitative difference which has not been evaluated by subtraction.

Influence of the theory of quantitative reasoning on this study. The design of my study was influenced by the theory of quantitative reasoning in three phases, namely task design, data collection, and data analysis. Drawing on the theory of quantitative reasoning as a theoretical framework for my study, I designed mathematical tasks (appendix A) that provided students with opportunities to reason about relationships between or among quantities. Task 2, for example, provided opportunities for students to reason about relationships among sales (number of computers sold), the discount, and the revenue generated when the Smith family business sells computers to a school. The tasks were also designed to provide opportunities for students to engage in performing quantitative operations. For example, students had the opportunity to reason about quantitative differences when determining marginal cost (e.g., the cost of producing the second unit in Task 2). My interview protocol (Appendix A) allowed me to engage students in reasoning about relationships among quantities during the data collection process. For example, one of my prompts in Task 4 asked students about how the company's total cost, total revenue, and profit is changing across the production and sales levels shown in the table that appears in the task. Finally, my data analysis phase focused on looking for evidence for when students created new quantities (engaging in quantitative operation), how they used these quantities to reason about relationships between or among quantities, and whether or not these quantities helped the students to solve the problems posed in the tasks.

Opportunity to Learn via Mathematics Textbooks

The concept of opportunity to learn (OTL) as it relates to mathematics instruction dates back to the early 1960s. Carroll (1963) defined OTL as the time allowed for learning a particular topic. To Carroll's definition of OTL, Schmidt and colleagues (Cogan & Schmidt, 2015; Schmidt & Burroughs, 2015; Schmidt, Burroughs, Zoido, & Houang, 2015) added that there is a relationship between what students have an opportunity to learn on a particular topic and their performance in that topic. Cogan and Schmidt (2015) defined OTL as "the idea that the time a student spends in learning something is related to what that student learns" (p. 207). That is, there is a direct correspondence between students' learning outcomes on a particular topic and the amount of time they spend learning about that topic. Husen (1997) defined OTL as "whether or not … students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (pp. 162-163). This study used Husen's definition of OTL which, according to Floden (2002), is also the most common definition of OTL used in the mathematics education research literature. In my study, I examined how students reason and solve a particular problem type, optimization, they had an opportunity to learn about via a business calculus textbook (Haeussler et al., 2011) and via course lectures.

Emphasizing the importance of mathematics textbooks in students' learning of mathematics, Reys, Reys, and Chavez (2004) argued "that the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Similar sentiments have been shared by other researchers (e.g., Alajmi, 2012; Kolovou et al., 2009). Begle (1973) asserted that "most student learning is directed by the text rather than the teacher" (p. 209). Begle's view of the significant role mathematics textbooks play in students' learning of mathematics is consistent with the views of other researchers such as Törnroos

(2005) and Wijaya et al. (2015). Charalambous, Delaney, Hsu, and Mesa (2010) presented a moderate viewpoint on the importance of mathematics textbooks in students' learning of mathematics. These researchers posited that "textbooks afford probabilistic rather than deterministic opportunities to learn mathematics" (p. 118). That is, students' opportunities to learn mathematics need not be limited to only textbooks as there may be other opportunities (e.g., classroom instruction) through which students can learn mathematics besides mathematics textbooks.

The role of mathematics textbooks as an opportunity to learn mathematics is well documented in the research literature on the learning of K-12 mathematics. Some of this research focuses on students' opportunities to learn mathematical topics such as linear functions and trigonometry (e.g., Wijaya, van den Heuvel-Panhuzen, & Doorman, 2015), addition and subtraction of fractions (e.g., Alajmi, 2012; Charalambous et al., 2010), probability (e.g., Jones & Tarr, 2007), statistics (e.g., Pickle, 2012), reasoning and proof (e.g., Stylianides, 2009; Thompson et al., 2012), proportional reasoning (e.g., Dole & Shield, 2008), and deductive reasoning (e.g., Stacey & Vincent, 2009). Research on students' opportunities to learn about other mathematics topics such as optimization and marginal change and at the upper secondary and undergraduate level is lacking.

One study at the undergraduate level by Mesa, Suh, Blake, and Whittemore (2012) examined the opportunity to learn about exponential functions, logarithmic functions, and the transformations of functions provided by college algebra textbooks. Mesa and colleagues analyzed the cognitive demands of examples and the representations of these examples given in ten textbooks. Five of these textbooks were used at community colleges, three textbooks were used at four-year institutions, and the other two textbooks were used at both community colleges and at four-year institutions. Mesa and colleagues found that “textbooks, independent of the type of institution in which they are used, present examples that have low cognitive demands, expect single numeric answers, emphasize symbolic and numerical representations, and give very few strategies for verifying correctness of the solutions” (p. 76). These results suggest that the opportunity to learn about a wide range, in terms of cognitive demand and problem representation (e.g., graphical), of exponential and logarithmic-related problems is limited if not totally absent in college algebra textbooks. It would be important to examine opportunities to learn about other context areas such as marginal change that are provided by undergraduate mathematics textbooks such as business calculus textbooks.

In a preliminary study at the undergraduate level, Mkhatshwa and Doerr (in press) examined the opportunities to learn about optimization problems provided by six textbooks that are commonly used in the teaching of business calculus in the United States. These researchers analyzed a total of 195 optimization tasks (examples and practice problems). There were 24, 29, 29, 56, 32, and 25 tasks respectively in the six textbooks. The textbook (Haeussler et al., 2011) used in my study has 29 tasks. A majority of the tasks in the six textbooks had unrealistic contexts; all the tasks had the exact amount of information needed to solve the task; and only three textbooks had cognitively demanding tasks (three tasks in one textbook, one task in another textbook, and six tasks in another textbook). None of the cognitively demanding tasks were in the textbook that was used in my study. Mkhatshwa and Doerr argued that these six business calculus textbooks offered limited opportunities to learn about solving a wide range (in terms of types of context, types of information, and cognitive demands) of optimization problems. In my study, I analyzed the opportunity to learn on two other topics (marginal change and quantitative reasoning) in addition to the optimization problems that were analyzed by Mkhatshwa and Doerr.

A related line of research on the concept of opportunity to learn (OTL) shows that there is a relationship between the opportunity to learn a particular content area and students' performance in that content area at the secondary school level (Cogan & Schmidt, 2015; Schmidt et al., 2015, Schmidt & Burroughs, 2015). Schmidt and Burroughs analyzed questions designed to measure students' opportunity to learn different mathematical content (e.g., change and relationships) that were given in the 2012 Programme for International Student Assessment's (PISA) study. More than 500,000 secondary school students from 62 different countries participated in the mathematics portion of the PISA study (Schmidt & Burroughs, 2015). These students were randomly selected using a stratified sampling technique. Schmidt and Burroughs found that "OTL is strongly related to students' performance" (p. 25) within schools, between schools, and between the 62 countries that participated in the mathematics portion of the PISA study. Schmidt and Burroughs added that students who had more exposure to certain mathematics content areas performed better in those content areas than did students who had less exposure to the same content areas.

Regardless of the level (elementary level, secondary level or undergraduate level), common themes that emerge from the research that has looked into the opportunity to learn mathematics via textbooks include: (1) types of context, (2) types of information, and (3) cognitive demands.

Types of context. The term, context, has been defined in several ways by researchers in mathematics education. My view on the meaning of context is consistent with that given by White and Mitchelmore (1996). These researchers posited that "in calculus, the context of an application problem may be a realistic or artificial "real-world" situation, or it may be an abstract, mathematical context at a lower level of abstraction than the calculus concept that is to be applied” (p. 81). White and Mitchelmore’s understanding of the term context is consistent with that of other researchers (e.g., Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2005). According to Wijaya et al. (2015), a mathematical task could have a relevant and essential (realistic) context, a camouflage context, or they could have no context (only mathematical symbols). Alajmi (2012) refers to mathematical tasks with no context as tasks that are situated in a “purely mathematics context” (p. 243). Tasks with a camouflage context “are merely dressed up bare problems, which do not require modeling because the mathematical operations needed to solve the task are obvious” (Wijaya et al., 2015, p. 45). Examples of tasks with each type of context identified above, namely relevant and essential, camouflage, and no context are given in the methods chapter.

Types of information. Several researchers (e.g., Maass, 2007; Maass, 2010; Wijaya et al., 2015) have identified three types of information that a mathematical task may have: matching, missing, and superfluous. A mathematical problem with matching information is one in which all the information required to solve the problem is stated in the problem statement. A mathematical problem has missing information if some of the information needed to solve the problem is not immediately available to the student, that is, the student has to deduce this information from the problem statement. A mathematical problem with superfluous information is one in which the problem statement not only contains the necessary information needed to solve the task but it also contains other extraneous or irrelevant information that may not be helpful in solving the posed problem. Wijaya et al. (2015) argued that:

> Providing more or less information than needed for solving a context-based task is a way to encourage students to consider the context used in the task and not just take numbers out of the context and process them mathematically in an automatic way. (p. 45)

Maass (2010) recommended that students should be given opportunities to deal with these three different types of information.

Types of cognitive demands. A related line of research (e.g., Charalambos et al., 2010; Kolovou et al., 2009; Mesa et al., 2012; Wijaya et al., 2015) has investigated the types of cognitive demands in tasks that are presented in mathematics textbooks. The types of cognitive demands are: reproduction, connection, and reflection. These types of cognitive demands are similar to the levels of cognitive demands discussed by Stein, Smith, Henningsen, and Silver (2000). Reproduction tasks are routine problems that require the lowest level of cognitive demand to solve. These problems can be easily solved using memorized mathematical algorithms such as the first derivative test when solving optimization problems. Connection tasks are non-routine in nature and may require the student to represent concepts in multiple representations: algebraically, numerically, graphically, and verbally. Reflection tasks require the highest level of cognitive demand to solve. These tasks "include complex problem situations in which it is not obvious in advance what mathematical procedures have to be carried out" (Wijaya et al., 2015, p. 46). Examples of tasks with each type of cognitive demand identified above, namely reproduction, connection, and reflection, are given in the methods chapter.

In summary, the literature on opportunity to learn via mathematics textbooks presented above suggests the need to examine features that tasks may have such as representation (e.g., algebraic, tabular, graphical, or verbal) in addition to types of context, types of information, and types of cognitive demands. Only one study (Mesa et al., 2012) examined the representations of tasks in mathematics textbooks. Given that much of the research on opportunity to learn mathematics via textbooks focused on content areas covered at the elementary and middle school level, I argue that there is a need to examine students' opportunity to learn mathematics via textbooks at the undergraduate level.

Calculus Learning

The birth of the calculus reform movement at a calculus conference held at Tulane University in the late 1980s led to a lot of research on students' learning of calculus (Garbier & Garnier, 2001; Hallet, 2006; Schoenfield, 1985; Tucker & Leitzel, 1995). One of the major problems with collegiate calculus instruction that led to the calculus reform movement was a growing trend of students developing proficiency in executing symbolic calculus procedures with very little or no conceptual understanding of the procedures, something that in turn limited their ability to solve real-world application problems (Tucker & Leitzel, 1995; Hallett, 2006; Garner & Garner, 2001). Garner and Garner (2001) stated that there was a great need for a "shift in emphasis from rote memorization and symbol manipulation to conceptual understanding and practical application" (p. 165) in the teaching and learning of college calculus.

Understanding the concept of a function and being able to engage in covariational reasoning is essential for students' learning of key ideas in calculus such as rate of change. Following is a review of research literature on students' conceptual understanding of the concept of function and the idea of covariational reasoning.

Students' understanding of functions. The importance of the concept of function in students' learning of calculus cannot be overemphasized. Oehrtman, Carlson, and Thompson (2008) argued that "the concept of function is central to undergraduate mathematics, foundational to modern mathematics, and essential in related areas of the sciences" (p. 27). Functions form an integral part of the middle and high school mathematics curriculum (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010;

NCTM, 2000). Students' difficulties with the concept of function from middle school through university level are well documented in the research literature (e.g., Beichner, 1994; Carlson, 1998; Habre & Abboud, 2006; Oehrtman, Carlson, & Thompson, 2008; Yerushalmy, 2000; Yerushalmy & Swidan, 2012). Following is a review of literature on students' understanding of the concept of function in multiple representations followed by a review of literature on students' understanding of real-valued functions of two variables.

Students' understanding of the concept of function in multiple representations. This section contains a review of research literature on students' understanding of the concept of function in multiple representations with respect to the following themes: (1) students' preference for the algebraic representation, (2) students' difficulties with graphs, and (3) students' difficulties with translating between representations of functions.

Students' preference for the algebraic representation. In his review of research literature about students' understanding of functions, Thompson (1994c) argued that "tables, graphs and expressions might be multiple representations of functions" (p. 23) to teachers and researchers but that they are not "multiple representations of anything to students" (p. 23). Evidence from research (e.g., Habre & Abboud, 2006; Haciomeroglu et al., 2010; Ibrahim & Rebello, 2012; Thompson, 1994b; Weber & Thompson, 2014) shows that students often have a stronger inclination towards the use of functions in algebraic form compared to other representations.

Habre and Abboud (2006) examined calculus students' understanding of functions and derivatives at a large university. Nine of the ten students who participated in this study were high achieving students. A qualitative analysis of two semi-structured interviews with each student revealed that a majority of the study participants had a stronger inclination for the function concept in algebraic form than in any other form. When the students were asked to talk about their understanding of the concept of a function, Habre and Abboud reported that "except for Mark and Denise, all interviewees thought of the function in one way or another as a formula" (p. 62). Even after receiving formal instruction about functions in multiple representations, the students' reasoning about functions was dominated by the algebraic representation.

Undergraduate students' preference to work with equations while solving mathematics problems has also been reported by Ibrahim and Rebello (2012) who studied how 19 engineering students enrolled in a calculus-based physics course solved calculus problems situated in the context of work and kinematics. The problems were presented in multiple representations, namely algebraic, textual, and graphical. Each student completed four written tasks in four different sessions that were videotaped over a two months period followed by a follow up interview that was audio taped. Analysis of the written tasks and transcripts of the interviews revealed that the students preferred "manipulating equations irrespective of the representational format of the task" (p. 1). For instance, in one of the problems that the students solved, Ibrahim and Rebello reported that "all 19 students utilized equations or calculus with the majority (17 in 19) of them not recognizing the applicability of a qualitative approach for attempting the problem" (p. 4). Ibrahim and Rebello stated that the students occasionally did not recognize powerful qualitative ways (such as sketching graphs) of thinking about the problems because of their tendency to prioritize an algebraic approach over a qualitative approach.

Results similar to that of Ibrahim and Rebello were also reported by Thompson (1994b) who investigated students' understanding of the Fundamental Theorem of Calculus (FTC). According to Thompson, students' difficulties with understanding the FTC are an immediate consequence of their limited understanding of functions. The research participants were a combination of 19 senior and graduate students majoring in mathematics and mathematics education. To gain information on students' understanding of the theorem, Thompson analyzed students' written responses in an exam given at the end of an experimental teaching unit that lasted for 15 hours spread over ten meetings. Thompson also conducted follow-up interviews with each of the 19 students. Consistent with the findings of other researchers (e.g., Habre & Abboud, 2006; Ibrahim & Rebello, 2012), Thompson found that:

> Students repeatedly made remarks that suggested a figural image of function-an image of a short expression on the left and a long expression on the right, separated by an equal sign. This was not the only image students could conjure, but it seemed to be many students' working image-what came to mind without conscious effort whenever "function" was mentioned. This often oriented them away from grappling with conceptual connections entailed in dealing with covarying quantities. (p. 268)

In other words, the students had a stronger preference for the function concept in algebraic form, something that in turn limited them from developing a rich conceptual understanding of the FTC. Thompson's findings together with those of other researchers (e.g., Habre & Abboud, 2006; Ibrahim & Rebello, 2012) shows that students prefer working with algebraic functions when solving calculus problems and that this preference tends to limit them from developing good understandings of important ideas in calculus such as the FTC.

Students' difficulties with graphs. Research that has looked at students' difficulties with graphs of functions shows that high school and undergraduate students have difficulty interpreting or constructing accumulation graphs from rate graphs and understanding rates that are changing from increasing to decreasing and vice versa (points of inflection) from graphs of functions (e.g., Beichner, 1994; Carlson et al., 2002; Tsamir & Ovodenko, 2013). The research further shows that students tend to have a pointwise understanding (seeing a function as a set of isolated points) of the concept of function which sometimes helps them avoid iconic translation (seeing the graph of a function as a picture of a physical event) (e.g., Monk, 1992).

Beichner's (1994) examination of students' understanding of kinematics graphs (position versus time, velocity versus time, and acceleration versus time graphs) revealed that interpreting rates of change from kinematics graphs is problematic both for high school and college students. A t-test analysis that was used to compare the performance of 524 high school and college students in a multiple choice written task about kinematics graphs showed that there was no difference in the performance of these two groups of students. The researcher also found that: (1) only 51% of the students could correctly determine velocity from a position versus time graph, (2) only 40% could determine acceleration from a velocity versus time graph, (3) only 49% could determine displacement from a velocity versus time graph, (4) only 23% could determine change in velocity from an acceleration versus time graph, and (5) only 38% could correctly match two kinematics graphs that showed the same information (e.g. match a velocity versus time graph with the corresponding acceleration versus time graph). These results indicate that high school and beginning undergraduate students have difficulty interpreting kinematics graphs and that these students have difficulty interpreting or constructing accumulation graphs from rate graphs and vice versa.

Research by Carlson et al. (2002) and Tsamir and Ovodenko (2013) shows that undergraduate students have difficulty understanding points of inflection from graphs of functions. In their investigation of students' understanding of inflection points, Tsamir and Ovodenko (2013) found that even undergraduate students who have completed a course in differential equations have numerous difficulties or misconceptions about the concept of an inflection point when presented in a graph. Each of the 52 students who participated in this study had completed at least a course in differential equations. The students responded to six written tasks aimed at assessing their understanding of inflection points followed by interviews with 15 of the students. In a task where the students were given graphs of functions and asked to mark inflection points, Tsamir and Ovodenko reported that "no student correctly identified all inflection points on the five graphs" (p. 416). The students were generally successful at identifying inflection points that lie on the horizontal axis, and tended to either ignore or not recognize other inflection points that do not lie on the horizontal axis. Twenty-seven percent (n=52) of the students incorrectly identified critical points where either a relative minimum or maximum occured as points of inflection. The findings of this study show that even undergraduate students who have taken several mathematics courses have difficulty identifying points of inflection from graphs of functions and that the concept of an inflection point is not well understood by these students.

Research by Monk and colleagues shows that the use of physical models has proven to be effective in revealing other students' difficulties with graphs of functions: pointwise understanding and iconic translation (Monk, 1992; Monk & Nemirovsky, 1994). Monk (1992) used a physical model of a ladder to investigate students' understanding of a function that models a dynamic situation. The students who participated in this study consisted of 12 freshmen who were enrolled in the first quarter of an introductory calculus course and eight other students who had completed a calculus sequence. The students were asked to draw a graph of a function that shows the rate of change of the vertical position of the ladder as it slid down a vertical wall. Data for the study included transcripts of task-based interviews and work done by students during the interviews.

Only nine students drew a correct graph. A majority of the students used the Pythagorean Theorem to calculate the height of the ladder for specific distances between the wall and the foot of the ladder, thus demonstrating a pointwise understanding of the function in this situation. Monk found that only three students created tables to record horizontal and vertical positions which they later used to construct the graph of the function they were asked to draw. While this is yet another demonstration of pointwise understanding of the function they were asked to draw, the researcher argued that sometimes a pointwise understanding of functions is "effective in helping students avoid iconic translation" (p. 191). Overall, only eight students demonstrated a global view (an understanding of the rate of change of the ladder's vertical position over time) while the rest demonstrated a pointwise understanding of the situation. I argue that it might be important to examine students' ability to construct accumulation graphs in a business or economic context such as the graph of a profit function when given the graph of a total cost function and total revenue function.

Findings of a cross sectional investigation of high achieving undergraduate students' understanding of the function concept conducted by Carlson (1998) shows that iconic translation is not only problematic for high school and beginning undergraduate students but also for undergraduate students who have completed a second semester of calculus (Monk, 1992; Monk & Nemirovsky, 1994). A total of 60 high achieving students; 30 precalculus students, 16 second semester calculus students, and 14 second semester graduate students participated in Carlson's (1998) study. After completing a written exam on the concept of function, Carlson conducted follow-up interviews with fifteen students, five from each group. These students were chosen in such a way that their responses in the exam were representative of other students' responses in their group.

In one of the items, the students were asked to comment on the position of two cars whose speed was given by two curves (one concave up and increasing and the other concave down and increasing) that intersect at a certain time. This task proved to be problematic for a majority of the undergraduate students but not for the graduates. Referring to the intersection of the two curves, 88% of the precalculus students either said that the cars were at the same position or that one was passing the other, 29% of the second semester calculus students gave similar interpretations, thus using iconic translation (Monk, 1992). All second semester graduate students correctly interpreted the graph. The findings of this study suggest that a majority of the precalculus students conflated accumulation (total distance travelled) with rate of change (the speed of the cars). I argue that while iconic translation may not occur in a business or economic context, undergraduate students are likely to conflate accumulation and rate of change in this context (e.g. conflating marginal cost with total cost).

Translating between representations of functions. Translating between representations of functions refers to the act of moving from one function representation to another function representation, such as representing a graphically defined function using an algebraic equation. A major finding of research that has looked at students' ability to translate from one function representation to another (e.g., Beichner, 1994; Carlson, 1998; Martinez-Plunell & Gaisman, 2012) is that this translation is problematic both for high school and undergraduate students. For instance, other findings of Beichner's (1994) study of high school and college students' understanding of kinematics graphs revealed that the students had difficulty translating between the graphical and verbal (textual) representations. Beichner found that only 39% could correctly select a textual description when given a kinematics graph while only 43% could correctly select a corresponding graph when given a textual motion description. These findings suggest that translating functions from a graphical representation to a verbal representation and vice versa is particularly difficult for high school and college students alike in the context of motion.

In her cross-sectional investigation of students' understanding of the concept of function, Carlson (1998) found that high achieving precalculus students "are not able to translate verbal function language to algebraic function notation" (p. 122) for linear functions. One of the items in the study asked the participants to give an example of a function all of whose values are equal. Carlson found that 7% (n=30) of the students gave a correct example. A common erroneous example given by students was $y = x$. Another student mentioned the graph of the absolute value function, $y = |x|$ as an example while two other students stated that an example of such a function is one in which all the input values are equal. The findings of this study suggest that translating linear functions from a verbal representation to an algebraic representation is particularly difficult for beginning undergraduate students.

Research by Martinez-Planell and Gaisman (2012) shows that translating real-valued functions of two variables from one representation to another is problematic for students who have completed a calculus sequence. These researchers examined students' understanding of real-valued functions of two variables. The participants in the study were 13 students of mixed abilities who had recently completed a multivariable calculus course. Each student participated in a semi-structured task-based interview. During the interviews, students were asked to represent numerically (table) and algebraically defined functions using graphs and to match graphs of functions with their algebraic formulas. Martinez-Planell and Gaisman reported that six students had "difficulties with conversions among different representations" (p. 374). Specifying the correct domain was a common difficulty that a majority of these students had when translating functions from one representation to another. Referring to one of the students who had difficulty translating functions in multiple representations, Martinez-Plunell and Gaisman stated that although this student "was able to represent points in three-dimensional space by doing a conversion from a tabular representation to a physical [verbal] representation of a function, he could not carry out conversions from a graphical representation to an algebraic one" (p. 375). Being able to translate real-valued functions of two or more variables from one representation to another is an important understanding that that students need in different fields such as economics.

In summary, the review of literature on students' understanding of functions shows that students have several difficulties or misconceptions about the concept of a real-valued function of a single variable. First, undergraduate students have a tendency to think of functions only as algebraic equations when solving calculus problems and that this tendency limits them from developing good understandings of important ideas in calculus such as average and instantaneous rates of change. Second, identifying or interpreting points of inflection from graphs of functions is problematic for undergraduate students. Third, translating functions from one representation to another in de-contextualized situations and in a motion context is problematic both for high school and undergraduate students.

Students' understanding of real-valued functions of two variables. Research on students' understanding of real-valued functions of two variables ($R_2 \rightarrow R_1$) is limited (Martinez-Planell & Gaisman, 2012; Weber & Thompson, 2014; Yerushalmy, 1997). Yerushalmy (1997) studied six Israeli secondary school students' reasoning while modeling a multivariable situation. These students were enrolled in an algebra class that followed an "innovative algebra curriculum" (p. 433) that was organized around three ideas central to the learning of the function concept, namely describing, comparing, and "generalizing various aspects of functions" (p. 433).

The researcher investigated how the students related three variables (cost, days, and number of kilometers driven) as well as how they described the various representations they used to relate the variables. A computer program was used in the teaching of the course to support students' understanding of the concept of function in multiple representations. Over a period of four one-hour meetings, the students worked in two groups to determine the relationship between the cost of renting a car from a rental car company that charges its customers "1000 shekels for a day and an additional 5 shekels per kilometer" (p. 435).

Students attempted to model the relationship among the three variables using three representations, namely real-valued algebraic functions of two variables, three-column tables, and a three dimensional graph. Some of the algebraic relationships created by the students were $f(n) = n * 100 + x * 5$ and $f(n, b) = (n * 100) + (b * 5)$ where the variable n represents number of days for renting a car while the variables x and b represent number of kilometers driven. Yerushalmy argued that sometimes the notations f, $f(n)$, and $f(n, x)$ meant one and the same thing to the students. Yerushalmy reported that the use of three-column tables helped some of the students to correctly determine an algebraic equation relating the three variables. Only one student attempted to represent the relationship in a three dimensional plane by combining a pair of two dimensional planes (cost versus day and cost versus kilometers driven). The findings of this study suggest that modeling multivariable situations using multiple representations of real-valued functions of two variables in the context of cost is particularly difficult for students. The findings of this study also suggest that the correct mathematical notation for representing these functions algebraically is not well understood by secondary school students.

Research by Weber and Thompson (2014) shows that even after receiving formal instruction about graphs of real-valued functions of two variables, constructing and interpreting these graphs is problematic for undergraduate students. These researchers investigated two undergraduate students' mental images (schemes) of real-valued functions of two variables that are algebraically and graphically defined. The students, Jesse and Lana, participated in a three week teaching experiment whose aim was to help them develop robust understandings of graphs and algebraic equations of real-valued functions of two variables. During each session of the teaching experiment, the students were encouraged to use a graphing computer program to visualize surfaces on a three dimensional plane and to graph three dimensional surfaces they were attempting to graph on paper. Neither of the students had seen real-valued functions of two variables before. At the time of participating in the teaching experiment, the students were concurrently enrolled in a first semester calculus course. The study reported on an activity that the students worked on during the experiment: constructing and interpreting the graph of $f(x,y) = x^2y^2$.

Analysis of interviews conducted with each student prior to the teaching experiment revealed that Jesse's understanding of a function was that of a "rule" (p. 76) that relates two variables and the graph of a function as a visual representation of the rule "for every point" (p. 76). Lana, on the other hand, understood a function as "two variables connected by an equal sign" and that a graph is a "picture" of an equation (p. 79). Weber and Thompson argued that Jesse had a covariation view of a function. When asked to explain how he was thinking about the function $f(x,y) = x^2y^2$, Jesse not only stated that "two variables vary in different directions" (p. 77) but he went on to explain what he would see in the zx, zy, and xy planes. Thus Jesse attended to three images simultaneously while thinking about the graph of a real-valued function of two variables. When Lana was asked about how she was thinking about the function $f(x,y) = x^2y^2$, unlike Jesse, there was no evidence of her imagining several variables changing in tandem as the surface is created. She stated that the equation "has squares" (p. 81) which made her "think of parabolas right away" (p. 81) and when probed about where the parabolas come from, she said that "algebra says they are there" (p. 81). Weber and Thompson argued that Jesse's covariation view of functions of one variable helped him to extend his understanding of real-valued functions of one variable to real-valued functions of two variables. The findings of this study suggest that a covariation view of the concept of function is a powerful and essential understanding that students hoping to understand real-valued functions of several variables need.

In their investigation of students' understanding of real-valued functions of two variables, Martinez-Planell and Gaisman (2012) reported that three students did not recognize the uniqueness of a function's output while eight students had weak understandings of restricted domains of real-valued functions of two variables. These researchers found that finding domains of functions "that were restricted to a specific region in the xy plane" (p. 365) was problematic for a majority of the students. Six students had difficulties describing and determining the range of real-valued functions of two variables in multiple representations, three students showed a lack of an understanding of the uniqueness of the output of real-valued functions of two variables, and two students had a conception of the "domain as x axis and range as y axis" (p. 374). Furthermore, Martinez-Planell and Gaisman reported that eight students had a strong conception of a real-valued function of two variables as a formula. Martinez-Planell and Gaisman argued that students' difficulties with real-valued functions of two variables are similar to students' difficulties with real-valued functions of one variable. The findings of this study suggest that even students who have taken multivariable calculus do not have robust understandings of properties of the concept of function, namely the domain, range, and the uniqueness of a function's output.

Together, the research on students' understanding of real-valued functions of two variables suggests that secondary school and undergraduate students have difficulty representing these functions using graphs (Martinez-Planell & Gaisman, 2012; Weber & Thompson, 2014; Yerushalmy, 1997). I argue that this difficulty would be compounded when students are asked to interpret multivariate graphs situated in real-world contexts such as the profit function in economics. I would expect that determining a meaningful domain and range of the graph of a profit function when given the graphs of the total cost and total revenue function to be particularly difficult for some undergraduate students.

Covariational reasoning. A solid understanding of the covariational relationship that exists between a function's input values and its output values is essential for students to understand average rates of change and instantaneous rate of change in pre-calculus and calculus respectively. According to Carlson et al. (2002), covariational reasoning refers to "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). For functions, this means being able to simultaneously attend to the changing output values of a function and the rate at which the output values are changing as the values of the input variable change on a given interval of the function's domain (Oehrtman et al., 2008). The last 25 years have seen a growing body of research on the topic of covariational reasoning as it relates to students' understanding of functions and their rates of change (e.g., Carlson, 1998; Carlson et al., 2002; Johnson, 2012; Monk, 1992; Monk & Nemirovsky, 1994; Oehhrtman et al., 2008; Thompson, 1994b; Zeytun, Cetinkaya, & Erbas, 2010).

A common finding of research studies that have looked at how students engage in covariational reasoning while modeling dynamic events is that high school, undergraduate, and graduate students have difficulty interpreting varying rates of change while modeling dynamic situations (e.g., Carlson, 1998; Carlson et al., 2002; Johnson, 2012; Monk; 1992, Monk & Nemorovsky, 1994; Thompson, 1994b). An important finding of Carlson's (1998) cross sectional investigation of high achieving students' understanding of the concept of function was that even second semester graduate students have difficulty understanding changing rates in an imagined dynamic situation. In a task that asked students to sketch a volume versus height graph by imagining a spherically shaped bottle filling with water, Carlson found that 28% (n=14) of the graduate students constructed graphs that were either strictly concave up or strictly concave down "for the portion of the graph corresponding to the spherical portion of the bottle" (p. 124). This was also found to be a common misconception for 74% (n=16) of the second semester calculus students. Overall, only 51% of the graduate students constructed correct graphs while only 13% of the second semester calculus students were able to do so. None of the pre-calculus students (n=30) constructed a correct graph. A majority of the pre-calculus students constructed increasing straight lines.

Later research (Carlson et al., 2002) confirmed Carlson's earlier finding that high achieving undergraduate mathematics majors have difficulty understanding changing rates in dynamic situations. Carlson and colleagues examined students' ability to reason about covarying quantities in continuously changing situations. The researchers stated that the 20 students who participated in their study were representative of students who had recently completed a second semester calculus course and earned a grade of A in the course. Each student was asked to graph the height as a function of the amount of water in a spherically-shaped bottle that was being filled with water at a constant rate. Only two of these students constructed correct graphs in this task while the rest of the students demonstrated difficulties that were similar to those exhibited by students in Carlson's (1998) study. A complete analysis of each student's response to the task and follow up interviews with six of the students revealed that "very few of these high-performing 2nd-semester calculus students were able to form accurate images of the continuously changing instantaneous rate for this dynamic event function" (p. 364). In particular, the students had difficulty understanding and interpreting the continuously changing rates at which the height was changing as the bottle was being filled with water. Taken together, these studies (Carlson, 1998; Carlson et al., 2002) suggest that poor covariational reasoning abilities limit undergraduate students from understanding continuously changing rates in a geometric context.

Other research suggests that secondary school students who have not taken calculus have powerful ways of applying covariational reasoning while reasoning about constant and varying rates of change (Johnson, 2012). Johnson investigated how a tenth grade student, Hannah, who had completed a year of algebra reasoned about rates of change by engaging the student in five weekly task-based interviews that had seven tasks one of which was the filling bottles task. For the filling bottles task, Hannah was given a graph that showed the relationship between the height of a liquid in a bottle and the volume of the liquid in the bottle as the "liquid was being dispensed into the bottle at a constant rate" (p. 318). She was asked to sketch an image of the bottle described in the task. While sketching the bottle, Hannah commented while moving her fingers on the graph that "as you go along" (p. 326) the volume "definitely increases" (p. 326). She added that "it gets smaller as you go to the cap of the bottle" (p. 326). Hannah created an acceptable graph. Johnson stated that the student "simultaneously coordinated variation in the intensity of change in volume with smooth chunks of increase in height" (p. 326). The findings of this study suggest that students who have not taken calculus may have powerful informal ways of reasoning about varying rates in dynamic situations. Throughout the interviews, Hannah demonstrated powerful mathematical understandings of rate of change by simultaneously attending to how one or two quantities covaried with another.

Research by Carlson, Larsen, and Lesh (2003) shows that the integration of a models and modeling perspective (Lesh & Doerr, 2003; Lesh & Zawojewski, 2007) in modeling dynamic events promotes the development of students' covariational reasoning abilities. In their investigation of 22 undergraduate students' reasoning about a modification of the bottle problem (Carlson et al., 2002) so that it adheres "to the six principles of model eliciting activities," Carslon et al. (2003, p. 469) found that all the students were successful in constructing an acceptable graph for the bottle problem. Analysis of video interviews taken while students worked on this task in groups of two to four students revealed that requiring students to verbalize their reasoning, getting and giving feedback to their peers, and requiring students to refine their graphs are some of the major factors that contributed to the students' success in this activity. Carlson and colleagues asserted that the activity was effective in developing the students' covariational reasoning abilities.

Other research on students' engagement in covariational reasoning while modeling a dynamic situation shows that the use of physical enactments of the situation enhances students' understanding of varying rates in these situations (Carlson, 1998; Carlson et al., 2002; Monk, 1992). In their investigation of students' ability to engage in covariational reasoning while modeling dynamic events, Carlson et al. (2002) asked the students in their study to describe the speed of the top of a ladder sliding down a vertical wall, a modification of task used by Carlson (1998) and Monk (1992). Eight students (40%) gave a correct response. Carlson and her colleagues posited that the students' success was a result of having visualized the ladder sliding down the wall, something that was later confirmed during the follow-up interviews. During the interview with two of the students who provided correct responses, the researchers observed that:

> When one of the students (Student B) was prompted to explain his correct response, he performed a physical enactment of the situation, using a pencil and book on a table. As he successively pulled the bottom of the pencil away from the book by uniform amounts, he explained, "as I pull the bottom out, the amount by which the top drops gets bigger as it gets closer to the table"…Student A provided a similar response, except that her enactment involved her hand and a book to model the situation. (Carlson et al., 2002, p. 371)

Carlson et al. asserted that the use of physical enactments of the situation by the students provided them with powerful tools that enhanced their understanding of the covariatioanl relationship of the vertical and horizontal speed of the ladder as it slid down the wall. This finding is consistent with that of other researchers who have used this task with undergraduate students (Carlson, 1998; Monk, 1992). In other contexts, such as in business or economics, this finding suggest that the use of simulations (e.g., spreadsheets and other business or economic softwares) could support students' understanding of a dynamic situation such as profit which changes in tandem with changes in cost, revenue, and number of items produced and sold respectively.

Research by Thompson (1994b) shows that failure to engage in covariational reasoning in dynamic situations limits students in developing deep and conceptual understandings of important concepts in calculus such as rates of change. While investigating students' understanding of the FTC, Thompson observed that the students (n=19) had a weak understanding of rates of change. This weak understanding was noted when students were asked to give the units of the average rate of change of volume for “a cooling object t hours after removing a heat source” (p. 265). Only seven of the 19 students correctly stated that the units would be cubic meters per hour. Four students gave units of amount of change such as hours, thus demonstrating a lack or limited understanding of the dependent variable and independent variable changing in tandem. This finding suggests that students’ tendency to conflate rate of change with amount of change in a physics context can be attributed to failure to recognize rate of change as a quantity that results from a co-variation of two quantities (input and output). Being able to distinguish between the amount of a substance and the rate of change of a substance is an important understanding that students in business or economics, for instance, need to be able to distinguish between marginal cost and total cost.

In summary, the literature reviewed in this subsection reveals that coordinating changes in two quantities that are changing simultaneously is particularly problematic for undergraduate students. The findings of the research suggest that the use of physical enactments of dynamic situations and model eliciting activities can be effective in revealing students’ difficulties with changing rates and that these activities can be used to promote students’ development of covariational reasoning abilities. I argue that there is a need to investigate undergraduate students’ covariational reasoning when modeling multivariable situations such as the profit function in economics which has multiple covariates, namely total cost, total revenue, and number of items produced and sold.

Learning in Context

This section contains a review of research literature on: (1) the importance of context in the teaching and learning of mathematics and (2) students’ understanding of context problems.

The importance of context in the teaching and learning of mathematics. This section contains a review of literature on the importance of context in students' learning of mathematics. In particular, I discuss different meanings of the term context as used in the mathematics education research literature followed by a synthesis of rationales given by mathematics education researchers and widely cited national curriculum documents for using real-world contexts in the teaching and learning of mathematics. These documents include, among others, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) that have been recently adopted by more than 45 states and the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000). The section concludes with a discussion of the role of real-world contexts in students' learning of mathematics from two perspectives, namely modeling activities and application problems.

Meanings of context. According to van den Heuvel-Panhuizen (2005), the term context, as used in educational settings, refers to either a learning environment or a characteristic of a mathematical task (problem) that is presented to students. The context of a learning environment could include students' opportunities to learn provided by the instructor (e.g. lectures) and teaching/learning materials in the classroom such as textbooks and manipulatives among other things. The context of a mathematical task, on the other hand, refers to "the words and pictures that help the students to understand the task, or concerning the situation or event in which the task is situated" (van den Heuvel-Panhuizen, 2005, p. 2). Borasi's (1986) explanation of the context of a mathematical task is consistent with that given by van den Heuval-Panhuizen (2005). For other researchers (e.g., Davis, 2007; Zandieh, 2000), the context of a mathematical task refers to the ways in which a concept such as instantaneous rate of change in the task could be represented i.e. algebraically, numerically, graphically, or textually (also called verbally).

According to Freudenthal (1993), the term context in mathematics could also refer to a domain of application, that is, the application of mathematical ideas in other disciplines besides mathematics. This includes, for instance, an interpretation of the concept of marginal change in economics as instantaneous rate of change in mathematics. Marrongelle (2004) stated that “in the mathematical problem solving literature, context typically refers to non-mathematical meanings present in the problem situation” (p. 258). Marrongelle added that such problems often require the solver to interpret and translate textually represented information given in the problem situation to “familiar mathematical form” (p. 258) such as using algebraic notation. White and Mitchelmore (1996) asserted that “in calculus, the context of an application problem may be a realistic or artificial “real-world” situation, or it may be an abstract, mathematical context at a lower level of abstraction than the calculus concept that is to be applied” (p. 81). White and Mitchelmore’s definition of context is consistent with Gravemeijer and Doorman’s (1999) view of the notion of context in calculus. A realistic or artificial situation is one that may not necessarily be real but rather a situation can that can be imagined by the solver of an application problem (Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2005).

Rationales for using contexts in the teaching and learning of mathematics. The use of real-world contexts in mathematics instruction and in the assessment of students’ understanding of mathematics has received considerable attention in national curriculum documents over the last 25 years. The *Common Core Learning Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) calls for the engagement of students in solving problems that are situated in real-world contexts. These standards stipulate that the teaching and learning of school mathematics should provide students with opportunities to analyze, explain, describe, and interpret mathematics concepts such as rates of change in real-world contexts. The *Principles and Standards for School Mathematics* (NCTM, 2000) state that "all instructional programs from pre-kindergarten through grade 12 should enable all students to analyze change in various contexts" (p. 40). Following is a review of research literature that discusses some of the rationales for using real-world contexts in mathematics instruction or in assessing students' understanding of mathematics.

Evidence from research shows that engaging students in solving mathematical problems that are situated in real-world contexts helps to reveal students' conceptual understandings and conceptual difficulties or misunderstandings of certain mathematical concepts and procedures (Ärlebäck, Doerr, & O'Neil, 2013; White & Mitchelmore, 1996; Wilhelm & Confrey, 2003; Yoon, Dreyfus, & Thomas, 2010). In their investigation of undergraduate students' understanding of average rates of change in the context of an exponential decay function modeling voltage drop across a discharging capacitor, Ärlebäck et al. (2013) found that students' reasoning about the context revealed that all the students were able to correctly interpret the vertical intercept of the function in the context of the problem situation. Ärlebäck et al. stated, with evidence, that "a focus on the context made visible students' reasoning about rates of change, including difficulties related to the use of language when describing changes in the negative direction" (p. 314). In essence, reasoning with the context of a discharging capacitor made visible students' understandings of rates of change in a physics context.

Other researchers (e.g., White & Michelmore, 1996; Michelsen, 2006) have shown that a focus on students' reasoning about the context of motion while solving calculus problems reveals that undergraduate students have impoverished understandings of the concept of a variable.

These researchers argue that students see variables as algebraic symbols that are to be manipulated instead of quantities that are to be related. Research by Monk and colleagues in the context of motion and volume shows that undergraduate students have a static view (pointwise view) of the concept of function instead of a dynamic view (global view) of the concept (Monk, 1992; Monk & Nemirovsky, 1994). Research by Carlson et al. (2002) shows that even high achieving undergraduate students who have completed a second semester course in calculus have weak understandings of instantaneous rates of change that are changing from increasing to decreasing and vice versa, that is, points of inflection in the real-world context of modeling dynamic situations such as the height of water in a spherically shaped bottle that is filling with water.

In light of the fact that much research has documented students' difficulties with transferring their knowledge of mathematics to other contexts, there is a need to find ways of helping students overcome these difficulties (Carlson et al., 2002; Herbert & Pierce, 211; Ibrahim & Rebello, 2012; Michelsen, 2006, White & Mitchemore, 1996; Wilhelm & Confrey, 2003). In his discussion of the need to integrate the teaching and learning of mathematics and science topics through modeling as a means of enhancing students' ability to transfer their understanding of mathematical ideas to new contexts, Michelsen (2006) argued that "teaching mathematics in relation to science supports students' learning by providing meaningful contexts in which the students can see the application of abstract mathematical concepts" (p. 273). Michelsen added that this motivates students to study mathematics. Consistent with Michelsen's argument on the importance of integrating real-world contexts in mathematics instruction, Yoon et al. (2010) stated that using real-world contexts in mathematics instruction gives "students the chance to see the utility of the mathematics they have learned" (p. 143) in real life. Taken together, Michelsen (2006) and Yoon et al.'s (2010) views about the use of real-world contexts in the teaching and learning of mathematics suggest that these contexts have a potential for enhancing students' ability to transfer their understanding of mathematical ideas to other fields of study.

A common finding of research that has looked at the role of contextual modeling (e.g., Kaiser & Sriraman, 2006) in mathematics instruction is that engaging students in modelling activities that are situated in real-world contexts reveals students' difficulties with certain mathematical concepts and procedures and that these activities have the potential to "motivate students to develop the mathematics needed to make sense of meaningful situation" (Ärlebäck et al., 2013, p. 316). Through the use of a sequence of modeling activities, namely model eliciting activities (MEAs), model exploration activities (MXAs), and model application activities (MAAs) in a six-week summer course, Ärlebäck and colleagues were successful in developing beginning engineering "students' abilities to describe and interpret rates of change in the context of exponential decay" (p. 314). Similar results were reported by Dominguez (2010) who engaged students in an MEA situated in an economic context. The use of MEAs with 22 pre-service teachers revealed the teachers' thought processes about co-varying quantities in the real-world contexts of kinematics and volume, something that prompted the researchers to start "developing model eliciting activities to promote students' development and understanding of the major conceptual strands of introductory calculus" (Carlson, Larsen, & Lesh, 2003, p. 478). In sum, the use of model eliciting activities has the potential to promote students' development and understanding of essential ideas (e.g., functions and rates of change) in the study of calculus.

Another well documented rationale in the mathematics education literature (Freudenthal, 1993; Gravemeijer & Doorman, 1999; Lesh, Hoover, Hole, Kelly, & Post, 2000; Michelsen, 2006; Yoon et al., 2010) for using real-world contexts in the teaching and learning of mathematics is that this provides students with opportunities to mathematise, “which means, turning a non-mathematical matter into mathematics, or a mathematically underdeveloped matter into more distinct mathematics” (Freudenthal, 1993, p. 72). Mathematising realistic context problems has the potential to help students develop deep and conceptual understandings of the mathematical ideas rooted in the problems (Lesh et al., 2000). I argue that the process of mathematising realistic situations enhances students’ development of robust understanding of mathematical concepts and that this process makes visible students’ conceptions of the formal mathematical ideas that are rooted in the situation.

