6-1-1992

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ILL–(TH)–92–7

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Gravity Coupled to Ising Matter

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ABSTRACT: We establish the phase diagram of three–dimensional quantum gravity coupled to Ising matter. We find that in the negative curvature phase of the quantum gravity there is no disordered phase for ferromagnetic Ising matter because the coordination number of the sites diverges. In the positive curvature phase of the quantum gravity there is evidence for two spin phases with a first order transition between them.

One of the results that propelled recent work on two dimensional quantum gravity was the discovery that dynamically triangulated random surfaces provide a convenient formulation of the theory and facilitate both analytic and numerical calculations. An analogous formulation of three dimensional quantum gravity has also been proposed [1] with some success [2]. This success is non–trivial because the proof for two dimensions that the number of triangulations is exponentially bounded as a function of the number of triangles does not carry over to three dimensions. In two dimensions, the classification of topologies in terms of genus is crucial in proving the bound. In three dimensions, there is no comparable topological classification scheme. Without the exponential bound, the naive result is to have factorial growth. This would mean that no chemical potential term could control the theory. However, numerical experiments suggest that there is an exponential bound, at least when the topology is restricted to S^3 [3].

Another difference in three dimensions is the need for a second chemical potential. In two dimensions, the Euler characteristic, χ, is

$$N_0 - N_1 + N_2 = \chi \tag{1}$$

(where N_i is the number of i–simplices). Furthermore, each triangle has three links and each link is shared by two triangles implying

$$3N_2 = 2N_1 \tag{2}$$

The most general action of the form

$$S = \sum g_i N_i \tag{3}$$

therefore has only one independent term. This can be taken to be a cosmological constant determining the mean number of triangles. In two dimensions it is further possible to restrict ourselves to a "microcanonical" ensemble which eliminates even this term. The corresponding equations in three dimensions are (for $\chi = 0$, which will be assumed from here on)

$$N_0 - N_1 + N_2 - N_3 = 0 \tag{4}$$

and

$$4N_3 = 2N_2 \tag{5}$$

leaving two independent terms in an action of the form (3). This action can be chosen to be

$$S = \alpha N_0 - \beta N_3 \tag{6}$$

so that β controls the number of tetrahedra (the volume) and α controls the number of nodes. In the continuum, β corresponds to the cosmological constant while α is related to the gravitational constant and hence the mean curvature. Unfortunately, there are no proofs of ergodicity for a microcanonical ensemble in three dimensions, so we are forced to utilize a canonical ensemble for our simulations. Of course, the existence of a continuum limit is a major issue. The question is whether or not there is a second order phase transition in the (α, β) plane (or in some larger space if actions of a more general form than (3) are considered). The coupling β must be adjusted so that the volume approaches infinity. This leaves α for our attention. It is reported that for $\alpha < \alpha^*$ (where $\alpha^* \sim$

$3.75 - 4.0$), the theory is in a negative curvature phase while for $\alpha > \alpha^*$ the mean curvature is positive [4]. Simulations suggest that the transition between them may be first order [5] implying that no continuum limit is available. It has been speculated that this negative result is related to the topological nature of pure three dimensional gravity in which there is no massless graviton.

Matter can be coupled to the gravity by placing it at the nodes (for instance) and adding an appropriate term to the action. In two dimensions, there is some understanding of the effects of coupling quantum gravity to matter, at least when the central charge is less than one. Knihznik, Polyakov, and Zamalodchikov (KPZ) [6] have argued that when the central charge is less than one, the scaling dimensions of the matter fields are simply "dressed" by quantum gravity. On the lattice this means that for a critical point of some matter theory on a fixed lattice, there is a corresponding second order phase transition for the matter on a dynamical lattice, but with different critical exponents. KPZ give an explicit relation for the two sets of scaling dimensions. In particular, the Ising model has a continuous phase transition when coupled to quantum gravity with exponents different from the usual ones. In this paper, we couple the Ising model to three dimensional quantum gravity [7]. It is possible that within this enlarged parameter space we may be able to find new critical regions where a continuum limit could be taken. There is no analog of the KPZ framework to guide us, so, clearly, a central issue is whether or not an Ising transition continues to exist and, if so, what its order is.

