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Entropy and the Approach to the Thermodynamic Limit in Three-Dimensional Simplicial Gravity

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Abstract

We present numerical results supporting the existence of an exponential bound in the dynamical triangulation model of three-dimensional quantum gravity. Both the critical coupling and various other quantities show a slow power law approach to the infinite volume limit.

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Introduction

Much interest has been generated recently in lattice models for euclidean quantum gravity based on dynamical triangulations [1, 2, 3, 4, 5, 6, 7, 8]. The study of these models was prompted by the success of the same approach in the case of two dimensions, see for example [9]. The primary input to these models is the ansatz that the partition function describing the fluctuations of a continuum geometry can be approximated by performing a weighted sum over all simplicial manifolds or triangulations T.

$$Z = \sum_{T} \rho\left(T\right) \tag{1}$$

In all the work conducted so far the topology of the lattice has been restricted to the sphere S^d. The weight function $\rho\left(T\right)$ is taken to be of the form

$$\rho\left(T\right) = e^{-\kappa_d N_d + \kappa_0 N_0} \tag{2}$$

The coupling κ_d represents a bare lattice cosmological constant conjugate to the total volume (number of d-simplices N_d) whilst κ_0 plays the role of a bare Newton constant coupled to the total number of nodes N_0.

We can rewrite eqn. 1 by introducing the entropy function $\Omega_d\left(N_d, \kappa_0\right)$ which counts the number of triangulations with volume N_d weighted by the node term. This the primary object of interest in this note.

$$Z = \sum_{N_d} \Omega_d\left(N_d, \kappa_0\right) e^{-\kappa_d N_d} \tag{3}$$

For this partition sum to exist it is crucial that the entropy function Ω_d increase no faster than exponentially with volume. For two dimensions this is known [10] but until recently the only evidence for this in higher dimensions came from numerical simulation. Indeed for four dimensions there is some uncertainty in the status of this bound [11, 12, 8].

With this in mind we have conducted a high statistics study of the three dimensional model at $\kappa_0 = 0$, extending the simulations reported in [13] by an order of magnitude in lattice volume. While in the course of preparing this manuscript we received a paper [14] in which an argument for the bound in three dimensions is given. Whilst we observe a rather slow approach to the asymptotic, large volume limit, our results are entirely consistent with the existence of such a bound. However, the predicted variation of the mean node number with volume is not seen, rather the data supports a rather slow power law approach to the infinite volume limit.

If we write $\Omega_3\left(N_3\right)$ as

$$\Omega_3\left(N_3\right) = a e^{\kappa_3^c\left(N_3\right) N_3} \tag{4}$$

the effective critical cosmological constant κ_3^c is taken dependent on the volume and a bound implies that $\kappa_3^c \rightarrow \text{const} < \infty$ as $N_3 \rightarrow \infty$. In contrast for a model where the entropy grew more rapidly than exponentially κ_3^c would diverge in the thermodynamic limit.

To control the volume fluctuations we add a further term to the action of the form $\delta S = \gamma\left(N_3 - V\right)^2$. Lattices with $N_3 \sim V$ are distributed according to the correct Boltzmann weight up to correction terms of order $O\left(\frac{1}{\sqrt{\gamma V}}\right)$ where we use $\gamma = 0.005$ in all our runs. This error is much smaller than our statistical errors and can hence be neglected.

Likewise, as a first approximation, we can set κ_3^c equal to its value at the mean of the volume distribution V which allows us to compute the expectation value of the volume exactly since the resultant integral is now a simple gaussian. We obtain

$$\langle N_3 \rangle = \frac{1}{2\gamma}\left(\kappa_3^3\left(V\right) - \kappa_3\right) + V \tag{5}$$

Equally, by measuring the mean volume $\langle N_3 \rangle$ for a given input value of the coupling κ_3 we can estimate $\kappa_3^c\left(V\right)$ for a set of mean volumes V. The algorithm we use to generate a Monte Carlo sample of three dimensional lattices is described in [15]. We have simulated systems with volumes up to 128000 3-simplices and using up to 400000 MC sweeps (a sweep is defined as V *attempted* elementary updates of the triangulation where V is the average volume).

