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Simon Catterall  
*Syracuse University*

Anosh Joseph  
*Syracuse University*

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# Lattice actions for Yang-Mills quantum mechanics with exact supersymmetry

Simon Catterall<sup>a\*</sup> and Anosh Joseph<sup>a†</sup>

<sup>a</sup>*Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA*

## Abstract

We derive lattice actions for Yang-Mills quantum mechanics for models with  $\mathcal{Q} = 4, 8$  and 16 supercharges which possess an exact supersymmetry at non-zero lattice spacing. These are obtained by dimensional reduction of twisted versions of the corresponding super Yang-Mills theories in  $D = 2, 3$  and 4 dimensions.

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\*smc@phy.syr.edu

†ajoseph@phy.syr.edu

## I. INTRODUCTION

Supersymmetric Yang-Mills theories are interesting both as playgrounds for understanding quantum field theory and as gauge theories which are conjectured to be dual to certain string theories [1, 2]. Typically these dualities between string and gauge theory require that the gauge theory be taken at strong coupling. This requirement motivates defining the theory on a lattice which would allow for strong coupling expansions and Monte Carlo simulation. Perhaps the simplest of these gauge-gravity dualities is exhibited by the conjectured equivalence of the type IIA string theory containing  $N$   $D0$ -branes and super Yang-Mills quantum mechanics with gauge group  $SU(N)$ . More specifically the large  $N$  limit of the gauge model at low temperature  $T$  or strong 't Hooft coupling is thought to provide a description of the black hole that arises in the low energy supergravity limit of the string theory. Initial investigations of this and related models have been reported in the literature [3, 4, 5]. The numerical work so far has employed actions that do not possess an exact supersymmetry [6]. The purpose of this work is to derive lattice actions which retain an exact supersymmetry at non-zero lattice spacing which could be used in similar Monte Carlo studies.

Unfortunately, conventional discretizations of supersymmetric theories break supersymmetry completely and the resultant lattice theories typically require a great deal of fine tuning in order that supersymmetry is recovered in the continuum limit [7, 8, 9]. However, it has been shown that in theories with a multiple of  $2^D$  supercharges, where  $D$  is the total number of spacetime dimensions, one or more supersymmetries can be retained provided the discretization scheme is chosen carefully. Two approaches have been successfully pursued based on either orbifolding a supersymmetric matrix model [10, 11, 12] or direct discretization of a reformulation of the continuum supersymmetric theory in terms of so-called *twisted* variables [13, 14, 15, 16, 17]. Recently, these two approaches have been shown to be equivalent [18, 19, 20, 21, 22].

The idea of twisting goes back to the seminal paper of Witten in which the twisted formulation was used to construct a topological field theory [23]. The process naturally exposes a nilpotent scalar supersymmetry with the topological sector of the twisted theory corresponding to operators invariant under the action of this scalar supersymmetry. In the context of creating supersymmetric lattice theories this projection to the  $Q$ -invariant subspace is dropped and the twisting (in flat

space) is simply regarded as a convenient change of variables - one more suitable for discretization. Indeed, the fermionic content is then encoded by a series of antisymmetric tensor fields which can be embedded as components of one or more Kähler-Dirac fields [24]. As was shown by Rabin [25], theories involving Kähler-Dirac fields may be discretized without inducing fermion doubling problems and indeed at the level of free field theory the resultant lattice theories are equivalent to staggered fermions [26].

In addition to a geometrical treatment of the fermions the twisted formulation has the merit of allowing the action to be written in  $\mathcal{Q}$ -exact form. Thus the problem of translating the  $\mathcal{Q}$ -invariance of the continuum theory to the lattice is replaced by the simpler requirement of keeping the scalar supercharge nilpotent when acting on the lattice fields. Typically this is a much simpler proposition and is the one adopted by all discretizations of the twisted theories considered so far [14, 15, 16, 17, 22, 27, 28].

In this paper, we start from the twisted forms of the gauge theories in two, three and four dimensions which possess  $\mathcal{Q} = 4, 8$  and 16 supersymmetries and dimensionally reduce them to one (Euclidean) dimension. The resultant continuum theories can be written in terms of multiples of the basic Kähler-Dirac field which in one dimension which contains one scalar and one vector field. We show also that each new scalar fermion is associated with an additional scalar supersymmetry which is inherited from the dimensional reduction. Discretization then proceeds using a prescription due to Sugino [27].

## II. THE FOUR SUPERCHARGE MODEL

Consider the continuum twisted form of the two dimensional  $\mathcal{N} = 2$  (Euclidean) super Yang-Mills model given in eg. [15]. The bosonic part of this theory contains two scalar fields  $\phi, \bar{\phi}$ , a vector  $A_\mu$  and a tensor field  $B_{\mu\nu}$ . The fermionic part consists of an anticommuting scalar field  $\eta$ , a vector  $\psi_\mu$  and a field  $\chi_{\mu\nu}$ . If  $\mathcal{Q}$  is a scalar supercharge obtained by *twisting* the original Majorana supercharges of the theory, the  $\mathcal{Q}$ -variation of the gauge fermion

$$\Lambda = \text{Tr} \int d^2x \left( \frac{1}{4} \eta[\phi, \bar{\phi}] + \chi_{\mu\nu} F_{\mu\nu} + \frac{1}{2} \chi_{\mu\nu} B_{\mu\nu} + \psi_\mu D_\mu \bar{\phi} \right) \quad (1)$$

will give us a twisted action

$$S = \beta \mathcal{Q} \Lambda, \quad (2)$$

where  $\beta$  is a coupling constant. Dimensional reduction of the action to one dimension will then yield a supersymmetric Yang-Mills quantum mechanics theory. Dimensionally reducing eqn. (1) with respect to the  $x^2$ -direction we find

$$\Lambda = \text{Tr} \int dx \left( \frac{1}{4} \eta[\phi, \bar{\phi}] + 2\chi_{12} D_1 A_2 + \chi_{12} B_{12} + \psi_1 D_1 \bar{\phi} + \psi_2 [A_2, \bar{\phi}] \right). \quad (3)$$

The scalar supercharge  $\mathcal{Q}$  acts on the component fields as follows

$$\begin{aligned} \mathcal{Q} A_1 &= \psi_1, \\ \mathcal{Q} A_2 &= \psi_2, \\ \mathcal{Q} \psi_1 &= -D_1 \phi, \\ \mathcal{Q} \psi_2 &= -[A_2, \phi], \\ \mathcal{Q} \phi &= 0, \\ \mathcal{Q} \chi_{12} &= B_{12}, \\ \mathcal{Q} B_{12} &= [\phi, \chi_{12}], \\ \mathcal{Q} \bar{\phi} &= \eta, \\ \mathcal{Q} \eta &= [\phi, \bar{\phi}]. \end{aligned} \quad (4)$$

Carrying out the  $\mathcal{Q}$ -variation on eqn. (3) and integrating over the multiplier field  $B_{12}$  we arrive at the action

$$\begin{aligned} S = \beta \text{Tr} \int dx & \left( - \sum_{i=1}^3 (D_1 \phi^i)^2 - \sum_{i < j, i, j=1}^3 [\phi^i, \phi^j]^2 - 2\psi_1^1 D_1 \eta^1 - 2\psi_1^2 D_1 \eta^2 + \psi_1^1 [\bar{\phi}, \psi_1^1] \right. \\ & \left. - \psi_1^2 [\phi, \psi_1^2] - \eta^1 [\phi, \eta^1] + \eta^2 [\bar{\phi}, \eta^2] + 2\psi_1^1 [\phi^3, \psi_1^2] - 2\eta^1 [\phi^3, \eta^2] \right). \end{aligned} \quad (5)$$

Notice that we have relabelled the fields:

OLD  $\rightarrow$  NEW

$$\psi_1 \rightarrow \psi_1^1$$

$$\chi_{12} \rightarrow \psi_1^2$$

$$\eta/2 \rightarrow \eta^1$$

$$\psi_2 \rightarrow \eta^2$$

$$A_2 \rightarrow \phi^3$$

It will also prove convenient to decompose the scalar fields  $\phi = -\phi^1 + i\phi^2$  and  $\bar{\phi} = -\phi^1 - i\phi^2$ .

