Physics   College of Arts and Sciences

2-28-1997

Minimal Dynamical Triangulations of Random Surfaces

Simon Catterall
Syracuse University

Mark Bowick
Syracuse University

Gudmar Thorleifsson
Syracuse University

Recommended Citation

Minimal Dynamical Triangulations of Random Surfaces

Mark J. Bowick, Simon M. Catterall
and *Gudmar Thorleifsson*

Department of Physics, Syracuse University
Syracuse, NY 13244-1130, USA

Abstract

We introduce and investigate numerically a *minimal* class of dynamical triangulations of two-dimensional gravity on the sphere in which only vertices of order five, six or seven are permitted. We show firstly that this restriction of the local coordination number, or equivalently intrinsic scalar curvature, leaves intact the fractal structure characteristic of generic dynamically triangulated random surfaces. Furthermore the Ising model coupled to *minimal* two-dimensional gravity still possesses a *continuous* phase transition. The critical exponents of this transition correspond to the usual KPZ exponents associated with coupling a central charge $c = \frac{1}{2}$ model to two-dimensional gravity.

That two-dimensional ($2D$) quantum gravity can be regularized and studied using *dynamical triangulations* (DT) is well known [1] and has lead to extensive study of the properties of dynamically triangulated random surfaces (DTRS). Indeed, many of the properties of critical spin systems on such lattices have been shown to follow from continuum treatments of central charge $c < 1$ theories coupled to $2D$-quantum gravity. Furthermore, this formulation has the advantage that it renders tractable the calculation of many features of these theories which are inaccessible to continuum methods. Foremost amongst these are questions related to the quantum geometry and fractal structure of the typical $2D$ manifolds appearing in the partition function [2].

It has often been speculated that some of these models should also find a realization in condensed matter physics as models of membranes and/or interfaces with *fluid* in-plane degrees of freedom. The problem in trying to map such problems onto dynamically triangulated models, however, has always been the occurrence of large vertex coordination numbers in an arbitrary random graph – real condensed matter systems typically place a cut-off on the number of interactions allowed by the microscopic degrees of freedom. It is possible that such cut-offs are unimportant in the

arXiv:hep-th/9605167v2 28 Feb 1997

critical region of these theories. As a crude analogy we note that the short-distance differences between different discrete polygonal decompositions of two-dimensional gravity are, without fine-tuning, irrelevant in the continuum limit [1].

In this letter we present numerical results for a new model (called the minimal dynamical triangulation or MDT model) which is based on random triangulations but with an additional constraint: the coordination number of any vertex in the lattice can only be five, six or seven. Such minimal models implement a severe form of the microscopic interaction cut-off required by condensed matter systems and correspond in gravitational language to the inclusion of only three possible states for the local curvature $-2\pi/7$, 0, $2\pi/5$. (The scalar curvature at a vertex with coordination number q_i is given by $R = 2\pi\frac{6-q_i}{q_i}$.) We have studied two systems: pure gravity as modelled by pure (minimal) triangulations and a set of Ising spins coupled to a dynamical lattice generated by minimal triangulations. We show in both cases that the critical behavior of the system appears to be identical to the conventional DTRS models – yet another striking indication of universality.

The dynamically triangulated version of $2D$–gravity is given by the partition function

$$Z(\mu) = \sum_A e^{-\mu A} Z(A), \tag{1}$$

$$Z(A) = \sum_{T \in \mathcal{T}} Z_M . \tag{2}$$

$Z(A)$ is the fixed area partition function, μ the cosmological constant and $\mathcal{T}$ the class of triangulations summed over. Z_M represents the partition function of some generic matter field living on the surface. A given choice of $\mathcal{T}$ corresponds to a particular discretization of the surfaces. Two classes commonly used are combinatorial and degenerate triangulations, $\mathcal{T}_C$ and $\mathcal{T}_D$. In the former triangles are glued together along edges so as to form closed manifolds, with the constraints that any two triangles can only have one edge in common and no triangle can be glued onto itself. For degenerate triangulations these constraints are relaxed – this amounts to allowing tadpoles and self-energy diagrams in the dual representation to triangulations. Clearly $\mathcal{T}_C \subset \mathcal{T}_D$.

