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Lattice Quantum Gravity : Review and Recent Developments

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We review the status of different approaches to lattice quantum gravity indicating the successes and problems of each. Recent developments within the dynamical triangulation formulation are then described.

1. Introduction

The problem of uniting the two pillars of twentieth century physics, namely general relativity and quantum mechanics, is arguably one of the most important and challenging goals of theoretical physics. Much effort has been expended on the task of formulating such a *quantum gravity* theory extending over a period of several decades. In this review we want to concentrate on the status of a relatively new approach to the problem based on replacing continuum spacetimes by simplicial approximations.

We will be working within a path integral approach in which the partition function for gravity is gotten by performing a functional integral over an appropriate set of spacetime geometries

$$Z = \int D[g] e^{-S(g)}. \quad (1)$$

The evaluation of this partition function presents several, well-known problems. Firstly, if we choose to work with positive definite metrics (Euclidean space), the Einstein-Hilbert action $S(g)$ is unbounded from below due to the conformal mode. Second, there exists no unique, physically well-motivated choice for the functional measure $D[g]$. Clearly at minimum the measure should ensure that we sum only over physically inequivalent metrics - those not related by reparametrizations.

The third problem relates to the use of perturbation theory - the Newton constant has mass dimension minus two which ensures that the

model is not perturbatively renormalizable. Indeed, taken at face value this implies that fluctuations of the geometry become ever more severe at smaller and smaller distance scales and it is not even clear that a continuum spacetime is an appropriate way of describing it. Finally, the sum over geometries should presumably include manifolds with varying topology. Unfortunately, there is no scheme for classifying four-manifolds by their topology unlike the situation in two dimensions. This means it is not even clear how to formulate such a sum over topologies. Hence we will ignore this problem throughout the rest of this talk.

Various methods have been proposed [1] to evade some or all of these problems - canonical methods have been devised to try to avoid the use of perturbation theory, supersymmetric extensions of the Einstein theory have been considered in the hope of cancelling loop divergences. In string theory (the most radical solution so far) the Einstein-Hilbert theory emerges as a low energy effective field theory. However, none of these approaches has been entirely successful and so another approach has been proposed based on a lattice approximation which we will now discuss.

2. Simplicial Quantum Gravity

The original idea of using simplicial lattices to approximate smooth manifolds is due to Regge [2]. The idea is that any manifold can be viewed locally as being composed of pieces of flat space appropriately glued together. In d dimensions

it is convenient to use as elementary building blocks d -simplices which are assembled together by identifying pairs of $(d - 1)$ dimensional faces. A (sub)simplex of size d is composed of $(d + 1)$ labels corresponding to its vertices. In two dimensions such objects are triangles, in three dimensions tetrahedra and in four hypertetrahedra. A simplicial complex K or triangulation is a set of such simplices together with a specific gluing. It is usual to impose a ‘manifold’ condition on this gluing; the neighborhood of any point should be homeomorphic to a d -dimensional ball. This effectively eliminates triangulations possessing degenerate subsimplices.

Two sorts of degrees of freedom are now apparent; the edge lengths of the basic simplices $\{l_i\}$ and the triangulation T itself. Using these variables Regge was able to write down discrete equivalents of the Einstein action, the volume element etc. While Regge’s original suggestion was confined to the solution of *classical* problems in general relativity it is natural to try to generalize it to the quantum domain by constructing a lattice partition function

$$Z_L = \sum_{K \in (T, \{l_i\})} \rho(T, l) e^{-S_L(T, l)}. \quad (2)$$

In practice this partition function is still difficult to handle and two further approaches have been taken; Regge Quantum Gravity (RQG) and Dynamical Triangulation (DT). In the former, the triangulation is held fixed and one simply integrates over link lengths l_i . The second approach treats all the link lengths as equal to some reparametrization invariant cutoff a and a sum is performed over all triangulations T .

3. Successes and Difficulties

3.1. RQG

First consider the RQG formulation. Its primary advantage is that it is possible to show that its lattice action reduces to the continuum action in some appropriate limit in which the mean simplex edge length is reduced to zero as the number of simplices is taken to infinity. It similarly possesses a weak field expansion in which contact can be made with continuum perturbation theory -

rather like the naive weak coupling limit of SU(3) lattice gauge theory. While this is encouraging it is of course insufficient for proving that we are really studying a quantum theory of gravity outside of perturbation theory. We will see evidence for this in the numerical results that have been obtained in two dimensions.