Modeling activities and application problems. Much research in mathematics education has advocated for the integration of real-world contexts in the teaching and learning of mathematics as a way of helping students recognize the importance of mathematics in solving real-world problems (e.g., Freudenthal, 1968; Lesh et al., 2000; Pollak, 1968). Yoon et al. (2010) argued that connecting mathematics and real-world contexts can be accomplished through the use of modeling activities and application problems. My study used Yoon et al.’s distinction between modeling activities and application problems. According to Yoon and colleagues, “modeling activities require students to develop a mathematical model by mathematising a real world situation, whereas application problems require students to apply a previously learned mathematical model to a real world context” (p. 142). In the mathematics education research literature, application problems are sometimes referred to as word problems (Ubuz & Ersoy, 1997) or story problems (Garner & Garner, 2001; Gerofsky, 1996).

Proponents of modeling activities such as Lesh and Doerr (2003) and Stillman (2012) argue that modeling activities are more beneficial than application problems in that engaging students in MEAs during instructional units offers students opportunities to deepen their conceptual understanding of mathematics ideas through the process of mathematising. In their review of research literature on the advantages and disadvantages of using either MEAs or application problems, Yoon et al. (2010) found that:

> When MEAs are implemented before any direct instruction on the topic, they serve their intended role of encouraging students to develop their own understandings through the process of mathematising. In contrast…when MEAs are implemented at the end of an instructional unit, they resemble application problems, in which students can apply what they have already been taught" (p. 142).

In light of Yoon and colleagues' distinction of modeling activities and application problems, my research study used application problems (Appendix A) instead of modeling activities. My study focused on investigating how students applied mathematical knowledge they had opportunities to learn in the textbook (Haeussler et al., 2011) and through course lectures to solve application problems that are situated in the real-world context of cost, revenue, and profit.

Students' understanding of context problems. Context problems are those that require reasoning about mathematical ideas such as functions, covariation, and rates of change in situations that are experientially real to students. Following is a review of research literature on what students' reasoning about context problems reveals about their difficulties or misconceptions about rates of change and rate-related concepts.

One major finding that emerges from research that has looked at students' reasoning about rates of change is that students have weak understandings of points of inflection both in real-world contexts (Carlson et al., 2002; Johnson, 2012) and in a purely mathematics context (Tsamir & Ovodenko, 2013). Research by Carlson et al. (2002) and Johnson (2012) shows that students have a poor understanding of points of inflection in the context of temperature and heat. Referring to students' difficulties in constructing a temperature rate graph, Carlson et al. stated that "their inability to note and represent the rate changing from increasing to decreasing (i.e., the inflection point), as shown by their concave up construction and remarks, suggested weaknesses in their understanding" (p. 370). Similar difficulties were exhibited by Hannah (Johnson, 2012) in a variation of the temperature problem given in Carlson et al.'s (2002) study.

Research by Tsamir and Ovodenko (2013) shows that in a purely mathematics context (i.e., absence of a real-world context) the concept of an inflection point is not well understood by students. In their study of students' understanding of inflection points, these researchers found that the students held "four erroneous images of inflection points: (1) $f'(x) = 0$ as a necessary condition, (2) $f'(x) \neq 0$ as a necessary condition, (3) $f''(x) = 0$ as a sufficient condition" (p. 409), and (4) the identification of critical points where either a relative minimum or maximum occurs as a point of inflection. Inflection points could determine crucial moments of a business such as when profit changes from increasing at an increasing rate to increasing at a decreasing rate. Hence, it would be important to investigate students' understanding of points of inflection in a business or economics context.

Other research studies show that secondary school and undergraduate students' robust understandings of rate of change in a motion context do not necessarily transfer to non-motion contexts (Cetin, 2009; Herbert & Pierce, 2008; Herbert & Pierce, 2011; Ibrahim & Rebello, 2012). Another finding of Ibrahim and Rebello's (2012) study of engineering students' strategies when solving problems in the contexts of kinematics and work respectively was that students' robust understandings of rate in the former context did not transfer to the latter context. For instance, in the problems where students were required to give detailed interpretations of equations:

> The data indicate that in kinematics all 19 students formulated an explanation of the physics depicted by the position equation, while in the topic of work, a high proportion of students (16 in 19) formulated a description focusing on apparent information and surface features of the representation. (p. 7)

A similar observation was made when the students were asked to interpret graphs in the two contexts. Based on the findings of this study, one may argue that undergraduate students have difficulty transferring their understandings of rate from one physics context (kinematics) to another (work).

In their study of 27 secondary school students' ability to transfer their understanding of rates of change from a motion context to another motion context and from a motion context to a context of cost, Herbert and Pierce (2008) found that the students were more successful in the former transfer (motion to motion) and less successful in the latter transfer (motion to cost). Collected data for the study consisted of students' written responses on two tests given during a teaching experiment on average rates of change, classroom observations, and transcriptions of follow up interviews with four students after completing the two tests. Herbert and Pierce found that the students were generally able to transfer their understanding of rate in a vertical motion context (speed of moving elevators) to a horizontal motion context (speed of a walking person). For one of the students, the researchers reported that this student was able to use his understanding of speed in the context of a moving elevator (model of) as a 'model for' understanding the horizontal motion context. A complete analysis of the data revealed that "a smaller majority of the students were able to use their, often incomplete, 'model of' rate of change" (p. 231) in a motion context "as a 'model for' reasoning about rate of change in a non-motion context" (p. 231). The findings of this study suggest that students' understandings of rate of change in a motion context do not transfer to non-motion contexts.

Similar findings were reported by Herbert and Pierce (2011) who examined the influence of context and representation on 20 secondary school students' reasoning about rates of change. The students' prior instruction on rates of change used two interactive multiple representation computer programs to broaden their understanding of constant and variable rates of change in the contexts of area versus height and distance versus time. Each student participated in a video recorded interview designed to assess their understanding of rate of change based on computer simulations displayed on the technologies described above. Herbert and Pierce found that nearly all the students were able to clearly explain their understanding of rate of change when presented with tasks about distance and time whereas about half of them could do so when presented with tasks about area and height. The researchers argued that although the students demonstrated a solid understanding of rates of change in the context of walking, this understanding did not transfer to the context of area and height. In other words, students' understandings of rate of change in a motion context did not transfer to a non-motion context.

Research by Cetin (2009) shows that projecting understandings of rates of change from a kinematics context onto a volume context is problematic for undergraduate students. Cetin examined science and engineering students' understanding of functions and their derivatives in the context of motion and volume respectively. Each of the 104 students who participated in the study was engaged in an untimed task-based interview consisting of motion-related and volume-related tasks. Referring to students' understanding of rate of change in the tasks that were situated in a motion context, Cetin reported that "subjects were successful by referring to their intuition in answering these questions" (p. 241). However, when the context was volume, only 43% showed evidence of understanding functions and their derivatives. The findings of this study together with those of other studies (e.g., Herbert & Pierce, 2008; Herbert & Pierce, 2011; Ibrahim & Rebello, 2012) suggest that while rates of change are often well understood in a motion context, projecting understandings of these concepts onto non-motion contexts is problematic for high school and college students. It might be important to examine students' ability to project their understanding of rates of change, marginal change in particular, in a purely mathematics context onto a business or economic context.

Other research suggests that students do not have to completely understand rates of change in a motion context to be able to understand rates of change in a non-motion context and vice versa (Wilhelm & Confrey, 2003). Wilhelm and Confrey studied four algebra I students' ability to project their understanding of average "rate of change in the context of motion onto the context of money" (p. 887). Motion detectors were used in a teaching experiment for teaching the concept of average rate in the former context and a banking software was used for teaching the same concept in the latter context. Analysis of clinical interviews conducted with each student at the end of the teaching experiment revealed that the students did "not have to completely understand the relationship between rate of change and accumulation graphs within a single context in order to be able to understand and project the concepts of rate of change and accumulation separately into multiple contexts" (p. 890). In this study, some of the students who seemed to have incomplete understandings of rate of change and accumulation in a motion context were able to project these concepts onto a banking context.

Another major idea from research is that representing rates of change in multiple representations is problematic for undergraduate students (Klymchuk et al., 2010; Villegas et al., 2009). Student A in Villegas et al.'s (2009) study of students' use of multiple representations when solving optimization problems was not successful in solving some of the problems because of failure to represent the derivative in multiple representations. Students' inability to set up an objective function for a routine optimization problem in Klymchuk et al.'s (2010) study was a result of students' inability to extract rate-related information given in text and writing it algebraically. When students were asked about challenges they had with this problem, some indicated that they "did not know how to convert the real life problem into one to solve mathematically" (p. 85). Taken together, the findings of Klymchuk et al. (2010) and Villegas et al. (2009) suggest that translating rates of change from one representation to another, especially from a verbal (textual) representation to an algebraic representation, is problematic for undergraduate students when solving optimization problems that are situated in a physics context. It would be important to investigate the nature of difficulties students have when asked to solve verbally-represented optimization problems that are situated in other contexts such as in economics.

Several studies have reported on undergraduate students' tendency to conflate the rate of change of a quantity with either the amount of change of the quantity (e.g., Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Mkhatshwa, 2014; Prince, Vigeant, & Nottis, 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006) or the accumulation of the quantity (e.g., Flynn, Davidson, & Dotger, 2014) when solving application problems that are situated in various contexts. Prince et al. (2012) used a concept inventory to investigate 373 engineering students' understanding of the concepts of heat and energy, before and after the students had received formal instruction in these concepts. Nine of the 36 items in the inventory targeted students' understanding of rates of change and amount of heat transfer. Following the last administration of the questionnaire, a convenience sample of 50 students responded to “open-ended conceptual questions” (p. 417) that were analyzed for common patterns in students’ reasoning about rate and amount of heat transfer. Analysis of student responses to the questions revealed that the students had “difficulty distinguishing between factors that influence the heat transfer rate vs. the amount of heat transferred in a given situation” (p. 427). A quantitative analysis of students’ responses on the rate versus amount of heat transfer questions revealed that there was no difference in students’ performance on these questions in the two administrations of the questionnaire. Hence, the findings of this study suggest that students’ difficulties with distinguishing between the rate of change and the amount of change of a quantity in a thermodynamics context persist even after students receive formal instruction in these concepts.

Students’ tendency to conflate the rate of change of a quantity with the amount of change of the quantity has also been observed in the context of differential equations (Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006). Through a semester-long teaching experiment, Rasmussen and King (2000) analyzed tasks that engaged undergraduate students in reasoning about approximate solutions to first order differential equations. Collected data for the study consisted of video recordings of classroom observations, copies of students’ written work, and video recordings of interviews with a group of three students (Jerry, Bill, and Sean) whose work was considered to be representative of students’ reasoning in the class. Analysis of students’ work while working on a task where they were asked to describe how they would use a first order differential equation to approximate the number of fish in a pond revealed that Bill and Jerry did not “conceive of the situation as rate, and therefore did not make a distinction between rate of change and change” (p. 168). That is, the students did not distinguish between the rate of change of the number of fish in the pond and the amount of change of the number of fish in the pond for a given time period. In another study situated in the context of differential equations but not directly related to students' learning, Rasmussen and Marrongelle (2006) reported that "students were not making a conceptual distinction between rate of change in the amount of salt and amount of salt" (p. 408). In particular, the students confused the amount of salt flowing in a tank at any given time with the rate at which the salt is flowing in. I argue for the need to investigate students' ability to distinguish between the amount of change of a quantity and the rate of change of the quantity in other context areas. In economics, for instance, this could mean investigating students' abilities to distinguish between the amount of profit generated in a given time period and the rate at which profit increased in the same time period.

A related line of research suggests that conflating rate of change and amount of change is common among students when dealing with functional situations where the input changes by a unit each time i.e. marginal change being per a unit of one (Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Mkhatshwa, 2014). Lobato and colleagues investigated 24 eighth grade students' understanding of average rate of change in a kinematics context. The students were given a numerical table showing time (in increments of 1 second) and altitude (height above ground measured in feet) of a remote-controlled airplane at the end of a 15-hour instructional unit on quadratic functions. They were asked to describe the speed of the airplane. A qualitative analysis of a clinical interview with one of the students revealed that the student treated marginal change as an amount of change and not as a rate of change when he indicated that the speed of the airplane would be in feet instead of feet per second. In a pilot study that examined undergraduate business calculus students' reasoning about marginal change in the context of cost, revenue, and profit (Mkhatshwa, 2014), nine of the ten students who participated in the study stated that the units of marginal cost, marginal revenue, and marginal profit would be dollars instead of dollars per unit of production. Because of the importance of the concept of marginal change in economics, I argue that there is a need for more research on students' understanding of marginal change in an economic context.

Research by Flynn et al. (2014) suggests that distinguishing between the rate of change of a quantity and the accumulation of the quantity in the context of water flow is particularly problematic for undergraduate students. These researchers used a rate and accumulation inventory to investigate the understandings of rate for 90 sophomore students enrolled in an engineering course. The inventory was administered twice: at the beginning and at the end of the course. Fifteen students were invited to participate in video and audio recorded interviews that were used to gain insight into the students' understanding of "groundwater flow and water flows on a green roof" (p. 2). Analysis of these transcripts revealed that the students confused rate processes with accumulation processes:

> For instance, several students stated that the rate of water flowing into the roof drain could be represented by an upward curve with no maximum, which suggest that they may have been representing the total amount of water accumulated over time rather than flow rate into the drain. (p. 3)

There is a need for more research that investigates what students' reasoning about other contexts that are experientially real reveal about their ability to distinguish between rates of change and accumulation. Taken together, these studies suggest that undergraduate students have difficulty distinguishing between the concepts of rate and amount of change as well as between rate and accumulation in context (Flynn et al., 2014; Lobato et al., 2012; Mkhatshwa, 2014; Prince, Vigeant et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006).

Research by Ärlebäck et al. (2013) and Orton (1983) suggest that interpreting negative rates of change is problematic both for high school and undergraduate students. Over a period of two consecutive summers, Ärlebäck and colleagues investigated students' ability to interpret and communicate their understanding of negative rates of change by exploring two models of changing phenomena: light intensity and a discharging capacitor. The 49 freshmen who participated in this study were enrolled in a six weeks modeling-based course designed to help prospective engineering majors develop a robust understanding of average rate of change. A qualitative analysis of students' written work from small groups and projects revealed that 29 students correctly "interpreted the values of the sequence of average rates of change while simultaneously referring to both the behavior of the function [exponential] and the context of the discharging capacitor" (p. 328). That is, 20 students had difficulty interpreting a sequence of negative and increasing rates of change of voltage in the context of a discharging capacitor.

Students' difficulties with interpreting negative rates of change were also reported by Orton (1983) who examined high school and college students' understanding of elementary calculus concepts in a mathematics context. Analysis of task-based interviews conducted with 60 high school students and 50 college students revealed that these students had difficulties interpreting negative and zero instantaneous rates of change, a problem that could be linked to a poor conceptual understanding of the derivative concept. Reporting about one task, Orton stated that "twelve students could not respond at all when asked to interpret $dy/dx = -2$, and a further ten students could only say "decreasing" or "decreasing gradient," or "similar", rather than "decreasing function" (p. 241). Being able to reason about negative rates of change is essential for business and/ economics majors hoping to understand the rate at which a company's deficit (negative profit) changes over a given time period.

Altogether, the findings of the literature reviewed in this section suggest that students' reasoning about application problems situated in real-world contexts reveals that average rates of change, instantaneous rates of change, and inflection points are not well understood by students. In particular, students have: (1) a tendency to confuse rate and amount of change, (2) a tendency to confuse rate and accumulation, (3) difficulty transferring their understanding of rates of change from a motion context to non-motion contexts, and (4) difficulty interpreting negative rates of change. In my study, I investigated some of these issues in the economic context.

Students' Understanding of Optimization Problems

Optimization is a mathematical technique used for finding minimum and maximum values of real-valued mathematical functions. Following is a review of research literature on students' understanding of problems concerned with finding minimum or maximum function values. There are generally two types of optimization problems in the research literature: (1) geometric (Brijlall & Ndlovu, 2013; Heid, 1988; Swanagan, 2012; Ubuz & Ersoy, 1997; Villegas, Castro, & Gutiérrez, 2009; White & Mitchelmore, 1996) and (2) non-geometric (Borgen & Manu, 2002; Klymchuk, Zverkova, & Sauerbier, 2010; Maharaj, 2013). The former refers to optimization problems whose method of solution requires the solver to recall some fundamental geometric rules and properties such as the Pythagorean Theorem, area of a rectangle, and volume of a rectangular prism while the latter refers to optimization problems whose method of solution does not require the solver to use geometric rules and properties.

Geometric optimization problems. The review of research on students' understanding of functions revealed that translating from one representation of a function to another is problematic for students. Evidence from research shows that translating rate-related information from one representation to another is one major difficulty that students have when solving geometric optimization problems (Swanagan, 2012; Villegas et al, 2009; White & Mitchelmore, 1996). White and Mitchelmore (1996) studied the conceptual knowledge of 40 undergraduate mathematics majors when solving application problems at the level of introductory calculus. Each student provided a written response to four problems, two of which were routine (common calculus textbook problem) geometric optimization problems. Analysis of students' performance on the optimization problems revealed that a majority of the students "could not identify and symbolize an appropriate variable by translating one or more quantities in the item to an appropriate symbolic form" (p. 89). White and Mitchelmore argued that students' difficulties in symbolizing instantaneous rates of change while solving optimization problems can be linked to the students' poor conceptual understanding of the concept of a variable. White and Mitchelmore found that most of the students in their study treated variables as algebraic symbols to be manipulated and not as quantities to be related. Being able to relate quantities such as marginal cost, marginal revenue, and marginal profit is an essential understanding that undergraduate business and economics majors need to be able to understand optimization problems that occur in business or economics.

Students' difficulties in translating rate-related information among different representations while solving optimization problems have also been reported by Villegas et al. (2009). These researchers studied how three senior undergraduate mathematics majors used multiple representations (verbal, tabular, algebraic, and graphic) when solving optimization problems. Following a think aloud protocol, the researchers asked each student to verbalize their reasoning while they individually solved three optimization problems. The first problem was about finding the minimum cost of laying an electrical cable from a power station to a factory across a river, the second problem was about finding the maximum area of a Norman window, and the third problem was about finding the maximum area of a rectangle inscribed in a triangle. Analysis of students' written work and transcriptions of each problem solving session revealed that identifying rates (instantaneous rates of change) in the problems they were given was a challenge and that representing these rates in multiple representations was problematic for the students. Villegas and colleagues found that one of these students frequently had "trouble understanding the word problems, drawing out the information from them and making translations from the word problem to any other representation" (p. 302). The findings of this study suggest that there is a link between students' success in solving optimization problems and their understanding of rates of change in multiple representations.

Other research shows that the algebraic procedure used to determine optimal function values (minimum or maximum) is not well understood by students (Brijlall & Ndlovu, 2013; Swanagan, 2012; White & Mitchelmore, 1996). Brijlall and Ndlovu (2013) studied the mental constructions of ten South African 12th grade students while solving three optimization problems in groups of students of mixed abilities. To better understand each group's written solution, follow-up interviews were conducted with one member of each group. Analysis of students' group work and transcriptions of the individual interviews revealed that the students were proficient in solving questions "where rules were required" (p. 15) but they had difficulty explaining the process of finding a minimum or maximum value when probed. For instance, in a task where the students were asked to find the minimum volume of a box with dimensions x units, $5x$ units, and $(9 - 2x)$ units, one group member during the follow-up interview correctly formulated the volume function, took the derivative and set it to zero, and then correctly found two critical numbers and stopped. The researchers posited that the students either did not know which critical number gives the minimum value or that they simply assumed that the critical numbers were the minimum values for the volume function.

Similar findings were reported by Swanagan (2012) who studied the relationship of five AP calculus students' understanding of the concept of the derivative and their ability to solve routine and non-routine (authentic and not a common problem to the students) optimization problems. Each student solved, on paper, three optimization problems: two routine and one non-routine problem. The first and second routine tasks included a task about finding the minimum cost for fencing a rectangular plot of land bounded by a straight road on one of its longer sides and another task about finding a point on the parabola, $y = x^2$, that is closest to the point (1,0). The third and non-routine task was about finding the "the maximum length of a pole that can be carried around a corner joining corridors of widths 8 ft and 4 ft" (p. 32). To better understand students' written responses to the tasks, the researcher conducted follow-up interviews with each of the students using a think aloud protocol. A qualitative analysis of both the students' written work and transcriptions of the follow-up interviews revealed that although nearly all the students showed some intuition on the process of finding optimal solutions yet only one student demonstrated an understanding that this process could sometimes yield several critical numbers and that some criteria had to be used in finding the optimal function values. Swanagan stated that the other four students appeared to be "relying on guesswork" (p. 102) while at other times some students did not realize that the process of finding minimum or maximum values could yield more than one minimum or maximum values. Together, the findings of this study and that of Brijlall and Ndlovu (2013) suggest that students tend to develop a procedural understanding that lacks a conceptual base of the algebraic processes used to solve optimization problems.

A common finding of research that has looked at students' understanding of optimization problems is that setting up the objective function, i.e. the function to be maximized or minimized, is problematic for both high school and undergraduate students (Klymchuk et al., 2010; Swanagan, 2012; Villegas et al., 2009; White & Mitchelmore, 1996). Another major finding of Swanagan's (2012) study was that setting up the objective function was a challenge for all the students in at least one of the three tasks they were asked to solve. For instance, in the task about finding a point on a parabola that is closest to a given point, one student simply used the equation of the parabola as the objective function and hence ended up with incorrect results. Another student used the perimeter function as the objective function in the task about finding the minimum cost of fencing a rectangular plot of land while in all three tasks some students used some version of the Pythagorean Theorem as the objective function. Similar difficulties were reported in White and Mitchelmore's (1996) study of students' conceptual knowledge of introductory calculus concepts. In a task where the students were asked to find the maximum area of a rectangle inscribed in a given parabola, White and Mitchelmore stated that as soon as some students saw the equation of the parabola they found its derivative without taking into consideration other important information such as the point where the parabola and rectangle intersect. That is, these students incorrectly used the equation of the parabola as the objective function.

Research by Heid (1988) shows that solving routine optimization problems is problematic for students who have completed an introductory traditional calculus course and for students who have completed a reformed introductory calculus course. Heid compared the conceptual understandings of two groups, a control group and an experimental group. The control group completed a traditional calculus course where a significant amount of time for the entire semester was spent on skill development such as developing proficiency in finding derivatives and solving routine textbook problems. The experimental group completed a reformed calculus course where most of the time in the semester was spent solving non-routine application problems and developing conceptual understanding of the derivative as a measure of rate of change with the aid of a computer program. A total of 135 students majoring in business, architecture, and life sciences participated in the study with 100 in the control group and 35 students in the experimental group.

The focus of the study was more on the impact of the sequencing of skills and concepts on students' understanding of fundamental concepts in calculus and less on students' understanding of optimization problems, but the final exam provided a source of data relevant to students' understanding of optimization problems. The final exam consisted of routine questions that required procedural knowledge (skill) rather than conceptual knowledge. Heid (1988) reported that students' performance in both groups was similar. There were two different questions about two different categories of optimization problems in the final exam; one question about finding optimal values (minimum or maximum) of a real-valued function of one variable and another question about finding optimal values of a real-valued function of two variables. In the former question, Heid reported that 16% of the students in the control group were successful in finding optimal values of the function while about 17% of the students in the experimental group were successful in the same question. In the question about finding optimal values for a function of two variables, Heid reported that 26% of the students in the control group were successful while only 11% of the students in the experimental group were successful in this question. These results suggest that finding optimal values for a function of two variables was difficult for both groups, especially the experimental group. The findings of this study suggests that even after receiving formal instruction on solving optimization problems, solving routine geometric optimization is problematic for undergraduate students. Hence, we can hypothesize that we can expect students to have even more difficulties when asked to solve non-routine optimization problems.

Taken together, the findings of the studies reviewed in this subsection revealed that students have difficulty in translating rate related information from a verbal representation to a symbolic representation, something that in turn limits their success in solving geometric optimization problems. There is a lack of research that has examined students' ability to translate rate related information from a verbal representation to either a numerical or graphical representation when solving optimization problems. Also, the research reviewed in this section shows that setting up objective functions is particularly difficult for students and that they have a poor conceptual understanding of the algebraic process used to determine optimal values of real-valued functions. As a result of students having weak understandings of the process involved in finding optimal solutions, even routine optimization problems are problematic for high school and college students.

Non-geometric optimization problems. The findings of some of the studies that have looked at students' understanding of non-geometric optimization problems reveal that students' difficulties with these problems are in many ways similar to those that students face when solving geometric optimization problems (Borgen & Manu, 2002; Klymchuk et al., 2010; Maharaj, 2013). One such difficulty is a poor conceptual understanding of the criteria for classifying optimal function values either as a minimum value or a maximum value, a result also reported in Brijlall and Ndlovu's (2013) study of 12th grade students' understanding of geometric optimization problems. Findings of a case study by Borgen and Manu (2002) involving a student who, on paper, "presented a picture-perfect solution to a calculus problem" (p. 151) revealed that although the student correctly applied an algebraic procedure she had learned for determining an optimal solution, her conceptual understanding of the procedure was poor. The student, Janet, was an undergraduate mathematics major who had previously completed a calculus course. She worked with a partner to find the critical point of the function $y = 2x^2 - x + 1$ in addition to stating that the function has a relative minimum value at the critical point.

Having successfully found the critical point (algebraically) and using a calculator to graph the function, the two students correctly determined that the function has a minimum value at the critical point. To justify why the function has a minimum value at the critical point, Janet incorrectly claimed that since the coefficient of the linear term in the function is negative, the function had to have a relative minimum value at the critical point. Janet confused the negative coefficient of the linear term of the function with the "the concept of the derivative and/or the equation of the tangent line" (p. 160).

Some studies that have looked at students' understanding of geometric optimization problems found that students have weak understandings of the algebraic process used to determine optimal solutions (Brijlall & Ndlovu, 2013; Swanagan, 2012; White & Mitchelmore, 1996). This difficulty has also been reported by Maharaj (2013) who examined 857 university students' understanding of the concept of the derivative when solving a non-geometric optimization problem in a life science context. The participants in this study were undergraduate students majoring in chemistry, physics, biology, zoology, and pharmacy at a large public university in South Africa. They were all enrolled in an introductory calculus course. Students' written responses to six multiple choice questions, two of which were optimization problems, were analyzed.

In one of the optimization problems, the students were given an algebraic equation that models the concentration of a certain drug in a person's bloodstream and asked to find the time at which the concentration of the drug is maximum in the body as well as the maximum concentration at this time. This equation had two critical numbers, one positive and the other negative. Maharaj (2013) reported that about 46% of the students were successful in this question. The researcher asserted that these students had developed an adequate schema for finding the optimal solutions of a function. Some of the students who were not successful in this question, not only chose the negative critical number as the time at which the concentration of the drug is maximum but they went on to substitute this number in the objective function and found an incorrect maximum blood concentration. The results of this study suggest that students' rote memorization of the process of finding optimal solutions algebraically limits them from recognizing optimal solutions that are not meaningful in a life sciences context. My study examined students' ability to determine, algebraically, meaningful optimal solutions (critical numbers or extrema) in an economic context.

Research by Klymchuk et al. (2010) shows that setting up an appropriate objective function is problematic for students when solving geometric and non-geometric optimization problems. Klymchuk and colleagues examined university students' perspectives about the difficulties they encounter when solving optimization problems in calculus. The study participants were 201 undergraduate students majoring in engineering at two research universities, one in New Zealand (92 students) and one in Germany (109 students). The students were asked to respond in writing to a mid-semester exam problem that asked them to show that the total cost of running a truck on a journey would be minimal if the truck maintains a certain constant speed throughout the journey. Klymchuk et al. reported that only ten students in the two institutions combined were successful in this task. To understand why students performed poorly in this task, the researchers gave the students in both institutions a questionnaire that asked them about the difficulties they had while trying to solve the problem and what they thought could be done to help them get better at setting up the objective function when solving optimization problems. The response rate was a little over 51% from the two institutions combined.

A qualitative analysis of the questionnaires revealed two findings about students' perspectives about optimization problems. First, a majority of the students expressed having difficulties understanding the wording of the problem and setting up the objective function for the problem. Common responses included comments like "the wording was ambiguous," "I did not know how to convert the real life problem into one to solve mathematically," and "I was confused because the result was given" (p. 85). Klymchuk and colleagues reported that the students indicated that they were familiar with problems asking them to find the velocity that would minimize the cost of running the truck instead of being asked to show that running the truck at a given velocity minimizes the total cost of running the truck as was the case in this problem. Second, a majority of the students indicated that having more opportunities to practice solving application problems in class and during recitations would help them improve their skills of solving optimization problems. My study examined students' difficulties with setting up the objective function (a profit function) when asked to solve a routine optimization problem in an economic context.

Research by Dominguez (2010) shows that even when students are successful in setting up the objective function and finding the optimal values, interpreting these optimal values is problematic for undergraduate students. Ninety-four calculus students working in groups of four or five students were asked to develop an economic model that can be used by the manager of a historic hotel to determine the daily rate per room that would maximize the hotel's profit in addition to finding the maximum profit corresponding to this rate. Analysis of observation notes and students' written responses to this task revealed that all the groups correctly constructed an objective function for the situation described in the statement of problem and that all the groups were successful in finding optimal values for each of the four objective functions that were constructed. Although they were successful at developing algebraic models for the situation, interpreting the optimal values obtained by using the model was particularly difficult for most of these students. I argue that, in part, students' difficulties with interpreting optimal solutions in an economic context could linked to the potential opportunities (e.g., applied calculus textbooks and classroom instruction) that students have to learn about context-based problems. Hence, there is a need to analyze these opportunities.

Taken together, the findings of the literature reviewed in this subsection revealed two things. First, undergraduate students have a poor conceptual understanding of the algebraic process of finding optimal solutions: (1) they cannot distinguish between meaningful and meaningless optimal values, (2) they have difficulty setting up objective functions, and (3) they have difficulty interpreting optimal values in an economics context. Second, there is a lack of research that examines students' reasoning about non-routine optimization problems, a gap addressed by my study.

Quantitative Reasoning

This study used the definition of quantitative reasoning proposed by Thompson (1993): the analysis of a situation in terms of the quantities and relationships among the quantities involved in the situation. According to Thompson, what is important in quantitative reasoning is not assigning numeric measures to quantities but rather reasoning about relationships between or among quantities. There are two types of research literature on the subject of quantitative reasoning: (1) empirical research (e.g., Lobato & Siebert, 2002; Moore & Carlson, 2012) and (2) non-empirical research (e.g., Ellis, 2011; Thompson, 1990; Thompson, 1994a; Thompson, 2011). In this study, the former refers to research studies that has human subjects (in particular, students) as participants and the latter refers to research that is theoretical in nature in that it does not have human subjects as participants.

Empirical research. Research on students' quantitative reasoning at the undergraduate level is scarce: much of the existing research that has looked at students' quantitative reasoning is at the secondary level. In what follows, I review two research studies (Lobato & Siebert, 2002; Thompson, 1993) whose findings paint a clear picture on students' difficulties with engaging in quantitative reasoning at the secondary level followed by a review of a study (Moore & Carlson, 2012) that has looked at students' quantitative reasoning at the undergraduate level.

Thompson (1993), through a four-day teaching experiment examined, six fifth grade students' ability to compare quantities using differences. Thompson found that these students had difficulty distinguishing between numerical and quantitative differences.

> These children were able to reason in terms of differences as quantitative operations-additive comparisons between two quantities-but it appeared that they did not distinguish between the quantitative operation of comparing two quantities additively and the arithmetic operation of subtraction. In fact, it was common for them to confound the two (although for some this was true more in the early part of the teaching experiment than in the later parts and in the interviews). (p. 203)

Thompson posited that students' "confounding of quantitative and arithmetical operations is symptomatic of all quantitative operations-ratio, additive and multiplicative combinations, and multiplicative compositions-not just of difference" (p. 203). In economics, for example, this means that students are likely not to distinguish between quantitative ratios (e.g., average cost as a ratio of quantities, namely total cost and number of units produced) and the arithmetic operation of division (e.g., average cost as a numerical value obtained by dividing two numerical values).

Lobato and Siebert (2002) conducted three semi-structured interviews with nine eight to tenth grade students over the course of a ten-day teaching experiment. The general purpose of the teaching experiment was to "create an environment in which students could develop sufficient quantitative reasoning to allow us to investigate the nature of successful performance in quantitatively complex situations" (p. 88). These researchers, however, only reported on how one of the students, Terry, an eighth grade student who had recently completed Algebra I, reasoned about a wheel chair ramp task which involved "determining how steep any ramp is (and what measurements you would need to take)" (p. 93) . Terry reasoned about the quantitative relationship among three quantities, namely the length, height, and steepness of a wheel chair ramp. Terry was chosen as the focus of the study because his interviews afforded "a detailed investigation of quantitative reasoning" (p. 92).

Analysis of the first and third interview revealed that Terry's reasoning about the relationship among the length, height, steepness of a wheel chair ramp changed between the first and the third interview. Terry progressed from considering the height of the ramp to be more important than the length of the ramp in determining the steepness of a wheel chair ramp in the first interview to considering both quantities (height and length of the ramp) to be equally important in determining the steepness of a ramp in the third interview.

> Terry's images of steepness and his understanding of the relationships among the quantities of steepness, height, and length changed significantly over the course of the two interviews. At the beginning of interview 1, Terry linked the measurement of the steepness of a ramp with a series of height measurements. At best, length was an implicit quantity, seemingly unconnected to the height and steepness…During Interview 3 Terry constructed a different quantitative relationship among steepness, height, and length. As a result of being able to extend the incline, Terry appeared to construct length as having equal status to height. Subsequently, he was able to create a new image of the ramp situation in which he could vary height and length independently of each other.

This finding suggests that Terry's initial failure to consider the height and length of a wheelchair ramp as co-varying quantities limited him from understanding the relationship among the height, length, and steepness of a wheelchair ramp. In my study, I examined how undergraduate students reason about relationships among quantities in an economic context.

Similar findings were reported by Moore and Carlson (2012) who examined how nine students, drawn from three sections of a precalculus course at a large public university, engaged in quantitative reasoning while reasoning about the volume of a box "formed by cutting equal-sized squares from each corner" (p. 51) of an 11 inch by 13 inch sheet of paper and folding the sides up. Each of these students participated in a task-based interview where they were asked to "write a formula that predicts the volume of the box from the length of the side of the cutout" (p. 51). Moore and Carlson found that, at first, a majority of the students did not recognize that the length and width of the box co-varies with the length of each square that is cut out from the sheet of paper. Consequently, the students conceived of the box as having a static base with dimensions 13 inches (length) by 11 inches (width) instead of a dynamic base with dimensions

(13-2x) inches by (11-2x) where x is the length of each square that is cut out from the sheet of paper.

> When initially responding to the box problem, students often did not conceive of the length and width of the box as distinct from the length and width of the paper. However, these students did conceive of the length and width of the box as measurable attributes, although they did not conceive of their values in terms of a relationship between varying length of the side of the cutout and the dimensions of the original sheet of paper. (p. 57)

Conceiving of "the length and width of the box as measurable attributes" shows that these students considered the length and width of the box as quantities and not as numerical values which, in turn, suggests that they considered the volume of the box to be a quantitative relationship among the length, width, and height of the box. The students were eventually successful in creating a correct formula for the volume of the box. This was after they were prompted by one of the researchers to re-read the problem in an effort to encourage the students to consider the context of the problem, that is, the process of making the box. Moore and Carlson argued that "it was only after the students imagined the process of making the box and considered how the relevant quantities of the situation changed in tandem that they created a correct volume formula" (p. 57). In general, this study shows that covariational reasoning is an essential understanding that students need if they are to be successful in relating co-varying quantities using algebraic equations.

Taken together, the findings of the studies reviewed in this section suggest that engaging in quantitative reasoning is not only problematic for students at the secondary level but also for students at the college level. Some of these studies (e.g., Lobato & siebert, 2002; Moore & Carlson, 2012) showed that covariational reasoning is an essential understanding that students at the secondary and undergraduate level need in order to reason quantitatively about relationships among quantities.

Non-empirical research. This section reports on the review of three research studies (Ellis, 2011; Thompson, 1990; Thompson, 2011) that have looked at the importance of quantitative reasoning from a theoretical perspective. This research elaborates on the idea of quantitative operations which was first discussed in the theory of quantitative reasoning that was presented at the beginning of this chapter.

In his early work on the theory of quantitative reasoning, Thompson (1990) argued that quantities can be related using quantitative operations. He also defined eight different types of quantitative operations. These quantitative operations, together with examples to illustrate each quantitative operation, are shown in Table 1.

Table 1. Definitions and examples of quantitative operations reproduced from Thompson (1990, pp. 11-12)

Operation	Example
Combine quantities additively	Unite two sets; consider two regions as one
Compare quantities additively	"How much more (less) of this is there than that?"
Combine quantities multiplicatively	Combine distance and force to get torque; combine linear dimensions to get regions; combine force applied and distance travelled to get work
Compare quantities multiplicatively	"How many times bigger is this than that?" "This is (multiplicatively) what part of that?" "How many of these in those?"

Generalize a ratio	"Suppose this comparison applies generally (i.e., suppose it were to continue at the same rate)."
Instantiate a rate	"Travel 50 miles per hour for 3 hours." "Travel 5 hours per mile for 6 miles"
Compose ratios	"Jim has 3 times as many marbles as Sally; Sally has 4 times as many marbles as Fred. Jim has so many times more marbles than Fred."
Compose rates	"A German mark is 75.53 Japanese yen. A US dollar is 1.88 marks. A dollar is some number of yen."

In his recent work on quantitative reasoning, Thompson (2011) argued that "quantitative operations are not the same as numerical operations (but they are related)" (p. 38).

> Quantitative operations are those operations of thought by which one constitutes situations quantitatively. Numerical operations are the operations by which one establishes numerical relationships among their measures. Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations children (and teachers) can confound them unproblematically. (p. 42)

Ellis (2011) provided an example to illustrate the distinction between a quantitative operation and a numerical operation: "one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division" (p. 216). In other words, quantitative operations are formed by performing operations (e.g., subtraction) on quantities while numerical operations are formed by performing operations on numerical values.

Ellis (2011) also argued that providing middle school students with opportunities to reason about quantities and relationships between quantities is a powerful way to introduce students to the concept of function.

> In this chapter I argue that reasoning directly with quantities and their relationships constitutes a powerful way to help students build beginning conceptions of function at the middle-school level. In particular, reasoning with quantities can directly support a covariation approach to function, while also providing a foundation for reasoning more flexibly with functional relationships later on. (p. 215)

Ellis's argument suggests that engaging middle-school students in quantitative reasoning supports them in developing a covariational view (Confrey & Smith, 1995) of the concept of function and provides a solid foundation for reasoning about functions and phenomena (e.g., exponential growth) that can be modeled using different families of functions (e.g., exponential functions) beyond middle-school level.

Knowledge Gaps in the Literature

The research literature reviewed in this chapter revealed several knowledge gaps regarding students' learning of calculus, learning mathematics in context, students' understanding of optimization problems, students' quantitative reasoning, and the opportunity to learn from mathematics from textbooks. First, while students' understanding of real-valued functions of a single variable is well documented, little research has attended to students' understanding of real-valued functions of two variables or more and their applications in real life situations. Second, of the research that has looked at students' understanding of real-valued functions of two variables, none has particularly investigated students' quantitative reasoning while modeling multivariable situations in an economic context. Much of the research that has looked at students' quantitative reasoning (e.g., Moore & Carlson, 2012) has been with real-valued functions of one variable and mainly in other contexts besides economics.

Third, research that has looked at the derivative (a continuous function) as an approximation of marginal change (a discrete quantity) is lacking. A related line of research (Lobato et al., 2012; Mkhatshwa, 2014) suggests that students treat marginal change as the difference (amount of change) and not as the difference quotient (rate of change per unit of one) in a kinematics and economics context respectively. I argue that, to some extent, students' confusion of marginal change and amount of change could be linked to the opportunities (e.g. textbooks and lectures) that students have to learn about these ideas. Because of the importance of the idea of marginal change in business or economics, there is a need to examine the treatment of marginal change (i.e. marginal change as a rate of change or marginal change as an amount of change) in business mathematics textbooks as well as how marginal change is presented to students during classroom instruction.

Fourth, there is a shortage of studies investigating students' understanding of non-geometric optimization problems especially those that are situated in real-world contexts such as economics. From the literature on students' understanding of geometric problems we know that high school and undergraduate students have difficulty: (1) setting up objective functions (e.g. Klymchuk et al., 2012; Swanagan, 2012; Villegas et al., 2009; White & Mitchelmore, 1996), (2) determining optimal values from critical numbers (e.g. Brijlall & Ndlovu, 2013; Swanagan), and

(3) symbolizing and relating variables (e.g. Villegas et al., 2009; White & Mitchelmore, 1996). I believe that these students' difficulties may show up when students solve non-geometric optimization problems that are situated in other contexts such as economics. Only one study (Dominguez, 2010) has particularly investigated students' understanding of non-geometric optimization problems in an economics context. Given the importance of calculus in the study of economics (Butler, Finegan, & Siegfried, 1998; Von Allmen, 1996), I argue that there is a need for more research that looks at students' understanding of non-geometric optimization problems in an economic context. I hypothesize that students might experience some difficulty in solving economic optimization problems that are situated in other context representations (graphical, tabular, and verbal) besides the algebraic context representation.

Fifth, research on the opportunity to learn mathematics via textbooks at the undergraduate level is scarce. Only one study by Mesa et al. (2012) has examined students' opportunity to learn about exponential and logarithmic functions at the undergraduate level. This research focused, among other things, representation of examples as well as the cognitive demands of those examples. There is a need for research on the opportunity to learn about various content areas such as optimization problems and marginal change at the undergraduate level. Also, none of the research on opportunity to learn mathematics via textbooks and the research on calculus learning has particularly focused on examining how students engage in quantitative reasoning when analyzing multivariable situations in an economic context.

My study sought to address some of the gaps identified in the research literature. In particular, my study examined business calculus students' quantitative reasoning when solving optimization problems that are situated in the economic context of cost, revenue, and profit. The following research questions guided me in this examination.

1. What opportunities to learn about (a) optimization problems, (b) the concept of marginal change and (c) quantitative reasoning in the context of cost, revenue, and profit do business calculus textbooks and classroom instruction provide to business calculus students?
2. How do business calculus students reason algebraically about optimization problems that are situated in the context of cost, revenue, and profit?
3. How do business calculus students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit?
4. What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit reveal about their quantitative reasoning?

Chapter 3 – Methodology

In this chapter, I discuss the design of my study, the participants, the data I collected, how the data were analyzed, and how I addressed the research questions.

Research Design

To answer my research questions, I conducted a qualitative study (Creswell, 2014) with university students enrolled in five sections of an undergraduate business calculus course. Two of the sections were taught in the spring semester of 2015 and three of the sections were taught in the fall semester of 2015. Two of the sections offered in the fall semester of 2015 were taught by the same professor while the other three sections (two in the spring of 2015 and one in the fall of 2015) were taught by three different professors. With a focus on examining students' algebraic reasoning, interpretation of marginal change, and quantitative reasoning when solving optimization problems that are situated in an economic context, 12 pairs of students were recruited to participate in task-based interviews. In conducting the interviews, I followed the principles and techniques (e.g., choosing tasks that are accessible to the participants) suggested by Goldin (2000). One of the principles, encouraging free problem solving, was unintentionally adapted when conducting the interviews in that sometimes I was quick to interject with a prompt or probe without giving the participants ample opportunity for free problem solving as described by Goldin (2000). The potential influence of adapting Goldin's free problem principle on the results and the interpretation of the results is discussed in the discussion and conclusions chapter.

There were four tasks in total (Appendix A). These tasks were designed to examine students' algebraic reasoning, interpretation of marginal change, and quantitative reasoning when solving optimization problems in an economic context. I classified each of the four tasks as either having a continuous representation or as having a discrete representation. The term continuous representation as used in this study refers to a mathematical task in which the given function(s) in the task is represented as continuous on some domain. Task 1 has a continuous representation because the given functions in the task, namely the total cost function and the total revenue function, are given algebraically as continuous on the domain [0,120]. Task 3 is another example of a task that has a continuous representation because the given functions in the task, namely the total cost function, the total revenue function, and the profit function, are given graphically as continuous on the domain [0,7]. Similarly, the term discrete representation as used in this study refers to a mathematical task in which the given function(s) in the task is represented as discrete on some domain. Task 2 is an example of a task that has a discrete (verbal) representation because the revenue function is discrete on the domain [0,1,…,N] where N is the maximum number of computers that the Smith family business can supply. Task 4 is another example of a task with a discrete representation because the numerical table as the given function in the task is discrete on the domain [400, 401,…, 405].

The findings of research by Borgen and Manu (2002) and Swanagan (2012) suggest that students have difficulty solving routine optimization problems algebraically. These difficulties include setting up the objective function in de-contextualized situations. Other research (e.g., Brijlall & Ndlovu, 2013) suggests that students have a poor conceptual understanding of the algebraic process used to find optimal values (absolute extrema) in a volume-minimization context. Informed by findings of this research, I designed Task 1 to examine students' algebraic reasoning when solving optimization problems that are situated in the economic context of cost, revenue, and profit. The purpose of Task 1 was to confirm students' ability as reported in the research literature to solve optimization problems algebraically that are similar to those given in a standard business calculus textbook in a profit-maximization context.