A priori different classes $\mathcal{T}$ do not have to lead to the same theory, although the two aforementioned do, at least for the case of Ising matter where the models are exactly soluble [3]. Clearly this would no longer be true if $\mathcal{T}$ only contained *one* triangulation – an extreme limit. The class of triangulation $\mathcal{T}_M$ we investigate in this letter is a subclass of $\mathcal{T}_C$ defined by allowing only vertices of order 5, 6 or 7. This is as far one can go in suppressing curvature fluctuations on the surface while still retaining its dynamical (fluid) nature.

We start by investigating the minimal model in the absence of matter ($Z_M = 1$ in Eq. 2). To see if this model belongs to the same universality class as pure gravity with triangulation space $\mathcal{T}_C$ or $\mathcal{T}_D$ we study its critical behavior numerically.
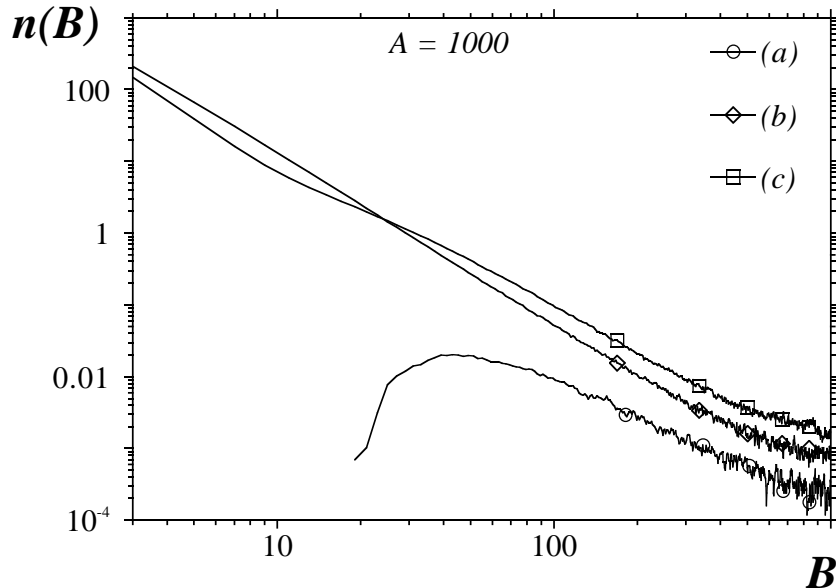
2

Figure 1: The measured distributions of baby universes for *(a)* the minimal class of triangulations, *(b)* combinatorial triangulations and *(c)* triangulations obtained by node decimation from *(a)*. All measurements are for triangulations of 1000 nodes.

More precisely we measure the string susceptibility exponent γ_s, defined through the scaling $Z(\mu) \sim (\mu_c - \mu)^{2-\gamma_s}$. The method we employ is that of calculating the distribution of *minimal baby universes* $n(B)$ on the ensemble of triangulations [4]. A baby universe of area B is a region of the triangulation joined to the bulk solely by a minimal neck. For the class $\mathcal{T}_M$ (and $\mathcal{T}_C$) a minimal neck is a loop consisting of three links. Using the expected asymptotic behavior of the fixed area partition function, $Z(A) \sim \exp(\mu_c A) A^{\gamma_s - 3}$, a simple argument relates $n(B)$ to γ_s [4, 5];

$$ n_A(B) \sim [B(A - B)]^{\gamma_s - 2}, \quad A \to \infty. \tag{3} $$

We have simulated the fixed area partition function $Z(A)$, using Monte Carlo methods, for triangulations consisting of 1000 and 4000 vertices [6]. Henceforth we will use the number of vertices as our measure of the area A. In Fig. 1 we show the measured distribution $n(B)$ for $A = 1000$. For comparison we include the corresponding distribution measured on the class of combinatorial manifolds $\mathcal{T}_C$. Fig. 1 also includes a third distribution (c) whose significance will be addressed shortly. The first thing we notice is that these distributions are very different for $B < 100$ – in particular there are no small baby universes for the class $\mathcal{T}_M$ as the curvature restriction puts a lower limit on the size of baby universes; our construction effectively smoothens the triangulation locally.

