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Stripes from (noncommutative) stars

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Abstract

We show that lattice regularization of noncommutative field theories can be used to study non-perturbative vacuum phases. Specifically we provide evidence for the existence of a *striped* phase in two-dimensional noncommutative scalar field theory.

Introduction

The mixing of UV and IR physics in noncommutative field theories is a intriguing phenomenon. One of the more spectacular manifestations of this mixing is the appearance of a so-called striped phase which breaks translational invariance in noncommutative ϕ^4 theory. The prediction of this phase was based on a self-consistent Hartree treatment of one-loop diagrams [1] or a one-loop renormalization group approach [2]. The purpose of the following note is to test the predictions using the non-perturbative framework of lattice computer simulations and at the same time test the viability of studying noncommutative field theories by numerical means. Here we consider the simplest possible such theory: a two-dimensional ϕ^4 theory. Two-dimensional chiral and Yang-Mills theories have also been studied recently [3, 4].

The model

Our starting point is noncommutative scalar field theory with the action

$$S = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g^2}{4} \phi^{*4} \right], \quad (1)$$

where $\phi^{*4} = \phi * \phi * \phi * \phi$ and where the star product is defined by

$$(\phi_1 * \phi_2)(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_{x_\mu} \partial_{y_\nu}} \phi_1(x) \phi_2(y) \Big|_{x=y} \quad (2)$$

$$= \frac{1}{\pi^2 |\det \theta|} \int \int d^2y d^2z \phi_1(y) \phi_2(z) e^{2i\theta_{\mu\nu}^{-1} (x-y)_\mu (x-z)_\nu}. \quad (3)$$

In particular we have

$$[x_\mu, x_\nu]_* = x_\mu * x_\nu - x_\nu * x_\mu = i\theta_{\mu\nu} = i\theta \varepsilon_{\mu\nu}. \quad (4)$$

It is known how to map formally the noncommutative theory into a matrix theory by the Weyl map. Let \hat{x}^μ and \hat{y}^ν be Hermitean operators satisfying $[\hat{x}_\mu, \hat{y}_\nu] = i\theta_{\mu\nu}$, which is the operator analogy of (4). One can define the Weyl map by

$$\hat{\phi} = \int d^2x \phi(x) \hat{\Delta}(x), \quad \hat{\Delta} = \int \frac{d^2k}{(2\pi)^2} e^{ik_\mu \hat{x}_\mu} e^{-ik_\nu x_\nu}. \quad (5)$$

and in this way the action (1) can be represented (formally, with a suitable normalization of the trace and suitable assumptions concerning the fall off of the functions $\phi(x)$ at infinity) as

$$S = \text{Tr} \left(\frac{1}{2} [\hat{\partial}_\mu, \hat{\phi}]^2 + \frac{1}{2} m^2 \hat{\phi}^2 + \frac{g^2}{4} \hat{\phi}^4 \right), \quad (6)$$

where $\hat{\partial}_\mu$ is an anti-hermitian derivative which together with \hat{x}_μ satisfy

$$[\hat{\partial}_\mu, \hat{x}_\nu] = \delta_{\mu\nu}, \quad [\hat{\partial}_m, \hat{\partial}_\nu] = ic_{\mu\nu}, \quad [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}. \quad (7)$$

The operators (or matrices) in (7) are all infinite dimensional. However, the associated Heisenberg algebras of exponentials of \hat{x}_μ (and $\hat{\partial}_\mu$) allow finite dimensional representations, as already noted by Weyl. This is the key to formulating the non-commutative theory on a finite periodic $N \times N$ lattice with lattice spacing a (see [5] for a more general formulation). The algebra (7) is replaced by

$$\hat{Z}_\mu \hat{Z}_\nu = e^{-2\pi i \Theta \varepsilon_{\mu\nu}} \hat{Z}_\nu \hat{Z}_\mu, \quad \hat{D}_\mu \hat{Z}_\nu \hat{D}_\mu = e^{2\pi i \delta_{\mu\nu}/N} \hat{Z}_\nu, \quad (8)$$

where

$$\hat{Z}_\mu = e^{2\pi i \hat{x}_\mu / aN}, \quad \hat{D}_\mu = e^{a\hat{\partial}_\mu}, \quad \Theta = \frac{2\pi\theta}{a^2 N^2}. \quad (9)$$

Periodicity on the lattice fixes the quantization of momentum p_μ and θ

$$p_\mu a = k_\mu = \frac{2\pi m_\mu}{N}, \quad \theta = \frac{na^2 N}{\pi} \left(\Theta = \frac{2n}{N} \right), \quad (10)$$

where $m_\mu \in 0, \dots, N-1$, and n is an integer. In the following we choose $n=1$ for simplicity.

