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Simulations of dynamically triangulated gravity  
– an algorithm for arbitrary dimension

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**Abstract**

Recent models for discrete euclidean quantum gravity incorporate a sum over simplicial triangulations. We describe an algorithm for simulating such models in arbitrary dimension. As illustration we show results from simulations in four dimensions.

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# Introduction

In recent years there has been considerable interest in studying statistical systems whose partition functions incorporate a sum over simplicial triangulations. Initially, efforts focused on two dimensional models which were proposed as discrete regularisations for string theory out of the critical dimension [1, 2, 3]. Plausibility arguments were given to suggest that the sum over lattices, in some scaling limit, would generate the effects of a nonperturbative inclusion of fluctuations in the worldsheet geometry. Later, a framework was developed [4, 5] which, in certain simple cases, allowed many of the important features of a system coupled to two dimensional gravity to be derived from a knowledge of the theory in flat space.

The results of these continuum analyses were in complete agreement with calculations and numerical simulations of the triangulated models [6, 7, 8, 9, 10, 11] and lent strong support to the lattice prescription. The continuum methods used to analyse the simple models (including pure two dimensional gravity) appear to break down for strings in physical dimensions. This has motivated a variety of numerical studies of the discrete models (which are well defined everywhere) with some interesting results [12, 13, 14, 15].

In addition, the basic idea of summing over simplicial triangulations to generate a path integral for quantum gravity has been extended to three [16, 17, 18, 19] and four dimensions [20, 21, 22, 23, 24]. Whilst these initial simulations are rather exploratory, the results for four dimensions are particularly exciting, as they seem to hint at a nonperturbative fixed point in the quantum theory. It is possible that the problems associated with the nonrenormalisability of Einstein gravity might be evaded in any continuum theory constructed in the vicinity of this new fixed point.

In this paper we present an algorithm for Monte Carlo simulation of these dynamically triangulated models, which is constructed in such a way as to make trivial the dependence of the code on the manifold dimension  $d$ .

## Model

We will be considering the problem of estimating a partition function of the form

$$Z(\kappa_0, \kappa_d) = \sum_{T(S^d)} e^{-S(\kappa_0, \kappa_d, T)} \quad (1)$$

The summation goes over all simplicial triangulations  $T$  of the sphere in  $d$  dimensions  $S^d$ . A simplicial triangulation is specified by a set of  $d$ -simplices (sets of  $d + 1$  labelled points) which are associated uniquely in pairs via their  $d - 1$ -dimensional faces, in such a way that the neighbourhood of any point is homeomorphic to a  $d$ -dimensional ball. In two dimensions  $d = 2$ , the fundamental building blocks are 2-simplices (triangles) which are glued together along their 1-dimensional faces (links) in such a way that two points (vertices) are connected by at most one link and the end points of all links are distinct. Similarly, a three dimensional simplicial manifold is built out of 3-simplices (tetrahedra) such that a given 2-dimensional face (triangle) is associated with exactly two tetrahedra. The restriction to manifolds ensures that every subsimplex is nondegenerate and unique.

Analogously, in four dimensions, the triangulation consists of hypertetrahedra associated in pairs via their tetrahedral faces. Again, the manifold condition effectively eliminates any degeneracies in the subsimplices.

The action  $S$  for dimensions  $d \leq 4$  can be taken to depend on only two coupling constants  $\kappa_0, \kappa_d$  related to the bare Newton and cosmological constants respectively. They are conjugate to the total number of  $d$ -simplices  $N_d$  and points  $N_0$  (0-simplices) for a given triangulation  $T$ .

$$S(\kappa_0, \kappa_d) = \kappa_d N_d - \kappa_0 N_0 \quad (2)$$

The numerical evaluation of this partition function  $Z$  (and expectation values computed from it) is effected by a Monte Carlo procedure which generates a random walk in the space of all such triangulations by a sequence of local ‘moves’ or deformations. The ones commonly used correspond to the replacement of an  $i$ -dimensional subsimplex (i.e a subset of  $i + 1$  points within a simplex) by its ‘dual’  $d - i$  subsimplex (see, for example, [25]). In order that this move preserve the manifold structure of the triangulation, there will be an associated change in the number and identity of neighbouring simplices and subsimplices. These moves have been shown to be ergodic (at least when  $d \leq 4$ ) in [25]. The latter statement implies that, at least in principle, a set of such moves can transform any such triangulation into any other of the same Euler character  $\chi$ . The topological invariant  $\chi$  is defined for triangulations as

$$\chi = \sum_{i=0}^d (-1)^i N_i \quad (3)$$

However, in the case of  $d = 4$ , it appears that the typical number of such moves may increase very rapidly with volume (number of  $d$ -simplices). This may place important constraints on the ‘practical’ ergodicity of the numerical simulations [26]. Indeed, there is some recent numerical evidence that in this case the triangulation space may grow factorially with volume [27].