Our results for $\kappa_3^c\left(V\right)$, computed this way, are shown in fig. 1 as a function of $\ln V$. The choice of the latter scale is particularly apt as the presence of a factorial growth in Ω_3 would be signaled by a logarithmic component to the effective $\kappa_3^c\left(V\right)$. As the plot indicates there is no evidence for this. Indeed, the best fit we could make corresponds to a *convergent* power law

$$\kappa_3^c\left(V\right) = \kappa_3^c\left(\infty\right) + aV^{-\delta} \tag{6}$$

If we fit all of our data we obtain best fit parameters $\kappa_3^c\left(\infty\right) = 2.087(5)$, $a = -3.29(8)$ and $\delta = 0.290(5)$ with a corresponding χ^2 per degree of freedom $\chi^2 = 1.3$ at 22% confidence (solid line shown). Leaving off the smallest lattice $V = 500$ yields a statistically consistent fit with an even better $\chi^2 = 1.1$ at 38% confidence. We have further tested the stability of this fit by dropping either the small volume data ($V = 500 - 2000$ inclusive), the large volume data ($V = 64000 - 128000$ inclusive) or intermediate volumes ($V = 8000 - 24000$). In each of these cases the fits were good and yielded fit parameters consistent with our quoted best fit to all the data. Furthermore, these numbers are consistent with the earlier study [13]. We are thus confident that this power law is empirically a very reasonable parameterisation of the approach to the thermodynamic limit. Certainly, our conclusions must be that the numerical data *strongly* favour the existence of a bound.

One might object that the formula used to compute κ_3^c is only approximate (we have neglected the variation of the critical coupling over the range of fluctuation of the volumes). This, in turn might yield finite volume corrections which are misleading. To check for this we have extracted κ_3^c directly from the measured distribution of 3-volumes $Q\left(N_3\right)$. To do this we computed a new histogram $P\left(N_3\right)$

$$P\left(N_3\right) = Q\left(N_3\right) e^{\kappa_3 N_3 + \gamma\left(N_3 - V\right)^2} \tag{7}$$

As an example we show in fig. 2 the logarithm of this quantity as a function of volume for $V = 64000$. The gradient of the straight line fit shown is an unbiased estimator of the critical coupling $\kappa_3^c\left(64000\right)$. The value of 1.9516(10) compares very favourably with the value $\kappa_3^c(64000) = 1.9522(12)$ obtained using eqn. 5. Indeed, this might have been anticipated since we might expect corrections to eqn. 5 to be of magnitude $O\left(V^{-(1+\delta)}\right)$ which even for the smallest volumes used in this study is again much smaller than our statistical errors.

In addition to supplying a proof of the exponential bound in [14] Boulatov also conjectures a relation between the mean node number and volume in the crumpled phase of the model (which includes our node coupling $\kappa_0 = 0$). This has the form

$$\langle N_0/V \rangle = c1 + c2\frac{\ln(V)}{V} \tag{8}$$

Our data for this quantity are shown in fig. 3. Whilst it appears that the mean coordination may indeed plateau for large volumes the approach to this limit seems not to be governed by the corrections envisaged in eqn. 8 – it is simply impossible to fit the results of the simulation with this functional form. Indeed, the best fit we could obtain corresponds again to a simple converging power with small exponent $\langle N_0/V \rangle \sim b + cV^{-d}$. The fit shown corresponds to using all lattices with volume $V \geq 8000$ and yields $b = 0.0045(1)$, $c = 1.14(2)$ and power $d = 0.380(3)$ ($\chi^2 = 1.6$). Fits to subsets of the large volume data yield consistent results.

Finally, we show in fig. 4, a plot of the mean intrinsic size of the ensemble of simplicial graphs versus their volume. This quantity is just the average geodesic distance (in units where the edge lengths are all unity) between two randomly picked sites. The solid line is an empirical fit of the form

$$L_3 = e + f\,(\ln V)^g \tag{9}$$

Clearly, the behaviour is close to logarithmic (as appears also to be the case in four dimensions [7]), the exponent $g = 1.047(3)$ from fitting all the data ($\chi^2 = 1.7$ per degree of freedom). This is indicative of the crumpled nature of the typical simplicial manifolds dominating the partition function at this node coupling. It is tempting to speculate that the true behaviour is simply logarithmic and the deviation we are seeing is due to residual finite volume effects.