A simple realization of the above scenario is obtained by choosing \hat{Z}_μ , $\mu=1, 2$ to be the standard $N \times N$ shift and clock matrices $(\hat{Z}_1)_{jk} = \delta_{j+1,k}$, $(\hat{Z}_2)_{jk} = (e^{4\pi i/N})^{j-1} \delta_{jk}$, $\hat{D}_1 = (\hat{Z}_2^\dagger)^{(N+1)/2}$, $\hat{D}_2 = (\hat{Z}_1)^{(N+1)/2}$, the lattice size N being odd.

In this way the lattice version of (3) becomes

$$\phi_1(x) * \phi_2(x) = \frac{1}{N^2} \sum_{y,z} \phi_1(y) \phi_2(z) e^{2i\theta_{\mu\nu}^{-1}(x-y)_\mu(x-z)_\nu}, \quad (11)$$

and the operator $\hat{\Delta}$ becomes an $N \times N$ matrix

$$\hat{\Delta}(x) = \sum_{m_1, m_2=1}^N \left(\hat{Z}_1^{m_1} \hat{Z}_2^{m_2} e^{-\pi i \Theta m_1 m_2} \right) e^{-2\pi i m_\mu x_\mu / aN}. \quad (12)$$

It generates an explicit map from the lattice field $\phi(x)$ defined at the $N \times N$ lattice points x to that $N \times N$ matrix $\hat{\phi}$ defined by (5),

$$\hat{\phi} = \frac{1}{N^2} \sum_x \phi(x) \hat{\Delta}(x), \quad \phi(x) = \frac{1}{N} \text{Tr} \hat{\phi} \hat{\Delta}(x) \quad (13)$$

such that the action is given by (6) with $N \times N$ matrices and the partition function is

$$Z(R, G) = \int d\hat{\phi} e^{-S[\hat{\phi}]}, \quad (14)$$

$$S[\hat{\phi}] = N \text{Tr} \left[\frac{1}{2} \sum_\mu \left(\hat{D}_\mu \hat{\phi} \hat{D}_\mu^\dagger - \hat{\phi} \right)^2 + R \hat{\phi}^2 + G \hat{\phi}^4 \right]. \quad (15)$$

The dimensionless lattice parameters are

$$R = \frac{1}{2}m^2a^2, \quad G = \frac{1}{4}g^2a^2. \quad (16)$$

The ordinary (commutative) ϕ^4 theory on the lattice has two different continuum limits when the space-time volume goes to infinity, one “trivial” associated with the fine tuning the approach to $R = 0$ and $G = 0$. It corresponds to the standard ϕ^4 two-dimensional field theory where only normal ordering is needed to renormalize the theory. The other limit is associated with the phase transition between a phase in which the symmetry $\phi \rightarrow -\phi$ is broken and a symmetric phase. It occurs for finite negative values of R and finite positive values of G and the continuum limit can be identified with the continuum limit of the Ising model, i.e. with a $c = 1/2$ conformal field theory.

In the noncommutative case the continuum limit of course requires an infinite lattice, i.e. we have to take N to infinity. In addition we are interested in a limit in which the lattice spacing can be viewed as scaling to zero while the dimensionful parameter θ stays finite. Two possible scenarios appear possible:

If non-commutativity can be considered as a perturbation imposed on a commutative theory then a kind of “double scaling limit” of the matrix model (15) suggests itself:

$$N \rightarrow \infty, \quad a^2 = \frac{\pi\theta}{N}, \quad a^2N^2 = \pi\theta N. \quad (17)$$

Thus the physical space-time diverges like N^{-1} and we can write the matrix model action (15) as

$$S[\hat{\phi}] = N\text{Tr} \left[\frac{1}{2} \sum_{\mu} \left(\hat{D}_{\mu} \hat{\phi} \hat{D}_{\mu}^{\dagger} - \hat{\phi} \right)^2 + \frac{\pi m^2 \theta}{2N} \hat{\phi}^2 + \frac{\pi g^2 \theta}{4N} \hat{\phi}^4 \right]. \quad (18)$$

The noncommutative parameter θ only appears in the dimensionless combinations $m^2\theta$ and $g^2\theta$ and implicitly in factors of N appearing in (18). This type of continuum limit seems to be most naturally associated with the trivial gaussian fixed point of the commutative theory. Note that this “double scaling” limit $N \rightarrow \infty$ is quite different from the usual planar limit of matrix models and thus also quite different from the double scaling limit of the 2d matrix models studied in the context of 2d quantum gravity.

If, on the other hand, non-commutativity alters the commutative theory profoundly one should follow the general strategy of non-perturbative lattice studies in order to identify a continuum limit: identify suitable physical observables usually related to a divergent correlation length of the lattice field. Fine-tuning the coupling constants such that the observables are constant when expressed in terms of the “physical” correlation length $a \cdot \xi$, where ξ is the correlation length

¹For other limits which allow for a finite physical space-time see [7, 6, 5].

in lattice units, will then lead to a relation between the coupling constants and the lattice spacing a eg. $a \sim (R - R_c)^\nu$ and allow us to identify the (potentially) non-perturbative renormalization of the coupling constants required to obtain a continuum theory. We can then potentially investigate another “double scaling” limit in which $N(R - R_c)^{2\nu} \sim \theta$ is held constant as $N \rightarrow \infty$ and $R \rightarrow R_c$. Such a phase transition would generalize the usual Ising transition of the commutative theory to the noncommutative case.

