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Singular Vertices and the Triangulation Space of the D-sphere

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Abstract

By a sequence of numerical experiments we demonstrate that generic triangulations of the $D-$sphere for $D > 3$ contain one *singular* $(D-3)-$simplex. The mean number of elementary $D-$simplices sharing this simplex increases with the volume of the triangulation according to a simple power law. The lower dimension subsimplices associated with this $(D-3)-$simplex also show a singular behaviour. Possible consequences for the DT model of four-dimensional quantum gravity are discussed.

1 Introduction

It has been well established that two dimensional quantum gravity can be recovered as the scaling limit of models of random triangulations, see for example [1]. Performing a sum over such simplicial manifolds generates the correct integral over physically inequivalent metrics. As a natural extension of these ideas it has been proposed that triangulations of higher dimensional manifolds can form the basis of a general regularization scheme for gravity [2]. In general, such simplicial manifolds are constructed by gluing together $D-$dimensional simplices across their $(D-1)-$dimensional faces so as to form a closed triangulation with some fixed topology. Additional manifold restrictions are commonly imposed to ensure that all simplices consist of a set of $(D+1)$ distinct labels and every subsimplex is unique.

The ansatz for the partition function in general dimension D then takes the form

$$Z = \sum_{V} e^{-\kappa V} \Omega_D\left(V\right), \tag{1}$$

where κ is a bare cosmological constant conjugate to the volume or total number of $D-$simplices V. The *microcanonical* partition function $\Omega_D\left(V\right)$ is given by

$$\Omega_D\left(V\right) = \sum_{T(V)} e^{-S(T,\kappa_i)}. \tag{2}$$

The sum over triangulations T is confined to those with volume V, with a weight determined by a discrete action $S\left(\kappa_i\right)$ governed by a set of couplings $\{\kappa_i\}$. In the case of four dimensions the simplest action contains only one such coupling κ_0 which can be identified with the bare (inverse) Newton constant. The corresponding analog of the Einstein-Hilbert action can be taken to be the total number of vertices N_0 in the triangulation.

Current interest in this model stems from the results of numerical simulations which have revealed a non-trivial phase structure in four dimensions. Between a crumpled phase with large negative curvature at small κ_0 and an elongated, branched polymer phase at large κ_0 there is evidence of a continuous phase transition. The existence of such a *non-perturbative* critical point offers the possibility of a continuum limit describing quantum gravity [3]. However, it would be fair to say that the nature of this continuum theory is only just beginning to be explored.

One of the major problems impeding progress in this direction has been the lack of any analytic methods for handling the sum over four-dimensional simplicial geometries. The structure of the triangulation space and its implications for the measure over simplicial geometries are unknown. In this paper we hope to make some progress in this direction using numerical simulation to identify a class of triangulations which dominate the microcanonical partition function $\Omega_D\left(V\right)$ for large volumes V.

This work is motivated by a recent observation that typical triangulations in the crumpled phase of the four-dimensional model are characterized by 2 highly degenerate or singular vertices [4]. Singular vertices are vertices that are shared by a large number of simplices – a number which diverges linearly with the total volume of the triangulation.

In Section 2 we describe our conjecture for the structure of the dominant $D-$dimensional triangulations together with supporting numerical results. Section 3 makes plausible why such configurations might be entropically favoured and uses a simple geometrical model to explain some of the observed volume dependencies. Section 4 contains a discussion of singular vertex dynamics and its practical implications for numerical simulations. Finally Section 5 contains a brief discussion of possible consequences of this singular structure. Specifically we discuss the issue of an exponential bound in four dimensions.

2 Structure of the Triangulation Space

Our Monte Carlo simulations employ a set of local, ergodic and topology preserving moves[1] (see for example [3, 6]). We have set the action S to zero so that the simulations explore equally all triangulations contributing to the partition function Eq. 2

Let us define the *local volume* associated with an $i-$simplex as the number of $D-$simplices containing that $i-$simplex. We then say that the $i-$simplex is *singular* if its local volume diverges with the total volume V (total number of $D-$simplices). Our results can then be summarized in a simple conjecture:

Conjecture 1 *The function $\Omega_D(V)$ is saturated as $V \to \infty$ by triangulations which contain one singular $(D-3)-$simplex.*

This is illustrated by Fig. 1 which shows the (normalized) distribution of $(D-3)-$simplices with a given local volume m, denoted by $P^{(D-3)}(m)$. The two plots correspond to four and five dimensions where the singular object is a link and a triangle respectively. The data in both cases come from simulations in which the total volume $V = 32000$. Clearly, both distributions possess an isolated peak in the tail corresponding to $(D-3)-$simplices which are common to a large number of $D-$simplices. Furthermore, we have observed that this peak corresponds to the presence *for each triangulation* of precisely one such singular $(D-3)-$simplex.

Fig. 2 shows the scaling of the mean local volume of this singular $(D-3)-$simplex with the total triangulation volume, again for $D = 4$ and 5. We have utilized lattices of size $V = 8000$ to $V = 64000$. For large volumes it can be seen that these results support the notion of a power-law divergence. Furthermore, the data is consistent with a unique, simple power growth given by the solid lines. These correspond to a power of $\frac{2}{3}$. The justification for the choice of this power will be discussed in Section 3.