One of the most exciting features of this approach has been the observation of a candidate continuous phase transition in the four-dimensional theory [3,4]. Current efforts focus on analyzing the scaling behavior of this model in the vicinity of the phase transition and extracting estimates for critical exponents.

The problems with this approach are four-fold. Firstly there are no exact solutions of the full quantum problem which can be tested against the predictions of continuum calculations. In two dimensions several models incorporating critical matter fields coupled to quantum gravity have been solved using CFT techniques. Unfortunately the RQG model cannot be solved exactly here to make detailed comparisons. In contrast we will see that this has been done for many of the DT models with results that show complete agreement with the continuum calculations.

Secondly, there remains a large ambiguity in the choice of measure for the link length integration. Commonly this measure is written as

$$D[g] \rightarrow \prod_i \int \frac{dl_i^2}{l_i^{2\sigma}}. \quad (3)$$

The question remains as to whether this is a suitably general choice of measure - specifically will such a *local* measure be sufficient and how should the power σ be chosen?

These problems are highlighted in two dimensions for the critical Ising model coupled to gravity. A sequence of simulations has revealed that the RQG approach simply fails to yield the correct *gravitationally dressed* Ising exponents [5,6].

Finally, it has been pointed out that a naive integration over all link lengths grossly overcounts physical degrees of freedom by including modes corresponding to general coordinate transformations in the continuum [7]. The authors argue that the correct continuum limit can only be obtained by utilizing an appropriate gauge fixing

procedure to eliminate the contributions of these gauge degrees of freedom.

Recently, an analytic calculation within the RQG framework has been completed in which the Fadeev-Popov determinant is computed for two-dimensional pure gravity [8]. The authors find that the Fadeev-Popov determinant when carefully regulated produces a non-local effective action which *precisely* matches the continuum conformal anomaly. It is the latter which is responsible for the dressing of the exponents. Any calculation which leaves this term out by not gauge fixing has no hope of generating the correct physics in two dimensions. It is not clear, however that the situation is necessarily so serious in higher dimensions.

3.2. DT

The inspiration for this approach really arose out of attempts to write down discrete regularizations for string theories; the latter being equivalent to two-dimensional quantum gravity coupled to a variety of matter fields [9–11]. Much of this work had been anticipated earlier by Weingarten in models of random surfaces embedded in hypercubic lattices [12].

One of the primary advantages of this approach is that in two dimensions a variety of exact solutions exist which may be compared with the solutions of continuum Liouville theory. A remarkable result has emerged from these studies; the DT correlation functions computed on arbitrary genus graphs agree with the corresponding quantities computed with continuum CFT techniques.

Furthermore, the critical behavior of these models has been studied numerically over the past few years and again very good agreement has been seen with the theoretical predictions [13]. The overall conclusion from this work has been that, at least for dimension two, the DT method affords a good prescription for regulating quantum gravity.

The potential problems for this approach in part derive from its nonperturbative formulation as a sum over all triangulations with equal weight - there is no weak coupling limit in which contact can be made with continuum perturbation theory. Indeed, the attractive feature of this for-

mulation - that it is purely geometric making no reference to coordinates and metric tensors also poses a problem; how do such classical quantities emerge from the model at large distance. Specifically we can ask the question: how do we recover the continuum metric tensor from the ensemble of triangulations? These are issues whose resolution will, we believe, prove crucial in determining the physical content of the higher dimensional models. The DT approach, while intuitively appealing, must remain essentially an ansatz for these latter models.

4. Summary of Background

There exist two closely related approaches to the problem of formulating a lattice theory of quantum gravity - one based closely on Regge's original proposal - RQG, and another called dynamical triangulation (DT) in which the dynamical variables correspond to abstract triangulations.

Both have seen some success. This is particularly true for the DT models in which a variety of exact results have effectively proven them equivalent to the continuum approaches wherever the latter can be solved. In both cases efficient numerical algorithms exist which can be used to probe the nonperturbative content of the theories. Both approaches show candidate phase transitions in four dimensions and it is an open question whether these lie in the same universality class.

From the rest of this talk, because of time constraints, I shall consider only the DT approach and refer the interested reader to the parallel sessions in this meeting for recent developments within the RQG framework [14–17].