The first scenario has the attractive feature that we know ahead of time the value of the non-commutativity parameter θ and the dependence of lattice spacing on the matrix size N . We will show numerical evidence which supports scaling according to this scenario although we stress that the existence of such a double scaling limit does not rule out the possible existence of additional continuum limits based on the second scenario.

Susceptibilities (i.e. correlation functions integrated over all space-time points) have been used in conventional lattice field theory to identify the couplings associated with a divergent correlation length, and the critical indices associated with the susceptibilities lead to a classification of the continuum Euclidean field theories. Susceptibilities are suitable observables also in the noncommutative case. In fact one has

$$\int d^d x \phi(x) * \phi(x) = \int d^d x \phi(x)\phi(x), \quad (19)$$

so they are maximally close to the commutative observables. Yet they may contain additional interesting information about the infrared behavior of the noncommutative propagators. In the computer simulations to be reported below we will indeed use (generalized) susceptibilities to identify potential phase transitions.

$$\begin{aligned} \chi_p &= \left\langle \int d^2 x e^{ip \cdot x} \phi(x) \int d^2 y e^{-ip \cdot y} \phi(y) \right\rangle \\ &= \langle \bar{\phi}(p) \bar{\phi}(-p) \rangle \end{aligned}$$

Physically, they are just the (mod square) of the Fourier component of the noncommutative field.

Stripes

The possibility of having a ground state which breaks translational invariance was first observed in [1]. The quadratic part of the effective action, obtained from (1) by a one-loop self-consistent Hartree calculation, is

$$\Gamma^{(2)}(p) = p^2 + m_R^2 + Ag^2 \ln \Lambda_p^2, \quad \Lambda_p^2 = \frac{1}{\theta^2 p^2 + 1/\Lambda^2}. \quad (20)$$

Λ denotes a UV cut-off, m_R^2 the renormalized mass term and A is some constant. The mixing of IR and UV modes is manifest in the form of Λ_p . In addition the

momentum dependence of Λ_p implies that $\Gamma^{(2)}(p)$ can have a minimum for

$$p_c^2 = Ag^2 - \frac{1}{\Lambda^2\theta^2}. \quad (21)$$

For sufficiently large coupling this analysis would predict that modes with momentum $p = p_c$ would be the first to become unstable as the mass parameter m_R were tuned to zero. Thus any phase transition would then be driven by condensation of a non-trivial momentum component of the field. This is to be contrasted with the usual Ising transition in which the $p = 0$ component of the field condenses to yield the broken symmetry phase. Using (16) and (17) this result can be written in lattice variables

$$\left(\frac{2\pi n_c}{N}\right)^2 = AG - \frac{1}{N^2} \quad (22)$$

Note that for small G the matrix size N has to be large ($N > 2\pi/\sqrt{AG}$) in order for (22) to have a solution. Conversely, we might expect an exotic vacuum state to exist for *any* coupling in the continuum limit $N \rightarrow \infty$.

In two dimensions it is usually difficult to have a spontaneous breaking of a continuous symmetry like translation invariance. The fluctuations associated with the corresponding massless Goldstone boson will spoil any long range order. However, the arguments leading to this conclusion are strictly speaking not valid for non-local field theories and there exist explicit examples of non-local field theories in two dimensions which exhibit spontaneous symmetry breaking² Since noncommutative field theories are non-local and show a mixing of IR and UV properties one cannot rule out that the simple one-loop arguments given above in favor of a non-translational invariant ground state could still be valid. In fact the renormalization group treatment in [2] predicts this ground state even in two dimensions.

In the following we will look for signals indicating the appearance of a non-trivial ground state in the lattice system (15).

The computer simulation

In these first explorative simulations we have used a simple metropolis algorithm for stimulating the matrix system (14)-(15). Specifically, the matrix field is updated by the addition of a random hermitian noise matrix η

$$\phi \rightarrow \phi + \epsilon\eta \quad (23)$$

We tune ϵ with N so as to achieve an acceptance rate close to 50%. Most of our runs employed 5×10^6 such matrix updates for a given set of values of the parameters m and g . If we only consider the quadratic part of (15) the system is *equivalent* to an ordinary gaussian lattice field theory via the map (13) and this has been a useful check of the computer code.

²One example is the so-called crumpling phase transition in crystalline membranes.

Let us now turn to the actual simulation with the noncommutative ϕ^{*4} term. We have looked in the (m,g) coupling constant space for signals of a nontrivial vacuum. We have not attempted to map the entire phase diagram but concentrated instead on the slice $\theta = 1.0$ and $g^2/4 = 1.0$ with varying physical mass parameter m . In Fig.