In terms of the relabelled fields we write the part of the action comprising the fermionic and Yukawa terms in the form

$$S_F + S_Y = \int dx \operatorname{Tr} \Psi^\dagger \Gamma^4 D_1 \Psi + \sum_{i=1}^3 \Psi^\dagger [\Gamma^i \phi^i, \Psi], \quad (6)$$

where

$$\Gamma^1 = \mathbf{1} \otimes \sigma_3, \quad \Gamma^2 = -i \mathbf{1} \otimes \mathbf{1}, \quad \Gamma^3 = \sigma_3 \otimes \sigma_1, \quad \Gamma^4 = \sigma_1 \otimes \mathbf{1}, \quad (7)$$

and the spinor

$$\Psi^\dagger = (\psi_1^1 \ \psi_1^2 \ \eta^1 \ \eta^2). \quad (8)$$

This form of the twisted theory can be related to the usual action for  $\mathcal{N} = 1$  super Yang-Mills theory reduced to one dimension by recognizing that the usual 4d Majorana matrices given by

$$\gamma^1 = i\sigma_1 \otimes \sigma_3, \quad \gamma^2 = i \mathbf{1} \otimes \mathbf{1}, \quad \gamma^3 = -i\sigma_1 \otimes \sigma_1, \quad \gamma^4 = -i\gamma^0 = i\sigma_1 \otimes \sigma_2 \quad (9)$$

may be transformed to the above  $\Gamma$  representation using the similarity transformation [30]

$$\Gamma^1 = iS\gamma^4\gamma^3S^{-1}, \quad \Gamma^2 = iS\gamma^4\gamma^4S^{-1}, \quad \Gamma^3 = iS\gamma^4\gamma^1S^{-1}, \quad \Gamma^4 = iS\gamma^4\gamma^2S^{-1}, \quad (10)$$

where

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \quad (11)$$

We see that the twisted fermions fill out the usual 4d Majorana spinor as expected.

Notice that the twisted theory is actually equivalent to a dimensional reduction of the usual  $\mathcal{N} = 1$  theory in four dimensional Minkowski space along one space and the *time* direction followed by a Wick rotation of the original temporal direction. The latter corresponds to  $A_t^{\text{Min}} \rightarrow \phi_2^{\text{Min}} \rightarrow i\phi_2^{\text{Eucl}}$ . The final Euclidean time direction is then associated with the Dirac matrix  $\Gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ . In this form it is easily discretized as we will see later while maintaining the antisymmetry of the discrete Dirac operator.

The twist decomposition of  $\mathcal{N} = 2$  theory gives rise to four supercharges - two scalars and two vectors. We have used only one scalar supercharge  $\mathcal{Q}$  in deriving the above continuum action from the gauge fermion  $\Lambda$ . Clearly the new scalar fermion that appears in the dimensionally reduced theory is related to the existence of this second scalar supersymmetry. Its action on the fields can be uncovered by noticing that the continuum action given in eqn. (5) is invariant under the field transformations

$$\begin{aligned}
\psi_1^1 &\rightarrow \psi_1^2 \\
\eta^1 &\rightarrow \eta^2 \\
\phi^3 &\rightarrow \phi^3 \\
\phi &\rightarrow -\bar{\phi}
\end{aligned}
\tag{12}$$

This set of field transformations is a symmetry of the continuum action. We can combine this with the action of the scalar supercharge  $\mathcal{Q}$  to derive an additional supersymmetry, say  $\mathcal{Q}'$  of the theory. It corresponds to the vector supercharge  $Q_2$  of the two dimensional parent theory before dimensional reduction. In the continuum the theory can then also be written in a  $\mathcal{Q}'$ -exact form. However, we will see that in our lattice construction this second supersymmetry is broken by terms of order the lattice spacing. However, using the arguments given in [4] it should be recovered without fine tuning in the continuum limit. This supersymmetry will transform component fields

of the continuum theory in the following way

$$\begin{aligned}
\mathcal{Q}' A_1 &= \psi_1^2, \\
\mathcal{Q}' \phi^3 &= \eta^1, \\
\mathcal{Q}' \psi_1^1 &= B_{12}, \\
\mathcal{Q}' \psi_1^2 &= D_1 \bar{\phi}, \\
\mathcal{Q}' \eta^1 &= -[\bar{\phi}, \phi^3], \\
\mathcal{Q}' \eta^2 &= -\frac{1}{2}[\phi, \bar{\phi}], \\
\mathcal{Q}' \bar{\phi} &= 0, \\
\mathcal{Q}' \phi &= -2\eta^2, \\
\mathcal{Q}' B_{12} &= -[\bar{\phi}, \psi_1^1]
\end{aligned} \tag{13}$$

This method for deriving additional twisted supersymmetries is described in some detail in [29].

Turning now to the lattice, it is straightforward to discretize this theory in a way which preserves the nilpotency of  $\mathcal{Q}$ . The prescription was given by Sugino [14] in the context of the two dimensional model. Here, we trivially extend it to discretization of a one dimensional model. We place all the fields on sites of a regular one dimensional lattice except the gauge field  $A_1(x)$ , which is replaced by a unitary variable  $U_1(x)$  living on the link  $(x, x + \hat{1})$ . We write the lattice gauge fermion  $\Lambda$  in terms of lattice variables

$$\begin{aligned}
\Lambda_L = \text{Tr} \sum_x \left( \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] + 2\chi_{12}(x) D_1^+ A_2(x) + \chi_{12}(x) B_{12}(x) + \psi_1(x) D_1^+ \bar{\phi} \right. \\
\left. + \psi_2(x) [A_2(x), \bar{\phi}(x)] \right),
\end{aligned} \tag{14}$$

where the forward difference operator is defined as

$$D_\mu^+ f(x) = U_\mu(x) f(x + \mu) U_\mu^\dagger(x) - f(x). \tag{15}$$



The scalar supersymmetry transformation rules on the lattice take the form [14]

$$\begin{aligned}
\mathcal{Q} U_1(x) &= \psi_1(x)U_1(x), \\
\mathcal{Q} A_2(x) &= \psi_2(x), \\
\mathcal{Q} \psi_1(x) &= \psi_1(x)\psi_1(x) - D_1^+ \phi(x), \\
\mathcal{Q} \psi_2(x) &= -[A_2(x), \phi(x)], \\
\mathcal{Q} \phi(x) &= 0, \\
\mathcal{Q} \chi_{12}(x) &= B_{12}(x), \\
\mathcal{Q} B_{12}(x) &= [\phi(x), \chi_{12}(x)], \\
\mathcal{Q} \bar{\phi}(x) &= \eta(x), \\
\mathcal{Q} \eta(x) &= [\phi(x), \bar{\phi}(x)].
\end{aligned} \tag{16}$$

These transformations reduce to their continuum counterparts in the limit of vanishing lattice spacing where, for example, the term quadratic in  $\psi_\mu$  is suppressed by an additional power of the lattice spacing. Notice that  $\mathcal{Q}^2$  is still nilpotent up to lattice gauge transformations. In dimensions two or greater this prescription is problematic in that it leads to a gauge action which possesses many vacua all but one of which are absent in the continuum theory. However, in one dimension this term is missing and the lattice prescription is well defined.

