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Black Hole Thermodynamics from Simulations of Lattice Yang-Mills Theory

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We report on lattice simulations of 16 supercharge $SU(N)$ Yang-Mills quantum mechanics in the 't Hooft limit. Maldacena duality conjectures that in this limit the theory is dual to IIA string theory, and in particular that the behavior of the thermal theory at low temperature is equivalent to that of certain black holes in IIA supergravity. Our simulations probe the low temperature regime for $N \leq 5$ and the intermediate and high temperature regimes for $N \leq 12$. We observe 't Hooft scaling and at low temperatures our results are consistent with the dual black hole prediction. The intermediate temperature range is dual to the Horowitz-Polchinski correspondence region, and our results are consistent with smooth behavior there. We include the Pfaffian phase arising from the fermions in our calculations where appropriate.

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I. INTRODUCTION

String theory has provided remarkable insight into the quantum physics underlying black holes. Much recent progress stems from conjectured dualities, which, in an appropriate limit, relate the finite temperature low energy supergravity limit of the string theory to strongly coupled thermal field theory. The entropy of the black holes that arise in these supergravity theories can then be computed in principle by counting microstates in their dual field theories. The pioneering calculations of black hole entropy in [1, 2] are examples where the dual field theory is a 2-d conformal field theory which allows computation of the entropy despite the strong coupling.

For a large number N of coincident Dp-branes in the 'decoupling' limit [3, 4] the dual field theory is $(1+p)$-dimensional strongly coupled maximally supersymmetric $SU(N)$ Yang-Mills theory, taken in the 't Hooft limit. The case of D3-branes yields the AdS–CFT correspondence. Analytic calculation of the corresponding black hole entropy of these theories has proven elusive despite interesting attempts [5].

Here we use lattice methods to study the thermal gauge theory and hence test these conjectured dualities. The simplest case for lattice work corresponds to D0-branes [6], where the dual is thermal 16 supercharge Yang-Mills quantum mechanics (the 'BFSS model' [7]). This theory has recently been numerically studied using a non-lattice formulation [8, 9]. Earlier analytic approaches used a variational method [10, 11]. Related zero temperature numerical works are [12, 13, 14].

In this letter we simulate the super quantum mechanics in the t'Hooft limit over a range of temperature and present preliminary results. We obtain intermediate temperature results for $N \leq 12$ and low temperature results for $N \leq 5$. We pay particular attention to the continuum limit and the behavior of the important Pfaffian phase arising from the fermions. More details of the method and results will be given in [15].

II. DUALITY AND BLACK HOLES

The type IIA string theory reduces to a supergravity theory for low energies compared to the string scale $(\alpha')^{-1/2}$. In this limit the thermal theory contains black holes with N units of D0-charge. Their energy, E, is a function of their Hawking temperature, T. Defining $\lambda = N g_s \alpha'^{-3/2}$ where g_s is the string coupling, we may write a dimensionless energy and temperature $\epsilon = E\lambda^{-1/3}$ and $t = T\lambda^{-1/3}$. One finds provided we take N large and $t \ll 1$ the black hole is weakly curved on string scales and the quantum string corrections are suppressed. The energy of this black hole can be precisely computed by standard methods [4] giving,

$$\epsilon = c\, N^2 t^{14/5} \qquad c = \left(\frac{2^{21}3^{12}5^2}{7^{19}}\pi^{14}\right)^{1/5} \simeq 7.41. \quad (1)$$

Duality posits that the thermodynamics of this black hole should be reproduced by the dual Yang-Mills quantum mechanics at the same temperature with $g_s\alpha'^{-3/2} = g^2_{YM}$ so λ is identified with the 't Hooft coupling.

In the large N limit, at high temperatures $t > 1$, the bound state of D0-branes is of order the size of the string scale, and hence all α' corrections are important. One should best think of the configuration dominating the partition function as a hot gas of D0-branes bound by strings. Horowitz and Polchinski have argued that the low temperature black hole and high temperature gas are the asymptotic descriptions and intermediate temperatures smoothly interpolate between these [16].