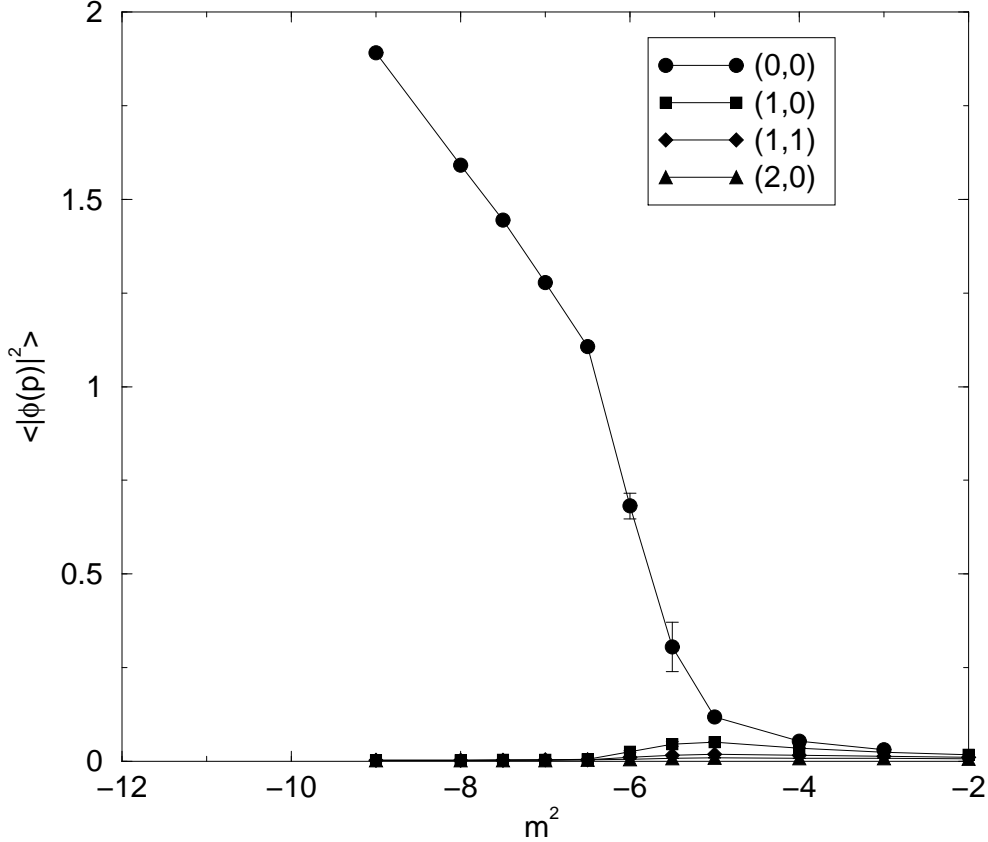


Figure 1: the average of the (mod square) of the Fourier amplitude for 4 lattice modes vs mass parameter m^2 (for lattice size $N = 9$). The 4 modes correspond to momentum states (24) where (m, n) is $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(2, 0)$.

(1) we show the generalized susceptibility χ_k corresponding to the (mod squared) Fourier component of the field $\phi(x)$ on an $N = 9$ lattice for various values of the lattice momentum

$$(k_1, k_2) = \left(\frac{2\pi m}{N}, \frac{2\pi n}{N} \right), \quad (p = k/a). \quad (24)$$

One sees a clear condensation of the usual $p = (0, 0)$ mode for values of $m^2 < -6$. Contrast this with the Fig. (2) which shows the same set of (unsubtracted) susceptibilities for $N = 23$. Now for $m^2 < -6$ we see that the dominant susceptibility

