10-10-2006

Twisted Supersymmetric Sigma Model on the Lattice

Simon Catterall
Syracuse University

Sofiane Ghadab
Syracuse University

Preprint typeset in JHEP style - HYPER VERSION SU-4252-833

Twisted supersymmetric sigma model on the lattice

Simon Catterall

Department of Physics, Syracuse University, Syracuse, NY13244, USA
E-mail: smc@phy.syr.edu

Sofiane Ghadab

Department of Physics, Syracuse University, Syracuse, NY13244, USA
E-mail: sghadab@phy.syr.edu

Abstract: In this paper we conduct a numerical study of the supersymmetric $O(3)$ non-linear sigma model. The lattice formulation we employ was derived in [1] and corresponds to a discretization of a *twisted* form of the continuum action. The twisting process exposes a *nilpotent* supercharge Q and allows the action to be rewritten in Q-exact form. These properties may be maintained on the lattice. We show how to deform the theory by the addition of potential terms which preserve the supersymmetry. A Wilson mass operator may be introduced in this way with a minimal breaking of supersymmetry. We additionally show how to rewrite the theory in the language of Kähler-Dirac fields and explain why this avenue does not provide a good route to discretization. Our numerical results provide strong evidence for a restoration of full supersymmetry in the continuum limit *without* fine tuning. We also observe a non-vanishing chiral condensate as expected from continuum instanton calculations.

Keywords: Lattice, Supersymmetry, Sigma Models, Kähler-Dirac.

Contents

1. Introduction

Supersymmetric field theories are an integral part of modern theories of particle physics. They provide a framework for solving the gauge hierarchy problem [2] by eliminating many divergences typical of quantum field theories through the cancellation between fermionic and bosonic loops. Moreover in the large N limit, it is known [3, 4] that supersymmetric gauge theories are related to quantum gravity and string theory. Two dimensional sigma models on the other hand are important because they have a rich mathematical structure, and moreover there exists a deep rooted analogy between them and four dimensional Yang Mills theories [5]. Indeed the former can serve as perfect theoretical laboratories to test methods and approaches developed for solving the problems of these far richer and more complicated theories. For example, the magnetic monopole and dyon in the gauge theory correspond to the kink and Q-kink solution of sigma models. In many cases the interesting physics lies in non-perturbative regimes which motivates use of the lattice to study these systems.

Unfortunately generic discretizations of supersymmetric theories break supersymmetry explicitly and necessitate fine tuning the couplings to a (usually large) number of lattice operators in order to approach a supersymmetric continuum limit. Recently, attention has turned to formulations which aim to preserve one or more exact supersymmetries at non-zero lattice spacing[1]. The hope is, this residual supersymmetry will shield us from the appearance of relevant operators in the lattice effective action which violate the full symmetry group thus reducing or possibly even eliminating such fine tuning problems. Two distinct approaches in this direction have been followed; in the first pioneered by Kaplan and collaborators, the lattice theory is constructed by orbifolding a certain supersymmetric matrix model and then extracting the lattice theory by expansion around some vacuum state [7, 8, 9, 10]. This has recently been extended to the the interesting case of gauge theories in two dimensions with matter fields interacting via a superpotential in [11]. The second approach relies on discretizing a *twisted* version of the supersymmetric theory. *Twisting* was first introduced by Witten [12] in the context of topological field theories. It consists of constructing a new rotation group from a combination of the original rotation group and part of the R-symmetry associated with the extended SUSY. The supersymmetric field theory is then reformulated in terms of fields which transform as integer representations of this new rotation group [13, 14, 15, 16]. In flat space one can think of the twisting as a merely exotic change of variables in the theory. When applied to the supersymmetry algebra, a scalar nilpotent supercharge is exposed. Furthermore as argued in [17, 18] the twisted superalgebra implies that the action rewritten in terms of these twisted fields is generically Q-exact. In this case it is straightforward to construct a lattice action which is Q-invariant provided *only* that we preserve the nilpotency of Q under discretization. An other approach in the same direction [15, 19], attempts to preserve all the twisted supercharges on the lattice by introducing a non-trivial commutation relation between the coordinates in superspace. However, this formulation remains controversial after a recent paper [20] pointed out an inconsistency appearing in this method when applied to a toy quantum mechanical model. Finally, although the orbifolding and the twisting approaches appear different, Ünsal [21] recently showed that the former approach reproduces the twisted Yang-Mills theories in the continuum limit.

The *twisting* approach to constructing lattice theories was initially developed for theories without gauge symmetry [22, 23]. An implementation for supersymmetric lattice gauge theories based on balanced topological field theories was given by Sugino [24]. An approach emphasizing the geometrical nature of the twist and employing Kähler-Dirac fermions was then developed by Catterall [17, 25] to construct super Yang-Mills theories in two and four dimensions. The use of Kähler-Dirac fermions for formulating lattice supersymmetry was first proposed in [26].

2. The 2D continuum action

As was shown in [1], the action of a general two dimensional sigma model with $\mathcal{N} = 2$

[1] We primarily discuss Euclidean formulations - for recent work on Hamiltonian approaches see [6]

supersymmetry may be written, using complex coordinates, in the twisted form

$$S = \beta \int d^2\sigma \Big(2h^{+-} g_{I\overline{J}} \partial_+ \phi^I \partial_- \phi^{\overline{J}} \\ - h^{+-} g_{I\overline{J}} \eta_+^I D_- \psi^{\overline{J}} - h^{+-} g_{\overline{I}J} \eta_-^{\overline{I}} D_+ \psi^J + \frac{1}{2} h^{+-} R_{I\overline{I}J\overline{J}} \eta_+^I \eta_-^{\overline{I}} \psi^J \psi^{\overline{J}} \Big) \tag{2.1}$$

with

$$D_+ \psi^J = \partial_+ \psi^J + \Gamma^J_{KL} \partial_+ \phi^K \psi^L \tag{2.2}$$

The requirement of $\mathcal{N} = 2$ supersymmetry forces the target manifold to be Kähler in which case $g_{IJ} = g_{\bar{I}\bar{J}} = 0$ and the only non-zero Christoffel symbols are Γ^I_{JK} and $\Gamma^{\bar{I}}_{\bar{J}\bar{K}}$ (see Appendix A). This action is invariant under four supersymmetries as expected for a theory with $\mathcal{N} = 2$ supersymmetry in two dimensions. In the twisted construction we focus on a single hermitian twisted supercharge Q whose action on the fields is given by

$$\begin{cases} Q\phi^I = \psi^I & Q\phi^{\bar{I}} = \psi^{\bar{I}} \\ Q\psi^I = 0 & Q\psi^{\bar{I}} = 0 \\ Q\eta^I_+ = B^I_+ - \Gamma^I_{KJ} \psi^J \eta^K_+ & Q\eta^{\bar{I}}_- = B^{\bar{I}}_- - \Gamma^{\bar{I}}_{\bar{K}\bar{J}} \psi^{\bar{J}} \eta^{\bar{K}}_- \\ QB^I_+ = -\Gamma^I_{JK} \psi^J B^K_+ - R^I{}_{K\bar{J}L} \psi^K \psi^{\bar{J}} \eta^L_+ & QB^{\bar{I}}_- = -\Gamma^{\bar{I}}_{\bar{J}\bar{K}} \psi^{\bar{J}} B^{\bar{K}}_- - R^{\bar{I}}{}_{\bar{J}L\bar{K}} \psi^{\bar{J}} \psi^L \eta^{\bar{K}}_- \end{cases} \tag{2.3}$$

The action of the remaining charges of the $N = 2$ twisted supersymmetry is given in appendix B. The field B^I_+ is an auxiliary field introduced to linearize the transformations and render the transformation nilpotent off-shell. It has been removed from the action in eqn. (2.1) by employing its equation of motion $B^I_+ = \partial_+ \phi^I$. The invariance of the action under this supercharge Q follows just from the nilpotency of the latter and the fact that the above action is Q-exact – that is it can be written as the Q-variation of some function which, borrowing from BRST gauge fixing terminology, can be termed a gauge fixing fermion [1].

3. The 2D lattice action

Surprisingly translating the action in (2.1) to the lattice is pretty straightforward. Indeed, as the twisted Q-symmetry makes no reference to derivatives of the fields its nilpotent property is preserved when continuum fields indexed by a continuous coordinate are replaced by lattice fields carrying a discrete index. We then obtain the supersymmetric lattice action by just replacing the continuum derivative by a symmetric finite difference

$$S = \beta \sum_x \Big(2h^{+-} g_{I\overline{J}} \Delta^s_+ \phi^I \Delta^s_- \phi^{\overline{J}} \\ - h^{+-} g_{I\overline{J}} \eta_+^I D_- \psi^{\overline{J}} - h^{+-} g_{\overline{I}J} \eta_-^{\overline{I}} D_+ \psi^J + \frac{1}{2} h^{+-} R_{I\overline{I}J\overline{J}} \eta_+^I \eta_-^{\overline{I}} \psi^J \psi^{\overline{J}} \Big) \tag{3.1}$$

where now,

$$D_+ \psi^J = \Delta^s_+ \psi^J + \Gamma^J_{KL} \Delta^s_+ \phi^K \psi^L \tag{3.2}$$

and $\Delta^s_\pm = \Delta^s_1 \pm i\Delta^s_2$ where $\Delta^s_\mu = \frac{1}{2}(\Delta^+_\mu + \Delta^-_\mu)$ and $\Delta^\pm$ are the usual forward and backward difference operators. However the lattice action in the form shown in eqn. (3.1) suffers from the usual fermion doubling problem. Indeed it is easy to see that the kernel of the (free) lattice Dirac operator constructed this way contains extra states which have no continuum interpretation. For instance consider the fermionic part in (3.1). If we ignore the interaction term in (3.2) then one finds that in Fourier space the kernel of Δ^s_+ corresponds to solving

$$0 = e^{ikx}[\sin k_1 + i \sin k_2] \tag{3.3}$$

Besides the solution $(k_1, k_2) = (0, 0)$, one also has $(0, \pi), (\pi, 0)$ and (π, π). The doubling problem is made worse because supersymmetry propagates it to the bosonic sector. There are two obvious lines of approach one may take to avoid this problem. One is rewrite the continuum theory in the language of Kähler-Dirac fields and try to utilize the discretization procedure developed in [17] to construct the lattice theory. This approach is reviewed in the next section where it is shown that such a procedure runs into difficulties.

The obvious alternative, which we have pursued here, is to add some form of Wilson mass term to lift the mass of the doubles up to the scale of the cut-off. However an ad hoc addition of such a Wilson term will break supersymmetry explicitly in an uncontrolled way thus spoiling our goal of preserving an element of SUSY on the lattice. Fortunately, it was shown [28, 29] that in the case where the Kähler manifold possesses some isometries, it is possible to add potential terms that involve the holomorphic Killing vectors V^I associated to those isometries in such a way as to keep the action Q-exact. By a careful choice of such potential terms we may add Wilson operators which accomplish the task of rendering the doublers heavy. In complex coordinates the possible terms are,

$$\Delta S = \beta \sum_x \left[\lambda^2 V^I V_I + \lambda^2 \psi^{\bar{I}} \nabla_{\bar{I}} V_J \psi^J - \frac{1}{4} h^{+-} \eta^{\bar{I}}_- \nabla_{\bar{I}} V_J \eta^J_+ + \text{h.c} \right] \tag{3.4}$$

Here, λ^2 is an arbitrary parameter. Here, we choose $\lambda = h^{+-} = \frac{1}{2}$. To keep the action Q-exact the action of the twisted supersymmetry must be changed.

$$\begin{cases} Q\phi^I = \psi^I \\ Q\psi^I = V^I \\ Q\eta^I_+ = B^I_+ - \Gamma^I_{JK}\psi^J\eta^K_+ \\ QB^I_+ = -\Gamma^I_{JK}\psi^J B^K_+ + \frac{1}{2} R^I{}_{JKL}\psi^K\psi^J\eta^L_+ + D_K V^I \eta^K_+ \end{cases} \tag{3.5}$$

Q is no longer nilpotent but Q^2 just amounts to a Lie derivative with respect to the Killing vector field. Indeed one can easily show that,

$$\begin{cases} Q^2\phi^I &= V^I \\ Q^2\psi^I &= \partial_J V^I \psi^J \\ Q^2\eta^I_+ &= \partial_J V^I \eta^J_+ \\ Q^2 B^I_+ &= \partial_J V^I B^J_+ \end{cases} \tag{3.6}$$

To introduce a mass term would require utilizing a Killing vector of the form

$$V^I = im\phi^I \tag{3.7}$$

It should be clear that the transformation induced by such a Killing vector corresponds to a infinitessimal phase rotation on the complex fields and can be seen to be a good symmetry of both the continuum *and* lattice actions. Many Kähler manifolds possess metrics invariant under such a phase rotation, for example, the CP^N theories and specifically the $CP^1 \sim O(3)$ model considered in detail later. In the latter case the theory actually possesses three isometries corresponding to the global $O(3)$ symmetry of the theory - the phase rotation associated with the Killing vector eqn. 3.7 corresponds to invariance under rotations about the z-axis.