Applying the lattice  $\mathcal{Q}$ -variation to eqn. (14) and integrating out the multiplier fields we get the lattice action

$$S = S_B + S_F + S_Y + S_R, \tag{17}$$

where the bosonic part of the action

$$S_B = \beta \text{Tr} \sum_x \left( - \sum_{i=1}^3 (D_1^+ \phi^i(x))^2 - \sum_{i < j, i, j=1}^3 [\phi^i(x), \phi^j(x)]^2 \right), \tag{18}$$

the fermionic kinetic term is

$$S_F = \beta \text{Tr} \sum_x -2 \left( \psi_1^1 D_1^+ \eta^1(x) + \psi_1^2(x) D_1^+ \eta^2(x) \right), \tag{19}$$

and the Yukawa part

$$\begin{aligned}
S_Y = \beta \text{Tr} \sum_x \left( \psi_1^1(x) [\bar{\phi}(x), \psi_1^1(x)] + 2\psi_1^1(x) [\bar{\phi}^3(x), \psi_1^2(x)] - \psi_1^2(x) [\phi(x), \psi_1^2(x)] \right. \\
\left. - \eta^1(x) [\phi(x), \eta^1(x)] - 2\eta^1(x) [\phi^3(x), \eta^2(x)] + \eta^2(x) [\bar{\phi}(x), \eta^2(x)] \right). \tag{20}
\end{aligned}$$

In addition, the lattice action picks up a residual part

$$S_R = -\beta \text{Tr} \sum_x \psi_1^1(x) \psi_1^1(x) D_1^+ \phi^1(x) + i \psi_1^1(x) \psi_1^1(x) D_1^+ \phi^2(x). \quad (21)$$

Here also we have rescaled and relabelled the fields as in the continuum case. Notice that the fields  $\phi$  and  $\bar{\phi}$  appear in an unusual way in the Yukawa part of the action:

$$\tilde{\phi}(x) = U_1(x) \phi(x + \hat{1}) U_1(x)^\dagger$$

and

$$\tilde{\bar{\phi}}(x) = U_1(x) \bar{\phi}(x + \hat{1}) U_1^\dagger(x).$$

This smearing of certain scalar Yukawa terms is due to the non-trivial  $\mathcal{Q}$ -transformations we defined on  $U_1(x)$  and  $\psi_1(x)$ . The residual part of the lattice action is also a consequence of these non-trivial transformations. Furthermore, the presence of these point smeared Yukawas and this residual piece in the lattice action break the  $\mathcal{Q}'$ -symmetry introduced earlier.

The fermion kinetic term can be expressed in the form

$$\mathbf{D}_1 = \begin{pmatrix} 0 & (\mathbb{I}_{2 \times 2}) D_1^+ \\ (\mathbb{I}_{2 \times 2}) D_1^- & 0 \end{pmatrix} \quad (22)$$

which has the same  $\Gamma^4$  structure as appeared in the continuum case and is explicitly antisymmetric as required. However, the Yukawa interactions cannot be put in such a simple form as a result of the appearance of terms depending on the smeared scalar fields as described above. Instead the Yukawa part takes the form

$$\Psi^\dagger [\tilde{\mathbf{O}}, \Psi], \quad (23)$$

where the matrix

$$\tilde{\mathbf{O}} = \begin{pmatrix} \tilde{\phi} & \tilde{\phi}^3 & 0 & 0 \\ \tilde{\phi}^3 & -\phi & 0 & 0 \\ 0 & 0 & \bar{\phi} & -\phi^3 \\ 0 & 0 & -\phi^3 & -\phi \end{pmatrix} \quad (24)$$

### III. THE EIGHT SUPERCHARGE MODEL

Here we consider the twisted version of  $\mathcal{N} = 4$  super Yang-Mills theory in three (Euclidean) dimensions. The theory contains eight supercharges. The bosonic part consists of a gauge potential  $A_\mu$ , where  $\mu = 1, 2, 3$ , two scalars  $\phi$  and  $\bar{\phi}$ , and two fields  $B_{\mu\nu}$  and  $W_{\mu\nu\lambda}$ . The fermionic part of the theory consists of one anti-commuting scalar  $\eta$ , a vector  $\psi_\mu$ , a tensor  $\chi_{\mu\nu}$  and the 3-form field  $\theta_{\mu\nu\lambda}$ .

The twisted action can again be written in a  $Q$ -exact form

$$S = \beta Q\Lambda, \quad (25)$$

where the gauge fermion  $\Lambda$  takes the form

$$\Lambda = \int d^3x \text{Tr} \left( \chi_{\mu\nu} (F_{\mu\nu} + \frac{1}{2} B_{\mu\nu} + D_\lambda W_{\lambda\mu\nu}) + \psi_\mu D_\mu \bar{\phi} + \frac{1}{4} \eta [\phi, \bar{\phi}] + \frac{1}{3!} \theta_{\mu\nu\lambda} [W_{\mu\nu\lambda}, \bar{\phi}] \right). \quad (26)$$

Dimensional reduction of this action to one dimension will give a supersymmetric quantum mechanics with eight supercharges. The gauge fermion  $\Lambda$ , after dimensionally reducing along  $x^2$ - and  $x^3$ -directions

$$\begin{aligned} \Lambda = \int d^3x \text{Tr} & \left( \chi_{12} D_1 A_2 + \chi_{13} D_1 A_3 + \chi_{23} [A_2, A_3] + \chi_{12} B_{12} + \chi_{13} B_{13} + \chi_{23} B_{23} \right. \\ & + \chi_{12} [A_3, W_{312}] + \chi_{13} [A_2, W_{213}] + \chi_{23} D_1 W_{123} + \psi_1 D_1 \bar{\phi} + \psi_2 [A_2, \bar{\phi}] \\ & \left. + \psi_3 [A_3, \bar{\phi}] + \frac{1}{4} \eta [\phi, \bar{\phi}] + \theta_{123} [W_{123}, \bar{\phi}] \right). \quad (27) \end{aligned}$$

Again it is straightforward to we write down the scalar supercharge transformation rules for the

component fields

$$\begin{aligned}
QA_\mu &= \psi_\mu, \\
Q\psi_1 &= -D_1\phi, \\
Q\psi_i &= -[A_i, \phi], \quad i \neq 1, \\
Q\phi &= 0, \\
Q\bar{\phi} &= \eta, \\
Q\eta &= [\phi, \bar{\phi}], \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}], \\
Q\chi_{\mu\nu} &= B_{\mu\nu}, \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda}, \\
Q\theta_{\mu\nu\lambda} &= [\phi, W_{\mu\nu\lambda}].
\end{aligned} \tag{28}$$