We introduce the matter by simply decorating the nodes of each diagram with spins, adding an Ising term to the action with coupling k, and tracing over spins $t_i = \pm 1$. Thus our total partition function Z is given by

$$Z\left(\alpha, \beta, k\right) = \sum_{\mathrm{T}, \emptyset = 0} \sum_{t_i} \exp\left(\alpha N_0 - \beta N_3 + k \sum_{<ij>} t_i t_j\right) \tag{7}$$

There are two important relations between the partition function of this new theory and the partition function of pure gravity. The first is due to the fact that the trace over spins contributes a factor 2^{N_0} even if the Ising coupling is zero:

$$Z(\alpha, \beta, k = 0) = Z_{pure\ grav}(\alpha + \ln 2, \beta) \tag{8}$$

In other words, in a grand canonical ensemble, the Ising spins renormalize the gravitational couplings even at zero coupling. For the new action, this implies that the curvature transition is in the range $\alpha^* \sim 3.06 - 3.31$. The second relation is due to the fact that once the spins are completely ordered, the Ising term just contributes a factor kN_1 to the action.

$$Z(\alpha,\beta,k) \sim Z_{pure\ grav}(\alpha+k,\beta-k) \qquad (\text{for k such that } <tt'>\sim N_1) \tag{9}$$

One implication of this last result is that starting with any α in the negative curvature phase, large enough k will take the system into the positive curvature phase. In the discussion that follows we always assume that β has been adjusted to fix the mean volume associated with the tetrahedra. The continuum limit will finally be taken by letting this mean value tend to infinity. This leaves us with a two dimensional phase diagram. (This can be done because the empirical exponential bound for the pure gravity implies an exponential bound in the presence of matter).

Before proceeding with a numerical analysis of the phase diagram it is useful to consider the Ising model in mean field theory on *quenched* lattices of the type found in the pure gravity theory. This will give a kind of zeroth order approximation to the phase diagram and will help direct the numerical work.

Take N_0 and N_3 to be fixed and make the standard mean field approximation

$$\sum tt' = q <t> \sum t \tag{10}$$

where tt' is the Ising term, $<t>$ is a spontaneous magnetization, and q is the average coordination number. As usual we may add an external magnetic field to facilitate the computation of expectation values. The partition function now reads

$$Z = (\text{const})\ \text{Tr}\ \exp([h+kq<t>]\sum t) \tag{11}$$

where h is the magnetic field and k is the Ising coupling. This is now a non-interacting system so the trace can be done to get

$$Z = (\text{const})\ (2\ \cosh[h+kq<t>])^{N_0} \tag{12}$$

The average spin can be calculated from this and is

$$<t> = \frac{1}{N_0}\frac{d\ln Z}{dh} = \tanh(h+kq<t>) \tag{13}$$

For $h=0$ and small $<t>$ (i.e. near the critical point) $<t> = kq<t>$, so the critical value of the Ising coupling is

$$k = 1/q. \tag{14}$$

Equations (10–14) are standard. What is new is the effect of the background lattice via the coordination number q. Each link contributes to the coordination number of two sites, so

$$2N_1 = N_0 q \tag{15}$$

Equations (14) and (15), together with $N_1 = N_0 + N_3$ (from (4) and (5)), give

$$k = \frac{1}{2}\frac{1}{(1 + (N_3/N_0))} \tag{16}$$

Now the problem boils down to the ratio of N_3/N_0 as N_3 goes to infinity. Ref. [4] finds that for large N_3, $< N_0 >= N_3^{(\delta)}$ with $\delta < 1$ in the negative curvature phase and $\delta = 1$ in the positive curvature phase. When $\delta < 1$, N_3/N_0 goes to infinity (which just means the average coordination number goes to infinity) and the critical coupling is at zero. This is what happens at all points in the negative curvature phase. When $\delta = 1$, i.e. $< N_0 >= N_3/c$, we get

$$k = \frac{1}{2}\frac{1}{(1 + c)} \tag{17}$$

Ref. [4] gets (in pure gravity) for $\alpha = 4$, $c = 3.368$. This gives $k = 0.114$ (which turns out to be very close to our numerical result).