[1] For a dimension independent implementation see [5].

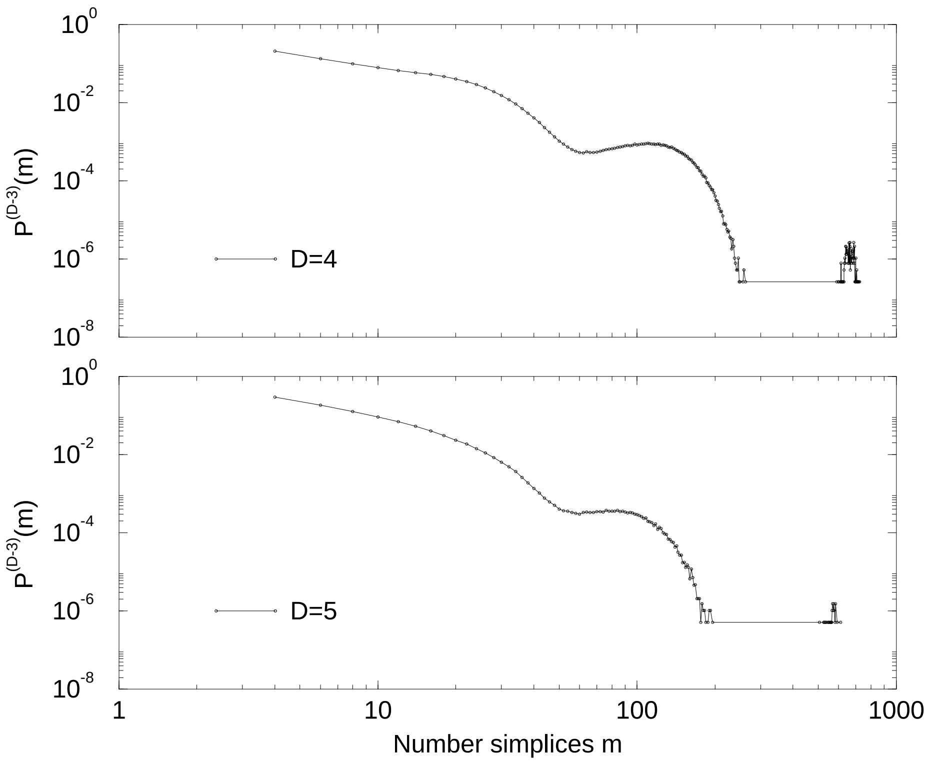

Figure 1: The normalized distribution of local volumes for $(D-3)-$simplices in four and five dimensions.

Associated with this *primary* $(D-3)-$simplex is a cascade of other singular simplices corresponding to all its possible subsimplices. Thus we observe precisely

$$\left(\begin{array}{c} D-2 \\ i+1 \end{array} \right)$$

secondary singular $i-$simplices, where $i = 0, \ldots, (D-4)$, whose local volumes diverge in the thermodynamic limit. Thus in four dimensions we see exactly two singular vertices corresponding to the endpoints of the original singular link. In five dimensions we have one singular triangle, three singular links corresponding to its edges and three singular vertices. Fig. 3 illustrates this for dimensions four (a) and five (b). This pattern continues in higher dimension, for example in six dimensions the dominant triangulations have four singular vertices.

In contrast to the primary singular simplex we find that the mean local volumes of these secondary singular simplices grow *linearly* with the volume of the triangulation. Fig. 4 shows a plot of the mean singular vertex volume for both four and five dimensions. The linear growth of the singular link volume in five dimensions is shown in Fig. 5. We have observed that each triangulation is symmetric with respect to exchange of two singular simplices of a given dimension - they have

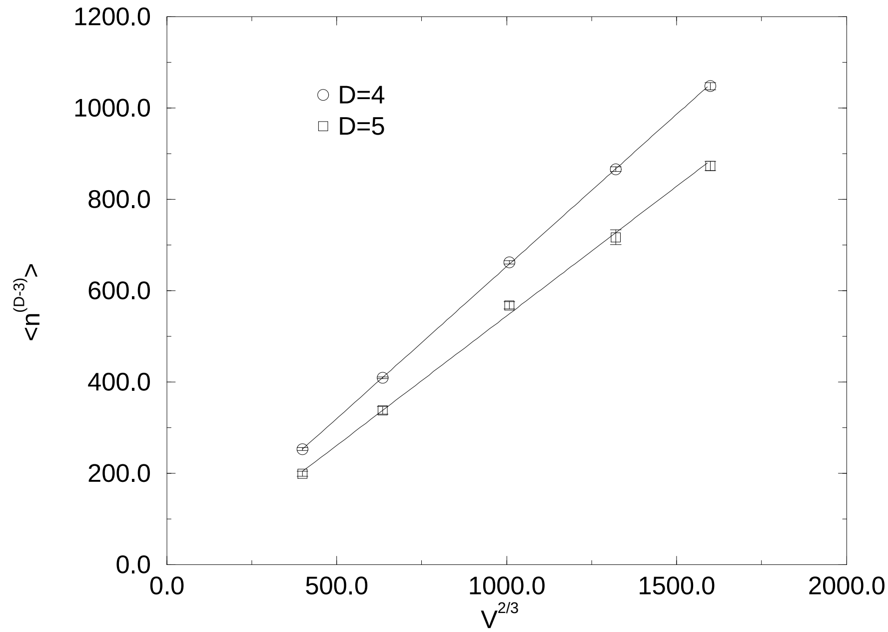

Figure 2: The mean local volume of the $(D-3)-$simplex vs the total volume V for four and five dimensions. Note that we plot the data as a function of $V^{\frac{2}{3}}$.

approximately the same local volumes. On the basis of this numerical evidence we can state another hypothesis.