To remove the doubles in lattice regularizations of such models we have employed such a twisted mass term with the constant mass parameter replaced by a Wilson operator $m \to m_W$ where

$$m_W = \frac{1}{2}\left(\Delta^+_+\Delta^-_- + \Delta^+_-\Delta^-_+\right) \tag{3.8}$$

This leads to the additional terms in the action

$$\Delta S = \beta \sum_x \left[\lambda^2 m_W\phi^I m_W\phi_I + \lambda^2\psi^{\overline{I}}\nabla_{\overline{I}}g_{J\overline{J}}m_W\phi^{\overline{J}}\psi^J - \frac{1}{4}h^{+-}\eta^{\overline{I}}_-\nabla_{\overline{I}}g_{J\overline{J}}m_W\phi^{\overline{J}}\eta^J_+ + \text{h.c}\right] \tag{3.9}$$

The generalized phase rotation associated with m_W *no longer* corresponds to an exact isometry of the metric and hence the modified action, while still Q-exact, is no longer exactly invariant under the generalized Q-symmetry. Novertheless we will argue that the twisted Wilson operator acts as a soft breaking term and hence should not affect the renormalization of the lattice theory for small enough lattice spacing.

The argument goes as follows. In the continuum one is used to thinking of mass terms as serving only to break supersymmetry *softly*. This expectation relies on the idea that any mass parameter will be small compared to the U.V cut-off in the theory. However, the generic Wilson operator in a lattice theory does *not* satisfy this property since the potential doublers in the theory pick up a mass from the Wilson term on the order of the cut-off. Thus addition of such Wilson terms generically will lead to a hard breaking of supersymmetry and it will be necessary to fine tune additional operators to recover a supersymmetric continuum limit. However, the twisted mass operator we use here does not have this property – in the limit of small a the propagators are dominated by contributions from a state near the origin of the Brillioun zone and a set of would be doublers. The latter contribution does not break supersymmetry since the doublers contribute like additional states with a *constant* twisted mass. Such a mass deformation preserves the Q-symmetry of the lattice action.

The soft character of the breaking can be seen in yet another way. For small lattice spacing a (large β) we can expand the bosonic action to quadratic order. Subsequently integrating over the bosons yields a determinant $\det(m_W^2 - D^S_+D^S_-)$. But at one loop this determinant is cancelled by an identical fermionic contribution $\det(M)$ where

$$M = \begin{pmatrix} m_W & -D_+^S \\ -D_-^S & m_W \end{pmatrix}$$

Thus the lattice theory appears both double free and supersymmetric at large β. These arguments lead us to expect the Q-symmetry to be restored at large coupling β without additional fine tuning. If this is the case then simple power counting arguments lead us to conclude that the only (marginally) *relevant* counter terms must take the Q-exact form

$$O = Q\left(\eta^{i\alpha} f_{ij}\left(\phi\right)\left(\partial_\alpha \phi^j + B^j\right)\right) \tag{3.10}$$

where we have reverted to a formulation using real fields. General covariance ensures that f_{ij} is a tensor which may then be taken to represent a quantum renormalization of the target space metric tensor. This counterterm structure would be consistent with a lattice model which exhibits $\mathcal{N} = 1$ supersymmetry in the continuum limit. The restoration of full $\mathcal{N} = 2$ supersymmetry appears to require additional constraints. Luckily, such constraints are present in the form of additional discrete symmetries of the lattice action. Consider the classical action in Kähler form given in eqn. 3.1. It is trivial to see that this action is also invariant under the finite transformations

$$\begin{aligned} \psi^I &\to i\psi^I \\ \psi^{\overline{I}} &\to -i\psi^{\overline{I}} \\ \eta_+^I &\to i\eta_+^I \\ \eta_-^{\overline{I}} &\to -i\eta_-^{\overline{I}} \end{aligned} \tag{3.11}$$

Actually, this additional symmetry arises from the Kähler structure of the target space appearing in the classical action [30]. This additional symmetry of the lattice model then ensures that only counterterms compatible with a Kähler target space survive in the quantum effective action. But as was shown in [30] any model with $\mathcal{N} = 1$ supersymmetry and a Kähler target space automatically possesses $\mathcal{N} = 2$ supersymmetry. Thus on the basis of these arguments we expect that no additional fine tuning is needed to regain the full supersymmetry of the continuum model which as will be seen later, is confirmed by our numerical results.

4. Kähler-Dirac reformulation

Recent work has emphasized the geometrical nature of certain twisted super Yang-Mills theories [17, 25]. In these constructions all fields appear as antisymmetric tensors with the twisted fermions arising as components of a geometrical object called a Kähler-Dirac field. The arguments leading to this geometrical interpretation are quite general, depending only in the number of supercharges and R-symmetry, and imply that it should be possible to formulate the twisted sigma models also in this language. Furthermore, the geometrical construction is key to a successful discretization of the Yang-Mills theory avoiding fermion doublers [27]. Thus such a reformulation of the sigma model potentially offers another way to build a supersymmetric lattice theory without encountering fermion doubling. As such it would offer an alternative to the addition of Wilson operators as described in the previous section. As we will see this reformulation can indeed be done in the continuum but an obstruction prevents any simple translation of the continuum theory to the lattice.

As shown in, for example, [25] the basic $\mathcal{N}=2$ twisted supermultiplet in two dimensions involves two Kähler-Dirac fields, a bosonic one $\Phi = (\phi, A_\mu, B_{12})$ whose p-form components commute and fermionic one $\Psi = (\lambda, \eta_\mu, \chi_{12})$ with grassmann valued forms. Initially we will consider a flat target space and consider only the continuum theory. The action of the nilpotent twisted supersymmetry is simply

$$\begin{array}{ll} Q\phi = \lambda & Q\lambda = 0 \\ Q\eta_\mu = A_\mu & QA_\mu = 0 \\ QB_{12} = 0 & Q\chi_{12} = 0 \end{array} \tag{4.1}$$

These transformations are essentially the same as the corresponding Yang-Mills variations except that the field A_μ plays the role of a multiplier field in the sigma model case (and hence has zero variation under Q). The twisted action can be written as the Q-variation of some gauge fermion Λ. The most general gauge fermion which is linear in the fermion fields, contains at most one derivative, and is not a Q-singlet is

$$\Lambda = \int d^2x \eta_\mu \left[c_1 A_\mu + c_2 \partial_\mu \phi + c_3 \partial_\nu B_{\mu\nu}\right] \tag{4.2}$$

Furthermore, by a simple rescaling of the fields I can set $c_1 = \frac{1}{2}$ and $c_2 = c_3 = 1$. Carrying out the Q-variation and integrating out the multiplier field A_μ (along the imaginary axis) leads to the action

$$S = \int d^2x \left[\frac{1}{2}\left(\partial_\mu\phi + \epsilon_{\mu\nu}\partial_\nu B_{12}\right)^2 - \eta_\mu\partial_\mu\lambda - \chi_{\mu\nu}\partial_{[\mu}\eta_{\nu]}\right] \tag{4.3}$$

As in [25] the twisted fermionic action is of Kälher-Dirac form. Writing $u^1 = \phi$, $u^2 = B_{12}$ the bosonic action can now be rewritten

$$S_B = \int d^2x \frac{1}{2}\left[\left(\partial_\mu u^i\right)^2 + \epsilon_{ij}\epsilon_{\mu\nu}\partial_\mu u^i \partial_\nu u^j\right] \tag{4.4}$$

where the implied summations over Roman indices run from $i = 1, 2$. This action can be simplified by introducing the projection operator

$$P^{+i\nu}_{j\mu} = \frac{1}{2}\left(\delta^i_j \delta^\nu_\mu + \epsilon^i_j \epsilon^\nu_\mu\right) \tag{4.5}$$

where the distinction between upper and lower indices is, for the present, immaterial. The bosonic action now reads

$$S_B = \int d^2x \left[P^{+i\nu}_{j\mu}\partial_\nu u^j\right]^2 \tag{4.6}$$

Proceeding in this way we can introduce a new grassman valued field which is the superpartner to u^i by defining $\psi^1 = \lambda$ and $\psi^2 = \chi_{12}$. Similarly the original field A_μ (and its partner η_μ) can be promoted to a field indexed by a Roman target index $A_\mu \rightarrow A_\mu^{+i}$ *provided* the field is required to be self-dual under the action of this projector i.e

$$A^i_\mu = A^{+i}_\mu = P^{+i\nu}_{j\mu} A^j_\nu \tag{4.7}$$

In terms of these new variables the Q-transformations are now

$$\begin{array}{ll} Qu^i = \psi^i & Q\psi^i = 0 \\ Q\eta^{+i}_\mu = A^{+i}_\mu & QA^{+i}_\mu = 0 \end{array} \tag{4.8}$$

Going to complex coordinates we recover our original sigma model variations for a flat two dimensional target space. It should be clear that the sigma model fields are essentially the the Hodge self-dual components of the original Kähler-Dirac fields. It should also be clear how to proceed to a more general target space. Clearly one must be able to introduce a tensor J^i_j, here just ϵ^i_j, which squares to minus the identity and is covariantly constant (so as not to disturb the form of the Q-transformations when applied to the self-dual fields). These requirements require the target manifold be Kähler with J^i_j its complex structure. Of course these were just the restrictions found in earlier constructions of the twisted sigma models [1]. Keeping the simple form of the Q-transformations the twisted sigma model action can be obtained as

$$S = Q \int d^2x \eta^{+i}_\mu \left[\partial_\mu u^j - \frac{1}{2} A^{+j}_\mu - \Gamma^j_{lk} \eta^{+l}_\mu \psi^k\right] g_{ij} \tag{4.9}$$

Having recast the continuum theory in this language we can return the problem of constructing a corresponding lattice model. We see an immediate problem. To avoid spectrum doubling we must discretize continuum derivatives with care. Specifically an exterior derivative must be replaced by a *forward* difference operator D^+ in the discrete theory while the adjoint of an exterior derivative by a *backward* difference operator D^- if the resulting lattice theory is to be free of spectrum doubling [27]. Thus the bosonic part of the action in eqn. (4.3) must be replaced with

$$S_B = \sum \frac{1}{2} \left(D^+\phi + D^- B_{\mu\nu}\right)^2 \tag{4.10}$$

Since the difference operators are no longer the same we can no longer introduce the projector P^+ and rewrite the theory in terms of self-dual fields. Thus the Kähler-Dirac approach cannot be used to construct twisted lattice theories without spectrum doubling. A similar phenomena is encountered in the Yang-Mills case where the self-dual nature of the fields in four dimensional $\mathcal{N} = 2$ super Yang-Mills prevents construction of a Q-exact and double free lattice theory.

