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Lattice Supersymmetry Via Twisting

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We describe how the usual supercharges of extended supersymmetry may be *twisted* to produce a BRST-like supercharge Q. The usual supersymmetry algebra is then replaced by a twisted algebra and the action of the twisted theory is shown to be generically Q-exact. In flat space the twisting procedure can be regarded as a change of variables carrying no physical significance. However, the twisted theories can often be transferred to the lattice while preserving the twisted supersymmetry. As an example we construct a lattice version of the two-dimensional supersymmetric sigma model.

1. 2D Twisting

In two dimensions theories with $N = 2$ supersymmetry admit a global symmetry: $SO(2) \times U(1)_L \times U(1)_R$ where the first factor reflects rotational invariance (in Euclidean space) with generator T and the $U(1)$ factors correspond to chiral symmetries with generators U_L and U_R. Correspondingly the four supercharges transform as

$$Q_{-+} = (-\frac{1}{2}, +1, 0) \qquad Q_{--} = (-\frac{1}{2}, -1, 0)$$
$$Q_{+-} = (+\frac{1}{2}, 0, +1) \qquad Q_{++} = (+\frac{1}{2}, 0, -1) \quad (1)$$

where the charges are taken in the spin-1/2 representation of $SO(2)$. The twist consists of decomposing the fields into representations of a new rotation group [1]

$$SO(2)' = \text{Diagonal subgroup}\,(SO(2) \times SO(2)_{L-R})$$

with new generator $T' = T + \frac{1}{2}(U_L - U_R)$. The supercharges now transform as

$$Q_{-+} = (0, +1, 0) \qquad Q_{--} = (-1, -1, 0)$$
$$Q_{+-} = (0, 0, +1) \qquad Q_{++} = (+1, 0, -1) \qquad (2)$$

Notice that by this procedure we have produced two scalar supercharges $Q_L = Q_{-+}$ and $Q_R = Q_{+-}$ and a spin 1 supercharge with components Q_{++} and Q_{--}. Thus we expect the field theory embodying this twisted supersymmetry will contain two anticommuting scalar fields and one anticommuting vector. To match these fields the twisted supermultiplet must also contain 2 commuting scalars and a commuting vector field. Furthermore, the original supersymmetry algebra

$$\begin{aligned} \{Q_{\alpha+}, Q_{\beta-}\} &= \gamma^{\mu}_{\alpha\beta} P_{\mu} \\ \{Q_{\alpha+}, Q_{\beta+}\} &= \{Q_{\alpha-}, Q_{\beta-}\} = 0 \end{aligned} \qquad (3)$$

where the spinor index $\alpha, \beta = +/-$ yields a corresponding twisted algebra. Defining $Q = Q_L + Q_R$ we can see that the latter takes the form

$$\begin{aligned} Q^2 &= 0 \\ \{Q, Q_{--}\} &= P_- \quad \{Q, Q_{++}\} = -P_+ \end{aligned} \qquad (4)$$

The twisted algebra makes it clear that the momentum operator is Q-exact; that is it may be written as the Q-variation of some other operator. This is a necessary (if not sufficient) condition for the entire energy-momentum tensor $T_{\mu\nu}$ of the twisted theory to be Q-exact. But since $T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$ Q-exactness of T implies that the twisted action is Q-exact too

$$S = Q\Psi(\Phi, \partial\Phi)$$

where Ψ is some function of the fields Φ. This is a powerful result – it implies that the lattice theory obtained by replacing Ψ by an appropriate lattice function $\Psi \rightarrow \Psi_L(\Phi, \Delta\phi)$ will be exactly invariant under the twisted SUSY *provided only* that

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I can preserve the nilpotent property of Q under discretization. In many cases this can indeed be done [2,3,4].