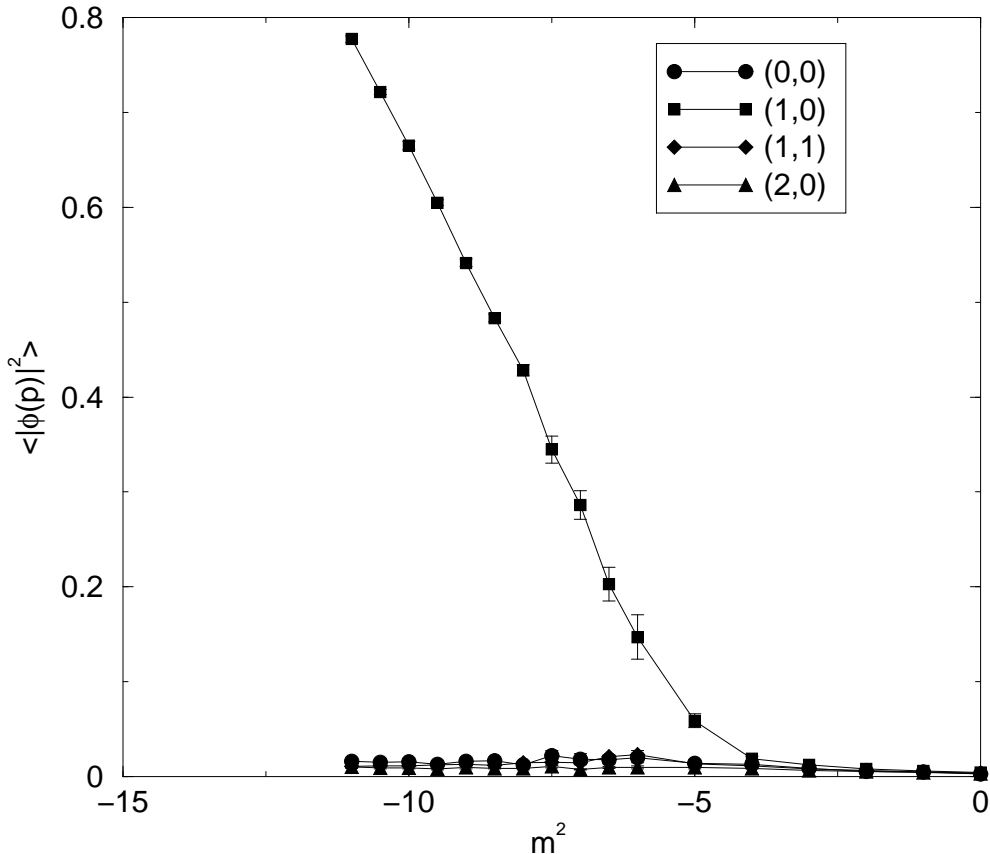


Figure 2: the average of the (mod square) of the Fourier amplitude for 4 lattice modes vs mass parameter m^2 (for lattice size $N = 23$). The 4 modes correspond to momentum states (24) where (m, n) is $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(2, 0)$.

corresponds to the lattice $(1, 0)$ mode – the vacuum in this region of the parameter space shows a spatially varying order parameter as predicted by the one loop calculation - a striped phase. Since the system still possesses a symmetry under exchange of axes we find that the condensation takes place along either the 1-axis or 2-axis randomly as the mass is varied. Thus, in our plots we show the average $\langle |\bar{\phi}(n, m)|^2 + |\bar{\phi}(m, n)|^2 \rangle$. Another way to illustrate graphically the existence of this phase is by considering the transverse-averaged correlator in position space. Fig. (3) shows a plot of $\langle \phi(x_1)\phi(0) \rangle$ for $m^2 = -10.0$. Notice that the correlator shows one complete oscillation within a lattice length. Since we are deep within the striped phase this correlator will essentially measure the behavior of the order parameter directly. Along the second direction the correlator is positive definite as shown in Fig. (4). This behavior is what we would expect of a striped phase. For the smaller lattice $N = 9$ we do not see any asymmetric behavior in the form of this position space correlator in the two directions, and indeed, as Fig. (5) indicates, the

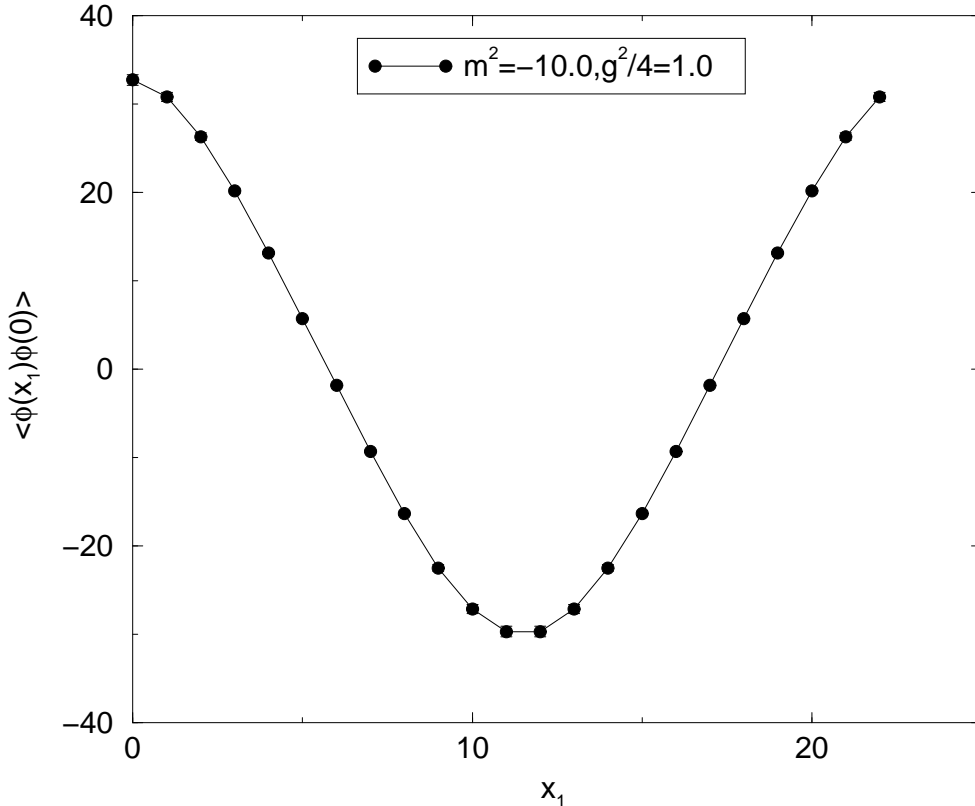


Figure 3: $\langle \phi(x_1)\phi(0) \rangle$ vs. x_1 for $m^2 = -10.0$ and $N = 23$