Task 2 was informed by the research of Lobato et al. (2012) and Mkhatshwa (2014). Lobato and colleagues found that students treated marginal change as an amount (the difference) and not as a rate of change (the difference quotient) in a kinematics context. Mkhatshwa (2014) reported similar findings in an exploratory study of business calculus students' understanding of marginal change that was situated in a decision-making context about marginal profit involving the addition of an extra jet by an airline. Task 2 helped me gain insight on students' interpretation of marginal change and quantitative reasoning in a revenue-maximization context. I designed Task 2 to examine students' quantitative reasoning about the relationship among three quantities, namely sales (the number of computers ordered by a small junior high school), the discount offered on orders of over 300 computers, and the revenue that is generated by a small family business that has been asked to supply the school with computers.

Evidence from research (e.g., Carlson, 1998; Carlson et al., 2002) showed that students have difficulty creating a graphical representation (a function) that shows how the input and output of a real-valued function of a single variable co-vary in a volume and heat context respectively. The purpose of Task 3 was to examine students' quantitative reasoning when analyzing a multivariable situation in a graphical context. I designed Task 3 to examine students' reasoning about relationships among quantities: creating the graph of a fourth quantity (profit) when given a graph of three quantities (number of units produced and sold, total cost, and total revenue) that co-vary. In addition to examining students' understanding of the relationship between marginal cost and marginal revenue at a profit-maximizing quantity (production and sales level at which profit is maximized) in Task 3, I also examined students' interpretation of marginal cost (the cost of producing the second unit).

The purpose of Task 4 was to examine how students interpret marginal change in a profit-maximization context. Research by Lobato et al. (2012) and Mkhatshwa (2014) suggest that students tend to think of marginal change as an amount of change (the difference) and not as a rate of change (the difference quotient). I designed Task 4 to help me gain insight into this issue by examining students' interpretation of marginal change (the cost of producing the 401st computer chip). This task also provided me with insight regarding students' quantitative reasoning, that is, students' understanding of the relationship among number of computer chips produced and sold, marginal cost, marginal revenue, and marginal profit. I also used each of the four tasks to examine new quantities (e.g., total revenue in Task 1) that students created and how they reasoned about these quantities when solving the problems posed in the four tasks. A complete description of each task is given in Appendix A.

I used a classroom observation protocol (Appendix B) to collect data that helped me understand the calculus knowledge students had an opportunity to learn in the business calculus course, through course lectures, prior to participating in the task-based interviews. My focus during classroom observations was on the definition or interpretation of marginal change (e.g., marginal change as a rate or marginal change as an amount of change), the type(s) of context (no context, camouflage context, relevant and essential context), the type(s) of information (matching, missing, superfluous), and the type(s) of cognitive demands (reproduction, connection, reflection) used in examples of optimization problems given during classroom instruction. A description of the types of context, types of information, and types of cognitive demands is shown in Appendix D. I observed three classes (only for the three sections taught in the fall of 2015) covering the following content areas: (1) the derivative, (2) the derivative as a rate of change (applications of rate of change to economics), and (3) applied maxima and minima. How and what the students were taught did, to some extent, have an influence on how they engaged with the mathematical ideas rooted in the four tasks. Details on this are discussed in the last chapter: Discussion and Conclusions. Each classroom observation lasted for as long as the meeting time for each section observed, that is, about 80 minutes.

I used a textbook analysis protocol (Appendix C) to collect data that helped me understand the calculus knowledge the students had an opportunity to learn in the business calculus course, through examples and assigned practice exercises given in the textbook (Haeussler, Paul, & Wood, 2011), prior to participating in the task-based interviews. According to Mkhatshwa and Doerr (in press), the textbook analyzed in this study is one of six widely used textbooks in the teaching of business calculus in the United States. My focus in analyzing the course textbook was on the definition or interpretation of marginal change given in the textbook (e.g., marginal change as a rate or marginal change as an amount of change), the type(s) of context (no context, camouflage context, relevant and essential context), the type(s) of information (matching, missing, superfluous), and the type(s) of cognitive demands (reproduction, connection, reflection) used in assigned or practice problems and examples of optimization problems given in the textbook. I also analyzed sections of the course textbook that dealt with the treatment of marginal change and optimization problems. I analyzed examples and assigned homework or practice problems corresponding to three content (topics) areas that are presented in the textbook: (1) the derivative, (2) the derivative as a rate of change (applications of rate of change to economics), and (3) applied maxima and minima. Students' reasoning about the four tasks (Appendix A) was, to a large extent, influenced by the opportunities that students had to learn about optimization problems and marginal change via the textbook. Nearly all the examples that were discussed in course lectures in the three sections that I observed were either the same or minor adaptations of the examples given in the textbook. Details on this are presented in the results chapter.

Benefits of the design. I have identified four advantages for using this design:

1. This design allowed me to conduct in-depth interviews that helped examine how students reason about economic quantities such as marginal cost, marginal revenue, and profit and how these quantities are related while solving optimization problems that are situated in a business or economic context. In essence, the use of this design enabled me to focus more on students' quantitative reasoning as opposed to just correct and incorrect answers.
2. The interviewing of pairs of students used in this design shifted the students' focus from the researcher to the tasks (Appendix A). Having two students interact with each other while engaging in the task-based interview, which is something fostered by this design, helped to reveal the students' detailed understanding of the mathematical ideas (e.g., marginal change in Task 3) rooted in the tasks. This could have been harder to achieve when interviewing individual students.
3. The use of this design allowed me to analyze opportunities (course lectures and a course textbook) the students had to learn about marginal change and how to solve optimization problems in an economic context. Analysis of the classroom observations (course lectures) and the course textbook: (1) provided possible explanations why students reasoned about the tasks the way they did in addition to (2) helping me understand the extent to which students were able to apply mathematical knowledge they were taught on how to solve realistic optimization problems in an economic context.
4. Implementing this design did not require a large number of participants which greatly minimized the challenge of recruiting many students to participate in the study especially

because some of the task-based interviews were administered towards the end of the semester, a time of the academic year when students are busy. Consequently, recruiting a large pool of students to participate in this study would have been a challenge.

Limitations of the design. I have identified four disadvantages of using this design:

1. The success of this design in producing rich data that was helpful in answering my research question depended, to a large extent, on two things: the quality of the tasks and my skills as an interviewer in conducting the interviews. To minimize the effect of this limitation on my research, I drew on my learning experiences from my previous research (Mkhatshwa, 2014) where I used a similar design. In particular, drawing from my experiences in my previous research helped me develop thought-revealing tasks (application problems) and also to apply effective interviewing skills.
2. The analysis and interpretation of the findings obtained by implementing this design may have been influenced by my bias as a researcher which includes my prior knowledge of some difficulties (through tutoring and teaching business calculus) similar to those examined in the task-based interview that students experienced when reasoning about context problems. The effect of this limitation was minimized by having other researchers scrutinize my data analysis on two occasions during our weekly mathematics education seminar.
3. In light of the small number of participants, findings of this study may not be externally valid, that is, they may not be generalized to other settings. Since my goal in this study was not to generalize findings from this study to other settings but rather to gain insights on how students interpret marginal change and how they engage in quantitative reasoning when solving context problems in economics, this was not an issue of concern.

4. I did not analyze the influence of an economics background on the students' reasoning about the tasks (Appendix A) they were given in this study. Students with a prior economics background might have had more learning experiences to draw on during the task-based interviews than did students who had only taken business calculus.

Participants

Participants of the study were 24 undergraduate students. Twenty-two of these students were business majors (e.g., management, marketing, accounting, etc.) while the other two students were considering majoring in business-related programs even though at the time of conducting this study they had not yet officially declared their majors. All the students had taken business calculus either in the spring semester of 2015 or in the fall semester of 2015. Table 2 shows a summary of the participants and their background data. The names appearing in Table 2 are pseudonyms.

Table 2. Summary of participants

Name	Major	Level	Taken Calculus Before	Taken AP Economics	Taken Some College Economics/ Business classes	Final Course Grade	GPA
Ivy	Business	Sophomore	Yes	No	Yes	A	3.863
Denise	Business	Sophomore	Yes	Yes	Yes	B	3.226
Ruth	Business	Sophomore	No	No	Yes	C	2.789
Eric	Undecided	Sophomore	No	No	Yes	F	2.374
Joy	Business	Sophomore	No	Yes	Yes	A	3.695
Nancy	Business	Sophomore	No	No	Yes	B	3.438
Nevaeh	Business	Sophomore	Yes	No	Yes	A	3.765
Zoe	Business	Sophomore	Yes	Yes	Yes	B	3.315
Mark	Business	Sophomore	Yes	Yes	Yes	A	3.667
Carlos	Business	Sophomore	No	Yes	Yes	B+	3.578
Kierra	Business	Sophomore	No	Yes	Yes	A-	3.388
Isaac	Business	Sophomore	No	No	Yes	A	3.748
Abby	Business	Freshman	No	No	Yes	A-	3.680
Shawna	Business	Freshman	No	No	No	A	3.375
Jacie	Business	Freshman	Yes	No	Yes	A-	3.216

Derby	Business	Freshman	Yes	No	Yes	A	3.471
Casey	Undecided	Freshman	No	No	Yes	A	3.370
Nikki	Business	Freshman	Yes	Yes	No	A	3.353
Sarah	Business	Freshman	No	No	Yes	A	3.750
Alan	Business	Freshman	No	Yes	Yes	C+	2.771
Fred	Business	Freshman	Yes	Yes	No	A	3.938
John	Business	Sophomore	Yes	Yes	Yes	B+	3.206
Yuri	Business	Sophomore	Yes	No	Yes	A	3.784
Kyle	Business	Sophomore	Yes	No	Yes	B+	3.458

In addition to taking business calculus: (1) twelve students had taken a calculus course (high school or college), (2) ten had taken AP economics (AP microeconomics and AP macroeconomics) at high school, and (3) twenty-one students had taken at least a college-level economics or business class (e.g., intermediate microeconomics, managerial accounting) prior to participating in this study. Fifty percent of the students took business calculus in the spring of 2015 semester while the other 50% (n=12) of the students took business calculus in the fall semester of 2015.

Together, there were four different professors who taught business calculus in the spring of 2015 and fall of 2015 semesters respectively. Each of the two sections of business calculus offered in the spring of 2015 semester were taught by two different professors while two of the three sections of business calculus offered in the fall of 2015 semester were taught by the same professor. The other section of business calculus offered in the fall of 2015 semester was taught by a different professor. All the professors have Ph.D.'s in mathematics, they taught from the same textbook, they covered the same content, and they used similar syllabi. Each professor created his/her semester exams and quizzes. Students in the two sections of business calculus taught in the spring semester of 2015 took a common final exam. All three sections of business calculus taught in the fall of 2015 semester also had a common final exam. The two common final exams, one given at the end of the spring of 2015 semester and the other given at the end of the fall of 2015 semester, were different in terms of the questions asked but similar in terms of the content tested. Each of the five sections met twice a week for 80 minutes during each semester. Students from each section also meet once a week for an additional 55 minutes every week for recitation with graduate teaching assistants.

The students in this study were recruited through a self-selection process at a medium-sized research university in the northeastern part of the United States. All the students were recruited via email using official class roasters (with student names and emails) obtained from their business calculus professors. The mathematical ideas examined in the task-based interview and the context of cost, revenue, and profit were familiar to all the students who participated in this study. The students were chosen based on: (1) their willingness to participate in the study and (2) their major (business or economics). The 24 students combined had an average GPA of 3.426. Judging by each student's course grade in business calculus and cumulative GPA shown in Table 2, a majority of the participants were above average performing students. Students' background information (Appendix E) such as major, academic year, whether or not they have taken an economics class before, and whether or not they have taken a calculus course before were collected at the time of recruitment.

Each student was compensated at a rate of $20 per hour for the amount of time spent during the task-based interview. Each interview, on average, lasted for about one hour fifteen minutes. There is some variation in the amount of time taken by each pair of students to complete each task. Table 3 lists the duration of time, in minutes, spent by each of the twelve pairs of students discussing each of the four tasks they were engaged in during each task-based interview session. Most of this time was spent on the first three tasks and the least amount of time was spent on the last task. It is rather surprising that, on average, students spent nearly as much time in Task 1 as they did in Task 2 and Task 3 even though Task 1 was a routine problem that all the students acknowledged having seen or even solved a problem similar to it prior to participating in this study. On the other hand, Task 4 was a task all the 24 participants had never seen and took the least amount of time to complete.

Table 3. Task duration in minutes

Student Pair	Task 1	Task 2	Task 3	Task 4	Average Completion Time for each Task by Pair
Ivy & Denise	11	22	19	13	16.25
Ruth & Eric	9	24	23	8	16
Joy & Nancy	29	19	24	11	20.75
Zoe & Nevaeh	24	20	22	13	19.75
Mark & Carlos	21	24	25	15	21.25
Kierra & Isaac	12	21	23	16	18
Abby & Shawna	21	27	16	8	18
Jacie & Derby	14	14	23	10	15.25
Nikki & Casey	27	14	23	7	17.75
Sarah & Alan	28	11	16	8	15.75
Fred & John	17	10	25	15	16.75
Yuri & Kyle	23	30	22	14	22.25
Average Completion Time for each of the Four Tasks by all the Pairs	19.67	19.67	21.75	11.5	

Data Collection

Data for the study came from three sources: task-based interviews which also included work written by students during each task-based interview session, classroom observations, and the textbook used in the course. I conducted task-based interviews with twelve pairs of undergraduate students using carefully designed tasks that are experientially real to the students. The students were paired based on their availability and willingness to participate in the task-based interviews. The tasks were piloted with two pairs of students from a business calculus course offered in the summer of 2015 prior to administration in the fall semester of 2015. Piloting the tasks helped me identify deficiencies in both the tasks and the task-based interview protocol which led to the improvement of the qualities of the tasks and interview protocol. One important improvement I made was to use three short paragraphs instead of two to improve the readability of the problem statement of Task 2. Thought-revealing prompts and probes were added, following the piloting of the task, in the interview protocol in an effort to elicit students' thinking about the relationship among quantities (e.g., number of computers sold, discount, and revenue generated in Task 2).

Six pairs of students were recruited from two sections of a business calculus course that was taught in the spring semester of 2015 and another six pairs of students were recruited from three sections of a business calculus course that was taught in the fall semester of 2015. In particular, three pairs of students were recruited from one section of the business calculus course taught by one professor in the fall of semester of 2015 and the other three pairs of students were recruited from the other two sections of the business calculus course taught by a different professor in the fall semester of 2015. The students were not paired by section but rather based on their availability (date and time) to participate in the interview. Each interview was video recorded and later transcribed for analysis.

Work written by students during each interview session was also collected as part of the data for the study. This data consisted of graphs of functions, equations, and textual descriptions of changing economic quantities such as profit. This data helped me get a better understanding of students' thinking about some of the mathematical ideas such as marginal change (in multiple representations) mentioned by the students during the task-based interview. Students also wrote some of their thoughts as they worked through the tasks. These written thoughts were helpful in understanding students' reasoning about the mathematical ideas rooted in the tasks. The video recordings were used for checking what students were referring to when they pointed at something during the task-based interview in cases when such information could not be easily obtained from the students' written work that was collected as part of the data for the study.

Following each interview, I wrote a brief memo where I documented my initial impressions about each pair of students' responses to each of the four tasks. Each memo also contained an elaboration of my jottings (brief notes) taken during the interview. Lastly, each memo served as a quick reminder of an interview session at a later date.

The two professors who taught business calculus in the fall of 2015 taught from the same textbook and their teaching approach was similar. I am not aware whether or not they shared teaching materials. I observed three classes of each of two sections of business calculus taught in the fall semester of 2015: three classes of one of the two sections of business calculus taught by one professor and three classes of another section that was taught by a different professor. During my first observation, I observed how each professor explained the concept of the derivative to students. The focus of my observation was on the professors' attention to the covariation of the input and output variables in their discussion of the difference quotient. During my second observation, I observed how each professor explained the concept of marginal change (as a rate of change-the difference quotient or as an amount-the difference) and the relationship between the derivative and marginal change (marginal change as the derivative or the derivative as an approximation of marginal change). During my third observation, my focus was on each professor's discussion of the algebraic procedure for finding optimal values (critical numbers and extrema) for an objective function. In particular, I observed each professor's effort(s) to encourage students to interpret critical numbers and extrema in context and to verify the type(s) of extrema (maximum/minimum) that the objective function has. Details of the areas of focus during each of my classroom observations can be seen in the classroom observation protocol which appears in Appendix B.

I used a textbook protocol (Appendix C) to collect data that helped me understand opportunities provided in the course textbook that students had to learn about quantitative reasoning (relationships among quantities such as relationships among number of units produced and sold, marginal cost, marginal revenue, and profit), marginal change, and solving optimization problem that are situated in a business or economic context using algebraic methods. This data collection focused on the treatment of the following topics: (1) the concept of the derivative, (2) marginal change, and (3) solving optimization problems using algebraic methods in the course textbook. I also collected information about the type(s) of context (no context, camouflage, relevant and essential), type(s) of information (matching, missing, superfluous), and cognitive demands (reproduction, connection, and reflection) used in examples or practice problems from the course textbook. All four professors taught from the same textbook. Hence, the students had similar learning opportunities to learn how to solve context-based tasks using algebraic methods from the textbook.

Data Analysis

Analysis of the collected data was done in three phases: (1) analysis of the business calculus course textbook, (2) analysis of classroom observations, and (3) analysis of task-based interviews. Following is a description of how data was analyzed in each of the three phases.

Analysis of the course textbook. The analysis of the textbook was done in two steps. In the first step, I used an adaptation of the textbook analysis framework developed by Wijaya (2015) (Appendix D) to analyze optimization tasks in the business calculus course textbook

(Haeussler et al., 2011). This analysis focused on the types of context, types of information, types of cognitive demands, and types of representation (algebraic, tabular, graphic, verbal). I coded a task: (1) as algebraic if an algebraic formula was given in the task, (2) as tabular if a numerical table was given in the task, (3) as graphic if a graph was given in the task, and (4) as verbal (textual) if there were no algebraic equations, numerical tables, and graphs that were given in the task. Since the focus of my study was on students' reasoning about application problems and not on modelling problems, I replaced the purpose of context based task dimension in the framework (Appendix D) with the representation of task dimension which I have described above. I analyzed a total of eleven optimization tasks (six optimization examples and five assigned optimization practice problems). Following is an illustration of how I used the textbook analysis framework to analyze three of the eleven optimization tasks:

> **Example 8 (Profit Maximization):** Suppose that the demand equation for a monopolist's product is $p = 400 - 2q$ and the average-cost function is $\bar{c} = 0.2q + 4 + \frac{400}{q}$, where q is the number of units, and both p and $\bar{c}$ are expressed in dollars per unit. Determine the level of output at which profit is maximized. (Haeussler et al., 2011, pp. 614-615).

I coded this example as: (1) algebraic because there are algebraic equations in the task, (2) having a camouflage context because the context can be ignored when solving this problem, (3) having matching information because it contains the exact amount of information needed to solve the problem posed, and (4) a reproduction task because this problem could be solved by simply recalling a memorized procedure. Similar problems were solved in the textbook prior to this example.

Example 6 (Maximizing TV Cable Company Revenue): The Vista TV Co. currently has 100, 000 subscribers who are each paying a monthly rate of $40. A survey reveals that there will be 1000 more subscribers for each $0.25 decrease in the rate. At what rate will maximum revenue be obtained, and how many subscribers will there be at this rate? (Haeussler et al., 2001, pp. 613-614).

I coded this example as: (1) verbal because there are no equations, numerical tables or graphs that are given in the task, (2) having a relevant and essential context because reasoning about the context of the problem is needed to solve the problem, (3) having matching information because it contains the exact amount of information needed to solve it, and (4) a connection task because to solve this problem, the solver must interpret the problem situation (to determine the revenue function) which is something that requires the solver to engage in simple mathematical reasoning.

Example 5 (Economic Lot Size): A company annually produces and sells 10,000 units of a product. Sales are uniformly distributed throughout the year. The company wishes to determine the number of units to be manufactured in each production run in order to minimize total annual setup costs and carrying costs. The same number of units is produced in each run. This number is referred to as the **economic lot size or economic order quantity.** The production cost of each unit is $20, and carrying costs (insurance, interest, storage, etc.) are estimated to be 10% of the value of the average inventory. Setup costs per production run are $40. Find the economic lot size. (Haeussler et al., 2001, pp. 612-613).

I coded this example as: (1) verbal because there are no equations, numerical tables or graphs that are given in the task , (2) having a relevant and essential context because reasoning about the context of the problem is needed to solve the problem, (3) having matching information because it contains the exact amount of information needed to solve it, and (4) a reflection task because the student must construct original mathematical approaches e.g., formulating the objective function (the total of the annual carrying costs and setup costs) is not straight forward.

Although there is no optimization task with no context given in the textbook, the following example, which I made up, illustrates what I mean by an optimization task with no context in this study:

Made up example (No Context): What value of x maximizes $y = -2x^2 + 3x + 2$? I coded this example as: (1) algebraic because there is an algebraic equation in the task, (2) having no context because the is no story line (camouflage or realistic) that goes with the problem, (3) having matching information because all the information needed to the maximum value of the given function is given in the statement of the problem, and (4) a reproduction task because this problem can be easily solved by performing explicit routine computations or procedures (e.g., completing the square in x, using first/second derivative test, or using a graphing calculator to determine the value of x that maximizes the given function).

In the second step, I analyzed the opportunities provided by the textbook through which students could learn about optimization and marginal change problems in a business or economic context. In particular, analysis of the course textbook in this step focused on three themes: (1) textbook conceptual opportunities to learn about optimization and marginal change problems, (2) textbook treatment of quantitative reasoning, and (3) textbook treatment of marginal change.

Textbook conceptual opportunities to learn about optimization problems are opportunities designed to help students develop a conceptual understanding of quantities (e.g., marginal cost) involved in the process of solving optimization problems in an economic context. Under this theme, I analyzed the opportunities provided in the textbook to encourage students to : (a) interpret quantities (e.g., critical numbers) in an economic context, (b) give appropriate units of quantities in an economic context (e.g., units of critical numbers in a profit maximization context), (c) distinguish reasonable critical numbers or extrema from those that are not reasonable, (d) verify mathematical results involving quantities in an economic context (e.g., verify that a particular critical number is a profit maximizing quantity), (5) the textbook's discussion of a procedure(s) for solving applied optimization problems, and (6) relative extrema versus absolute extrema optimization problems.

Textbook treatment of quantitative reasoning refers to opportunities provided in the textbook to engage students in analyzing economic situations in terms of quantities and relationships among quantities. Under this theme, I analyzed (a) the textbook's explanation of quantities such as marginal cost, and (b) the textbook's explanation of relationships among quantities such as number of units produced or sold, cost, revenue, and profit (e.g., with an increase in production and sales of units, costs per unit decrease while both revenue and profit per unit increase gradually).

Textbook treatment of marginal change refers to opportunities provided in the textbook that students had to learn about the concept of marginal change. Under this theme, I analyzed: (a) the definition or interpretation of marginal change given in the textbook (e.g. as a rate of change (difference quotient), as a change in amount (the difference), as an amount, or as both a rate of change and change in amount, (b) the relationship between marginal change and the derivative (marginal change being the derivative or the derivative being an approximation of marginal change) given in the textbook, (c) ways used to represent marginal change in the textbook (e.g. algebraically, tabular, graphically, or verbally), (d) the relationship between extrema and marginal change given in the textbook (e.g. maximum profit occurs when MR=MC provided total revenue exceeds total cost), (e) the textbook's attention to interpreting marginal change-related results, and (f) the textbook's attention to (assigning of) units for marginal change.

Analysis of classroom observations. The analysis of classroom observations was done in two steps. In the first step, I adapted (by replacing purpose of context-based task with types of representation as explained in the previous section) the textbook analysis framework (Wijaya, 2015) (Appendix D) to analyze optimization examples given by each of the two professors who taught business calculus in the fall semester of 2015, herein referred to as Professor A and Professor B. The purpose of doing this was to see how the opportunities students had to learn about optimization problems in an economic context, through course lectures, compare with the opportunities they had to learn about optimization problems through optimization examples given in the textbook. This analysis focused on the types of context, types of information, types of cognitive demands, and types of representations. I analyzed a total of five economic optimization examples given in class: two examples presented in Professor A's class and three examples that were presented in Professor B's class.

In the second step, I analyzed the opportunities students had, through course lectures to learn about optimization problems and marginal change in a business or economic context in each of the two sections I observed: one taught by Professor A and the other taught by Professor B. Analysis of course lectures focused on three themes: (1) classroom conceptual opportunities to learn about optimization problems, (2) attention to quantitative reasoning, and (3) discussion of marginal change.

Classroom conceptual opportunities are opportunities designed to help students develop a conceptual understanding of quantities (e.g., marginal cost) involved in the process of solving optimization problems in an economic context. Under this theme, I discussed opportunities students had, through examples given in course lectures, to learn about the importance and need to: (a) interpret quantities (e.g. critical numbers) in an economic context, (b) give appropriate units of quantities (e.g., units of critical numbers in a profit maximization context), (c) distinguish between reasonable critical numbers or extrema from those that are not reasonable, and (d) verify mathematical results involving quantities in an economic context (e.g., verifying that a particular number of units is the profit-maximizing quantity). This also includes a clear presentation of the purpose of each step of a procedure(s) introduced as a guide (if any) for students when solving applied optimization problems and a presentation of relative extrema optimization problems versus absolute extrema optimization problems.

Attention to quantitative reasoning refers to opportunities students had, through course lectures, to engage in analyzing economic situations in terms of quantities and relationships among quantities. Under this theme, I discussed (a) the professors' explanation of quantities such as marginal cost and (b) the professors' explanation of relationships among quantities such as number of units produced or sold, cost, revenue, and profit (e.g. with increase in production and sales of units, costs per unit decrease while both revenue and profit per unit increase gradually).

Discussion of marginal change refers to opportunities students had, through classroom instruction, to learn about the concept of marginal change. Under this theme, I analyzed (a) the definition or interpretation of marginal change given by the professors during lectures (e.g., as a rate of change (difference quotient), as a change in amount (the difference), as an amount, or as both a rate of change and change in amount (or amount)), (b) the relationship between marginal change and the derivative given by the professors during lectures (marginal change being the derivative or the derivative being an approximation of marginal change), (c) ways used by the professors to represent marginal change such as when giving examples or when lecturing (e.g., algebraically, tabular, graphically, or verbally) (d) the relationship between extrema and marginal change given by the professors when giving examples during lectures (e.g., maximum profit occurs when MR=MC provided total revenue exceeds total cost), (e) the professors' emphasis on the need for students to interpret marginal change-related results, and (f) the professors' use of units of marginal change in examples given in lectures.

Analysis of task-based interviews. Data analysis in this phase proceeded in two stages. In the first stage, I used a priori structure, that is, pre-determined codes. In particular, I carefully read through each interview transcript and coded instances where students reasoned about: (1) new quantities (e.g., diminishing marginal returns), (2) relationships between or among quantities (e.g., marginal profit equals marginal revenue minus marginal cost), (3) representations of relationships among quantities (e.g., using graphs and algebraic equations), (4) quantities in context (e.g., critical numbers in a profit-maximization), (5) how to solve each problem, and (6) how to verify the correctness or reasonableness of their solution(s). I also coded unanticipated instances that were of interest such as unanticipated solution strategies. Table 4 shows a sample of codes and examples from the data.

Table 4. Sample codes and examples from the data

Codes	Definition of Code	Examples
Discrete reasoning	Treating continuous quantities as if they were discrete quantities when reasoning about relationships among several quantities in an economic context.	Sarah and Alan reasoned discretely in Task 3 when they created a discrete profit graph despite the fact that the total cost function and total revenue function (given in the task) which they used to create the profit graph are continuous functions.
Marginal change as a rate	Interpreting marginal change as a rate. This includes giving units of marginal cost (MC)	Researcher: What are the units of these numbers [MR and MC values in Task 4, Appendix A]?

	and marginal revenue (MR) in dollars per unit.	Yuri: Dollars per unit Researcher: Why dollars per unit? Yuri: Marginal revenue is additional, extra revenue per unit
Marginal change as a consecutive relationship between one value and the next value.	Refers to the use of phrases or words such as “from one to two, ” “next, ” and “additional” without computing specific values when interpreting marginal change	Researcher: How much does it cost this company to produce the second unit, not the first two units? Fred: The cost of two units is 400 John: It’s only a hundred Fred: Oh yah, that’s going from the first to the second, so you are saying?
Verifying extrema	Using a formal procedure (e.g., the first derivative test) or an informal procedure (e.g., graphing a profit function on a graphing calculator and using the maximum function in a graphing calculator) to determine the profit maximizing quantity.	Researcher: How would you convince someone that that profit [pointing at the maximum profit that Sarah and Alan found] is the maximum profit? Sarah: Because, because, I don’t know Alan: It shows here [pointing at the critical number 40 in the number line for the first derivative test], it’s maximum at 40 [units], because it’s [profit] increasing and then decreasing, whenever it [profit] goes from increasing to decreasing you have a max [maximum profit] and that’s at 40 so that proves that the maximum profit is at 40 [units]

In the second stage, I looked for patterns (common understandings or difficulties with concepts or procedures) in students' responses to the tasks. I also looked for differences in students' reasoning about the mathematical ideas rooted in the tasks. In Task 1, for example, this consisted of looking for common understandings or difficulties in: (1) setting up the objective function, (2) finding critical numbers using algebraic methods, (3) interpreting critical numbers in a profit-maximization context, (4) interpreting extrema in a revenue-maximization context, (5) determining extrema from critical numbers, and (6) verifying extrema as minimum or maximum values of the objective function.

Looking for patterns in students' reasoning about Task 2 consisted of looking for common understandings or difficulties around: (1) thinking about the relationship among the number of computers ordered, the discount, and the revenue that is generated (e.g. thinking about how the discount affects revenue when the school orders more than 300 computers), (2) reasoning about marginal change in a problem situation that is verbally (textually) represented, and (3) giving a reasonable advice in a revenue-maximization context.

Looking for patterns in students' responses to Task 3 consisted of looking for common understandings or difficulties around: (1) reasoning quantitatively about relationships among four quantities, namely the number of units produced and sold, total cost, total revenue, and profit, (2) marginal analysis from a graph (reasoning about marginal cost and marginal revenue at a critical number), (3) determining critical numbers or absolute extrema from a graph, and (4) making a reasonable decision in a profit-maximization context.

Looking for patterns in students' responses to Task 4 consisted of looking for common understandings or difficulties around: (1) thinking about the relationship among number of units (computer chips) produced and sold, marginal cost, marginal revenue, and marginal profit (or profit), (2) marginal analysis from a numerical table (e.g., interpreting the significance of MR=MC in a profit-maximization context), (3) understanding of marginal change (giving units of marginal cost and marginal revenue), (4) interpreting marginal change (the cost of producing the 401st computer chip), and (5) making a reasonable decision in a profit-maximization context. I then used the common difficulties or understandings (including differences) in students' reasoning found in the second stage of the task-based interview analysis to address my research questions. Details about how I did this are given in the following section.

Addressing the Research Questions

The analysis of the textbook, the analysis of course lectures, and the analysis of the task-based interview data (phase three) led to answers to the research questions for my study. To answer the first research question:

1. What opportunities to learn about (a) optimization problems, (b) the concept of marginal change and (c) quantitative reasoning in the context of cost, revenue, and profit do business calculus textbooks and classroom instruction provide to business calculus students?

I used results obtained by using the four dimensions of the analytical framework (Appendix D) to describe the opportunities that students have to learn about optimization problems, marginal change, and quantitative reasoning via the textbook and via course lectures. I discussed students' opportunities to learn about optimization problems via the textbook and via course lectures in terms of types of context, types of information, types of cognitive demand, and types of representations of the optimization examples that were given in the textbook and in course lectures. I also discussed other opportunities to learn about optimization problems that were presented in the textbook and in course lectures. These opportunities included, among others, interpreting critical numbers in context and stating units of critical numbers and extrema. Opportunities to learn about marginal change consisted of definitions or interpretations of marginal change (e.g., marginal cost) as well as the use of units in the presentation of marginal change examples given in the textbook and in course lectures. Opportunities to learn about quantitative reasoning included the analysis of economic situations, in the textbook and in course lectures, in terms of quantities and relationships between or among quantities.

To answer the second research question:

2. How do business calculus students reason algebraically about optimization problems that are situated in the context of cost, revenue, and profit?

I used patterns of students' understanding or difficulties when solving Task 1. These patterns included common understandings or difficulties in: setting up objective functions, interpreting critical numbers in context (e.g., interpreting a profit maximizing quantity), interpreting extrema in context, verifying extrema as a minimum or maximum value of an objective function, and determining which critical number gives a minimum or maximum value of the objective function.

To answer the third research question:

3. How do business calculus students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit?

I used patterns of students' understandings or difficulties when solving Task 1, Task 2, Task 3, and Task 4. For Task 1, these patterns included students' common understanding of the derivative of the total cost function given in the task as marginal cost. For task 2, these patterns included common understandings or difficulties in reasoning about marginal change in a verbal [textual] context, and giving a reasonable advice in a revenue-maximization context. For task 3, these patterns included common understandings or difficulties in: (1) determining marginal change from a graph (e.g., the cost of producing the second unit), (2) being able to see marginal change from a graph (e.g. showing the cost of producing the second unit on a graph), and (3) understanding the relationship between maximum/minimum profit and marginal change (e.g., maximum profit occurs at a production and sales level where marginal cost equals marginal revenue provided total revenue exceeds total cost at that level). For task 4, these patterns included students' common understandings or difficulties in: (1) giving units of marginal change, (2) interpreting marginal change (e.g., the cost of producing the 401st computer chip), and (3) reasoning quantitatively about the relationship among the number of computer chips produced and sold, marginal cost, marginal revenue, and profit.

To answer, the fourth research question:

4. What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit reveal about their quantitative reasoning?

I used patterns of students' understandings or difficulties when reasoning about the four tasks. These patterns included common understandings or difficulties around: (1) relationships among quantities in multivariable situations (e.g., the relationship among sales (number of computers sold), discount, and revenue in Task 2) and (2) new quantities (e.g., average cost in Task 1) that students created and used (or did not use) in an attempt to solve the problems posed in the four tasks.

Chapter 4 – Results: Opportunity to Learn and Algebraic Reasoning

In this chapter, I present the results from the analysis of the data I collected with the purpose of answering the first two research questions on opportunity to learn and algebraic reasoning:

1. What opportunities to learn about (a) optimization problems, (b) the concept of marginal change and (c) quantitative reasoning in the context of cost, revenue, and profit do business calculus textbooks and classroom instruction provide to business calculus students?
2. How do business calculus students reason algebraically about optimization problems that are situated in the context of cost, revenue, and profit?

In the first section of this chapter, I address my first research question by presenting results from the analysis of opportunities to learn (via the textbook and course lectures) about solving optimization problems, the concept of marginal change, and quantitative reasoning in the context of cost, revenue, and profit. In the second section of this chapter, I address my second research question by presenting results from the analysis of students' algebraic reasoning when solving optimization problems in the context of cost, revenue, and profit.

Opportunity to Learn

To examine the opportunity to learn about optimization problems, marginal change, and quantitative reasoning in the context of cost, revenue, and profit via the textbook and course lectures, I analyzed examples and assigned practice problems that are presented in the textbook and examples that were presented in course lectures. There are four results from this analysis. First, the opportunity to learn to solve a range of realistic and cognitively demanding optimization problems in an economic context via the textbook (examples and practice problems) and examples given in class were limited. Second, several conceptual opportunities to learn about optimization problems were presented through optimization tasks (examples and assigned practice problems) given in the textbook and through examples given in course lectures. Third, the presentation of the concept of marginal change both in the textbook and in course lectures was largely vague and procedural. Fourth, opportunities for students to engage in quantitative reasoning, especially about relationships between or among economic quantities, via the textbook and in course lectures were limited.

Opportunity to learn solving a range of realistic and cognitively demanding optimization problems. There are eight optimization examples given in the textbook (Appendix H), six of which are situated in an economic context and the other two examples (example 4 and example 7 in Appendix H) are situated in non-economic contexts. Since one major reason for analyzing the textbook was to examine opportunities students had to learn about solving optimization problems in an economic context, the two non-economic optimization problems were not included in the analysis. Two economic examples were given in course lecture A. These examples were exactly the same as two examples (example 1 and example 2 in Appendix H) given in the textbook. Three economic examples (Appendix G) were given in course lecture B. These examples were minor adaptations of three examples given in the textbook (example 2, example 3, and example 8 in Appendix H). Since the examples given in course lectures were generally similar to the examples given in the textbook, the results of the analysis of optimization problems given in the textbook and optimization problems given in course lectures are similar. Hence, I only report the findings from the analysis of optimization examples given in the textbook.

Table 5. Results from analyzing optimization examples in the textbook

Number of Economic Examples	Type of Context	Type of Information	Type of Cognitive Demand	Representation of Example
6	No context: 0	Matching: 6 (100%)	Reproduction: 4 (67%)	Algebraic: 3 (50%)
	Camouflage context: 4 (67%)	Missing: 0	Connection: 1 (17%)	Tabular: 0
	Relevant and essential context: 2 (33%)	Superfluous: 0	Reflection: 1(17%)	Graphical: 0
				Verbal: 3 (50%)

Analysis of optimization examples in the textbook focused on four dimensions of analysis, namely types of context, types of information, types of cognitive demand, and representation of example. These dimensions of analysis are described in the task analysis framework (Wijaya, 2015) that appears in Appendix D. The results of the analysis of optimization examples are summarized in Table 5.

There are four major results from the analysis of optimization examples. First, a majority (67%) of the optimization examples have a camouflage context (examples 1, 2, 3 and 8 in Appendix H), only a few had a realistic (relevant and essential) context (examples 5 and 6 in Appendix H), and none of the examples had no context (an example of an optimization task with no context is given in the methods chapter). This result means that business calculus students have limited exposure, via the textbook, to realistic optimization problems that are situated in an economic context. Second, all six examples (examples 1, 2, 3, 5, 6, and 8 in Appendix H) have the exact amount of information students need to solve the problems posed in the examples. Consequently, students do not have to make sense of the context (if any) of the examples in order to either deduce missing information or identify important information (in the case of superfluous information) that is necessary to solve the problems posed in the problem statements of the examples.

Third, the opportunity to learn how to solve optimization problems with higher cognitive demands (reflection tasks) via examples given in the textbook is extraordinarily low: only one reflection task (example 5 in Appendix H) was given in the textbook. Most of the examples (examples 1, 2, 3, and 8 in Appendix H) were reproduction tasks. That is, they were tasks of lower cognitive demand.

Fourth, three of the six optimization problems were represented algebraically (examples 2, 3, and 8 in Appendix H) while the other three examples were represented verbally/textually (examples 1, 5, and 6 in Appendix H). This suggests that the textbook does not provide opportunities for students to reason about economic optimization problems from either a tabular or graphical representation. Taken together, the results from analyzing optimization examples suggests that students' opportunity to learn to solve a wide range of optimization problems in an economic context both from the course textbook and through examples given in course lectures is limited.

Students in the course were assigned five practice problems (problems 1 through 5 in Appendix H) that were selected from the textbook. All five practice problems were situated in an economic context. Analysis of assigned optimization practice problems focused on types of context, types of information, types of cognitive demand, and representation of practice problem. These dimensions of analysis are explained in the textbook analysis framework which appears in Appendix D. The results of the analysis are summarized in Table 6.

Table 6. Results from analyzing assigned optimization practice problems in the textbook

Number of Practice Problems	Type of Context	Type of Information	Type of Cognitive Demand	Representation of Practice Problem
5	No context: 0	Matching: 5 (100%)	Reproduction: 3 (60%)	Algebraic: 3 (60%)
	Camouflage context: 2 (40%)	Missing: 0	Connection: 1 (20%)	Tabular: 0
	Relevant and essential context: 3 (60%)	Superfluous: 0	Reflection: 1 (20%)	Graphical: 0
				Verbal: 2 (40%)

To some extent, the results from analyzing assigned optimization problems are similar to those obtained from the analysis of optimization examples. All the practice problems (problems 1 through 5 in Appendix H) contain the exact amount of information needed to solve the problems. The opportunity to learn to solve optimization problems with higher cognitive demands (reflection tasks) via practice problems given in the textbook is extraordinarily low: only one reflection task (problem 4 in Appendix H) was assigned as a practice problem in the textbook. A majority of the practice problems (problems 1 through 3 in Appendix H) were reproduction tasks. That is, they were tasks of lower cognitive demand.

It is, however, worth noting that a majority of the assigned practice problems (problem 2, problem 4, and problem 5 in Appendix H) have a realistic (relevant and essential) context so that the student has to reason about the context of these tasks in the process of solving the problems posed in these tasks. This result means that business calculus students had more exposure through homework problems to realistic optimization problems that are situated in an economic context than they did with optimization examples given in the textbook. Finally, a majority of the practice problems (problem 1, problem 3, and problem 5 in Appendix H) were algebraic. Hence, business calculus students had limited opportunities to practice solving optimization problems that are represented in multiple ways such as using graphs or numerical tables. These results suggest that the opportunity to learn to solve a wide range of realistic and cognitively demanding optimization problems in an economic context via assigned optimization practice problems in the textbook is limited.

Conceptual opportunities to learn about optimization problems. As earlier defined, conceptual opportunities to learn about optimization problems are opportunities provided in the textbook and in course lectures that are designed to help students develop a conceptual understanding of quantities (e.g., marginal cost) involved in the process of solving optimization problems in an economic context. In this section, conceptual understanding includes attention to: (1) the interpretation of quantities (e.g. critical numbers) in an economic context, (2) giving appropriate units of quantities in an economic context (e.g., units of critical numbers in a profit maximization context), (3) distinguishing between reasonable critical numbers or extrema from those that are not reasonable, (4) verifying mathematical results involving quantities in an economic context (e.g., verifying that a particular number of units is the profit-maximizing quantity), (5) the explanation of a procedure(s)/guide(s) given in the textbook or in course lectures that was intended to help students when solving applied optimization problems in an economic context, and (6) relative extrema versus absolute extrema optimization problems.

Interpreting quantities in an economic context. Analysis of optimization tasks (examples and assigned practice problem in Appendix H) given in the textbook as well as optimization examples (Appendices F and G) given in class revealed that the textbook and course lectures generally encouraged students to interpret critical numbers and extrema in context. Critical numbers and extrema were interpreted in the six economic-based optimization examples given in the textbook. For example, in their concluding remarks regarding the solution to Example 5 in Appendix H, Haeussler et al., 2011 stated that “the number of production runs is $10{,}000/632.5 \approx 15.8$” (p. 613). Haeussler and colleagues added that “for practical purposes, there would be 16 lots, each having the economic lot size of 625 units” (p. 613), thus Haeussler interpreted both the critical number [16 lots] and the extreme value [625 units] in the context of the problem i.e., economic lot size.

Also, students were required to interpret critical numbers and extrema in the five assigned optimization problems (all economic-based) that appeared at the end of the applied extrema minima and maxima section in the textbook. For example, the problem statement in problem 2 (Appendix H): “what rate will yield maximum revenue, and what will this revenue be?” (Haeussler et al., 2001, p. 618) implies that to answer the question, students had to interpret the critical number (monthly rate in dollars per month) that will yield maximum revenue and the extreme value (the maximum revenue in dollars). Interpretations, in context, of critical numbers and extrema were also given in all the optimization examples given in course lectures A and B. These interpretations were similar to those given in the textbook.

Encouraging students to include units. Analysis of optimization tasks given in the textbook and in class revealed that the textbook and course lectures rarely encouraged students to include units for critical numbers and extrema when solving optimization problems. Units for critical numbers were included in only three of the six economic optimization examples presented in the textbook while units of extrema (e.g., dollars for maximum profit) were included in only two of these examples. For example, when answering the question posed in example 6 (Appendix H), Haeussler et al. (2011) stated that “the monthly rate is $\$40 - \$7.50 = \$32.50$” (p. 614) and that “the number of subscribers at this rate is $100{,}000 + 30(1000) = 130{,}000$” (p. 614), thus Haeussler and colleagues stated both the units of the critical number (dollars) and the units of the extreme value of the revenue function (number of subscribers that yields maximum revenue). It is, however, confusing that the units of the monthly rate were only stated as dollars and not as dollars per month. When answering the question posed in optimization example 2 (Appendix H), Haeussler and colleagues did not include units of either the critical number (revenue maximizing quantity) or the extrema (maximum revenue). Instead, these authors concluded:

> Thus, 40 is the only critical value. Now we see whether this gives a minimum. Examining the first derivative for $0 \leq q < 40$, we conclude that $\frac{dr}{dq} > 0$, so r [revenue] is increasing. If $q > 40$, then $\frac{dr}{dq} < 0$, so r is decreasing. Because to the left of 40 we have r increasing, and to the right r is decreasing, we conclude that $q = 40$ gives the absolute maximum revenue, namely $r(40) = \frac{80(40)-(20)^2}{4} = 400$ (p. 611)

In the above section of the textbook, Haeussler et al. only verified that 40 is the value of q that maximizes the revenue function but they did not specifically say that q is the number of units that maximizes revenue and that 400 is the maximum revenue in dollars.