2. 2D Sigma Model

To make the foregoing arguments more concrete let us construct the twisted version of the 2D supersymmetric sigma model. Consider a set of commuting fields $\phi^i(\sigma)$ which act as coordinates on some N-dim target space with metric $g_{ij}(\phi)$. To ensure the model has a twisted SUSY we will introduce scalar grassmann fields $\lambda^i(\sigma)$ and vector grassman fields $\eta_{i\mu}(\sigma)$ and another commuting vector field $B_{i\mu}(\sigma)$. We then postulate the following variations of the fields under Q

$$\begin{aligned}
Q\phi^i &= \lambda^i \\
Q\lambda^i &= 0 \\
Q\eta_{i\mu} &= \left(B_{i\mu} - \eta_{j\mu}\Gamma^j_{ik}\lambda^k\right) \\
QB_{i\mu} &= \left(B_{j\mu}\Gamma^j_{ik}\lambda^k - \frac{1}{2}\eta_{j\mu}R^j_{ilk}\lambda^l\lambda^k\right)
\end{aligned}$$

It is lengthy but straightforward [4] to verify that $Q^2 = 0$ on all fields. To derive the twisted action consider the Q-exact form

$$S = \alpha Q \int_\sigma \eta_{i\mu}\left(N^{i\mu}(\phi) - \frac{1}{2}g^{ij}B^\mu_j\right)$$

If we carry out the Q-variation and integrate out B we find

$$\begin{aligned}
S &= \alpha \int_\sigma \left(\frac{1}{2}g_{ij}N^{i\mu}N^j_\mu - \eta_{i\mu}\nabla_k N^{i\mu}\lambda^k \right. \qquad (5)\\
&+ \left. \frac{1}{4}R_{jlmk}\eta^{j\mu}\eta^l_\mu\lambda^m\lambda^k\right) \qquad (6)
\end{aligned}$$

To generate a kinetic term for the ϕ fields it is necessary to choose

$$N^{i\mu} = \partial^\mu\phi^i$$

Finally to make contact with usual sigma model we need to impose *self-duality* conditions on the vector fields in the model. This amounts to replacing η by $P^{(+)}\eta$ where

$$P^{i\mu\,(+)}_{j\nu} = \frac{1}{2}\left(\delta^i_j\delta^\mu_\nu + J^i_j\epsilon^\mu_\nu\right)$$

and J is a covariantly constant matrix $\nabla_k J^i_j = 0$. Manifolds possessing such a tensor field are termed Kähler. It can be shown that the resultant model can we written in complex coordinates as

$$\begin{aligned}
S &= \alpha\int d^2\sigma\left(2h^{+-}g_{I\overline{J}}\partial_+\phi^I\partial_-\phi^{\overline{J}}\right. \\
&- h^{+-}g_{I\overline{J}}\eta^I_+D_-\lambda^{\overline{J}} - h^{+-}g_{\overline{I}J}\eta^{\overline{I}}_-D_+\lambda^J \\
&+ \left.\frac{1}{2}h^{+-}R_{I\overline{I}J\overline{J}}\eta^I_+\eta^{\overline{I}}_-\lambda^J\lambda^{\overline{J}}\right)
\end{aligned}$$

Assembling the anticommuting fields into Dirac spinors

$$\Psi^I = \begin{pmatrix} \lambda^{\overline{I}} \\ \frac{1}{2i}\eta^{\overline{I}}_- \end{pmatrix} \qquad \overline{\Psi}_I = \begin{pmatrix} \lambda^I \\ \frac{1}{2i}\eta^I_+ \end{pmatrix}$$

the quadratic part of the anticommuting action can be written

$$\overline{\Psi}_I\gamma\cdot D\Psi^I$$

where the Dirac operator in chiral basis is

$$\begin{pmatrix} 0 & iD_+ \\ -iD^\dagger_+ & 0 \end{pmatrix}$$

Notice that the continuum Q-symmetry makes no reference to derivatives of the fields and so is trivially preserved under discretization. Thus it appears that we need only replace continuum derivatives by symmetric finite differences inside N to generate a Q-invariant lattice model with the correct classical continuum limit. Of course such a procedure will produce both bosonic and fermionic doubles. To remove these we need to add a Wilson term to the lattice action in such a way as to preserve supersymmetry. For many Kähler manifolds this can be done and the doubles eliminated – we refer for details of this construction to Sofiane Ghadab's talk at this conference.