5. The DT Model

In four dimensions the DT limit of the partition function eqn.2 can be written

$$Z = \sum_{T(S^4)} e^{\kappa_0 N_0 - \kappa_4 N_4}, \quad (4)$$

where the coupling κ_0 is conjugate to the total number of vertices N_0 and corresponds to a

bare (inverse) Newton constant. Likewise κ_4 is a lattice cosmological constant coupled to the total volume or number of four-simplices N_4 . We sum over a class of triangulations of the four-sphere which satisfy a certain ‘manifold-restriction’. namely, the boundary of the region in T enclosing any vertex is homeomorphic (‘looks-like’) S^3 . To be concrete imagine a two-dimensional triangulation and focus on one point within that triangulation. If that point were to be removed (together with its links to neighbors) a polygonal hole would be created whose boundary would be a bona fide triangulation of S^1 - a piecewise linear circle.

A set of ‘moves’ or local retriangulations of such d -dimensional triangulated manifolds has long been known to exist [18] and to satisfy an ergodic property: any triangulation can be transformed into any other by an appropriate sequence of such moves [19]. In d dimensions there are $(d + 1)$ types of move which may be labelled by the order i of a subsimplex central to the move. If this randomly chosen subsimplex is associated with $(d + 1 - i)$ simplices it is possible to try to replace it with a ‘dual’ $(d - i)$ -subsimplex provided that the manifold condition is not violated. By utilizing these moves inside a metropolis algorithm it is possible to sample the dominant triangulations in the partition function see eg [20]. This allows a wide range of nonperturbative investigations to be carried out which are currently inaccessible to analytic methods. We will discuss the results of some recent numerical studies which focus on a variety of topics; the nature of the quantum geometry, the development of MCRG methods for DT systems and detailed studies of the phase transition and correlation functions in four dimensions. Finally we will discuss some of the open issues and future research directions.

6. Recent Progress in Two Dimensions

6.1. Quantum Geometry

There are many quantities of physical interest which are not currently accessible to analytic study even in ‘exactly solved’ models such as pure $2d$ gravity. One of the most interesting of these concerns the nature of the quantum geometry, for

example, does it exhibit simple fractal properties, what are the typical geodesic paths etc.

These questions may be at least partially answered by looking at the distribution of geodesic paths as a function of lattice volume $n(r, V)$

$$n(r, V) = \frac{1}{V} \left\langle \sum_{ij} \delta(d_{ij} - r) \right\rangle \quad (5)$$

The function $n(r, V)$ counts the number of points which may be reached from some origin in r steps along links of a random triangulation with volume V . If $n(r, V)$ has the following simple behavior

$$n(r, V) \sim r^{d_H - 1}, \quad V \rightarrow \infty, \quad (6)$$

we can say that the ensemble of triangulated manifolds exhibits a simple *fractal* structure with Hausdorff dimension d_H . It turns out that direct fits at fixed volume to eqn. 6 are unable to reveal a convincing power law regime because of the presence of large finite volume corrections. In the light of this we have used a finite size scaling ansatz to extract the physics hidden in $n(r, V)$ [21]. Specifically, we assume that $n(r, V)$ can be written in the form

$$n(r, V) = V^{1 - \frac{1}{d_H}} f\left(\frac{r}{V^{\frac{1}{d_H}}}\right) \quad (7)$$

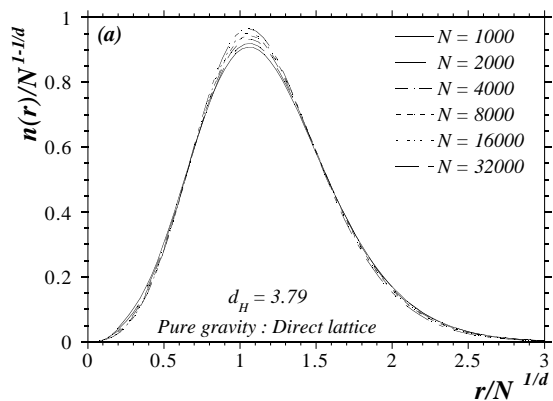


Figure 1. Scaling plot for $n(r, V)$ for pure gravity

We show in fig. 1 a plot of the function $f\left(r \frac{r}{V^{\frac{1}{d_H}}}\right)$ obtained from lattice sizes varying

from $V = 500 - 32000$ in the case of pure gravity. Clearly the data collapse almost perfectly onto a single scaling curve. The optimal d_H may be estimated by a suitable fitting procedure and here corresponds to $d_H = 3.8(1)$ - very consistent with a recent analytic calculation which predicts $d_H = 4$ [22]. Notice that the numerical calculations provide more than just a direct confirmation of this fractal dimension - they show graphically the emergence of *nonperturbative* length scale in these quantum gravitational systems which is power-like related to the total volume. Similar numerical results have been reported by the Copenhagen group [23].