This mean field argument suggests the phase diagram of fig. 1. The dashed line is just eqn. (9) with the knowledge that the negative curvature phase orders the spins for arbitrarily small couplings. This line then separates the negative curvature phase from the positive curvature phase. The thick line is equations (16) and (17) with input about the relationship between N_0 and N_3 in the two phases. Since mean field theory is an expansion in $1/q$, when $q = \infty$ it should be exact. In the negative curvature phase, therefore, the Ising spins order for arbitrarily small positive values of the Ising coupling. The only part of the phase diagram in question is the positive curvature phase part of the Ising line. We address this question numerically.

The gravity sector of the theory is simulated using update moves that have been shown to be ergodic [8] and have become standard. Barycentric subdivision, also called the (1,4) move, takes a tetrahedra, inserts a node in the middle and connects this node to the other four vertices creating a total of four tetrahedra. The analog of the bond flip in two–dimensions, here the (2,3) move, removes the triangle shared by two tetrahedra, introduces a link between the two nodes that are not shared and places three new triangles around this new link (one for each of the shared nodes) in such a way that there are now three tetrahedra.

There is a fine tuning problem associated with this theory. If the number of simplicial manifolds regarded as a function of the number of tetrahedra behaves as

$$\#(N_3) \sim (N_3)^a e^{cN_3} \tag{18}$$

with a negative (such as happens for two dimensions), then the grand canonical partition function is dominated by large volumes if the chemical potential is slightly too small and by small volumes if the chemical potential is slightly too large. In numerical simulations this manifests itself as the tendency for the system to evolve to one of the two extremes no matter how carefully the couplings are adjusted. Several approaches to this difficulty have been used. One group confined their simulations to a range of N_3 [4]. We prefer to retain an action formulation of the problem and add a gaussian term to the action as proposed in [2] (except that we do not constrain N_0):

$$\delta S = -\gamma(N_3 - M_3)^2 \tag{19}$$

The desired volume M_3 is put in by hand. The coupling β is then adjusted so that $< N_3 >= M_3$. The new coupling γ is made small so that it does not influence the physics of interest. This new term in the action effectively ensures that the probability distribution for N_3 is approximately gaussian about the mean value M_3, with a characteristic width determined by γ. The length of the simulation is then tuned with γ so that the system has time to wander back and forth across the distribution many times. Otherwise, the system may not reach equilibrium. In practice, $\gamma = 0.005$ is a practical value.

With the moves and action now defined, it is important that detailed balance must be built into the update probabilities [9]. For pure gravity, two lists are made, one with all sets of two or three tetrahedra for which the 23 or 32 move is allowed and another with all sets of one or four tetrahedra for which the 14 or 41 move is allowed. Each list consists of items in one of two possible states which we will call A and B. There is some number, n, of items in the list. Picking a state at random and updating it produces a modified list with n' items. Detailed balance is maintained if the updating probability is

$$P(A \to B) = \frac{1}{1 + \frac{n'}{n} e^{S_A - S_B}} \tag{20}$$

Adding the Ising matter requires some modifications. Except for its effect on the action, no modification is necessary for the 23 or 32 moves. However, the update probabilities must be modified for the 14 and 41 moves because the list now has three possible states (no node, node spin up, node spin down) rather than two. Call the spin states $B1$ and $B2$. A suitable modification of the update probabilities is

$$
\begin{aligned}
P(A \to B1) &= \frac{n e^{S_{B1}}}{n' e^{S_A} + n(e^{S_{B1}} + e^{S_{B2}})} \\
P(B1 \to A) &= \frac{n' e^{S_A}}{n' e^{S_A} + n(e^{S_{B1}} + e^{S_{B2}})} \\
P(B1 \to B2) &= \frac{n e^{S_{B2}}}{n' e^{S_A} + n(e^{S_{B1}} + e^{S_{B2}})}
\end{aligned} \tag{21}
$$

and three more expressions with 1 and 2 interchanged. It is most convenient to consider the total probability of introducing a new node, $P(A \to B1) + P(A \to B2)$, and to determine the spin value (using Metropolis) after deciding whether or not to put in the new node.