After integrating out the multiplier field  $B_{\mu\nu}$ , and using the Bianchi identity, the bosonic part of the action can be written in the following form

$$\begin{aligned}
S_B &= \beta \int dx \operatorname{Tr} \left( - (D_1 A_2)^2 - (D_1 A_3)^2 - [A_2, A_3]^2 - (D_1 W_{123})^2 - [A_2, W_{231}]^2 - [A_3, W_{312}]^2 \right. \\
&\quad - (D_1 \phi^1)^2 - (D_1 \phi^2)^2 - [A_2, \phi^1]^2 - [A_2, \phi^2]^2 - [A_3, \phi^1]^2 - [A_3, \phi^2]^2 \\
&\quad \left. - [\phi^1, \phi^2]^2 - [\phi^1, W_{123}]^2 - [\phi^2, W_{123}]^2 \right)
\end{aligned} \tag{29}$$

where we have decomposed the fields  $\phi = \phi^1 + i\phi^2$  and  $\bar{\phi} = \phi^1 - i\phi^2$ .

Relabelling the fields

$$\begin{aligned}
\phi^3 &= A_2 \\
\phi^4 &= A_3 \\
V^1 &= W_{123}
\end{aligned} \tag{30}$$

the bosonic part of the action becomes

$$S_B = \beta \int dx \operatorname{Tr} \left[ - \sum_{i=1}^4 (D_1 \phi^i)^2 - (D_1 V^1)^2 - \sum_{i < j, i, j=1}^4 [\phi^i, \phi^j]^2 - \sum_{i=1}^4 [\phi^i, V^1]^2 \right]. \tag{31}$$

Dimensional reduction of the fermionic kinetic part of the action will give pure kinetic part  $S_F$  corresponding to the  $x^1$ -direction together with Yukawa couplings  $S_{FY}$  from the  $x^2$ - and  $x^3$ -directions

$$S_F = -2\beta \int dx \operatorname{Tr} \left[ \chi_{12} D_1 \psi_2 + \chi_{13} D_1 \psi_3 + \chi_{23} D_1 \theta_{123} + \frac{\eta}{2} D_1 \psi_1 \right], \quad (32)$$

$$S_{FY} = 2\beta \int dx \operatorname{Tr} \left[ \chi_{12} [A_2, \psi_1] - \chi_{23} [A_2, \psi_3] + \chi_{13} [A_2, \theta_{123}] - \frac{\eta}{2} [A_2, \psi_2] + \chi_{13} [A_3, \psi_1] \right. \\ \left. + \chi_{23} [A_3, \psi_2] - \chi_{12} [A_3, \theta_{123}] - \frac{\eta}{2} [A_3, \psi_3] \right]. \quad (33)$$

The Yukawa part of the action being

$$S_Y = \beta \int dx \operatorname{Tr} \left( -\frac{\eta}{2} [\phi, \frac{\eta}{2}] - \chi_{12} [\phi, \chi_{12}] - \chi_{13} [\phi, \chi_{13}] - \chi_{23} [\phi, \chi_{23}] + \psi_1 [\bar{\phi}, \psi_1] \right. \\ \left. + \psi_2 [\bar{\phi}, \psi_2] + \psi_3 [\bar{\phi}, \psi_3] + \theta_{123} [\bar{\phi}, \theta_{123}] + 2\frac{\eta}{2} [\theta_{123}, W_{123}] + 2\chi_{12} [W_{123}, \psi_3] \right. \\ \left. - 2\chi_{13} [W_{123}, \psi_2] + 2\chi_{23} [W_{123}, \psi_1] \right). \quad (34)$$

Relabelling the fermionic fields

$$OLD \longrightarrow NEW$$

$$-\psi_1 \longrightarrow \psi_1^1$$

$$\chi_{12} \longrightarrow \psi_1^2$$

$$\chi_{13} \longrightarrow \psi_1^3$$

$$\theta_{123} \longrightarrow \psi_1^4$$

$$-\frac{\eta}{2} \longrightarrow \eta^1$$

$$\psi_2 \longrightarrow \eta^2$$

$$\psi_3 \longrightarrow \eta^3$$

$$\chi_{23} \longrightarrow \eta^4$$

the action for the supersymmetric Yang-Mills quantum mechanics can be written in a more compact form

$$S = \beta \int dx \operatorname{Tr} \Psi^\dagger \Gamma^6 D_1 \Psi + \sum_{i=1}^4 \Psi^\dagger [\Gamma^i \phi^i, \Psi] + \Psi^\dagger [\Gamma^5 V^1, \Psi], \quad (35)$$

where the spinor

$$\Psi^\dagger = (\psi_1^1 \psi_1^2 \psi_1^3 \psi_1^4 \eta^1 \eta^2 \eta^3 \eta^4), \quad (36)$$

and the  $\Gamma$ 's

$$\begin{aligned} \Gamma^1 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3, & \Gamma^2 &= -i \, 1 \otimes 1 \otimes 1, & \Gamma^3 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_1, \\ \Gamma^4 &= -\sigma_3 \otimes \sigma_1 \otimes 1, & \Gamma^5 &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2, & \Gamma^6 &= -\sigma_1 \otimes 1 \otimes 1. \end{aligned}$$

Dimensional reduction of the eight supercharge action leads to four scalar fermions and thus four scalar supercharges. So far we have made use of only one of them, the supercharge  $\mathcal{Q}$ , to construct the action. The remaining scalar supersymmetries may be revealed using the scalar supercharge  $\mathcal{Q}$  and additional symmetries of the action. For example, to obtain another scalar supersymmetry say  $\mathcal{Q}'$ , we look at the invariance of the action under the field transformations

$$V^1 \rightarrow -V^1, \quad \phi^3 \rightarrow -\phi^3, \quad (37)$$

along with

$$\begin{aligned} \psi_1^1 &\rightarrow \psi_1^4, & \eta^1 &\rightarrow \eta^4, \\ \psi_1^2 &\rightarrow \psi_1^3, & \eta^2 &\rightarrow \eta^3, \\ \psi_1^3 &\rightarrow \psi_1^2, & \eta^3 &\rightarrow \eta^2, \\ \psi_1^4 &\rightarrow \psi_1^1, & \eta^4 &\rightarrow \eta^1. \end{aligned} \quad (38)$$

The scalar supersymmetry,  $\mathcal{Q}'$  associated with this invariance of the action is

$$\begin{aligned} \mathcal{Q}'\psi_1^1 &= [\phi, V^1], & \mathcal{Q}'\eta^1 &= B_{23}, & \mathcal{Q}'B_{12} &= [\phi, \psi_1^3], & \mathcal{Q}'\phi &= 0, \\ \mathcal{Q}'\psi_1^2 &= B_{13}, & \mathcal{Q}'\eta^2 &= [\phi, \phi^4], & \mathcal{Q}'B_{13} &= [\phi, \psi_1^2], & \mathcal{Q}'\bar{\phi} &= -\eta^4, \\ \mathcal{Q}'\psi_1^3 &= B_{12}, & \mathcal{Q}'\eta^3 &= -[\phi, \phi^3], & \mathcal{Q}'B_{23} &= [\phi, \eta^1], & \mathcal{Q}'\phi^3 &= -\eta^3, \\ \mathcal{Q}'\psi_1^4 &= D_1\phi, & \mathcal{Q}'\eta^4 &= [\phi, \bar{\phi}], & \mathcal{Q}'A_1 &= \psi_1^4, & \mathcal{Q}'\phi^4 &= \eta^2, \\ \mathcal{Q}'V^1 &= \psi_1^1. \end{aligned}$$