The numerical approach can be trivially extended to study $c > 0$ theories coupled to quantum gravity. An example of a scaling plot for the critical Ising system is shown in fig. 2. The conclusion from these studies can be simply summarized; for $c < 1$ no backreaction of the critical matter field is visible in the measured Hausdorff dimension which remains at four, for $c > 1$ scaling is still observed with a Hausdorff dimension which decreases monotonically towards two with increasing c - the value for branched polymers. These latter results were obtained in [23] together with corresponding results for another fractal dimension - the spectral dimension.

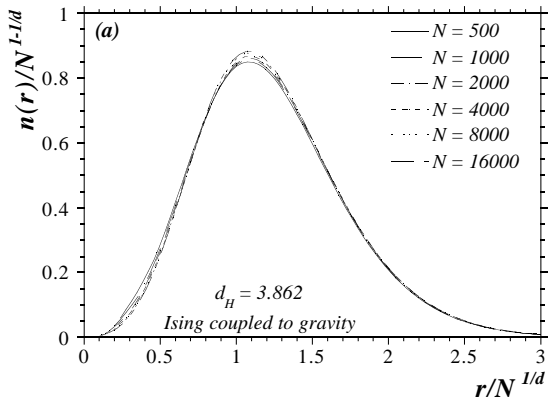


Figure 2. Scaling plot for $n(r, V)$ for Ising+gravity

Preliminary results in four dimensions have in-

dicated that a similar scaling behavior may also occur there and constitutes further evidence for critical behavior in the model [24].

6.2. MCRG for 2d gravity

Much of the numerical work in 2d gravity has centered around the application of finite size scaling to extract critical exponents see eg [13]. In conventional statistical mechanical systems the validity of such techniques depends on the existence of a renormalisation group (RG) which governs the flow of couplings under changes in scale. It is tempting to believe that a similar RG structure must underlie the DT models and is responsible for the observed scaling.

In the continuum formulations of quantum gravity the very issue of a renormalisation group is a difficult one to formulate since the theory (in the absence of a cosmological constant) possesses no length scale. In contrast the DT formulation contains an invariant cut-off corresponding to the elementary triangle edge length. The latter may be traded in for the number of triangles N if the physical volume is held fixed.

A successful RG transformation would be important both conceptually and as a powerful new tool with which to compute critical points and critical exponents for systems coupled to quantum gravity. We have undertaken a systematic exploration of the features of one such blocking transformation used in conjunction with Monte Carlo simulation [25].

In conventional lattice field theory a RG or block transformation acts so as to reduce the number of degrees of freedom by replacing the original (bare) theory by one defined on a coarser lattice. If this new theory is to preserve the essential long distance physics there will be an associated flow in the coupling constants of the model. Notice that for the usual field theory models on regular (flat space) lattices the blocking transformation trivially preserves the lattice structure and the main problem is to devise a blocking scheme for the matter fields.

Quantum gravity is quite different - it is not possible to exactly preserve the features of any given fine triangulation under blocking. Indeed the lattice itself now carries dynamical degrees of

freedom. Thus the transformation we envisage will replace a given triangulation $T(N)$ by some triangulation $T'(N')$ corresponding to a blocking factor $b = N/N' < 1$.

$$T'(N') = R(T(N)) \quad (8)$$

Clearly the choice of the transformation R is of crucial importance. Presumably it should have the property of preserving certain aspects of the long distance geometry. Recently various proposals have been put forward for accomplishing this in the case of pure gravity. The first of these approaches due to Renken [26] associates nodes of the block lattice with a randomly selected subset of the fine lattice points. The block triangulation is then chosen so as to preserve *locally* the relative geodesic distances of these block nodes.

Another approach advocated by Krzywicki et al. [27] constructs a block triangulation by eliminating a class of extremal baby universes associated with the original random lattice. For a discussion of baby universes see later. Since the baby universe distribution is intimately connected to the fractal structure of the ensemble of triangulated manifolds it is hoped that this transformation will (approximately) preserve the fractal structure. This approach has recently been applied to four dimensions resulting in a prediction for the β -function corresponding to the node coupling κ_0 in the vicinity of the transition [28]. However it remains to be shown whether this method can yield the correct gravitationally dressed exponents in two dimensions.