None of the five optimization practice problems (Appendix H) specifically requested students to give units of critical numbers or extrema. Only in one of the two optimization examples (Appendix F) given in course lecture A were units of extrema given:

> Of the two optimization examples that were given in course lecture A, units for extrema were only included in one example (dollars for minimum total cost in optimization example 1, Appendix F). In both optimization examples, units for critical numbers (length of fence needed to minimize the cost of fencing a rectangular storage area in example 1, Appendix F, and number of units that must be produced to minimize average cost in example 2, Appendix F) were not given. Students were also not encouraged,

verbally or through a written comment on the chalkboard, to include units for critical numbers and extrema when solving applied optimization problems. (Thembinkosi, Classroom Observation 3A, 12/01/2015).

Units for critical numbers and extrema were not given in any of the three examples (Appendix G) that were given in course lecture B. Hence, the importance and need for students to include units of critical numbers and extrema when solving contextualized optimization problems was not emphasized in course lectures.

Distinguishing between reasonable critical numbers and extrema and those that are not reasonable. An objective function can have more than one critical number or extremum, some of which may not be reasonable based on the context of the problem. For example, if a revenue function has two critical numbers (number of items that must be sold to maximize revenue), one positive and the other negative, it is reasonable to keep the positive critical number and discard the negative one as number of units sold cannot be a negative number. Analysis of optimization tasks in the textbook and in class revealed that students had limited opportunities to work with objective functions that have more than one critical number and that when they did, deciding between reasonable and unreasonable critical numbers did not go beyond discarding negative critical numbers and keeping positive critical numbers. Three of the six economic-based optimization examples given in the textbook (examples 1, 3, and 5 in Appendix H) have objective functions (e.g., the total cost function in example 1, Appendix H) that have two critical numbers, one positive and the other negative. In all three examples, the negative critical number was always discarded. In these three examples, the domain of the objective function was given as a justification for discarding the unreasonable (negative) critical number. As a result of discarding the unreasonable (negative) critical numbers, the objective functions yielded only one extremum (e.g., minimum total cost). None of the five optimization practice problems (problems 1 through 5 in Appendix H) had objective functions with more than one critical number. Hence, the practice problems did not give students enough practice dealing with objective functions where they have to decide which critical numbers are reasonable and which are not.

Students were encouraged, through examples given in class, to write and consider the domain of the objective function when determining which critical numbers are reasonable and which ones are not. Identifying the appropriate domain for the objective function when solving an applied optimization problem is key to recognizing unreasonable critical numbers or extrema. In course lecture A, for instance, the importance of identifying the correct domain for the objective function was emphasized through a verbal and a written explanation on the chalkboard about what the appropriate domain of each of the objective functions in examples 1 and 2 (Appendix F) should be. Specifying the domain of the objective function was mentioned as a rationale for discarding the unreasonable critical numbers in optimization example 1 and 2 (Appendix F). Students were verbally encouraged to always write the domain of the objective function when solving applied optimization problems.

The challenge of having to decide which critical numbers are reasonable and which ones are not did not come up in course lecture B. This is because all of the objective functions (e.g., profit function in optimization example 3, Appendix G) in the three optimization problems (Appendix G) that were given in course lecture B had only one reasonable critical number. However, the importance of identifying the correct domain for the objective function when solving applied optimization problems was emphasized in course lecture B. In all the three optimization examples (examples 1 through 3 in Appendix G), the appropriate domain was stated. Students were asked, verbally, to include the appropriate domain of the objective function when solving applied optimization problems. Hence, I argue that a criterion (specifying the domain of the objective function) for identifying reasonable critical numbers from those that are not was also discussed in course lecture B, even though this was done in an implicit manner.

Verifying mathematical results. In this study, to verify a mathematical result means reasoning about whether or not a critical number (e.g., profit-maximizing quantity) is indeed the input to an objective function (e.g., profit function) that yields an absolute maximum value (e.g., maximum profit) of the function. Analysis of the six economic optimization tasks in the textbook and optimization examples given in lectures revealed that the textbook and course lectures encouraged students (especially via examples) to verify their results, using various methods (first derivative test, second derivative test, comparing values of objective function at critical points including endpoints of the domain for the objective function), when solving applied extrema problems in an economic context. Verification of critical numbers was carried out in all six examples. For example, the textbook's presentation of the solution to example 6 in Appendix H:

> **Solution:** Let x be the number of \$0.25 decreases. The monthly rate is then $40 - 0.25x$, where $0 \leq x \leq 160$ (the rate cannot be negative), and the number of new subscribers is $1000x$. Thus, the total number of subscribers is 100,000+$1000x$. We want to maximize the revenue, which is given by
>
> $$\begin{aligned} r &= (number\ of\ suscribers)(rate\ per\ subscriber) \\ &= (100{,}000 + 1000x)(40 - 0.25x) \\ &= 1000(100 + x)(40 - 0.25x) \\ &= 1000(4000 + 15x - 0.25x^2) \end{aligned}$$
>
> Setting $r' = 0$ and solving for x, we have
>
> $$r' = 100(15 - 0.5x) = 0$$

$$x = 30$$

Since the domain of r is the closed interval [0,160], the absolute maximum value of r must occur at $x = 30$ or at one of the endpoints of the interval. We now compute r at these endpoints:

$$r(0) = 1000(4000 + 15(0) - 0.25(0)^2) = 4{,}000{,}000$$

$$r(30) = 1000(4000 + 15(30) - 0.25(30)^2) = 4{,}225{,}000$$

$$r(60) = 1000(4000 + 15(160) - 0.25(160)^2) = 0$$

Accordingly, the maximum revenue occurs when $x = 30$. (Haeussler et al., 2011, pp. 613-614).

This solution shows that the value of the revenue function at the critical number $x = 30$ was compared with the values of the revenue function at the endpoints of the domain [0, 60] of the revenue function in an effort to verify that $x = 30$ is the revenue maximizing quantity. However, none of the assigned optimization problems required students to verify that a critical number maximizes/minimizes a given objective function. For example, students were not asked to show that the critical number (130 units in problem 3, Appendix H) would maximize the profit for the monopolist mentioned in the problem. The first or second derivative test was used to verify that critical numbers minimized or maximized (depending on the problem) the objective function in the optimization examples that were given in course lectures: two examples (appendix F) in course lecture A and three examples (Appendix G) in course lecture B. For instance, the first derivative test was used in optimization example 1 (Appendix G) given in course lecture B to verify that the critical number $q = 25$ is the number of units that must be sold to maximize revenue.

Explanation of a procedures for solving applied optimization problems. The textbook gives a five-step guide for solving applied optimization problems (Haeussler, Paul, & Wood, 2011, p. 611):

Step 1: When appropriate, draw a diagram that reflects the information in the problem.

Step 2: Set up an expression for the quantity that you want to maximize or minimize.

Step 3: Write the expression in step 2 as a function of one variable, and note the domain of this function. The domain may be implied by the nature of the problem itself.

Step 4: Find the critical values of the function. After testing each critical value, determine which ones gives the absolute extreme value you are seeking. If the domain of the function includes endpoints, be sure to also examine function values at these endpoints.

Step 5: Based on the results of step 4, answer the question(s) posed in the problem.

Each step is illustrated using the first of the six economic-based examples given in the textbook (example 1 in Appendix H). For example, when illustrating step 5 using example 1 in Appendix H, Haeussler et al. (2011) stated that "the questions posed in the problem must be answered…120 ft of the $3 fencing and 180 ft of the $2 fencing are needed. The minimum cost can be obtained from the cost function…and is C(120)=720" (p. 610). In particular, Haeussler and colleagues answered the first question by stating the amount of each type of fence (120 ft and 180 ft respectively) so that the total cost of the fence will be a minimum. They also answered the second question regarding the minimum cost by stating that the minimum cost would be 720 (without units). Even though step 1 was illustrated by drawing a rectangular sketch of the storage area mentioned in optimization example 1 (Appendix H), none of the assigned optimization practice problems or examples that are given in the course textbook or examples (besides example 1, Appendix H) that were given in lectures (Appendix F and G) have or require the drawing of diagrams that may be appropriate for economics problems.

In the rest of the examples, the steps were not explicitly shown in the process of working through each example as is the case in the first example. The five-step guide for solving applied optimization problems does not provide details on the purpose of each step, hence it is not conceptual in my view. For example, while noting the domain of the objective function is emphasized in step 3 of the five-step guide, no details are given on the importance of noting the domain of the objective function e.g., determining unreasonable critical numbers such as a negative number of units sold when calculating revenue. This five-step guide promotes one specific way of thinking about optimization problems, that is, algebraically. It does not support students' reasoning about applied optimization problems that are represented in non-algebraic ways (e.g., using graphs and numerical tables).

A similar but nine-step guide was given in course lecture A:

Step 1: Read the question carefully

Step 2: When appropriate, draw a diagram that reflects the information in the problem

Step 3: Label variables

Step 4: Set up an expression for the quantity that you want to minimize or maximize

Step 5: Write the expression in step 4 as a function of one variable using the given relations e.g., the area of a rectangle in the cost-minimization problem

Step 6: Write the domain of the function

Step 7: Find the critical points

Step 8: Compare the values of f at critical points. If the domain involves the endpoints, then examine the values of the function at this point

Step 9: Give your answer

As in the textbook, this nine step guide was illustrated using example 1 in Appendix F which is the same example (example 1 in Appendix H) used to illustrate the five step guide in the textbook. As in the textbook, details on the purpose of each of the nine steps were not given. This includes, for example, an explanation why drawing a diagram (when appropriate) might be helpful in step 2. There is no guide (procedure) on how to solve applied optimization problems that was given in course lecture B.

Relative extrema and absolute extrema optimization problems. In this study, a relative extrema optimization problem is one in which the domain of the objective function is an open interval such as in example 3 (Appendix H) where the domain of the objective function (average cost function) is the open interval $(0, \infty)$. An absolute extrema optimization problem, on the other hand, is one in which the domain of the object function is a closed interval such as example 6 (Appendix H) where the domain of the objective function (revenue function) is the closed interval [0, 160]. Analysis of optimization tasks in the textbook and in course lectures (Appendices F and G) revealed that the opportunity to learn solving absolute extrema optimization problems in an economic context is low. For example, in the six economic optimization examples given in the textbook, only one example (example 6 in Appendix H) is an absolute extrema optimization problem. Only one practice problem (problem 2 in Appendix H) of the five practice problems in Appendix H is an absolute extrema problem. This problem is a variation of the only economic absolute extrema example (example 6 in Appendix H) given in the textbook. In essence, there is no variation in terms of context and cognitive demand in the two economic absolute extrema optimization problems that students were exposed to via textbook examples and practices problems. Also, none of the optimization examples given in course lectures A and B (Appendix F and G respectively) were absolute extrema optimization problems.

Textbook and lecture treatment of marginal change. As stated in the methods chapter, textbook and lecture treatment of marginal change refers to opportunities provided in the textbook and through course lectures that students had to learn about the concept of marginal change. Analysis of marginal change related tasks (Appendix I) and examples that were given in course lectures A and B (Appendices F and G) which I observed revealed that: (1) the definition of marginal change and the interpretation of marginal change given in both the textbook and during classroom instruction were not consistent with each other, (2) the importance of giving units when solving marginal change related tasks received little attention both in the textbook and in course lectures, (3) the textbook and lecture presentation of marginal change was more procedural and less conceptual, and (4) the discussion of the relationship between marginal change and profit (maximum or minimum) received little attention in the textbook and no attention in course lectures.

Definition and interpretation of marginal change. The definition of marginal change given in the textbook is not consistent with the interpretation of marginal change given in the same textbook. Marginal cost is defined as the derivative of the total cost function which means that marginal cost is a rate with units such as dollars per item. Haeussler at al. (2011) stated: “we interpret marginal cost as the approximate cost of one additional unit of output” (p. 513). This interpretation means that marginal change is an amount with units such as dollars. This inconsistency between the definition and interpretation of marginal change in the textbook can be seen in some of the marginal change examples given in the textbook. For example, when presenting their solution to marginal change example 1 (Appendix I), Haeussler et al. interpreted $c'(50) = \$3.75$, a rate, as an amount in dollars. The same definition and interpretation of marginal change was given in course lectures A and B. This finding suggests that the presentation of the concept of marginal change both in the textbook and in course lectures was inconsistent in that the same concept (marginal change) is treated as both an amount and a rate at the same time.

Units of marginal change. Including units of marginal change was not emphasized when solving marginal change related problems in the textbook and in course lectures. Analysis of the seven marginal change related tasks (Appendix I) in the textbook revealed that units of marginal change (dollars per unit) were not included in the two examples given in the textbook (examples 1 and 2 in Appendix I) while only one practice problem (problem 5 in Appendix I) out of five assigned practice problems in the textbook required students to give units of marginal change which would have to be given as a rate (dollars per year). Units of marginal change (dollars per unit) were given in the only two examples (examples 4 and 5 in Appendix G) given in course lecture B where it was necessary to state units of marginal change. The other three marginal change examples (examples 1, 2 and 3 in Appendix G) given in course lecture B only required students to find algebraic formulas for the derivative: a formula for the derivative of the total cost function, a formula for the derivative of the revenue function, and a formula for the derivative of the average cost function. Units of marginal change (dollars per unit) were not given in the two marginal change examples (example 1 and example 2 in Appendix F) given in course lecture A. In general, the textbook and course lectures barely encouraged students to include units of marginal change when solving marginal change related tasks.

Procedural and conceptual understanding of marginal change. The presentation of marginal change in both the textbook and in course lectures appears to have valued a procedural understanding of marginal change over a conceptual understanding of marginal change. A majority of the marginal change related tasks in the textbook and examples given in course lectures were algebraic, something that has the potential to promote a procedural understanding of marginal change. More specifically, of the seven optimization tasks (examples 1 and 2, and practice problems 1 through 5 in Appendix I), only one task (practice problem 5) required interpretation of marginal change (the dollar amount by which a certain machine depreciates on a yearly basis). The rest of the tasks, such as marginal change practice problem 1 (Appendix I), only required students to find the formula for the derivative of some function (e.g., total cost function in problem 1 below) and then evaluate it at a given quantity (number of units).

Problem 1. $c = 500 + 10q$ is the total cost of producing q units of a product. Find the marginal-cost function. What is the marginal cost when $q = 100$? (Haeussler et al., 2011, p. 516).

Of the five marginal change examples given in course lecture B, only two examples (examples 4 and 5 in Appendix G) were interpretation questions while the other three examples (examples 1 through 3 in Appendix G) were computational questions. Also, of the two marginal change examples given in course lecture A, only one example (example 2 in Appendix F) was an interpretation question while the other example (a question with four parts) was a computational task. The marginal change tasks given in the textbook and during classroom instruction did not offer students opportunities to reason about the concept of marginal change in multiple representations (e.g., from numerical tables and graphs) and only limited opportunity to reason about it algebraically.

Relationship between marginal change and profit. The relationship between marginal change (marginal cost and marginal revenue) and profit (maximum or minimum) received little attention in the textbook and no attention at all in course lectures. There are two instances in the textbook where the relationship between marginal change and maximum profit is discussed. The first instance occurred in the concluding section on solving applied optimization problems (shown in Appendix J). In this section of the textbook, Haeussler et al. (2011) stated that maximum profit occurs at a production and sales level where marginal cost equals marginal revenue provided total revenue is greater than total cost at that level. There is, however, no discussion of what it means for marginal cost to equal marginal revenue at production and sales level when total cost is less than total revenue, that is, where profit is minimal.

The second instance where the relationship between marginal change and maximum profit comes up is through assigned practice problem 1 in Appendix H:

Problem 1. **(Profit)** For XYZ Manufacturing Co., total fixed costs are $1200, material and labor costs combined are $2 per unit, and the demand equation is

$$p = \frac{100}{\sqrt{q}}$$

> What level of output will maximize profit? Show that this occurs when marginal revenue is equal to marginal cost. What is the price at profit maximization?

In particular, the above task requires that students evaluate the marginal cost and the marginal revenue function, separately, at the profit-maximizing quantity. These separate evaluations should yield the same result to verify that marginal cost equals marginal revenue at the profit maximizing quantity.

Textbook and classroom treatment of quantitative reasoning. To reiterate, this study used the definition of quantitative reasoning proposed by Thompson (1993): analyzing a situation in terms of the quantities and relationships among the quantities involved in the situation. Analysis of the textbook and course lectures revealed that: (1) reasoning about economic quantities (e.g., marginal cost) received considerable attention in the textbook and during lectures while (2) reasoning about relationships between or among quantities (e.g., relationship among marginal cost, marginal revenue, and marginal profit) received little attention in the textbook and in course lectures.

Reasoning about quantities. Since the examples presented in course lectures are generally the same as the examples and practice problems presented in the textbook, I only refer to the textbook for evidence on students' opportunity to reason about economic quantities. Nearly all the optimization and marginal change related situations (tasks) presented in the textbook and in course lectures involved reasoning about different economic quantities. In particular, five of the six economic optimization problems presented in the textbook involved analyzing economic situations with a focus on one quantity (e.g., analyzing average cost in example 2, Appendix F) and four of the five optimization practice problems (Appendix H) involved analyzing economic situations with a focus on one quantity (e.g., analyzing total cost in problem 5, Appendix H). Also, all the marginal change examples involved analyzing economic situations with a focus on one quantity (e.g., analyzing marginal revenue in example 2, Appendix I). All the marginal change examples involved analyzing economic quantities with a focus on one quantity (e.g., analyzing marginal cost in problem 3, Appendix I). Hence, students had plenty of opportunities to reason about economic quantities in the textbook and in course lectures.

Reasoning about relationships between or among quantities. Relationships between or among economic quantities (e.g., relationship between total cost, total revenue, and profit) were rarely presented in the textbook and in course lectures. Of the six economic optimization problems presented in the textbook only one example (example 8 in appendix H) involved analyzing an economic situation (profit) with a focus on the relationship among three quantities, namely the demand for a product, average cost, and profit). Two of the five optimization practice problems (problem 1 and problem 3 in Appendix H) given in the textbook involved analyzing economic situations with a focus on relationships among economic quantities. Problem 1 in Appendix H, for instance, emphasizes the relationship among marginal cost, marginal revenue, and maximum profit.

The other, and final instance, where a relationship among quantities is presented in the textbook was in an expository section of the textbook which is shown in Appendix J. Commenting about the relationship between marginal cost, marginal revenue, and marginal profit and referring to the last figure in the expository section that appears in Appendix J, Haeussler et al. (2011) stated that:

> For production up to $q1$, the revenue from additional output would be greater than the cost of such output, and the total profit would increase. For output beyond $q1$, $MC > MR$, and each unit of output would add more to total costs than to total revenue. Hence, total profits would decline. (p. 616)

This example highlights the fact that as a firm approaches maximum profit, the number of units produced and sold increases and that this happens at the same time that total cost, total revenue, and total profit increases respectively. This process is reversed after the profit maximizing quantity. Overall, the opportunities to reason about relationships among quantities in an economic context via the textbook were rare. The relationship among total cost, total revenue, and profit was briefly discussed in optimization example 3 that was given in course lecture B when formulating the profit function. Besides this instance, relationships between or among economic quantities were not discussed in course lecture B. There was no discussion of relationships between or among the economic quantities that were discussed in any of the optimization and marginal change examples (Appendix F) that were given in course lecture A. Hence, the opportunity to reason about relationships between or among economic quantities via the textbook and course lectures was limited.

Summary of opportunity to learn results. In summary, analysis of the textbook and course lectures revealed that the presentation of problems (e.g., optimization and marginal change related problems) is largely algebraic which has the potential to promote a procedural understanding of optimization problems and the concept of marginal change. A majority of these problems are reproduction tasks and have the exact amount of information that students need to solve them. This suggests that students might experience difficulties when reasoning about non-algebraic and cognitively demanding tasks such as those that were given to students in this study (with the exception of Task 1 in Appendix A). Verification of critical numbers (e.g., as a profit maximizing quantity), using formal techniques such as using the first derivative test, is generally encouraged in the textbook. The presentation of marginal change as both a rate (the difference quotient) and an amount (the difference) in the textbook and in course lectures might be confusing to students. There were almost no opportunities (in the textbook and in course lectures) for students to reason about important relationships between or among economic quantities such as the relationship between marginal cost and marginal revenue at a profit maximizing quantity. Finally, given that the presentation of optimization problems and marginal change in course lectures closely followed the presentation of these topics in the textbook, this shows that the textbook had a major influence on the instructors on their teaching of the above mentioned topics.

Algebraic Reasoning

To examine students' algebraic reasoning when solving optimization problems in the economic context of cost, revenue, and profit, I analyzed students' verbal responses and written work to the problem posed in Task 1 which appears in Appendix A. In general, this analysis revealed that reasoning about this optimization problem was problematic for a majority of the students. This is despite the fact that all the students acknowledged having seen and even solved a problem similar to the one posed in Task 1. Of the 12 pairs of students who attempted the problem posed in Task 1, only three pairs of students correctly solved this problem and one of those pairs made a computational error. In what follows, I discuss how the students reasoned about: (1) the context of the problem, (2) the profit function, (3) critical numbers and extrema, (4) verifying extrema, and (5) relative extrema versus absolute extrema.

Students' reasoning about the context of Task 1. Eight pairs of students, at one point or another, reasoned about the context of Task 1 in their attempt to solve the problem posed in the task. In what follows, I discuss how two pairs who are representative of the eight pairs of students reasoned about the context of the task. Nikki and Casey are one of the pairs of students who reasoned about the context of Task 1 in their attempt to solve the problem posed in the task, that is, to find maximum profit. The following excerpt, which occurred at the beginning of working on Task 1, illustrates how Nikki reasoned about the context of the task while reasoning about the manufacturer's profit. Prior to this excerpt, Nikki had determined the profit to be $1.33 by evaluating the second derivative of the total cost function given in the task, which she considered to be the profit function, at 120 units, the maximum number of units the manufacturer can produce and sell per year.

Researcher: Nikki, you got one point three three [1.33] as your maximum profit and you said it was not correct. What did you expect?

Nikki: I feel like that the profit should be a bigger number

Researcher: Like?

Nikki: I feel like it should be over ten thousand [pointing at the 10,000 fixed cost in the total cost function, $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$]

Researcher: Why?

Nikki: Because that's the total cost [pointing at the fixed costs of 10,000 in the total cost function]. If you want to earn profit which means the number [1.33] should be higher than total cost, than the cost [pointing at the 10,000 fixed cost]

By reasoning about the fixed cost of $10,000 of the manufacturer, Nikki recognized that something was likely not right about her maximum profit of $1.33. In particular, Nikki recognized that it is unlikely for a company that produces in the thousands of jackets to make a maximum profit of only $1.33. Before giving up on this task, Nikki and Casey indicated that they needed the revenue function in order to be able to find the maximum profit. In the following excerpt, which occurred at the end of the task, Nikki and Casey talk about how they might have determined the maximum profit if they were given an algebraic form of the revenue function. Prior to this excerpt, Nikki and Casey had indicated that they needed an algebraic equation for the total revenue in order to find the maximum profit.

Researcher: If the revenue function was given just like the cost function is given, what would you do with it?

Nikki: I feel like you would take the derivative for some reason because that's how you gonna find the profit. I don't know but I know you don't just like do addition or subtraction [writing P=R-C], you have to like do the derivative

Casey: I don't remember

Nikki: I'm not sure. It's [this task] like beyond calculus, like business calculus level, right?

Casey: No, I think we did it but I don't remember how to do it

Nikki: I don't know but, why is it so hard?

Casey: I think we did it but I don't remember. The step by step guide [guideline for solving applied extrema problems given in her class] is like you find the derivative and then one value [possibly critical number]

Nikki: Do you want to try another problem [task 2]? I have no clue for this one [Task 1] and I feel like we should try something else

Casey: I feel like if I saw what I did in my notes I would be like alright

In the above conversation, Casey and Nikki both reasoned correctly when they, at different times, indicated that in order to find the maximum profit algebraically, they would have to take the derivative, presumably of the profit function. They, however, appeared to be uncertain about the significance of taking the derivative other than as a procedure they learned in class or in the textbook. Nikki and Casey gave up on the task and decided that they wanted to try another task.

Yuri and Kyle are the second pair of students who also reasoned about the context of Task 1 while solving the problem posed in the task. These students determined the domain of the profit function by considering the context of the task. After they had found the profit function, algebraically, Yuri and Kyle correctly determined the two critical numbers (40 and 80) of the profit function using algebraic methods. They then used the first derivative test to determine which of the two critical numbers is the profit maximizing quantity. In the following excerpt, which occurred towards the end of working on Task 1, Yuri and Kyle used the context of the task (minimum and maximum number of units that can be produced and sold) to correctly determine the minimum and maximum possible test values that can be used in the first derivative test based on the context of the task. Prior to this excerpt, Yuri and Kyle had verified that 40 units is the profit maximizing quantity by using the test values 10 and 81 in the first derivative test. The excerpt begins with the researcher asking them about how they chose these test values.

Researcher: You chose 10 for the first one [test value]. Could you have chosen like minus fifteen [-15] instead?

Yuri: No

Researcher: Why?

Yuri: Quantity [number of units produced] should be positive

Researcher: What do you think Kyle?

Kyle: No

Researcher: Why?

Kyle: Like he said quantity can't be negative in a real life situation

Researcher: You also chose 81 for the other test value. Could you have chosen 150?

Yuri: No

Kyle: No

Researcher: Why?

Kyle: Because the most you can produce is one twenty [120 units]

By imagining Task 1 as a real life situation, Yuri and Kyle used the context of production in the task to restrict the domain of the profit function to the closed interval, [0, 120]. In other words, Yuri and Kyle used the context of the task to determine the minimum and maximum number of units that can be produced by the manufacturer: 0 units and 120 units, respectively. The rest of the pairs of students generally ignored the context of the task when solving the problem posed in the task. In sum, eight pairs of students reasoned about the context of Task 1 while trying to solve the problem posed in the task. One of the representative pairs of students, Nikki and Casey, used the context of the task to reason about reasonable amounts of profit that the company mentioned in the task can make. The second representative pair of students, Yuri and Kyle, used the context of the task to reason about the domain of the profit function, that is, the minimum and maximum number of units that can be produced by the company mentioned in the task.

Students' reasoning about the profit function. Only six pairs of the students who participated in this study were successful in setting up the profit function for the optimization problem posed in Task 1. Sarah and Alan, who are representative of these students correctly determined, algebraically, the profit function needed to solve the problem posed in the task. At first, Sarah and Alan incorrectly treated the demand equation ($p = q^2 - 100q + 3200$) as the profit function. They proceeded to find the critical number (50) of the demand equation, algebraically. They then used the first derivative test to check if the critical number was a profit maximizing quantity. Instead, the first derivative test showed that the critical number is a profit minimizing quantity. In the following excerpt, which occurred early in the discussion of the task, Sarah realized that the demand equation is not the profit function. This occured shortly after they had tested the critical number of the demand equation to see if it is a profit maximizing quantity.

Sarah: But we did like something wrong somewhere [after the first derivative test showed that the critical number of the demand equation ($p = q^2 - 100q + 3200$) is a profit minimizing quantity]. I think we need to back up, profit, so this [derivative of demand equation] is wrong because p, in this situation is price and not profit, profit is revenue minus cost [writing P=R-C in Figure 1]

Alan: Yah

Sarah: And revenue is p times q, so we have to times [multiply] this [demand equation] by q [to get the revenue function] and then subtract the cost [total cost function from the revenue function] to get the profit function

Researcher: Alan, what do you think about what she just said?

Alan: Yah, I think that's right, but do we need to use this [derivative of demand equation], so this [derivative of demand equation] is irrelevant, right?

Sarah: Yah, all this [derivative and critical numbers of demand equation] is irrelevant. So this becomes [writing out the profit function, $\mathrm{P} = \frac{1}{3}\mathrm{q}^3 - 60\mathrm{q}^2 + 3200\mathrm{q} - 10{,}000$ in Figure 1]. Ok, so this is the profit function [pointing at the profit function]. I think we now do the derivative of this [profit function] and then do this [sign chart for the first derivative test]

Alan: Oh, ok, and then we put the critical points [critical numbers] here [first derivative test]

Sarah: Yeah

Alan: Oh, ok

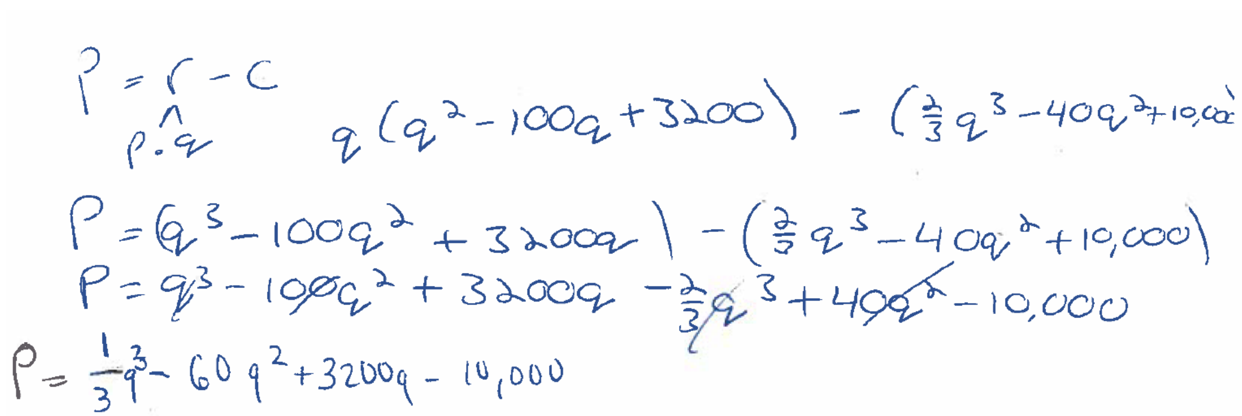


Figure 1. Sarah and Alan's work leading to the determination of the profit function.

Not only did Sarah correctly set up the profit function but she also convinced Alan that the profit function she found is correct and that they should not use the derivative of the demand equation to determine maximum profit.

Three other pairs of students correctly stated what needed to be done in order to set up the profit function even though, eventually, they were not successful in setting up the profit function. In particular, these students verbalized the relationship between cost, revenue, and profit but they struggled to express this relationship algebraically. Isaac and Kierra are representative of these students. Isaac said that "profit equals revenue minus cost." At another time, Kierra stated that "they are not telling us how much each unit is being sold for, so without that piece of information we can't figure out the revenue for every time we are selling something." Essentially, Kierra and Isaac knew how to set up the profit function (subtracting the total cost from the total revenue) but they struggled to use the information given in the problem statement of Task 1 to determine an algebraic representation of the revenue function, something that could have been done by multiplying the demand equation ($\mathrm{p} = \mathrm{q}^2 - 100\mathrm{q} + 3200$) by the number of units sold, q.

Another pair of students who incorrectly determined the profit function, Ivy and Denise, created a new quantity, which they understood to be the profit function, by dividing the total cost function by the number of units produced, q. That is, Ivy and Denise incorrectly referred to the average cost function as the profit function. In the following excerpt, which occurred at the beginning of working on the task, Denise and Ivy reasoned about what they needed to do in order to solve the problem posed in the task; finding maximum profit. The excerpt begins with Denise giving her thoughts about what they needed to do. Prior to this excerpt, Denise had suggested equating the total cost function to the demand equation both of which were given in the task.

Denise: Part of me thinks we need to divide the cost [$\mathrm{c} = \frac{2}{3}\mathrm{q}^3 - 40\mathrm{q}^2 + 10{,}000$] by q because it's already cubed and then we have that [cost function] over q and then

Researcher: So if you divide the cost by q, what do you get?

Denise: I forget, there is a whole thing, it starts with one thing, if you multiply by q then it goes to something, if you divide by q it goes to something else, like I forgot. I don't think we set them [cost function and demand equation] to each other

Ivy: Isn't it profit if you divide cost by q, won't that be profit?

Denise: Oh, maybe [dividing the cost function by q and writing $\frac{2}{3}q^2 - 40q + \frac{10,000}{q} = \text{profit (?)}$]

Ivy: I feel like it would be profit [writing $\frac{2}{3}q^2 - 40q + \frac{10,000}{q} \rightarrow \text{profit}$], because if you divide cost by the number of things made, won't that be profit?

Even though Ivy and Denise appeared not to be sure if the profit function they have determined is correct, throughout their reasoning about Task 1, they kept referring to it as the profit function. In the above excerpt, Denise tried to recall economic quantities that are formed by multiplying one quantity by another. In particular, when she said "if you multiply by q then it goes to something," she might have been thinking of the revenue function which they could have found by multiplying the demand equation by the number of units produced and sold. When she said "if you divide by q it goes to something else," she might have been thinking of average cost which is obtained by diving the total cost function by the number of units produced.

Another pair of students, Jacie and Derby, incorrectly referred to the revenue function as the profit function. In the following excerpt, which occurred at the beginning of working on Task 1, Derby reasoned about how to determine the profit function. Prior to this excerpt, these students, had, in response to the researcher's question acknowledged having seen a problem similar to the one given in Task 1 prior to participating in this interview.

Derby: So, yesterday I did a problem like this [Task 1] in WeBWork [online homework environment] like for c [total cost equation] don't you find the derivative of that one [total cost function] and for p [demand equation] don't you divide by q? Wait, this one [demand equation] you divide by q and this one [total cost function]

Jacie: Yes, this is the demand equation [pointing at the demand equation: $p = q^2 - 100q + 3200$], then profit, is that p [demand equation] times q?

Derby: Yah, so that's why you multiply this one [demand equation] by q

Jacie: Yah [writing pro=pq in Figure 2] so profit equals p times q.

Derby: [writing an expression for the profit: $q(q^2 - 100q + 3200)$ in Figure 2]

Researcher: And what's that [pointing at the expression written by Derby]

Jacie: Profit function

Derby: That's the profit function [equating the expression $q(q^2 - 100q + 3200)$ to the letter p in Figure 2]. And this one [pointing at the total cost function], you divide it by q, do you remember exactly why? I do remember doing this problem, you just divide by q

Researcher: What do you get when you divide it by q?

Derby: [smiling] It's a good question

Jacie: Because then we will only get c over q, that's not really anything. So this is profit [pointing at the revenue function: $p = q(q^2 - 100q + 3200)$ in Figure 2]

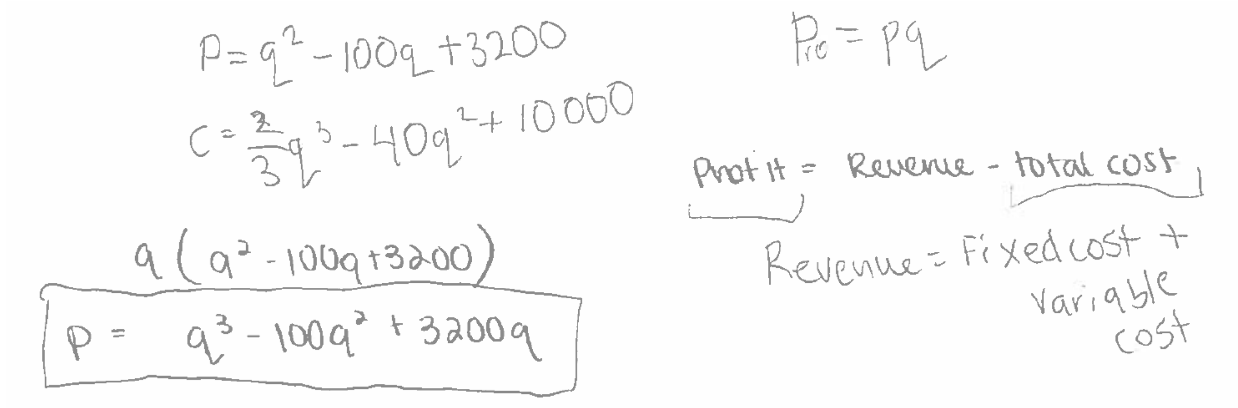


Figure 2. Jacie and Derby's work leading to the determination of the profit function.

In Jacie and Deby's view, the revenue function they created is the profit function. Derby referred to the expression for profit as an equation, which suggest that an expression is the same as an equation in her view. Her statement that "that's not really anything" when one divides the total cost function by the number of units produced to get a new quantity suggests that this quantity, which is actually the average cost function, is a meaningless quantity to Derby.

Another pair of students, Ruth and Eric, equated the total cost function to the demand equation (Figure 3), made one side of the resulting equation to zero by moving terms to one side, and presumably treated the nonzero side ($\frac{2}{3}q^2 - 39q + 100q + 6{,}800$ in Figure 3) as the profit function. They then evaluated the nonzero side at 120 units and got -30,800 as a result which they claimed is the maximum profit they were asked to determine in the task. In the following excerpt, which occurred in the middle of working on Task 1, Eric and Ruth reasoned, following a question from the researcher, about how they would answer the question posed in the task based on their work in Figure 3. Prior to this excerpt, Ruth and Eric had evaluated the expression $\frac{2}{3}q^2 - 39q + 100q + 6{,}800$ in Figure 3 at 120 (the maximum number of units the manufacturer can produce and sell per year) to get a result of -30,800.

Researcher: How would you answer the question? The question is, find the maximum profit

Ruth: It [profit] can't be a negative. I mean it can be a negative that means they are losing profit

Eric: If this was an exam, I would just say that the maximum profit is 30,800 just leave it and go to the next one

Ruth: Yeah, throw away the negative

Researcher: Tell me one more time, what did you do to get to that result?

Ruth: We just made each equation equal to each other, made all the common terms like the q squared and q's, put them together, plug in one twenty [maximum number of units that can be produced by the manufacturer] for q and we got thirty thousand eight hundred

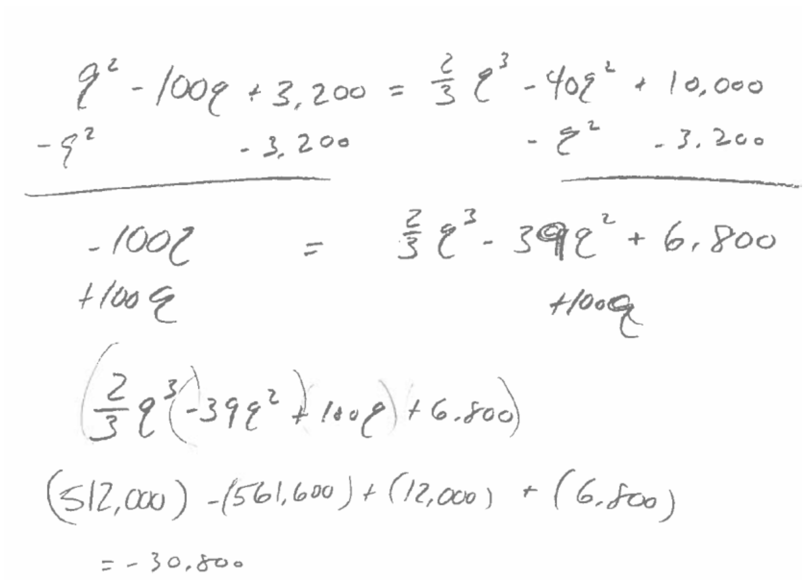


Figure 3. Ruth and Eric's work leading to the determination of maximum profit.

Even though Eric and Ruth never explicitly referred to the expression

$\frac{2}{3}q^2 - 39q + 100q + 6{,}800$ as the profit function, yet by stating that the result they got by evaluating this expression at 120 units is the maximum profit implies that, in their view, this expression is the profit function.

In summary, a majority of the students knew how to set up the profit function even though some of them had difficulties in determining an algebraic form for the profit function. Nearly all the students created algebraic quantities that helped them to solve the problem posed in the task. At other times, however, students created quantities that were meaningless to them. For example, Derby indicated that the quantity obtained by dividing the total cost function by the number of units produced is “not really anything” when, in fact, this quantity is generally referred to as the average cost function in economics.

Students’ reasoning about critical numbers and extrema. Analysis of students’ reasoning about the optimization problem posed in Task 1 revealed that only a few students (five pairs) were successful in determining the critical numbers for the profit function. In what follows, I present results from the analysis of the multiple ways used by these students to determine the critical numbers, followed by a presentation of results from the analysis of how three of these pairs of students correctly interpreted critical numbers in context, followed by a presentation of results from the analysis of how two other pairs of students treated critical numbers as absolute extrema.

Students’ use of multiple ways to determine critical numbers. There were three methods that were used by five pairs of students to determine the critical numbers (maximizing or minimizing quantities) of the profit function for the problem posed in Task 1. Four pairs of students used algebraic methods to determine the critical numbers of the profit function. In particular, three pairs of students correctly found the critical numbers by factoring the derivative of the profit function, equating each factor to zero and solving for the critical numbers. Another pair of students, Joy and Nancy, correctly used the quadratic formula to determine the critical numbers. These students, however, made a computational error while evaluating the discriminant of the quadratic formula which resulted in them getting incorrect critical numbers. Nevaeh and Zoe were the only pair of students who used a non-algebraic method to determine the critical numbers of the profit function. These students correctly determined the critical numbers by using a graphing method: graphing the derivative of the profit function and determining its zeros. Prior to graphing the derivative, Nevaeh had indicated that "finding the max and min, you have to do the first derivative test and then. I feel like we need boundaries of some sort." Nevaeh and Zoe correctly determined the endpoints (0 units and 120 units) of the domain for the profit function. They referred to these endpoints as bounds of the profit function. In the following excerpt, which occurred in the middle of working on Task 1, Nevaeh reasoned about how she found the critical numbers in addition to stating, following a question from the researcher, her understanding of what these numbers represent. Prior to this excerpt, Neveah had determined the critical numbers of the profit function to be 40 and 80 respectively.

Researcher: You just came up with x=40 and x=80 [critical numbers], how did you get these numbers?

Nevaeh: The graph on the calculator looks like this [showing the researcher a concave up parabola which was the graph of the derivative of the profit function] and I did second function, trace, zero and tried to find the zeros.

Researcher: So which one [function] did you graph on the calculator?

Neveah: This [pointing at the derivative of the profit function which is denoted as π' in Figure 4]

Researcher: So what does that [the zeros] give you?

Nevaeh: It gives me the place where the graph crosses the x-axis

Researcher: What do those numbers [x=40 and x=80] represent?

Neveah: I guess those will be our new bounds, I think [deleting the older bounds, 0 and 120]

Researcher: Why are you deleting the older bounds?

Nevaeh: They don't seem to make sense to me anymore. It doesn't make sense, like this [pointing at 120 units in Task 1] is the maximum they can sell, you know, just putting [pointing at 120 units in Task 1], it doesn't make sense

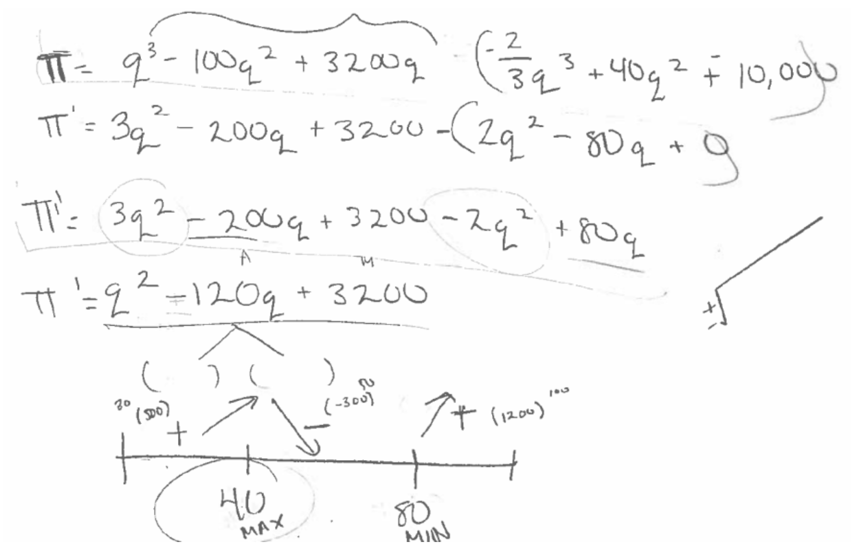


Figure 4. Nevaeh's work leading to the determination of 40 units as the profit maximizing quantity.

In the above excerpt, Nevaeh used a sequence of graphing calculator steps to determine the zeros of the derivative of the profit function. Saying that the bounds 0 and 120 do not make sense anymore meant that Nevaeh did not consider these bounds to be important when finding maximum profit for the problem posed in Task 1. Zoe did not say anything about the new bounds.

Interpreting critical numbers in context. Three of the five pairs of students who were successful in determining critical numbers for the profit function for the problem posed in Task 1 also correctly interpreted the critical numbers. Joy and Nancy are representative of these students. In the following excerpt, which occurred toward the end of the task, Joy and Nancy interpreted the critical numbers for the profit function, following a question from the researcher. Prior to this excerpt, Joy and Nancy made a computational error (which they never realized) using the quadratic formula, which resulted in them getting the incorrect critical numbers, 11.9375 and 268.06, instead of 40 and 80 respectively.

Researcher: What do these numbers [pointing at the critical numbers 11.9375 and 268.06] mean?

Joy: Possible values that would just give you maximum profit

Researcher: Possible values of what?

Joy: Of q

Nancy: The number of units

Joy's statement that the critical numbers are "possible values that would just give maximum profit" suggests an awareness that something else might be going on or perhaps that something else need to be checked. Joy and Nancy checked, by evaluating the profit function at the critical numbers (11.9375 and 268.06), which of the critical numbers they found is the profit maximizing quantity. They concluded that 11.9375 is the profit maximizing quantity. Joy and Nancy, however, did not check whether or not profit is maximized at the endpoints of the domain of the profit function.