The lattice action is obtained by the  $\mathcal{Q}$ -variation of the gauge fermion

$$\begin{aligned}
\Lambda = \sum_x \text{Tr} & \left( \chi_{12}(x) D_1^+ A_2(x) + \chi_{13}(x) D_1^+ A_3(x) + \chi_{23}(x) [A_2(x), A_3(x)] + \chi_{12}(x) B_{12}(x) \right. \\
& + \chi_{13}(x) B_{13}(x) + \chi_{23}(x) B_{23}(x) + \chi_{12}(x) [A_3(x), W_{312}(x)] + \chi_{13}(x) [A_2(x), W_{213}(x)] \\
& + \chi_{23}(x) D_1^- W_{123}(x) + \psi_1(x) D_1^+ \bar{\phi}(x) + \psi_2(x) [A_2(x), \bar{\phi}(x)] + \psi_3(x) [A_3(x), \bar{\phi}(x)] \\
& \left. + \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] + \theta_{123}(x) [W_{123}(x), \bar{\phi}(x)] \right), \tag{39}
\end{aligned}$$

where the backward difference operator is defined as

$$D_\mu^- g_\mu(x) = g_\mu(x) - U_\mu^\dagger(x - \mu) g_\mu(x - \mu) U_\mu(x - \mu), \tag{40}$$

and the  $\mathcal{Q}$ -transformation rules for fields on lattice are similar to the  $\mathcal{Q} = 4$  model

$$\begin{aligned}
\mathcal{Q}U_1(x) &= \psi_1(x) U_1(x) & \mathcal{Q}A_\mu(x) &= \psi_\mu(x) \\
\mathcal{Q}\psi_1(x) &= \psi_1(x) \psi_1(x) - D_1^+ \phi(x), & \mathcal{Q}\psi_i(x) &= -[A_i(x), \phi(x)], \quad i \neq 1, \\
\mathcal{Q}\eta(x) &= [\phi(x), \bar{\phi}(x)], & \mathcal{Q}B_{\mu\nu}(x) &= [\phi(x), \chi_{\mu\nu}(x)] \\
\mathcal{Q}\chi_{\mu\nu}(x) &= B_{\mu\nu}(x), & \mathcal{Q}W_{\mu\nu\lambda}(x) &= \theta_{\mu\nu\lambda}(x) \\
\mathcal{Q}\theta_{\mu\nu\lambda}(x) &= [\phi(x), W_{\mu\nu\lambda}(x)], & \mathcal{Q}\phi(x) &= 0, \\
\mathcal{Q}\bar{\phi}(x) &= \eta(x).
\end{aligned} \tag{41}$$

Carrying out the  $\mathcal{Q}$ -variation we write down the action

$$S = S_B + S_F + S_Y + S_R. \tag{42}$$

In terms of the relabelled fields

$$\begin{aligned}
\psi_1^1(x) &= -\psi_1(x), & \eta^1(x) &= -\frac{\eta}{2}(x), \\
\psi_1^2(x) &= \chi_{12}(x), & \eta^2(x) &= \psi_2(x), \\
\psi_1^3(x) &= \chi_{13}(x), & \eta^3(x) &= \psi_3(x), \\
\psi_1^4(x) &= \theta_{123}(x), & \eta^4(x) &= \chi_{23}(x), \\
\phi^3(x) &= A_2(x), & \phi^4(x) &= A_3(x), \\
V^1(x) &= W_{123}(x), & \phi(x) &= \phi^1(x) + i\phi^2(x).
\end{aligned}$$

the bosonic part takes the form

$$\begin{aligned}
S_B = \beta \sum_x \text{Tr} & \left[ - \left( D_1^+ \phi^3(x) + D_1^+ \phi^4(x) + [\phi^3(x), \phi^4(x)] \right)^2 - \left( D_1^- V^1(x) - [\phi^3(x), V^1(x)] \right. \right. \\
& + [\phi^4(x), V^1(x)] \left. \right)^2 - \left( D_1^+ \phi^1(x) \right)^2 - \left( D_1^+ \phi^2(x) \right)^2 - [\phi^3(x), \phi^1(x)]^2 \\
& - [\phi^3(x), \phi^2(x)]^2 - [\phi^4(x), \phi^1(x)]^2 - [\phi^4(x), \phi^2(x)]^2 - [\phi^1(x), \phi^2(x)]^2 \\
& \left. - [\phi^1(x), V^1(x)]^2 - [\phi^2(x), V^1(x)]^2 \right]. \tag{43}
\end{aligned}$$

Notice that unlike the continuum case this cannot be further simplified since the lattice theory does not possess an exact Bianchi identity. The fermionic kinetic term can be expressed in a matrix form similar to the  $\Gamma^6$  matrix in the continuum, with the same spinor structure as eqn. (36),

$$\mathbf{D}_1 = \begin{pmatrix} 0 & (\mathbb{I}_{4 \times 4}) D_1^- \\ (\mathbb{I}_{4 \times 4}) D_1^+ & 0 \end{pmatrix} \tag{44}$$

The Yukawa part of the action  $S_Y$  can also be expressed in a matrix form

$$\Psi^\dagger [\tilde{\mathbf{O}}, \Psi], \tag{45}$$

where

$$\tilde{\mathbf{O}} = \begin{pmatrix} \tilde{\phi} & -\tilde{\phi}^3 & -\tilde{\phi}^4 & 0 & 0 & 0 & 0 & -\tilde{V}^1 \\ -\tilde{\phi}^3 & -\phi & 0 & -\phi^4 & 0 & 0 & V^1 & 0 \\ -\tilde{\phi}^4 & 0 & -\phi & \phi^3 & 0 & -V^1 & 0 & 0 \\ 0 & -\phi^4 & \phi^3 & \bar{\phi} & V^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & V^1 & -\phi & \phi^3 & \phi^4 & 0 \\ 0 & 0 & -V^1 & 0 & \phi^3 & \bar{\phi} & 0 & \phi^4 \\ 0 & V^1 & 0 & 0 & \phi^4 & 0 & \bar{\phi} & -\phi^3 \\ -\tilde{V}_1 & 0 & 0 & 0 & 0 & \phi^4 & -\phi^3 & -\phi \end{pmatrix} \tag{46}$$

We cannot decompose the Yukawa matrix as given in the continuum case because of the specific form of the latticization procedure we have chosen.