Conjecture 2 *While the primary singular $(D-3)-$simplex has a local volume which grows as some power $p \sim \frac{2}{3}$ of the total volume, the secondary singular simplices have local volumes which grow linearly with volume.*

We have also recorded the mean number of $D-$simplices $V_{\rm ns}$ which are *not* associated with any of the singular vertices. The number of these again increases linearly with volume $V_{\rm ns} = c_{\rm ns} V$. Since the total volume is fixed at V there is a relationship between the local volumes $\omega_i = c_i V$ of singular $i-$simplices and the non-singular simplices $c_{\rm ns} V$. In the infinite volume limit (where the $(D-3)-$simplex does not contribute) it is straightforward to verify that the following relation holds between the coefficients c_i.

$$1 - c_{\rm ns} = \sum_{i=0}^{D-4} \left(\begin{array}{c} D-2 \\ i+1 \end{array} \right) (-1)^i \, c_i \; . \tag{3}$$

In four dimensions the measured $c_{\rm ns} = 0.279(2)$ which is to be compared with its value computed from the above relation, $c_{\rm ns} = 0.266(3)$. Given the systematic

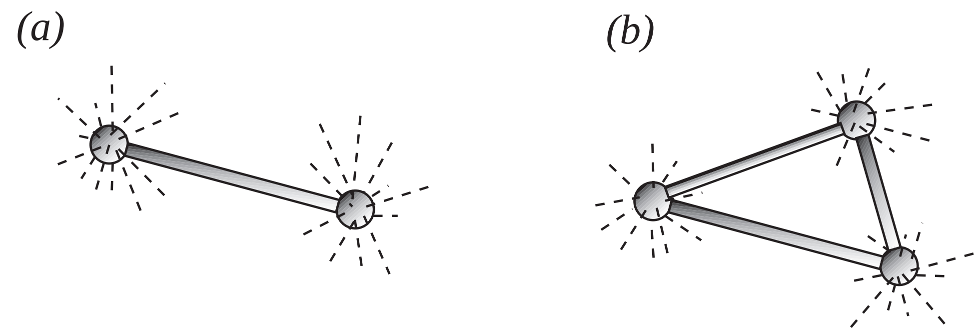

Figure 3: The singular structure in a) $D = 4$ b) $D = 5$. The balls correspond to singular vertices which overlap along singular links and triangles.

errors present in these fits we regard this as quite satisfactory agreement. Notice that this is completely consistent with the observation that the singular link volume increases sublinearly as $V \to \infty$. If that were not the case the righthand side of Eq. 3 would receive another contribution from the links. It is also satisfied in five dimensions where the measured value $c_{\rm ns} = 0.045(1)$ is statistically consistent with the value estimated using this relation, $c_{\rm ns} = 0.058(6)$.

3 Entropy Considerations

Given these conjectures about the nature of the configuration space, is it possible to understand why this very special class of triangulations is entropically favored? Why for instance are there no singular $(D-2)-$simplices? In this section we give some heuristic arguments for this, and also try to explain the nature of the observed power-law divergence of the singular $(D-3)-$simplices.

Consider the local volume associated with a particular $i-$simplex. It is composed of a set of $D-$simplices each of which contains the $(i+1)$ vertex labels of the $i-$simplex in question. Take the set of vertex labels associated to these local volume $D-$simplices and remove the $(i+1)$ labels of the common $i-$simplex. The remaining vertex labels then constitute a triangulation of a $(D-i-1)-$sphere. This sphere is the boundary of the dual to the simplex - a $(D-i)-$dimensional volume. The volume of this sphere is just proportional to the original $i-$simplex local volume.

For example, in three dimensions, the dual to a link is an area element whose boundary is a triangulation of the circle. The vertices defining this circle are just those obtained from the simplices making up the local volume of the link excluding the endpoints of the link itself. By definition, the link's local volume is then just proportional to the number of vertices on the circle. This is illustrated in Fig. 6 which shows the simplices making up the local volume of a link in three dimensions, its dual area and the associated $1-$sphere - the triangulated circle.