Clearly, there are  $d + 1$  possible moves which may be labeled by the dimension of the subsimplex  $i$  central to the move. The definition of the  $i$ -move requires that the order of a given  $i$ -subsimplex (the number of  $d$ -simplices associated with it) be exactly  $d + 1 - i$ . Such a subsimplex will be referred to as a *legal* subsimplex. Subsequent upon finding such a legal subsimplex it is necessary to test whether substituting it by its ‘dual’ will lead to a bona fide triangulation which satisfies the manifold restriction. Such a move is referred to as geometrically allowed. Finally, for such a geometrically allowed move, the change in the action is computed and the update subjected to a metropolis test. Together with an explicit detailed balance condition, this procedure, under repeated iteration, will guarantee that the configurations approach a static distribution governed by the Boltzmann weight. The details of this algorithm are given in the next section.

## Algorithm

The recipe for generating legal moves is as follows. Select a move type  $i$  at random. Then choose a simplex and one of its  $i$ -subsimplices ( $(i + 1)$  vertex labels) also at random. Using

a local procedure find the order of this subsimplex (i.e the number of simplices to which it is associated). If  $O(i) \neq (d+1-i)$  go back and select another move type.

The details of the neighbour search are organised as follows. Denote the labels of the  $(d+1)$  points making up a simplex containing the subsimplex  $i$  in question by  $a_0 \dots a_d$ . Examine all neighbour simplices which are associated with this simplex by any face containing the  $i$ -subsimplex. There are  $d-i$  of these and each contains one vertex which is not in the original simplex. If the move is to be ‘legal’ (i.e the subsimplex  $i$  is of order  $d+1-i$ ) then this extra vertex must be the same in all these  $d-i$  cases and can be denoted  $a_{d+1}$ . The slight exception to this picture corresponds to barycentric node insertion ( $i=d$ ) where the extra vertex is a new label and no searching is required.

It is now convenient to relabel the  $d+2$  vertices central to the move in such a way that the  $i+1$  points that define the subsimplex are arranged from  $a_0$  to  $a_i$ , the content of the  $i$ -move may then be seen from the following construction

$$\overbrace{a_0 \dots a_i} a_{i+1} \dots a_d a_{d+1} \rightarrow a_0 \dots a_i \overbrace{a_{i+1} \dots a_d a_{d+1}} \quad (4)$$

The  $d+1-i$  initial state simplices (the lefthand side of this equation) are constructed by pairing the common subsimplex vertices (indicated by the brace) with  $d-i$  selected from the  $d-i+1$  other vertices. The final state is now gotten by identifying the points  $a_{i+1} \dots a_d a_{d+1}$  as the new common subsimplex vertices (as indicated by the shift of the brace). The vertices needed to make up the  $i+1$  final state simplices are just  $i$  selected from the  $i+1$  remaining  $a_0 \dots a_i$ .

In order that the new simplicial complex still corresponds to a triangulation of a manifold, it is necessary to check that the potential new simplices and subsimplices introduced by such a move are not already present in the triangulation. In effect, this means that the extra vertex  $a_{d+1}$  must not already exist in any simplex associated to the subset  $a_{i+1} \dots a_d$ . To check for this a local search is carried out on all simplices which contain this subset. The nearby simplices are explored by moving out on faces containing this subset, with simplices being flagged and removed from the search list when they have been examined once.

Once this manifold condition has been checked, the update is treated by the usual Metropolis test and the triangulation updated if necessary. In order that the simulation produce the correct Boltzmann probability density we have chosen to incorporate strict detailed balance into the algorithm (see [23]). Denoting the probability of transition between one state or triangulation  $\alpha$  to another  $\beta$  via some subsimplex move  $i$  by  $\tau(i, \alpha, \beta)$ , detailed balance requires

$$P(\alpha) \tau(i, \alpha, \beta) = P(\beta) \tau(d-i, \beta, \alpha) \quad (5)$$

$P(\alpha)$  is the usual factor

$$P(\alpha) = e^{-S(\alpha)} \quad (6)$$

In practice, the transition probability factors into a product of probabilities to select the initial and final states –  $\eta(i, \alpha) \phi(d-i, \beta)$ , together with a piece  $t(i, \alpha, \beta)$  dependent on the change in action  $S$ . Here, a choice of initial state produces a unique final state so  $\phi = 1$ .