For larger baby universes the distributions look much more alike. To measure γ_s

3

Table 1: Measured values of γ_s for minimal dynamical triangulations (without matter) after applying varying levels of node decimation with a blocking factor of $b = 2$.

$A^{(k)}$	$A^{(0)} = 1000$		$A^{(0)} = 4000$	
	k	γ_s	k	γ_s
4000			0	-0.530(26)
2000			1	-0.544(32)
1000	0	-0.644(48)	2	-0.530(18)
500	1	-0.619(26)	3	-0.504(9)
250	2	-0.574(49)	4	-0.478(36)

we fit the tail of $n(B)$ to Eq. 3. This gives $\gamma_s = -0.64(5)$ ($A = 1000$) and $-0.53(3)$ ($A = 4000$). This agrees with the exact value for pure gravity $\gamma_s = -0.5$. The corresponding value obtained using $\mathcal{T}_C$ is $\gamma_s = -0.501(4)$ ($A = 1000$) and $-0.504(4)$ ($A = 4000$). It is easy to understand why the values for minimal triangulations are less accurate – finite-size effects are more pronounced since more data for small values of B has to be excluded before we reach the asymptotic regime. Yet the result indicates that the minimal triangulated model is in the same universality class as the full-blown DTRS model.

Additional evidence for this claim is obtained by applying a recently proposed Monte Carlo renormalization group method for blocking dynamical triangulations (*node decimation* [8]) to the model. Each triangulation is blocked by removing vertices at random and in the process the restrictions on the curvature are dropped so that the model flows into the wider class of combinatorial triangulations. In this way we can ascertain whether the model after blocking is the same as that obtained with the triangulation class $\mathcal{T}_C$ – in particular we would like to know if the RG transformation generates a flow towards a fixed point, accessible from combinatorial triangulations, and corresponding to pure gravity.

Curve (c) of Fig. 1 is the measured distribution $n(B)$ for surfaces of 1000 vertices obtained from surfaces of 4000 vertices by node decimation. It is apparent that this distribution is much closer to the distribution for combinatorial triangulations than the minimal one. This we take as an indication that the model flows towards the same non-trivial pure gravity fixed point as combinatorial manifolds. This evidence can be strengthened by measuring the exponent γ_s on the set of triangulations obtained with blocking. This we show in Table 1 for various degrees of blocking. Agreement with the pure gravity value clearly improves with blocking. This confirms that the model, after blocking, is closer to the pure gravity fixed point.

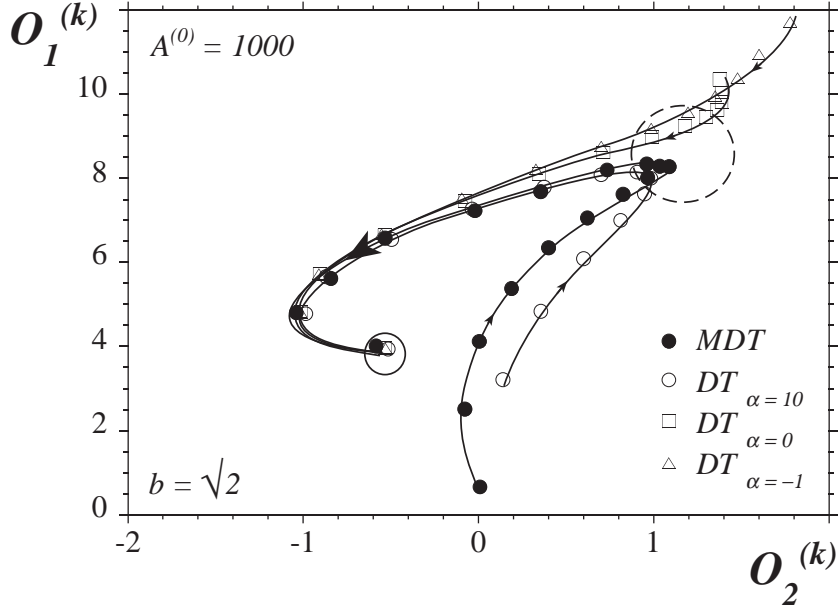The RG flows above can be visualized by looking at the expectation values of