We can now ascribe a local entropy to an $i-$simplex with local volume ω_i

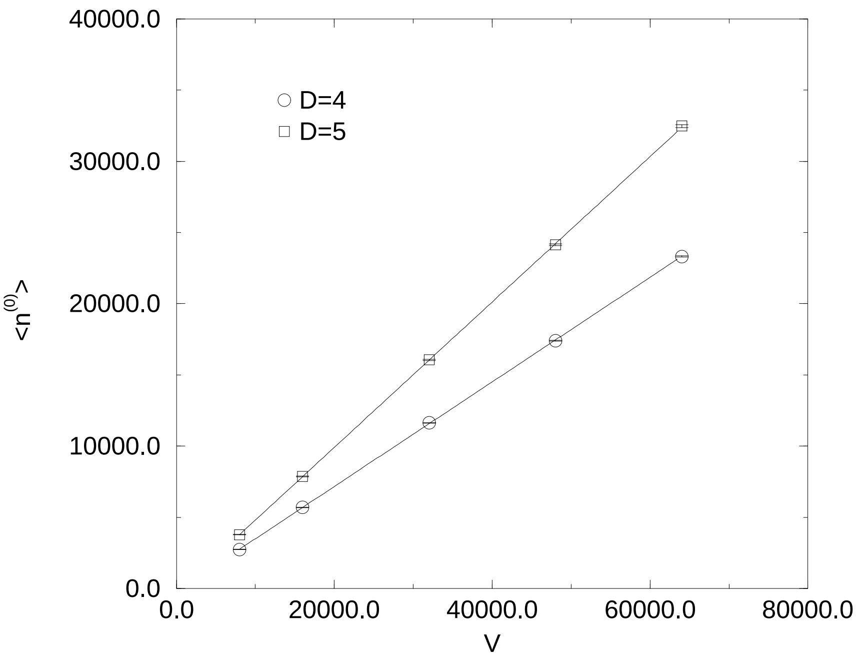

Figure 4: The mean local volume of singular vertices vs total volume V for four and five dimensions.

by counting the number of ways of gluing together the ω_i simplices containing it. Each of these gluings corresponds to a distinct triangulation of the associated dual $(D-i-1)-$sphere. This allows us to map the problem of counting the number of ways of achieving a certain local volume by gluing together $D-$simplices into the enumeration of all the possible triangulations of the dual $(D-i-1)-$sphere. Specifically, the local entropy of the $i-$simplex with local volume ω_i is just determined by the number of triangulations of the associated dual $(D-i-1)-$sphere with volume ω_i.

For $(D-2)-$simplices the relevant sphere is S^1. There is only one distinct way of arranging the simplices in its local volume. Equivalently, there is a unique triangulation of S^1 for any local volume ω_{D-2}. Thus the local entropy of a $(D-2)-$simplex does not increase as its local volume increases. It is *not* entropically favoured for such a $(D-2)-$simplex to acquire a large local volume. Indeed, the number of $(D-2)-$simplices possessing large local volumes falls off (approximately) exponentially fast. This is seen in Fig. 7 which shows the (normalized) distribution of $(D-2)-$simplices common to m $D-$simplices, denoted by $P^{(D-2)}(m)$. The data corresponds to four dimensions but similar results are obtained in dimensions three

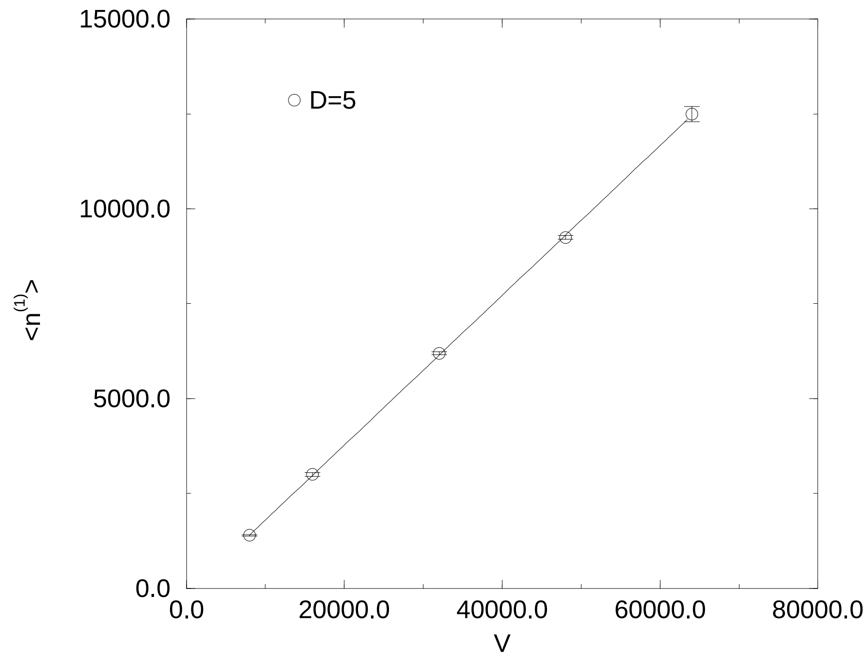

Figure 5: The mean local volume of singular links vs total volume V for five dimensions.

through six. Notice that the curvature density is associated to $(D-2)-$simplices and hence never receives any singular contributions.