3. 4D Twisting

Theories with $N = 2$ SUSY in four dimensions possess the global symmetry group

$$SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)$$

The supercharges then transform like

$$\begin{aligned} Q^i_\alpha &: (\frac{1}{2},0,\frac{1}{2}) \\ \overline{Q}_{i\dot{\alpha}} &: (0,\frac{1}{2},\frac{1}{2}) \end{aligned} \quad (7)$$

As in two dimensions we obtain the twisted theory by replacing the original symmetry group by $SU(2)_R \times SU(2)'$ where

$SU(2)' = \text{Diagonal subgroup}\,(SU(2)_L \times SU(2)_I)$

This implies that the isospin index can be treated like a spinor index

$$\begin{aligned} Q^i_\alpha &\rightarrow Q^\beta_\alpha \\ \overline{Q}_{i\dot{\beta}} &\rightarrow G_{\alpha\dot{\beta}} \end{aligned} \quad (8)$$

The trace of Q^α_α is then just our new nilpotent charge Q and the twisted supercharges now transform as

$$\begin{aligned} G_{\alpha\dot{\beta}} &: (\frac{1}{2},\frac{1}{2}) - - \text{vector} \\ Q_{(\alpha\beta)} &: (1,0) - - \text{selfdual tensor} \\ Q &: (0,0) - -\text{scalar} \end{aligned} \quad (9)$$

Corresponding to this we expect the twisted theory to contain an anticommuting vector ψ_μ, antisymmetric, self-dual tensor $\chi_{\mu\nu}$ and scalar η together with their commuting counterparts. The original supersymmetry algebra

$\{Q^i_\alpha, \overline{Q}_{j\dot{\beta}}\} = \delta^i_j P_{\alpha\dot{\beta}}$

the becomes the twisted algebra:

$\{Q, G_{\alpha\dot{\beta}}\} = P_{\alpha\dot{\beta}}$ and $\{Q,Q\} = 0$

Notice, once again that the momentum operator P is again Q-exact! Hence (in most cases) the action is also Q-exact

$S_{\text{twisted}} = Q\Psi$

Actually the algebra $Q^2 = 0$ can be generalized in a very useful way – if we are dealing with a gauge theory we can allow $Q^2 =$ gauge transformation without spoiling the Q-invariance of the action. As we will see in our example, twisted super Yang Mills theories are precisely of this type. Indeed the twisted supermultiplet contains a set of commuting fields A_μ,$B_{\mu\nu}$ and $\overline{\phi}$ together with the twisted fermion fields ψ_μ, $\chi_{\mu\nu}$ and η. All fields must be taken in the adjoint of some gauge group. We postulate the following Q-transformations [5]

$$\begin{aligned} QA_\mu &= \psi_\mu \\ Q\psi_\mu &= iD_\mu\phi \\ Q\chi_{\mu\nu} &= B_{\mu\nu} \\ QB_{\mu\nu} &= [\phi,\chi_{\mu\nu}] \\ Q\phi &= 0 \\ Q\overline{\phi} &= \eta \\ Q\eta &= [\phi,\overline{\phi}] \end{aligned}$$

Again, it is not hard to show that $Q^2 = \delta^\phi_G$ an infinitesimal gauge transformation with parameter ϕ. The Q-exact form of the twisted SYM action is just

$$S = Q\int \frac{1}{4}\eta[\phi,\overline{\phi}] + \chi^{\mu\nu}\left(B_{\mu\nu} + F^+_{\mu\nu}\right) - i\psi_\mu D_\mu\overline{\phi}$$

In [6] Sugino has shown how to generalize the continuum Q-transformations to the lattice in which the vector field A_μ is replaced by a link field U_μ. The resulting theory is gauge invariant, Q-symmetric and yields the (twisted) super Yang Mills theory in the naive continuum limit.

Finally, we would like to mention the recent work by D'Adda et al. [7] which points out the intimate connection between the twisting procedure we have described and the representation of fermions in terms of Dirac-Kähler fields.

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