The action has a residual part also

$$S_R = \beta \text{Tr} \sum_x \psi_1^1(x) \psi_1^1(x) D_1^+ \phi^1(x) - i \psi_1^1(x) \psi_1^1(x) D_1^+ \phi^2(x). \tag{47}$$



#### IV. THE SIXTEEN SUPERCHARGE MODEL

Starting from the form of the four dimensional gauge fermion  $\Lambda$  of twisted  $\mathcal{N} = 4$  super Yang-Mills theory given in [28] we find

$$\begin{aligned} \Lambda = \int d^4x \text{Tr} & \left[ \chi_{\mu\nu} \left( F_{\mu\nu} + \frac{1}{2} B_{\mu\nu} - \frac{1}{4} [W_{\mu\lambda\rho}, W_{\nu\lambda\rho}] + \sqrt{2} D_\lambda W_{\lambda\mu\nu} \right) + \psi_\mu D_\mu \bar{\phi} + \frac{1}{4} \eta [\phi, \bar{\phi}] \right. \\ & \left. + \frac{1}{2} \frac{1}{3!} \theta_{\mu\nu\lambda} [W_{\mu\nu\lambda}, \bar{\phi}] - \frac{1}{3!} \kappa_{\mu\nu\lambda\rho} D_{[\mu} W_{\nu\lambda\rho]} + \frac{1}{2} \frac{1}{4!} \kappa_{\mu\nu\lambda\rho} C_{\mu\nu\lambda\rho} \right]. \end{aligned} \quad (48)$$

To construct a 16 supercharge Yang-Mills quantum mechanics, we dimensionally reduce eqn. (48) with respect to the  $x^2$ -,  $x^3$ - and  $x^4$ -directions. The reduced scalar supercharge transformation rules are

$$\begin{aligned} \mathcal{Q}A_\mu &= \psi_\mu, & \mathcal{Q}\psi_1 &= -D_1\phi, \\ \mathcal{Q}\psi_i &= -[A_i, \phi], \quad i \neq 1, & \mathcal{Q}\phi &= 0, \\ \mathcal{Q}\bar{\phi} &= \eta, & \mathcal{Q}\eta &= [\phi, \bar{\phi}], \\ \mathcal{Q}B_{\mu\nu} &= [\phi, \chi_{\mu\nu}], & \mathcal{Q}\chi_{\mu\nu} &= B_{\mu\nu}, \\ \mathcal{Q}W_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda}, & \mathcal{Q}\theta_{\mu\nu\lambda} &= [\phi, W_{\mu\nu\lambda}], \\ \mathcal{Q}C_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}], & \mathcal{Q}\kappa_{\mu\nu\lambda\rho} &= C_{\mu\nu\lambda\rho}. \end{aligned} \quad (49)$$

After  $\mathcal{Q}$ -variation and integrating out the multiplier fields we find the action

$$S = S_B + S_F + S_Y + S_R. \quad (50)$$

In terms of the relabelled fields

$$\begin{aligned} \psi_1^1 &= \psi_1, & \eta^1 &= \eta, & V_1^1 &= W_{123}, \\ \psi_1^2 &= \chi_{12}, & \eta^2 &= \psi_2, & V_1^2 &= W_{124}, \\ \psi_1^3 &= \chi_{13}, & \eta^3 &= \psi_3, & V_1^3 &= W_{134}, \\ \psi_1^4 &= \theta_{123}, & \eta^4 &= \chi_{23}, & \phi^1 &= W_{234}, \\ \psi_1^5 &= \chi_{14}, & \eta^5 &= \psi_4, & \phi^2 &= A_2, \\ \psi_1^6 &= \theta_{124}, & \eta^6 &= \chi_{24}, & \phi^3 &= A_3, \\ \psi_1^7 &= \theta_{134}, & \eta^7 &= \chi_{34}, & \phi^4 &= A_4, \\ \psi_1^8 &= \kappa_{1234}, & \eta^8 &= -\theta_{234}, & \phi &= \phi^5 + i\phi^6, \end{aligned} \quad (51)$$

the bosonic part of the action takes the form

$$S_B = \beta \text{Tr} \int dx \left( - \sum_{i=1}^6 (D_1 \phi^i)^2 - \sum_{i=1}^3 (D_1 V_1^i)^2 - \sum_{i < j, i, j=1}^6 [\phi^i, \phi^j]^2 \right. \\ \left. - \sum_{i < j, i, j=1}^3 [V_1^i, V_1^j]^2 - \sum_{i=1}^3 \sum_{j=1}^6 [V_1^i, \phi^j]^2 \right), \quad (52)$$

where we have made use of integration by parts and the Bianchi identity in simplifying the original expression. If we were to relabel the fields  $V_1^i, i = 1 \dots 3$  as additional scalars this would be the usual bosonic action of  $\mathcal{N} = 1$  super Yang-Mills in  $D = 10$  reduced to one dimension.

The fermion kinetic term and Yukawa interactions can then be put in the compact form

$$S_{F+Y} = \beta \text{Tr} \int dx \Psi^\dagger \Gamma^{10} D_1 \Psi + \sum_{i=1}^6 \Psi^\dagger [\Gamma^i \phi^i, \Psi] + \sum_{j=7}^9 \Psi^\dagger [\Gamma^j V_1^j, \Psi], \quad (53)$$

where the spinor

$$\Psi^\dagger = (\psi_1^1 \psi_1^2 \psi_1^3 \psi_1^4 \psi_1^5 \psi_1^6 \psi_1^7 \psi_1^8 \eta^1 \eta^2 \eta^3 \eta^4 \eta^5 \eta^6 \eta^7 \eta^8), \quad (54)$$

and the  $\Gamma$ 's,

$$\begin{aligned} \Gamma^1 &= -\sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2, & \Gamma^2 &= \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1, & \Gamma^3 &= \sigma_3 \otimes \mathbf{1} \otimes \sigma_1 \otimes \sigma_3, \\ \Gamma^4 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3, & \Gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, & \Gamma^6 &= -i\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, \\ \Gamma^7 &= -\sigma_2 \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_1, & \Gamma^8 &= -\sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_1, & \Gamma^9 &= -\sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbf{1}, \\ \Gamma^{10} &= \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \end{aligned}$$

As we have seen dimensional reduction of the 16 supercharge action leads to eight scalar fermions and hence also eight scalar supercharges among which we have made use of only one. Using the scalar supercharge  $\mathcal{Q}$  and the symmetries of the action, one can derive the remaining supersymmetries. For example, to obtain another scalar super symmetry say  $\mathcal{Q}'$ , we look at the invariance of the action under the field transformations

$$\begin{aligned} V_1^1 &\rightarrow -V_1^1, & \phi^1 &\rightarrow -\phi^1, \\ V_1^2 &\rightarrow -V_1^2, & \phi^3 &\rightarrow -\phi^3, \end{aligned} \quad (55)$$

along with

$$\begin{aligned}
\psi_1^1 &\rightarrow \psi_1^4, & \psi_1^5 &\rightarrow \psi_1^8, & \eta^1 &\rightarrow \eta^4, & \eta^5 &\rightarrow \eta^8 \\
\psi_1^2 &\rightarrow \psi_1^3, & \psi_1^6 &\rightarrow \psi_1^7, & \eta^2 &\rightarrow \eta^3, & \eta^6 &\rightarrow \eta^7 \\
\psi_1^3 &\rightarrow \psi_1^2, & \psi_1^7 &\rightarrow \psi_1^6, & \eta^3 &\rightarrow \eta^2, & \eta^7 &\rightarrow \eta^6 \\
\psi_1^4 &\rightarrow \psi_1^1, & \psi_1^8 &\rightarrow \psi_1^5, & \eta^4 &\rightarrow \eta^1, & \eta^8 &\rightarrow \eta^5.
\end{aligned} \tag{56}$$