III. LATTICE IMPLEMENTATION

The 16 supercharge $SU(N)$ Yang-Mills quantum mechanics arises from dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills in 10-d. The 10-d gauge field reduces to the 1-d gauge field A and 9 scalars, X^i, $i = 1, \ldots, 9$ and the 10-d Majorana-Weyl fermion to 16 single component fermions, Ψ_α, $\alpha = 1, \ldots, 16$. All fields transform in the adjoint of the gauge group. In order to simulate the theory we must integrate out the fermions giving rise to a Pfaffian. The continuum Euclidean path integral, $Z = \int dAdX \mathrm{Pf}\left(\mathcal{O}\right) e^{-S_{bos}}$, is then given by,

$$
\begin{aligned}
S_{bos} &= \frac{N}{\lambda} \mathrm{Tr} \oint^R d\tau \left\{ \frac{1}{2}(D_\tau X_i)^2 - \frac{1}{4}\left[X_i, X_j\right]^2 \right\} \\
\mathcal{O} &= \gamma^\tau D_\tau - \gamma^i \left[X_i, \cdot\right] \qquad (2)
\end{aligned}
$$

The γ^τ, γ^i are the Euclidean Majorana-Weyl gamma matrices, and we choose a representation where, $\gamma^\tau = \begin{pmatrix} 0 & \mathrm{Id}_8 \\ \mathrm{Id}_8 & 0 \end{pmatrix}$. We take Euclidean time to have period R.

We have a choice of fermion boundary conditions. Thermal boundary conditions correspond to taking the fermions antiperiodic on the Euclidean time circle and correspond to a temperature $t = \lambda^{-1/3}/R$. We will also employ periodic fermions and the continuum partition function is then an index, with t the inverse volume. The Pfaffian is in general complex [17]. It is important in principle to include the phase of the Pfaffian in the Monte-Carlo simulation, and we discuss this later.

We discretize this continuum model as,

$$
\begin{aligned}
S_{bos} &= \frac{NL^3}{\lambda R^3} \sum_{a=0}^{L-1} \mathrm{Tr}\left[\frac{1}{2}\left(D_+ X_i\right)_a^2 - \frac{1}{4}[X_{i,a}, X_{j,a}]^2 \right] \\
\mathcal{O}_{ab} &= \begin{pmatrix} 0 & (D_+)_{ab} \\ (D_-)_{ab} & 0 \end{pmatrix} - \gamma^i \left[X_{i,a}, \cdot\right] \delta_{ab} \qquad (3)
\end{aligned}
$$

where we have rescaled the fields $X_{i,a}$ and $\Psi_{i,\alpha}$ by powers of the lattice spacing $a = R/L$ where L is the number of lattice points to render them dimensionless. We have introduced a Wilson gauge link field U_a, and taken covariant difference operators $(D_- W)_a = W_a - U_a^\dagger W_{a-1} U_a$, $(D_+ W)_a = U_a W_{a+1} U_a^\dagger - W_a$. Notice that the fermionic operator is free of doublers and is manifestly antisymmetric. This lattice action is finite in lattice perturbation theory and hence will flow *without fine tuning* to the correct supersymmetric continuum theory as the lattice spacing is reduced [6, 15].

We use the RHMC algorithm [18] to sample configurations using the absolute value of the Pfaffian. The phase may be re-incorporated in the expectation value of an observable $\mathcal{A}$ by reweighting as $< \mathcal{A} >= \frac{\sum_m \left(\mathcal{A} e^{i\phi}\right)}{\sum_m \left(e^{i\phi}\right)}$. Here $e^{i\phi(\mathcal{O})}$ is the phase of the Pfaffian and the sum runs over all members of our phase quenched ensemble.

