From Twisted Supersymmetry to Orbifold Lattices

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ABSTRACT: We show how to derive the supersymmetric orbifold lattices of Cohen et al. [1, 2] and Kaplan et al. [3] by direct discretization of an appropriate twisted supersymmetric Yang-Mills theory. We examine in detail the four supercharge two dimensional theory and the theory with sixteen supercharges in four dimensions. The continuum limit of the latter theory is the well known Marcus twist of $\mathcal{N} = 4$ Yang-Mills. The lattice models are gauge invariant and possess one exact supersymmetry at non-zero lattice spacing.

Contents

1. Introduction

The problem of putting supersymmetry on the lattice is an old one going back at least 25 years (see the review [4] and references therein). However most of the older work utilized discretization schemes that break supersymmetry completely at the classical level. With a few notable exceptions like $\mathcal{N} = 1$ super Yang-Mills in $D = 4$ such an approach generically leads to fine tuning problems – the couplings to a set of induced SUSY violating operators must be tuned carefully to zero as the lattice spacing is reduced [5]. In low dimensions this fine tuning may be manageable since the theories are super-renormalizable and hence all divergences occur in low orders of perturbation theory [6, 7].

Recently, however, the field has seen a resurgence of activity due to the realization that a certain subclass of theories could be discretized while preserving a fraction of the continuum supersymmetries [8, 1, 2, 3, 9, 10, 11]. Two main approaches have been followed; obtaining a lattice theory by orbifolding a supersymmetric matrix model (for a good review of this approach see [12]) and direct discretization of a reformulation of theory in terms of twisted fields[1]. The twisting procedure goes back to Witten [14] in his seminal construction of topological field theories but actually had been anticipated in earlier lattice work using Kähler-Dirac fields [15]. The precise connection between the Kähler-Dirac fermion mechanism and topological twisting was found by Kawamoto and collaborators [16, 17].

[1]A third approach based on deformation of the IIB matrix model appears to provide an independent construction of the $\mathcal{N} = 4$ theory which also preserves a scalar supersymmetry [13]

While the orbifold constructions are essentially unique [18] various approaches to discretization of the twisted theories have been advocated in [19, 20, 21, 22, 23, 24]. Recent work by Damgaard, Matsuura and Takimi has indicated that there are, in fact, strong connections between these twisted theories and the orbifold models [25, 26, 27]. This had already been anticipated by Unsal who showed that the naive continuum limit of the sixteen supercharge orbifold model in four dimensions led to the Marcus twist of $\mathcal{N}=4$ Yang-Mills [28].

In this paper we complete this web of interconnections by showing that the orbifold actions can be obtained by direct discretization of an appropriate twist of the supersymmetric Yang-Mills theory. We consider first the two dimensional theory with four supercharges which has been extensively discussed in the literature and for which numerical simulations have already been attempted [29, 30, 31]. We then show how to rewrite the continuum Marcus twist of $\mathcal{N}=4$ as the dimensional reduction of a very simple five dimensional theory which is almost of the same form as the two dimensional theory. This simple five dimensional structure allows us to use the geometric discretization prescription employed in two dimensions to write down a supersymmetric lattice theory corresponding to this Marcus twist of $\mathcal{N}=4$ Yang-Mills. The resulting theory is nothing more than the $\mathcal{Q}=16$ orbifold lattice theory in four dimensions.

2. Four supercharge theory in two dimensions

2.1 Continuum twisted theory

Following the arguments given in [32, 17, 16] the continuum theory is first rewritten in *twisted* form. The bosonic sector of the twisted theory comprises a single *complexified* gauge connection $\mathcal{A}$ and a scalar auxiliary field d. Fermionic degrees of freedom are naturally embedded as components of a single complex Kähler-Dirac field $\Psi=(\eta,\psi_\mu,\chi_{\mu\nu})$ whose components are antisymmetric tensor fields. We will take all these fields as living in the adjoint of a $U(N)$ gauge group. The twisted theory naturally possesses a nilpotent scalar supercharge $\mathcal{Q}$ whose action on these fields is given by

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned} \tag{2.1}$$

Notice that in this formulation of the twisted theory all the physical bosonic degrees of freedom are carried by the complex gauge field. The scalar supercharge that is employed here corresponds to a complex combination of the scalar supercharge Q employed in earlier twisted lattice constructions [19, 33, 20, 21] and its 2-form dual Q_{12}. Notice that this supersymmetry implies that the fermions are complex which is natural in a Euclidean theory. As in previous constructions the twisted action in two dimensions can be written in $\mathcal{Q}$-exact form $S = \beta\mathcal{Q}\,\Lambda$ where Λ is

$$\Lambda = \int \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right) \tag{2.2}$$

and we have introduced the complexified covariant derivatives (we employ an antihermitian basis for the generators of $U(N)$)