In practice, an attempted update starts with a random selection of the move type  $i$  followed by a random selection of a simplex. The probability then of selecting a given  $i$ -subsimplex is then  $\frac{O(i)}{N_d(\alpha)}$  where  $O(i)$  is the order of the subsimplex in the triangulation  $\alpha$ . For a legal move  $O(i) = (d + 1 - i)$ . If the  $i + 1$  vertices are then drawn at random from this simplex, the total probability of selection is

$$\eta(i, \alpha) = \frac{1}{d + 1} \frac{1}{N_d(\alpha)} \frac{(d + 1 - i)}{\binom{d+1}{i+1}} \quad (7)$$

It is elementary to then see that the inverse move  $\eta(d - i, \beta)$  differs only by the number of simplices  $N_d(\beta)$ . Thus eqn. 5 reads

$$e^{-S(\alpha)} \frac{1}{N_d(\alpha)} t(i, \alpha, \beta) = e^{-S(\beta)} \frac{1}{N_d(\beta)} t(d - i, \beta, \alpha) \quad (8)$$

This relation is then satisfied by choosing the reduced transition matrix  $t(i, \alpha, \beta)$  to have the simple form

$$t(i, \alpha, \beta) = \frac{1}{1 + \left(1 + \frac{(2i-d)}{N_d(\alpha)}\right) e^{S(\beta)-S(\alpha)}} \quad (9)$$

The change in action only depends on the order  $i$  of the subsimplex move (since the total change in the number of simplices is  $2i - d$ ).

$$S(\beta) - S(\alpha) = \kappa_d(2i - d) - \kappa_0(\delta_{i,d} - \delta_{i,0}) + \gamma(2i - d)(2(N_d - V) + 2i - d) \quad (10)$$

The supplementary term with coefficient  $\gamma$  acts to control the volume fluctuations so that with a tuning of  $\kappa_d$  we can simulate a quasi microcanonical ensemble of fixed volume  $V$ . Typically the coupling  $\gamma$  is taken small  $\gamma = 0.005$ . We have verified that expectation values computed at small  $\gamma$  are independent of  $\gamma$  but possess statistical errors that grow as  $\gamma \rightarrow 0$ .

To simulate a lattice with volume  $V$  we tune the bare cosmological coupling  $\kappa_d$  during equilibration according to a formula which follows from steepest descent evaluation of the partition function

$$\delta\kappa_d = 2\gamma(\langle N_d \rangle - V) \quad (11)$$

## Data structures and practical considerations

The code is written in C in order to handle dynamic memory allocation. A structure of type `SIMPLEX` is defined which contains both an array of labels for its vertices and an array of pointers to the neighbour simplices. The pointer to a neighbour simplex is stored with a local array index identical to the vertex which does not appear in the face separating the two simplices. In addition each simplex contains a logical flag which may be set and unset during simplex searching operations to prevent a given simplex being used more than once. Finally the sum of its labels is also stored, as this allows for a fast calculation of the opposing vertex of a new simplex neighbour across a given face [23].

Sequences of functions allocate and delete simplices dynamically and update the pointer fields of simplices neighbour to a move. To handle the node insertion and deletion moves, a stack of ‘used’ vertex labels is maintained. If a node insertion is attempted, the new label is drawn from the top of this stack, unless the stack is empty in which case the total node number is incremented. Conversely deleted nodes are placed on the stack. The stack itself is managed as a linked list.

The total storage is of order  $4(2d + 12)V$  bytes for a  $V$  simplex simulation. This equates to approximately 0.6 Mbyte for an 8000 simplex lattice in four dimensions.

The update time for  $V = 8000$ ,  $d = 4$  at  $\kappa_0 = 0.0$  is of order 4000 microseconds per *accepted* elementary move on a HP-735 workstation. One Monte Carlo sweep is defined as  $V$  attempted, legal subsimplex moves. At  $V = 8000$  our sweep time is 1.4 secs – this equates to 180 microseconds per attempted move and hence our average acceptance rate is approximately 5%. Sweep times decrease monotonically with increasing  $\kappa_0$  due to the decreased connectivity of the lattice yielding significantly faster local search times. At  $\kappa_0 = 2.4$  (close to criticality for  $V = 8000$ ) the sweep time is just 0.9 secs with a correspondingly smaller update time per accepted move.

The CPU time per attempted move increases with volume in  $d = 4$  and for small  $\kappa_0$ : for  $V = 4000$  it is 120 microsec, at  $V = 8000$  it is 180 microsec and for  $V = 16000$  it has reached 230 microsec. However for sufficiently large  $\kappa_0$  it is essentially constant.