5. Simulations

Here we revisit the $O(3)$ sigma model discussed in detail in [1]. The $O(3)$ sigma model has the advantage of being simple and can be reformulated as a twisted model with a CP^1 target space. The metric, connection and curvature take the form,

$$g_{u\overline{u}} = \frac{1}{2\rho^2} \tag{5.1}$$

$$\Gamma^u_{uu} = g^{\overline{u}u}\partial_u g_{\overline{u}u} = -\frac{2\overline{u}}{\rho} \tag{5.2}$$

$$R_{\overline{u}u\overline{u}u} = g_{\overline{u}u}\partial_{\overline{u}}\Gamma^u_{uu} = -\frac{1}{\rho^4} \tag{5.3}$$

where $\rho = 1 + u\overline{u}$. From (3.1), (3.4) and (3.8) the (effective) lattice action for a flat base space is given by (here $I = \bar{I} = 1$),

$$\begin{aligned} S = \sum_x \Big\{ \beta \Big[& \frac{1}{\rho^2}\Delta^S_+ u \Delta^S_- \overline{u} + \frac{1}{\rho^2}(m_W u)(m_W \overline{u}) - \frac{1}{2\rho^2}\eta D^S_- \overline{\psi} - \frac{1}{2\rho^2}\overline{\eta} D^S_+ \psi - \frac{1}{2\rho^4}\overline{\eta}\eta\overline{\psi}\psi \\ & + i\overline{\psi}[\frac{1}{2\rho^2}m_W + m_W\frac{1}{2\rho^2} - \frac{\bar{u}}{\rho^3}(m_W u) - \frac{u}{\rho^3}(m_W \bar{u})]\psi \\ & - \frac{i}{4\rho^2}\overline{\eta}[\frac{1}{2\rho^2}m_W + m_W\frac{1}{2\rho^2} - \frac{\bar{u}}{\rho^3}(m_W u) - \frac{u}{\rho^3}(m_W \bar{u})]\eta \Big] + \ln[\beta/(2\rho^2)] \Big\} \end{aligned} \tag{5.4}$$

where a factor of two has been absorbed into the coupling β and we have simplified our notation by replacing $\phi^1 \to u$,$\psi^1 \to \psi$, $\eta^1_+ \to \eta$ and $\overline{\eta}^{\bar{1}}_- \to \overline{\eta}$. The explicit form of the lattice covariant derivative is

$$D^S_+ = \Delta^S_+ - \frac{2\overline{u}}{\rho}\left(\Delta^S_+ u\right) \tag{5.5}$$

Note that the new term $\ln[\beta/(2\rho^2)]$ missing in [1], emerges from integrating out the auxiliary fields B^i_+ [2]. To proceed further it is convenient to introduce an other auxiliary field σ to remove the quartic fermion term. Explicitly we employ the identity

$$\beta^V \int D\sigma e^{-\alpha\left(\frac{1}{2}\sigma\overline{\sigma} + \frac{\sigma}{2\rho^2}\overline{\eta}\psi + \frac{\overline{\sigma}}{2\rho^2}\eta\overline{\psi}\right)} = e^{\frac{\alpha}{2\rho^4}\overline{\eta}\eta\overline{\psi}\psi} \tag{5.6}$$

where V is the number of lattice sites. Thus the partition function of the lattice model can be cast in the form

$$Z = \int DuD\sigma D\eta D\psi e^{-S(u,\sigma,\eta,\psi)} \tag{5.7}$$

where the action is now given by

$$S = \beta\sum_x \left[\frac{1}{\rho^2}\Delta^S_+ u\Delta^S_-\overline{u} + \frac{1}{\rho^2}(m_W u)(m_W\overline{u}) + \frac{1}{2}\sigma\overline{\sigma} - \frac{1}{\beta}\ln(2\rho^2) + \overline{\Psi}M(u,\sigma)\Psi\right] \tag{5.8}$$

where we have assembled the twisted fields into Dirac spinors

$$\overline{\Psi} = \begin{pmatrix} \frac{\overline{\eta}}{2i} \\ \overline{\psi} \end{pmatrix} \qquad \Psi = \begin{pmatrix} \frac{\eta}{2i} \\ \psi \end{pmatrix}$$

[2] We thank Joel Giedt and Erich Poppitz for pointing out this factor

and the Dirac operator $M(u,\sigma)$ in this chiral basis is

$$M(u,\sigma) = \begin{pmatrix} \frac{1}{2\rho^2}m_W - \frac{\bar{u}}{\rho^3}(m_W u) + \text{h.c} & \frac{1}{\rho^2}(\Delta_+^S - \frac{2\overline{u}}{\rho}\left(\Delta_+^S u\right) + \sigma) \\ (\Delta_-^S + \frac{2u}{\rho}\left(\Delta_-^S \bar{u}\right) - \bar{\sigma})\frac{1}{\rho^2} & \frac{1}{2\rho^2}m_W - \frac{\bar{u}}{\rho^3}(m_W u) + \text{h.c} \end{pmatrix} \tag{5.9}$$

Notice the extra terms depending on the target space connection appearing along the diagonal which correspond to the twisted Wilson mass operator. These were not present in our previous paper [1].

In order to be able to simulate this model, one needs to rewrite the effective action in a form that doesn't involve the grassman fields. For that we integrate out the Dirac field Ψ generating,

$$\det{}^{\frac{1}{2}}(\beta^2 M(u,\sigma)^\dagger M(u,\sigma)) \tag{5.10}$$

The effective action is now given by,

$$S = \beta S_B(u,\sigma) - \ln(2\rho^2) - \frac{1}{2}\text{Tr}\ln\left(\beta^2 M^\dagger(u,\sigma)M(u,\sigma)\right) \tag{5.11}$$

Clearly, the form of the fermion effective action we employ does not take into account any non-trivial phase associated with the fermion determinant - our simulation generates the phase quenched ensemble. We later examine the phase explicitly.

To simulate this model we used the RHMC algorithm developed by Clark and Kennedy [31]. The first step of this algorithm replaces the fermion determinant by an integration over auxiliary *commuting* pseudofermion fields F and $F^\dagger$ in the following way,

$$det^{\frac{1}{2}}(\beta^2 M(u,\sigma)^\dagger M(u,\sigma)) = \int DFDF^\dagger e^{-\frac{1}{\beta}\sum F^\dagger (M^\dagger M)^{-\frac{1}{2}} F} \tag{5.12}$$

The key idea of RHMC is to use an optimal (in the minimax sense) rational approximation to this inverse fractional power.

$$\frac{1}{x^{\frac{1}{2}}} \sim \frac{P(x)}{Q(x)}$$

where

$$P(x) = \sum_{i=0}^{N-1} p_i x^i, \quad Q(x) = \sum_{i=0}^{N-1} q_i x^i$$

Notice that we restrict ourselves to equal order polynomials in numerator and denominator. In practice it is important to use a partial fraction representation of this rational approximation,

$$\frac{1}{x^{\frac{1}{2}}} \sim \alpha_0 + \sum_{i=1}^{N} \frac{\alpha_i}{x+\beta_i} \tag{5.13}$$

The coefficient α_i, β_i for $i = 1, ..., N$ can be computed offline using the Remez algorithm. Furthermore, the coefficients can be shown to be real positive. Thus the linear systems are well behaved and, unlike the case of polynomial approximation, the rational fraction approximations are robust, stable and converge rapidly with N. The resulting pseudofermion action becomes

$$S_{PF} = \frac{1}{\beta}\left[\alpha_0 F^\dagger F + \sum_i^N F^\dagger \frac{\alpha_i}{M^\dagger M + \beta_i} F\right] \tag{5.14}$$

In principle this pseudofermion action can now be used in a conventional HMC algorithm to yield an *exact* simulation of the original effective action [32]. This algorithm requires that we compute the pseudofermion forces. For example, the additional force on the scalar field u due to the pseudofermions takes the form,

$$f_u = \frac{\partial S_{PF}}{\partial u} = -\sum_i^N \alpha_i \chi^{\dagger i} \frac{\partial}{\partial u}\left(M^\dagger M\right)\chi^i$$

where the vector χ^i is the solution of the linear problem

$$(M^\dagger M + \beta_i)\chi^i = F$$

The final trick needed to render this approach feasible is to utilize a *multi-mass* solver to solve all N sparse linear systems simultaneously and with a computational cost determined primarily by the smallest β_i. We use a multi-mass version of the usual Conjugate Gradient (CG) algorithm [33]. Note that for our simulations we have used $N = 12$ and an approximation that gives us an absolute bound on the relative error of 8×10^{-6} for eigenvalues of $M^\dagger M$ ranging from 10^{-6} to 10.0 which conservatively covers the range needed.

5.1 Spectrum

As a check of supersymmetry, we have studied the boson and fermion propagators projected to zero spatial momentum, namely, $G^B(t) = \mathrm{Re}(< \overline{u}(t)u(0) >)$ and $G^F_{ij}(t) =< \overline{\Psi}_i(t)\Psi_j(0) >$. One can then extract the lowest lying mass states by fitting these two point functions to the form $a + b\cosh(m^B(t - T/2))$ for bosons and $A(t - T/2)\delta_{ij} + iB(t - T/2)\epsilon_{ij}$ for fermions respectively. The quantities A and B are even and odd functions of their argument and we have taken the Dirac gamma matrix in the time direction to correspond to the usual Pauli matrix σ_2. To calculate the fermion masses m^F_{00}, m^F_{11}, m^F_{01} and m^F_{10} we have used the simple functions $a\cosh(m^F_{00}(t - T/2))$, $a\cosh(m^F_{11}(t - T/2))$, $a\sinh(m^F_{01}(t - T/2))$ and $a\sinh(m^F_{10}(t-T/2))$ to fit $\mathrm{Re}G^F_{00}(t)$, $\mathrm{Re}G^F_{11}(t)$, $\mathrm{Im}G^F_{01}(t)$ and $\mathrm{Im}G^F_{10}(t)$ respectively. Figure 1 shows the bosonic and fermionic masses versus coupling β in the range 0.5 to 10.0 for different lattice sizes. While the masses are quite different at small couplings, the mass degeneracy required by supersymmetry is recovered as we approach the continuum limit corresponding to large coupling β.

5.2 Ward Identities

Of prime interest and a stringent test of supersymmetry are the supersymmetric Ward identities. Consider first the scalar supersymmetry Q. The Ward identities corresponding to Q

are simply expectation values of the form $< QO >$. The simplest of these corresponds to the action itself, indeed as the latter is Q-exact [1], it is clear that,

$$
\begin{aligned}
-\frac{\partial \ln Z}{\partial \beta} &= \frac{1}{Z}\int [D\Phi] e^{-\beta S} S \\
&= \frac{1}{Z}\int [D\Phi] e^{-\beta S} Q\Lambda =< Q\Lambda >
\end{aligned}
\tag{5.15}
$$

which should be zero if the action is supersymmetric[3]. On the other hand using the effective action in (5.11),

$$
-\frac{\partial \ln Z}{\partial \beta} = 0 =< S_B > -\frac{2L^2}{\beta} \tag{5.16}
$$

Figure 2 shows a plot of $\frac{\beta}{2L^2} < S_B >$ for a range of couplings β and different lattice sizes. While there are deviations from unity at small coupling these seem to disappear as the coupling β is increased beyond $\beta \sim 4.0$ and yield very strong support to exact Q-symmetry in the continuum limit.

We have also looked at other Ward identities. Consider the local operator O of the form

$$
O = h^{+-} g_{I\overline{J}(x)} [\eta^I_+(x)\partial_-\phi^{\overline{J}}(y) + \eta^{\overline{J}}_-(x)\partial_+\phi^I(y)] \tag{5.17}
$$

The operator O is chosen in such a way that QO is local and leads to a 2-point function for the fermions. It is also Lorentz invariant in the base space and invariant under reparametrizations of the target space (at least in the continuum limit where the base difference operator becomes a true derivative). These constraints ensure the resultant supersymmetric Ward identity is satisfied non-trivially. In the $O(3)$ case $< QO >= 0$ leads to

$$
\begin{aligned}
< \tfrac{1}{2\rho^2(x)}\partial_+ u(x)\partial_-\overline{u}(y) > + < \tfrac{1}{2\rho^2(x)}\partial_+ u(y)\partial_-\overline{u}(x) > = \tfrac{1}{2} < \tfrac{1}{2\rho^2(x)}\eta(x)\partial_-\overline{\psi}(y) > + \\
\tfrac{1}{2} < \tfrac{1}{2\rho^2(x)}\overline{\eta}(x)\partial_+\psi(y) >
\end{aligned}
\tag{5.18}
$$

Figures 3, 4 and 5 show plots of (5.18) projected to zero spatial momentum for different lattice sizes and coupling $\beta = 0.5, 4$ and 10.0 respectively. While for small coupling β the Ward identities are violated they are clearly satisfied for large coupling and vanishing lattice spacing. Notice the additional, at first sight rather startling feature; for $\beta \geq 4.0$ the functions are approximately independent of the coordinate separation $|x - y|$! Actually this result follows from the Q-exactness of the theory. A continuum theory which is Q-exact has an energy momentum tensor $T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$ which is also Q-exact. Furthermore, it then follows that the correlation function of two Q-invariant operators is independent of the metric and

[3] Note that, this tells us that Z is independent of β, as long as β is not zero, and thus one can evaluate Z in the large-β limit. Such a limit corresponds to the semi-classical approximation. Such an approximation is exact in our case (this is true for Witten type theories in general)

hence also of their separation [34]. Using the Q-variation of the η_+ field in eqn. (2.3) and the equation of motion $B_+ = \partial_+ u$ we see that

$$\partial_+ u = Q\eta - \frac{2\overline{u}}{\rho}\psi\eta \tag{5.19}$$

The second term becomes small for large coupling β since $u \sim \beta^{-\frac{1}{2}}$ and hence in this limit the correlator is dominated by the leading Q-exact piece yielding the constant correlator as we observe. Thus the Ward identity reveals not only that the scalar supersymmetry is restored for large β but that the energy-momentum tensor and action of the theory are also Q-exact as expected from the continuum theory.