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + \mathcal{A}_\mu = \partial_\mu + A_\mu + iB_\mu \\ \overline{\mathcal{D}}_\mu &= \partial_\mu + \overline{\mathcal{A}}_\mu = \partial_\mu + A_\mu - iB_\mu \end{aligned} \tag{2.3}$$

Doing the $\mathcal{Q}$-variation and integrating out the field d yields

$$S = \int \mathrm{Tr}\,\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right) \tag{2.4}$$

The bosonic terms can be written

$$\begin{aligned} \overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} &= (F_{\mu\nu} - [B_\mu, B_\nu])^2 + \left(D_{[\mu}B_{\nu]}\right)^2 \\ \frac{1}{2}\left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right]^2 &= -2\left(D_\mu B_\mu\right)^2 \end{aligned} \tag{2.5}$$

where $F_{\mu\nu}$ and D_μ denote the usual field strength and covariant derivative depending on the real part of the connection A_μ. After integrating by parts the term linear in $F_{\mu\nu}$ cancels and the final bosonic action reads[2]

$$S_B = \int \mathrm{Tr}\,\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2\right) \tag{2.6}$$

Notice that the imaginary parts of the gauge field have transformed into the two scalars of the super Yang-Mills theory! This is further confirmed by looking at the fermionic part of the action which can be rewritten in 2×2 block form as

$$\begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.7}$$

which is easily recognized as the dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions in which a chiral representation is employed for the fermions. As usual the scalar fields B_μ arise from the gauge fields in the reduced directions.

2.2 Lattice theory

The transition to the lattice theory is straightforward; we employ the geometrical discretization scheme proposed in [19]. For completeness we summarize it here. In general continuum p-form fields are mapped to lattice fields defined on p-subsimplices of a general simplicial

[2] The bosonic action is real positive definite on account of the antihermitian basis that we have chosen

lattice. In the case of hypercubic lattices this assignment is equivalent to placing a p-form with indices $\mu_1 \ldots \mu_p$ on the link connecting $\mathbf{x}$ with $(\mathbf{x} + \boldsymbol{\mu}_1 + \ldots + \boldsymbol{\mu}_p)$ where $\boldsymbol{\mu}_i, i = 1 \ldots p$ corresponds to a unit vector in the lattice. Actually this is not quite the full story; each link has two possible orientations and we must also specify which orientation is to be used for a given field. A positively oriented field corresponds to one in which the link vector has positive components with respect to the coordinate basis.

Continuum derivatives on such a hypercubic lattice are represented by lattice difference operators acting on these link fields. Specifically, covariant derivatives appearing in curl-like operations and *acting on positively oriented fields* are replaced by a lattice gauge covariant forward difference operator whose action on lattice scalar and vector fields is given by

$$\begin{aligned} \mathcal{D}^{(+)}_\mu f(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x}) f(\mathbf{x}+\boldsymbol{\mu}) - f(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}) \\ \mathcal{D}^{(+)}_\mu f_\nu(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x}) f_\nu(\mathbf{x}+\boldsymbol{\mu}) - f_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\nu}) \end{aligned} \tag{2.8}$$

where $\mathbf{x}$ denotes a two dimensional lattice vector and $\boldsymbol{\mu} = (1,0)$, $\boldsymbol{\nu} = (0,1)$ unit vectors in the two coordinate directions. Here, we have replaced the continuum *complex* gauge fields $\mathcal{A}_\mu$ by non-unitary link fields $\mathcal{U}_\mu = e^{i\mathcal{A}_\mu}$. The backward difference operator $\overline{\mathcal{D}}^{-}_\mu$ replaces the continuum covariant derivative in divergence-like operations and its action on (positively oriented) lattice vector fields can be gotten by requiring that it be adjoint to $\mathcal{D}^+_\mu$. Specifically its action on lattice vectors is

$$\overline{\mathcal{D}}^{(-)}_\mu f_\mu(\mathbf{x}) = f_\mu(\mathbf{x})\overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(\mathbf{x}-\boldsymbol{\mu}) f_\mu(\mathbf{x}-\boldsymbol{\mu}) \tag{2.9}$$