Alternatively, we can fit the data as a linear combination of the form

$$L_3 = e + f\ln V + g\ln\ln V \tag{10}$$

This gives a competitive fit with $e = -1.45(4)$, $f = 1.438(4)$ and $g = -0.55(3)$ with $\chi^2 = 1.6$. One might be tempted to favour this fit on the grounds that it avoids the problem of a power close to but distinct from unity. However, the situation must remain ambiguous without further theoretical insight.

To summarise this brief note we have obtained numerical results consistent with the existence of an exponential bound in a dynamical triangulation model of three dimensional quantum gravity. Thus, practical numerical studies can reveal the bound argued for in [14]. Our results also favour the existence of a finite (albeit large ~ 200) mean coordination number in the infinite volume limit in the crumpled phase. However, the nature of the finite volume corrections to the latter appear very different from those proposed in [14]. Indeed, both for the critical coupling and mean coordination number we observe large power law corrections with small exponent. Finally, we show data for the scaling of the mean intrinsic extent with volume which suggests a very large (possible infinite) fractal dimension for the typical simplicial manifolds studied.

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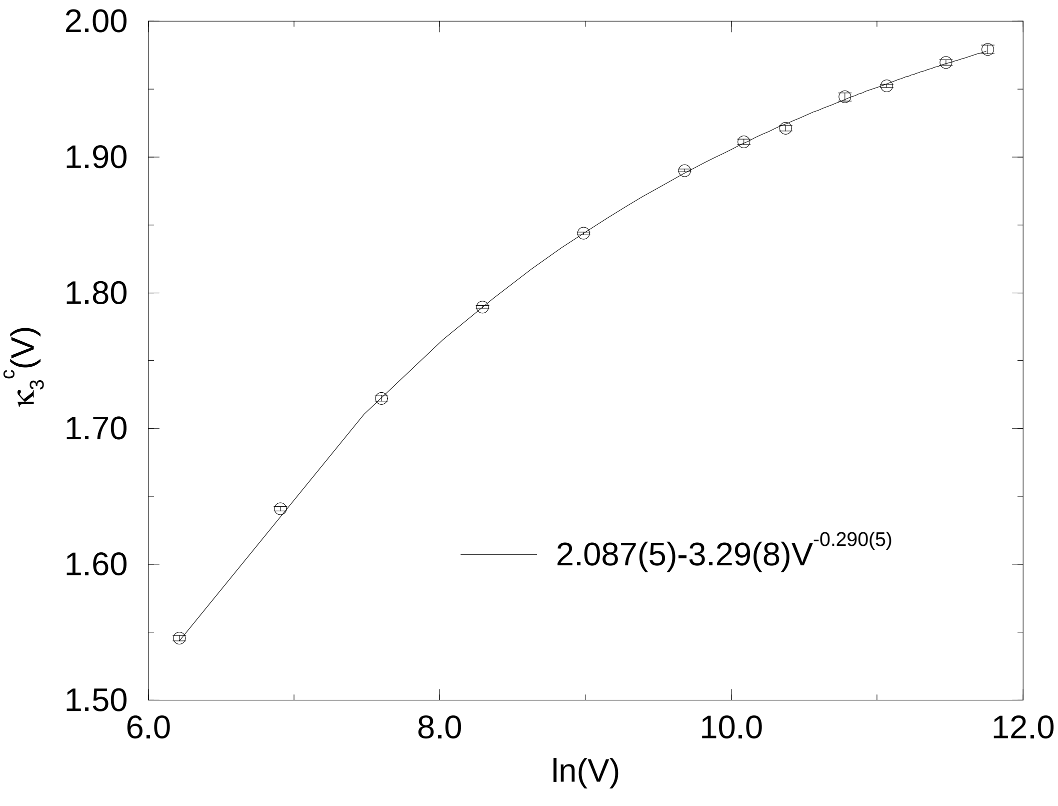