We have also looked at the Ward identities following from the other supercharges of the twisted $N = 2$ supersymmetry (see Appendix B for the action of these other supersymmetries). Indeed for the vector supercharges $G_\pm$ we have studied

$$< h^{+-}(G_+ O_- + G_- O+) >= 0 \tag{5.20}$$

where,

$$O_- = g_{I\overline{J}}(x)\psi^I(x)\partial_-\phi^{\overline{J}}(y) \tag{5.21}$$

$$O_+ = g_{I\overline{J}}(x)\psi^{\overline{J}}(x)\partial_+\phi^I(y) \tag{5.22}$$

For $O(3)$, eqn. (5.20) gives,

$$< \frac{1}{2\rho^2(x)}\partial_+ u(x)\partial_-\overline{u}(y) >= -\frac{1}{2} < \frac{1}{2\rho^2(x)}\overline{\psi}(x)\partial_+\eta(y) > \tag{5.23}$$

Figures 6, 7 and 8 show plots of eqn. 5.23 projected to zero spatial momentum for different lattice sizes and coupling. As for the scalar supercharge Q the Ward identities due to the vector supercharges are broken for small coupling but appear to be restored without fine tuning for small lattice spacing and large coupling. Again for large enough coupling these correlation functions appear independent of the temporal coordinate. This again follows from the fact that asymptotically we are examining the correlator of two Q-exact operators; $Q\eta$ and $Qu = \psi$.

We have additionally examined a simpler Ward identity. Namely

$$\frac{1}{2}\langle \eta(x)\bar{\psi}(y)\rangle = \langle \partial_+ u(x)\bar{u}(y)\rangle \tag{5.24}$$

This Ward identity arises as a result of applying the scalar supercharge Q to the operator $O' = \eta(x)\bar{\phi}(y)$, i.e. $\langle QO'\rangle = 0$. Figure 9 shows a numerical calculation of eqn. (5.24) projected to zero spatial momentum. While supersymmetry is broken at small β coupling it is clearly restored as we approach the continuum limit, confirming the results we obtained with the other Ward identities.

5.3 Fermionic Condensate

In the previous section we considered Ward identities that lead to fermionic 2-point functions. Here we consider Ward identities involving 4-point functions of the fermion fields. As we will see, the physics here is richer and more complicated as these quantities can receive contributions from instantons. Calculations in the continuum theory [35, 36] predict the presence of a vacuum condensate $< \frac{1}{2\rho^2}\overline{\psi}\psi >$ similar to the gluino condensate in $\mathcal{N}=1$ super Yang-Mills theory. Since $\psi = \Psi_L$ a single chiral component of the Dirac field we see that the presence of such a condensate constitutes an anomalous breaking of the chiral symmetry of the theory.

To see how this comes about in the twisted theory consider the following correlation function

$$C(x-y) = < \Theta(x)\Theta(y) > \tag{5.25}$$

where

$$\Theta(x) = J_{I\overline{J}}\psi^{\overline{J}}\psi^I(x) \tag{5.26}$$

Here, J_I^J is the complex structure in the target manifold which locally takes the form $i\delta_I^J$. It is straightforward to verify that $\Theta(x)$ is invariant under Q. Hence, as we argued earlier the correlation function $C(x-y)$ should actually be independent of $|x-y|$; $C(x-y) = C$ a constant. Further, using cluster decomposition, it can be shown that any non-vanishing value for C implies a non-vanishing value for the condensate $\langle\Theta(x)\rangle = \langle\Theta\rangle = \pm\sqrt{C}$.[4] At first glance the classical chiral symmetry of the theory would prohibit such a non-zero value for C. However quantum anomalies can and do spoil this symmetry. As is well known, the theory admits non-trivial classical solutions called instantons. These are given by solutions of the equations

$$\partial_+ u = 0 \tag{5.27}$$

The simplest single instanton is hence given by the analytic function

$$u(\overline{z}) = \frac{\alpha}{z+\beta} \tag{5.28}$$

where α and β are complex constants. There are four real zero modes associated to variation of these parameters and hence, by supersymmetry, there will be four real fermion zero modes localized in the vicinity of such an instanton. Examination of the equation $D_+\Psi = 0$ in such a background reveals these zero modes to be chiral and such a condensation will hence break chiral symmetry. Furthermore, since Θ is topological its expectation value can be computed *exactly* in the semi-classical limit corresponding to a one loop computation in such an instanton background. Notice that the number of fermion zero modes is just sufficient to saturate the four point function we are considering. The result is given in [35]. A lattice

[4] A word of caution here – [37, 38] shows that cluster decomposition *fails* in the strong coupling instanton calculation of the analogous condensate in $\mathcal{N}=1, D=4$ super Yang-Mills theory. Here, the topological character of C allows it to be calculated *exactly* at weak coupling

calculation of this quantity is potentially very interesting as it yields information on both the $U(1)$ anomaly, the presence of approximate lattice instantons and, as we will see, the dynamical breaking of supersymmetry.

In our lattice regulated $O(3)$ model we have hence measured the four point function

$$< \Theta >=< \frac{1}{\rho^2(0)}\overline{\psi}(0)\psi(0)\frac{1}{\rho^2(t)}\overline{\psi}(t)\psi(t) > \tag{5.29}$$

The plots in figure 10. shows (5.29) projected to zero spatial momentum for different lattice sizes and different values of the coupling β. Clearly in the large β limit $< \Theta(t) >$ is approximately independent of t confirming the presence of a condensate.

Of course on a torus only instanton/anti-instanton pairs can exist but provided these are well separated and the discretization errors small we can still expect a condensation of four approximate zero modes in the vicinity of such a lattice instanton yielding a corresponding contribution to the local value of C. In practice the value of the correlator is not constant as we will find contributions to C also from the anti-instanton. This is particularly true for small lattice volumes as can be seen in the figures 10. There are also explicit lattice supersymmetry breaking effects visible in the data at $\beta = 0.5$. Notice that the value of the condensate for a fixed lattice volume initially grows as β is increased but eventually turns over and decreases. We can understand this as the result of finite volume effects which become large for large β and suppress such instanton-like configurations. Notice that the correlation function we plot is dimensionless and hence contains powers of the lattice spacing $a(\beta)$. We have not measured this quantity directly so it hard to assess from figure 10 whether this condensate survives the continuum limit. We will try to address this question in the next section.

We have argued that the presence of such a condensate is signal for the anomalous breaking of chiral symmetry but it can also be viewed as an order parameter for supersymmetry breaking. To see this notice that the condensate of the $O(3)$ model can actually be obtained as the Q-variation of the following operator.

$$O_{\rm cond} = \frac{\overline{u}(x)}{\rho(x)\rho^2(y)}\psi(x)\overline{\psi}(y)\psi(y) \tag{5.30}$$

Thus a non-zero value for this quantity is equivalent to the statement $< QO >\neq 0$ and indicates a dynamical breaking of supersymmetry driven by instantons rather along the lines originally envisaged by Witten [39]. However it was pointed out in [35, 36] that this operator $O_{\rm cond}$ is *not* invariant under the global $O(3)$ symmetry of the model. It is conjectured that supersymmetry is only violated in the *unphysical* $O(3)$ non-invariant sector of the theory. Specifically, the expectation is that any operator QO containing a multiple of a four-fermion term and for which O is not $O(3)$ invariant would develop a vacuum expectation value different from zero whereas for an $O(3)$ invariant O it would remain zero. For a general $\mathcal{N} = 2$ sigma model the notion of $O(3)$ invariance would then be replaced by the requirement that the operator be invariant under all possible isometries of the target metric. It would be interesting to investigate these issues in more detail.

5.4 Phase

Up to this point we have neglected a possible phase arising in the calculation of fermion determinant. It is not hard to show that the determinant is real since generically the eigenvalues occur in complex conjugate pairs. The exception occurs with purely real eigenvalues. As one of these crosses the origin the sign of the determinant will change. Indeed figure 11. indicates that this phase does undergo weak fluctuations and one has to check for its effect on the simulation. As usual we can always compensate for neglecting the phase factor in the simulation by re-weighting all observables by the phase factor according to the simple rule

$$< O >= \frac{< O e^{i\alpha(\Phi)} >_{\alpha=0}}{< e^{i\alpha(\Phi)} >_{\alpha=0}} \tag{5.31}$$

Hence we examined the Ward identities we have studied in the previous section now weighted with this phase factor. For example table 1 shows the mean re-weighted bosonic action together with the phase quenched value for different lattice sizes and a range of couplings β. While re-weighting typically amplifies the estimated errors it does not appear to change the mean value for this observable at least within statistical errors. This seems to indicate that the phase fluctuates approximately independently of the remaining part of the measured observable leading to an, at least approximate factorization in the reweighted observable

$$< O >_{full} = \frac{< O e^{i\alpha} >_{\alpha=0}}{< e^{i\alpha} >_{\alpha=0}} \sim \frac{< O >_{\alpha=0} < e^{i\alpha} >_{\alpha=0}}{< e^{i\alpha} >_{\alpha=0}} = < O >_{\alpha=0} \tag{5.32}$$

5.5 Continuum Limit

Figure 12 shows a plot for the dimensionless ratio value $\frac{<1/\rho^2 \overline{\psi}\psi>}{m_B}$ (m_B is the lightest boson mass) for different lattice sizes and for different values of the coupling. One can see that an approximate plateau occurs for sufficiently small lattice spacing which we interpret as the onset of continuum physics. The separation of the curves from different lattice volumes we consider to be a measure of possible finite volume effects. Modulo these finite volume questions we take this as evidence that the magnitude of the condensate is non-zero in the continuum limit.

6. Conclusions

This paper is devoted to a study of the $\mathcal{N} = 2$ supersymmetric sigma model regulated on a lattice. The lattice formulation we employ was derived in an earlier paper [1] and results from a discretization of a *twisted* version of the continuum theory. The twisting process is to be viewed as a change of variables (we are in flat space) and has the merit of exposing a nilpotent scalar supersymmetry Q which can be preserved under discretization. In this paper we extend our previous work by introducing a modified Wilson operator in the form of a twisted mass term which is used to remove doubler modes from our lattice theory without spoiling the Q-exact property of the lattice action. The introduction of such a mass term is

β	$\beta < S_b >$	$\beta < S_b >_W$
0.5	145.523 ± 0.286	144.464 ± 3.858
1.0	147.369 ± 0.281	148.764 ± 21.547
2.0	135.51 ± 0.281	139.749 ± 53.707
3.0	128.275 ± 0.242	122.731 ± 8.747
4.0	127.272 ± 0.329	126.933 ± 2.093
5.0	127.326 ± 0.213	127.342 ± 1.274
10.0	127.552 ± 0.299	127.164 ± 1.967

β	$\beta < S_b >$	$\beta < S_b >_W$
0.5	321.541 ± 0.568	321.461 ± 8.984
1.0	310.143 ± 0.391	305.595 ± 17.114
2.0	291.275 ± 0.360	284.338 ± 40.365
3.0	286.719 ± 0.352	276.315 ± 24.448
4.0	285.546 ± 0.326	283.904 ± 5.489
5.0	285.627 ± 0.340	285.307 ± 3.386
10.0	287.163 ± 0.285	286.943 ± 2.483

β	$\beta < S_b >$	$\beta < S_b >_W$
0.5	283.02 ± 2.166	285.369 ± 4.369
1.0	458.823 ± 1.707	459.028 ± 7.847
2.0	509.266 ± 0.628	504.264 ± 38.233
3.0	503.627 ± 0.784	496.215 ± 46.938
4.0	507.243 ± 0.609	505.991 ± 13.837
5.0	507.412 ± 1.028	506.915 ± 14.778
10.0	511.561 ± 0.570	511.715 ± 5.386

Table 1: $\beta < S_b >$ for quenched and unquenched phase on 8x8, 12x12 and 16x16 lattices.

compatible with supersymmetry if it takes the form of a holomorphic Killing vector in the target space [40] and corresponds to the addition of central charges in the supersymmetry algebra. The replacement of a simple mass term with a Wilson difference operator breaks supersymmetry softly and is not expected to lead to additional fine tuning. We study the $CP^1 \sim O(3)$-model in detail using the RHMC algorithm on lattices as large as 16^2 over a range of coupling $\beta = 0.5 \to 10.0$. Our results for both the spectrum and Ward identities provide strong evidence for a restoration of *full* supersymmetry at large β and small lattice spacing *without additional fine tuning*. Notice that while the Ward identities are examples of trivial topological observables the mass spectrum we measure is non-topological. Further investigations of such physical observables are underway and will published elsewhere [41]. We are also able to see a non-zero chiral condensate and this appears to persist into the continuum limit as expected from the chiral anomaly. It would be very interesting to extend this work to the general CP^{N-1} models where the numerical results could be compared with exact results on the low lying mass spectrum [42].