Fig. 1 illustrates a typical execution profile for the code with  $d = 4$  and  $V = 8000$ , giving the percentage CPU time spent in the most important routines which are labelled according to

1. Searching for legal subsimplices
2. Checking the geometric constraints
3. Computing the metropolis test
4. Updating the lattice structures

Clearly, the program is dominated by the searching required to check that a proposed move does not violate the geometric restriction to manifolds.

## Characteristic output

Fig. 2 is a histogram illustrating  $A(i)$  the number of accepted moves of type  $i$  per sweep for a four dimensional lattice of volume  $V = 8000$  at zero node coupling. The manifest symmetry about  $i = d/2$  is a crude check of detailed balance – there are as many moves of type  $i$  as inverse moves of type  $d - i$ .

For the same run, fig. 3 shows  $L(i)$  the average number of legal subsimplices of type  $i$  encountered per sweep. Clearly, by the definition of the triangulated manifold, 3 and 4 subsimplices are always legal, whilst at this value of  $\kappa_0 = 0$  there are relatively few legal nodes and links (i.e 5-fold coordinated vertices and 4-fold coordinated links). This is to be contrasted with fig. 4 where the same quantity is plotted for  $\kappa_0 = 2.4$ . The number of legal nodes has increased by nearly a factor of five.

In fig. 5 we show the fraction of legal subsimplices  $P(i)$  for which an update would lead to an geometrically acceptable triangulation. Again the lattices are four dimensional with mean volume  $V = 8000$  at zero node coupling. In the case of move 0 (node deletion) this is possible with unit probability  $P(0) = 1$ , but for subsequent moves  $P(i)$  decreases until for moves of type 4 (node insertion) it again reaches unity.

As evidence of the two phase structure mentioned in the introduction we show in fig. 6 a plot of the node susceptibility  $\chi$  as a function of node coupling  $\kappa_0$  and for a variety of four dimensional lattice volumes  $V = 500 - 8000$ .

$$\chi = \frac{1}{V} (\langle N_0^2 \rangle - \langle N_0 \rangle^2) \quad (12)$$

There appears to be a growing peak which shifts and narrows with increasing volume. For conventional statistical mechanical systems this would be taken as an indicator of a phase transition. This quantity is sensitive to the presence of long range correlations in the geometric curvature. The node coupling being (inversely) related to a bare Newton gravitational constant, this is taken as evidence that there may be a nontrivial fixed point in the theory about which it may be possible to have a consistent quantum theory of euclidean gravity.

## Conclusions

We have described, in some detail, an algorithm to simulate models for quantum gravity based on dynamical triangulations. We have demonstrated that the necessary procedures can be implemented in such a way that their dependence on dimension is trivial - a single compact code can be written in which the dimension is simply input as a parameter.

Furthermore, we have illustrated the utility of such a code by recording the results of some high statistics studies of the four dimensional theory. Numerical evidence is presented to support a phase transition.

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## References

- [1] D. Boulatov, V. Kazakov, I. Kostov and A. Migdal, Nucl. Phys. B275 (1986) 641.
- [2] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433.
- [3] F. David, Nucl. Phys. B257 (1985) 543.
- [4] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.
- [5] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.
- [6] M. Agishtein and A. Migdal, Nucl. Phys. B350 (1991) 690.
- [7] J. Jurkiewicz, A. Krzywicki, B. Petersson and B. Soderberg, Phys. Lett. B213 (1988) 511.
- [8] S. Catterall, J. Kogut and R. Renken, Phys. Rev. D45 (1992) 2957.
- [9] C. Baillie and D. Johnston, Mod. Phys. Lett. A7 (1992) 1519.
- [10] V. Kazakov, Phys. Lett. B119 (1986) 140.
- [11] V. Kazakov and A. Migdal, Nucl. Phys. B311 (1989) 171.
- [12] S. Catterall, J. Kogut and R. Renken, Phys. Lett. B292 (1992) 277.
- [13] C. Baillie and D. Johnston, Phys. Lett. B286 (1992) 44.
- [14] G. Thorleifsson, Lattice 92, Nucl. Phys. B (Proc. Suppl.) in press.
- [15] S. Catterall, R. Renken and J. Kogut, ILL-(TH)-93-5, Nucl. Phys. B in press.
- [16] N. Godfrey and M. Gross, Phys. Rev. D43 (1991) R1749.
- [17] J. Ambjørn and S. Varsted, Nucl. Phys. B373 (1992) 557.
- [18] M. Agistein and A. Migdal, Mod. Phys. Lett. A6 (1991) 1863.
- [19] D. Boulatov and A. Krzywicki, Mod. Phys. Lett. A6 (1991) 3005.
- [20] J. Ambjørn and J. Jurkiewicz, Phys. Lett. B278 (1992) 42.
- [21] S. Varsted, UCSD/PTH 92/03.
- [22] M. Agistein and A. Migdal, Mod. Phys. Lett. A7 (1992) 1039., Nucl. Phys. B385 (1992) 395.
- [23] B. Brugmann and E. Marinari, Phys. Rev. Lett. 70 (1993) 1908.
- [24] S. Catterall, J. Kogut and R. Renken, CERN-TH.7149/94, Phys. Lett. B in press.
- [25] M. Gross and S. Varsted, Nucl. Phys. B378 (1992) 367.