correlator is of conventional (positive definite) form over the entire lattice. This is consistent with the idea that for small N , there are no solutions to (20) and the minimum of the quadratic effective action is still found for mode $(0, 0)$.

We have also examined the system for $N = 39$. Fig. (6) and Fig. (7) show the corresponding correlators along the two lattice axes for $m^2 = -10.0$. From these two pictures we infer that the condensed mode is now $(2, 1)$. Thus, rather than stripes we actually see a checkerboard-like arrangement for the vacuum state in this case. The fact that the lattice mode number increases with larger lattice size for fixed physical parameters is encouraging as this is the behavior that is required if this effect is to survive the large volume limit $N \rightarrow \infty$. Indeed, (20) indicates that the mode number n should increase like $n \sim \sqrt{N}$ for $N \rightarrow \infty$ to ensure that the physical momentum at which condensation occurs remains fixed.

Discussion

We have shown that it is indeed possible to observe and investigate non-trivial phenomena like the appearance of “stripes” in noncommutative field theory within

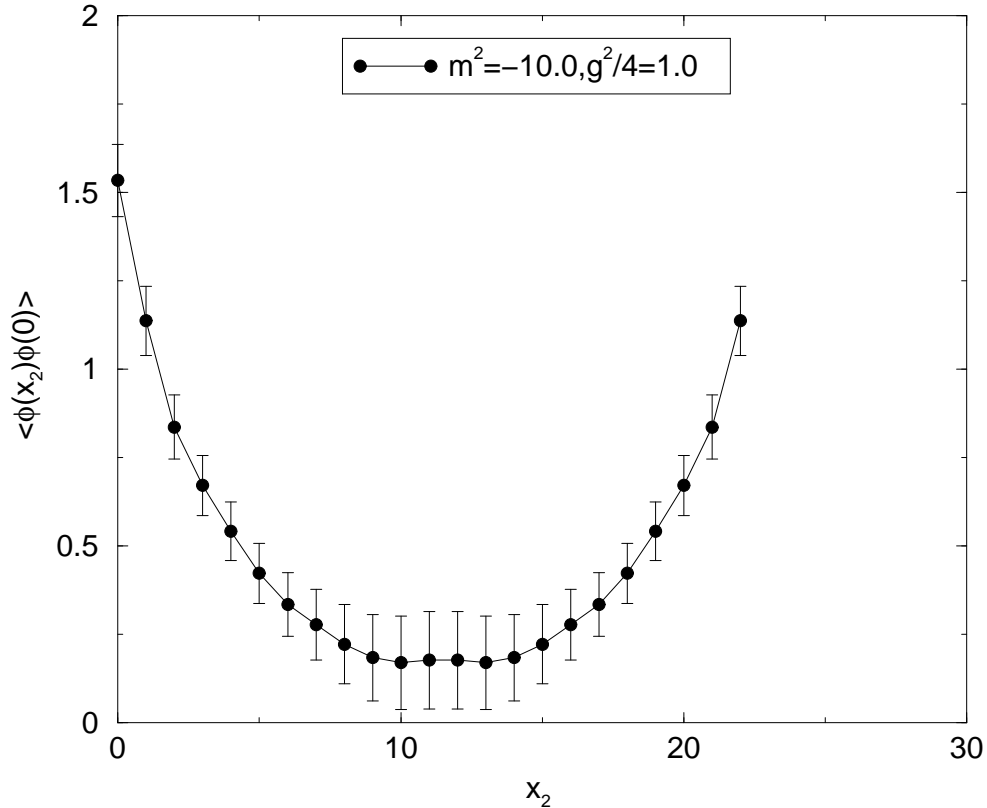


Figure 4: $\langle \phi(x_2)\phi(0) \rangle$ vs. x_2 for $m^2 = -10.0$ and $N = 23$

the non-perturbative framework of lattice simulations.