7. Acknowledgements

This work was supported in part by DOE grant DE-FG02-85ER40237.

A. Appendix: Notations

A real system is denoted by x^1, x^2 which combine to give holomorphic coordinates,

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2 \tag{A.1}$$

Then a vector V^μ with components (V^1, V^2) has holomorphic components,

$$V_+ = \frac{1}{2}(V_1 - iV_2), \quad V_- = \frac{1}{2}(V_1 + iV_2) \tag{A.2}$$

The Dirac matrices γ^μ are given by (Chiral basis),

$$(\gamma^1)_\alpha{}^\beta = \sigma^1, \quad (\gamma^2)_\alpha{}^\beta = \sigma^2 \tag{A.3}$$

Spinor indices are raised and lowered by the matrix $C_{\alpha\beta} = \sigma^1$,

$$(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_\alpha{}^\tau C_{\tau\beta} \tag{A.4}$$

thus for a vector V^μ,

$$\gamma^\mu_{++} V_\mu = V_1 - iV_2 = V_+ \tag{A.5}$$
$$\gamma^\mu_{--} V_\mu = V_1 + iV_2 = V_- \tag{A.6}$$
$$\tag{A.7}$$

For a Kähler manifold, the metric is given by,

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K \tag{A.8}$$

where K is a Kähler potential. (Note that $g_{\bar{I}\bar{J}} = g_{IJ} = 0$ for a almost complex manifold). The non vanishing components of the Christoffel symbols are then,

$$\Gamma^I_{JK} = g^{I\bar{L}} \partial_J g_{K\bar{L}}, \quad \Gamma^{\bar{I}}_{\bar{J}\bar{K}} = g^{\bar{I}L} \partial_{\bar{J}} g_{L\bar{K}} \tag{A.9}$$

This implies the only non-trivial components of the Riemann tensor are,

$$R_{\bar{I}J\bar{K}L} = g_{\bar{I}M} \partial_{\bar{K}} \Gamma^M_{JL}, \ or \quad R^I{}_{J\bar{K}L} = \partial_{\bar{K}} \Gamma^I_{JL}, \tag{A.10}$$

B. Appendix: Twisted $\mathcal{N} = 2$ supersymmetry algebra

The algebra of $\mathcal{N} = 2$ supersymmetry in 2 space dimensions is given by [28],

$$\begin{array}{ll} \{Q_{\alpha+}, Q_{\beta-}\} = \gamma^\mu_{\alpha\beta} P_\mu, & [R, Q_{\alpha\pm}] = \pm\frac{1}{2} Q_{\alpha\pm} \\ \{Q_{\alpha+}, Q_{\beta+}\} = \{Q_{\alpha-}, Q_{\beta-}\} = 0, & [J, P_\mu] = -i\epsilon_\mu{}^\nu P_\nu \\ [Q_{\alpha a}, P_\mu] = [P_\mu, P_\nu] = 0, & [R, P_\mu] = 0 \\ [J, Q_{\pm a}] = \pm\frac{1}{2} Q_{\pm a}, & [J, R] = [J, J] = [R, R] = 0 \end{array} \tag{B.1}$$

where J is the generator of Lorentz $SO(2)$ symmetry and R is the generator of the internal $SO(2)$ symmetry. In order to obtain some scaler charges, one perform a twisting that changes the spin of the above charges. In order to do this we need to redefine the Lorentz generator. Indeed if we take (see Appendix A for our notations),

$$\tilde{J} = J + R \tag{B.2}$$

with respect to this new generator one finds that Q_{+-} and Q_{-+} behave as scalars while the pair Q_{++} and Q_{--} as vectors. To make manifest the new Lorentz structure of each of the generators we define,

$$\begin{aligned}
&Q_L = Q_{+-} \\
&Q_R = Q_{-+} \\
&Q_{++} = \gamma^{\mu}_{++} G_{\mu} = G_{+} \\
&Q_{--} = \gamma^{\mu}_{--} G_{\mu} = G_{-}
\end{aligned}$$

it is clear from (B.1) that,

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0 \tag{B.3}$$

The algebra (B.1) in terms of the new Lorentz generator $\tilde{J}$ and the following redefined operators,

$$Q = Q_L + Q_R, \quad M = Q_L - Q_R \tag{B.4}$$

is

$$\begin{aligned}
&Q^2 = M^2 = \{Q, M\} = [Q, P_{\mu}] = [M, P_{\mu}] = 0 \\
&\{Q, G_{\mu}\} = P_{\mu} \\
&[Q, \tilde{J}] = [M, \tilde{J}] = 0 \\
&\{M, G_{\mu}\} = -i\epsilon_{\mu}{}^{\nu} P_{\nu} \\
&[\tilde{J}, P_{\mu}] = -i\epsilon_{\mu}{}^{\nu} P_{\nu} \\
&[\tilde{J}, G_{\mu}] = -i\epsilon_{\mu}{}^{\nu} G_{\nu} \\
&[P_{\mu}, P_{\nu}] = \{G_{\mu}, G_{\nu}\} = [\tilde{J}, \tilde{J}] = 0
\end{aligned} \tag{B.5}$$

in addition, the action of the R generator on the new twisted charges is,

$$\begin{aligned}
&[R, Q] = -\tfrac{1}{2} M \\
&[R, M] = -\tfrac{1}{2} Q \\
&[R, G_{\mu}] = -\tfrac{i}{2}\epsilon_{\mu}{}^{\nu} G_{\nu} \\
&[R, R] = [R, \tilde{J}] = [R, P_{\mu}] = 0
\end{aligned} \tag{B.6}$$

Let our field content be $\phi^i, B^i_{\alpha}, \psi^i$ and η^i_{α} or on the complex manifold ϕ^I, B^I_{+}, ψ^I and η^I_{+}. The R-transformations of these fields is given by

$$\begin{aligned}
&[R, \phi^I] = 0, && [R, \phi^{\bar{I}}] = 0 \\
&[R, \psi^I] = \tfrac{1}{2}\psi^I, && [R, \psi^{\bar{I}}] = -\tfrac{1}{2}\psi^{\bar{I}} \\
&[R, \eta^I_{+}] = \tfrac{1}{2}\eta^I_{+}, && [R, \eta^{\bar{I}}_{-}] = -\tfrac{1}{2}\eta^{\bar{I}}_{-} \\
&[R, B^I_{+}] = B^I_{+}, && [R, B^{\bar{I}}_{-}] = -B^{\bar{I}}_{-}
\end{aligned} \tag{B.7}$$

More importantly the transformation of our fields under the twisted charges Q, M, G_+ and G_- is given by, for Q,

$$\begin{aligned}
&Q\phi^I = \psi^I && Q\phi^{\bar{I}} = \psi^{\bar{I}} \\
&Q\psi^I = 0 && Q\psi^{\bar{I}} = 0 \\
&Q\eta_+^I = B_+^I - \Gamma^I_{KJ}\psi^J\eta_+^K && Q\eta_-^{\bar{I}} = B_-^{\bar{I}} - \Gamma^{\bar{I}}_{\bar{K}\bar{J}}\psi^{\bar{J}}\eta_-^{\bar{K}} \\
&QB_+^I = -\Gamma^I_{JK}\psi^J B_+^K - R^I{}_{K\bar{J}L}\psi^K\psi^{\bar{J}}\eta_+^L && QB_-^{\bar{I}} = -\Gamma^{\bar{I}}_{\bar{J}\bar{K}}\psi^{\bar{J}}B_-^{\bar{K}} - R^{\bar{I}}{}_{\bar{J}L\bar{K}}\psi^{\bar{J}}\psi^L\eta_-^{\bar{K}}
\end{aligned} \tag{B.8}$$

for M

$$\begin{aligned}
&M\phi^I = -\psi^I && M\phi^{\bar{I}} = \psi^{\bar{I}} \\
&M\psi^I = 0 && M\psi^{\bar{I}} = 0 \\
&M\eta_+^I = -(B_+^I - \Gamma^I_{KJ}\psi^J\eta_+^K) && M\eta_-^{\bar{I}} = B_-^{\bar{I}} - \Gamma^{\bar{I}}_{\bar{K}\bar{J}}\psi^{\bar{J}}\eta_-^{\bar{K}} \\
&MB_+^I = \Gamma^I_{JK}\psi^J B_+^K - R^I{}_{K\bar{L}J}\psi^J\psi^{\bar{L}}\eta_+^K && MB_-^{\bar{I}} = -\Gamma^{\bar{I}}_{\bar{J}\bar{K}}\psi^{\bar{J}}B_-^{\bar{K}} + R^{\bar{I}}{}_{\bar{K}L\bar{J}}\psi^{\bar{J}}\psi^L\eta_-^{\bar{K}}
\end{aligned} \tag{B.9}$$

for G_+

$$\begin{aligned}
&G_+\phi^I = \tfrac{1}{2}\eta_+^I && G_+\phi^{\bar{I}} = 0 \\
&G_+\psi^I = -\tfrac{1}{2}(B_+^I - \Gamma^I_{KJ}\psi^J\eta_+^K) && G_+\psi^{\bar{I}} = 0 \\
&G_+\eta_+^I = 0 && G_+\eta_-^{\bar{I}} = 0 \\
&G_+B_+^I = -\tfrac{1}{2}\Gamma^I_{JK}B_+^J\eta_+^k && G_+B_-^{\bar{I}} = \partial_+\eta_-^{\bar{I}} - \tfrac{1}{2}R^{\bar{I}}{}_{\bar{K}L\bar{J}}\psi^{\bar{J}}\eta_+^L\eta_-^{\bar{K}}
\end{aligned} \tag{B.10}$$

and finally for G_-,

$$\begin{aligned}
&G_-\phi^I = 0 && G_-\phi^{\bar{I}} = \tfrac{1}{2}\eta_-^{\bar{I}} \\
&G_-\psi^I = 0 && G_-\psi^{\bar{I}} = -\tfrac{1}{2}(B_-^{\bar{I}} - \Gamma^{\bar{I}}_{\bar{K}\bar{J}}\psi^{\bar{J}}\eta_-^{\bar{K}}) \\
&G_-\eta_+^I = 0 && G_-\eta_-^{\bar{I}} = 0 \\
&G_-B_+^I = \partial_-\eta_+^I - \tfrac{1}{2}R^I{}_{K\bar{L}J}\psi^J\eta_-^{\bar{L}}\eta_+^K && G_-B_-^{\bar{I}} = -\tfrac{1}{2}\Gamma^{\bar{I}}_{\bar{J}\bar{K}}B_-^{\bar{J}}\eta_-^{\bar{k}}
\end{aligned} \tag{B.11}$$

References

[1] S. Catterall & S. Ghadab, JHEP 05, 044(2004), hep-lat/0311042.
S. Catterall, Nucl.Phys.Proc.Suppl.140:751-753,2005, hep-lat/0409015.
S. Ghadab, Nucl.Phys.Proc.Suppl.140:760-762,2005, hep-lat/0409102.

[2] S. Weinberg, The Quantum Theory of Fields III (Cambridge University press, 2000).