[26] A. Nabutovsky and R. Ben-Av, PUPT-1327.

[27] S. Catterall, J. Kogut and R. Renken, CERN-TH.7197/94, Phys. Rev. Lett in press.

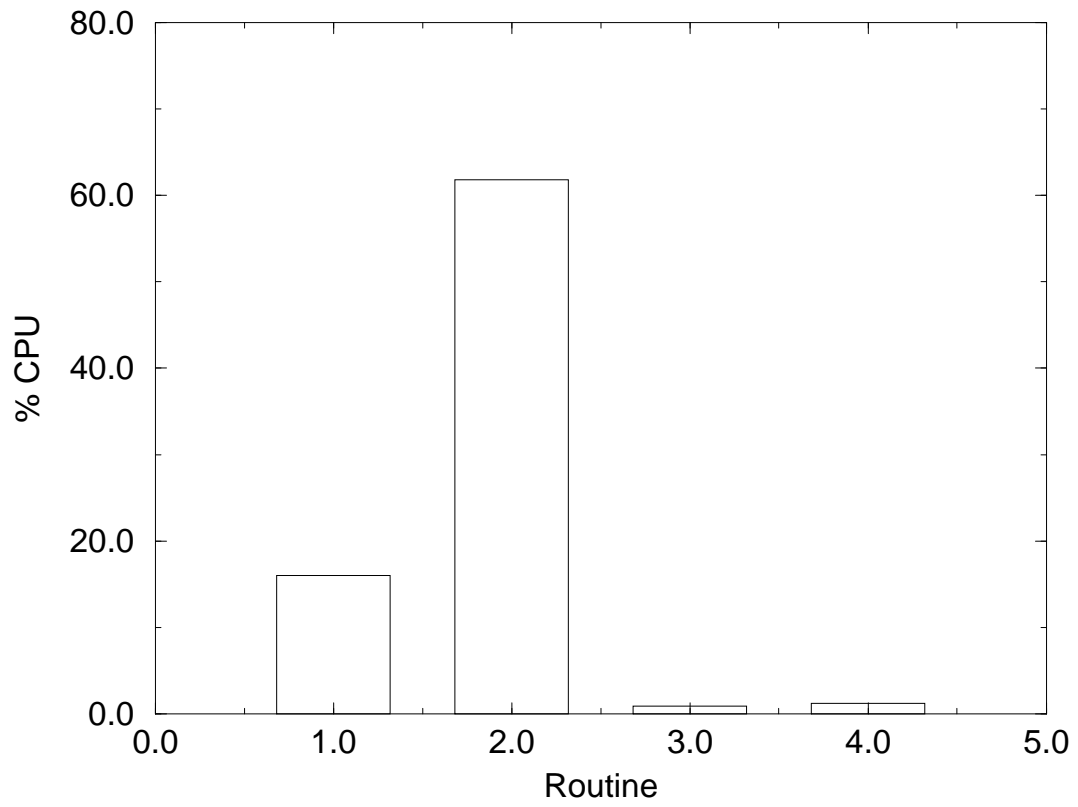


Figure 1: Execution profile  $d = 4, V = 8000, \kappa_0 = 0$