The situation is very different for $(D-3)-$simplices. The local entropy associated to such an object is again given by the number of ways of gluing together

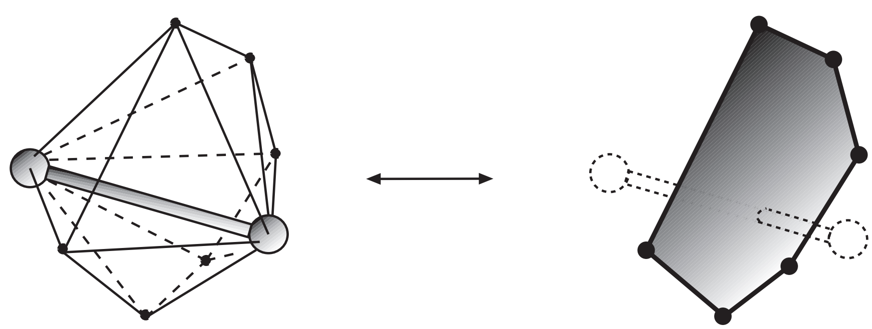

Figure 6: The dual area and its bounding triangulated circle for a link in three dimensions.

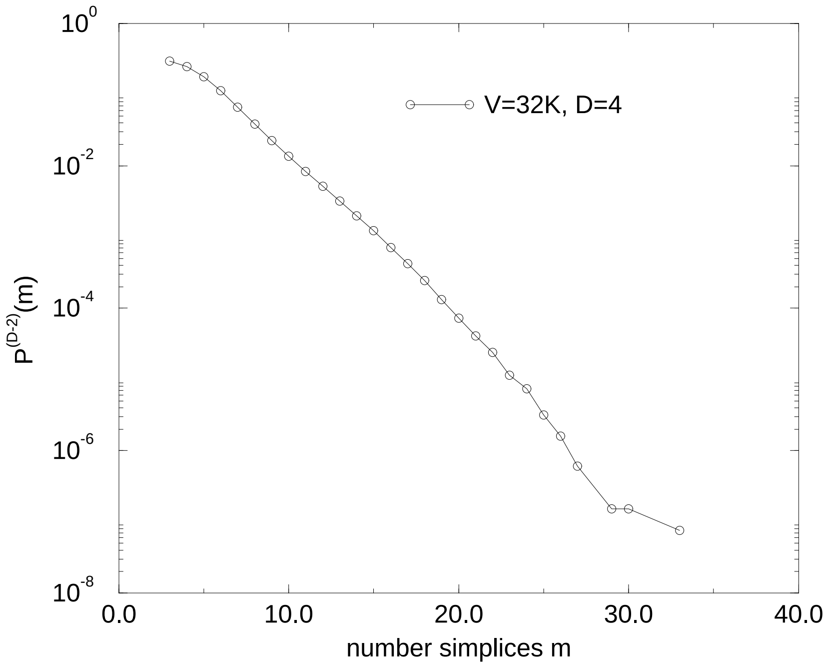

Figure 7: The normalized distribution of local volumes for $(D-2)-$simplices in four dimensions.

the simplices constituting its local volume. By our arguments this corresponds exactly to the number of triangulations of the sphere S^2 with area equal to the local volume ω_{D-3}. This grows exponentially with the local volume ω_{D-3}.

Similarly, for $i-$simplices with $i = D-4, \ldots, 0$ the local entropy is determined by the number of triangulations of the sphere $S^{D-i-1} = S^3, S^4, \ldots, S^{D-1}$ containing ω_i faces. This is known to increase *at least* as fast as exponentially with local volume ω_i. Thus, in contrast to $(D-2)-$simplices, simplices of dimension $i = 0, \ldots, (D-3)$ can maximize their local entropy by acquiring large local volumes.

If we start out with some arbitrary triangulation of fixed volume and perform a random set of local moves it is clear that individual $i-$simplices with $i = 0, \ldots, D-3$ are unstable to growing their local volumes. We can imagine qualitatively that individual simplices *compete* with each other subject to the constraint that the topology and total number of simplices remains fixed. Is it possible to understand why a *single* $(D-3)-$simplex will eventually dominate? While we cannot construct a rigorous argument that this should necessarily be so the following line of reasoning renders it, we believe, at least plausible.

Suppose we have a configuration with some number n, not necessarily $(D-2)$, potential singular vertices. If n is smaller than $(D-2)$ the system can increase its entropy by acquiring more singular vertices. What stops the number n growing arbitrarily? The entropy of each vertex increases with local volume; thus these vertices will want to grow their local volumes as large as possible. Ultimately, this means that the potential singular vertices will want to get as close as possible to each other so that they can share simplices. This overlapping of local volumes will be maximal when the potential singular vertices form the vertices of some simplex S. The overlaps between vertex volumes are then associated with subsimplices of the simplex S. These subsimplices too can gain entropy by becoming singular - their local volumes acquiring a finite fraction of all the simplices in the triangulation.

Thus the simplex S which results from the intersection of such singular simplices will itself become singular. But we have seen that singular simplices of dimension $(D-2)$ and greater are not entropically favoured. Thus the degeneration process stops when S has dimension of $(D-3)$ - it becomes the primary, singular $(D-3)-$simplex. Then the number of singular vertices cannot increase beyond $(D-2)$. This qualitative argument is able to account for at least the local stability of the configurations that are seen in the numerical simulations. It makes credible the notion that these configurations are at least *local* maxima of the total entropy.