4

Figure 2: The RG flow of curvature operators in the (O_1, O_2)–plane. The trajectory for minimal triangulations is marked with solid circles, while open symbols correspond to combinatorial manifolds with varying degrees of coupling to a logarithmic curvature term. The dashed circle indicates a non-trivial fixed point, while the solid one is a zero-volume sink.

a subset of operators such as

$$O_1^{(k)} = \left\langle \sum_i (q_i^{(k)} - 6)^2 \right\rangle \quad \text{and} \quad O_2^{(k)} = \left\langle \sum_{<ij>} (q_i^{(k)} - 6)(q_j^{(k)} - 6) \right\rangle, \qquad (4)$$

where q_i is the curvature of a node i, k is the blocking level and the first sum is over nodes while the second is over nearest neighbors. Different trajectories in the (O_1, O_2)–plane can be accessed by adding an irrelevant operator to the bare action and varying its coupling constant. We have chosen $\alpha \sum_i \log q_i$. This is shown in Fig. 2. By changing the coupling constant α we effectively change the trajectories identifying the location of the non-trivial fixed point. And indeed the trajectory for the minimal triangulations flows directly towards the pure gravity fixed point before it veers off and follows a universal trajectory into a zero-volume sink.

Our findings thus far are consistent with the exact solution of two-dimensional R^2 gravity given in [9]. These authors solve a matrix model of dually weighted planar graphs which allows the incorporation of the higher-curvature operator R^2. They find that the R^2 operator is non-perturbatively irrelevant in the continuum limit. The model always reduces at large length scales to a model of pure gravity

– there is no transition to a "flat" phase. For technical reasons the solution of [9] is restricted to lattices with *even* coordination number vertices. There is thus no direct mapping to the model we treat. Nevertheless in both cases it is found that higher-curvature contributions to the effective action are irrelevant.

In the second part of this letter we investigate the Ising model coupled to minimal dynamical triangulations. In analogy to the situation in conventional dynamical triangulation we expect that the addition of Ising spins to the graphs leads to the possibility of a spin-ordering phase transition at some critical value of the temperature β_c [10]. Performing simulations on triangulations ranging from 250 to 8000 vertices we indeed observe all the usual signs of such a transition; a peak in the spin susceptibility which narrows and grows with increasing lattice volume, a non-zero magnetization for large coupling and a cusp in the specific heat.

We have attempted to determine the infinite volume critical coupling β_c using an extrapolation of the pseudo-critical coupling $\beta_c(A)$ associated with the peaks in the specific heat and the derivative of Binder's cumulant g_r. In both cases the location of the peaks is expected to approach β_c as $|\beta_c - \beta_c(A)| \sim A^{-1/\nu d_H}$. This fit is made possible by an independent determination of νd_H – the height of the peak in $\partial g_r/\partial \beta$ scales as $A^{1/\nu d_H}$. In Fig. 3 we show the scaling of the pseudo-critical couplings together with the infinite volume extrapolation. Our best estimate of the critical coupling from this is $\beta_c = 0.2663(3)$.

It is interesting to compare this value to the critical temperature of the Ising model on other classes of triangulations: $\beta_c \approx 0.2747$ (fixed triangular lattice), $\beta_c \approx 0.2163$ (combinatorial), and $\beta_c \approx 0.1603$ (degenerate). It is clear that magnetic ordering occurs more readily as the triangulation space is enlarged.