Here we have examined another blocking method based on a *local* direct decimation of the initial lattice. The algorithm is extremely simple. First pick a point at random in the fine lattice. If we attempt to remove this node then unless the original node coordination was three we will be left with a polygonal ‘hole’ in the triangulation. However by the addition of suitable links it is possible to retriangulate the interior of this hole. We choose one of the many possible retriangulations at random.

By iterating this procedure an arbitrary number of times we can produce a block lattice of any volume. In practice, this procedure is effected by randomly flipping the links around a

selected node until its coordination number becomes equal to three. At this point any original curvature associated with the node has been smeared out over its neighbors. We then remove the three-fold node. Thus this method can be seen to closely preserve the local curvature.

In the case when a spin configuration decorates the original lattice we have experimented with two methods for blocking the lattice. The first proceeds as for pure gravity, the lattice blocking being unaffected by the spin configuration. In the second method we subject the link flipping to a standard metropolis test using the critical Ising action. The former appears to give more stable critical exponent estimates. The block spins are simply arrived at by a variation on the majority rule: a given block spin is determined by examining its neighbors on the original lattice (excluding surviving block nodes) and assigning a spin according to the sign of the sum of their spins. I refer the reader to Thorleifsson’s talk at this meeting [25] and reference [29] for further details and here merely show results for the exponent γ_{str} obtained after blocking, both for pure gravity and gravity coupled to Ising spins figs. 3 and 4.

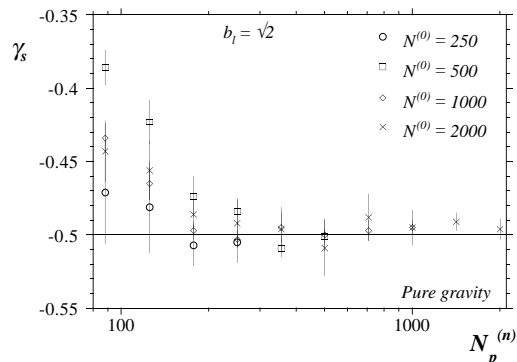


Figure 3. γ_{str} vs blocking level for pure gravity

Clear evidence for fixed point behavior is seen with the correct gravitational exponent. A more

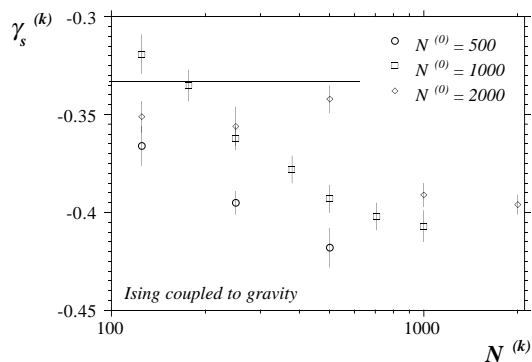


Figure 4. γ_{str} vs blocking level for Ising+gravity

stringent test can be made by looking at the dressing of the Ising exponents extracted in the usual MCRG manner from an estimated T -matrix built out of a truncated operator basis. Figure 5 shows estimates of the Ising exponent $\frac{1}{\nu d_H}$ and fig. 6 the exponent $\frac{\delta}{\delta+1}$ for the critical Ising plus gravity system as a function of blocking level.

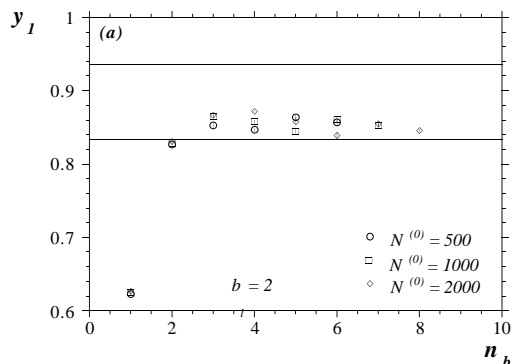


Figure 5. $\frac{1}{\nu d_H}$ vs blocking level for Ising+gravity

These numbers correspond to an even spin operator basis consisting of the simple energy bond operator, the conjugate bond operator and the

product of the four spins around the boundary of two neighbour simplices. The odd spin operator basis is formed from the simple single spin operator and the product of spins around a triangle.

The theoretical prediction governed by the KPZ formula is shown by the lower horizontal line while the solid line above indicates the Onsager flat space limit. Very encouragingly we see the numerical results are both stable under blocking and close (in most cases statistically consistent with) the KPZ predictions. This is highly non-trivial of the correctness of our blocking procedure.