[3] E. J. Martinec and V. Sahakian, Phys. Rev. D **59**, 124005 (1999) [arXiv:hep-th/9810224].
L. Susskind, arXiv:hep-th/9805115.
O. Aharony, J. Marsano, S. Minwalla and T. Wiseman, Class. Quant. Grav. **21**, 5169 (2004) [arXiv:hep-th/0406210].
N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998) [arXiv:hep-th/9802042].

[4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[5] A. M. Polyakov, Phys. Lett. 59B (1975) 79.

[6] M. Harada and S. Pinsky, Phys. Rev. D **71**, 065013 (2005) [arXiv:hep-lat/0411024].
M. Harada and S. Pinsky, Phys. Rev. D **70**, 087701 (2004) [arXiv:hep-lat/0408026].
M. Harada and S. Pinsky, Phys. Lett. B **567**, 277 (2003) [arXiv:hep-lat/0303027].

[7] D. B. Kaplan, E. Katz and M. Unsal, JHEP **0305**, 037 (2003) [arXiv:hep-lat/0206019].
A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, JHEP **0308**, 024 (2003) [arXiv:hep-lat/0302017].
A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal JHEP **0312**, 031 (2003) [arXiv:hep-lat/0307012].

[8] D. B. Kaplan and M. Unsal, JHEP **0509**, 042 (2005) [arXiv:hep-lat/0503039].

[9] M. Unsal, JHEP **0511**, 013 (2005) [arXiv:hep-lat/0504016].

[10] J. Nishimura, S. J. Rey and F. Sugino, JHEP **0302**, 032 (2003) [arXiv:hep-lat/0301025].
J. Giedt, E. Poppitz and M. Rozali, JHEP **0303**, 035 (2003) [arXiv:hep-th/0301048].
J. Giedt, Nucl. Phys. B **668**, 138 (2003) [arXiv:hep-lat/0304006].
J. Giedt, Nucl. Phys. B **674**, 259 (2003) [arXiv:hep-lat/0307024].

[11] M. G. Endres and D. B. Kaplan, [arxiv:hep-lat/0604012].

[12] E. Witten, Commun. Math. Phys. **117**, 353 (1988
Commun. Math. Phys. **118**, 411 (1988).

[13] S. Catterall, JHEP **0305**, 038 (2003) [arXiv:hep-lat/0301028].

[14] J. Kato, N. Kawamoto and Y. Uchida, Int. J. Mod. Phys. A **19**, 2149 (2004) [arXiv:hep-th/0310242].

[15] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, Nucl. Phys. B **707**, 100 (2005) [arXiv:hep-lat/0406029].

[16] J. Kato, N. Kawamoto and A. Miyake, Nucl. Phys. B **721**, 229 (2005) [arXiv:hep-th/0502119].

[17] S. Catterall, JHEP **0411**, 006 (2004) [arXiv:hep-lat/0410052].

[18] S. Catterall, Nucl. Phys. Proc. Suppl. **140**, 751 (2005) [arXiv:hep-lat/0409015].

[19] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata Phys.Lett.B633:645-652,2006 [arXiv:hep-lat/0507029]

[20] F. Bruckmann and M. de Kok, arXiv:hep-lat/0603003.

[21] M. Unsal, arXiv:hep-th/0603046.

[22] S. Catterall and E. Gregory, Phys. Lett. B **487**, 349 (2000) [arXiv:hep-lat/0006013].

[23] S. Catterall and S. Karamov, Phys. Rev. D **65**, 094501 (2002) [arXiv:hep-lat/0108024].

[24] F. Sugino, JHEP **0401**, 015 (2004) [arXiv:hep-lat/0311021].
F. Sugino, JHEP **0403**, 067 (2004) [arXiv:hep-lat/0401017].
F. Sugino, JHEP **0501**, 016 (2005) [arXiv:hep-lat/0410035].
F. Sugino, Phys. Lett. B **635**, 218 (2006) [arXiv:hep-lat/0601024].

[25] S. Catterall, JHEP **0506**, 027 (2005) [arXiv:hep-lat/0503036].
S. Catterall, PoS LAT2005:006,2005 [arXiv:hep-lat/0509136].
S. Catterall, arXiv:hep-lat/0510054.
S. Catterall, JHEP **0603**, 032 (2006) [arXiv:hep-lat/0602004].

[26] S. Elitzur, E. Rabinovici and A. Schwimmer, Phys. Lett. B119 (1982) 165.

[27] J.F. Rabin, Nucl. Phys. B **201**, 315 (1982).

[28] J.M.F. Labastida and P.M. Llatas, Nucl. Phys. B379(1992)220-258, [arXiv:hep-th/9112051]

[29] J.M.F. Labastida and P.M. Llatas, Phys. Lett. B271(1991)101.

[30] L. Alvarez-Gaume, D.Z. Freedman, Comm. Math. Phys. 80 (1981) 443

[31] M. A. Clark, A. D. Kennedy and Z. Sroczynski, Nucl. Phys. Proc. Suppl. **140**, 835 (2005) [arXiv:hep-lat/0409133].

[32] S. Duane, A. D. Kennedy, B. J. Pendleton and D. Roweth, Phys. Lett. B **195**, 216 (1987).

[33] B. Jegerlehner, arXiv:hep-lat/9612014.

[34] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys.Rept. **209**, 129(1991).

[35] V.A. Novikov et al., Phys. Rept. **116**, 103(1984).

[36] H. Bohr, E. Katznelson and K.S. Narain, Nucl. Phys. B **238**, 407(1984).

[37] T. Hollowood, V. Khoze, W. Lee and M. Mattis, Nucl. Phys. B570 (2000) 241.

[38] N. Michael Davis, T. Hollowood, V. Khoze and M. Mattis, Nucl. Phys. B559 (1999) 123.

[39] E. Witten, Nucl. Phys. B**202**, 253(1982).

[40] L. Alvarez-Gaume and D.Z. Freedman, Commun.Math.Phys.91:87,1983.

[41] S. Catterall and S. Ghadab, in preparation.