The nilpotent scalar supersymmetry now acts on the lattice fields as

$$\begin{aligned} \mathcal{Q}\, \mathcal{U}_\mu &= \psi_\mu \\ \mathcal{Q}\, \psi_\mu &= 0 \\ \mathcal{Q}\, \overline{\mathcal{U}}_\mu &= 0 \\ \mathcal{Q}\, \chi_{\mu\nu} &= \mathcal{F}^{L\dagger}_{\mu\nu} \\ \mathcal{Q}\, \eta &= d \\ \mathcal{Q}\, d &= 0 \end{aligned} \tag{2.10}$$

Here we written the lattice field strength as

$$\mathcal{F}^L_{\mu\nu} = \mathcal{D}^{(+)}_\mu \mathcal{U}_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})\mathcal{U}_\nu(\mathbf{x}+\boldsymbol{\mu}) - \mathcal{U}_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\nu}) \tag{2.11}$$

which reduces to the continuum (complex) field strength in the naive continuum limit and is automatically antisymmetric in the indices (μ, ν).

Notice that this supersymmetry transformation implies that the fermion fields ψ_μ have the same orientation as their superpartners the gauge links $\mathcal{U}_\mu$ and run from $\mathbf{x}$ to $(\mathbf{x} + \boldsymbol{\mu})$. However, the field $\chi_{\mu\nu}$ must have the same orientation as $\mathcal{F}^{L\dagger}_{\mu\nu}$ and hence is to be assigned to the negatively oriented link running from $(x + \boldsymbol{\mu} + \boldsymbol{\nu})$ down to $\mathbf{x}$ i.e parallel to the vector

$(-1,-1)$. This link choice also follows naturally from the matrix representation of the Kähler-Dirac field Ψ

$$\Psi = \eta I + \psi_\mu \gamma_\mu + \chi_{12}\gamma_1\gamma_2 \tag{2.12}$$

which associates the field χ_{12} with the lattice vector $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 = \boldsymbol{\mu} + \boldsymbol{\nu}$. We will see that the negative orientation is crucial for allowing us to write down gauge invariant expressions for the fermion kinetic term. Finally, it should be clear that the scalar fields η and d can be taken to transform simply as site fields.

These link mappings and orientations are conveniently summarised by giving the gauge transformation properties of the lattice fields

$$\begin{aligned}
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x})\\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\boldsymbol{\mu})\\
\chi_{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\boldsymbol{\mu}+\boldsymbol{\nu})\chi_{\mu\nu}G^\dagger(\mathbf{x})\\
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x})\\
\overline{\mathcal{U}}_\mu(\mathbf{x}) &\to G(\mathbf{x}+\boldsymbol{\mu})\overline{\mathcal{U}}_\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned} \tag{2.13}$$

Notice that this choice of link and orientation for the twisted lattice fields maps exactly into their r-charge assignments in the orbifolding approach [1]. Furthermore, the above $\mathcal{Q}$-variations and field assignments are equivalent to the approach described in [24] *provided* that we set the fermionic shift parameter a in that formulation to zero and consider only the corresponding scalar superymmetry.

The lattice gauge fermion now takes the form

$$\Lambda = \sum_{\mathbf{x}} \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{D}^{(+)}_\mu \mathcal{U}_\nu + \eta\overline{\mathcal{D}}^{(-)}_\mu \mathcal{U}_\mu - \frac{1}{2}\eta d\right) \tag{2.14}$$

It is easy to see that in the naive continuum limit the lattice divergence $\overline{\mathcal{D}}^{(-)}_\mu \mathcal{U}_\mu$ equals $[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]$. Notice that with the previous choice of orientation for the various fermionic link fields this gauge fermion is automatically invariant under lattice gauge transformations. There is no need for the doubling of degrees of freedom necessary in previous approaches to geometric discretization [19, 33]. In those constructions the nature of the gauge fermion and the scalar supercharge led to the presence of explicit Yukawa interactions in the theory. These in turn required the lattice theory to contain fermion link fields of both orientations and hence led to a doubling of degrees of freedom with respect to the continuum theory. In the twist described in this paper the Yukawa interactions are embedded into the complexified covariant derivatives and successive components of the Kähler-Dirac field representing the fermions can be chosen with alternating orientations leading to a Kähler-Dirac action which is automatically gauge invariant without these extra degrees of freedom.

Acting with the $\mathcal{Q}$-transformation shown above and again integrating out the auxiliary field d we derive the gauge and $\mathcal{Q}$-invariant lattice action

$$S = \sum_{\mathbf{x}} \mathrm{Tr}\,\left(\mathcal{F}^{L\dagger}_{\mu\nu}\mathcal{F}^L_{\mu\nu} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_\mu \mathcal{U}_\mu\right)^2 - \chi_{\mu\nu}\mathcal{D}^{(+)}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}^{(-)}_\mu \psi_\mu\right) \tag{2.15}$$

But this is precisely the orbifold action arising in [1] with the modified deconstruction step described in [34] and [25]. The two approaches are thus entirely equivalent.