We have here looked at the very simplest model, a two-dimensional scalar field theory simply to verify that phenomena like the appearance of a “striped phase” can be observed. Our data show the appearance of such a phase in an appropriate double scaling limit. Notice that the issue of possible I.R divergences in two dimensions does not preclude such a study - the lattice regulates any such divergences automatically and the issue returns only when the continuum limit is taken. Our current data do not address this issue – much larger volumes must be simulated in order to determine whether this phase truly survives the large N limit. Nevertheless, we feel that our simulations lend nonperturbative, albeit qualitative support to the analysis of [1] and [2]. However, it is clearly of more interest to do this in higher dimensional theories. The lattice formalism has been developed [5] and just awaits use. Indeed, while this paper was in preparation we received a preprint [8] in which the same model was examined in $2 + 1$ dimensions with similar conclusions.

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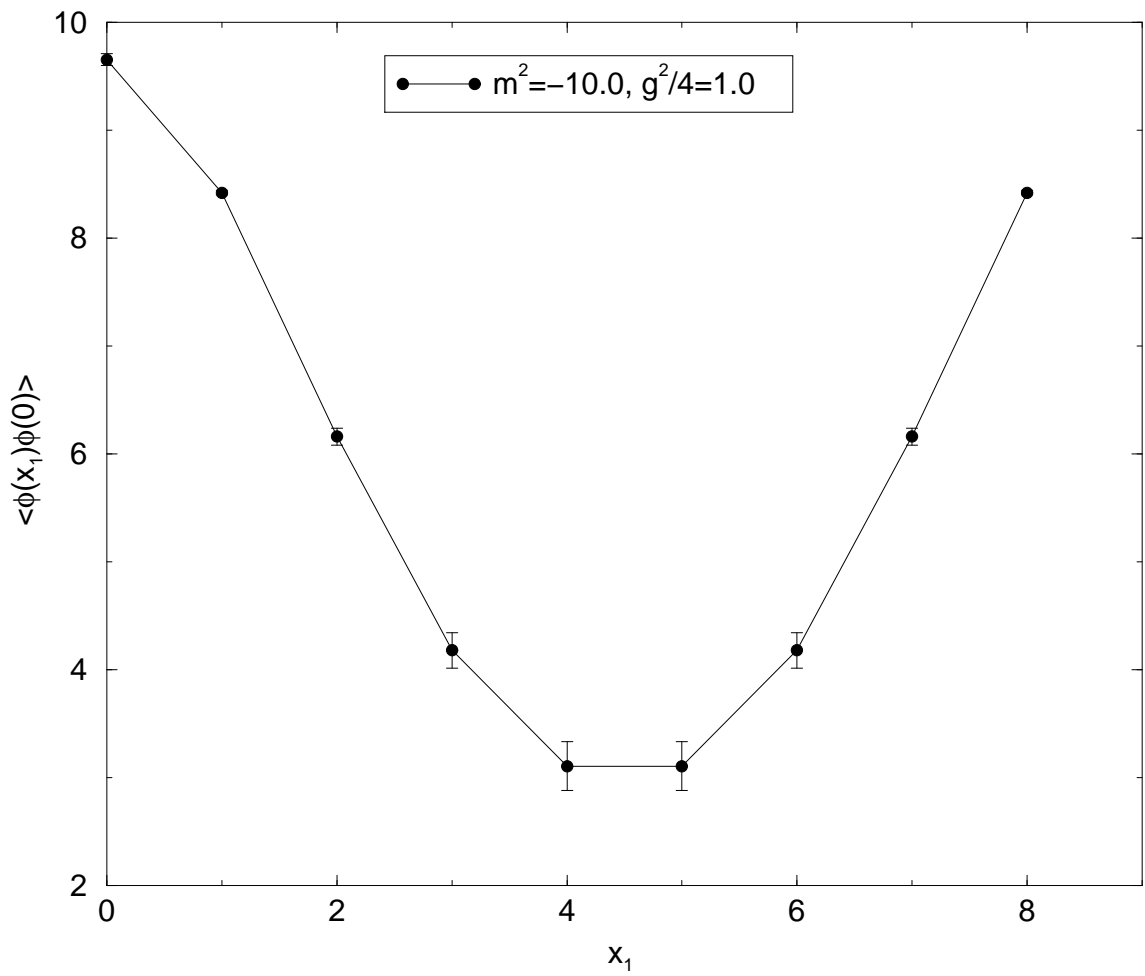


Figure 5: $\langle \phi(x_1)\phi(0) \rangle$ vs. x_1 for $m^2 = -10.0$ and $N = 9$