We find in practice that the RHMC simulation of the thermal theory at low temperature, $t \lesssim 1$, exhibits an instability corresponding to the scalar fields moving out along the flat directions of the classical potential. Hence the algorithm never thermalizes and cannot be used to approximate the path integral. This has been observed before [9]. We believe this divergence is a lattice artifact that is related to the discretization of the fermion operator. In previous work [6] we have simulated the 4 supercharge quantum mechanics over a range of t, using a Weyl representation for the fermions where one obtains a real positive determinant. However, we have also tried using a Majorana representation where one obtains

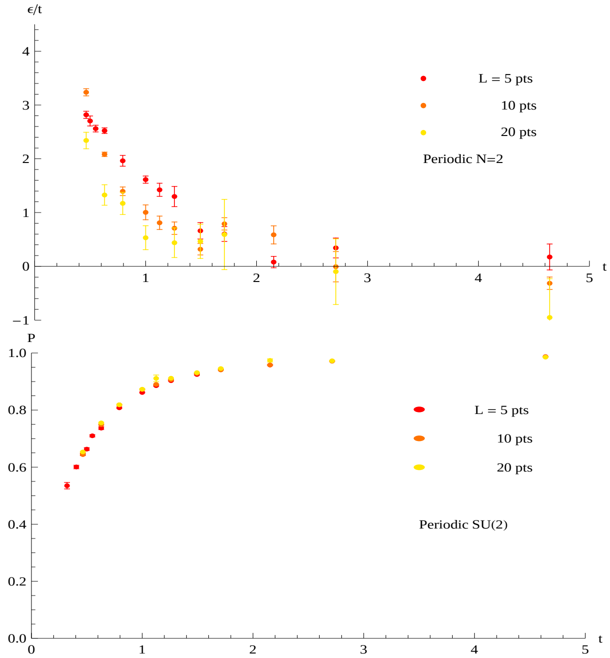

FIG. 1: *Top:* Plot showing ϵ/t verses dimensionless temperature t for the periodic $SU(2)$ theory for various numbers of lattice points. *Bottom:* Plot of the Polyakov loop against temperature for the same theory.

a Pfaffian which we have discretized in analogy with the 16 supercharge case discussed here. While no divergence of the scalars was observed in the Weyl simulations over a large range of t [6], the Majorana implementation has the same instability we observe in the 16 supercharge case for $t \lesssim 1$. Since both representations are equivalent in the continuum limit this implies that the instability is not a property of the continuum theory as is claimed in [9] but merely an artefact of finite lattice spacing. More details will be given in [15].

We find no such problem simulating the periodic theory at small t. At low temperature we expect the thermal and periodic theory to be similar, and the configurations that dominate the path integral will be similar. Hence in order to simulate the thermal theory at low temperature, $t \lesssim 1$, we have employed a reweighting of the periodic theory. We can expect to get good results for the thermal theory at low temperature by computing expectation values using the periodic theory, and reweighting as,

$$
< \mathcal{A} >_T = \frac{\sum_m^{(P)} \left(\mathcal{A}\, \mathrm{Pf}(\mathcal{O}_T)/|\mathrm{Pf}(\mathcal{O}_P)|\right)}{\sum_m^{(P)} \left(\mathrm{Pf}(\mathcal{O}_T)/|\mathrm{Pf}(\mathcal{O}_P)|\right)} \qquad (4)
$$

where $\sum_m^{(P)}$ is a sum over the phase quenched ensemble generated for the periodic theory, $\mathcal{O}_P$ and $\mathcal{O}_T$ are the periodic and thermal fermion operators respectively, and $< \ldots >_T$ is the expectation value for the thermal theory.

IV. RESULTS

We have simulated the thermal and periodic theories concentrating on the range $0.3 < t < 5$. We have focused