The scalar supersymmetry,  $\mathcal{Q}'$  associated with this invariance of the action is

$$\begin{aligned}
\mathcal{Q}'\psi_1^1 &= [\phi, V_1^1], & \mathcal{Q}'\eta^1 &= B_{23}, & \mathcal{Q}'B_{12} &= -[\phi, \psi_1^3], & \mathcal{Q}'\phi^1 &= \eta^5, \\
\mathcal{Q}'\psi_1^2 &= B_{13}, & \mathcal{Q}'\eta^2 &= -[\phi, \phi^3], & \mathcal{Q}'B_{13} &= [\phi, \psi_1^2], & \mathcal{Q}'\phi^2 &= \eta^3, \\
\mathcal{Q}'\psi_1^3 &= B_{12}, & \mathcal{Q}'\eta^3 &= [\phi, \phi^2], & \mathcal{Q}'B_{14} &= [\phi, \psi_1^8], & \mathcal{Q}'\phi^3 &= -\eta^2, \\
\mathcal{Q}'\psi_1^4 &= -D_1\phi, & \mathcal{Q}'\eta^4 &= [\phi, \bar{\phi}], & \mathcal{Q}'B_{23} &= [\phi, \eta^1], & \mathcal{Q}'\phi^4 &= \eta^8, \\
\mathcal{Q}'\psi_1^5 &= C_{1234}, & \mathcal{Q}'\eta^5 &= [\phi, \phi^1], & \mathcal{Q}'B_{24} &= [\phi, \eta^7], & \mathcal{Q}'V_1^1 &= -\psi_1^1, \\
\mathcal{Q}'\psi_1^6 &= [\phi, V_1^3], & \mathcal{Q}'\eta^6 &= B_{34}, & \mathcal{Q}'B_{34} &= [\phi, \eta^6], & \mathcal{Q}'V_1^2 &= \psi_1^7, \\
\mathcal{Q}'\psi_1^7 &= -[\phi, V_1^2], & \mathcal{Q}'\eta^7 &= B_{24}, & \mathcal{Q}'A^1 &= \psi_1^4, & \mathcal{Q}'V_1^3 &= \psi_1^6, \\
\mathcal{Q}'\psi_1^8 &= B_{14}, & \mathcal{Q}'\eta^8 &= [\phi, \phi^4], & \mathcal{Q}'\phi &= 0, & \mathcal{Q}'\bar{\phi} &= \eta^4, \\
\mathcal{Q}'C_{1234} &= [\phi, \psi_1^5].
\end{aligned} \tag{57}$$

In principle Ward identities corresponding to these additional scalar supersymmetries can be computed in lattice Monte Carlo simulations to test for a restoration of full supersymmetry in the continuum limit.

Finally, to construct the lattice version of the theory, we again write down the dimensionally reduced  $\Lambda$  on a lattice. The  $\mathcal{Q}$ -transformation rules for fields on lattice are similar to the  $\mathcal{Q} = 4$  and  $\mathcal{Q} = 8$  models

$$\begin{aligned}
\mathcal{Q}U_1(x) &= \psi_1(x)U_1(x) & \mathcal{Q}A_\mu(x) &= \psi_\mu(x) \\
\mathcal{Q}\psi_1(x) &= \psi_1(x)\psi_1(x) - D_1^+\phi(x), & \mathcal{Q}\psi_i(x) &= -[A_i(x), \phi(x)], \quad i \neq 1, \\
\mathcal{Q}\eta(x) &= [\phi(x), \bar{\phi}(x)], & \mathcal{Q}B_{\mu\nu}(x) &= [\phi(x), \chi_{\mu\nu}(x)] \\
\mathcal{Q}\chi_{\mu\nu}(x) &= B_{\mu\nu}(x), & \mathcal{Q}W_{\mu\nu\lambda}(x) &= \theta_{\mu\nu\lambda}(x) \\
\mathcal{Q}\theta_{\mu\nu\lambda}(x) &= [\phi(x), W_{\mu\nu\lambda}(x)], & \mathcal{Q}C_{\mu\nu\lambda\rho}(x) &= [\phi(x), \kappa_{\mu\nu\lambda\rho}(x)] \\
\mathcal{Q}\kappa_{\mu\nu\lambda\rho}(x) &= C_{\mu\nu\lambda\rho}(x), & \mathcal{Q}\phi(x) &= 0, \\
\mathcal{Q}\bar{\phi}(x) &= \eta(x).
\end{aligned} \tag{58}$$

Carrying out the  $\mathcal{Q}$ -variation we write down the action

$$S = S_B + S_F + S_Y + S_R. \tag{59}$$

In terms of the relabelled fields

$$\begin{aligned}
\psi_1^1(x) &= \psi_1(x), & \eta^1(x) &= \eta(x), & V_1^1(x) &= W_{123}(x), \\
\psi_1^2(x) &= \chi_{12}(x), & \eta^2(x) &= \psi_2(x), & V_1^2(x) &= W_{124}(x), \\
\psi_1^3(x) &= \chi_{13}(x), & \eta^3(x) &= \psi_3(x), & V_1^3(x) &= W_{134}(x), \\
\psi_1^4(x) &= \theta_{123}(x), & \eta^4(x) &= \chi_{23}(x), & \phi^1(x) &= W_{234}(x), \\
\psi_1^5(x) &= \chi_{14}(x), & \eta^5(x) &= \psi_4(x), & \phi^2(x) &= A_2(x), \\
\psi_1^6(x) &= \theta_{124}(x), & \eta^6(x) &= \chi_{24}(x), & \phi^3(x) &= A_3(x), \\
\psi_1^7(x) &= \theta_{134}(x), & \eta^7(x) &= \chi_{34}(x), & \phi^4(x) &= A_4(x), \\
\psi_1^8(x) &= \kappa_{1234}(x), & \eta^8(x) &= -\theta_{234}(x), & \phi(x) &= \phi^5(x) + i\phi^6(x),
\end{aligned} \tag{60}$$

the bosonic part takes the form

$$\begin{aligned}
S_B = \beta \int dx \text{Tr} & \left[ - \left( D_1^+ \phi^2 - [\phi^1, V_1^3] + [\phi^3, V_1^1] - [\phi^4, V_1^2] \right)^2 - \left( D_1^+ \phi^3 - [\phi^1, V_1^2] + [\phi^4, V_1^3] \right. \right. \\
& - [\phi^2, V_1^1] \left. \right)^2 - \left( D_1^+ \phi^4 - [\phi^1, V_1^1] + [\phi^2, V_1^2] - [\phi^3, V_1^3] \right)^2 - \left( [\phi^2, \phi^3] - [V_1^3, V_1^2] \right. \\
& + D_1^- V_1^1 - [\phi^4, \phi^1] \left. \right)^2 - \left( [\phi^2, \phi^4] - [V_1^3, V_1^1] + [\phi^3, \phi^1] - D_1^- V_1^2 \right)^2 - \left( [\phi^3, \phi^4] \right. \\
& - [V_1^2, V_1^1] + D_1^- V_1^3 - [\phi^2, \phi^1] \left. \right)^2 + \frac{1}{2} \left( D_1^+ \phi^1 \right)^2 + \frac{1}{2} [\phi^2, V_1^3]^2 + \frac{1}{2} [\phi^3, V_1^2]^2 \\
& + \frac{1}{2} [\phi^4, V_1^1]^2 - \left( (D_1^+ \phi^5)^2 + (D_1^+ \phi^6)^2 + [\phi^2, \phi^5]^2 + [\phi^2, \phi^6]^2 + [\phi^3, \phi^5]^2 \right. \\
& + [\phi^3, \phi^6]^2 + [\phi^4, \phi^5]^2 + [\phi^4, \phi^6]^2 \left. \right) - [\phi^5, \phi^6]^2 - \left( [\phi^5, \phi^1]^2 + [\phi^6, \phi^1]^2 + [\phi^5, V_1^3]^2 \right. \\
& \left. + [\phi^6, V_1^3]^2 + [\phi^5, V_1^2]^2 + [\phi^6, V_1^2]^2 + [\phi^5, V_1^1]^2 + [\phi^6, V_1^1]^2 \right) \left. \right]. \tag{61}
\end{aligned}$$