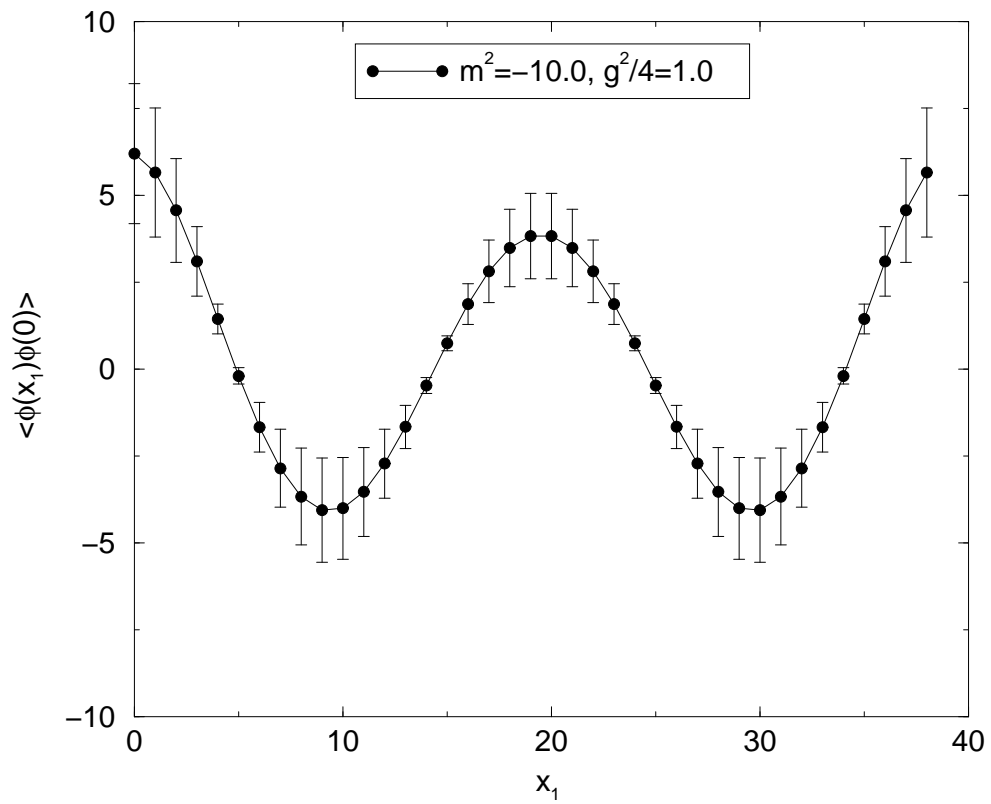


Figure 6: $\langle \phi(x_1)\phi(0) \rangle$ vs x_1 for $m^2 = -10.0$ and $N = 39$

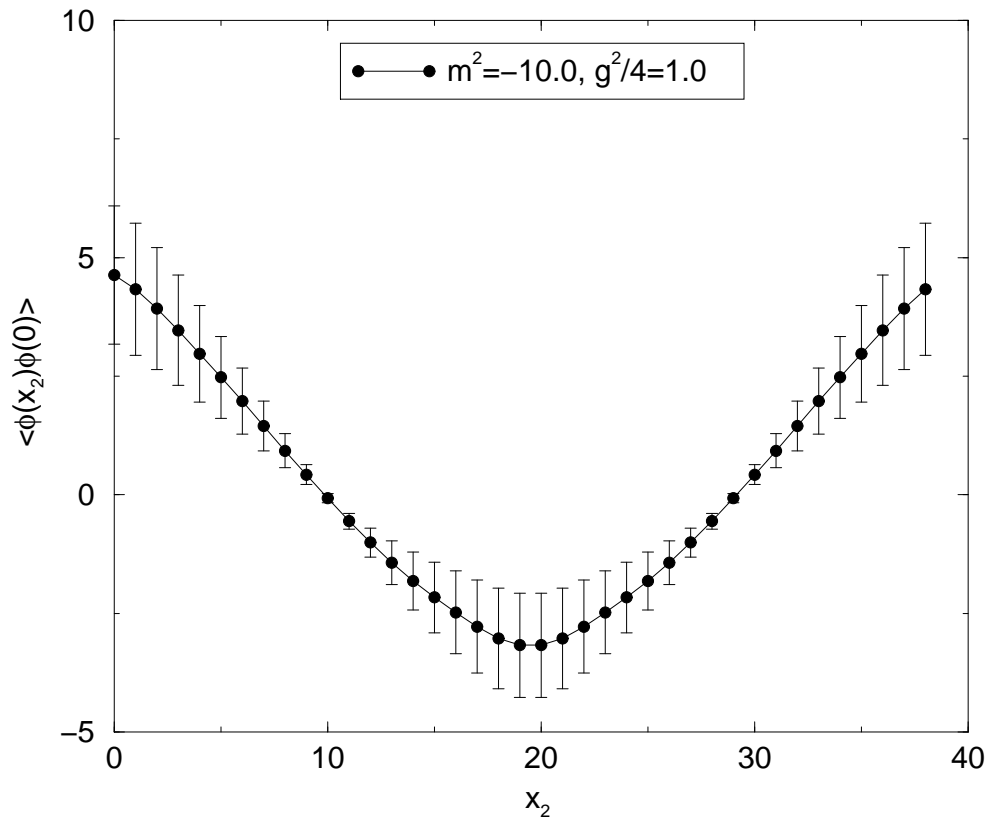


Figure 7: $\langle \phi(x_2)\phi(0) \rangle$ vs x_2 for $m^2 = -10.0$ and $N = 39$