To obtain the critical exponents of the model we use standard finite size scaling. We have already discussed νd_H. In the same way α is obtained from the scaling of the peak in the specific heat: $C_V = c_0 + c_1 A^{\alpha/\nu d_H}$. From the scaling of the magnetization $\mathcal{M}$ and the spin susceptibility χ at the critical point we get β and γ respectively. The results are shown in Table 2. There we also demonstrate the finite size effects present by systematically excluding data for small volume. The results are compared to the corresponding values for the Onsager solution of the Ising model on a fixed $2D$ lattice, and the KPZ values for the Ising model coupled to $2D$-gravity. It is clear that for the Ising model coupled to the minimal class of triangulations the values agree with those of KPZ. An analytic formulation of a model with vertices of restricted coordination number has recently been given [12]. This model treats squares with coordination number three, four or five. Although discrete loop equations can be derived they have yet been solved.

In this letter we have provided strong evidence that a new class of triangulated models, the MDT models, exhibit the same critical behavior as the conventional dynamically triangulated random surface models of two-dimensional quantum gravity. We have demonstrated this both in pure gravity, where the critical properties are encoded in the string susceptibility γ_s, and in the critical Ising model coupled
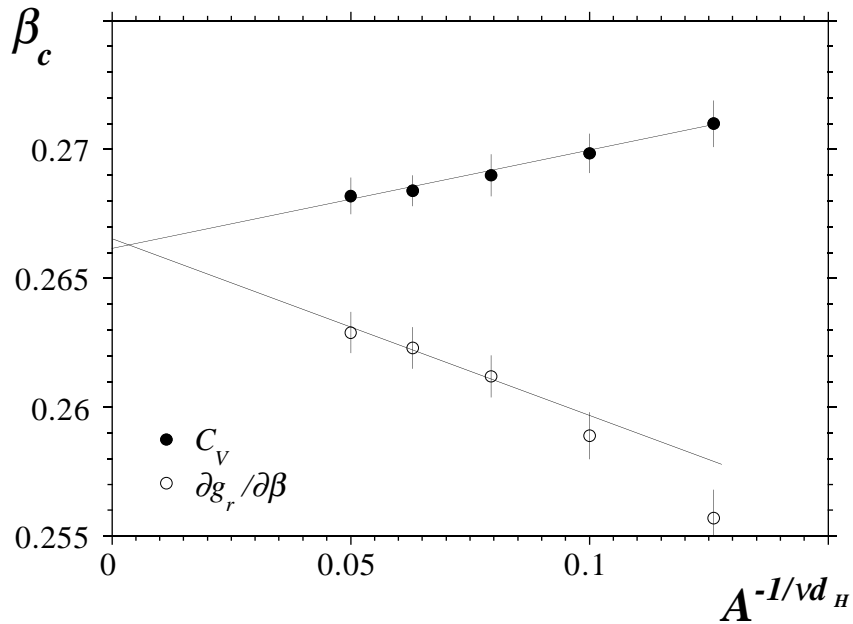
Figure 3: The infinite volume extrapolation of the pseudo-critical couplings obtained from the peaks in the specific heat and the derivative of Binder's cumulant. The data points correspond to $A = 500, 1000, 2000, 4000$ and 8000. For the Binder's cumulant fit only the three largest volumes are included in the extrapolation since finite-size effects are more pronounced. The rescaling uses the measured value $\nu d_H = 2.9$.

Table 2: The measured critical exponents. νd_H and α are obtained from the scaling of the peaks in $\partial g_r/\partial \beta$ and C_V respectively. β and γ are measured from scaling at β_c, in which case the errors are dominated by the uncertainty in the location of β_c.