[42] T. Hollowood and J. Evans, Phys. Lett. B343 (1995) 198

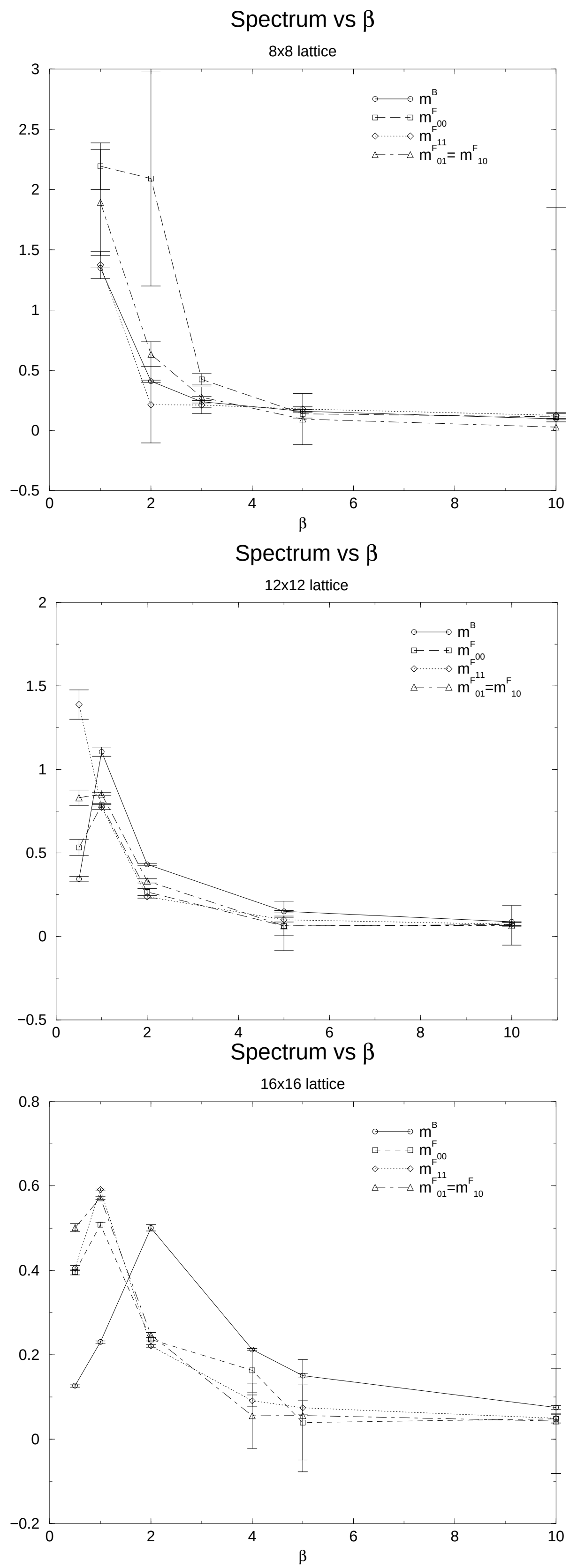

Figure 1: mass spectrum for 8x8, 12x12 and 16x16 lattices

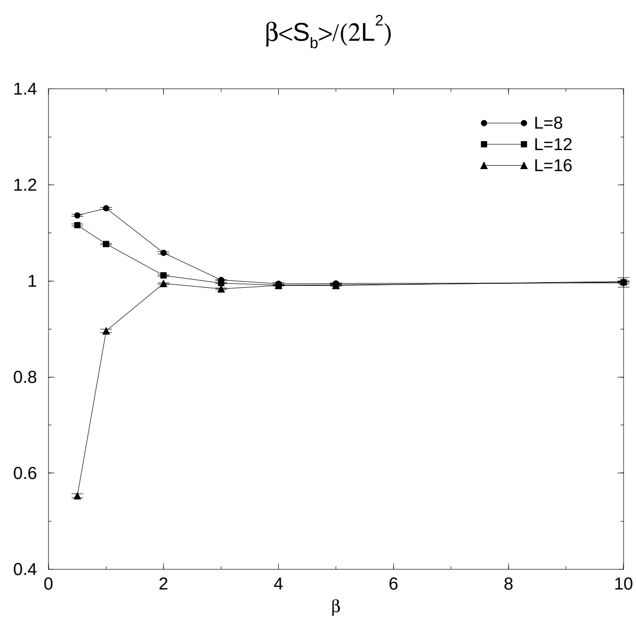

Figure 2: Simple Ward Identity

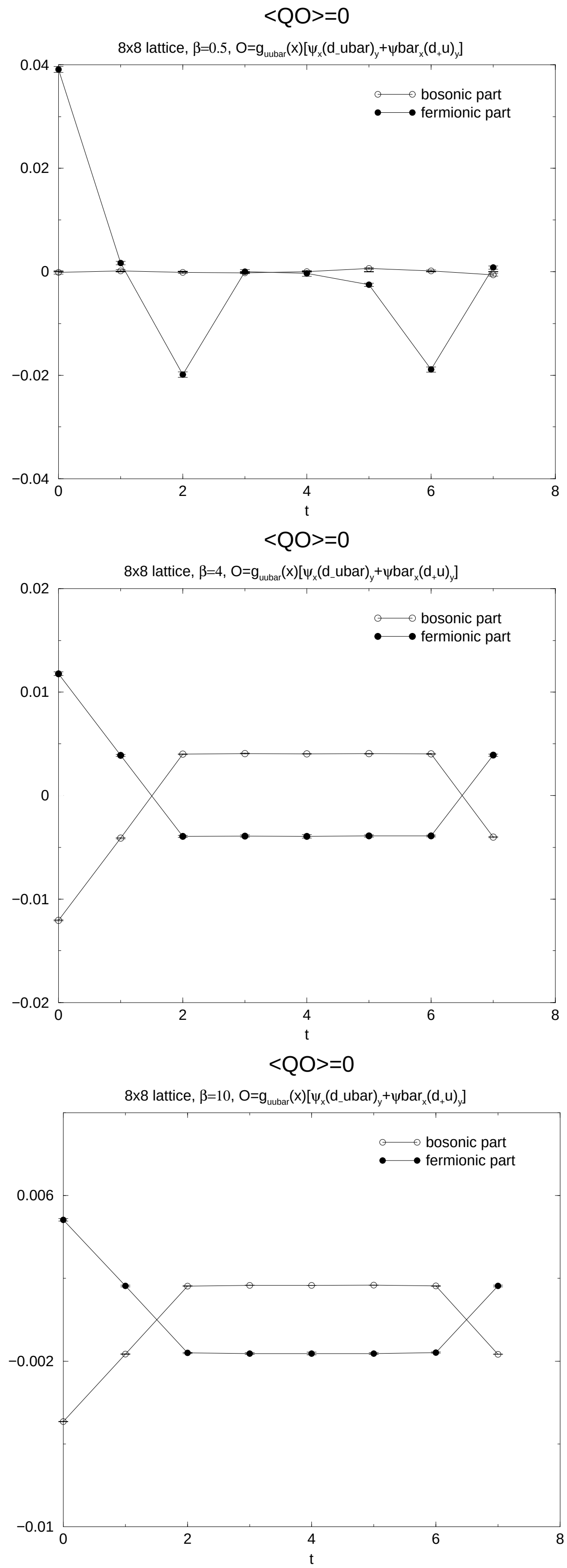

Figure 3: $< QO >= 0$ for 8×8 lattice

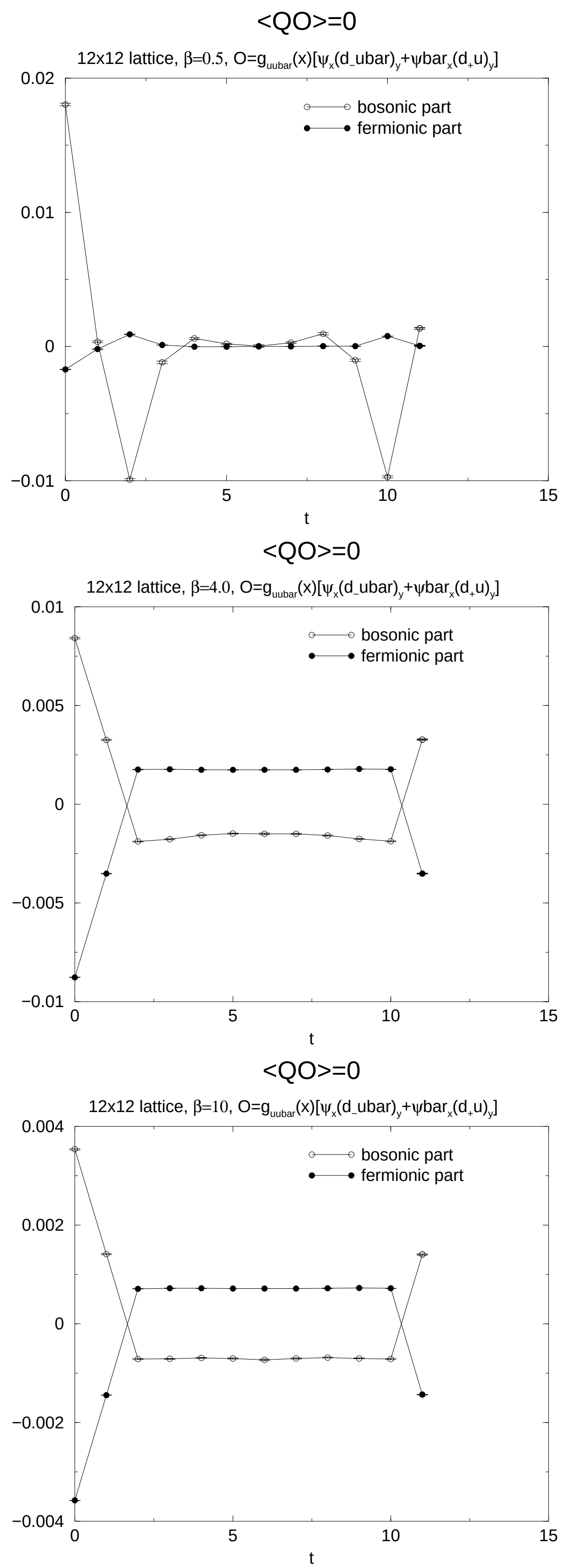

Figure 4: $< QO >= 0$ for 12×12 lattice

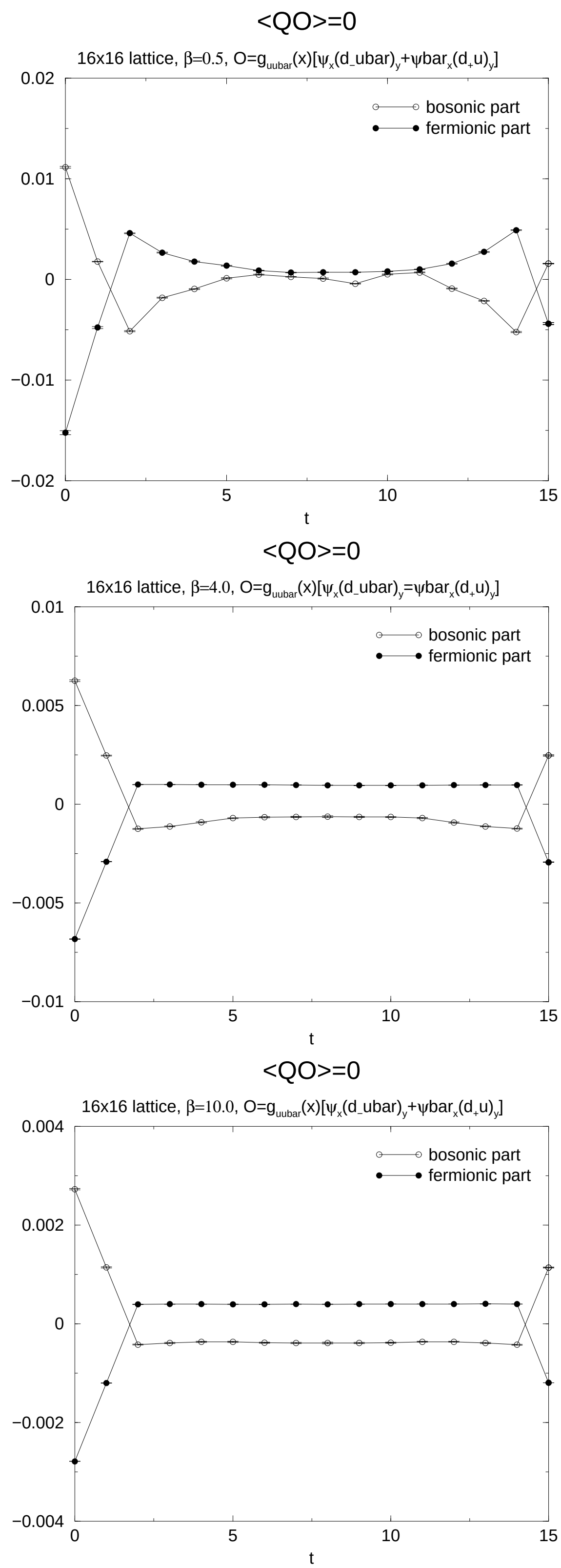

Figure 5: $< QO >= 0$ for 16×16 lattice

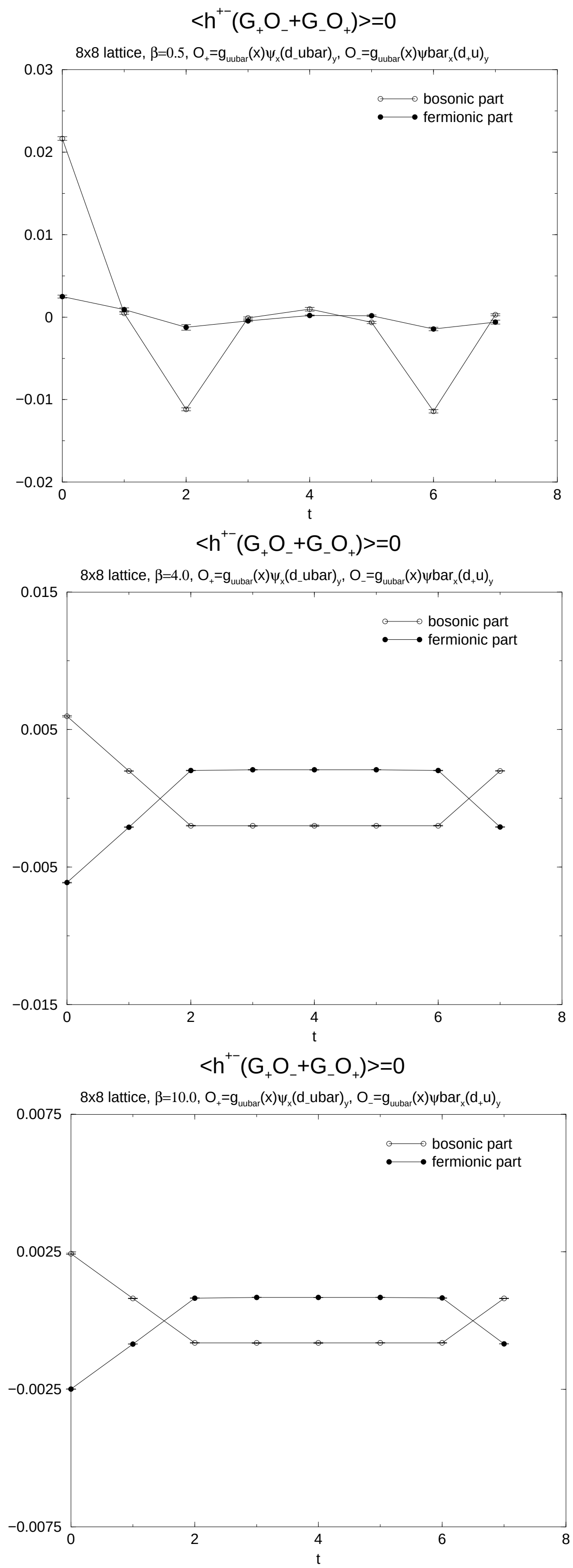

Figure 6: $< h^{+-}(G_+O_- + G_-O_+) >= 0$ for 8×8 lattice