Of course, there are other ways to maintain detailed balance. One can choose from lists of moves which are not necessarily allowed and then test for this during the accept/reject step. In the approach above, it is most time consuming to predict the number of elements that will be in the list of allowed (23) moves if a contemplated (23) move is made. It is more efficient to postpone part of this calculation to the accept/reject step (the part that checks whether or not a candidate simplex already exists or not) while keeping the simpler restrictions on the list (such as requiring links to have the correct coordination number).

Calculations were performed for values of α in the range 3.5 to 4.0 and for values of the Ising coupling between 0 and 0.2. As k is varied in this range the nearest neighbor spin expectation value varies from 0 to around 0.7 (i.e. from complete disorder to fairly ordered). The Ising component of the specific heat was used to search for a possible spin ordering transition and the Ferrenberg–Swendsen approach was used to cover a range of k (and sometimes α) from a single simulation. At $\alpha = 3.5$, on small lattices (up to $N_3 = 1600$), no peak was observed in the specific heat as a function of the Ising coupling and correlation times remained small. Instead, the specific heat was observed to rise to a plateau as k was increased. For large k, nearest neighbor spins are strongly correlated and the fluctuation in the number of nodes contributes significantly to the specific heat. This is the origin of the plateau. On larger lattices, $N_3 = 3200$ and $N_3 = 6400$, a peak appears above the plateau value and grows (along with an increased correlation time) suggesting the existence of a phase transition. Figure 2 shows the peak just beginning to appear for 3200 tetrahedra.

Very long runs (over half a million sweeps) coupled with the Ferrenberg–Swensen method allowed us to calculate the kurtosis ($1 - < E^4 > /3 < E^2 >^2$ where $E = k \sum tt$) of the distribution of the spin correlation for couplings near that of the specific heat peak [11]. Tables 1 and 2 give a sample of the results. The minimum value was near 0.48 for both volumes while the value at the peak of the specific heat increased with increasing volume. The most naive extrapolation to infinite volume still leaves the kurtosis significantly below its gaussian value (2/3) indicating a first order transition. The peak of the specific heat itself roughly doubles as the volume is doubled, another hint that the transition is first order.

What happens as α is increased? For significantly greater values of α we were unable to find a peak rising above the plateau. It is possible that the first order transition ends and is replaced with a smooth crossover behavior, but it appears more likely that it is simply necessary to go to larger lattices in order to continue to see the transition.

In conclusion, the ferromagnetic sector of the phase diagram of three dimensional quantum gravity coupled to Ising spins appears to have three components (see figure 1). Phase 1 has ordered Ising spins and negative curvature. Phase 2 has ordered Ising spins and positive curvature. Phase 3 has disordered Ising spins and positive curvature. Near the gravitational transition, the transition from phase 2 to phase 3 appears to be first order. For larger α the nature of the transition is uncertain. Further calculations on larger lattices are needed to resolve this question and to verify the order of the transition.

This work was supported, in part, by NSF grant PHY 87–01775. Some calculations were performed on the Florida State University Cray Y–MP and at the National Center for Supercomputer Applications in Illinois. Also, we acknowledge National Science Foundation Support through the Materials Research Laboratory at the University of Illinois, grant NSF–DMR–20538.

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FIGURE CAPTIONS

[1] The phase diagram suggested by quenched mean field theory.

[2] The specific heat, C, versus the Ising coupling for 3200 tetrahedra.

k	C	κ
0.110	0.125	0.653
0.115	0.135	0.566
0.118	0.136	0.495
0.120	0.136	0.479
0.122	0.135	0.495
0.125	0.132	0.548
0.129	0.125	0.612

Table 1. Ferrenberg–Swendsen estimates of the specific heat, C, and kurtosis, κ, as a function of Ising coupling for 3200 tetrahedra. The actual coupling for the run was 0.120.

k	C	κ
0.0790	0.029	0.653
0.0820	0.062	0.568
0.0844	0.154	0.474
0.0866	0.248	0.557
0.0900	0.124	0.613
0.0920	0.092	0.608
0.0960	0.079	0.631

Table 2. Ferrenberg–Swendsen estimates of the specific heat, C, and kurtosis, κ, as a function of Ising coupling for 6400 tetrahedra. The actual coupling for the run was 0.090.