We can use this geometrical formulation to show very easily that the lattice theory exhibits no fermion doubling problems. The simplest way to do this is merely to notice that the lattice action at zero coupling $\mathcal{U} \to I$ conforms to the canonical form required for no doubling by the theorem of Rabin [35]. Explicitly, discretization of continuum geometrical actions will not encounter doubling problems if continuum derivatives acting in curl-like operations are replaced by forward differences in the lattice theory while continuum derivatives appearing in divergence-like operations are represented by backward differences on the lattice. More precisely the continuum exterior derivative d is mapped to a forward difference while its adjoint $d^\dagger$ is represented by a backward difference.

However we can also see this by simply examining the the form of the fermion operator arising in this construction.

$$\begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -\mathcal{D}_2^{(+)} & \mathcal{D}_1^{(+)} \\ \mathcal{D}_1^{(-)} & \mathcal{D}^{(-)} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.16}$$

Clearly the determinant of this operator in the free limit is nothing more than the usual determinant encountered for scalars in two dimensions and hence possesses no extraneous zeroes that survive the continuum limit.

3. Sixteen supercharge theory in four dimensions

3.1 Continuum twisted theory

In the $\mathcal{Q} = 4$ theory in two dimensions the physical degrees of freedom were encoded in a complex gauge field. The same idea applied to the $\mathcal{Q} = 16$ theory naturally leads us to consider a theory of complex gauge fields $\mathcal{A}_a, a = 1 \ldots 5$ in *five dimensions*. Paralleling the four supercharge theory we introduce an additional auxiliary bosonic scalar field d and a set of five dimensional antisymmetric tensor fields to represent the fermions $\Psi = (\eta, \psi_a, \chi_{ab})$. This latter field content corresponds to considering just one of the two Kähler-Dirac fields used to represent the 32 fields of the five dimensional theory. Again, a nilpotent symmetry relates these fields

$$\begin{aligned}
\mathcal{Q}\, \mathcal{A}_a &= \psi_a \\
\mathcal{Q}\, \psi_a &= 0 \\
\mathcal{Q}\, \overline{\mathcal{A}}_a &= 0 \\
\mathcal{Q}\, \chi_{ab} &= -\overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\, \eta &= d \\
\mathcal{Q}\, d &= 0
\end{aligned} \tag{3.1}$$

and we may write down the same $\mathcal{Q}$-exact action that was employed in two dimensions $S = \beta\mathcal{Q}\Lambda$ with

$$\Lambda = \int \mathrm{Tr}\ \left(\chi_{ab}\mathcal{F}_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d\right) \tag{3.2}$$

where we have again employed complexified covariant derivatives. Carrying out the $\mathcal{Q}$-variation and subsequently integrating out the auxiliary field as for the $\mathcal{Q} = 4$ theory leads to the action

$$S = \int \mathrm{Tr}\ \left(-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a\right) \tag{3.3}$$

Actually in this theory there is another fermionic term one can write down which is also invariant under this supersymmetry taking the form

$$S_{\mathrm{closed}} = -\frac{1}{2}\int \epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \tag{3.4}$$

The invariance of this term is just a result of the Bianchi identity $\epsilon_{abcde}\mathcal{D}_c\mathcal{F}_{de} = 0$. The final action we will employ is the sum of the $\mathcal{Q}$-exact piece and this $\mathcal{Q}$-closed term. The coefficient in front of this term is determined by the requirement that the theory reproduce the Marcus twist of $\mathcal{N} = 4$ Yang-Mills.

Clearly to make contact with a twist of $\mathcal{N} = 4$ in four dimensions we must dimensionally reduce this theory along the 5th direction. This will yield a complex scalar $\phi = A_5 + iB_5$ and its superpartner $\overline{\eta}$. The 10 five dimensional fermions χ_{ab} naturally decompose into a 2-form $\chi_{\mu\nu}$ and vector $\overline{\psi}_\mu$ in four dimensions.