It is also possible to understand the origin of the power-law divergence of the $(D-3)-$simplex. The dual $2-$sphere associated with this simplex is the boundary of the overlap of two $3-$spheres dual to the secondary $(D-4)-$simplices. If we assume that the simplices associated with these $3-$spheres are uniformly distributed over the surface of the spheres, then simple geometry allows us to compute the number on the boundary of the overlap – the local volume ω_{D-3}.

Introduce a length scale or radius for the $3-$sphere by equating the classical volume formula for a $3-$sphere with the number of $D-$simplices in the local volume of a $(D-4)-$simplex – $\omega_{D-4} = c_{D-4}\, V$. Uniform distribution of simplices implies that the radius of the $2-$sphere is linearly related to the radius of this $3-$sphere. Using this we can obtain a prediction for the volume of the $2-$sphere or equivalently the $(D-3)-$simplex local volume ω_{D-3}:

$$\omega_{D-3} = 3\pi \left(\frac{c_{D-4}}{2\pi^2}\right)^{\frac{2}{3}} V^{\frac{2}{3}}. \tag{4}$$

This equation predicts both the volume dependence and coefficient c_{D-3} of the singular simplex divergence (using as input the measured coefficient c_{D-4}). We have already seen that a $\frac{2}{3}$ power of the volume is very consistent with the observed scaling of ω_{D-3}. In four dimensions, the predicted value of $c_{D-3} = 0.661(2)$ which compares very well with its value estimated from a last squares fit to the $(D-3)-$simplex data, $c_{D-3} = 0.667(5)$. In five dimensions $c_{D-3} = 0.437(2)$ from Eq. 4, while our best (using the largest three volumes) fit estimate is $c_{D-3} = 0.49(2)$.

These quantitative tests lend strong support to our basic geometrical picture. Essentially these triangulations are formed by taking $D-2$ singular vertices with approximately equal volumes $c_0 V$ and gluing them together to form a singular $(D-3)-$simplex. A large fraction of all the $D-$simplices have then been used to create this special $D-$ball. The remainder are used to help glue faces of this $D-$ball together in order to create a triangulation with the correct S^D topology. In the context of the crumpled phase of DT gravity this basic structure forms a non-perturbative background about which small fluctuations in triangulation occur.

4 Singular vertex dynamics

To what extent can we trust the results of these numerical experiments? Is it possible that these configurations are not truly dominant but act as local stationary points of the entropy which trap the configurations and effectively break the ergodicity of the algorithms? We have tried to address these questions by performing a number of tests.

Current algorithms used in Monte Carlo simulation of these models rely on a sequence of $D+1$ local moves or re-triangulations which are known to be ergodic on the full space of triangulations T (at least for $D<5$ [6]). However, to approach the continuum limit in a regular fashion lattice simulations are restricted to the microcanonical ensemble T_V characterized by $\Omega_D(V)$. Actually to allow the elementary moves to be carried out it is necessary that the volume be allowed to fluctuate by at least $+/-D$. Typical simulation strategies have relaxed this restraint still further and allowed the volume to fluctuate about the target volume by some amount $\Delta V > D$.

Unfortunately, the ergodic properties of the algorithm when thus restricted to a fixed volume slice T_V are unknown. One simple scenario might be that the system possesses 'volume barriers' $B(V)$ of all sizes up to some volume dependent limit,

$$B(V) \leq B_{\max}(V) \ . \qquad (5)$$

We might then expect a practical breakdown of ergodicity if the allowed fluctuation volume ΔV becomes less than $B_{\max}(V)$. It is possible that such an effect might be important in effectively trapping configurations in the vicinity of one of these singular triangulations. We have investigated this issue by performing simulations with a variety of ΔV. In order to keep control of the systematic error associated with the finite volume V we have chosen to take measurements only when the volume of the triangulation lies within some distance δ of the target volume V. In practice we have set $\delta = 10$. A breaking of ergodicity would be signaled by a dependence of expectation values on ΔV.

Table 1 summarizes our results in the case of $D=4$ for a volume $V=4000$ (similar results have been obtained in $D=3$ and at other lattice sizes). We observe no statistically significant dependence in the mean vertex number $\langle N_0 \rangle$ and mean intrinsic extent $\langle L \rangle$ on ΔV over a wide range in ΔV. This is a very encouraging and, in principle, non-trivial result. It is in agreement with earlier studies [7, 8]

ΔV	$\langle L\rangle$	$\langle N_0\rangle$
6	9.168(5)	226.7(2)
10	9.156(6)	226.1(7)
14	9.187(14)	226.9(7)
20	9.153(20)	226.1(7)
32	9.157(10)	226.6(3)
45	9.137(19)	226.7(8)
63	9.156(15)	226.6(8)
100	9.165(6)	226.5(1)
200	9.166(9)	226.3(4)

Table 1: The dependence of expectation values on fluctuation parameter ΔV for $D = 4$ and $V = 4000$.

which have have not shown any evidence of ergodicity breaking in four and five dimensions.