Treating critical numbers as extrema. Two pairs of students, at one point or another, incorrectly interpreted critical numbers as extrema. These students incorrectly referred to the profit maximizing or minimizing quantities as the maximum and minimum profit. Abby and Shawna are representative of these students. In the following excerpt, which occurred towards the end of working on Task 1, Abby and Shawna interpreted the critical number 40. Prior to this excerpt, Abby had expressed being confused about where they would use the 120 units given in the task which is the maximum number of units the manufacture can make and sell per year. This was after they had correctly determined, using the first derivative test, that the critical number 40 they had previously found is a profit maximizing quantity. The excerpt begins with the researcher telling Abby to assume the 120 units was not given in an effort to focus her attention on giving an interpretation for the critical number 40.

Researcher: Suppose the 120 wasn't there

Abby: Then I would say 40 is the answer [$40 is the maximum profit]

Shawna: I wouldn't

Researcher: Shawna, why?

Shawna: Because I feel like we need to plug in 40 into one of these [pointing at the total cost function and the demand equation] but I don't know which one. Maybe we would have to take the derivative again [second derivative of profit function] and plug in 40

Abby: It's asking for the maximum profit

Shawna: I know but what's all this [pointing at the notations for average cost, derivative of total cost, and second derivative of total cost: $\bar{c}$, c', and c'']. This is what we learned yesterday.

Abby: I don't remember what part this is.

Shawna: Me either but I'm guessing

Researcher: What's your final thought on this?

Shawna: Ok, 40 [dollars] is our answer

Since the students were asked to find the maximum profit for the problem posed in Task 1, saying that "40 is their answer," means that Abby treated the profit maximizing quantity of 40 units as the maximum profit. Seven pairs of students never found or even talk about critical numbers in their reasoning about Task 1. One of these seven pairs of students, Ruth and Eric (who's reasoning about Task 1 was presented in the previous section on students' reasoning about the profit function) only evaluated the expression $\frac{2}{3}q^2 - 39q + 100q + 6{,}800$ in Figure 3 to determine the maximum profit without saying anything about critical numbers.

Students' reasoning about verifying extrema. Analysis of students' reasoning about the optimization problem posed in Task 1 revealed that verifying extrema (maximum profit) was particularly difficult for a majority of the students. In particular, only four pairs of students attempted to verify that the maximum profit they found was indeed the maximum profit. Three of these pairs of students correctly used the first derivative test to verify that the maximum profit they found was indeed maximum and the other pair of students incorrectly stated that maximum profit would occur when the manufacturer produces the maximum number of units (120) he can produce. Alan and Sarah are representative of the three pairs of students who used the first derivative test. In the following excerpt, which occurred towards the end of Task 1, Alan and Sarah, following a question from the researcher, reasoned about how they would convince someone that the maximum profit they found is indeed the maximum profit. Prior to this excerpt, Alan and Sarah had, in addition to correctly determining the critical numbers (40 and 80) algebraically, also determined using the first derivative test (Figure 5) that the critical number 40 is the profit maximizing quantity.

Researcher: How would you convince someone that that profit [pointing at the maximum profit that Sarah and Alan found] is the maximum profit?

Sarah: Because, because, I don't know

Alan: It shows here [pointing at the critical number 40 in Figure 5], it's maximum at 40 [units], because it's [profit] increasing and then decreasing, whenever it [profit] goes from increasing to decreasing you have a max [maximum profit] and that's at 40 so that proves that the maximum profit is at 40 [units]

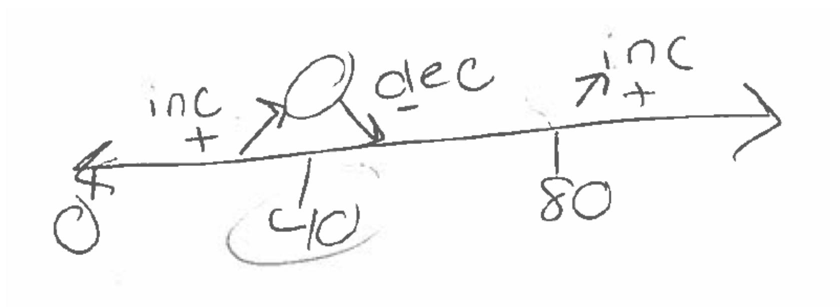


Figure 5. Sarah and Alan's work leading to the determination of the critical number 40 as the profit maximizing quantity.

In verifying that the profit they found is maximum, Alan explained the first derivative test they used to find the profit-maximizing quantity (40 units). He evaluated the first derivative of the profit function at a test value less than 40. This evaluation yielded a positive result which he understood to mean that profit increases when production and sales are increased from zero units to 40 units. This is shown by the positive sign and up-pointing arrow on the left-hand side of 40 in Figure 5. Alan repeated this process with a test value greater than 40 but less than 80 and found a negative result which he understood to mean that profit decreases when production and sales are increased from 40 units up to 80 units. This is shown by the negative sign and down-pointing arrow on the right-hand side of the number 40 in Figure 5. Finally, Alan evaluated the derivative of the profit function at a test value greater than 80, found a negative result which he understood to mean that profit increases when production and sales are increased above 80 units. This is shown by the positive sign and up-pointed arrow on the right-hand side of the number 80 in Figure 5.

Another pair of students, Kierra and Isaac, incorrectly stated that maximum profit would occur when the manufacturer produces the maximum number of units (120) he can produce. Kierra stated that “if at most, if the uppermost limit is a hundred and twenty [units], you can’t sell a unit more and by not selling a unit more you can’t generate a bigger profit.” Kierra’s way of verifying that the profit they got was indeed maximum suggests that she assumed that increase in production results in increase in total revenue and total cost in such a way that profit is maximized when the manufacturer produces and sells the maximum number of units he can possibly produce, 120 units. Other students simply said they “don’t know” how they could verify that the profit they claimed was maximum was indeed the maximum profit the manufacturer can generate.

Relative extrema versus absolute extrema. Analysis of students’ reasoning about the optimization problem posed in Task 1 revealed that none of the five pairs of students who found critical numbers considered this problem to be an absolute extrema problem: they all considered it as a relative extrema problem. The students did not evaluate the profit function at the endpoints of the domain [0, 120] of the profit function in their determination of maximum profit. Instead, they only evaluated the profit function at the profit-maximizing quantity which they identified as 40 units (or 11.9375 units in the case of Joy and Nancy). Had it been the case that maximum profit occurred at one of the endpoints of the domain of the profit function, say at 120 units, only one of the pairs of students, Kierra and Isaac, might have correctly determined the maximum profit. In general, a majority of the students who participated in this study tended to disregard information about the endpoints of the domain of the profit function when determining maximum profit. In the following excerpt, which occurred at the end of working on Task 1, Joy and Nancy reasoned about the 120 units given in the task as the maximum number of units the manufacturer can produce and sell per year. The excerpt begins with the researcher asking Joy and Nancy if there is something else they had to say about Task 1 in an effort to conclude the discussion of this task.

Researcher: I think we are done with this task [Task 1], is there anything else you would like to say about it?

Joy: The only thing I can say is the 120 units [from the problem statement] because we didn't use that [120 units]

Nancy: Yeah

Joy: But I'm not sure where it would be

Nancy: Well, there used to be always extra numbers in the problems we used to have

Joy: Maybe, that's true

In the above except, Joy is concerned about the 120 units (maximum number of units that the manufacturer can produce) even though she is not sure how she and Joy could have used the 120 units in the process of finding the maximum profit. Joy, however, seems to consider the 120 units as extraneous information that does not need to be taken into consideration when determining maximum profit for the problem posed in Task 1.

Summary of findings regarding students' algebraic reasoning when solving optimization problems. Almost all the students, at one point or another, correctly reasoned about the context of the problem posed in Task 1. This included being able to recognize reasonable results and those that are not. For example, Nikki correctly reasoned that it is unlikely that a manufacturer that produces and sells thousands of jackets would make a profit of only $1.33. Nearly half of the students had difficulty setting up the profit function even though a majority of the students had an idea of what they needed to do to be able to set up the profit function. A majority of the students could verbalize how the quantities, namely cost, revenue and profit are related (profit equals revenue minus cost) but they had difficulty representing this relationship among these quantities algebraically which was key to solving the problem posed in the task. Fewer than half of the students correctly determined and interpreted critical numbers in context. A majority of these students also correctly determined the maximum profit from the critical numbers. A few students treated critical numbers as extrema. Providing proof that the maximum profit they found (for those who did) was indeed maximum was problematic for the majority of the students. Finally, none of the students who were successful in determining critical numbers for the profit function in Task 1 considered the problem posed in Task 1 to be an absolute extrema problem: these students did not check if profit is maximized at the endpoints of the domain [0,120] of the profit function. Instead, once students had determined (e.g., using the first derivative test) that the critical number 40 is the profit maximizing quantity, they only evaluated the profit function at 40 units to get the maximum profit. That is, they did not bother checking if profit is, perhaps, maximized at a production and sales level of 120 units.

Chapter 5 – Results: Interpretation of Marginal Change and Quantitative Reasoning

In this chapter, I present the results from the analysis of the data I collected with the purpose of answering the third and fourth research questions on interpretation of marginal change and quantitative reasoning:

3. How do business calculus students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit?
4. What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit reveal about their quantitative reasoning?

In the first section of this chapter, I present results from the analysis of students' interpretation of marginal change in the economic context of cost, revenue, and profit. In the second section of this chapter, I present results from the analysis of students' quantitative reasoning when dealing with multivariable situations in the context of cost, revenue, and profit.

Interpretation of Marginal Change

To examine how students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit, I analyzed students' verbal responses and written work to the problems posed in the four tasks shown in Appendix A. In general, my analysis revealed that the 12 pairs of students who participated in this study interpreted marginal change differently, at different times, and in different representations (continuous or discrete) during the task-based interviews. In particular, (1) ten pairs of students interpreted marginal cost as total cost, (2) two pairs of students interpreted marginal cost as a difference between two quantities, (3) three pairs of students interpreted marginal change as a consecutive relationship between one value and the next value, and (4) three pairs of students interpreted marginal change as the derivative.

Interpreting marginal cost as total cost. At one point or another, ten pairs of students incorrectly interpreted marginal cost as total cost in different contexts. In particular, four pairs of students interpreted marginal cost as total cost in a continuous representation (Task 3) and in a discrete representation (Task 4). Three other pairs of students interpreted marginal cost as total cost only in a continuous representation (Task 3). Another three pairs of students interpreted marginal cost as total cost only in a discrete representation (Task 4).

Interpreting marginal cost as total cost in a continuous representation and in a discrete representation. There were no pairs of students where both partners consistently interpreted marginal cost as total cost in Task 3 and in Task 4. If however, one of the partners in a pair of students consistently interpreted marginal cost as total cost in Task 3 and in Task 4, and the other partner interpreted marginal cost as total cost in one of the tasks, the entire pair was considered to have interpreted marginal cost as total cost in the continuous representation (Task 3) and in the discrete representation (Task 4) since my unit of analysis is a pair of students and not individual student's interpretations of marginal change. Yuri and Kyle are representative of the four pairs of students who interpreted marginal cost as total cost in a continuous representation (Task 3) and in a discrete representation (Task 4).

In the following excerpt, which occurred towards the end of discussing Task 3, Yuri and Kyle reasoned about the cost of producing the second unit and the cost of producing the first two units from the total cost graph shown in Task 3 (Appendix A). The excerpt begins with a question from the researcher about the cost of producing the second unit. Prior to this excerpt, Yuri and Kyle had been reasoning about how marginal cost and marginal revenue compare at the profit maximizing quantity (which they had earlier identified as five units).

Researcher: How much does it cost this company to produce the second unit, not the first two units?

Yuri: Four hundred

Researcher: Kyle?

Kyle: Four hundred

Researcher: And how much does it cost them to produce the first two units?

Yuri: Seven hundred

Researcher: And how did you get seven hundred?

Yuri: Yah, seven hundred

Researcher: And how did you get seven hundred?

Yuri: Three hundred plus four hundred

Researcher: What's the three hundred for?

Kyle: The first unit, the cost of the first unit

Researcher: And the four hundred?

Kyle: The cost of the second unit

In the above excerpt, Yuri and Kyle incorrectly interpreted marginal cost (the cost of producing the second unit) as total cost (the cost of producing the first two units). More generally, it appears that these students interpreted the total cost graph $C(n)$ in Task 3 (Appendix A) to be a marginal cost graph. In other words, for Yuri and Kyle, the value of $C(n)$ represents marginal cost (the cost of producing the n^{th} unit) and not the total cost of producing the first n units for C, a total cost function.

Yuri and Kyle's interpretations of marginal cost differed from each other when they reasoned about the MC (marginal cost) and MR (marginal revenue) values shown in the table that appears in Task 4. In particular, Kyle interpreted marginal cost as total cost while Yuri interpreted marginal cost as a rate. In the following excerpt, Yuri and Kyle respond to the researcher's question about the units of the MC and MR values in the table shown in Task 4. Prior to this excerpt, Yuri and Kyle had been giving rationales to justify their claim that the cost of producing the 401st computer chip (which they both said would be "fifty four") is an estimated cost and not the exact cost of producing the 401st computer chip.

Researcher: What are the units of these numbers [MR and MC values]?

Kyle: Dollars

Researcher: Yuri?

Yuri: Dollars per unit

Kyle: or cents

Researcher: Yuri, why did you say dollars per unit?

Yuri: Marginal revenue is additional, extra revenue per unit

Researcher: Tell me more about that

Yuri: [Silence]

Saying that "marginal revenue is additional, extra revenue per unit" in the above excerpt suggests that Yuri sees marginal change as a rate, the rate, $\frac{R(q+1)-R(q)}{1}$, for a total revenue function $R(q)$. Kyle's initial response that the units of the marginal cost and marginal revenue are "dollars" suggests that he was interpreting marginal cost and marginal revenue as total cost and total revenue respectively. This was confirmed later in the interview when Kyle stated that the company mentioned in Task 4 breaks even (i.e., total cost equals total revenue) at a production and sales level of 402 computer chips in the table shown in Task 4 (Appendix A) when, in fact, marginal cost equals marginal revenue. However, when he added "or cents" after Yuri had said that the units of marginal cost and marginal revenue would be "dollars per unit," Kyle was either agreeing with Yuri that the units were dollars per unit or cents per unit, or he was still interpreting marginal cost as total cost. Regardless of the units given by each student for marginal cost and marginal revenue, the fact that Kyle and Yuri assigned units to the marginal cost and marginal revenue values suggests that these students also interpreted marginal cost and marginal revenue as quantities and not as numerical values. Taken together, Yuri and Kyle's interpretation of marginal change suggests that Yuri's interpretation of marginal change varied with representation (marginal cost as total cost in a continuous representation (Task 3) and marginal cost as a rate in discrete representation (Task 4)) while Kyle's interpretation of marginal change did not change with representation (marginal cost as total cost in Task 3 and in Task 4).

Interpreting marginal cost as total cost only in a continuous representation. Three pairs of students interpreted marginal cost as total cost only in a continuous representation (Task 3). The pair of students, Joy and Nancy, is representative of these three pairs of students. In the following excerpt, initiated by a question from the researcher, Joy and Nancy reasoned about marginal cost (the cost of producing the second unit) and total cost (the cost of producing the first two units) from the graph of a total cost function (Task 3). Prior to this excerpt, the students had been reasoning about how marginal cost and marginal revenue compare at the profit maximizing quantity (which they had earlier identified to be 4.5 units).

Researcher: How much does it cost this company to produce the second unit?

Joy: So then to make it, it would be [drawing a vertical line where q=2 on the q-axis to the total cost curve and then a horizontal line to the vertical axis on the graph shown in Figure 6]

Nancy: Four hundred dollars

Joy: Four hundred dollars

Researcher: How much does it cost the company to make the first two units?

Nancy: The first is two fifty plus four hundred, the cost of the first one is 300, right?

Joy: Yeah, plus the cost of the second is 400, so seven hundred [writing 300+400=700]

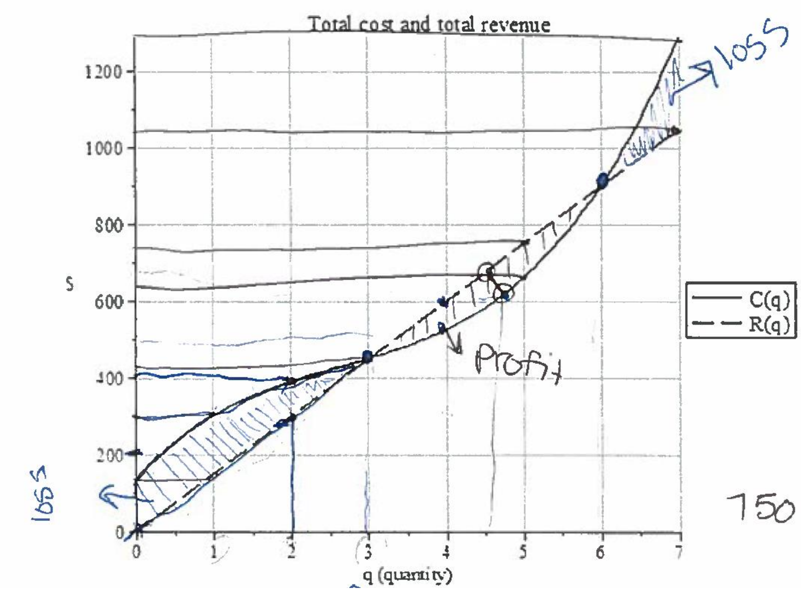


Figure 6. Graph used by Joy and Nancy to determine the cost of producing the second unit. When asked about the cost of the second unit, Joy drew the vertical line where q=2 which appears in Figure 6 which Nancy then read as four hundred dollars. Joy restated the quantity of four hundred dollars, which was taken by the researcher to be their understanding of the cost of the second unit. However, $400 according to the graph (Figure 6) is the total cost of producing

the first two units, not the cost of producing the second unit. The researcher then asked about the cost of producing the first two units and Joy and Nancy, consistent with their interpretation of the total cost of producing the first two units as the cost of producing the second unit, gave the accumulated cost of producing the first two units (sum of the cost of producing first unit and the cost of producing the first two units) which was taken by the researcher to be their understanding of the cost of producing the first two units. Hence, Joy and Nancy's reasoning about the cost of producing the second unit shows that they interpreted marginal cost (the cost of producing the second unit) as total cost (the cost of producing the first two units).

Joy and Nancy, however, did not interpret marginal cost as total cost when reasoning about the cost of producing the 401^{st} computer chip in the discrete representation presented in Task 4. In the following excerpt, Joy and Nancy reasoned about how to determine the cost of producing the 401^{st} computer chip using the information given in the table that appears in Task 4. The excerpt begins with a question from the researcher about the cost of producing the second unit. Prior to the excerpt that follows, Joy and Nancy were asked by the researcher about the units of the MC and MR values in the table shown in Task 4. Both students stated that the units would be "dollars."

Researcher: How much does it cost this company to produce the four hundred and first computer chip?

Nancy: Fifty four dollars

Joy: Yeah, because cost is what you will be paying for it

Researcher: How did you get the fifty four dollars?

Joy: Because if you look at 401, it says 54 [pointing at the MC value of 54 in the column where q=401 in the table shown in Task 4]

When asked about the cost of producing the 401st computer chip, Joy and Nancy said it would be "fifty four dollars" which according to the table shown in Task 4 is the cost of producing 402nd computer chip. In other words, Joy and Nancy interpreted marginal cost at 401 units as the cost of producing the 401st computer chip and not as the additional cost to produce one more computer chip after the 401st computer chip which, in this case, would have been the 402nd computer chip. Across the two tasks (Task 3 and Task 4) Joy and Nancy's interpretation of marginal cost was inconsistent in that in Task 3 they interpreted marginal cost (cost of producing the second unit) as total cost (cost of producing the first two units) while in Task 4 they interpreted marginal cost (cost of producing the 401st computer chip) as another marginal cost (cost of producing the 402nd computer chip).

Interpreting marginal cost as total cost only in a discrete representation. Three pairs of students interpreted marginal cost as total cost only in a discrete representation (Task 4). Denise and Ivy are representative of these pairs of students. Before I discuss how these students interpreted marginal change in Task 4, I present an analysis of their interpretation of marginal change in the continuous representation given in Task 3. Prior to the excerpt given below, Denise and Ivy had, at the researcher's prompt, reasoned about how marginal cost and marginal revenue compare at a profit maximizing quantity which they identified as 4.75 units using the graph given in Task 3. In their reasoning about marginal cost and marginal revenue at the profit maximizing quantity, Denise had indicated that she "really doesn't remember what marginal is" while Ivy thought that "marginal is the difference between revenue and cost." They concluded their reasoning about marginal cost and marginal revenue at the profit maximizing quantity by saying that they "don't know what it [marginal means]." To further examine Denise and Ivy's interpretation of marginal change in the continuous representation (Task 3), the researcher asked

Denise and Ivy about the cost of producing the second unit in Task 3. In the following excerpt, Denise and Ivy reasoned about marginal cost (the cost of producing the second unit) and total cost (the cost of producing the first two units).

Researcher: How much does it cost this company to produce the second unit, not the first two units?

Ivy: So it would be like the second cost minus the first cost

Researcher: Can you write that down for me?

Denise: The cost for two is four hundred dollars [reading off C(2) from the graph given in Task 3], the cost for one is three hundred dollars [reading off C(1) and writing cost 1&2=400, cost 1=300, cost 2=$100] so $100

Researcher: How did you get $100?

Ivy: The cost of one and two minus the cost of just one

Researcher: How about the cost for two units?

Ivy: It would be $400

To determine the marginal cost (the cost of producing the second unit), Denise and Ivy calculated the difference between the total cost of producing the first two units and the cost of producing the first unit. Even though Denise and Ivy did not explicitly state the units of the cost of producing the first unit and the units of the cost of producing the first two units, yet by saying that the $100 is "the cost of one and two minus the cost of just one" suggests that they interpreted marginal cost (the cost of producing the second unit) as a quantitative difference and not as a numerical difference.

When prompted to reason about marginal cost in the discrete representation (Task 4), Denise and Ivy interpreted marginal cost as total cost and they interpreted it as a numerical difference. The following excerpt illustrates how Denise and Ivy reasoned about marginal cost (the cost of producing the 401st unit) in the process of which they interpreted marginal cost as total cost and marginal cost as a numerical difference. Prior to this excerpt, Denise and Ivy had reasoned about how marginal cost, marginal revenue, and marginal profit are changing over the production and sales level shown in Figure 7 in a way that was understood by the researcher to be consistent with a view of marginal cost as total cost and marginal revenue as total revenue.

Researcher: How much does it cost this company to produce the 401st computer chip?

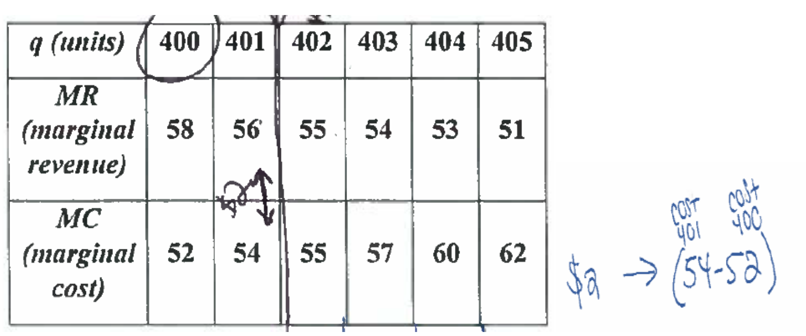
Ivy: Two dollars

Denise: Two dollars

Researcher: How did you get that?

Denise: Because it cost to make the first one fifty two dollars [pointing at the MC value when q=400 in Figure 7]. For the second one, fifty four [pointing at MC value when q=401 in Figure 7] and the difference between 54 and 52 is two, so we said two dollars

Ivy: [Writing down the calculation (on the right of Figure 7) for the cost of producing the 401st computer chip]



q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62

Figure 7. Diagram used by Ivy and Denise to calculate the cost of producing the 401st computer chip.

To determine the cost of producing the 401st computer chip, Denise and Ivy interpreted the marginal cost at a production level of 400 computer chips as the total cost for producing 400 computer chips and the marginal cost at a production level of 401 computer chips as the total cost of producing the 401 computer chips. They then subtracted the marginal cost at a production level of 401 units from the marginal cost at a production level of 400 units to get a difference of "two dollars" which was taken by the researcher to be their understanding of the cost of producing the 401st computer chip. The combination of Denise's statement in the above excerpt that "the difference between 54 and 52 is two, so we said two dollars" and Ivy's calculation of the cost of producing the 401st computer chip on the left of Figure 7 further suggests that Denise and Ivy interpreted marginal cost as a numerical difference and not as a quantitative difference even though they identified this difference of two, from 54-52, as having units of dollars.

In summary, given that six pairs of students interpreted marginal cost differently in Task 3 than they did in Task 4, it would appear that students' interpretations of marginal change varied with representation. For instance, Ivy and Denise interpreted marginal cost as a quantitative difference in a continuous representation (Task 3) but then they interpreted it as total cost in a discrete representation (Task 4).

Interpreting marginal cost as a difference between two quantities. Two pairs of students interpreted marginal cost as a difference between two quantities when reasoning about two different representations: a continuous representation (Task 3) and a discrete representation (Task 4). Zoe and Nevaeh are representative of these two pairs of students. In the following excerpt, which occurred towards the end of discussing Task 3, Zoe and Nevaeh reasoned about marginal cost (the cost of producing the second unit) after they were asked by the researcher about this cost. Prior to this excerpt, Zoe and Nevaeh had reasoned about marginal cost and marginal revenue at the profit maximizing quantity of 4.5 units.

Researcher: How much does it cost the company to produce the second unit?

Zoe: Like from here [pointing at the point (1,C(1)) in Figure 8 to here [pointing at the point (2,C(2)) in Figure 8]. Is that what you are asking?

Nevaeh: [joining the two points identified by Zoe using a line segment] So it will be like a difference in cost between these two [pointing at the points (1,C(1)) and (2, C(2) in Figure 8] units, that's how much it cost to produce that second unit. So it will be four hundred minus three hundred [writing 400-300=100]. So it cost a hundred dollars to produce the second unit

Zoe: Yeah

In the above excerpt, Zoe and Nevaeh determined marginal cost (the cost of producing the second unit) by calculating the difference between the total cost of producing the first two units and the cost of producing the first unit. Saying that the cost of producing the second unit will be a "difference in cost between these two [pointing at the point (2,C(2)) in Figure 8]" suggests that Nevaeh interpreted marginal cost (the cost of producing the second unit) as a difference between two quantities. Even though Nevaeh concluded that the cost of producing the second unit would be "a hundred dollars," her unitless calculation, 400-300=100, suggests that she might have been interpreting the cost of producing the second unit as a difference between two numerical values.

When asked to show the cost of producing the second unit in Figure 8, Zoe and Nevaeh's interpretation of marginal cost shifted from being a difference (of either two quantities or two values) to being the length of a line segment in Figure 8. This is illustrated in the following excerpt which occurred immediately after the previous excerpt where Nevaeh and Zoe reasoned about the cost of producing the second unit. The excerpt begins with the researcher asking Nevaeh and Zoe to show him the cost of producing the second unit in Figure 8.

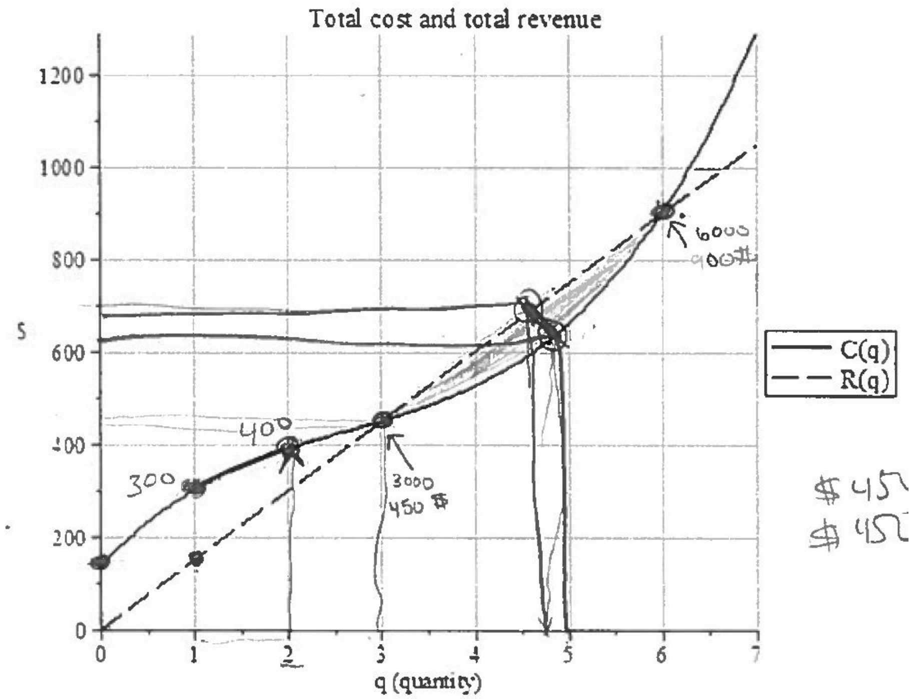


Figure 8. Zoe and Nevaeh's graphical illustration of the cost of producing the second unit.

Researcher: Can you show me the cost for producing the second unit on the graph?

Nevaeh: Yeah, this is the 300 and this is the 400 [pointing at the point labeled 300 in Figure 8 and the point labeled 400 in Figure 1]. Here is the difference [joining the points (1, C(1)) and (2, C(2) in Figure 8 using a line segment]. This distance [moving her hand over the length of the line segment joining the points (1, C(1) and (2, C(2)) in Figure 8] is too much to calculate.

Researcher: If you calculated this distance, what would it give you?

Zoe: The change between unit 1 and unit 2

Researcher: Change in what?

Zoe: Price

Researcher: Nevaeh?

Nevaeh: Yes, change in price

Zoe and Nevaeh recognized that the length of the line segment which they thought would represent the cost of producing the second unit would not be easy to calculate as shown in the above excerpt. Somehow, this was not a conflict for them even though they had earlier calculated the cost of producing the second unit and determined it to be "a hundred dollars". This suggests that sometimes students interpret marginal change differently within the same situation (Task 3). In particular, this shows that even when reasoning about the same representation (Task 3), the concept of marginal change has different meanings for students.

Later in the interview, Zoe and Nevaeh interpreted marginal change as a difference between two quantities when reasoning about marginal cost (the cost of producing the 401st computer chip) in the discrete representation presented in Task 4. In the following excerpt, which occurred midway through discussing Task 4, Zoe and Nevaeh interpreted marginal cost (the cost of producing the 401st computer chip) as a difference between two quantities in response to the researcher's question. Prior to this excerpt, Zoe and Nevaeh had been reasoning about how total cost and total revenue were changing as production increases from 400 units to 405 units in Figure 9 and concluded that "total cost is increasing" while "total revenue is decreasing."

Researcher: How much does it cost this company to produce the 401st computer chip, not 401 computer chips, the 401st computer chip?

Nevaeh: Oh we just find the difference, two dollars [writing $2]

Researcher: How did you get two dollars?

Nevaeh: Fifty four dollars minus the fifty two dollars

Researcher: Zoe, let me hear from you.

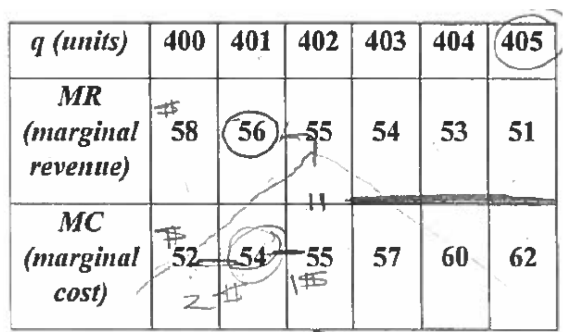
Zoe: Yeah, like because it's the same column [pointing at the third row in Figure 9]

Researcher: So what does the 54 represent?

Zoe: The total, oh no, not total, it's like how much this one costs them but the two dollars is representing the change

Nevaeh: Yeah, the change in the cost from the four hundred to the four hundred and first computer chip

Zoe: Yeah



q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62

Figure 9. Diagram used by Zoe and Nevaeh to reason about the cost of producing the 401st computer chip.

In the above excerpt, Nevaeh and Zoe interpreted marginal cost (the cost of producing the 401st computer chip) as the difference between marginal cost at 401 (Figure 9) and the marginal cost at 400 (Figure 9). Nevaeh's statement that the two dollars represents "the change in cost from the four hundred to the four hundred and first computer chip" shows that she interpreted marginal cost (the cost of producing the 401st computer chip) as a difference between two quantities. Zoe's statement, on the other hand, that "the two dollars is representing the change" could either mean that she interpreted the cost of producing the 401st computer chip as a numerical difference or as a quantitative difference (earlier in their reasoning about Task 4, Task Zoe and Nevaeh identified the MR and MC numbers in Figure 9 as quantities with units of dollars). In sum, Zoe and Nevaeh interpreted marginal cost (the cost of producing the second unit in Task 3 and the cost of producing the 401st computer chip in Task 4) as a difference between two quantities in a continuous representation (Task 3) and in a discrete representation (Task 4).

Interpreting marginal change as a consecutive relationship between one value and the next value. The coding category, consecutive relationship between one value and the next value, refers to the use of phrases or words such as "from one to two," "next," and "additional" without computing specific values when interpreting marginal change (e.g., marginal cost). Three pairs of students interpreted marginal change as a consecutive relationship between one value and the next value. Two of these pairs of students interpreted marginal change as a consecutive relationship between one value and the next value in a continuous representation (Task 3) and in a discrete representation (Task 4) while the other pair of students interpreted marginal change as a consecutive relationship between one value and the next value only in a discrete representation (Task 2).

Interpreting marginal change as a consecutive relationship between one value and the next value in a continuous representation and in a discrete representation. John and Fred are representative of the two pairs of students who interpreted marginal change as a consecutive relationship between one value and the next value in a continuous representation and in a discrete representation. The following excerpt, which occurred towards the end of Task 3, illustrates how John and Fred first interpreted marginal change as a consecutive relationship between one value and the next value while reasoning about the cost of producing the second unit. The excerpt begins with a question from the researcher about the cost of producing the second unit. Prior to this excerpt, John and Fred had been reasoning about how marginal cost and marginal revenue compare at a production and sales level of five units in Figure 10.

Researcher: How much does it cost this company to produce the second unit, not the first two units?

Fred: The cost of two units is 400

John: It’s only a hundred

Fred: Oh yah, that’s going from the first to the second, so you are saying?

Researcher: How much does it cost the company to produce the second unit?

John: A hundred

Fred: That’s basically the marginal cost, right?

Researcher: Yes

Fred: Then it’s a hundred

Researcher: How did you get the hundred?

John: From one to two [units], it [cost] increases by a hundred

Researcher: Where can I see the hundred on this graph [pointing at Figure 10]?

John: This is three hundred [showing the cost of producing the first unit on the vertical axis in Figure 10] and this is 400 [showing the cost of producing the first two units on the vertical axis in Figure 10], it increases by one hundred [marking the cost of producing the second unit using an upward arrow at q=1 in Figure 10]

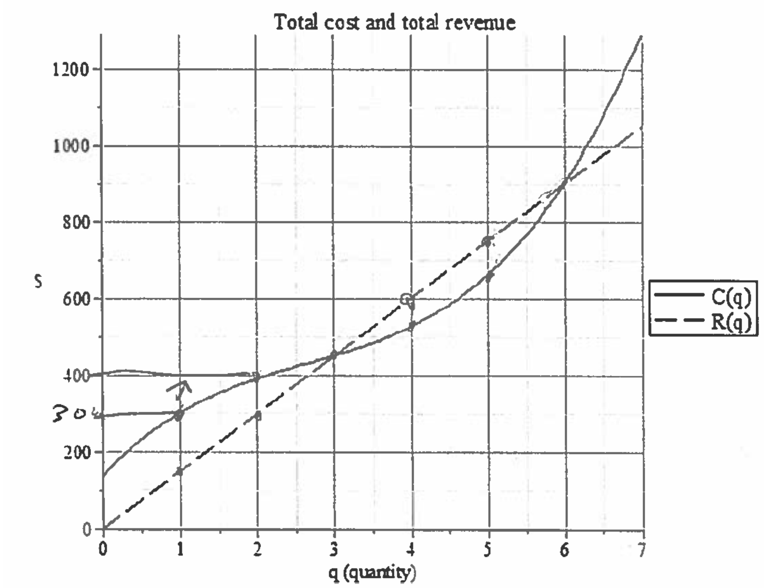


Figure 10. John's graphical illustration of the cost of producing the second unit.

The combination of John's statement that "from 1 to 2 [units], it [cost] increases by a hundred" and Fred's statement that "that's going from the first to the second" shows that John and Fred interpreted marginal cost (the cost of producing the second unit) as a consecutive relationship between the first unit and the first two units.

Later in the interview, while reasoning about the same task (Task 3), John and Fred's interpretation of marginal cost shifted from being a consecutive relationship between one value and the next value to being the cost of producing one more unit. The following excerpt, which occurred towards the end of Task 3, illustrates how John and Fred interpreted marginal cost as the cost of producing one more unit while reasoning about how marginal cost and marginal revenue compares at a production and sales level of three units in Figure 10.

Researcher: What can you say about marginal cost and marginal revenue at three units?

Fred: The marginal revenue is greater than the marginal cost

John: I agree

Researcher: What did you say again?

Fred: At three [units], the cost of producing one more unit is less than the revenue of producing one more unit

Researcher: Marginal cost is less than marginal revenue?

John: Right

In the above excerpt, John and Fred talked about marginal cost at three units as "the cost of producing one more" and marginal revenue at three units as "the revenue from producing one more unit," thus John and Fred interpreted marginal cost at three units as additional cost for producing one more unit.

John and Fred reverted to interpreting marginal change as a consecutive relationship between one value and the next value while reasoning about marginal cost (the cost of producing the 401st computer chip) in Task 4. The following excerpt, which occurred midway through discussing Task 4, illustrates how John and Fred reasoned about the cost of producing the 401st computer chip. The excerpt begins with a question about the cost of producing the 401st computer chip which was posed by the researcher. Prior to this excerpt, John and Fred had been reasoning about the units of the MC (marginal cost) and MR (marginal revenue) values shown in the table that appears in Task 4 which they concluded would be "dollars."

Researcher: How much does it cost to produce the 401st computer chip?

John: fifty two

Fred: The four hundred and first?

Researcher: Yes

John: You do, because this [pointing at the MC value at 400 units in the table shown in Task 4] is for the next one, this will be for the next one, wouldn't it? The marginal cost of four hundred [pointing at the MC value at 400 units in the table shown in Task 4], that's the marginal cost of producing the four hundred and first, I think

Fred: Ok ok, I get what you are saying. You are right, the marginal cost is fifty two

John's repeated use of the phrase "for the next one" while reasoning about the cost of producing the 401^{st} computer chip in the above excerpt shows that he interpreted marginal cost (the cost of producing the 401^{st} computer chip) as a consecutive relationship between one value and the next value. Both John and Fred stated that the cost of producing the 401^{st} computer chip would be "fifty two," the marginal cost at 400 units, which is consistent with their interpretation of marginal cost as a consecutive relationship between one value and the next value.

Interpreting marginal change as a consecutive relationship between one value and the next value only in a discrete representation. Joy and Nancy are the only pair of students who interpreted marginal change as a consecutive relationship between one value and the next value only in a discrete representation (Task 2). The following excerpt, which occurred early in the discussion of Task 2, illustrates how Joy and Nancy interpreted marginal change as a consecutive relationship between one value and the next value in the context of supplying computers and receiving revenue. The excerpt, begins with a question from the researcher about the revenue that the Smith family will receive when they supply the junior high school with 300 computers. Prior to this excerpt, Joy and Nancy had been reasoning about finding an algebraic equation they could use to determine the revenue that the Smith family will get when they supply the school with any number of computers.

Researcher: How much revenue do they [Smith family] get if they supply the school with 300 computers?

Joy: Twenty seven thousand

Researcher: How did you figure out that? Can you show me how you got that?

Joy: [writing 900*300=27,000]

Nancy: It's two hundred and seventy

Joy: Another zero, two hundred and seventy?

Nancy: Yeah

Joy: [adding an extra zero on her figure to get 270,000]

Researcher: What if they sell 301 computers, how much revenue do they get?

Nancy: It would be this number [pointing at the 270,000] plus

Joy: Plus nine hundred minus two point five times one

Nancy: Yeah

Researcher: Can you just write that down for me?

Nancy: [writing 270,000+(900-2.5(1)]

Researcher: Why did you put a one here [pointing at the 1 that is multiplied by 2.5 in the expression that Nancy wrote]?

Nancy: Because there is only one laptop, because it's for every additional over [the first three hundred laptops]

In the above excerpt, Joy and Nancy made reference to the revenue from selling the first 300 laptops (270,000) when talking about the additional revenue (900-2.5(1)) from selling the 301st laptop. This is the only instance in the entire interview where Joy and Nancy can be said to have interpreted marginal change as a consecutive relationship between one value and the next value.

As I reported earlier in the section on interpreting marginal cost as total cost, Nancy and Joy interpreted marginal cost as total cost when reasoning about Task 3. These students gave the cost of producing the 402nd computer chip in Task 4 as the cost of producing the 401st computer chip. In sum, Nancy and Joy's interpretation of marginal change shifted across different tasks.

Interpreting marginal change as the derivative. Three pairs of students interpreted marginal change as the derivative while reasoning about Task 1 and Task 4. In particular, two of these pairs of students interpreted marginal change as the derivative only in a continuous representation (Task 1) and the other pair of students interpreted marginal change as the derivative only in a discrete representation (Task 4). None of these three pairs of students consistently interpreted marginal change as the derivative in both tasks (Task 1 and Task 4) or even in the other tasks (Task 2 and Task 3).

Interpreting marginal change as the derivative only in a continuous representation. Alan and Sarah are one of the two pairs of students who interpreted marginal change as the derivative while reasoning about how to solve the problem posed in Task 1. The following excerpt, which occurred early in the interview, illustrates how Sarah and Alan reasoned about what they needed to do in order to answer the question posed in Task 1. Prior to this excerpt, the researcher had asked Sarah and Alan if they had seen a problem similar to the one given in Task 1 (to which they both answered in the affirmative) and what they needed to do in order to solve the problem posed in Task 1.

Sarah: Take the derivative of the demand equation [$p = q^2 - 100q + 3200$]

Researcher: What do you get when you take the derivative of the demand equation?

Alan: Is it the marginal?

Sarah: That would be the marginal cost

Researcher: What is marginal cost?

Sarah: The derivative of the total cost [function], right?

Alan: Yah yah, you are right

In the above excerpt, Alan wondered if taking the derivative of the demand equation would give them the "marginal" while Sarah stated that by taking the derivative of the demand equation what they will get "would be the marginal cost." Alan's wondering about the derivative of the demand equation being "the marginal" and Sarah's assertion that marginal cost is "the derivative of the total cost" were taken by the researcher to be the students' interpretations of the derivative of the demand equation by Alan (or the total cost function by Sarah) as marginal cost. Alan and Sarah, did not reason any further about the idea of marginal cost in solving the problem posed in Task 1. It would appear that they associated the act of taking the derivative of an equation (e.g., the demand equation in this case) with the term "marginal."

Abby and Shawna are the second pair of students who interpreted the derivative as marginal change while reasoning about how they would create the profit function in Task 1. The following excerpt, which occurred at the beginning of Task 1, illustrates how Abby and Shawna reasoned about how to get started in solving the problem posed in the task. Prior to this excerpt, Abby and Shawna had answered in the affirmative to the researcher's question on whether or not they had seen or even solved a problem similar to the one posed in Task 1.

Shawna: Do we have to find the derivative for something?

Abby: Yah, we have to find the derivative, ok wait. Marginal is just the derivative [pointing at the total cost function, $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$] but average is like you have to divide [total cost function] by q [number units produced]. This is like you take the derivative two times

Shawna: That's for concave up and concave down though

Abby: I feel like that's also finding the maximum

Shawna: The maximum profit is taking it [derivative] once but I don't know which one [moving her pen over the total cost function, $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$ and over the demand equation, $p = q^2 - 100q + 3200$]. I am guessing this one [pointing at the total cost function] because the 3s will cancel out.

In the above excerpt, Abby's remark that "marginal is just the derivative" while pointing at the total cost function suggests that she was interpreting the term marginal as the derivative of the total cost function. There is, however, no evidence that Shawna also thought that way even though she did not object to Abby's remark that "marginal is just the derivative." Instead, Shawna related the action of differentiating a function twice with the idea of concavity when she said that "taking the derivative two times" is for "concave up and concave down." Like Alan and Sarah, Abby and Shawna did not use the idea of "marginal" in their reasoning about Task 1 beyond what can be seen in the above excerpt.

Interpreting marginal change as the derivative only in a discrete representation. Joy and Nancy are the only pair of students who interpreted marginal change as the derivative while reasoning about the units of the MC (marginal cost) values and MR (marginal revenue) values in in Figure 11. The following excerpt, which occurred towards the end of Task 4, illustrates how Joy and Nancy reasoned about the units of the MC and MR values in Figure 11. The excerpt begins with a question from the researcher about the units of the MC and MR values in Figure 11. Prior to this excerpt, Joy and Nancy had been reasoning about how the profit of the company mentioned in Task 4 is changing across the production and sales levels shown in Figure 11.

Researcher: What do you think are the units of these numbers [pointing at the MR and MC values]?

Nancy: Oh, dollars

Joy: Dollars

Researcher: How do you know it's dollars?