$$\begin{aligned}
\mathcal{A}_a &\rightarrow \mathcal{A}_\mu \oplus \phi \\
\mathcal{F}_{ab} &\rightarrow \mathcal{F}_{\mu\nu} \oplus \mathcal{D}_\mu\phi \\
\left[\overline{\mathcal{D}}_a, \mathcal{D}_a\right] &\rightarrow \left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right] \oplus \left[\overline{\phi}, \phi\right] \\
\psi_a &\rightarrow \psi_\mu \oplus \overline{\eta} \\
\chi_{ab} &\rightarrow \chi_{\mu\nu} \oplus \overline{\psi}_\mu
\end{aligned} \tag{3.5}$$

where we will employ the convention that Greek indices run from one to four and are reserved for four dimensional tensors while Roman indices refer to the original five dimensional theory. The reduced action takes the form

$$\begin{aligned}
S = \int \mathrm{Tr}\ \Big(&-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right]^2 + \frac{1}{2}\left[\overline{\phi}, \phi\right]^2 + (\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}_\mu\phi) - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} \\
&- \overline{\psi}_\mu\mathcal{D}_\mu\overline{\eta} - \overline{\psi}_\mu\left[\phi, \psi_\mu\right] - \eta\overline{\mathcal{D}}_\mu\psi_\mu - \eta\left[\overline{\phi}, \overline{\eta}\right] - \chi^*_{\mu\nu}\overline{\mathcal{D}}_\mu\overline{\psi}_\nu - \chi^*_{\mu\nu}\left[\overline{\phi}, \chi_{\mu\nu}\right]\Big)
\end{aligned} \tag{3.6}$$

where the last two terms arise from dimensional reduction of the $\mathcal{Q}$-closed term and χ^* is the Hodge dual of χ, $\chi_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\chi_{\rho\lambda}$. Up to trivial rescalings this is the action (with gauge parameter $\alpha = 1$) of twisted $\mathcal{N} = 4$ Yang-Mills in four dimensions written down by Marcus

[36][3]. This twisted action is well known to be fully equivalent to the usual form of $\mathcal{N}=4$ in flat space. Here, we have shown how to derive this theory by dimensional reduction of a rather simple five dimensional theory employing a complex gauge field and integer spin twisted fermions. It will be the basis of our lattice formulation to which we now turn.

3.2 Lattice theory

The discretization scheme we use is precisely the same as for two supercharge theory. Complex five dimensional gauge fields are replaced by complex gauge links $\mathcal{U}_a, a = 1\ldots 5$. The $\mathcal{Q}$-supersymmetry is essentially the same as in the continuum and remains nilpotent

$$\begin{aligned}
\mathcal{Q}\,\mathcal{U}_a &= \psi_a \\
\mathcal{Q}\,\overline{\mathcal{U}}_a &= 0 \\
\mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= \left(\mathcal{F}^L_{ab}\right)^\dagger \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned} \tag{3.7}$$

where the lattice field strength $\mathcal{F}^L_{ab}$ is given by eqn. 2.11 as before. The chief difficulty remaining is to decide how the continuum tensor fields are to be assigned to lattice links after dimensional reduction to four dimensions. For the moment let us base our discretization scheme around a hypercubic lattice. Then the gauge links $\mathcal{U}_\mu \equiv \mathcal{U}_a, a = 1\ldots 4$ should live on elementary coordinate directions in the unit hypercube. This then implies that the superpartners of those gauge links ψ_μ should also live on those links and be oriented in the same fashion i.e running from $\mathbf{x}$ to $(\mathbf{x}+\boldsymbol{\mu})$. We will adopt the notation that these four basis vectors are labeled $\boldsymbol{\mu}_a, a = 1\ldots 4$.

However the assignment of ψ_5 is not immediately obvious – a naive assignment to a site field would result in *two* fermionic scalars which is not what is expected for a four dimensional Kähler-Dirac field. The same line of reasoning suggests in fact that we associate ψ_5 with the 4-form component of that field. An independent line of argument confirms this; the field ψ_5 is part of the vector component of a five dimensional Kähler-Dirac field and is thus associated with the five dimensional gamma matrix Γ^5. This is usually represented by the chiral matrix of the four dimensional theory $\Gamma^5 = \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and suggests a 4-form interpretation for the field in the four dimensional theory. As for two dimensions this motivates assigning the lattice field to the body diagonal of the unit hypercube. Actually we must be careful; this same assignment will also apply to the field $\mathcal{U}_5$. To construct the bosonic action we need to able to apply $\mathcal{D}^{(+)}_a$ to this link field and stay within the unit hypercube. Thus we choose the fields to be oriented in the *opposite direction* corresponding to the basis vector $\boldsymbol{\mu}_5 = (-1,-1,-1,-1)$. Notice that this assignment ensures that $\sum_{a=1}^5 \boldsymbol{\mu}_a = \mathbf{0}$ which will be seen to be crucial for constructing gauge invariant quantities.