Figure 1: Critical coupling vs volume

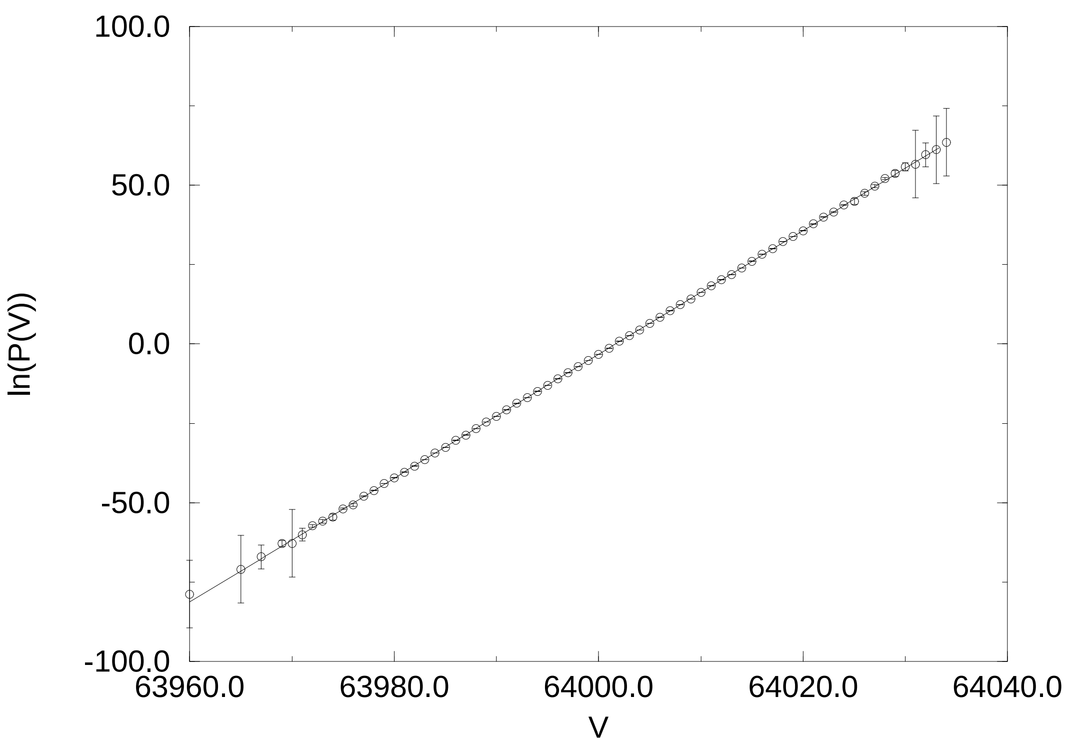

Figure 2: Modified distribution of 3-volumes