Joy: Because revenue and cost is dealing with money

Researcher: But that's marginal cost and marginal revenue, is it the same thing?

Nancy: Yah

Researcher: Joy?

Joy: I think, if you like take the, like if you take the derivative of the revenue it gives you the marginal revenue, so then like

Nancy: You have to have the same units [pointing at the MR and MC values in the table shown in Table 4] to get this profit [pointing at the profit row in the Table 4], so profit is always in dollars, so you can't have anything [pointing at the MR and MC values in the table] other than dollars to get profit

Joy: Yah

q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62
Profit	6	2	0	-3	-7	-11

Figure 11. Diagram used by Joy and Nancy to reason about the units of marginal cost and marginal revenue.

Joy's statement that "if you take the derivative of the revenue, it gives you the marginal revenue" suggests that she was interpreting the marginal revenue as the derivative of the revenue function. There is, however, no evidence that Nancy also thought the same way even though she did not object to Joy's statement about the derivative of the revenue function being the marginal revenue. Joy and Nancy calculated differences (which they labeled as profit in the third row in Figure 11) between marginal revenue and marginal cost which means they also interpreted marginal cost as total cost and marginal revenue as total revenue.

Summary of students' interpretations of marginal change. In summary, ten pairs of students, at one point or another, interpreted marginal cost (the cost of producing the second unit) as total cost (the cost of producing the first two units) while reasoning about the cost of producing the second unit in Task 3, a continuous representation. A majority of these students further interpreted marginal cost as total cost when they interpreted the profit maximizing quantity of 402 units in Task 4 (a discrete representation) as a break-even quantity. These students stated that the company makes no profit at a production and sales level of 402 units where marginal cost equals marginal revenue. Saying that the company makes no profit at a production and sales level of 402 units means that the students interpreted marginal cost at a production level of 402 units as total cost for producing 402 units and marginal revenue at a sales level of 402 units as total revenue from selling 402 units. Taken together, these results show that students' view of marginal cost as total cost did not vary with different task representations. Also, a majority of these students interpreted marginal change as a numerical difference and not as a quantitative difference in that they, for example, they did not consider units when determining the cost of producing the second unit in Task 3. To determine the cost of producing the second unit, these students calculated the difference between the total cost of producing the first two units and the cost of producing the first unit without considering the cost of producing the first two units and the cost of producing the first unit as quantities, each with units of dollars, but rather as numerical values without units. A majority of these students also considered the result of this calculation, which they called the cost of producing the second unit, to be unit-less in that they did not talk about it as a quantity with units of dollars but rather as a numerical value.

Three pairs of students of students, in different contexts and representations, interpreted marginal change as a consecutive relationship between one value and the next value. For example, John interpreted marginal change as a consecutive relationship between one value and the next value when he used the words "that's going from the first to the second" when talking about the cost of producing the second unit in a profit maximization context and continuous representation (Task 3). Fred interpreted marginal change as a consecutive relationship between one value and the next value when he repeatedly used the words "for the next one" when talking about the cost of producing the 401^{st} computer chip in a profit maximization context and discrete representation (Task 4). Joy and Nancy interpreted marginal change as a consecutive relationship between one value and the next value in a revenue maximization context and discrete representation (Task 2). Taken together, these results suggest that the students' interpretation of marginal change as a consecutive relationship between one value and the next value not only varied with context but also with the representations of the functions given in each of the four tasks.

Three pairs of students associated the term “marginal” (e.g., marginal cost) with the act of taking the derivative of a function (e.g., total cost function in Task 1) algebraically. For example, while reasoning about Task 1, Sarah and Alan indicated that “the marginal cost” would be the derivative of the total cost function. They, however, did not use the idea of a marginal cost as they further reasoned about the task. Joy’s statement when reasoning about Task 4 that “if you take the derivative of the revenue, it gives you marginal revenue” also suggests that she associated taking the derivative of the revenue with marginal revenue. Like Sarah and Alan, Joy did not use the idea of marginal revenue as the derivative of the revenue as she reasoned further about Task 4. Taken together, students’ association of the term marginal with taking the derivative algebraically suggests that students’ understanding of the relationship between the idea of marginal change and the concept of the derivative is only limited to the world of algebra. In other words, these students demonstrated more of a procedural understanding of the idea of marginal change and almost no conceptual understanding of the idea of marginal change as it relates to the concept of the derivative.

Students’ interpretations of marginal change tended to change within and across tasks. For example, Nevaeh and Zoe at one time interpreted the cost of producing the second unit in Task 3 as the difference between two quantities but at another time within the same task, these students interpreted the cost of producing the second unit as the length of a line segment on a graph. Denise and Ivy, for example, interpreted marginal change differently across two tasks. They interpreted marginal cost as a difference between two quantities in Task 3 but then they interpreted marginal cost as total cost in Task 4. Taken together, these results suggest that the students who participated in this study had weak understandings of the concept of marginal change and that their interpretations of marginal change varied in different contexts and representations of economic situations.

Quantitative Reasoning

This study used Thompson's (1993) definition of quantitative reasoning: analyzing a situation in terms of the quantities and relationships among the quantities involved in the situation. According to Thompson, what is important in quantitative reasoning is not assigning numeric measures to quantities but rather reasoning about relationships between or among quantities. The term, reasoning quantitatively, as used in this study, refers to how students described and represented relationships between or among quantities and how they created and used new quantities to solve the problems they were given. This study used the definition of a quantity proposed by Ärlebäck et al. (2013): "a quantity is the result of conceiving a quality (an attribute) of an object to have an explicit or implicit unit that enables a process of measurement" (p. 317). Examples of quantities in this study include total cost, total revenue, profit, marginal cost, marginal revenue, marginal profit, and number of units produced and sold. As defined earlier, the term, discrete reasoning, as used in this study, refers to the treatment of continuous quantities as if they were discrete quantities when reasoning about relationships among several quantities in an economic context.

To examine students' quantitative reasoning when solving optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit, I analyzed students' verbal responses and written work to the four tasks in Appendix A. This analysis revealed that: (1) eleven pairs of students created new quantities (the accumulation of the discount, diminishing marginal returns, and the rate at which the revenue is increasing) which they used to reason about relationships among sales, the discount, and the revenue in a revenue maximization context (Task 2), (2) ten pairs of students reasoned discretely about relationships among several quantities (number of units produced and sold, total cost, total revenue, and profit) while creating a profit graph in a continuous representation (Task 3), and (3) two pairs of students interpreted a sequence of positive and decreasing quantitative differences (differences between marginal revenue and marginal cost) to mean that marginal profit is increasing while reasoning about a profit maximization context (Task 4).

Creating and reasoning about new quantities in a revenue maximization context. To solve Task 2, students had to reason about the context of the task which is captured in the following statement:

> For any supply of more than 300 laptops, the school will receive a $2.50 discount per computer (on the whole order) for every additional computer over 300 supplied (Task 2, Appendix A).

Of the eleven pairs of students who created new quantities while reasoning about the problem posed in Task 2, six pairs of students reasoned with the context of the task but not as intended: they applied the discount only on the additional number of computers over 300 ordered. These students created a new quantity, the rate at which the revenue is increasing, which they used to reason about the relationship among sales (number of computers sold), the discount, and the revenue.

Another five pairs of students reasoned with the context of the task. Consequently, these students applied the discount on the whole order. They also created a new quantity, the accumulation of the discount, which they used to reason about the relationship among sales, the discount, and the revenue. The only pair of students, Abby and Shawna, who did not create any quantity with which to reason about relationships among sales, the revenue, and the discount stated that they needed to have the demand and supply equations (which were not given in Task 2) in order to solve the problem posed in the task. In what follows, I discuss how the six pairs of students who applied the discount only on the additional computers ordered over 300 reasoned about the new quantity they created (the rate at which the revenue is increasing), followed by a discussion of how the other five pairs who applied the discount on the whole order reasoned about the new quantity they created (the accumulation of the discount).

Applying the discount only to the additional computers over 300 ordered. Six pairs of students reasoned with the context of Task 2 but not as intended when solving the problem posed in the task: advising the Smith family business on whether or not to sign the contract. These students applied the discount only to the additional computers over 300 ordered by the school. Students' failure to apply the discount to the whole order (as was intended in the task) may likely be the result of lack of experience with everyday situations where the discount is applied on the whole order as described in Task 2. It may also be a result of limited exposure to optimization problems where reasoning with the context of a task (as intended) is key to solving the problem in the task. As shown in the results section on opportunity to learn that was presented earlier, a majority of the optimization problems that the students in this study were exposed to, through the textbook and course lectures, had a camouflage context. Students can ignore the context of a camouflage problem and still be successful in solving the problem (Wijaya et al., 2015). Kierra and Isaac are representative of the six pairs of students who applied the discount only to the additional computers over 300 ordered. These students created a new quantity (the rate at which the revenue is increasing) which they used to reason about the relationship among the quantities:

sales, the discount, and the revenue. Kierra and Isaac are the only pair of students (out of the six pairs of students) who created a graph (Figure 12) to show the relationship among these quantities.

When explaining their understanding of Task 2 following a question from the researcher, Kierra and Isaac both indicated that each computer will sell for $900 for any order of at most 300 computers and that for any order of more than 300 computers, the additional computers over 300 sold will each sell for $897.50 which is $2.50 (the discount) less than the selling price of a computer for any order of at most 300 computers. By applying the discount only on the additional computers over 300 sold, Kierra and Isaac came to the conclusion that the relationship between the revenue and sales is such that the revenue will continue to increase as more computers are sold. That is, there will never be a time when the discount will decrease the Smith family's revenue.

In the following excerpt, which occurred just after Kierra and Isaac had explained their understanding of Task 2, Kierra and Isaac reasoned about how the relationship between the number of computers that are sold and the revenue could be represented using a graph (Figure 12). The excerpt begins with Kierra asking if the Smith family business will know, at the time of signing the contract, the number of computers that the school would want to order.

Kierra: Question, will the family know how many computers the school will buy at the time they sign the contract?

Researcher: Yes

Isaac: Well I think it's, I think, I kind of picture it as almost like a graph

Researcher: How would it look like as a graph?

Kierra: Like it would, for the first 300 [computers] it [revenue] would go up at the same rate like at one rate of $900 and then like once it hits that 300, like 301, it slightly changes the slope of the line of going up of each laptop by $897.50 so I guess like, there is never gonna be a time when like they will be losing money. They will just be like slightly gaining money, nine hundred minus $2.50

Researcher: You said something about the slope changing at like 300, right?

Isaac: Yeah, because like anything above 300 is when they give that discount of $2.50

Kierra created a new quantity (the rate at which the revenue is going up) when she stated that the revenue "would go up at the same rate like one rate of $900." Her statement that "there is never gonna be a time when like they be losing money" suggests that she viewed the application of the discount on orders of more than 300 computers as insignificant in that the revenue continues to increase no matter the size of the order. Kierra and Isaac recognized that graphically, the effect of applying the discount will be a slight change in the slope of the graph of the revenue function after 300 computers as shown in Figure 12. In the following excerpt, which occurred after the above excerpt, Isaac and Kierra continued to reason about the change in slope of the graph of the revenue function. The excerpt begins with the researcher probing Kierra and Isaac about the change in slope in the graph of the revenue function.

Researcher: So how will the slope change?

Isaac: [silent]

Kierra: [writing something]

Researcher: Can you just show me how the graph will look like?

Isaac: Oh, it will go [sketching the vertical and horizontal axis in Figure 12]

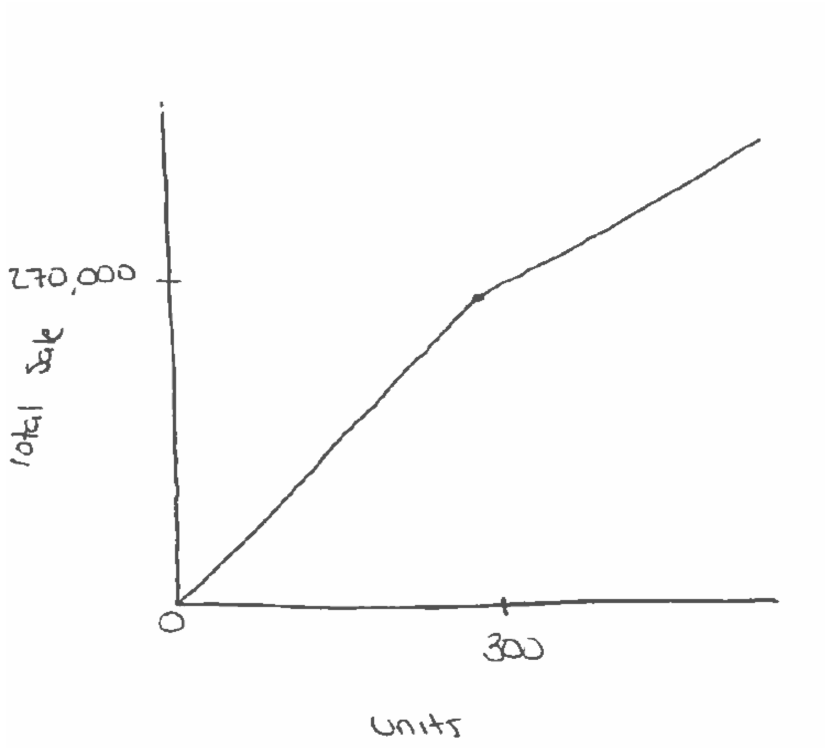


Figure 12. Kierra and Isaac's illustration of the relationship among the number of computers sold, the revenue, and the discount.

Researcher: So what goes on the horizontal axis?

Isaac: It will be the units [writing 'units' on the horizontal axis on Figure 12]

Researcher: And what goes on the vertical axis?

Isaac: It will be, I guess the sale, like the contract price like the sale price not per unit but the total sale [writing 'total sale' on the vertical axis on Figure 12]

Kierra: [Watching and listening to Isaac]

Isaac: [Sketching Figure 12] and so it will go up like a steep rate of a, for each and like up here this would be whatever 900 times three hundred is [marking the point (300, 270000) on his graph] and then from here [the point (300, 270000)] like at that point, anything above that like the slope would be less steep because it's going, because they give that $2.50 discount so just like over exaggerated [meaning Figure 12 is not accurate] it would go up like at a lesser slope [sketching the less steep line segment in Figure 12] but obviously it would still be pretty close to that line [steeper line segment] because it's only $2.50

Researcher: And what would you call that graph if you were to give it a name?

Isaac: I don't know [smiling], I took, I took both of these classes last year, I can't remember

Researcher: What is it that you said it will keep going?

Isaac: Oh like their, their profit will keep going, oh like the family, like the revenue will keep going. They will never hit a point where those units, the sale units would like cause them to go like to lose money on the deal, I guess

Figure 12 is a graphical representation of the relationship among the number of computers sold, the revenue, and the discount. According to Figure 12, the more units (computers) that are sold, the more revenue that the Smith family will make. The effect of the discount in the relationship is indicated by the change in the steepness of the revenue function at 300 units in Figure 12. The absence of units for the numbers on the vertical and horizontal axis labeled as "total sales" and "units" respectively may either suggest that Isaac interpreted the number of computers sold and the revenue received as numerical values instead of quantities or that they did not consider units to be an important part of quantitative reasoning. Isaac's statement that the revenue "will go up like a steep rate" as more computers are sold up to 300 suggests that he also created a new quantity, the rate at which the revenue will go up. Isaac continued to reason about this new quantity when he indicated that “the slope would be less steep” when he referred to how the revenue continues to increase as more computers are sold over 300.

Applying the discount to the whole order. Two pairs of students are representative of the five pairs of students who, as a result of reasoning with the context of Task 2 as intended, applied the discount to the whole order. These students created a new quantity (the accumulation of the discount and diminishing marginal returns) which they used to reason about the relationship among sales, the discount, and the revenue in a revenue maximization context (Task 2). The first pair of students (who are representative of four other pairs of students in this group of five), Yuri and Kyle, created and reasoned about the accumulation of the discount as a new quantity. In addition to reasoning about the accumulation of the discount, the second pair of students, John and Fred, created and reasoned about another quantity, the diminishing marginal returns.

In addition to verbalizing the relationship among sales, revenue, and the discount, Yuri used an algebraic equation shown in Figure 13 to represent this relationship and Kyle used a graphing calculator to graph Yuri’s revenue equation. Kyle’s graph was a concave down parabola. Yuri and Kyle described the discount as “cumulative.” They explained what a cumulative discount is by giving examples. Kyle indicated that “if you buy three hundred and two you get five dollars off each computer.” Yuri added that “if you sold three hundred and ten computers, you get twenty five dollars discount per computer you sell.” With the understanding of the discount as cumulative, Yuri and Kyle calculated the discount from selling 310, 350, and 400 computers respectively. They found that the revenue of 271250 from selling 310 computers is the same as the revenue from selling 350 computers and that the revenue of 260000 from selling 400 computers is less than the revenue from selling 310 computers. Yuri and Kyle then advised the Smith family to consider the size of the order placed by the school prior to signing the contract. They indicated that an order of "315 computers" will generate more revenue "than if they sold 415 computers." This suggests that Yuri and Kyle recognized that, in the long run, the accumulation of the discount will cease to increase the Smith family's revenue and will, instead, result in loss of revenue. In giving their advice, Yuri and Kyle explicitly talked about the two quantities: sales (number of computers sold) and the revenue which they referred to as "money" while the other quantities, namely the discount and accumulation of the discount were not explicitly referred to.

In the following excerpt, which occurred immediately after they gave their advice to the Smith family, Yuri and Kyle reasoned about the maximum revenue the Smith family could get by supplying computers to the school. The excerpt begins with the researcher asking about the maximum revenue, in an effort to probe Yuri and Kyle about how they were thinking about the accumulation of the discount.

Researcher: Do they [Smith family business] ever get any maximum revenue?

Kyle: Yes

Researcher: When?

Yuri: We need a function [algebraic function for computing revenue]

Researcher: You need a function?

Yuri: Yes

Researcher: Can you find a function?

Yuri: Yes

Researcher: The function you are finding is function for what?

Kyle: Revenue [doing something on the calculator]

Yuri: [writing the revenue function, $[900-(q-300)2.5]q=R$, $q \geq 300$ in Figure 13]

Researcher: What did you get?

Yuri: If it's bigger than 300 [number of computers supplied], then we can get this function for revenue [pointing at the revenue function, $[900-(q-300)2.5]q=R$, $q \geq 300$ in Figure 13]

Researcher: Kyle what did you get?

Kyle: I'm having trouble

The use of the word "maximum" in the researcher's question about maximum revenue in the above excerpt appears to have prompted Yuri to reason algebraically about the relationship between the two quantities, namely number of computers sold and the revenue generated. The researcher's question prompted Yuri to find an algebraic function, the revenue function denoted by the letters R and r in Figure 13, which shows the relationship between R (which is the revenue) and q (which is the total number of computers sold). Yuri's statement that "if it's bigger than 300, then we get this function for revenue" suggests an awareness that there is, perhaps another revenue function, besides the one shown in Figure 13, that can be used to determine revenue for orders that are at most 300 computers. That is, the discount or the accumulation of the discount only has an effect on revenue on orders of more than 300 computers. Yuri concluded by saying that the Smith family business should sign the contract if the school orders at most 330 computers. He, however, did not talk about much of his work in Figure 13 such as when he took the derivative of the revenue function.

Kyle, on the other hand, indicated "having trouble" in finding an algebraic function that relates the quantities: the number of computers sold, the revenue, and the discount. He then used a graphical approach (graphing Yuri's revenue function on his calculator) to represent the relationship between the revenue and the number of computers sold. When the researcher asked him about his graph, he said, "I did second calc max to find the maximum and the max quantity is three hundred and thirty for a revenue of two hundred and seventy two, two hundred and fifty." Not stating the units of the revenue maximizing quantity (330 computers) and the maximum revenue ($272,250) suggests that Kyle did not consider units to be an import part of quantitative reasoning. He then advised the Smith family to sign the contract if the school orders 330 computers "since that's the most money they can make on this deal [contract]."

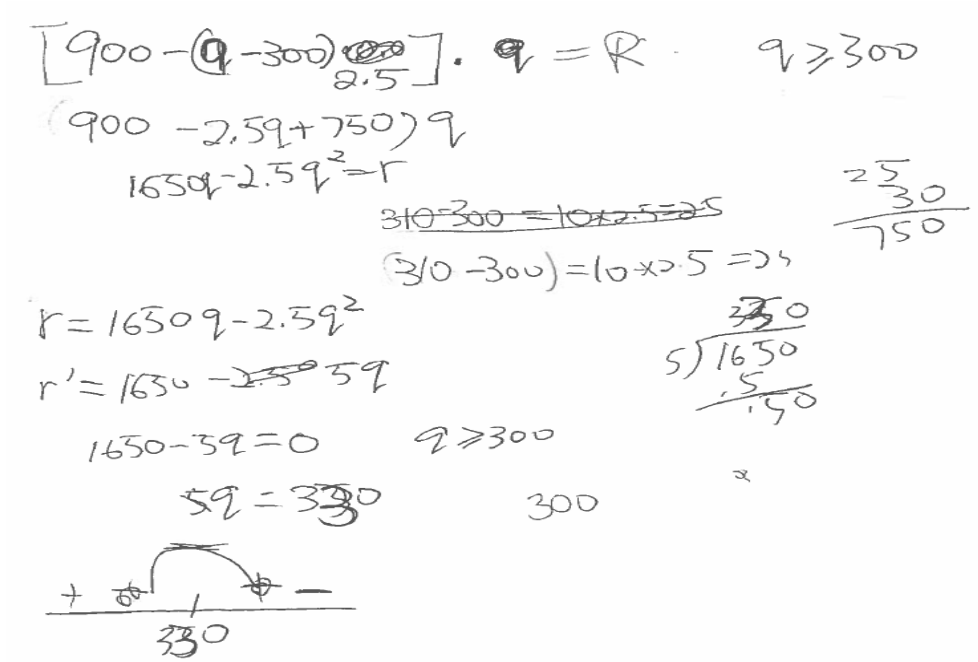


Figure 13. Yuri's representation of the relationship between the revenue and the number of computers sold using an algebraic equation.

In sum, even though there are several quantities involved (number of computers sold, revenue, and the discount), Yuri and Kyle tended to talk, explicitly, about the relationship between the number of computers sold and the revenue while the discount and the accumulation of the discount were often not explicitly referred to in the relationship. Also, Yuri and Kyle almost never explicitly mentioned units when talking about the discount or the revenue which means that they might not have considered the use of units to be important when reasoning about quantities and relationships between or among quantities.

John and Fred are the second pair of students who, in addition to reasoning about the accumulation of the discount also created and reasoned about a new quantity which they referred to as "diminishing marginal returns," when reasoning about the relationship among sales, revenue, and the discount. Since their reasoning about the accumulation of the discount is the same as that of Yuri and Kyle, in what follows, I only report on how they reasoned about the new quantity they created, diminishing marginal returns. In the following excerpt, which occurred within the first three minutes of working on Task 2, John and Fred explained what they mean by "diminishing marginal returns" and they drew on their understanding of the economic context to argue that selling more computers will result in more revenue being generated until the point of "diminishing marginal returns" is reached. The excerpt begins with Fred, in response to the researcher's question, talking about the revenue the Smith family business will get when they sell 310 computers. Prior to this excerpt, John and Fred had reasoned about the effect of the revenue on orders of 301 and 302 computers respectively.

Fred: So at 310 computers you still make $1,250 more than you would when selling 300 computers

John: We have to have that point where it starts going down again

Fred: Right, so, what is it called? Diminishing returns

John: We need, it would be easier with an equation

Researcher: You said diminishing returns?

Fred: Yah

Researcher:	What does that mean?
Fred:	Diminishing marginal returns, it means after a certain amount like it just decreases
Researcher:	What decreases?
Fred:	The revenue
Researcher:	It's called marginal?
Fred:	Marginal diminishing return. Were you gonna guess and check [asking John who was calculating revenue at different sales levels]?
John:	I don't know

Fred's statement that "at 310 computers you still make $1,250 more than you would when selling 300 computers" suggests that he understood the revenue as a quantity increases with an increase in the number of computers (another quantity) sold from 300 to 310. The combination of John's statement on the need "to have that point where it starts going down again" and Fred's reasoning about "diminishing returns" suggests that these students understood revenue to be initially increasing with more sales and that at some point the revenue reaches a maximum and subsequently begins to decrease when more computers are sold beyond the point of "diminishing returns." Even though John and Fred did not verbally say anything about the discount (as a quantity) and how it relates to the number of computers sold and the revenue received, it can be argued that they did relate the discount to the number of computers sold and the revenue generated. This is because when calculating the revenue from selling 310 computers (310*875) which they found to be 271250, John and Fred discounted each computer by $25 (from $900 to $875).

In an effort to find the point of diminishing returns using the "guess and check" approach suggested by Fred in the above excerpt, John and Fred calculated revenue from selling 320, 330, 350, and 400 computers respectively. After calculating this revenue, John and Fred determined the point of "diminishing returns" to be 330 computers. The excerpt begins with the researcher asking John and Fred about the calculations of revenue received from selling 320, 330, 350, and 400 computers respectively.

Researcher: What have you done so far?

John: We have done 300 and we did 400, and then we did 310 and then we did 320 and then we did 350 and now we just did 330

Researcher: And now he [Fred] is doing?

John: 331

Fred: Right, so now, 331 doesn't work out [revenue from selling 331 computers is less than the revenue they got from selling 330 computers] so now I'm checking 329

John: 321 [meaning 331] goes down?

Fred: Yah, 331 goes down so now I'm checking 329, yep we found the maximum, at 330 computers they are maximizing their revenue so

John: At 330?

Fred: Yah

John: At 329, it's lower?

Fred: Yah, at 329 it's lower [revenue is lower than at 330 computers] but it's still rising so the point of diminishing marginal returns is when you sell

> the 331st computer. I'm sorry, the diminishing marginal returns is 330 computers so after 330 computers everything goes down afterwards

To determine the number of computers (330) that that must be sold in order to maximize revenue, John and Fred calculated revenue for different orders of computers using a guess and check approach. John and Fred's observation that the revenue is "lower" but "still rising" when 329 computers are sold, that the revenue is maximum when 330 computers are sold, and that the revenue "goes down" when 331 computers are sold suggests that John and Fred observed changes in revenue with an increase in the number of computers sold. In particular, John and Fred noted that revenue increases with sales less than 330 computers and that revenue decreases with sales over 330 computers.

The quantity they referred to as the point of "diminishing marginal returns" is actually the revenue-maximizing quantity, which is a sales level of 330 computers in this case. John and Fred concluded by advising the family to sign the contract if the school orders "330 computers or less." This shows that their quest to determine the numerical value for the "diminishing marginal returns" helped them to solve the problem posed in Task 2, that is, to advise the Smith family business on whether or not to sign the contract. Unlike the other four pairs represented by Yuri and Kyle, John and Fred did not represent their reasoning about the relationships among sales, the revenue, and the discount using graphs and equations.

Taken together, analysis of students' reasoning about the multivariable context in Task 2, revealed several things. First, students created and reasoned about new quantities ("e.g., the diminishing marginal returns created by John and Fred) which helped them to solve the problem posed in the task. Second, students rarely paid attention to units (especially units of revenue and the discount) while reasoning about the relationship among sales, the discount, and the revenue, which suggests that they did not consider units to be an important part of the quantities involved in the task. Since any quantity has units associated with it, I argue that units of quantities need to be made explicit in quantitative reasoning. Third, in addition to verbalizing relationships among sales, the discount, and the revenue, students often used other ways (e.g., the sales versus units graph by Kierra and Isaac) to represent these relationships.

Reasoning discretely about quantities in a continuous representation. As defined earlier, the term continuous representation as used in this study refers to a mathematical task in which the given function(s) in the task is continuous on some domain. For example, Task 3 uses a continuous representation because the given functions in the task, namely the total cost function and the total revenue function are represented as continuous on the domain [0,7]. To re-state, the term discrete reasoning refers to the treatment of continuous quantities as if they were discrete quantities when reasoning about relationships among several quantities in an economic context. Treating the continuous quantities (number of units produced and sold, total cost, total revenue, and profit) in Task 3 as if they were discrete quantities when creating the profit graph is an example of discrete reasoning.

Ten pairs of students reasoned discretely about relationships among several quantities (number of units produced and sold, total cost, total revenue, and profit) while creating a profit graph in Task 3. These students created a continuous profit graph. Denise and Ivy are representative of the ten pairs of students. Sarah and Alan were the only pair of students who reasoned discretely about the relationships among the quantities (number of units produced and sold, total cost, total revenue, and profit) and created a discrete profit graph. Another pair of students, Abby and Shawna, did not reason about relationships among the quantities in Task 3. These students indicated that they needed to have an algebraic form of the profit function in order to create the profit graph. In what follows, I discuss how Denise and Ivy reasoned discretely while creating the continuous profit graph in Figure 15 followed by a discussion of how Sarah and Alan reasoned discretely while creating the discrete profit graph in Figure 16.

Creating a continuous profit graph. Denise and Ivy created a continuous profit graph (Figure 15). Prior to creating this graph, these students reasoned about relationships among number of units produced and sold, total cost, total revenue, and profit. In the following excerpt, which occurred at the beginning of Task 3, Denise and Ivy compared revenue and cost in order to determine production and sales levels where the company (Winter Store) mentioned in the task is in debt, breaks-even, and where it makes a profit. The excerpt begins with Denise verbalizing her thoughts, after reading the problem statement of the task, on what she thinks they have to do to create a profit graph.

Denise: Ok, so we have to create a graph but first we can identify points like here and there [making circles around the first intersection and second intersection of the total cost function and the total revenue function in Figure 14]

Researcher: What's happening at those points?

Denise: Cost and revenue are equal. They are at equilibrium which is that point when cost and revenue are equal [looking at Ivy]? We don't want the cost ever to be greater than the revenue

Ivy: Yah, so like right here [pointing at the region bounded by cost and revenue between the two intersection points of the total cost and total revenue in Figure 14], it's a good time because revenue is greater than cost

Denise: Yeah, that's good. That means profit when revenue is greater than the cost

Researcher: How about the other areas [regions]?

Denise: No, we don't want that. That's where it costs more to make the jackets than to sell them, so they are gonna be in debt. We don't want that.

Ivy: [labeling, in Figure 14, the region where Winter Store makes a profit with the inequality revenue>cost and the other areas where the store makes a loss with the inequality cost>revenue]

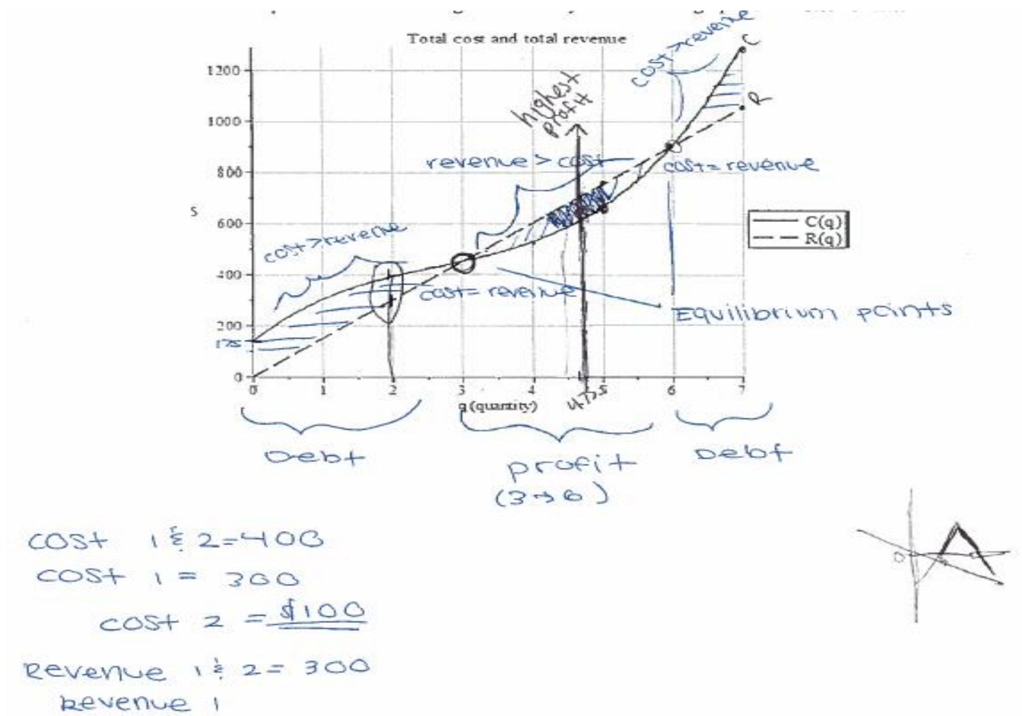


Figure 14. Graph used by Denise and Ivy to show production and sales levels where the company is in debt, at equilibrium, and makes a profit.

Denise and Ivy's description of where Winter Store is in "debt," "at equilibrium," and making "profit" suggests that these students reasoned about profit as a comparative relationship between two quantities: total cost and total revenue. Their understanding of "debt" is when total cost exceeds total revenue, profit is when total revenue exceeds total cost, and "equilibrium" when total cost and total revenue are equal. In the following excerpt, which occurred after the above excerpt, Denise created a sketch of the profit graph shown on the bottom right corner of Figure 14 which she eventually crossed out.

Denise: Could we do like a graph like this where like profit would start here [pointing at the vertical intercept of the graph she sketched in Figure 14] and this is zero [marking the origin of the graph in Figure 14] and this is zero because now they are like making profit when they get to three [moving her pen over the graph she has sketched] but then it's back down

Ivy: But then it goes back down. So would it be like, kind of thing.

Denise: [Looking at the graph she had created]. No that wouldn't work actually [crossing out her graph]

Ivy: I mean it could be a parabola

Researcher: Why wouldn't it be this [pointing at Denise's crossed out graph]?

Ivy: Because this is showing that profit is increasing, profit is decreasing but, I guess. Well revenue starts steady, the cost starts to rise more? Each one of the points on the graph is gonna be the revenue minus the cost

According to Denise's canceled graph, as the number of units that are produced and sold increases, the company's profit increases, reaches a maximum, and then decreases. It is unclear why she crossed out her graph since she did not give any rationale for doing that. Ivy appears to have imagined a profit graph that is shaped like a parabola. She also imagined that "each one of the points on the graph is gonna be the revenue minus cost." Saying that each point on the profit graph is "revenue minus cost" suggest that in Ivy's view each point on the profit graph should be a difference between revenue and cost. With the help of Denise, Ivy proceeded to calculate profit in Figure 15 at each of the production and sales levels marked 0 to 7. She then created the profit graph shown in Figure 15 by plotting and joining the points using a continuous curve. Using a continuous curve to join the points she plotted in Figure 15 suggests that Ivy understood the quantities (total cost, total revenue, and profit) to be continuous on the closed interval [0,7]. However, a closer look at her profit graph suggests that she did not attend to how the quantities (total cost, total revenue, and profit) co-vary with each other as production and sales increase from zero to three units. This is because according to the graph that appears in Task 3 (Appendix A), profit is decreasing between zero units and one unit and profit increases steadily between one unit and 3 units. Based on Ivy's profit graph shown in Figure 15, however, profit is increasing between zero and three units. Hence, Ivy must have treated the quantities (total cost, total revenue, and profit) as discrete and not as continuous when creating the profit graph.

When asked by the researcher about the vertical and horizontal axes of the graph in Figure 15, Ivy stated that "this will be y [labelling the vertical axis as y revenue-cost (profit)] and that will be x which is quantity [labeling her horizontal axis as x (quantity)]." According to Ivy's graph, profit increases as production and sales increase up to about five units and then decreases afterwards.

In summary, Denise and Ivy's reasoning about Task 3 revealed that they understood profit to be a difference between two quantities, namely total cost and total revenue. They, however, never explicitly mentioned units when talking about profit. This may suggest that these students did not consider the use of units to be important when reasoning about relationships among quantities. Also, besides calculating differences between total cost and total revenue, using these differences to generate points, and using a curve to join the points to create the pointwise profit graph in Figure 17, these students never talked about how the quantities (total cost, total revenue, and profit) are continuously changing in tandem as production and sales of units are increased from zero to seven units. This suggests that the students imagined the domain of the profit function, [0,7] to be discrete instead of being continuous.

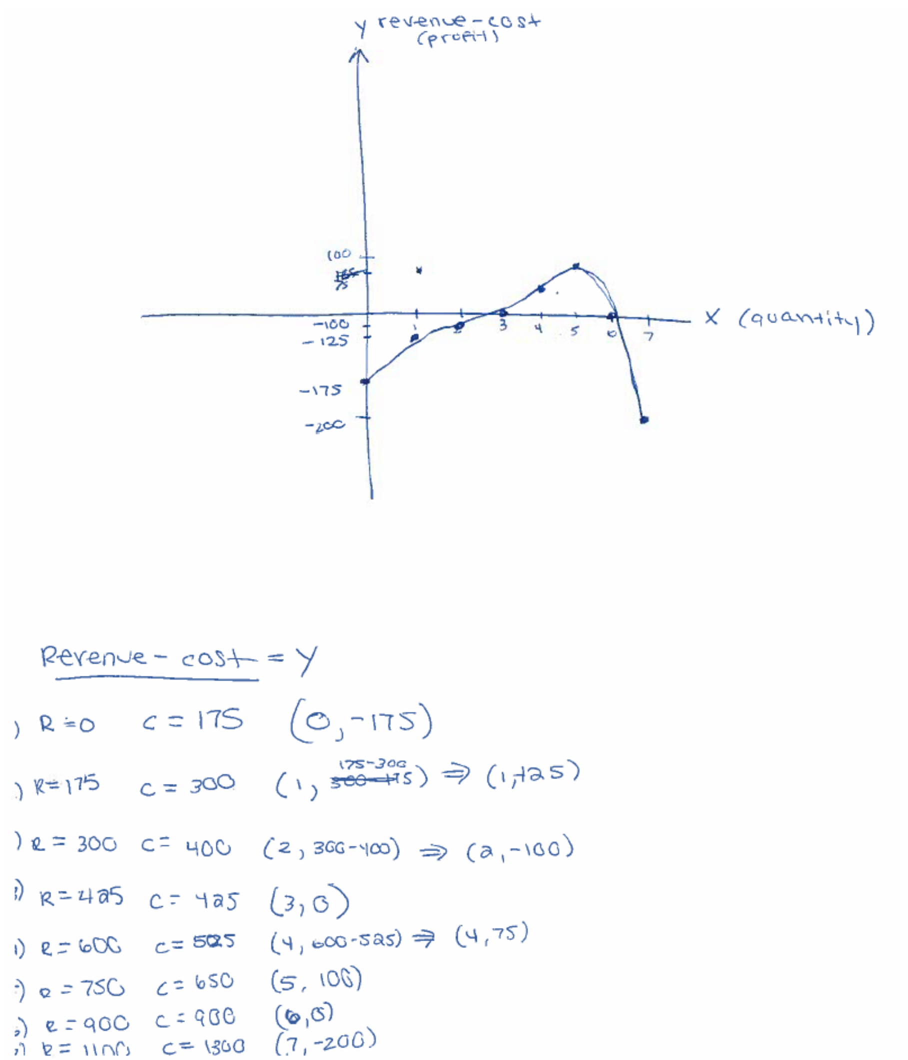


Figure 15. Ivy's profit calculations and profit graph.

Creating a discrete profit graph. Sarah and Alan are the only pair of students who created a discrete profit graph (Figure 16). Like Denise and Ivy, these students considered differences between total revenue and total cost when determining profit at each of the production and sales levels shown in the graph that appears in Task 3. They then plotted these differences as points to get the profit graph shown in Figure 16. Unlike Denise and Ivy who, after plotting points on the profit graph (Figure15) joined the points using a continuous curve, Sarah and Alan left their graph as only points as shown in Figure 16.

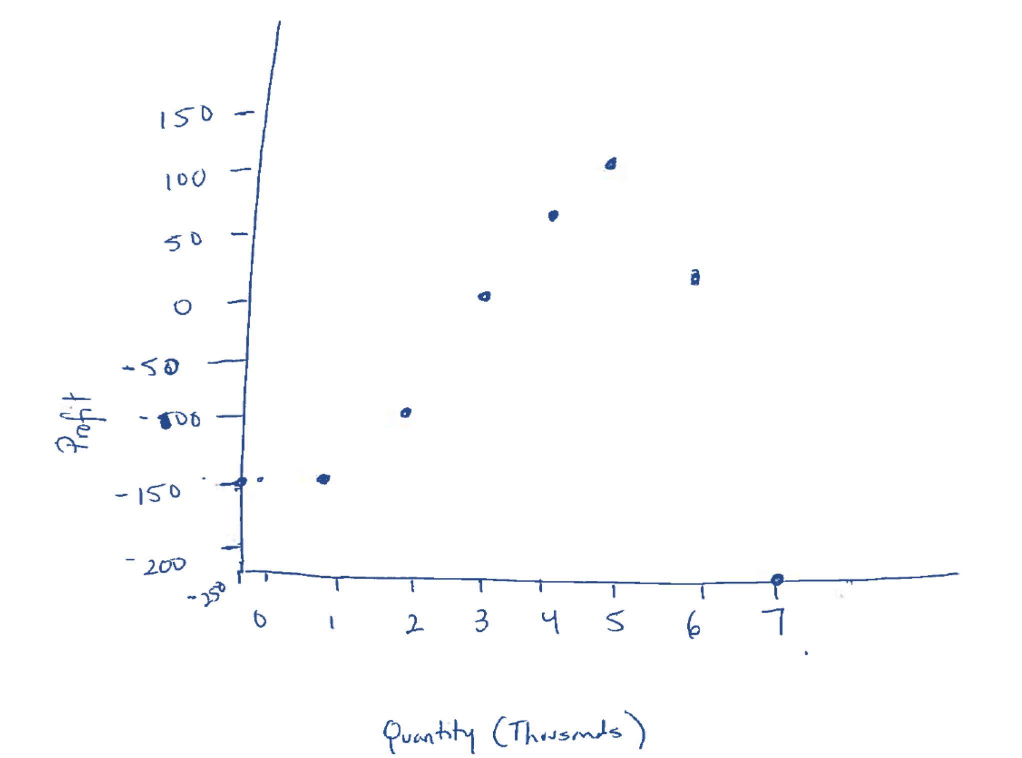


Figure 16. Alan and Sarah's profit graph.

Despite the fact that the total cost and total revenue in the graph shown in Task 3 were given as continuous functions, Sarah and Alan left their profit graph as only points, a discrete graph. This suggests that when creating the profit graph, these students focused on the discrete nature of the economic context in the task, in particular, that number of units produced and sold is discrete, and therefore decided that the profit function had to be discrete as well. At one point during the interview, Alan spontaneously said that "the quantity [number of units produced and sold] is discrete". A focus on the context of the task seems to have shifted the students' attention from the continuous representation of the quantities in the task. This finding suggests that when creating graphical representations of relationships among several quantities in an economic context, students tend to focus only on the context of the task: the representation of quantities in the task is not taken into consideration.

When asked about what would go on the vertical axis of the graph shown in Figure 16, Alan simply said “profit” without specifying the units of profit as dollars and when asked about what would go on the horizontal axis of the graph, Sarah simply said “quantity, units” after which Alan labelled the horizontal axis as quantity (in thousands). The absence of dollars as units in the vertical axis of the profit graph in Figure 16 may have been a result of Alan having thought that it is obvious that profit is measured in dollars and hence the units of profit did not have to be stated. Like Denise and Ivy, these students never talked about how the quantities in the task continuously changed in tandem as production and sales of units increases from zero units to seven units. They considered the domain of the profit function, [0,7] to be discrete and not as continuous, which led to them creating the discrete profit in Figure 16.

Interpreting a sequence of quantitative differences. Two pairs of students interpreted a sequence of positive and decreasing quantitative differences (D=MR-MC) to mean that marginal profit is increasing. Mark and Carlos are representative of these pairs of students. In the following excerpt, which occurred towards the end of working on Task 4, Mark and Carlos reasoned about how marginal cost, marginal revenue, and marginal profit are changing across the production and sales levels shown in Figure 17. The excerpt begins with Carlos explaining how marginal cost and marginal revenue are changing as you “move from left to right” in Figure 17 in response to the researcher. Prior to this excerpt, Mark and Carlos had advised the company to increase production and sales of computer chips up to 402 units and then decrease production and sales afterwards, and they had stated that the units of the MC (marginal cost) and MR (marginal revenue) values in Figure 17 would be "dollars."

Carlos: As you go from left to right, marginal revenue is decreasing, marginal cost is increasing,

Mark: Increasing

Carlos: and marginal profit from 400 to 402 is increasing and from 402 on it's going to be

Mark: Decreasing

Carlos: Decreasing, like the negative

Mark: Negative, yeah

Carlos: So marginal profit is like increasing or like going to be positive for 400th unit, 401st unit, and then at 402 it gonna just be zero or there is none because they [MC and MR in Figure 17] all equal each other

Mark: It should be like this [sketching the graph in Figure 18], and then at 403, 404 and 405, it's going to be negative because marginal cost is greater than the marginal revenue.

Researcher: So the curve represents? [Asking about Figure 18]

Carlos: The peak of the curve represents

Mark: Marginal profit

Carlos: Yes, this [pointing at the graph in Figure 18] could represent marginal profit because the graph at first like marginal profit is increasing, increasing, this could be like 400, 401 and at 402 marginal profit is zero because marginal cost equals marginal revenue and then it starts going negative because marginal cost is greater than marginal revenue

q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62

MR=MC

Figure 17. Diagram used by Mark and Carlos to reason about the relationship among number of units produced and sold, marginal cost, marginal revenue, and profit.