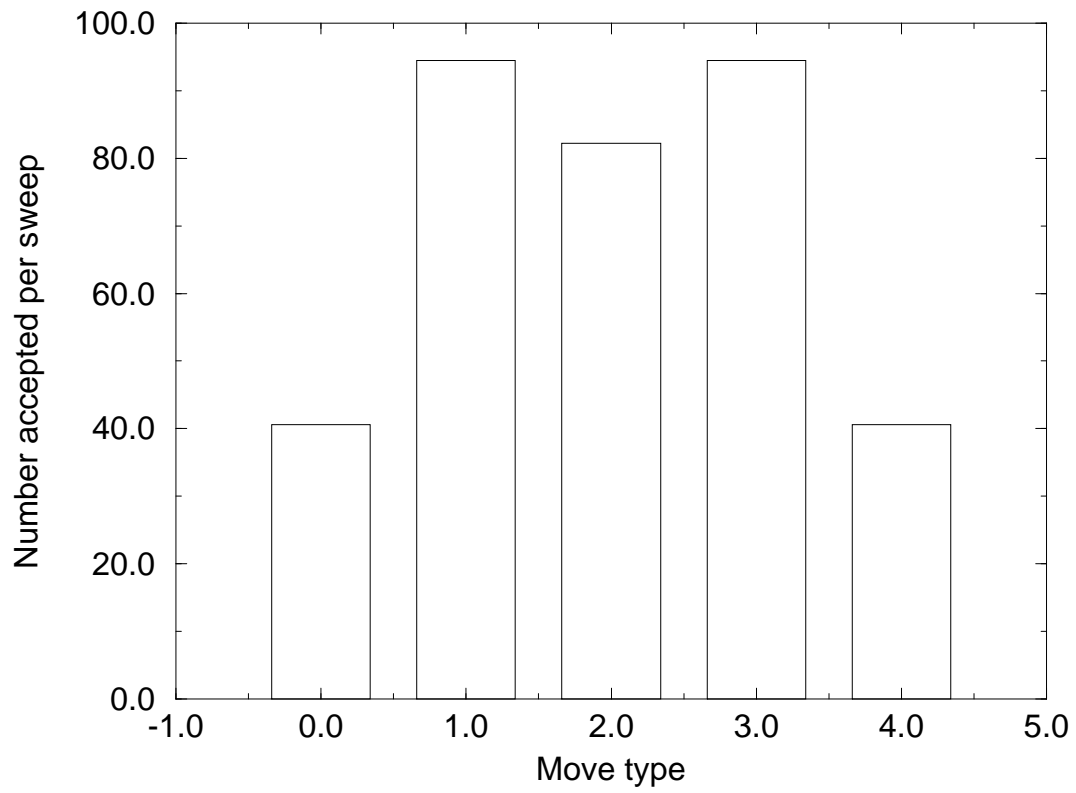


Figure 2: Number accepted i-moves per sweep  $d = 4$ ,  $V = 8000$ ,  $\kappa_0 = 0$

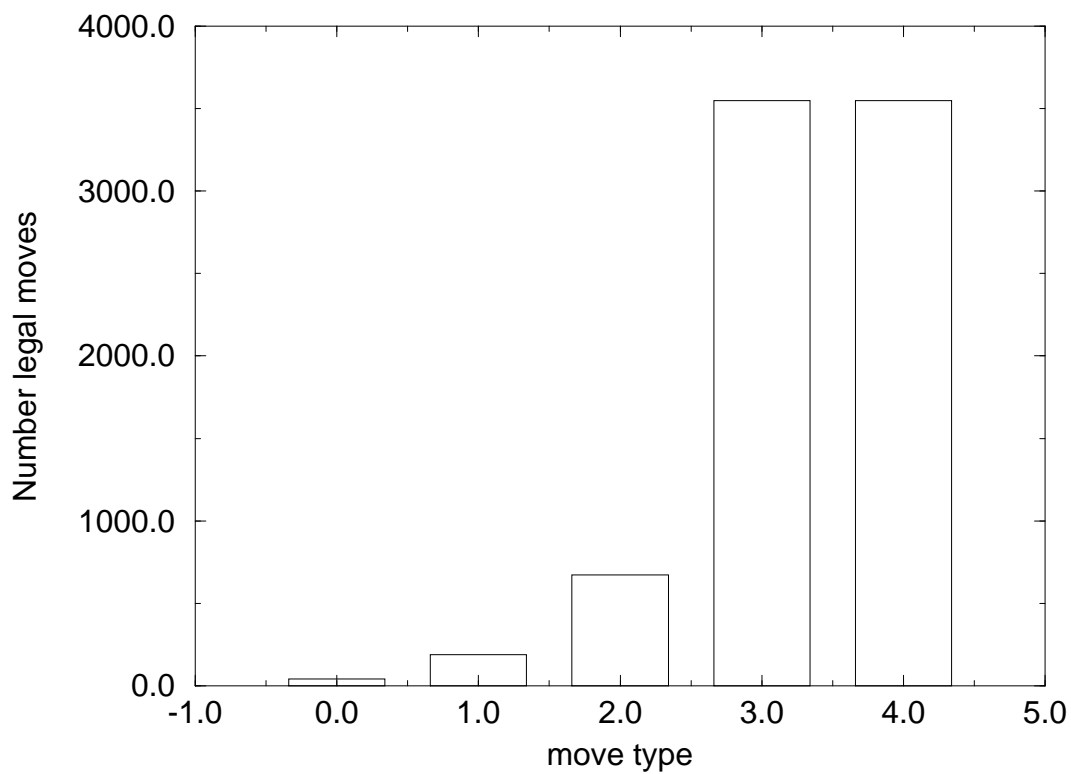


Figure 3: Number legal subsimplices per sweep  $d = 4$ ,  $V = 8000$ ,  $\kappa_0 = 0$