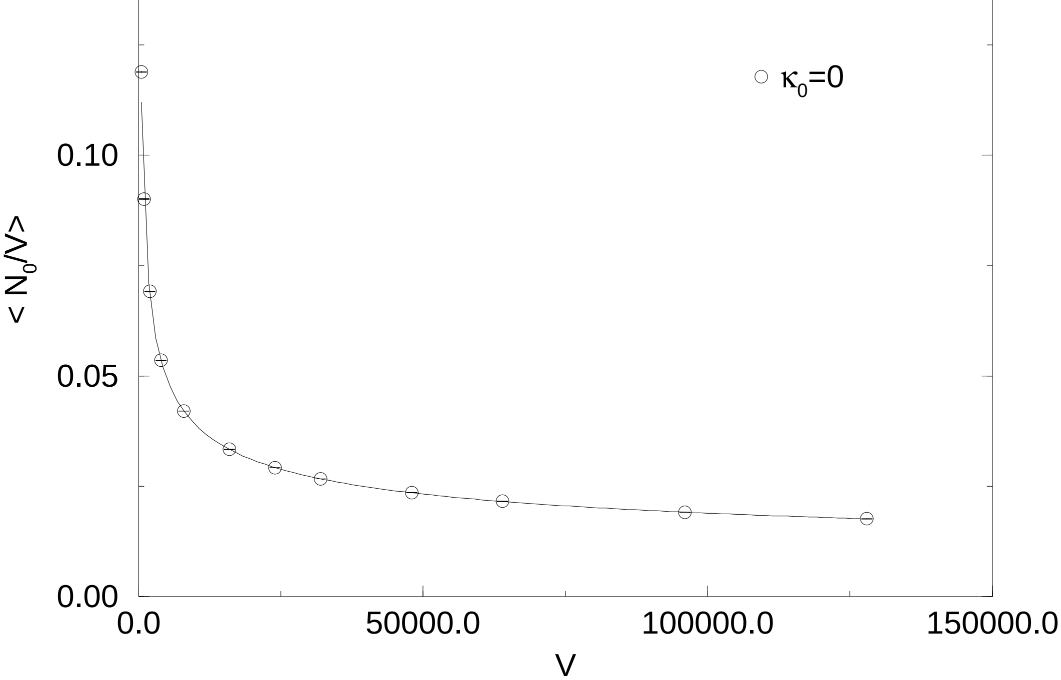

Figure 3: Number of nodes per unit volume

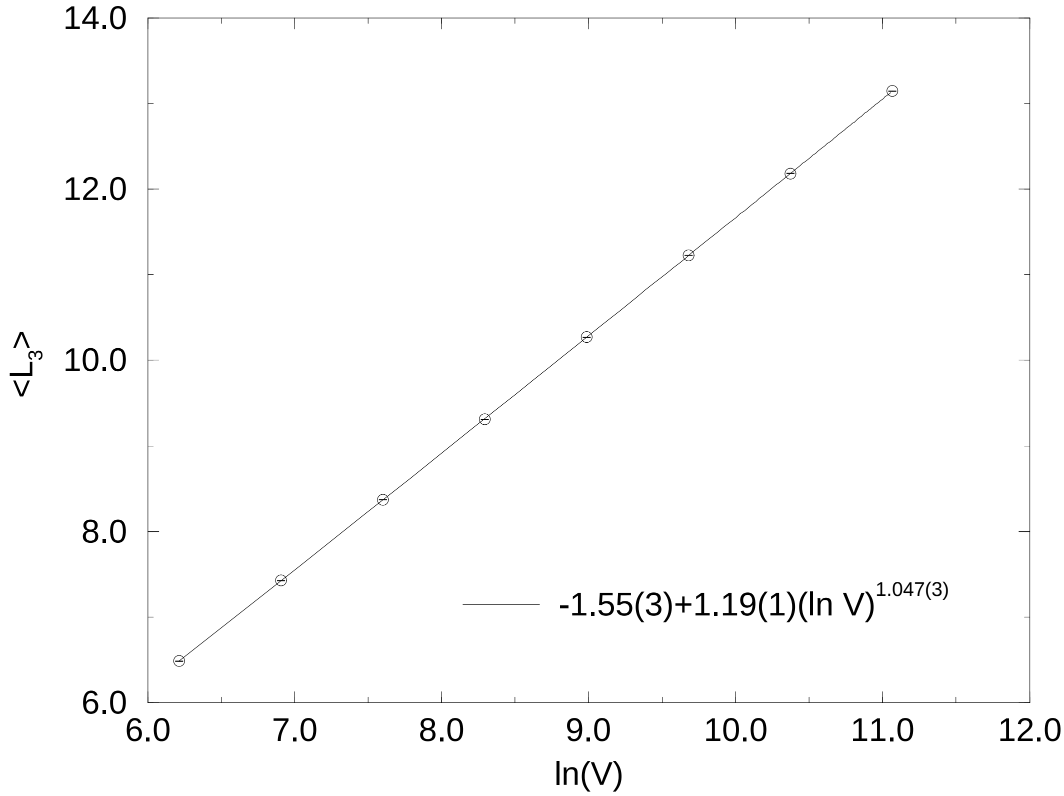

Figure 4: Mean intrinsic extent