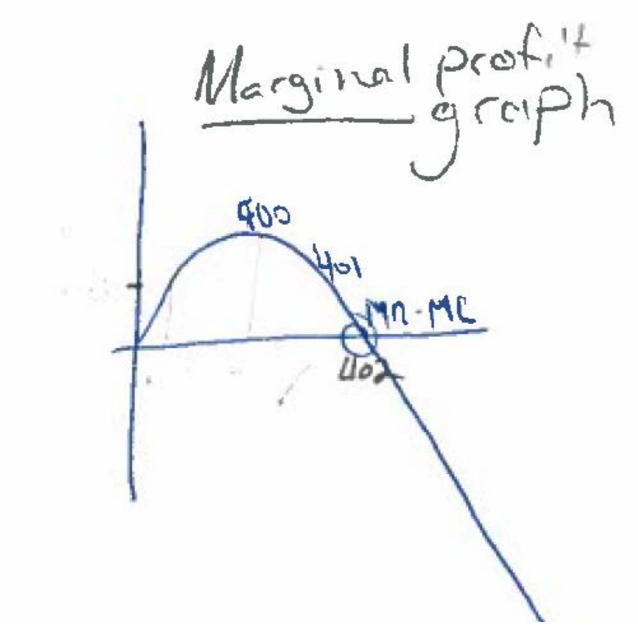


Figure 18. Graph drawn by Mark to illustrate how marginal profit changes across the production and sales levels shown in Figure 17.

When talking about how the quantity, marginal profit, is changing across the production and sales levels shown in Figure 17, Carlos indicated that "marginal profit from 400 to 402 is increasing and from 402 on it's going to be decreasing." Since marginal profit is not explicitly shown in Figure 17, when talking about marginal profit increasing or decreasing, Carlos was referring to differences between marginal revenue and marginal cost across the production and sales levels shown in Figure 17. Mark added that marginal profit is "decreasing, like the negative" as production and sales of computer chips are increased from 403 to 405. Carlos then echoed Mark's words that marginal profit is "negative" when production and sales of computer chips is increased from 403 to 405 computer chips. Mark proceeded to give a rationale for why he said marginal profit is negative at a production and sales level of 403, 404, and 405 units respectively: "because marginal cost is greater than marginal revenue." This suggests that Mark and Carlos interpreted the quantity, marginal profit, to be a difference between marginal revenue and marginal cost. With this interpretation of marginal profit, Mark and Carlos must have mentally determined marginal profit to be $6 (by calculating the difference, 58-52 in Column 2 of Figure 19) at a production and sales level of 400 units, marginal profit to be $2 (by calculating the difference, 56-54 in column 2 of Figure 19), and so on. Claiming that marginal profit is increasing "from 400 to 402," as Carlos and Mark did, suggests that these students interpreted the decreasing and positive sequence ($6, $2, $0) of marginal profit values from 400, 401, and 402 units respectively to mean that marginal profit is increasing. Mark's statement (echoed by Carlos) that marginal profit is "decreasing, like negative" when referring to marginal profit at the production and sales levels of 403, 404, and 405 units respectively suggests that these students interpreted a decreasing and negative sequence of negative values of marginal change to mean that marginal profit is decreasing. Mark's profit graph (Figure 18) is consistent with an interpretation of marginal profit that is decreasing across all the production and sales levels shown in Figure 17. This is in contrast with their claim that marginal profit is increasing up to the production and sales level of 402 computer chips and then decreasing afterwards.

Summary of students' quantitative reasoning. Taken together, my analysis of students' quantitative reasoning about multivariable situations in a revenue maximization context

(Task 2) and in a profit maximization context (Task 3 and Task 4) revealed several things. First, while reasoning about a revenue maximization context, students created new quantities (e.g., the diminishing marginal return quantity created by John and Fred) which they used to reason about the problem. The creation of these quantities also helped them to solve the problem posed in the task: giving reasonable advice to the Smith family business on whether or not to sign the contract.

Second, only five pairs of students (e.g., Yuri and Kyle) reasoned with the context of Task 2 as intended. These students applied the discount on the whole order which led to them to conclude that at some point (sales level of 330 computers), the discount will cease to increase the revenue generated by the Smith family and hence it is important not to supply more computers beyond that point. Six other pairs of students also reasoned with the context of Task 2 but not as intended. These students (e.g., Kierra and Isaac) applied the discount only on the additional computers sold over 300. They concluded that there will never be a time when the discount will not increase the revenue generated by the Smith family business. Failure to reason with the context of the task as intended may have been a result of a lack of familiarity with real life situations where the discount is applied as described in the problem statement of Task 2. That is, the context of the task may not have been realistic to the students.

Third, a majority of the students reasoned discretely about relationships among several quantities (number of units produced and sold, total cost, total revenue, and profit) while creating a profit graph in a continuous representation (Task 3). This may, in part, be attributed to the fact that the context in all the tasks is discrete. It can be argued that students' focus on the context of Task 3 shifted their attention away from the representation of the task, that is, the continuous total cost and total revenue function respectively in the task which is why they reasoned discretely and in one case created a discrete profit graph instead of a continuous graph.

Fourth, students often used different ways to represent relationships between or among several quantities in different contexts. For example, the pair of students, Kierra and Isaac used a graph to represent the relationship among the number of computers sold, the discount, and the revenue in a revenue maximization context (Task 2). Yuri and Kyle used a graph and an algebraic equation to represent to represent the relationship between number of computers sold and the revenue. Mark and Carlos used a graph to represent the relationship among number of units produced and sold, marginal cost, and marginal revenue in a profit maximization context (Task 4).

Fifth, two pairs of students interpreted a sequence of positive and decreasing values of marginal profit as production and sales of computer chips is increased from 400 to 402 units in the table shown in Task 4 to mean that marginal profit is increasing at these production and sales levels. They interpreted a negative and decreasing sequence of marginal profit values to mean that marginal profit is decreasing. This finding suggests that interpreting sequences of positive and decreasing quantities in an economic context is problematic for undergraduate students.

Chapter 6 – Discussion and Conclusions

This was a qualitative study with a two-fold purpose: (1) to examine opportunities provided by a business calculus textbook and classroom instruction for business calculus students to learn about optimization problems, marginal change, and quantitative reasoning in the economic context of cost, revenue, and profit and (2) to examine business calculus students' interpretation of marginal change and quantitative reasoning when solving optimization problems that are situated in the economic context of cost, revenue, and profit. I analyzed a business calculus textbook (Haeussler et al., 2011), business calculus lectures, and task-based interviews (Goldin, 2000) conducted with 12 pairs of students to answer my research questions:

1. What opportunities to learn about (a) optimization problems, (b) the concept of marginal change and (c) quantitative reasoning in the context of cost, revenue, and profit do business calculus textbooks and classroom instruction provide to business calculus students?
2. How do business calculus students reason algebraically about optimization problems that are situated in the context of cost, revenue, and profit?
3. How do business calculus students interpret marginal change when solving optimization problems that are situated in the context of cost, revenue, and profit?
4. What do business calculus students' responses to optimization problems involving multiple covariates that are situated in the context of cost, revenue, and profit reveal about their quantitative reasoning?

Following is a discussion of the main findings of my study. These findings are discussed in light of the related literature. I then discuss the limitations of this study and possible areas for future research, followed by a discussion of the implications of this study. The chapter concludes with final remarks that highlight the major contributions of this study.

Discussion of Findings

Opportunity to learn via the textbook and course lectures. Research on the opportunity to learn via mathematics textbooks is well documented at the pre-college level (e.g., Alajmi, 2012; Kolovou et al., 2009; Wijaya et al., 2015). However, little is known about the opportunity to learn mathematics via textbooks at the undergraduate level, a gap in knowledge that this study sought to narrow.

The first finding from this study is that both the textbook and course lectures provided students with limited opportunities to practice solving a wide range, in terms of cognitive demands (reproduction, connection, and reflection) and representations (algebraic, graphic, tabular, and verbal) of different types of optimization and marginal change problems in an economic context. Analysis of the textbook and course lectures revealed that the presentation of problems (e.g., optimization and marginal change related problems) was largely algebraic. This has the potential to promote a procedural understanding of mathematical ideas such as the concept of marginal change. This finding is consistent with that of Mesa et al. (2012) who examined the opportunity to learn about the concept of a function from ten college algebra textbooks from different learning institutions. These researchers found that "textbooks, independent of the type of institution in which they are used, present examples that have low cognitive demands, expect single numeric answers, emphasize symbolic and numerical representations, and give very few strategies for verifying correctness of the solutions" (p. 76). Consistent with the finding of Mesa et al., a majority of the optimization and marginal change tasks given in the textbook are algebraic, expect single numeric answers, and are tasks of low cognitive demand (reproduction tasks). It is, therefore, not a surprise that reasoning about non-algebraic and cognitively demanding tasks such as those used in this study (Tasks 2, 3, and 4 in Appendix A) was problematic for a majority of the students. Contrary to the findings of Mesa and colleagues, verifying the correctness of mathematical solutions (e.g., verifying that a certain critical number is a revenue maximizing quantity such as in Example 8 in Appendix H) was generally encouraged in the textbook and in course lectures. Unlike in Mesa et al.'s study, none of the tasks (Appendix H and Appendix I) in the textbook emphasized numerical representations (numerical tables).

A second finding from this study is that the opportunity for students to learn how to solve optimization and marginal change tasks (Appendix H and Appendix I) that have different types of contexts and information is limited in the textbook and in course lectures. All the optimization and marginal change tasks presented in the textbook and in course lectures had matching information while only a few problems had a realistic and essential context. A majority of the optimization and marginal change tasks given in the textbook and in course lectures had a camouflage context. Other researchers (e.g., Wijaya et al., 2015) have argued against engaging students in solving tasks with a camouflage context as such tasks do not encourage students to consider and reason with the context of a problem when solving applications problems. While reasoning about Task 1, a task with a camouflage context, a majority of the students did not consider the context of the problem. Consequently, they indicated that the manufacturer in the task could produce either a negative number of units or more than the number of units that are possible to produce per year when reasoning about the first derivative test. Maass (2010) recommended that students should be given tasks with different types of information especially tasks with either missing or superfluous information as such tasks encourage students to consider the context when solving application problems.

A third finding from this study is that the concept of marginal change was poorly presented in the textbook and in course lectures. In both the textbook and in course lectures, marginal change was defined as a rate (the difference quotient) and interpreted as an amount (the difference), something that is likely to be confusing to students. It is not surprising that all but one of the students who participated in this study treated marginal cost as an amount (difference) when they indicated that the units of marginal cost and marginal revenue in Task 4 would be dollars instead of dollars per unit. The only student who stated that the units of marginal revenue would be in dollars per unit explained that “marginal revenue is additional, extra revenue per unit.” This suggests that this student considered marginal revenue as a rate with units of dollars per unit and not as a difference with units of dollars. Units of marginal change were not given in the two examples given in the textbook (Appendix I) and in the examples given in course lecture B (Appendix F). Units of marginal change (in dollars per unit) were given in two of the five marginal change examples given in course lecture B (examples 4 and 5 in Appendix G).

A fourth finding from this study related to opportunity to learn via the textbook and course lectures is that opportunities to reason about important relationships between or among quantities were extraordinarily low in the textbook and never discussed in course lectures. The fundamental principle of economics which states that maximum profit occurs at a production and sales level when marginal cost equals marginal revenue provided total cost is greater than total revenue at that level received little attention in the textbook and no attention at all in course lectures. There were two instances where this principle was discussed, one at the end of an expository section in the textbook and the other in one practice problem given in the textbook. I argue that the scarcity of opportunities, in the textbook and in course lectures, that are intended to expose students to this principle may be the reason why a majority of the students who participated in this study incorrectly indicated that there is no profit when marginal cost equals marginal revenue at a production and sales level of 402 computer chips in Task 4, when, in fact, profit is maximized at this level. In part, this incorrect reasoning by the students that there is no profit at a production and sales level of 402 units could be attributed to a poor understanding of the idea of marginal change, which, in turn, could be attributed to the poor presentation of marginal change in the textbook and in course lectures.

In summary, a major finding of this study with regard to the opportunity to learn mathematics via textbooks and course lectures at the undergraduate level is that the presentation of topics in course lectures closely followed the presentation of topics in the textbook. In particular, the presentation of optimization problems and the concept of marginal change in the course lectures I observed closely followed the presentation of optimization problems and marginal change in the textbook. That is, the textbook had a major influence on the instructors on their teaching of optimization problems and the concept of marginal change. In essence, students' opportunities to learn mathematics via classroom instruction are similar to those they have to learn mathematics via mathematics textbooks.

Students' algebraic reasoning when solving optimization problems. Research on students' algebraic reasoning about routine optimization problems in non-economic contexts is well documented in the research literature (e.g., Brijlall & Ndlovu, 2013; Swanagan, 1996; White & Mitchelmore, 1996). In what follows, I report on three findings related to students' algebraic reasoning when solving the routine optimization problem posed in Task 1. The first two findings are confirmatory in that they have been previously reported in other studies, however, the third finding has not been previously reported in other studies, which makes it a new contribution.

First, nearly half of the students had difficulty setting up the profit function (objective function) even though a majority of the students had an idea of what they needed to do to be able to set up the profit function. In particular, a majority of the students could verbalize how the quantities, namely cost, revenue and profit are related (profit equals revenue minus cost) but they had difficulty representing this relationship among these quantities algebraically which was key to solving the problem posed in the task. To a large extent, students' difficulty in setting up the profit function was finding the revenue function, which in itself, was a result of not understanding what the demand equation given in the task represented. Some students interpreted the demand equation as a profit function, others as price, and yet others interpreted it as a product. Students' difficulties in setting up the objective function when solving optimization problems in non-economic contexts have also been reported by other researchers (e.g., Klymchuk et al., 2010; Swanagan, 2012; Villegas et al., 2009; White & Mitchelmore, 1996). The five AP calculus students studied by Swanagan (2012) had difficulty setting up the objective function when asked to determine the maximum area for a rectangular plot of land and only ten (out of 201) engineering students in Klymchuk's (2010) study were successful in setting up the objective function in a physics context. Essentially, my study confirmed that setting up the objective function in an economic context is problematic for undergraduate students. A new insight gained from my study is that even when students are unsuccessful in setting up the objective function algebraically, sometimes they are able to verbalize the quantities involved in setting up the objective function.

Second, only five pairs of the students who participated in this study were successful in determining the critical numbers for the profit function in Task 1; a majority of the students did not even get to the point of determining critical numbers as a result of the difficulty in setting up the profit function algebraically. However, only three of these five pairs correctly interpreted one of the critical numbers as the number of units that must be sold to maximize profit. The other two pairs interpreted critical numbers as extrema. These students indicated that the number of units that must be sold to maximize profit is actually the maximum profit. Treating critical numbers as extrema has also been reported by Brijlall and Ndlovu (2013). These researchers found that the high school students they studied treated the critical numbers (dimensions of a box that will result in a box with minimal volume) as if they were possible minimum values for the volume of the box. It is surprising that although the students in this study were generally encouraged to interpret critical numbers through the examples given in the textbook and in course lectures, only a few of the students correctly interpreted critical numbers in this study.

Third, nearly all the students treated the absolute extrema problem in Task 1 as if it were a relative extrema problem. The students who were successful in finding an algebraic form of the profit function in addition to determining the critical numbers did not check to see if profit is maximized at the endpoints of the domain [0,120] of the profit function. Instead, once they determined (e.g., using the first derivative test) that the critical number 40 is the profit maximizing quantity, they only evaluated the profit function at 40 units to get the maximum profit. That is, they did not check whether or not profit is potentially maximized at a production and sales level of 120 units, one of the endpoints of the domain of the profit function. Treating the absolute extrema problem as if it were a relative extrema problem in Task 1 may be directly related to the limited opportunities the students had to learn about optimization problems via the textbook and course lectures respectively. Only two of the eleven optimization tasks in the textbook were absolute extrema problems. None of the optimization examples given in course lectures was an absolute extrema problem. There is no research that has reported on students' treatment of absolute extrema problems as if they were relative extrema problems, hence this is a new finding.

Students' interpretation of marginal change when solving optimization problems. Understanding marginal change is vital in several fields such as marketing, managerial accounting, supply chain management, finance, and economics. However, there is a dearth of research on students' interpretation of marginal change at all levels. Analyses of students' responses to the four tasks in Appendix A revealed four significant findings related to undergraduate students' interpretation of marginal change in an economic context.

First, ten pairs of students incorrectly interpreted marginal cost as total cost in different profit maximization contexts (Task 3 and Task 4). Four pairs of students interpreted marginal cost as total cost in Task 3, a continuous representation, and in Task 4, a discrete representation. These students gave the total cost of producing the first two units in Task 3 as the marginal cost (the cost of producing the second unit). They also indicated that profit is zero when marginal cost equals marginal revenue in Task 4, thus interpreting marginal cost as total cost and marginal revenue as total revenue. Three other pairs of students interpreted marginal cost as total cost only in Task 3. Another three pairs of students interpreted marginal cost as total cost only in Task 4. Taken together, this finding suggests that students tend to interpret marginal cost as total cost in profit maximization contexts regardless of the representations (continuous or discrete) of the tasks in which the contexts are given.

Second, nearly all the students who participated in this study interpreted marginal change as an amount (the difference) and not as a rate of change per unit of one (the difference quotient). These students stated that the units of marginal cost and marginal revenue in Task 4 would be dollars. Only one student interpreted marginal change as a rate of change per unit of one (the difference quotient). This student stated that the units of marginal cost and marginal revenue in Task 4 would be dollars per unit. Similar results were reported by Lobato et al. (2012) in a kinematics context. Essentially, my study has shown that in an economic context, students tend to interpret marginal change as the difference and not as the difference quotient. As discussed earlier, to some extent, students' interpretation of marginal change as both a difference and a difference quotient can be attributed to the opportunities in the textbook and in course lectures, they had to learn about marginal change. Marginal change was defined as a rate per unit of one (the difference quotient) and interpreted as an amount (the difference) both in the textbook and in course lectures.

Third, a majority of the students' interpretations of marginal change tended to change within and across the four tasks they were given. One pair of students, John and Fred, interpreted marginal change as a consecutive relationship between one value and the next value in Task 3 and as a difference between two quantities in the same task. Another pair of students, Nevaeh and Zoe, interpreted marginal cost (the cost of producing the second unit) in Task 3 as a difference between two quantities but at another time within the same task, these students interpreted the cost of producing the second unit as the length of a line segment on a graph. Joy and Nancy interpreted marginal change as a consecutive relationship between one value and the next value in Task 2 but then they interpreted marginal cost as total cost in Task 3. Denise and Ivy interpreted marginal change as a difference between two quantities in Task 3 but then they interpreted marginal cost as total cost in Task 4. Taken together, these results suggest that students have weak understandings of the concept of marginal change and that their interpretations of this concept varied in different contexts and representations of economic situations.

Fourth, three pairs of students interpreted marginal change as the derivative of some function while reasoning about two profit maximization contexts, Task 1 and Task 4. One of these pairs of students, Sarah and Alan, indicated that "marginal cost" would be the derivative of the total cost function while reasoning about Task 1. Another pair of students indicated that "if you take the derivative of the revenue, it gives you marginal revenue" while reasoning about Task 4. In light of the opportunities that students had to learn about the concept of marginal change, it is not surprising that some of the students who participated in this study interpreted marginal change as the derivative. The presentation of marginal change in both the textbook and in course lectures promoted a procedural understanding of marginal change over a conceptual understanding of marginal change. Opportunities for students to interpret marginal change (e.g., marginal cost) in the textbook and in course lectures were limited. A vast majority of the marginal change tasks given in the textbook and in course lectures did not require students to go beyond finding derivatives algebraically (e.g., derivative of the total cost function to get a marginal cost function) and evaluating those derivatives at different inputs (e.g., evaluating a marginal cost function at a production level of 50 units).

Students' quantitative reasoning when solving optimization problems. Much of the existing research literature that has looked at students' quantitative reasoning is at the secondary level and scattered. A few research studies (e.g., Ärlebäck et al., 2013; Moore & Carlson, 2012; Moore et al., 2014) have looked at students' quantitative reasoning at the undergraduate level.

Analysis of students' verbal responses and written work to the four tasks in Appendix A revealed four findings related to students' quantitative reasoning at the undergraduate level.

First, creating new quantities while reasoning about a revenue maximization context (Task 2) and a profit maximization context (Task 1) was common among the students who participated in this study. Eleven pairs of students created new quantities which they reasoned with and which helped them to solve the problem posed in Task 2: advising the Smith family on whether or not to sign the contract. Fred and John, who are one of the eleven pairs of students, created and reasoned about a new quantity (the point of diminishing marginal return) which is the maximum number of computers (330) that must be sold in order for the Smith family business to get maximum revenue. Their quest to determine this quantity helped them give reasonable advice to the Smith family: sign the contract if the school orders at most 330 computers. The creation of this quantity also helped the students understand and further reason about the relationship among sales (the number of computers sold), the revenue, and the discount. On the other hand, only six pairs of students created new quantities (e.g., the profit function created by Sarah and Alan in Task 1) which helped them solve the problem posed in Task 1. Other students, such as Denise and Ivy, created quantities that did not help them solve the problem posed in the task: finding maximum profit for the manufacturer mentioned in the task. Denise and Ivy created a new quantity, which they referred to as the profit function, by dividing the total cost function by the number of units produced. These students later indicated that they needed numbers to plug in their profit function and when they could not determine how these numbers could be found, they gave up on the task claiming that they were "bad at math."

Second, nearly all the students who participated in this study reasoned discretely about relationships among several quantities (number of units produced and sold, total cost, total revenue, and profit) while creating a profit graph in a continuous representation (Task 3). Since the context of the task (producing and selling units of jackets) is discrete, it may have been that the students' focus on the discreteness of the context of the task shifted their attention away from the continuous representation of the task: the continuous functions (total cost and total revenue) in the task. Consequently, when creating the profit graph, the students in this study did not seem to realize that these quantities (number of units produced and sold, total cost, total revenue, and profit) co-varied and that they were continuous. Similar findings have been reported in other non-economic contexts (e.g., Lobato & Siebert, 2002; Moore & Carlson, 2012). Moore and Carlson (2012) engaged nine precalculus students in creating a formula for the volume of a box "formed by cutting equal-sized squares from each corner" (p. 51)." These researchers observed that "it was only after the students imagined the process of making the box and considered how the relevant quantities of the situation changed in tandem that they created a correct volume formula" (p. 57). It was only when the student studied by Lobato and Siebert (2012) understood the length and height of a wheelchair ramp to be co-varying quantities that he began to reason correctly about the steepness of a wheelchair ramp. Essentially, this finding shows that reasoning about relationships involving co-varying quantities in an economic context is problematic for students. Failure to understand co-varying quantities resulted in students treating the continuous quantities (number of units produced and sold, total cost, total revenue, and profit) in Task 3 as if they were discrete.

Third, a few of the students who participated in this study demonstrated a lack of understanding of what it means for the quantity, marginal profit, to be decreasing. These students interpreted a sequence of positive and decreasing quantitative differences (D=MR-MC) as production and sales of computer chips is increased from 400 units to 402 units in Task 4 to mean that marginal profit is increasing at these production and sales levels. They also interpreted a sequence of negative and decreasing quantitative differences as production is increased from 403 to 405 units in Task 4 to mean that marginal profit is decreasing. This suggests that in the students' view, a quantity (marginal profit in this case) is only decreasing if it is negative, otherwise the quantity is increasing. In contrast, research by Arleback et al., (2013) found that 20 of the 49 students they studied had difficulty interpreting a sequence of negative and increasing rates of change of voltage in the context of a discharging capacitor. Taken together, this finding and that of Ärlebäck and colleagues suggests that interpreting, in context, decreasing sequences that describe the behavior of a quantity is problematic for undergraduate students.

Fourth, some of the students who participated in this study used different ways to represent relationships between or among quantities in addition to verbalizing these relationships. Kierra and Isaac used a graph to represent the relationship among the number of computers sold, the discount, and the revenue in a revenue maximization context (Task 2), Yuri and Kyle used a graph and an algebraic equation to represent the relationship between the number of computers sold and the revenue, and Mark and Carlos used a graph to represent the relationship between number of units produced and sold, marginal cost, and marginal revenue in a profit maximization context (Task 4). Given that quantitative reasoning received little attention in the textbook and in course lectures, it is rather surprising that the students in this study verbally reasoned about relationships between or among quantities in addition to using different ways to represent these relationships.

Limitations of this Research and Future Research

Like any research study, this study has limitations that need to be pointed out. The first limitation has to do with the design of my study. In this study, I only examined examples given in course lectures (Appendix F and Appendix G) and examples and practice problems given in the course textbook (Haeussler et al., 2011) as opportunities for students to learn about solving optimization problems, the concept of marginal change, and quantitative reasoning in an economic context respectively. I did not examine weekly online homework problems.

Second, the examination of students' quantitative reasoning carried out in this study only focused on the creation of new quantities by students, how these quantities helped (or did not help) students in solving the problems posed in the tasks, how students talked about relationships between or among quantities, and how students represented relationships between or among quantities in multiple ways such as using graphs and algebraic equations. Examining how students engage in covariational reasoning while modeling multivariable situations such as when creating the profit graph in Task 3, is an important aspect of covariational reasoning which I did not explore in this study. Future research might look at the nature of covariational reasoning exhibited by students when modeling multivariable situations in different context areas including economics.

Third, although the data I collected using task-based interviews is rich in that I was able to use it to answer my research questions, the data could have been richer had I consistently followed my interview protocol. For example, when asking students about the cost of the company mentioned in Task 3 to produce the second unit, sometimes I asked, "how much does it cost the company to produce the second unit," and yet at other times I asked, "how much does it cost the company to produce the second unit not the first two units" which I later recognized are two different ways of asking about the cost of the second unit. The former approach is preferred in that it only asks about the cost of producing the second unit whereas the latter approach makes a distinction (for the students) between the cost of producing the second unit and the cost of producing the first two units. Using two different ways to ask about the cost of producing the second unit made it harder to know who among my participants could clearly distinguish between marginal cost (the cost of producing the second unit) and total cost (the cost of producing the first two units).

Fourth, with regard to the examination of students' reasoning about the concept of marginal change, the scope of this study was limited to the interpretation of marginal change. I did not, however, examine the relationship between the derivative and marginal change: the derivative (a continuous function) as an approximation of marginal change (a discrete quantity). Future research might look at this relationship.

Fifth, even though some of the participants in this study had taken some business or economics classes beyond calculus (or they had taken AP economics at high school) and other students had only taken (or were taking) business calculus, this study did not look at how these two groups of students compared in their reasoning about important ideas in the tasks such as the concept of marginal change. It would be of interest for future research to examine how students who have an economics or business background draw on this background to reason about some of the ideas rooted in the tasks such as the concept of marginal change (marginal cost, marginal revenue, and marginal profit) and how their reasoning about marginal change compares with that of students who have only taken business calculus and lack an economics or business background.

Sixth, in light of the fact that a self-selection process was used to recruit the students who participated in this study, it is important to highlight the limitations of this process on the sample of students who were recruited. One limitation is that only interested students chose to participate in this study so that the sample of students who participated in this study were not representative of all the students who took business calculus in the spring or fall semester of 2015. Consequently, the findings of this study cannot be generalized to all the students who took business calculus in the spring or fall semester of 2015. Given that a majority of the students who participated in this study were high achieving students (Table 2), little is known about the average students or even the low achieving students' reasoning about optimization problems in an economic context.

Implications of this Study

The findings of this study have implications for textbook authors and textbook selection committees, business calculus instructors, economics instructors, and theory.

Implications for textbook authors and textbook selection committees. A vast majority of the optimization and marginal change tasks presented in the textbook had a camouflage context and matching information. These tasks were largely algebraic in nature in addition to being tasks of low cognitive demand. I recommend that business calculus textbook authors include a much broader range of economic-based optimization and marginal change examples and practice problems in terms of types of context, types of information, types of cognitive demands, and representations of tasks to maximize the learning opportunities provided by their textbooks. Also, given that this study revealed that students had limited opportunities to learn about absolute extrema problems in an economic context, authors of business calculus textbooks might want to consider adding more of such problems in their textbooks.

Opportunities to reason about important relationships in economics such as the relationship between marginal cost and marginal revenue at a profit-maximizing quantity received little attention in the textbook that was analyzed in this study. My recommendation is that business calculus textbook authors need to include more discussions of such relationships in their textbooks if students are to be proficient in reasoning about relationships between or among economic quantities by the time they complete a business calculus course. The recommendations I made for textbook authors above also apply for textbook selection committees. In particular, textbook selection committees need to select textbooks that will provide students with a wide range of opportunities to: (1) learn about solving a wide range of optimization problems, (2) support students to develop a deep and conceptual understanding of the concept of marginal change, and (3) reason about important relationships among quantities in economics.

Implications for business calculus instructors. According to Reys et al. (2004) "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). This was the case in this study as the examples given in course lectures (Appendix F and Appendix G) were either the same or minor adaptations of those given in the textbook (Appendix H). Also, the definition and interpretation of marginal change presented in course lectures was the same definition and interpretation of marginal change presented in the textbook. Given the limitations, highlighted in the preceding paragraphs, of the examples and practice problems in the course textbook, business calculus instructors may need to supplement the examples and practice problems given in business calculus textbooks to include: (1) tasks with realistic contexts, (2) tasks with superfluous (or missing) information, (3) tasks of higher cognitive demands, and (4) tasks with multiple representations (e.g., using graphs and numerical tables) in order to maximize students' opportunity to learn from such tasks which are rare in the textbook that was analyzed in this study.

Students' difficulties with some of the concepts that were examined through the task-based interviews have implications for business calculus instructors. The presentation of marginal change during classroom instruction was limited to the use of algebra. Opportunities to learn about marginal change from other representations such as graphs and numerical tables during classroom instruction and in the textbook examples and exercises were totally absent. This might be the reason why in Task 4 (a numerical representation of marginal change), some of the students in this study interpreted marginal revenue as the derivative of the revenue function and not as the revenue generated per additional computer chip that is sold. Business calculus instructors need to provide opportunities for students to reason about marginal change from graphs and numerical tables in addition to the use of algebra. Given that the presentation of marginal change during classroom instruction closely followed the presentation of marginal change in the textbook, this study shows the influence of the textbook on business calculus instructors. To maximize the opportunity to learn about marginal change during classroom instruction, business calculus instructors may need to adapt the presentation of marginal change in the textbook to include opportunities for students to reason about realistic and cognitively demanding marginal change problems.

A majority of the students interpreted marginal cost as total cost. Business calculus instructors need to be clear in their presentation of marginal change during classroom instruction that these quantities (marginal cost and total cost) are not the same. The distinction between these quantities can be emphasized through carefully designed homework assignments on marginal change. The distinction between marginal cost and total cost was not made clear in the examples and practice problems on marginal change that were given in the textbook which means that business calculus instructors may either have to create their own problems or adapt those given in the textbook to include opportunities for students to reason about these two quantities.

A majority of the students interpreted marginal change differently in different contexts and representations of tasks. For example, some students interpreted marginal change as an amount (a difference between two values) in Task 3, a graphical and profit maximization context. These same students interpreted marginal change as a rate in Task 4, a numerical representation and profit maximization context. There is a need for business calculus instructors to help students develop a robust understanding of the concept of marginal change that would be invariant in different contexts and representations of tasks. Marginal change was defined as a rate and interpreted as an amount in the textbook. Business calculus instructors need to notice the discrepancy in the definition and interpretation of marginal change given in the textbook and address it during classroom instruction.

The only way students reasoned about relationships between or among quantities in this study was to create new quantities. This may have been influenced by the opportunities they had to reason about relationships between or among quantities during classroom instruction and in the textbook. There were no opportunities for students to reason about important relationships such as the relationship among marginal cost, marginal revenue, and profit that were presented during classroom instruction and only two such opportunities in the textbook: one at the end of an expository section and another as a practice problem. Given the importance of quantitative reasoning skills for economics or business majors, business calculus instructors need to provide opportunities for students to engage in reasoning about quantities, relationships among quantities, and representing relationships among economic quantities in multiple ways (e.g., using graphs and numerical tables).

Implications for economics instructors. In light of the importance of quantitative skills in the study of economics, the findings of this study have implications for economics instructors.

A majority of the students in this study could easily and correctly find derivatives of functions (e.g., the derivative of the total cost function). However, they had difficulty interpreting calculus based results such as critical numbers (e.g., profit maximizing quantity in Task 1). These students had weak understandings of the concept of marginal change, a concept that has important applications in economics in addition to having had a limited exposure to reasoning about important relationships among economic quantities such as the relationship among marginal cost, marginal revenue, and profit. Being aware and mindful of students' difficulties with interpreting calculus-related concepts such marginal change and critical numbers might be helpful for economics instructors in their teaching of economics classes (e.g., mathematical microeconomics) where knowledge of calculus is crucial.

Implications for theory. The findings of this study have implications for the theories of learning used to frame this study. In the theory of realistic mathematics education (RME), horizontal mathematising is concerned with moving from the world of life to the world of mathematical symbols. This includes developing a mathematical model (e.g., a profit function) from a textual problem. As an extension of RME, there is a need to add reverse horizontal mathematising which will be concerned with moving from the world of mathematical symbols back to the world of life. This includes interpreting critical numbers and extrema for a mathematical model in a real-world context.

The theory of quantitative reasoning deals with explicit quantities and relationships between or among quantities. Students in this study sometimes reasoned about quantities implicitly; they did not mention the units associated with quantities explicitly when reasoning about relationships between or among quantities. For example, when talking about the advice they would give to the Smith family in Task 2 on whether or not to sign the contract, several students talked about the number of computers sold and the revenue generated; the discount and its units were treated as implicit quantities. A few researchers (e.g., Lobato & Siebert, 2002; Moore & Carlson, 2012) have written about students' use of implicit quantities in quantitative reasoning. There is a need to address the issue of how students talk about implicit quantities and how they represent these quantities when reasoning about relationships among quantities in the theory of quantitative reasoning.

Final Remarks

With this study, I sought to add to the knowledge base in three research areas: (1) the role of undergraduate mathematics textbooks in students' learning about optimization problems, marginal change, and quantitative reasoning in an economic context, (2) students' interpretation of marginal change in an economic context, and (3) students' quantitative reasoning: reasoning about quantities and relationships between or among quantities in an economic context. In what follows, I highlight the major contributions of my study with respect to the aforementioned research areas.

Results from the analysis of the textbook and the course lectures that were observed in this study suggest that undergraduate mathematics textbooks play a significant role in determining what students learn. The content (e.g., definition and interpretation of marginal cost) presented in the course lectures that were observed in this study was generally the same as the content presented in the textbook that was analyzed in this study. Also, the optimization and marginal change examples that were given in course lectures were either the same examples given in the textbook or they were minor adaptations of the examples given in the textbook.

Engaging students in solving optimization problems in the economic context of cost, revenue, and profit revealed that students interpret marginal change differently in different contexts and representations of tasks. For example, some students interpreted marginal change as a consecutive relationship between one value and the next value in a revenue-maximization context and discrete representation (Task 2). These students interpreted marginal cost as total cost in a profit maximization context and continuous representation (Task 3).

This study also explored students' quantitative reasoning when analyzing multivariable situations in an economic context. A review of the literature on quantitative reasoning indicates that there is no research that has previously engaged students in reasoning about relationships among several variables in any context area including economics. To reason about relationships among quantities while analyzing multivariable situations in an economic context, the students who participated in this study created new quantities. These quantities helped them answer the questions posed in the tasks. For example, one pair of students created a new quantity, diminishing marginal returns, which helped them answer the question posed in Task 2: advising the Smith family business on whether or not to sign the contract to supply a school with computers. The creation of this quantity also helped the students correctly reason about the relationship among sales (number of computers sold), discount, and revenue. Other students represented relationships between or among quantities using multiple ways (e.g., using graphs and algebraic equations) in addition to making verbal statements about the relationships.

Vertical and horizontal mathematising are important features of the theory of realistic mathematics education (RME). In light of RME, as one of the theoretical frameworks guiding this study, I found that engaging in vertical mathematising (manipulation of mathematical symbols) was problematic for nearly half of the students in this study. These students had difficulty formulating the profit function (objective function) algebraically from the demand equation and total cost function given in Task 1. One pair of students that was successful in setting up the profit function made a computational error while evaluating the discriminant of the quadratic formula which resulted in them getting incorrect critical numbers, which is another example of vertical mathematising. There were almost no opportunities for students to engage in horizontal mathematising (translating text to algebraic symbols) as the demand equation and total cost function were given in algebraic form in Task 1. Only one student (Yuri) engaged in horizontal mathematising in Task 2 when he determined the revenue function algebraically from the text of the problem statement of the task.

Performing quantitative operations (creating new quantities), reasoning about relationships between or among quantities, and representing relationships between or among quantities are important characteristics of the theory of quantitative reasoning (TQR), another theoretical framework guiding this study. In light of these characteristics, several students created new quantities which they used to reason about important relationships in this study. Fred and John, for example, created the diminishing marginal return quantity which helped them analyze the relationship among sales (number of computers sold), the discount, and the revenue in Task 2. Creating and reasoning about this quantity also helped them solve the problem posed in the task: advising the Smith family on whether or not to sign the contract to supply a school with computers. At times, students created quantities that did not help them to solve the problem posed in Task 1. Other students used different ways to represent relationships among quantities. Kierra and Isaac used a graph to represent the relationship among sales, the discount, and the revenue. Yuri and Kyle used an algebraic equation and a graph to represent the relationship among sales, the discount, and the revenue. Carlos and Mark used a graph to represent the relationship among number of units produced and sold, marginal cost, marginal revenue, and marginal profit.

Adapting Goldin's (2000) free problem solving principle might have influenced the results of this study and the interpretation of the results. For example, my interjection with prompts or probes when students were reasoning about the cost of producing the second unit (marginal cost) in Task 2 might have resulted in the loss of additional information on students' interpretation of marginal change. When asking about the cost of producing the second unit, I sometimes interjected to clarify that I wanted them to determine the cost of producing the second unit (marginal cost) and not the cost of producing the first two units (total cost). Interjecting this way may have enabled students to recognize that the cost of producing the second unit and the cost of producing the first two units are two distinct quantities, a distinction that was meant to be made by the students without the researcher's help. This made it hard to determine with certainty the students who knew that these quantities were distinct from those who did not. Essentially, I could not rely only on students' responses about the cost of producing the second unit as the only source of evidence to support the claim that a majority of the students in this study confused marginal cost with total cost. To further examine students' reasoning about marginal cost and total cost, I asked them about the relationship between profit, marginal cost, and marginal revenue in Task 4. The reliability of the task-based interview protocol (examining the same understanding through multiple questions) helped to establish the validity of the claim that a majority of the students in this study confused marginal cost with total cost.

In conclusion, my study makes four significant contributions on the opportunity to learn calculus via textbooks and course lectures, students' algebraic reasoning, students' interpretation of marginal change, and students' quantitative reasoning at the undergraduate level respectively. First, students' difficulties when reasoning about optimization and marginal change problems are related to the opportunities they have to learn about these topics in calculus textbooks and in course lectures. Second, students tend to treat absolute extrema problems as if they were relative extrema problems when solving applied optimization problems in an economics context. Third, students' interpretation of marginal change varies with the context and representation of the tasks they are given. Fourth, creating new quantities helps students to reason about relationships among quantities when analyzing multivariable situations in an economic context.

Appendix A: Task-based Interview Protocol

Task 1

(Haeussler, Paul, & Wood, 2010, p. 617)

A manufacturer can produce at most 120 units of a certain product each year. The demand equation for the product is $p = q^2 - 100q + 3200$ and the manufacture's total cost function is $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$, where q denotes the number of units that are produced and sold. Find the maximum profit.

Anticipations and possible prompts and probes:

Anticipation: Although, I expect students to be able to solve this problem with ease, they might not have a conceptual understanding of the algebraic procedure for solving this problem.

Possible prompts and probes:

- Have you seen a problem like this before?
- In your own words, explain to me what is going on in this problem.
- What does taking the derivative do to your function [objective function]?
- What do these numbers [critical numbers] mean?
- What are the units of these numbers [critical numbers]?
- How can you convince someone that this number(s) [critical number(s)] will result in maximum profit?
- How can you convince someone that this is maximum profit?
- What does the sign of the number you got by evaluating your equation [first derivative of objective function/second derivative of the objective function] tell you about the profit?

Task 2

(Adapted from Hughes-Hallet et al., 2002)

The Smith family runs an electronics business in Southern California. The family is considering signing a contract to supply a small junior high school with laptops, the exact number to be determined by the principal of the school later. For any supply of up to 300 laptops, the price per laptop will be $900.

For any supply of more than 300 laptops, the school will receive a $2.50 discount per computer (on the whole order) for every additional computer over 300 supplied.

The Smith family would like you to advise them whether or not to sign the contract. They want to make sure that they make the most amount of revenue possible from this contract. What advice can you give to the Smith family on whether or not to sign the contract and why?

Anticipations and possible prompts and probes:

Anticipation: Students might experience difficulty getting started on the problem
Possible prompts and probes:

- In your own words, tell me what's going on in this problem.
- How might you advise the Smith family?
- What are you basing your advice on?
- How can you convince the Smith family that this is the maximum revenue they can get?

Anticipation: Students might not realize the relationship between quantities in the problem.
Possible prompts and probes:

- How does the discount affect the revenue?
- Can you figure out the revenue when the school orders 300 computers?
- Will the Smith family get more revenue or less revenue when 310 computers are ordered?
    - How do you know that?
    - Tell me more about that

Anticipation: Students might not explain in detail about when the Smith family's revenue will be maximized.
Possible prompts and probes:

- Will there ever be a time when the discounts will not increase the Smith family's revenue?
    - How do you know that?
    - Tell me more about that

Anticipation: Students might offer conflicting advice

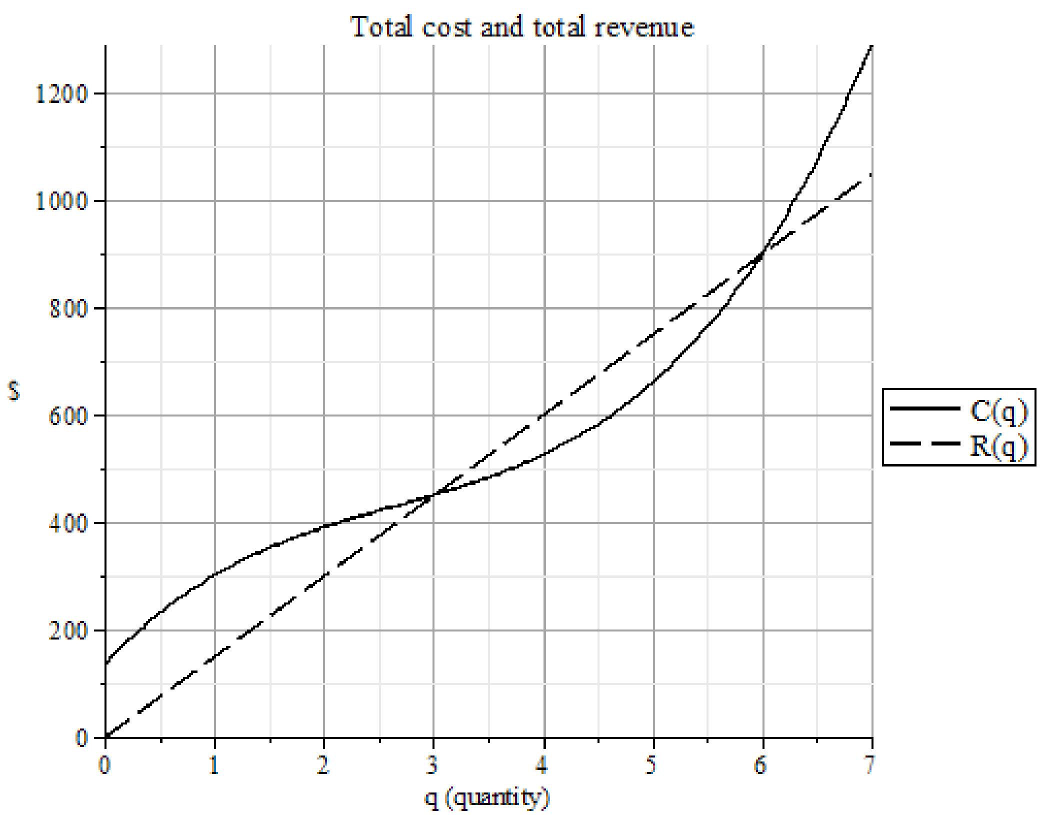
What to do in this case: I will have each student take a moment to write down their advice and then read it to the other student. I will then ask what each student thinks about the other students' advice. If the conflict persists, I will accept each students' advice and move on.

<u>Task 3</u>

(Adapted from Hughes-Hallet et al., 2002)

Winter Store is a medium sized company that manufactures and sells high-quality winter jackets in northeastern United States. The following graph shows Winter Store's total cost and total revenue when they make and sell up to 7000 jackets. A unit represents a batch of 1000 jackets.

The store manager knows that the store owner is a visual person and as such he likes graphs. The store manager would like to be able to take a graph to the owner of the store that shows the profit of the store in order to recommend the number of jackets the store should manufacture and sell in order to maximize profit. The store manager has asked you to create a graph for the store owner.



<u>Possible prompts and probes:</u>

Anticipation: Students might not be able to see the profit on the given graph.
Possible prompts and probes:

- What is going on in this graph?
- How much profit does the company generate by making and selling 1 unit?