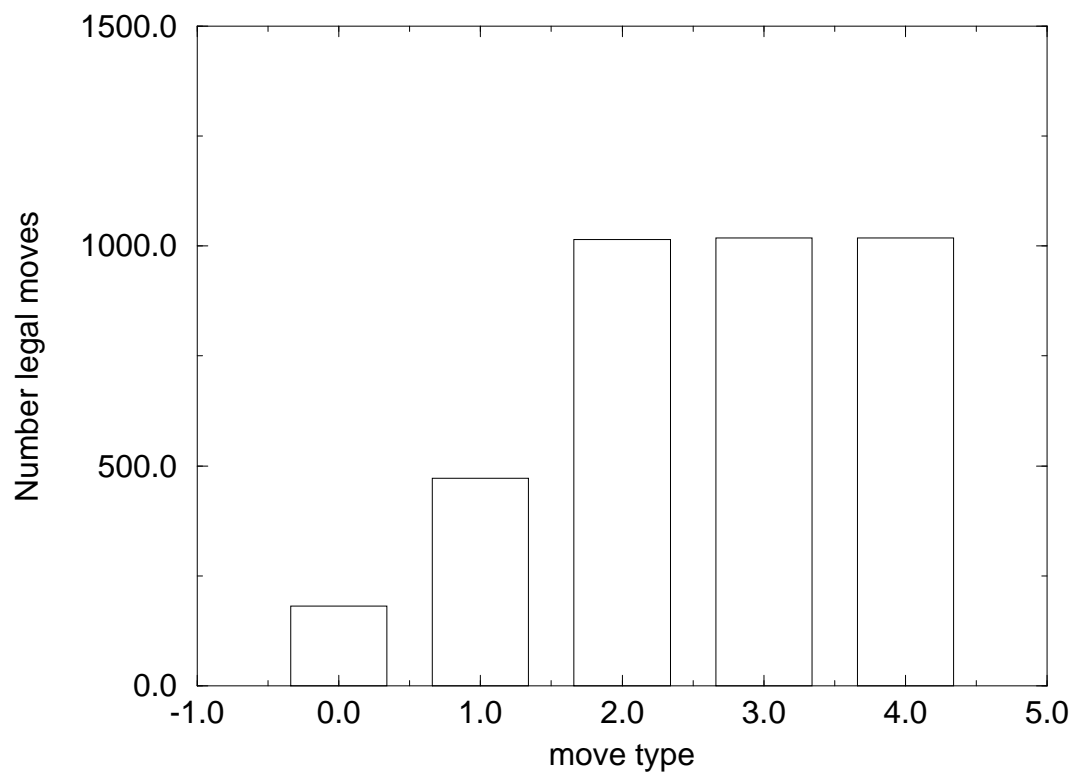


Figure 4: Number legal subsimplices per sweep  $d = 4$ ,  $V = 8000$ ,  $\kappa_0 = 2.4$