However, we have observed *very* long autocorrelation times in both observables which can easily obscure this result if only short runs are employed. The upper graph of Fig. 8 illustrates this with a plot of the Monte Carlo time series for the mean extent $\langle L\rangle$ for a lattice of size $V = 4000$ with $\Delta V = 10$. We can see that the system makes occasional excursions to 'super-crumpled' states with small extent and remains trapped there for many tens of thousands of sweeps before it can tunnel back. The typical timescale between such events is of order one million sweeps![2]

We have observed the same problems over wide ranges in the fluctuation volume ΔV. The frequency of such large fluctuation events seems independent of this parameter. The lower graph of Fig. 8 contains a plot of the two vertices with the largest local volumes for the same Monte Carlo history. It is clear that these rare fluctuation events are associated to the appearance and disappearance of singular vertices. For small volumes it appears that the system can sometimes have zero or a single singular vertex - contrary to the claims made in the previous section which state that configurations with two such vertices are dominant. However, that claim is true only for $V \to \infty$ and it is clear that for small volumes tunneling between distinct free energy minima (labeled by differing numbers of singular vertices) can occur. However, our simulations revealed no evidence that the tunneling time depends on ΔV.

We have also observed long transient effects in trying to equilibrate larger volumes. Fig. 9 illustrates a typical run for $V = 8K$ in four dimensions. We show both $\langle L\rangle$ and the two most singular vertices. It is clear that the system appears to

[2] One sweep corresponds to V attempted local moves.

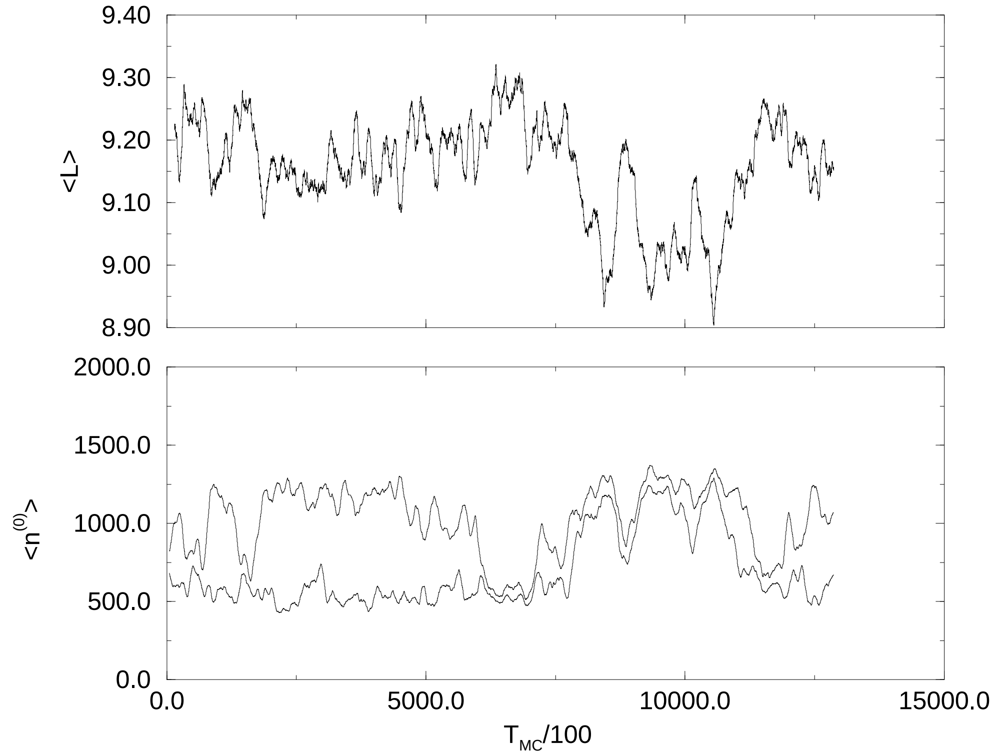

Figure 8: The MC time series for the mean intrinsic extent $\langle L \rangle$ (top) and the local volumes associated to the two most singular vertices (bottom) for $V = 4000$ and $D = 4$.

settle down into an equilibrium state with small fluctuations after perhaps a few tens of thousands of sweeps. This state contains precisely one singular vertex. However it is clearly metastable and after a further few hundred thousand sweeps undergoes a rapid transition to a more crumpled state possessing two singular vertices. We have not managed to observe any subsequent reverse tunneling. This transient behaviour has been observed for many different fluctuation volumes ΔV and a variety of initial state configurations. Similar behaviour has also been seen in five and six dimensions at small to intermediate volumes. It is also consistent with recent findings reported by Hotta et al [4] who observe a relaxation to two singular vertices in four dimensions independent of start configuration.

In three dimensions we observe no singular vertices and no corresponding tunneling or metastable behaviour. In this sense three dimensions appears qualitatively different from four and higher dimensions.