A	νd_H	α	β	γ
250-8000	2.857(24)	-0.806(41)	0.474(20)	2.117(54)
500-8000	2.890(33)	-0.922(58)	0.488(21)	2.095(52)
1000-8000	2.907(25)	-0.977(87)	0.500(18)	2.070(55)
Onsager	2	0(log)	1/8	7/4
KPZ	3	-1	1/2	2

to gravity where the Ising exponents yield a sensitive test of the critical behavior of the model. Our results suggest that putting rather severe cut-offs on the vertex coordination number does not affect the universal behavior of the model. This is a highly non-trivial observation and implies that it may indeed be possible to find physical $2D$ systems which exhibit this universal behavior [11].

For quantum gravity it implies that large fluctuations in the curvature at the scale of the lattice cut-off are unimportant for determining the continuum structure of $2D$ quantum gravity – small curvature defects are sufficient. The pure gravity model may be interpreted, similarly to [9], as a gas of strength ± 1 curvature defects (or vortices) in flat space. In the same way the Ising-type model consists of an interacting gas of Ising spins and curvature defects – the gravitational dressing of the Ising spins being a function of their interaction with these discrete defects. In this context, it may be possible to get some insight into the physical origins of this dressing within this simple model. A study of the geometry of spin clusters and its relation to the curvature defect distribution is certainly warranted.

References

[1] F. David, *Simplicial Quantum Gravity and Random Lattices*, (hep-th/9303127), Lectures given at Les Houches Summer School on Gravitation and Quantization, Session LVII, Les Houches, France, 1992; J. Ambjørn, *Quantization of Geometry*, (hep-th/9411179), Lectures given at Les Houches Summer School on Fluctuating Geometries and Statistical Mechanics, Session LXII, Les Houches, France 1994; P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. 254 (1995) 1.

[2] S. Catterall, G. Thorleifsson, M. Bowick and V. John, Phys. Lett. B354 (1995) 58; J. Ambjørn, J. Jurkiewicz and Y. Watabiki, Nucl. Phys. B454 (1995) 313.

[3] V.A. Kazakov, Phys. Lett. A119 (1986) 140: D. Boulatov and V.A. Kazakov, Phys. Lett. B186 (1987) 379; Z. Burda and J. Jurkiewicz, Acta. Phys. Polon. B20 (1989) 949.

[4] S. Jain and S.D. Mathur, Phys. Lett. B286 (1992) 239; J. Ambjørn, S. Jain and G. Thorleifsson, Phys. Lett. B297 (1993) 34; J. Ambjørn and G. Thorleifsson, Phys. Lett. B323 (1994) 7.

[5] Strictly speaking surfaces obtained by cutting along a minimal neck are not in $\mathcal{T}_M$ since they have punctures and the vertices on the boundary do not obey the same restrictions as those in the interior. This distinction is not important for our numerical measurements.

[6] The triangulations are updated using the standard link-flip algorithm. Its ergodicity on the class of minimal triangulations is, though, an open issue. The usual proof of ergodicity for DTRS[7] is based on the reduction of dual ϕ^3 graphs to a canonical form with the aid of the appropriate Schwinger-Dyson equations for amplitudes. This proof does not extend directly to the minimal class of triangulations.

[7] D.V. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, Nucl. Phys. B275 (1986) 641.

[8] G. Thorleifsson and S.M. Catterall, Nucl. Phys. B461 (1996) 350.

[9] V.A. Kazakov, M. Staudacher and T. Wynter, Nucl. Phys. B471 (1996) 309.

[10] J. Jurkiewicz, A. Krzywicki, B. Petersson and B. Soderberg, Phys. Lett. B213 (1988) 511.

[11] One such physical system may be a *liquid ferromagnet*. We caution, however, that the Ising model coupled to minimal dynamical geometry would not be a planar liquid ferromagnet because of the highly fractal nature of the intrinsic two-dimensional geometry in this system. It remains a challenge to identify an appropriate physical system to which our model applies.

[12] S. E. Carroll, M.E. Ortiz and W. Taylor, Nucl. Phys. B. 468 (1996) 383.

9