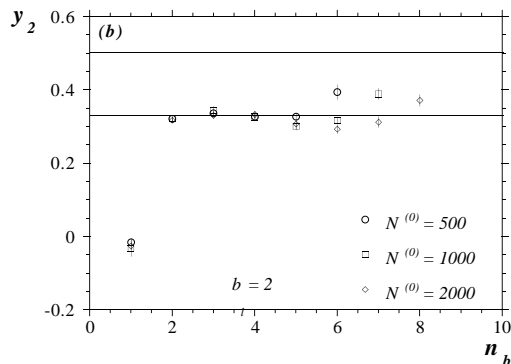


Figure 6. $\frac{\delta}{\delta+1}$ vs blocking level for Ising+gravity

It is clearly important to check the efficacy of this blocking scheme on other solved gravity coupled models and to extend it to the case of scalar fields. One of the prime advantages this method possesses over the usual finite size scaling is that the critical coupling is an end result of the calculation - it is not needed as input to determine critical exponents. Both $c > 1$ models and the crumpling transition furnish examples of suitable arenas for application of these ideas.

7. Recent Progress in Four Dimensions

7.1. Critical Properties

All the groups that have studied this model report evidence for two phases separated by a continuous phase transition. For small node couplings κ_0 the model is in a crumpled phase with a typical mean intrinsic extent growing only logarithmically with volume and negative total curvature. For large κ_0 the model enters a branched polymer phase characterized by a mean size varying as the square root of the volume (Hausdorff dimension two) and positive mean curvature. A phase transition separates these regimes revealed by a node susceptibility whose peak value scales in power-like fashion with the volume.

These calculations have been greatly strengthened in the past year by a calculation of Ambjørn and Jurkiewicz [30] in which a new algorithm was used to extend the range of lattice sizes to much larger values than previously (64K simplices). The basis of their global algorithm termed ‘baby universe surgery’ resides in the identification of minimal neck baby universes. Once one of these has been located it may be moved globally to another region of the manifold by randomly selecting a simplex on the ‘mother’ universe and gluing the baby at that point by identifying the minimal neck with the boundary of that simplex. The authors show that this procedure is very efficient in reducing critical slowing down associated with the geometry near the phase transition and within the branched polymer phase. The location of baby universes also allows them to measure the baby universe distribution and hence the exponent γ_{str} .

The computed value is consistent with $\gamma_{str} = \frac{1}{2}$ throughout the branched polymer phase. This provides yet another clue that the dominant configurations are indeed true branched polymers. An analysis of the geodesic path distribution, defined similarly as in two dimensions, shows very accurate scaling throughout this region and provides another accurate estimate for the Hausdorff dimension $d_H = 2$.

By tracking the deviation of the exponent γ_{str} from one half it is also possible to read off a new estimate for the pseudo-critical coupling $\kappa_0^c(V)$

at finite volume. It is found that

$$\kappa_0^c(V) - \kappa_0^c(\infty) \sim V^{-\delta}, \quad (9)$$

with $\delta = 0.47(3)$. This is a significant improvement over past estimates which have estimated this pseudo-critical point by following the volume dependence of the peak in the node susceptibility.

Finally, by fitting the geodesic path distribution to a suitable ansatz in the crumpled phase of the model it is possible to extract a ‘massgap’ $\Delta m(\kappa_0)$ for the model. As the phase transition is approached it is found that

$$\Delta m \sim (\kappa_0 - \kappa_0^c)^\epsilon, \quad (10)$$

with critical exponent $\epsilon = 0.50(1)$

Baby universes have also been used in another way to analyze the critical properties of this 4d phase transition, In [28] Burda, Kownacki and Krzywicki employed an extension of their blocking method for two dimensional quantum gravity to four dimensions. The essence of the method is to identify baby universes of a certain size and remove them from the triangulated manifold closing the minimal neck with a simple simplex and discarding the simplices associated with the original ‘baby’. This reduction of the volume can be traded for a change in length scale and hence an RG transformation. By reasoning that this elimination of extremal baby universes at least approximately preserves the fractal structure of the ensemble of manifolds it is hoped that it is a truly ‘apt’ transformation. The authors are able to compute the beta-function for the node coupling κ_0 close to the critical point and indeed observe a IR unstable zero very close to the peak in the node susceptibility. This provides further confirmation of the transition and also indicates that the fixed point is not, as has been recently speculated [31], an IR stable fixed point associated with the conformal mode.