To summarize this section. We have looked for evidence that the singular states are metastable as a consequence of ergodicity breaking in the simulation algorithm. Such a breaking would be signaled by a dependence of expectation values on the fluctuation parameter ΔV. Over a wide range of this parameter we observe

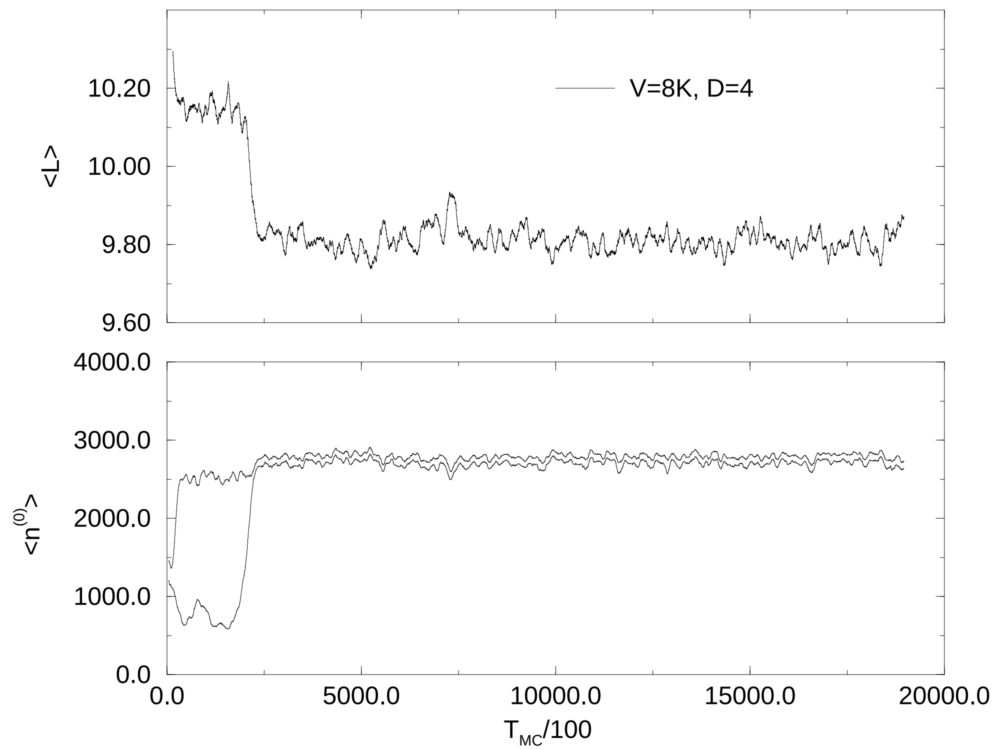

Figure 9: The same as the previous figure for $V = 8000$.

no such dependence. While we observe long autocorrelation times at small volume associated with singular vertex dynamics, this behaviour appears to disappear for large volume and we are led to conclude that the singular states do indeed saturate $\Omega_D(V)$ in the thermodynamic limit.

5 Possible consequences

Our numerical results imply that the typical triangulations as $V \to \infty$ are singular configurations - they consist of a set of $D-2$ singular vertices assembled into a singular $(D-3)-$simplex. The local volume associated to the $(D-3)-$simplex increases as a fractional power $p \sim \frac{2}{3}$ of the total volume. In contrast, the local volume associated to its secondary subsimplices increases in proportion to the volume. We have argued in section 3 that this structure is at least a local maximum of the entropy function for triangulations with fixed volume. The numerical results of section 4 imply that it appears to be a global maximum. The question arises whether this structural information can be used to cast light on a variety of other issues in DT gravity.

Specifically, in four dimensions can we learn anything about the possibility of an exponential bound in the microcanonical partition function; i.e. $\Omega_4(V) \sim$

$\exp \mu V$? Such a bound is needed to take the thermodynamic limit. A proof for triangulated manifolds has so far eluded all efforts (although a related proof for metric ball coverings does exist [9]). Direct numerical estimates of $\Omega_4(V)$, while consistent with such a bound, are unfortunately plagued with large finite size effects which require rather delicate analysis [11, 12, 13, 14].

In four dimensions the important simplicial manifolds consist of two elementary $4-$balls containing the singular vertices joined along a common link. Approximately two thirds of the total volume is locked up in these balls, which become independent in the $V \to \infty$ limit.

Thus the triangulation of the four-sphere contains within it two independent $3-$sphere boundaries carrying a large fraction of the total volume. Provided the number of triangulations of the $3-$sphere is bounded exponentially (which is believed to be true from previous numerical simulations [10]), the proof of the $4D$ bound rests on showing that the number of ways these balls can be glued together, using the remaining one third of the volume, is exponentially bounded. This seems to be an easier task than to show that the triangulation space is exponentially bounded for arbitrary triangulations. However we have not been able to prove this or its obvious generalizations to higher dimensions.

Indeed, there is one question which we do not understand concerning this structure. Why does the primary singular $(D-3)-$simplex only diverge sublinearly with volume in contrast to the linear divergence of all lower dimension singular simplices? Does this signal a different volume behaviour of the entropy function for S^2 as compared with S^r, $r = 3, \ldots$ or is it a simple consequence of the constraints which are present? Further work, both analytic and numerical, is needed to understand the consequences of this and related features of the structure presented here.

Acknowledgments

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