[3]It is also the twist of $\mathcal{N}=4$ YM used in the Geometric Langlands program [37]

As for two dimensions we will summarize these link and orientation assignments by writing down a set of gauge transformations for the fields

$$\begin{aligned}
\eta(\mathbf{x}) &\rightarrow G(\mathbf{x})\eta(\mathbf{x})G^{\dagger}(\mathbf{x}) \\
\psi_a(\mathbf{x}) &\rightarrow G(\mathbf{x})\psi_a(\mathbf{x})G(\mathbf{x}+\boldsymbol{\mu}_a) \\
\chi_{ab}(\mathbf{x}) &\rightarrow G(\mathbf{x}+\boldsymbol{\mu}_a+\boldsymbol{\mu}_b)\chi_{ab}(\mathbf{x})G^{\dagger}(\mathbf{x}) \\
\mathcal{U}_a(\mathbf{x}) &\rightarrow G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^{\dagger}(\mathbf{x}+\boldsymbol{\mu}_a) \\
\overline{\mathcal{U}}_a(x) &\rightarrow G(\mathbf{x}+\boldsymbol{\mu}_a)\overline{\mathcal{U}}_\mu(\mathbf{x})G^{\dagger}(\mathbf{x}) \qquad (3.8)
\end{aligned}$$

$$(3.9)$$

In this form the reader will see that the gauge transformations of fields in this four dimensional theory follow almost exactly the same form as their cousins in two dimensions. Notice also that these link choices and orientations match exactly the r-charge assignments of the orbifold action for the sixteen supercharge theory in four dimensions [3].

However, one should note that with these conventions not all fields lie in the positively oriented unit hypercube. The problematic fields all possess a tensor index $a = 5$. However they can be mapped into the hypercube by a simple lattice translation. The transformation is

$$\begin{aligned}
\chi_{5\mu}(\mathbf{x}-\boldsymbol{\mu}_5-\boldsymbol{\mu}) &\rightarrow \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho}\theta_{\nu\lambda\rho}(\mathbf{x}) \\
\psi_5(\mathbf{x}-\boldsymbol{\mu}_5) &\rightarrow \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\kappa_{\mu\nu\lambda\rho}(\mathbf{x}) \qquad (3.10)
\end{aligned}$$

where we have relabeled the mapped fields so as to match their corresponding link assignment in the unit hypercube. Notice that $\chi_{5\mu}$ contains the field $\theta_{\nu\lambda\rho}$ which plays the role of the 3-form component of a four dimensional Kähler-Dirac field. The 2-form and 4-form components are then supplied by $\chi_{ab}, a, b = 1 \ldots 4$ and $\kappa_{\mu\nu\lambda\rho}$. Furthermore the $\theta_{\nu\lambda\rho}$ and $\kappa_{\mu\nu\lambda\rho}$ fields have positive and negative orientation. Thus, as for two dimensions, successive components of the resultant fermionic Kähler-Dirac field alternate in orientation which will be the key to writing down gauge invariant fermion kinetic terms. Clearly any expression which is summed over all lattice points will be invariant under such a translation and we will use this freedom later to recast the lattice action in a way which makes clear why the fermionic action does not suffer from doubling problems.

The advantage of the 5D variables is that they allow easy comparison with the analogous orbifold expressions and are compatible with our previous expressions for gauge covariant finite differences which can now be written in the general form

$$\mathcal{D}_c^{(+)} f_d(\mathbf{x}) = \mathcal{U}_c(\mathbf{x})f_d(\mathbf{x}+\boldsymbol{\mu}_c) - f_d(\mathbf{x})U_c(\mathbf{x}+\boldsymbol{\mu}_d) \qquad (3.11)$$

$$\mathcal{D}_c^{(-)} f_c(\mathbf{x}) = f_c(\mathbf{x})\overline{\mathcal{U}}_c(\mathbf{x}) - \overline{\mathcal{U}}_c(x-\boldsymbol{\mu}_c)f_c(\mathbf{x}-\boldsymbol{\mu}_c) \qquad (3.12)$$

Using these ingredients the lattice action arising from the $\mathcal{Q}$-exact piece of the continuum action takes the form

$$S = \sum_{\mathbf{x}} \mathrm{Tr}\, \left(\mathcal{F}_{ab}^{L\dagger} \mathcal{F}_{ab}^{L} + \frac{1}{2} \left(\overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a \right)^2 - \chi_{ab} \mathcal{D}_{[a}^{(+)} \psi_{b]} - \eta \overline{\mathcal{D}}_a^{(-)} \psi_a \right) \tag{3.13}$$