- How about when the company makes and sells four units?
- How can the store owner see the maximum profit in the graph above?
- How can the store owner see the maximum profit in the graph you have created?

Anticipation: students might not spontaneously speak about marginal change and as well as the relationship between marginal change and maximum profit while working on this task.

Possible prompts and probes:

- What can you say about marginal cost and marginal revenue where the profit is maximum?
  - How do you know that?
- What is the company's cost for producing the 2^{nd} unit? How about the cost for 2 units?
- How can you find the company's cost for producing the 2^{nd} unit?
- How can you see the company's cost for producing the 2^{nd} unit on the graph above?
- What can you say about the company's total cost, revenue, and profit at a production and sales level of 3 units?
- What can you say about the company's marginal cost, marginal revenue, and marginal profit at a production and sales level of 3 units?

Task 4

The following table shows the marginal revenue and marginal cost at various production and sales levels for SciTech, a company that specializes in producing and selling computer chips. The company knows that total revenue is greater than total cost at all the production and sales levels shown on the table.

What advice can you give to the management of the company about when to increase or decrease production and sales of computer chips?

q (units)	**400**	**401**	**402**	**403**	**404**	**405**
MR (marginal revenue)	**58**	**56**	**55**	**54**	**53**	**51**
MC (marginal cost)	**52**	**54**	**55**	**57**	**60**	**62**

Possible prompts and probes:

Anticipation: Students might lack a conceptual understanding of the quantities shown on the table in addition to experiencing difficulty getting started on the problem

Possible prompts and probes:

- Explain to me what is going on in this table.
- What are the units of the MR and MC values given on the table?
  - How do know the units are what you say they are?
- What do you think the fourth column of this table means in terms of profit?

Anticipation: In solving this problem students might not talk about the relationship between total cost, total revenue and profit as well as marginal change and its relationship to profit. They may also be able to distinguish between exact marginal change and approximate marginal change.

Possible prompts and probes:

- How is the company's profit changing over the production and sales levels shown in the table?
- What can you say about how the company's total cost, revenue, and profit are changing over the production and sales levels shown in the table?

- What can you say about how the company's marginal cost, marginal revenue, and marginal profit are changing over the production and sales levels shown in the table?
- How much does it cost this company to produce the 401st computer chip?
    - Is this the exact cost or an approximation? How do you know that?
- Which is cheaper producing the 401st computer chip or the 402nd computer chip? How do you know that?
- How much profit does the company generate by making and selling the 401st computer chip?

Description of Tasks

In the following table, the symbol π denotes profit, R denotes total revenue, C denotes total cost, $\bar{c}$ denotes average cost, p denotes the selling price per unit sold, q denotes the number of units produced or sold, MP denotes marginal profit, MC denotes marginal cost, and MR denotes marginal revenue.

	Purpose of Task	Quantities Involved	Relationships of Quantities	Representations of Quantities	Context of Quantities
Task 1	Examine students' abilities to solve optimization problems similar to those given in their textbook. Examine students' abilities to create and reason about new quantities when solving routine optimization problems in an economic context. Examine students' reasoning about relationships among quantities: number of units produced and sold, total cost, total revenue, and profit.	$\pi, R, C, p,$ and q	$\pi = R - C$ $R = pq$ $C = \bar{c}q$	Algebraic	Profit-maximization
Task 2	Examine students' interpretation of marginal change (e.g., revenue from selling the 301^{st} computer).	$R, p, q,$ and MR	$R = pq$	Textual (verbal)	Revenue-maximization

	Examine students' reasoning about the relationship among the quantities: number of computers ordered, the discount, and the revenue that is generated.				
Task 3	Examine students' interpretation of marginal change (e.g., the cost of producing the second unit). Examine students' reasoning about relationships among the quantities: number of units produced and sold, total cost, total revenue, and profit.	$\pi, R, C, q, p, MC, MR,$ and MP	$\pi = R - C$ $MP = MR - MC$	Graphical	Profit-maximization
Task 4	Examine students' interpretation of marginal change (e.g., the cost of producing the 401st computer chip). Examine students' reasoning about relationships among the quantities: number of computer chips produced and sold, marginal cost, marginal revenue, and profit.	$\pi, MP, MC,$ and MR	$MP = MR - MC$ Maximum profit will occur when $MR = MC$ Since $R > C$	Tabular	Profit-maximization

Appendix B: Classroom Observation Protocol

Date of observation: ____________________ Section observed: ____________________

Start time: ____________________________ End time: __________________________

Mathematical topic of the lecture: ____

MC: marginal cost **MR:** marginal revenue **C:** cost **R:** Revenue

Part I: Optimization Examples

Context of in-class examples

- What type(s) of context did the professor use in the optimization examples he gave?

___ No context (purely mathematical context i.e. algebraic symbols only)

___Camouflage context

___Relevant and essential context (experientially real)

Types of information (in-class examples)

- What type(s) of information did the professor use in the optimization examples he gave?

___Matching information

___Missing information

___Superfluous information

Levels of cognitive demands of in-class examples

- What level (s) of cognitive demand was used in the optimization examples given by the professor?

___Reproduction

___Connection

___Reflection

Representations of examples of optimization problems

- What representations were used by the professor to represent different examples of optimization problems?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Purpose of in-class example(s)

- What was the purpose of the optimization example(s) given by the professor?

___Application (The example(s) was given after the explanation section)

___Modeling (The example(s) was is given before the explanation section)

Frequency

- How many in-class optimization examples did the professor do?

___How many economic context examples did the professor do?

___How many non-economic context examples did the professor do?

Conceptual emphasis

- What did the professor do that shows an emphasis on conceptual understanding of the mathematical ideas such as critical numbers and extrema during the lecture?

___He included units of critical numbers or extrema when giving examples

___He talked about the importance of identifying the appropriate domain for the objective function

___He wrote down the appropriate domain of each objective function in the examples he did

___He asked students to state the appropriate domain of each objective function when solving contextualized optimization problems

___He showed students an algebraic procedure for solving optimization problems, how to use the procedure, and also explained what each step of the procedure does.

___He asked students for interpretations of critical numbers in context.

___He asked students for interpretations of extrema in context.

___He interpreted critical numbers in context when giving examples.

___He interpreted extrema in context when giving examples.

___He verified if a relative extrema was a minimum or maximum value of the objective function (e.g., using the first or second derivative test).

___He asked students to verify if a relative extrema is a minimum value or maximum value of the objective function.

___He asked students for interpretations of the significance of MR=MC (when R>C).

___He asked students for interpretations of the significance of MR=MC (when C>R).

Procedural emphasis

- What did the professor do that shows an emphasis on procedural understanding of the mathematical ideas such as critical numbers and extrema during the lecture?

___He did not include units of critical numbers or extrema when giving examples

___He did not talk about the importance of identifying the appropriate domain for the objective function

___He did not write down the appropriate domain of each objective function in the examples he did

___He did not ask students to state the appropriate domain of each objective function when solving contextualized optimization problems

___He showed students an algebraic procedure for solving optimization problems, how to use the procedure but never explained what each step of the procedure does.

___He did not ask students for interpretations of critical numbers in context.

___He did not ask students for interpretations of extrema in context.

___He did not interpret critical numbers in context when giving examples.

___He did not interpret extrema in context when giving examples.

___He did not verify if a relative extremum was a minimum or maximum value of the objective function.

___He did not ask students to verify if a relative extremum is a minimum value or maximum value of the objective function.

___He did not ask students for interpretations of the significance of MR=MC (when R>C).

___He did not ask students for interpretations of the significance of MR=MC (when C>R).

Part II: Marginal Change

Discussion of marginal change

- How did the professor explain/interpret marginal change (e.g. marginal cost)?

___He interpreted marginal change as a rate of change (the difference quotient)

___He interpreted marginal change as an amount or change in amount (the difference)

___He interpreted marginal change as both a rate and an amount (or change in amount)

Conceptual understanding of marginal change

- What did the professor do to promote students' conceptual understanding of the idea of marginal change?

___He included units of marginal change in the examples about marginal change that he gave in class.

___ He asked students for the interpretation of marginal change (e.g., marginal revenue)

___He asked students for units of marginal change (e.g. marginal cost) in the examples about marginal change that he worked out in class.

___He asked students to always include units of marginal change when answering marginal change-related questions on exams or homework problems.

___ He stated that the units of marginal change would be dollars per unit.

Procedural understanding of marginal change

- What did the professor do to promote students' procedural understanding of the idea of marginal change?

___He did not include units of marginal change in the examples about marginal change that he gave in class.

___He did not ask asked students for units of marginal change (e.g. marginal cost) in the examples about marginal change that he worked out in class.

___He asked students to always include units of marginal change when answering marginal change-related questions on exams or homework problems.

Relationship between marginal change and the derivative

- How did the professor talk about marginal change and its relationship to the derivative?

___Marginal change is the derivative (e.g. marginal cost is the derivative of the cost function)

___Marginal change can be approximated using the derivative

___He never talked about the relationship between marginal change and the derivative

Representations of marginal change

- What representations were used by the professor to illustrate or represent marginal change?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Part III: Covariation

The derivative and covariation

- How did the professor attend to covariation of quantities while discussing the derivative and marginal change?

___He talked about the derivative as a covariation of two quantities (explained the difference quotient)

___He talked about how quantity produced and sold, revenue, cost, and profit co-vary.

___He talked about how cost and revenue change to determine profit.

___He talked about how changes in marginal cost and marginal revenue impacts profit.

___He never talked about covariation of quantities.

Reasoning about covariation in multiple representations

- What representations were used by the professor when giving examples about the derivative in class?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Appendix C: Textbook Protocol

MC: marginal cost **MR:** marginal revenue **C:** cost **R:** Revenue

Part I: Textbook Examples

Context of textbook examples

- What type(s) of context was used in the optimization examples given in the textbook?

___ No context (purely mathematical context i.e. algebraic symbols only)

___Relevant and essential context (experientially real)

___Camouflage context

Types of information in textbook examples

- What type(s) of information was used in the optimization examples given in the textbook?

___Matching information

___Missing information

___Superfluous information

Levels of cognitive demands of textbook examples

- What level (s) of cognitive demand was used in the optimization examples given in the textbook?

___Reproduction

___Connection

___Reflection

Purpose of textbook examples

- What was the purpose of the optimization examples given in the textbook?

___Application (The example(s) was given after the explanation section)

___Modeling (The example(s) was is given before the explanation section)

Representations of examples of optimization examples (problems) in the textbook

- What representations were used in the textbook to represent different examples of optimization problems?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Frequency

- How many optimization examples were given in the textbook?

___How many economic context examples were given in the textbook?

___How many non-economic context examples were given in the textbook?

Part II: Textbook Exercises (Assigned/Practice)

Context of textbook exercises

- What type(s) of context was used in the optimization exercises given in the textbook?

___No context (purely mathematical context i.e. algebraic symbols only)

___Relevant and essential context (experientially real)

___Camouflage context

Types of information in textbook exercises

- What type(s) of information was used in the optimization exercises given in the textbook?

___Matching information

___Missing information

___Superfluous information

Levels of cognitive demands of textbook exercises

- What level (s) of cognitive demand was used in the exercises given in the textbook?

___Reproduction

___Connection

___Reflection

Representations of optimization problems (exercises) in the textbook

- What representations were used in the textbook to represent different exercises of optimization problems?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Frequency

- How many optimization exercises were given in the textbook?

___How many economic context exercises were assigned from the textbook?

___How many non-economic context excises were assigned from the textbook?

Part III: Conceptual Understanding of Optimization Problems

Conceptual emphasis

- How does the textbook support students' conceptual understanding of optimization problems in its presentation of mathematical ideas such as critical numbers and extrema?

___Each step of the algebraic procedure for solving optimization problems is explained (i.e. what each step does) prior to giving examples.

___Critical numbers in textbook examples are interpreted in context.

___Exercises in the textbook require students to interpret critical numbers in context.

___Extrema in the textbook examples are interpreted in context.

___Exercises in the textbook require students to interpret extrema in context.

___Extrema in textbook examples are verified as a minimum or maximum value of the objective function (e.g. using the first derivative test).

___Exercises in the textbook require students to verify extrema as minimum or maximum values of the objective function.

___Units of marginal change (e.g. marginal cost) are given in textbook examples.

___Exercises require students to give units of marginal change.

___The textbook discusses the significance of MC=MR (when R>C)

___The textbook discusses the significance of MC=MR (when C>R)

___The textbook discusses the possibility of an objective function having multiple critical numbers

___The textbook discusses how to determine if a critical number is meaningful (reasonable) when the objective function has multiple critical numbers

Part IV: Procedural Understanding of Optimization Problems

Procedural emphasis

- How does the textbook support students' procedural understanding in its presentation of mathematical ideas such as critical numbers and extrema?

___The steps of the algebraic procedure for solving optimization problems are not explained (i.e. what each step does).

___Critical numbers in textbook examples are not interpreted in context.

___Exercises in the textbook do not require students to interpret critical numbers in context.

___Extrema in the textbook examples are not interpreted in context.

___Exercises in the textbook do not require students to interpret extrema in context.

___Extrema in textbook examples are not verified as a minimum or maximum value of the objective function (e.g. using the first derivative test).

___Exercises in the textbook do not require students to verify extrema as minimum or maximum values of the objective function.

___Units of marginal change (e.g. marginal cost) are not given in textbook examples.

___Exercises do not require students to give units of marginal change.

___The textbook does not discuss the significance of MC=MR (when R>C)

___The textbook does not discuss the significance of MC=MR (when C>R)

___The textbook does not discuss the possibility of an objective function having multiple critical numbers

___The textbook does not discuss how to determine if a critical number is meaningful (reasonable) when the objective function has multiple critical numbers

Part V: Marginal Change

Discussion of marginal change

- How is marginal change (e.g. marginal cost) explained/ interpreted in the textbook?

___As a rate of change (the difference quotient)

___As an amount or change in amount (the difference)

___As both a rate and an amount (or change in amount)

Relationship between marginal change and the derivative

- How is the relationship between marginal change and the derivative discussed in the textbook?

___Marginal change is the derivative (e.g. marginal cost is the derivative of the cost function).

___Marginal change can be approximated using the derivative.

___The textbook does not discuss the relationship between marginal change and the derivative.

Representations of marginal change

- What representations were used in the textbook to illustrate or represent marginal change?

___Algebraic

___Tabular

___Graphical

___Verbal (textual)

Part VI: Textbook Treatment of Covariation

Covariation

- How does the textbook attend to covariation of quantities while discussing the derivative and marginal change?

___The derivative is explained as a covariation of two quantities (includes an explanation of the difference quotient) in the textbook.

___The textbook discusses how quantity produced and sold, revenue, cost, and profit co-vary.

___The textbook discusses how cost and revenue change to determine profit.

___The textbook discusses how changes in marginal cost and marginal revenue impacts profit.

___The textbook does not discuss covariation of quantities.

Appendix D: Textbook Analysis Framework

Task Characteristic	Sub-category	Explanation
Type of context	No context	-Refers to only mathematical objects, symbols, or structures.
	Camouflage context	-Experiences from everyday life or common sense reasoning are not needed. -The mathematical operations needed to solve the problems are already obvious. -The solution can be found by combining all numbers given in the text.
	Relevant and essential context	-Common sense reasoning within the context is needed to understand and solve the problem. -The mathematical operation is not explicitly given. -Mathematical modeling is needed.
Purpose of context-based task	Application	-The task is given after the explanation section.
	Modeling	-The task is given before the explanation section.
Type of information	Matching	-The task contains exactly the information needed to find the solution.
	Missing	-The task contains less information than needed so students need to find the missing information.
	Superfluous	-The task contains more information than needed so students need to select information.
Type of cognitive demand	Reproduction	-Reproducing representations, definitions, or facts. -Interpreting simple and familiar representations. -Memorization or performing explicit routine computations/procedures.
	Connection	-Integrating and connecting across content, situations, or representations. -Non-routine problem solving. -Interpretation of problem situations and mathematical statements. -Engaging in simple mathematical reasoning.
	Reflection	-Reflecting on, and gaining insight into, mathematics. -Constructing original mathematical approaches. -Communicating complex arguments and complex reasoning.

Textbook Analysis Framework Reproduced from Wijaya, 2015

Appendix E: Background Information

Neatly PRINT

Your name: ____

Cell phone #_____________________

Email: __________________________

Office Use Only:
Prof. _________
Room: _______
Mtg. times: _______________

Please check all that apply, as you feel comfortable.

1. My gender is: ____ male ____ female
2. My major is: ____ economics ____ business (e.g. marketing)

   ____ Other (please specify: _____________________)
3. I am a: ____ freshman ____ sophomore

   ____ junior ____ senior
4. Have you taken calculus before? ____ yes ____ no

   If **YES**, what year? ____________

   If **YES**, where? _____ high school _____ college
5. Have you taken AP economics before? ____ yes ____ no

   If **YES**, what year? ____________
6. What business or economics class (or classes) are you currently taking?

   ____
7. Does your experience in the business or economics class (or classes) that you are currently taking (or have taken in the past, if any) help you learn some of the concepts in the course? ____ yes ____ no

   If **YES**, please specify the concepts: _________________________________

8. Please indicate on the following table the days and times that would be best to meet with you:

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Morning							
Afternoon							
Evening							

Appendix F: Optimization and Marginal Change Examples Given in Course Lecture A

Optimization Examples

Example 1. For insurance purposes, a manufacturer plans to fence in a $10{,}800\ ft^2$ rectangular storage area adjacent to a building by using a building as one side of the enclosed area. The fencing parallel to the building faces a highway and will cost $3 per foot installed, whereas the fencing for the other two sides costs $2 per foot installed. Find the amount of each type of fence so that the total cost of the fence will be a minimum. What is the minimum cost?

Example 2. A manufacture's total cost function is given by

$$c = c(q) = 0.25q^2 + 3q + 400$$

where *c* is the total cost of producing q units. At what level of output will average cost per unit be a minimum? What is this minimum?

Marginal Change Examples

Example 1. Let $C(q) = 10{,}000 + \sqrt{q}$

a. Find the marginal cost function
b. Find the cost of [producing] 10,000 units
c. Find the marginal cost when 10,000 units are produced
d. Find the approximate cost for producing 10,001 units

Example 2. Interpret $C'(10.000) = 0.05$

Appendix G: Optimization and Marginal Change Examples Given in Course Lecture B

Optimization Examples

Example 1. Given the demand function $p = -7q + 350$ where q is quantity [number of units sold] and p is the price that consumers will pay when q units are demanded, we want to find the price that maximizes revenue. What's the maximum revenue?

Example 2. Given total cost $C = 0.06q^2 + 10q + 294$, find the quantity which minimizes average cost.

Example 3. Given the demand equation $p = -2q + 256$ and total cost

$C = 48q + 700$, find the output level (q) and price (p) which maximizes profit.

Marginal Change Examples

Example 1. $C = 0.2q^2 + 4q + 50$. Find C' [marginal cost function]

Example 2. $R = q(15 - \frac{q}{15})$. Find R' [marginal revenue function]

Example 3. $\bar{C} = 5 + \frac{2000}{q}$. Find C' [marginal cost function]

Example 4. What does $C'(10) = 6$ mean?

Example 5. What does $R'(10) = 4$ mean?

Appendix H: Textbook Optimization Examples and Assigned Practice Optimization Problems

Optimization Examples (Reproduced from Haeussler et al., 2011, pp 609-616)

Example 1. **(Minimizing the Cost of a Fence)** For insurance purposes, a manufacturer plans to fence in a $10{,}800\ ft^2$ rectangular storage area adjacent to a building by using a building as one side of the enclosed area. The fencing parallel to the building faces a highway and will cost $3 per foot installed, whereas the fencing for the other two sides costs $2 per foot installed. Find the amount of each type of fence so that the total cost of the fence will be a minimum. What is the minimum cost?

Example 2. **(Maximizing Revenue)** The demand equation a manufacturer's product is

$$p = \frac{80-q}{4} \quad 0 \leq q \leq 80$$

where q is the number of units and p is the price per unit. At what value of q will there be maximum revenue?

Example 3. **(Minimizing Average Cost)** A manufacture's total cost function is given by $c = c(q) = 0.06q^2 + 10q + 294$ where c is the total cost of producing q units. At what level of output will average cost per unit be a minimum? What is this minimum?

Example 4. **(Maximization Applied to Enzymes)** An Enzyme is a protein that acts as a catalyst for increasing the rate of a chemical reaction that occurs in cells. In a certain reaction, an enzyme is converted to another enzyme called the product. The product acts as a catalyst for its own formation. The rate R at which the product is formed (with respect to time) is given by

$$R = kp(l - p)$$

where l is the total initial amount of both enzymes, p is the amount of the product enzyme, and k is a positive constant. For what values of p will R be maximum?

Example 5. **(Economic Lot Size)** A company annually produces and sells 10,000 units of a product. Sales are uniformly distributed throughout the year. The company wishes to determine the number of units to be manufactured in each production run in order to minimize total annual setup costs and carrying costs. The same number of units is produced in each run. This number is referred to as the **economic lot size or economic order quantity.** The production cost of each unit is $20, and carrying costs (insurance, interest, storage, etc.) are estimated to be 10% of the value of the average inventory. Setup costs per production run are $40. Find the economic lot size.

Example 6. **(Maximizing TV Cable Company Revenue):** The Vista TV Co. currently has 100, 000 subscribers who are each paying a monthly rate of $40. A survey reveals that there will be 1000 more subscribers for each

$0.25 decrease in the rate. At what rate will maximum revenue be obtained, and how many subscribers will there be at this rate?

Example 7. **(Maximizing the Number of Recipients of Health-Care Benefits)** An article in a sociology journal stated that if a particular health-care program for the elderly were initiated, then t years after its start, n thousand elderly people would receive direct benefits, where

$$n = \frac{t^3}{3} - 6t^2 + 32t \quad 0 \leq t \leq 12$$

For what value of t does the maximum number receive benefits?

Example 8. **(Profit Maximization)** Suppose that the demand equation for a monopolist's product is $p = 400 - 2q$ and the average-cost function is $\bar{c} = 0.2q + 4 + \frac{400}{q}$, where q is the number of units, and both p and $\bar{c}$ are expressed in dollars per unit. Determine the level of output at which profit is maximized.

Assigned Practice Optimization Problems (Reproduced from Haeussler et al., 2011, pp 617-619) (Problems 1, 2, 3, 4, and 5 below are Problems 17, 19, 27, 32, and 33 in the textbook)

Problem 1. **(Profit)** For XYZ Manufacturing Co., total fixed costs are $1200, material and labor costs combined are $2 per unit, and the demand equation is

$$p = \frac{100}{\sqrt{q}}$$

What level of output will maximize profit? Show that this occurs when marginal revenue is equal to marginal cost. What is the price at profit maximization?

Problem 2. **(Revenue)** A TV cable company has 6400 subscribers who are each paying $24 per month. It can get 160 more subscribers for each $0.50 decrease in the monthly fee. What rate will yield maximum revenue, and what will this revenue be?

Problem 3. **(Profit)** The demand equation for a monopolist's product is

$$p = 600 - 2q$$

And the total-cost function is

$$c = 0.2q^2 + 28q + 200$$

Find the profit-maximizing output and price, and determine the corresponding profit. If the government were to impose a tax of $22 per unit on the manufacturer, what would be the new profit-maximizing output and price? What is the profit now?

Problem 4. **(Cost of Leasing Motor)** The Kiddie Toy Company plans to lease an electric motor that will be used 80,000 horsepower-hours per year in manufacturing. One horsepower-hour is the work done in 1 hour by a 1-horsepower motor. The annual cost to lease a suitable motor is $200, plus $0.40 per horsepower. The cost per horsepower-hour of operating the motor is $0.008/N, where N is the horsepower. What size motor, in horsepower, should be leased in order to minimize cost?

Problem 5. **(Transportation Cost)** The cost of operating a truck on a thruway (excluding the salary of the driver) is

$$0.165 + \frac{s}{200}$$

Dollars per mile, where s is the (steady) speed of the truck in miles per hour. The truck driver's salary is $18 per hour. At what speed should the truck driver operate the truck to make a 700-mile trip most economical?

Appendix I: Textbook Marginal Change Examples and Assigned Practice Marginal Change Problems

Marginal Change Examples (Reproduced from Haeussler et al., 2011, pp 513-514)
(Examples 1 and 2 below are Examples 7 and 8 in the textbook)

Example 1. **(Marginal cost)** If a manufacturer's average-cost function is

$$\bar{c} = 0.0001q^2 - 0.2q + 5 + \frac{5000}{q}$$

Find the marginal-cost function. What is the marginal cost when 50 units are produced?

Example 2. **(Marginal Revenue)** Suppose a manufacturer sells a product at $2 per unit. If q units are sold, the total revenue is given by

$$r = 2q$$

The marginal revenue function is

$$r'(q) = 2$$

which is a constant function. Thus, the marginal revenue is 2 regardless of the number of units sold. This is what we would expect, because the manufacturer receives $2 for each unit sold.

Assigned Practice Marginal Change Problems (Reproduced from Haeussler et al., 2011, p. 616)

(Problems 1, 2, 3, 4, and 5 below are problems 13, 19, 21, 23, and 30 in the textbook)

Problem 1. $c = 500 + 10q$ is the total cost of producing q units of a product. Find the marginal-cost function. What is the marginal cost when $q = 100$?

Problem 2. $\bar{c} = 0.01q + 5 + \frac{5000}{q}$ represents average cost per unit, which is a function of the number of units produced. Find the marginal-cost function and the marginal cost for $q = 100$.

Problem 3. $\bar{c} = 0.00002q^2 - 0.01q + 6 + \frac{20{,}000}{q}$ represents average cost per unit, which is a function of the number of units produced. Find the marginal-cost function and the marginal cost for the following values of q: $q = 100, q = 500$

Problem 4. $r = 0.8q$ represents total revenue and is a function of the number of units sold. Find the marginal-revenue function and the marginal revenue for the following values of q:
$q = 9, q = 300, q = 500$

Problem 5. (Depreciation) Under the straight-line method of depreciation, the value v of a certain machine after t years have elapsed is given by

$$v = 120{,}000 - 15{,}500t$$

where $0 \leq t \leq 6$. How fast is v changing with respect to t when $t = 2$? $t = 4$? at any time?

Appendix J: Relationship between Marginal Change and Profit

a. Determine the level of output at which profit is maximized.
b. Determine the price at which maximum profit occurs.
c. Determine the maximum profit.
d. If, as a regulatory device, the government imposes a tax of \$22 per unit on the monopolist, what is the new price for profit maximization?

Solution: We know that

$$\text{profit} = \text{total revenue} - \text{total cost}$$

Since total revenue r and total cost c are given by

$$r = pq = 400q - 2q^2$$

and

$$c = q\bar{c} = 0.2q^2 + 4q + 400$$

the profit is

$$P = r - c = 400q - 2q^2 - (0.2q^2 + 4q + 400)$$

so that

$$P(q) = 396q - 2.2q^2 - 400 \quad \text{for } q > 0 \tag{6}$$

a. To maximize profit, we set $dP/dq = 0$:

$$\frac{dP}{dq} = 396 - 4.4q = 0$$
$$q = 90$$

Now, $d^2P/dq^2 = -4.4$ is always negative, so it is negative at the critical value $q = 90$. By the second-derivative test, then, there is a relative maximum there. Since $q = 90$ is the only critical value on $(0, \infty)$, we must have an absolute maximum there.

b. The price at which maximum profit occurs is obtained by setting $q = 90$ in the demand equation:

$$p = 400 - 2(90) = 220$$

c. The maximum profit is obtained by evaluating $P(90)$. We have

$$P(90) = 396(90) - 2.2(90)^2 - 400 = 17{,}420$$

d. The tax of \$22 per unit means that for q units, the total cost increases by $22q$. The new cost function is $c_1 = 0.2q^2 + 4q + 400 + 22q$, and the new profit is given by

$$P_1 = 400q - 2q^2 - (0.2q^2 + 4q + 400 + 22q)$$
$$= 374q - 2.2q^2 - 400$$

Setting $dP_1/dq = 0$ gives

$$\frac{dP_1}{dq} = 374 - 4.4q = 0$$
$$q = 85$$

Since $d^2P_1/dq^2 = -4.4 < 0$, we conclude that; to maximize profit, the monopolist must restrict output to 85 units at a higher price of $p_1 = 400 - 2(85) = \$230$. Since this price is only \$10 more than before, only part of the tax has been shifted to the consumer, and the monopolist must bear the cost of the balance. The profit now is \$15,495, which is less than the former profit.

Now Work Problem 13 ◁

This discussion leads to the economic principle that when profit is maximum, marginal revenue is equal to marginal cost.

We conclude this section by using calculus to develop an important principle in economics. Suppose $p = f(q)$ is the demand function for a firm's product, where p is price per unit and q is the number of units produced and sold. Then the total revenue is given by $r = qp = qf(q)$, which is a function of q. Let the total cost of producing q units be given by the cost function $c = g(q)$. Thus, the total profit, which is total revenue minus total cost, is also a function of q, namely,

$$P(q) = r - c = qf(q) - g(q)$$

Let us consider the most profitable output for the firm. Ignoring special cases, we know that profit is maximized when $dP/dq = 0$ and $d^2P/dq^2 < 0$. We have

$$\frac{dP}{dq} = \frac{d}{dq}(r - c) = \frac{dr}{dq} - \frac{dc}{dq}$$

Consequently, $dP/dq = 0$ when

$$\frac{dr}{dq} = \frac{dc}{dq}$$

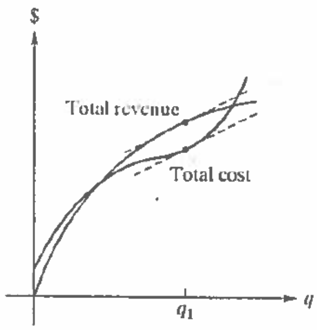


FIGURE 13.66 At maximum profit, marginal revenue equals marginal cost.

That is, at the level of maximum profit, the slope of the tangent to the total-revenue curve must equal the slope of the tangent to the total-cost curve (Figure 13.66). But dr/dq is the marginal revenue MR, and dc/dq is the marginal cost MC. Thus, under typical conditions, to maximize profit, it is necessary that

$$\text{MR} = \text{MC}$$

For this to indeed correspond to a maximum, it is necessary that $d^2P/dq^2 < 0$:

$$\frac{d^2P}{dq^2} = \frac{d^2}{dq^2}(r - c) = \frac{d^2r}{dq^2} - \frac{d^2c}{dq^2} < 0 \quad \text{equivalently} \quad \frac{d^2r}{dq^2} < \frac{d^2c}{dq^2}$$

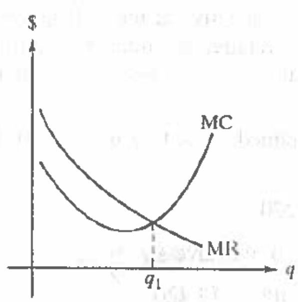


FIGURE 13.67 At maximum profit, the marginal-cost curve cuts the marginal-revenue curve from below.

That is, when MR = MC, in order to ensure maximum profit, the slope of the marginal-revenue curve must be less than the slope of the marginal-cost curve.

The condition that $d^2P/dq^2 < 0$ when $dP/dq = 0$ can be viewed another way. Equivalently, to have MR = MC correspond to a maximum, dP/dq must go from + to −; that is, it must go from $dr/dq - dc/dq > 0$ to $dr/dq - dc/dq < 0$. Hence, as output increases, we must have MR > MC and then MR < MC. This means that at the point q_1 of maximum profit, *the marginal-cost curve must cut the marginal-revenue curve from below* (Figure 13.67). For production up to q_1, the revenue from additional output would be greater than the cost of such output, and the total profit would increase. For output beyond q_1, MC > MR, and each unit of output would add more to total costs than to total revenue. Hence, total profits would decline.

References

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics*, *79*(2), 239-261.

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2013). A modeling perspective on interpreting rates of change in context. *Mathematical Thinking and Learning*, *15*(4), 314-336.

Barnes, H. (2004). Realistic mathematics education: Eliciting alternative mathematical conceptions of learners. *African Journal of Research in Mathematics, Science, and Technology Education*, *8*(1), 53-64.

Beichner, R. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics, 62*(8), 750–755.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior, 22*(4), 479-495.

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68*(1), 19-35.

Borasi, R. (1986). On the nature of problems. *Educational Studies in Mathematics*, *17*(2), 125-141.

Borgen, K. L., & Manu, S. S. (2002). What do students really understand? *The Journal of Mathematical Behavior, 21*(2), 151-165.

Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimisation problems in Calculus: A South African case study. *South African Journal of Education*, *33*(2), 1-18.

Butler, J. S., Finegan, T. A., & Siegfried, J. J. (1998). Does more calculus improve student learning in intermediate micro-and macroeconomic theory? *Journal of Applied Econometrics*, *13*(2), 185-202.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. *CBMS Issues in Mathematics Education*, *7*, 114-162.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Carlson, M., Larsen, S., & Lesh, R. A. (2003). Integrating a model and modeling perspective with existing research and practice. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 465–478). Mahwah, NJ: Lawrence Erlbaum Associates.

Carroll, J. (1963). A model of school learning. *Teachers College Record*, 64, 723–733.

Çetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *PRIMUS*, *19*(3), 232-244.

Charalambous, C. Y., Delaney, S., Hsu, H. Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117-151.

Christensen, W. M., & Thompson, J. R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics-Physics Education Research*, *8*(2), 1-5.

Cogan, L. S. & Schmidt, W. H. (2015). The concept of opportunity to learn (OTL) in international comparisons of education. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 207-216). New York: Springer.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, CA: Sage Publications.

Davis, J. D. (2007). Real-world contexts, multiple representations, student-invented terminology, and y-intercept. *Mathematical Thinking and Learning*, *9*(4), 387-418.

Dole, S., & Shield, M. J. (2008). The capacity of two Australian eighth-grade textbooks for promoting proportional reasoning. *Research in Mathematics Education, 10*(1), 19–35.

Dominguez, A. (2010). Single Solution, Multiple Perspectives. In Lesh, R., Galbraith, P. L., Haines, C. R., & Hurford, A. (Eds.), *Modeling Students' Mathematical Modeling Competencies: ICTMA 13* (pp. 223-233). New York: Springer.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In C. Jinfa & E. Knuth *Early algebraization: A global dialogue from multiple perspectives* (pp. 215-235). Heidelberg: Springer.

Floden, R. E. (2002). The measurement of opportunity to learn. In A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of* *educational achievement* (pp. 231–266). Washington, D.C.: National Academy Press.

Flynn, C. D., Davidson, C. I., & Dotger, S. (2014). Engineering student misconceptions about rate and accumulation processes. In *Proceedings of the Zone 1 Conference of the American Society for Engineering Education, Conference Proceedings.*

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, *1*(1), 3-8.

Freudenthal, H. (1993). Thoughts in teaching mechanics didactical phenomenology of the concept of force. *Educational Studies in Mathematics*, *25*(1&2), 71–87.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer Academic Publishers.

Gamer, B. E., & Gamer, L. E. (2001). Retention of concepts and skills in traditional and reformed applied calculus. *Mathematics Education Research Journal*, *13*(3), 165-184.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics*, 36-45.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Laurence Erlbaum Associates.

Gordon, S. P. (2008). What's wrong with college algebra? *Primus*, *18*(6), 516-541.

Gravemeijer, K. P. E. (1994). *Developing realistic mathematics education.* The Netherlands, Utrecht: Freudenthal Institute.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111-129.

Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, *25*(1), 57-72.

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning*, *12*(2), 152-176.

Haeussler, E. F., Paul, R. S., & Wood, R. J. (2011). *Introductory mathematical analysis for business, economics, and the life and social sciences* (13th ed). Upper Saddle River, NJ: Pearson Prentice Hall.

Hallett, D. H. (2006). What have we learned from calculus reform? The road to conceptual understanding. *Mathematical Association of America Notes*, *69*, 1-4.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education*, 3-25.

Herbert, S., & Pierce, R. (2008). An 'emergent model' for rate of change. *International Journal of Computers for Mathematical Learning*, *13*(3), 231-249.

Herbert, S., & Pierce, R. (2011). What is rate? Does context or representation matter? *Mathematics Education Research Journal*, *23*(4), 455-477.

Hughes-Hallett, D., Lock, P. F., Gleason, A. M., Flath, D. E., Lovelock, D., Quinney, D.,…Osgood, B. G. (2002). *Applied Calculus* (2nd ed.). Hoboken, NJ: Wiley.

Husén, T. (Ed.). (1967). *International study of achievement in mathematics: A comparison of twelve countries* (Vol. 2). New York: John Wiley & Sons.

Ibrahim, B., & Rebello, N. S. (2012). Representational task formats and problem solving strategies in kinematics and work. *Physical Review Special Topics-Physics Education Research*, *8*(1), 010126(1-19).

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, *31*(3), 313- 330.

Jones, D. L., & Tarr, J. E. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*, *6*(2), 4-27.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *Zentralblatt für Didaktik der Mathematik, 38*(3), 302-310.

Klymchuk, S., Zverkova, T., Gruenwald, N., & Sauerbier, G. (2010). University students' difficulties in solving application problems in calculus: Student perspectives. *Mathematics Education Research Journal*, *22*(1), 81–91.

Kolovou, A., van den Heuvel-Panhuizen, M., & Bakker, A. (2009). Non-routine problem solving tasks in primary school mathematics textbooks-a needle in a haystack. *Mediterranean Journal for Research in Mathematics Education,* 8(2), 31–68.

Lesh, R. A., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on* *mathematics problem solving, learning and teaching* (pp. 3–33). Mahwah, NJ: Lawrence Erlbaum Associates

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 763-804). Charlotte, NC: Information Age Publishing.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior, 21*(1), 87-116.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning, 14*(2), 85-119.

Maass, K. (2007). Modelling tasks for low achieving students—first results of an empirical study. In D. PittaPantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME 5* (pp. 2120–2129). Cyprus: Larnaca.

Maass, K. (2010). Classification scheme for modelling tasks. *Journal für Mathematik-Didaktik, 31*(2), 285–311.

Maharaj, A. (2013). An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education, 33*(1), 1-19.

Marrongelle, K. A. (2004). How students use physics to reason about calculus tasks. *School Science and Mathematics*, *104*(6), 258-272.

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, *81*(3), 365-384.

Mesa, V., Suh, H., Blake, T., & Whittemore, T. (2012). Examples in college algebra textbooks: Opportunities for students' learning. *Primus*, *23*(1), 76-105.

Michelsen, C. (2006). Functions: a modelling tool in mathematics and science. *Zentralblatt für Didaktik der Mathematik, 38*(3), 269-280.

Mkhatshwa, T. P. (2014). *Marginal analysis in the context of cost, revenue, and profit: An exploratory case study.* Unpublished manuscript, Department of Mathematics, Syracuse University, Syracuse, USA.

Mkhatshwa, T., & Doerr, H. M. (in press). Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study. *Conference on Research in Undergraduate Mathematics Education, the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: RUME.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes, Vol.25* (pp. 175-193). Washington, DC: Mathematical Association of America.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. *CBMS Issues in Mathematics Education*, *4*, 139-168.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors. http://www.corestandards.org/Math/

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235-250.

Pickle, M. C. C. (2012), *Statistical content in middle grades mathematics textbooks.* Unpublished doctoral dissertation, University of South Florida, USA.

Pollak, H. (1968). On some of the problems of teaching applications of mathematics. *Educational Studies in Mathematics, 1*(1), 24-30.

Prince, M., Vigeant, M., & Nottis, K. (2012). Development of the heat and energy concept inventory: Preliminary results on the prevalence and persistence of engineering students' misconceptions. *Journal of Engineering Education*, *101*(3), 412-438.

Rasmussen, C. L., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(2), 161-172.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, *37*, 388-420.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership*, *61*(5), 61-66.

Schmidt, W. H., & Burroughs, N. A. (2015). Puzzling out PISA: What can international comparisons tell us about American education? *American Educator*, *39*(1), 24-31.

Schmidt, W. H., Burroughs, N. A., Zoido, P., & Houang, R. T. (2015). The role of schooling in perpetuating educational inequality: An international perspective. *Educational Researcher*, *44*(7), 371-386.

Schoenfeld, A. H. (1995). A brief biography of calculus reform. *UME trends*, *6*(6), 3-5.

Smith, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York, NY: Lawrence Erlbaum Associates.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics*, *72*(3), 271–288.

Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* New York: Teachers College Press.

Stillman, G. (2012). Applications and modelling research in secondary classrooms: what have we learnt? In Cho, S. J. (Ed.), *The 12th International Congress on Mathematical Education* (pp. 902-921). Seoul, South Korea: Convention and Exhibition Center.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning*, *11*(4), 258–288.

Swanagan, B. S. (2012). *The impact of students' understanding of derivatives on their performance while solving optimization problems.* (Doctoral dissertation). University of Georgia, Georgia.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education*, *43*(3), 253-295.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic*. Unpublished manuscript, Center for Research in Mathematics & Science Education, San Diego State University, San Diego, USA.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of* *multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2), 229-274.

Thompson, P. W. (1994c). Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education*, *1*, 21-44.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation*, *31*(4), 315-327.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction-the Wiskobas Project*. The Netherlands, Dordrecht: Reidel.

Tsamir, P., & Ovodenko, R. (2013). University students' grasp of inflection points. *Educational Studies in Mathematics*, *83*(3), 409-427.

Tucker, A. C., & Leitzel, J. R. C. (Eds.), (1995). *Assessing calculus reform efforts.* Washington, DC: The Mathematical Association of America.

Ubuz, B., & Ersoy, Y. (1997). The effect of problem-solving method with handout material on achievement in solving max-min word problems. *The Journal of Mathematical Behavior, 16*(1), 75-85.

van den Heuvel-Panhuizen, M. (2000). Mathematics education in the Netherlands: A guided tour. *Freudenthal Institute Cd-rom for ICME9*. Utrecht: Utrecht University.

van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics, 25*(2), 2-9.

van den Heuvel-Panhuizen, M. (2010). Reform under attack--forty years of working on better mathematics education thrown on the scrapheap? no way!. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia*. Fremantle: MERGA.

Villegas, J. L., Castro, E., & Gutierrez, J. (2009). Representations in problem solving: A case study with optimization problems. *Electronic Journal of Research in Educational Psychology, 7*(1), 279–308.

Von Allmen, P. (1996). The effect of quantitative prerequisites on performance in intermediate microeconomics. *Journal of Education for Business*, *72*(1), 18-22.

Weber, E., & Dorko, A. (2014). Students' and experts' schemes for rate of change and its representations. *The Journal of Mathematical Behavior*, *34*, 14-32.

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics*, *87*(1), 67-85.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*, 79-95.

Wijaya, A. (2015). *Context-Based Mathematics Tasks in Indonesia: Toward Better Practice and Achievement*. Utrecht: FIsme Scientific Library.

Wilhelm, J. A., & Confrey, J. (2003). Projecting rate of change in the context of motion onto the context of money. *International Journal of Mathematical Education in Science and Technology*, *34*(6), 887-904.

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education*, *28*(4), 431-466.

Yerushalmy, M. (2000). Problem solving strategies and mathematical resources: A longitudinal view on problem solving in a function based approach to algebra. *Educational Studies in Mathematics*, *43*(2), 125-147.

Yerushalmy, M., & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representation environment. *Educational Studies in Mathematics*, *80*(3), 287-306.

Yoon, C., Dreyfus, T., & Thomas, M. (2010). How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity. *Mathematics Education Research Journal*, *22*(2), 141-157.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-127.

Zeytun, A. S., Cetinkaya, B., & Erbas, A. K. (2010). Mathematics teachers' covariational reasoning levels and predictions about students' covariational reasoning abilities. *Educational Sciences: Theory and Practice*, *10*(3), 1601-1612.

Zulkardi, Z. (1999). *How to design mathematics lessons based on the realistic approach.* Retrieved from http://www.fisme.science.uu.nl/en/wiki/index.php/Realistic_Mathematics_Education

VITA

Name of Author: Thembinkosi Peter Sydney Mkhatshwa

Degrees Awarded:

Master of Science in Teaching and Curriculum (Mathematics Education), 2013, Syracuse University, USA

Master of Arts in Mathematics, 2010, Marshall University, USA

Bachelor of Science in Mathematics & Computer Science, 2008, University of Swaziland, Swaziland

Teaching Experience:

Graduate Teaching Assistant, Syracuse University, 2012-2016

Graduate Teaching Assistant, Oklahoma State University, 2010-2012

Graduate Teaching Assistant, Marshall University, 2008-2010

Journal Publications:

Mkhatshwa, T., & Mummert, A. (2011). Modeling super-spreading events for infectious diseases: Case study SARS. *IAENG International Journal of Applied Mathematics, 41*(2), 82-88.

Conference Proceedings:

Mkhatshwa, T., & Doerr, H. M. (in press). Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study. *Conference on Research in Undergraduate Mathematics Education, the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: RUME.

Mkhatshwa, T. P., & Doerr, H. M. (2015). Students' understanding of marginal change in the context of cost, revenue, and profit. In Krainer, K., & N. Vondrová (Eds.), *Proceedings of the 9th Congress of European Research in Mathematics Education* (pp. 2201-2206). Prague: Czech Republic.

Mkhatshwa, T., & Doerr, H. (2015). Students' reasoning about marginal change in an economic context. In T. Fukawa-Connelly (Ed.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 772-773). Pittsburgh, PA: RUME.

Professional Development:

Future Professoriate Program (2013-2015), Syracuse University, USA