7.2. Two-Point Functions

The last year has also witnessed substantial progress in the understanding of two-point correlation functions in 4d gravity. De Bakker and Smit argue that the definition of a ‘connected’ correlator is somewhat subtle in the situation where a sum over geometries is being performed

[32]. They present numerical results which support the idea that it is necessary to subtract a *distance dependent* one-point function to obtain a simple behavior for the two-point function.

One way to understand this procedure is to consider the definition of a connected function down geodesic paths

$$C_Q(r) = \left\langle \frac{1}{n(r)} \sum_{ij} \delta Q_i \delta Q_j \delta(d_{ij} - r) \right\rangle, \quad (11)$$

where

$$n(r) = \sum_{ij} \delta(d_{ij} - r) \quad (12)$$

$$\delta Q_i = Q_i - \bar{Q}. \quad (13)$$

The quantity Q_i can be any local operator associated with simplex i and d_{ij} is the (dual lattice) geodesic distance between simplices i and j . Smit et al. consider the case where Q is the curvature density. In flat space the expectation values \bar{Q} are just constants; however in order that eqn. 11 factorizes at large distance to a form $\langle Q_i Q_j \rangle - \bar{Q}^2$ it is necessary to allow the expectation value \bar{Q} to depend on distance down the geodesic path r . Specifically,

$$\bar{Q} = \bar{Q}(r) = \left\langle \frac{1}{n(r)} \sum_{ij} Q_i \delta(d_{ij} - r) \right\rangle \quad (14)$$

If the operator is completely uncorrelated with the geometry (the situation in flat space for example) this expression reduces to the usual one. However, in general, when we sum over triangulations it is perhaps natural to assume that the detailed distribution of matter field configurations will depend on geometry and the average of some operator on geodesic circles may not be the same as its bulk expectation value. This appears to be the case and data is shown illustrating that when such a subtraction is made the correlation function simplifies to yield a simple power law behavior. Indeed close to the transition it appears that

$$C_R(r) \sim \frac{1}{r^4} \quad (15)$$

which is consistent with the exchange of two gravitons.

7.3. Open Issues

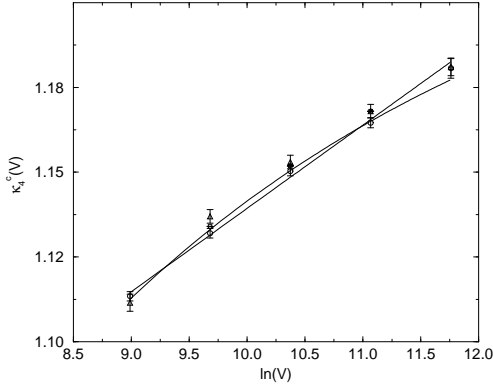
Over the previous two years there has been a considerable amount of activity devoted to the question of an exponential bound in DT quantum gravity. The issue is this; does the number of triangulations of, for example, the d -sphere, grow no faster than exponentially with the volume. In two dimensions this is known to be true and is the basis for taking the continuum limit - the bare cosmological constant is tuned to compensate for this exponential growth leading to a power divergence of the mean volume and susceptibility.

In higher dimensions there are no rigorous proofs of such a bound and one must turn to numerical simulation to attempt to answer this important question. In three dimensions the results of these studies have been pretty convincing [33,34] - at large volumes the partition function indeed saturates at exponential growth. However in four dimensions the results have been more controversial. In [35] evidence supporting an unbounded scenario was presented. However the results of simulations at larger volumes were shown to be consistent with such a bound [36,37]. Figure 7 shows a plot of the effective critical cosmological constant $\kappa_4^c(V)$ at node coupling $\kappa_0 = 0$ with results from the three groups working on this issue.

A bound would be indicated by a effective coupling that becomes independent of volume at large volume. This is not seen at the lattice volumes accessible to experiment ($V \leq 128000$ simplices here). So the question of the bound translates into whether suitable empirical fits to the data favor a coupling which becomes constant for infinite volume or merely keeps increasing.

Two scenarios are usually used for the fits; a logarithmic divergence of $\kappa_4^c(V)$ with volume or a weak power law convergence. The issue is simply which fit describes the data best and how robust is that fit - how confident are we that the fit is stable over a wide range in volume.

The first point to note is that the data from different groups are all statistically consistent with each other. The straight line and curve represent the best fits to our data assuming the log and power scenario respectively. The quality of the

Figure 7. $\kappa_4^c(V)$ vs $\log V$ at $\kappa_0 = 0$

fits as revealed in the chi square per degree of freedom are comparable at order unity and our conclusion must be that it is *impossible* to distinguish confidently between the two possibilities from this data alone.