There is one remaining subtlety in this identification. Exactly how does the $\mathcal{Q}$-closed term remain supersymmetric under discretization ? A natural lattice analog of $\overline{\mathcal{D}}_c \chi_{ab}$ is given by

$$\overline{\mathcal{D}}_c^{(-)} \chi_{ab}(\mathbf{x}) = \chi_{ab}(\mathbf{x}) \overline{\mathcal{U}}_c(\mathbf{x} - \boldsymbol{\mu}_c) - \overline{\mathcal{U}}_c(\mathbf{x} + \boldsymbol{\mu}_a + \boldsymbol{\mu}_b - \boldsymbol{\mu}_c) \chi_{ab}(\mathbf{x} - \boldsymbol{\mu}_c) \tag{3.14}$$

Using this it is straightforward to write down a gauge invariant lattice analog of the continuum $\mathcal{Q}$-closed term

$$S_{\mathrm{closed}} = -\frac{1}{2} \sum_{\mathbf{x}} \mathrm{Tr}\, \epsilon_{abcde} \chi_{de}(\mathbf{x} + \boldsymbol{\mu}_a + \boldsymbol{\mu}_b + \boldsymbol{\mu}_c) \mathcal{D}_c^{(-)} \chi(\mathbf{x} + \boldsymbol{\mu}_c) \tag{3.15}$$

Notice that the ϵ-tensor forces all indices to be distinct and the gauge invariance of this result follows from the fact that $\sum_{i=1}^{5} \boldsymbol{\mu}_i = \mathbf{0}$. It is easy to see that it is equal to third fermionic term of the orbifold action appearing in eqn. (3.18) of reference [3].

In the continuum the invariance of this term under $\mathcal{Q}$-transformations requires use of the Bianchi identity. Remarkably, the lattice difference operator satisfies a similar identity (see [38] for the four dimensional result)

$$\epsilon_{abcde} D_c^{(+)} \mathcal{F}_{de}^{L} = 0 \tag{3.16}$$

Thus the discretization of the $\mathcal{Q}$-closed term in eqn. 3.15 is indeed invariant under the lattice $\mathcal{Q}$-transformation given in eqn. 3.7. This completes the proof of the equivalence. The connection between the naive continuum limit of the orbifold lattice and the Marcus twist of $\mathcal{N} = 4$ super Yang-Mills was shown earlier by Unsal [28]; in this paper we make this connection explicit by discretizing the latter theory in a way which maintains the scalar supersymmetry and obtain the orbifold action directly.

Finally to obtain the hypercubic lattice discretization of the continuum Marcus theory requires setting $\mathcal{U}_5 = \phi$ a complex field with vanishing expectation value. Notice though that this discretization contains elementary links of varying length. Actually the lattice action we have derived is clearly supersymmetric for arbitrary deformations of the lengths and orientations of the five basic vectors $\boldsymbol{\mu}_a, a = 1 \ldots 5$. Thus it is possible to consider the symmetric situation in which the lattice *in spacetime* is constructed from a unit cell in which these basis vectors are equivalent – they point out from the center of a four-dimensional hypertetrahedron to its five vertices. These vectors $e_i, i = 1 \ldots 5$ are given explicitly in [3]. At the same time we must set $\mathcal{U}_5$ to the exponential of a complex matrix to maintain symmetry with the other link fields $\mathcal{U}_\mu, \mu = 1 \ldots 4$. This construction necessitates introducing a map between the abstract lattice used to build the supersymmetric theory and spanned by the integer component vectors $\mathbf{x} = (n_1, n_2, n_3, n_4)$ and the physical spacetime coordinates $\mathbf{R}$. Explicitly $\mathbf{R} = \sum_{i=1}^{5} n_i \boldsymbol{e}_i$.

Such a lattice has the point group symmetry S^5 which is much larger than the S^4 symmetry of the hypercubic lattice - a factor which may prove to be important when examining the restoration of rotational invariance and the other supersymmetries of the continuum $\mathcal{N} = 4$ theory.