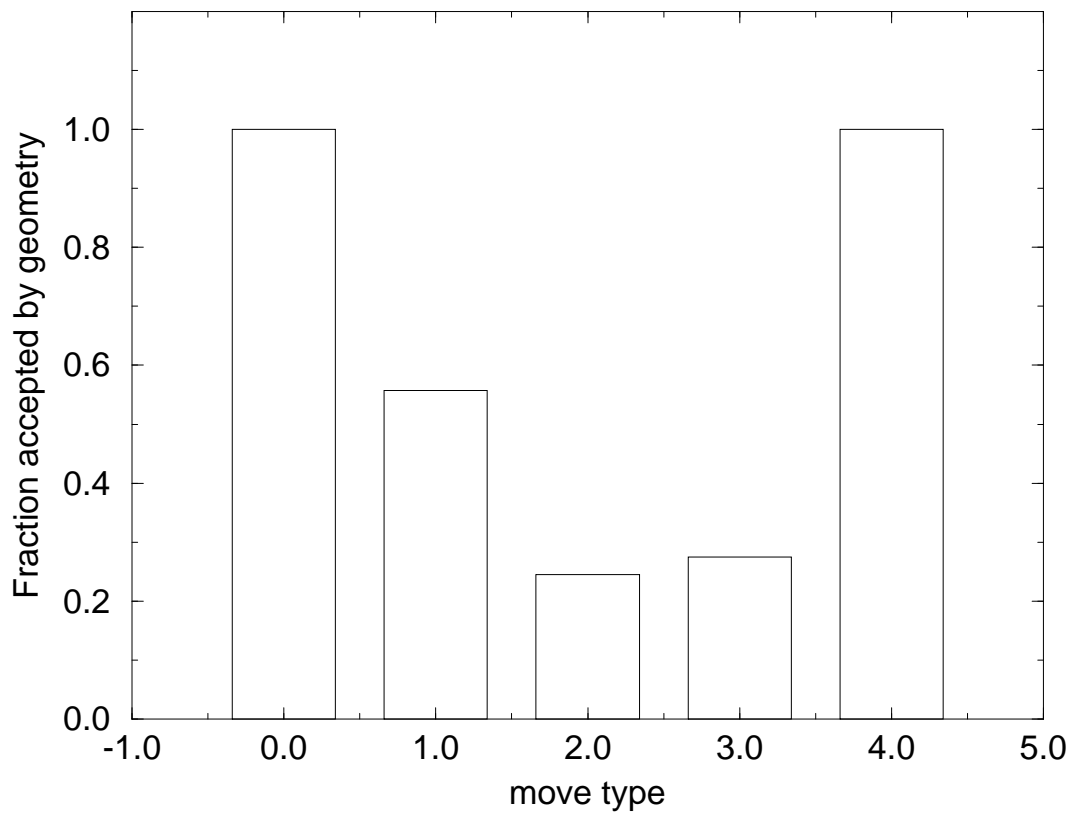


Figure 5: Fraction legal i-moves allowed geometrically,  $d = 4$ ,  $V = 8000$

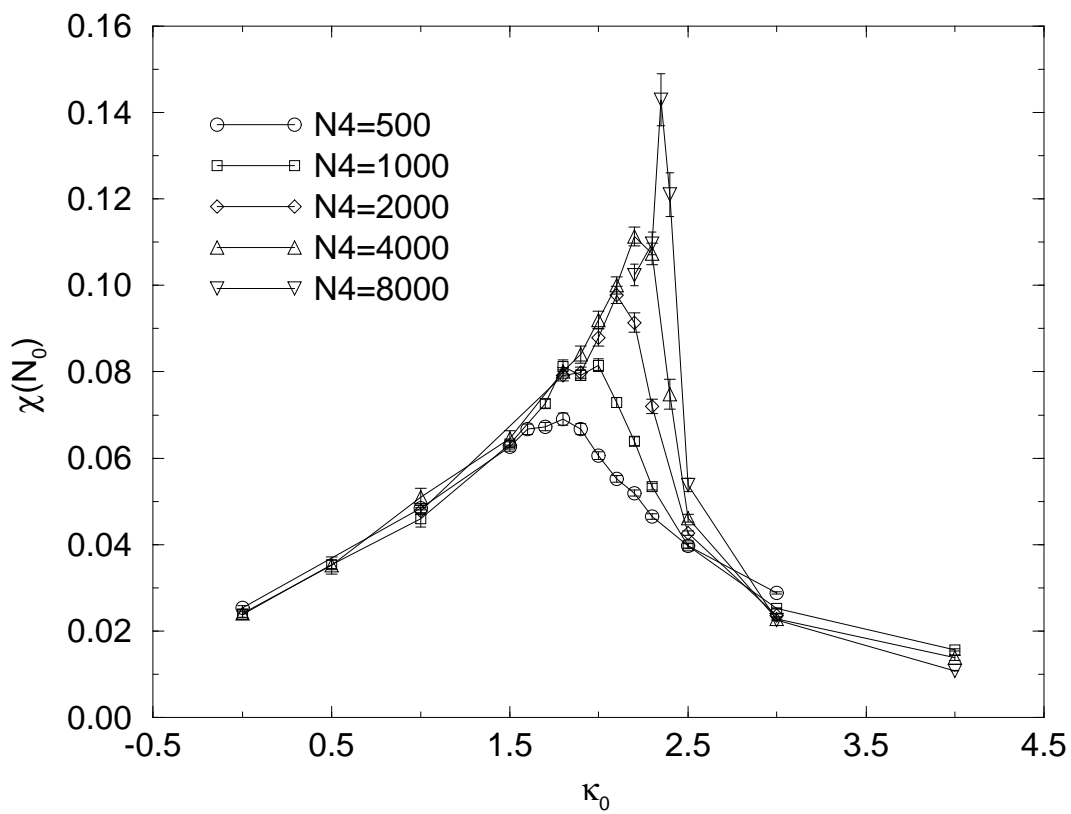


Figure 6: Node susceptibility  $d = 4$