Faced with this problem we have turned to the analysis of other quantities in the hope of resolving the issue. One such quantity is the distribution of baby universes. A baby universe is defined as a portion of the triangulation which is separated from the bulk by a so-called ‘minimal neck’. In d dimensions the latter is defined to be $d + 1$ $(d - 1)$ -simplices which form the boundary of a d -simplex *not* already present in the triangulation. The fraction of such baby universes with volume B is a sensitive test of the subleading behavior of the microcanonical partition function and hence yields information on the exponential bound.

We have conducted a careful study of this distribution measured on lattices with volumes $V = 500 - 8000$ and fitted the data to one of the two following forms corresponding to logarithmic divergence and power law convergence.

$$\log P(B) = a + \beta(B + \delta) \log(B + \delta) + \beta'(V - B + \delta) \log(V - B + \delta) \quad (16)$$

$$\log P(B) = a' + \beta'(B + \delta)^{1-\gamma} + \beta'(V - B + \delta)^{1-\gamma}. \quad (17)$$

The constant δ is inserted as a phenomenological parameter to reflect sub-leading finite size corrections and a and a' reflect an ambiguity in overall normalization. In practice we have removed the largest contribution to the latter by dividing the measured number of baby universes by the volume V

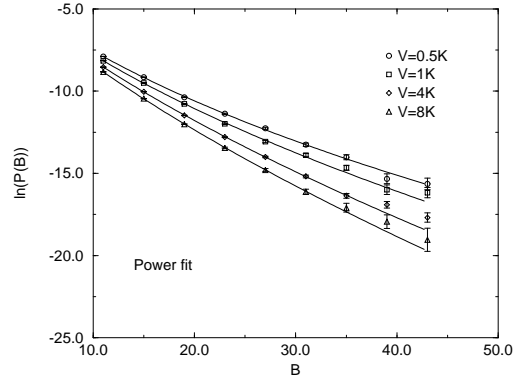
Figure 8. $\log P(B)$ vs B with power fit

Figure 8 shows the data fitted according to the power scenario Eq. (7). The best fit in this case yields $a' = -0.2(15)$, $\beta' = -1.38(5)$ and $\delta = 3(2)$ with $\chi^2 = 6.3/6$ assuming $\gamma = 0.25$ as before. The log gives a marginally worse fit with $\beta = 0.056(1)$ but still is consistent with the data. At face value it remains hard to differentiate between the two situations. However, notice that the extracted value of $\beta = 0.056(1)$ from the log fit is more than twice its estimated value from the fits for the effective critical coupling $\beta = 0.025(1)$. In contrast the estimate for $\beta' = -1.38(5)$ from the power fit is quite close

to its value estimated from the critical coupling (figure 7) $\beta' = -1.23(4)$. The relative proximity of the two estimates is particularly impressive considering that one is derived from the behavior of baby universes with size less than 8000 simplices while the other is extracted from the critical coupling at volumes much greater than 8000. Furthermore, it is clear that the power fit would still hold good if we set $\delta = a' = 0$ so that such a fit (with a truly minimal number of parameters) would do much better than the logarithm.

In conclusion, the numerical results which have been obtained by three groups, although not definitive, are very consistent with the existence of an exponential bound in the dynamical triangulation model of $4d$ quantum gravity. The evidence for this comes both from fits to the volume dependence of the critical coupling, and an analysis of the baby universe distribution in the crumpled phase.

Although individually these quantities are not very conclusive, it is remarkable how consistent results are obtained if we assume a weak power convergence. Clearly, it is important to strengthen these conclusions both by simulating intermediate lattice volumes and perhaps via a high statistics simulation at say volume $V = 16000$ directed at probing further into the tail of the baby universe distribution.

Finally I should like to mention the work of the Japanese group [38] who have observed that ‘singular’ vertices are present in four dimensional DT. These are vertices which are common to a fixed fraction of the total number of simplices. There appear to be two such vertices in 4d whose distribution of simplex coordination is well separated from the ‘bulk’ distribution corresponding to all other vertices in the triangulation. I refer the interested reader to the parallel session given at this meeting for further details and merely make two remarks here. Firstly, it appears that the dynamics of these singular vertices plays an important role in the longest autocorrelation times observed for these systems [39]. Furthermore, a crude suppression of these objects appears to remove the phase transition. Secondly, no such vertices exist in three dimensions which is probably the first qualitative indication

that the physics of three-dimensional and four-dimensional DTs can be quite different.

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