3.3 Absence of fermion doubling

Finally this geometric approach makes it easier to understand why this lattice theory does not suffer from doubling problems. We will analyze this question in the context of the hypercubic lattice discretization. Clearly most of the fermionic kinetic terms manifestly satisfy the double free discretization prescription given by Rabin [35]. The only difficult terms arise when one or more tensor indices of the fields equal $a = 5$. Expressions involving these fields are not located wholly in the positively oriented unit hypercube and must be translated into the hypercube before they can examined from the perspective of this prescription. As an example, consider the term

$$\sum_{\mathbf{x}} \mathrm{Tr}\ \chi_{5\mu} D^{(+)}_{\mu} \psi_5 = \sum_{\mathbf{x}} \mathrm{Tr}\ \chi_{5\mu}(\mathbf{x}) \left(\mathcal{U}_\mu(\mathbf{x})\psi_5(\mathbf{x}+\boldsymbol{\mu}) - \psi_5(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\mu}_5)\right) \tag{3.17}$$

We first shift the coordinates $\mathbf{x} \to (\mathbf{x} - \boldsymbol{\mu}_5 - \boldsymbol{\mu})$ and then use the previous change of variables given in eqn. 3.10 to rewrite this as

$$\frac{1}{3!} \sum_{\mathbf{x}} \mathrm{Tr}\ \theta_{\nu\lambda\rho}(\mathbf{x}) \left(\mathcal{U}_\mu(\mathbf{x} - \boldsymbol{\mu}_5 - \boldsymbol{\mu})\kappa_{\mu\nu\lambda\rho}(\mathbf{x}) - \kappa_{\mu\nu\lambda\rho}(\mathbf{x}-\boldsymbol{\mu})\mathcal{U}_\mu(\mathbf{x}-\boldsymbol{\mu})\right) \tag{3.18}$$

In the limit of zero coupling $\mathcal{U} = I$ this takes the form

$$\frac{1}{3!} \sum_{\mathbf{x}} \mathrm{Tr}\ \theta_{\nu\lambda\rho}(\mathbf{x}) D^{(-)}_{\mu} \kappa_{\mu\nu\rho\lambda}(\mathbf{x}) \tag{3.19}$$

which now has the correct canonical form to exclude doubles according to the theorem of Rabin [35]. Notice that the original forward difference has become a backward difference operator after the change of variables.

The only other term requiring this more careful analysis arises from the $\mathcal{Q}$-closed term. The problematic term looks like

$$\epsilon_{abcd5}\chi_{5d}(\mathbf{x}+\boldsymbol{\mu}_a+\boldsymbol{\mu}_b+\boldsymbol{\mu}_c)\left(\chi_{12}(\mathbf{x}+\boldsymbol{\mu}_c)\overline{\mathcal{U}}_c(\mathbf{x}) - \overline{\mathcal{U}}_c(\mathbf{x}+\boldsymbol{\mu}_a+\boldsymbol{\mu}_b)\chi_{ab}(\mathbf{x})\right) \tag{3.20}$$

Using the result $\sum_{a=1}^{5} \boldsymbol{\mu}_a = \mathbf{0}$ this can be written for zero coupling ($\mathcal{U} \to I$) as

$$\frac{1}{3!}\theta_{abc}(\mathbf{x}) D^{(+)}_{[c}\chi_{ab]}(\mathbf{x}) \tag{3.21}$$

where the presence of the epsilon symbol ensures the complete antisymmetrization of the derivative. This final form has the form required by Rabin's theorem. The theory is thus manifestly free of doubles.

4. Conclusions

In this paper we have shown how to derive the supersymmetric orbifold lattices of Cohen et al. [1] and Kaplan et al. [3] by geometrical discretization of the continuum twisted supersymmetric Yang-Mills theory. This connection is not unexpected – Unsal showed earlier [28] that the naive continuum limit of the $\mathcal{Q} = 16$ orbifold theory in four dimensions corresponded to the Marcus twist of $\mathcal{N} = 4$ and more recent work by Damgaard et al. [25] and Takimi [27] have exhibited the strong connections between discretizations of the twisted theory and orbifold theories. Our new work makes the connection complete – the two approaches are in fact identical provided one chooses the exact lattice supersymmetry carefully and uses the geometric discretization proposed in [19]. In fact, as was pointed out in [26] this lattice theory is essentially equivalent to the one proposed in [24] *provided* that the fermionic shift parameter employed in that model is chosen to be zero and we restrict our attention solely to the corresponding scalar supercharge.

The case of $\mathcal{Q} = 16$ is particularly interesting. We have shown that the continuum theory can be recast as the dimensional reduction of a very simple five dimensional theory. The $\mathcal{Q}$-exact part of the action is essentially identical to the two dimensional theory with the primary difference between the two theories arising because of the appearance of a new $\mathcal{Q}$-closed term which was not possible in two dimensions. Nevertheless discretization proceeds along the same lines, the one subtlety being the lattice link assignment of the fifth component of the complex gauge field after dimensional reduction. The key requirement governing discretization is that successive components of the Kähler-Dirac field representing the fermions have opposite orientations. This allows the fermionic action to be gauge invariant without any additional doubling of degrees of freedom. It seems likely that all the orbifold actions in various dimensions can be obtained in this manner.

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