Notice that here also this cannot be further simplified since the lattice theory does not possess an exact Bianchi identity. The fermionic kinetic term can be expressed in a matrix form similar to the  $\Gamma^{10}$  matrix in the continuum, with the same spinor structure as eqn. (54),

$$\mathbf{D}_1 = \begin{pmatrix} 0 & (\mathbb{I}_{8 \times 8}) D_1^- \\ (\mathbb{I}_{8 \times 8}) D_1^+ & 0 \end{pmatrix} \tag{62}$$

The Yukawa part of the action  $S_Y$  can also be expressed in a matrix form

$$\Psi^\dagger [\tilde{\mathbf{O}}, \Psi], \tag{63}$$

where  $\tilde{\mathbf{O}}$  is a  $16 \times 16$  matrix:

$$\left( \begin{array}{cccccccccccccccc}
\tilde{\phi} & \tilde{\phi}^2 & \tilde{\phi}^3 & 0 & \tilde{\phi}^4 & 0 & 0 & \tilde{\phi}^1 & 0 & 0 & 0 & \tilde{V}_1^1 & 0 & \tilde{V}_1^2 & \tilde{V}_1^3 & 0 \\
\tilde{\phi}^2 & -\phi & 0 & -\phi^3 & 0 & -\phi^4 & -\phi^1 & 0 & 0 & 0 & V_1^1 & 0 & V_1^2 & 0 & 0 & V_1^3 \\
\tilde{\phi}^3 & 0 & -\phi & \phi^2 & 0 & \phi^1 & -\phi^4 & 0 & 0 & -V_1^1 & 0 & 0 & V_1^3 & 0 & 0 & -V_1^2 \\
0 & -\phi^3 & \phi^2 & \bar{\phi} & -\phi^1 & 0 & 0 & \phi^4 & -V_1^1 & 0 & 0 & 0 & 0 & V_1^3 & -V_1^2 & 0 \\
\tilde{\phi}^4 & 0 & 0 & -\phi^1 & -\phi & \phi^2 & \phi^3 & 0 & 0 & -V_1^2 & -V_1^3 & 0 & 0 & 0 & 0 & V_1^1 \\
0 & -\phi^4 & \phi^1 & 0 & \phi^2 & \bar{\phi} & 0 & -\phi^3 & -V_1^2 & 0 & 0 & -V_1^3 & 0 & 0 & V_1^1 & 0 \\
0 & -\phi^1 & -\phi^4 & 0 & \phi^3 & 0 & \bar{\phi} & \phi^2 & -V_1^3 & 0 & 0 & V_1^2 & 0 & -V_1^1 & 0 & 0 \\
\tilde{\phi}^1 & 0 & 0 & \phi^4 & 0 & -\phi^3 & \phi^2 & -\phi & 0 & -V_1^3 & V_1^2 & 0 & -V_1^1 & 0 & 0 & 0 \\
0 & 0 & 0 & -V_1^1 & 0 & -V_1^2 & -V_1^3 & 0 & -\phi & -\phi^2 & -\phi^3 & 0 & -\phi^4 & 0 & 0 & -\phi^1 \\
0 & 0 & -V_1^1 & 0 & -V_1^2 & 0 & 0 & -V_1^3 & -\phi^2 & \bar{\phi} & 0 & \phi^3 & 0 & \phi^4 & \phi^1 & 0 \\
0 & V_1^1 & 0 & 0 & -V_1^3 & 0 & 0 & V_1^2 & -\phi^3 & 0 & \bar{\phi} & -\phi^2 & 0 & -\phi^1 & \phi^4 & 0 \\
\tilde{V}_1^1 & 0 & 0 & 0 & 0 & -V_1^3 & V_1^2 & 0 & 0 & \phi^3 & -\phi^2 & -\phi & \phi^1 & 0 & 0 & -\phi^4 \\
0 & V_1^2 & V_1^3 & 0 & 0 & 0 & 0 & -V_1^1 & -\phi^4 & 0 & 0 & \phi^1 & \bar{\phi} & -\phi^2 & -\phi^3 & 0 \\
\tilde{V}_1^2 & 0 & 0 & V_1^3 & 0 & 0 & -V_1^1 & 0 & 0 & \phi^4 & -\phi^1 & 0 & -\phi^2 & -\phi & 0 & \phi^3 \\
\tilde{V}_1^3 & 0 & 0 & -V_1^2 & 0 & V_1^1 & 0 & 0 & 0 & \phi^1 & \phi^4 & 0 & -\phi^3 & 0 & -\phi & -\phi^2 \\
0 & V_1^3 & -V_1^2 & 0 & V_1^1 & 0 & 0 & 0 & -\phi^1 & 0 & 0 & -\phi^4 & 0 & \phi^3 & -\phi^2 & \bar{\phi}
\end{array} \right)$$

Notice that here also we cannot decompose the Yukawa matrix as given in the continuum case because of the specific form of the latticization procedure we have chosen.

Here also the lattice action picks up a residual part

$$S_R = \beta \text{Tr} \sum_x \psi_1^1(x) \psi_1^1(x) D_1^+ \phi_1(x) - i \psi_1^1(x) \psi_1^1(x) D_1^+ \phi_2(x). \quad (64)$$

## V. CONCLUSIONS

In this paper we have derived twisted actions for  $\mathcal{Q} = 4, 8$  and 16 supercharge Yang-Mills quantum mechanics by dimensionally reducing the corresponding  $\mathcal{Q}$ -exact actions in  $D = 2, 3$  and 4 dimensions. The dimensional reduction is done in the continuum but the final theories may be translated to the lattice using a simple discretization prescription that preserves the  $\mathcal{Q}$ -supersymmetry.

We show that the reduced twisted theories contain two, four or eight one dimensional Kähler-Dirac fields respectively. Correspondingly the continuum models contain additional nilpotent scalar supercharges all but one of which are softly broken by the discretization procedure adopted here. These lattice theories should prove useful in studies of the holographic correspondence between maximally supersymmetric Yang-Mills quantum mechanics and D0-branes in type IIA string theory. It would also be interesting to compare these lattice actions with those derived by orbifold methods as detailed in [8, 10].

## VI. ACKNOWLEDGEMENTS

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