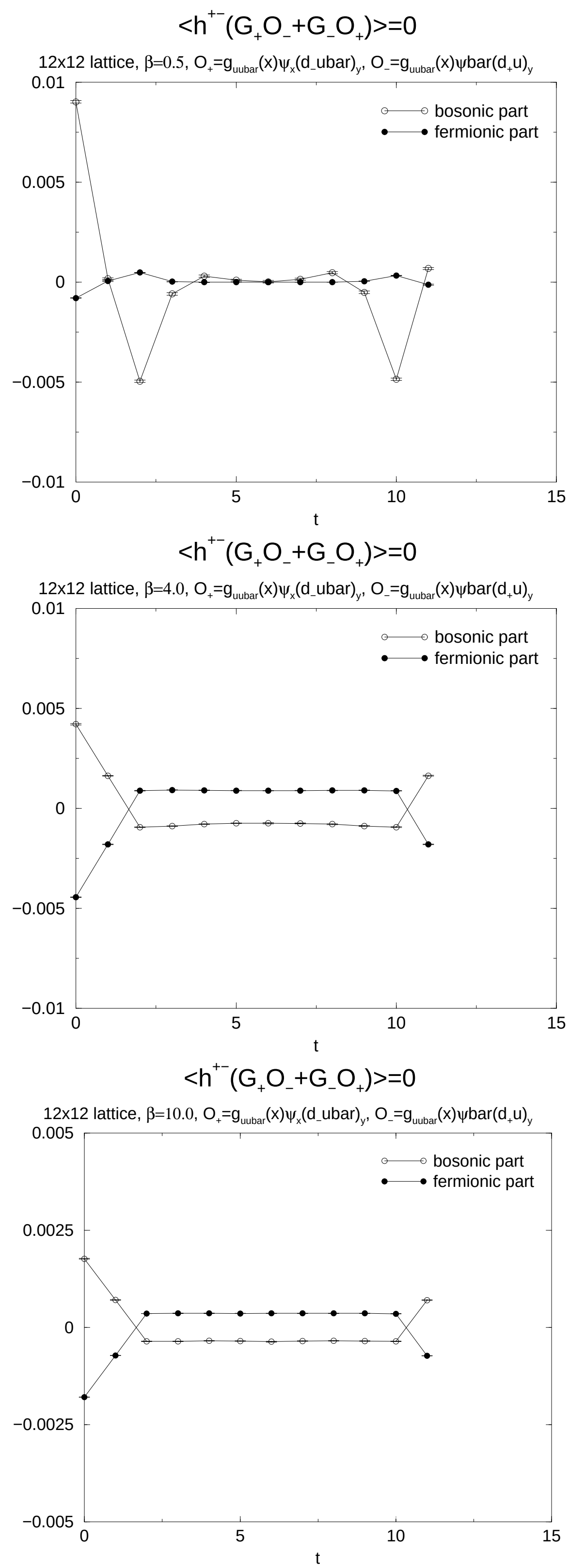

Figure 7: $< h^{+-}(G_+O_- + G_-O_+) >= 0$ for 12×12 lattice

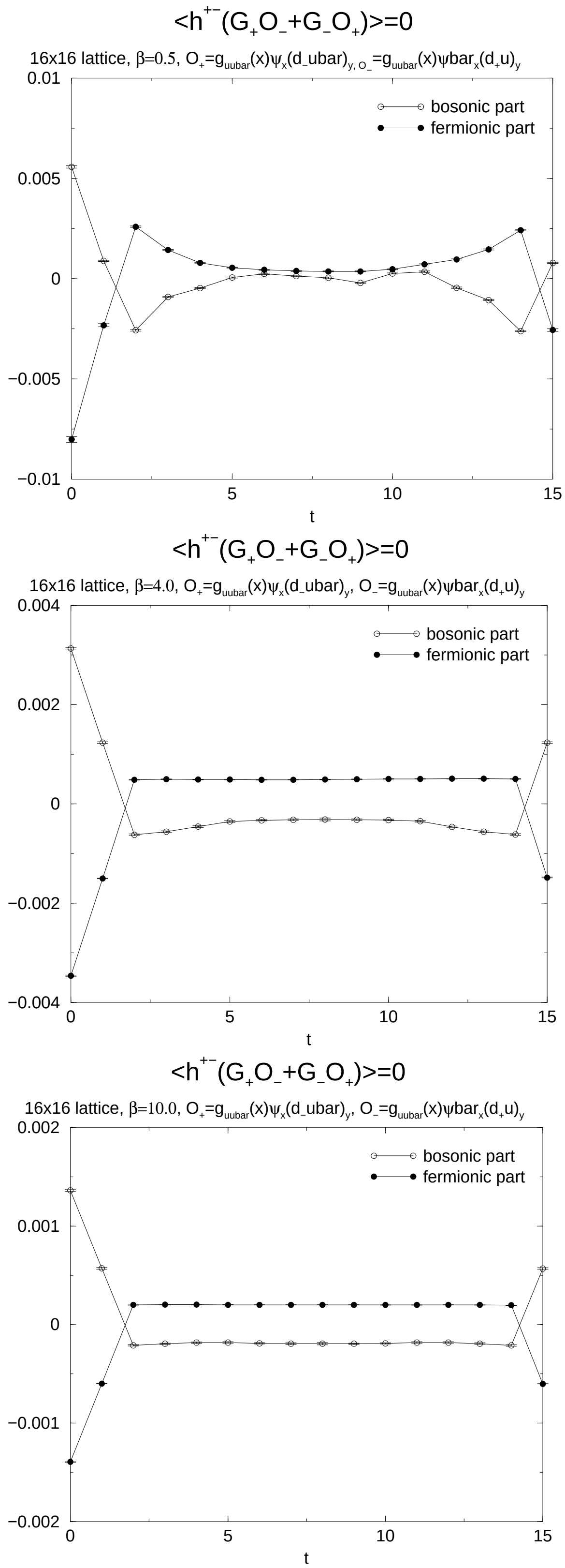

Figure 8: $< h^{+-}(G_+O_- + G_-O_+) >= 0$ for 16×16 lattice

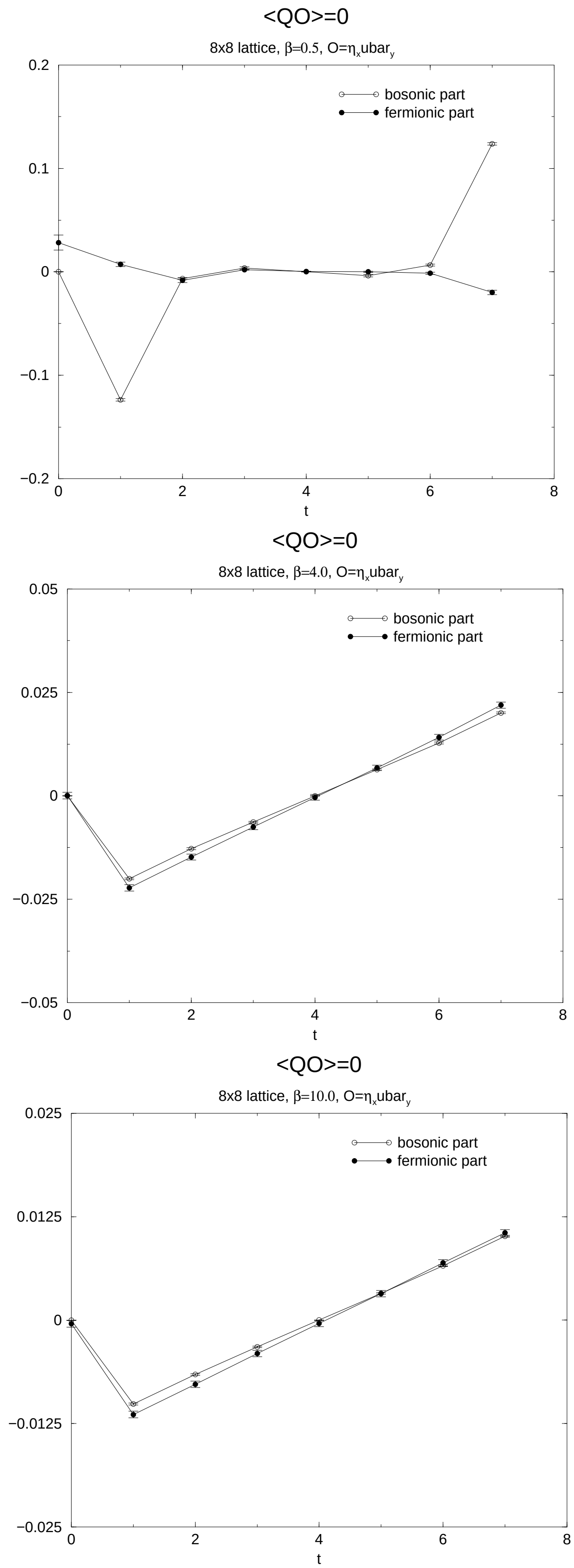

Figure 9: $< Q(\eta_x \bar{u}_y) >= 0$ for 8×8 lattice

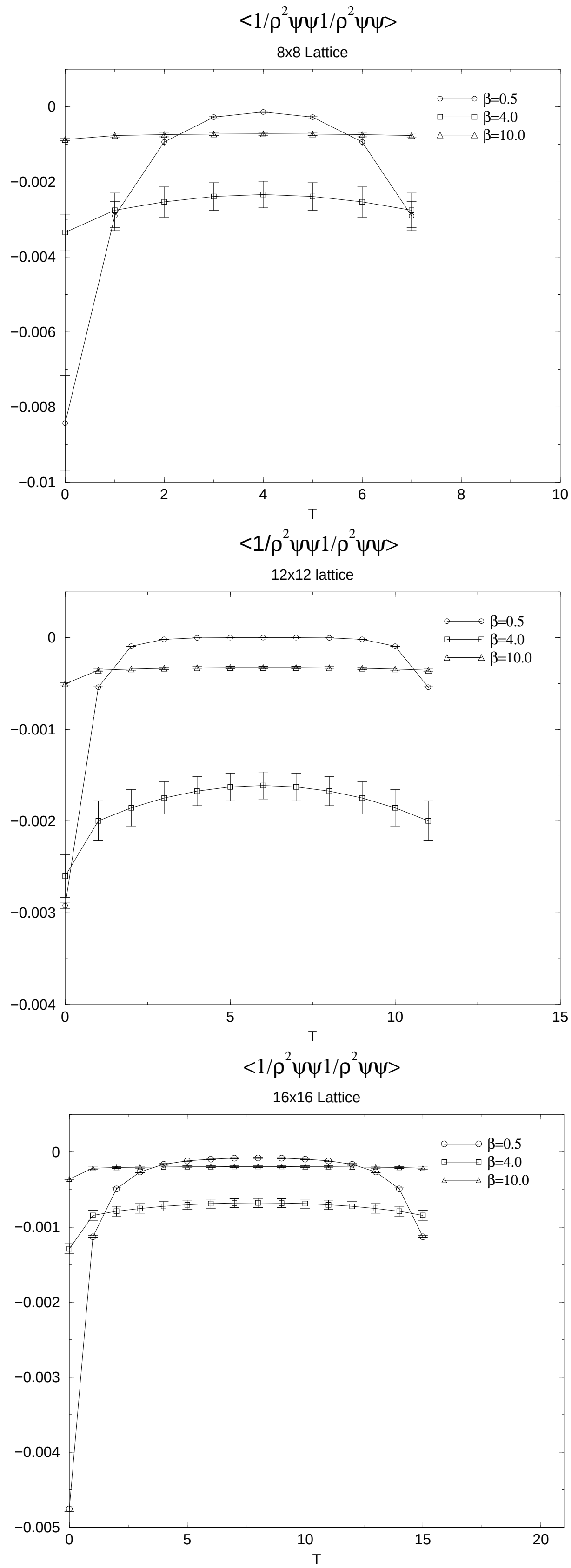

Figure 10: condensate for 8x8, 12x12 and 16x16 lattices

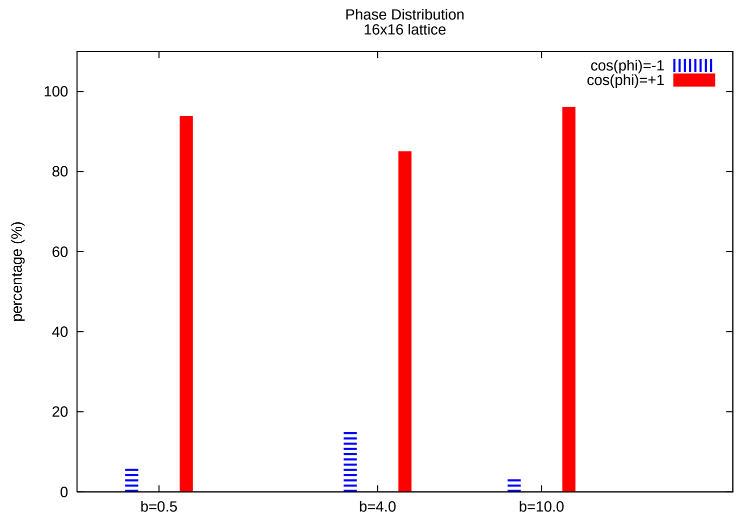

Figure 11: Distribution of phase on 16x16 lattice

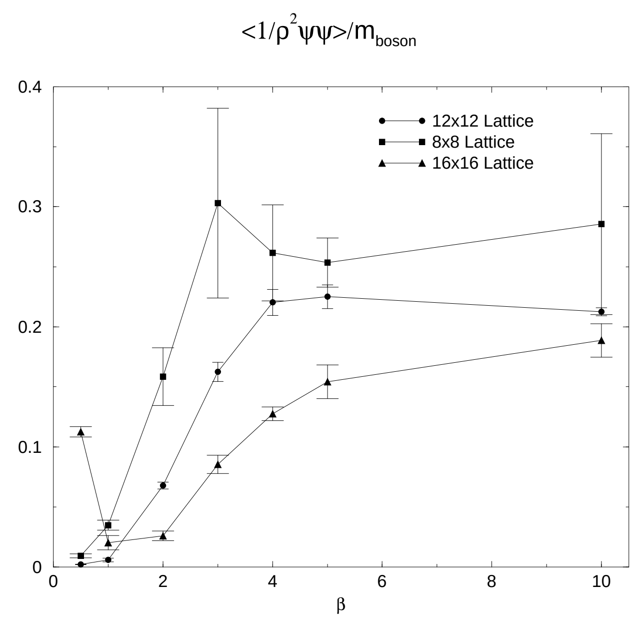

Figure 12: $< 1/\rho^2\overline{\psi}\psi > /m^B$