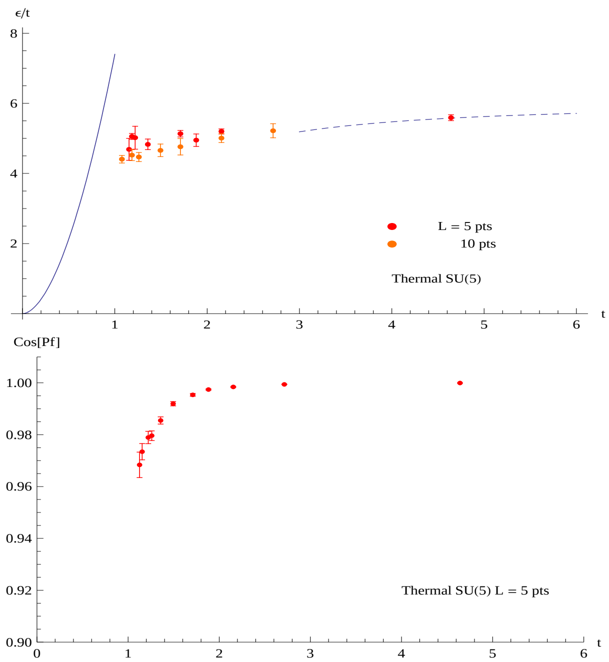

FIG. 2: *Top:* Plot of dimensionless energy ϵ/t verses dimensionless temperature t for the thermal $SU(5)$ theory with 5 and 10 lattice points. *Bottom:* Plot of the cosine of the Pfaffian argument for the thermal $SU(5)$ theory with 5 points.

on two observables - the mean energy, ϵ, and absolute value of the trace of the Polyakov loop, P. In the Yang-Mills theory these are given by [6, 15]

$$\begin{aligned} <\epsilon/t> &= \frac{3}{N^2}\left(\frac{9}{2}L(N^2-1) - <S_{bos}>\right) \\ P &= \frac{1}{N} < |\mathrm{Tr}\prod_{a=0}^{L-1} U_a| > . \end{aligned} \tag{5}$$

The inclusion of $1/N^2, 1/N$ in these definitions is to ensure these quantities are finite in the t'Hooft limit for a deconfined phase. In the periodic case, since Z is an index, it should not depend continuously on the inverse volume t, and hence in the continuum $\epsilon = 0$.

To check for a restoration of supersymmetry we have computed ϵ/t in the periodic theory for a variety of lattice sizes $L = 5, 10$ and 20. The upper plot of figure 1 shows ϵ/t for $SU(2)$. For large t the index ϵ is already consistent with zero for $L = 5$, while at small t it appears to approach zero as L increases. Notice that while this quantity is a sensitive test of the restoration of supersymmetry in the continuum limit $L \to \infty$ other observables such as P shown in the lower plot are relatively insensitive to the number of lattice sites for $L \geq 5$.

We have also examined the continuum limit of the thermal theory. In figure 2 we show $L = 5$ and 10 data for the thermal energy for $SU(5)$ (in the phase quenched approximation – which we discuss shortly). As noted above, we find a lattice instability for the thermal theory with $t \lesssim 1$ (with some dependence on N and L). However,

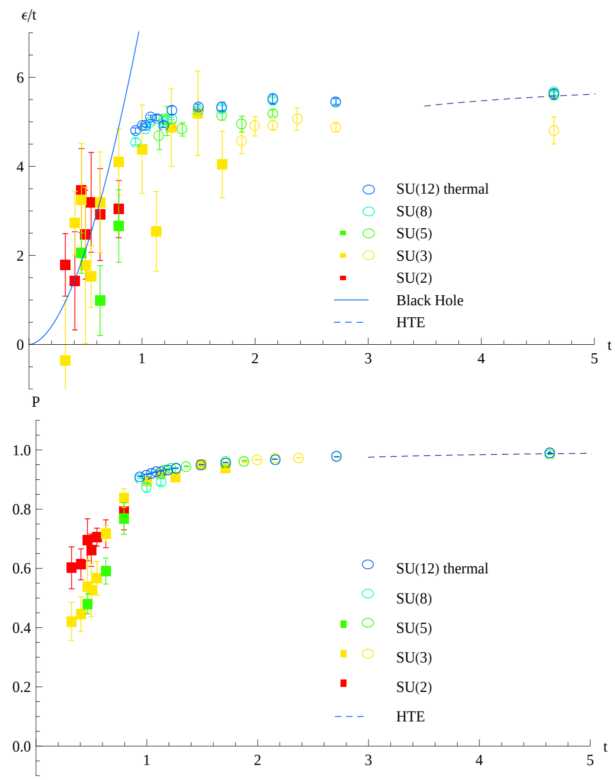

FIG. 3: *Top:* A plot of the dimensionless energy ϵ/t verses dimensionless temperature t. Data shown is generated in two ways. For temperatures larger than $t \sim 1$ we simulate the thermal theory for $N = 2, 3, 5, 8, 12$ with 5 points. The low temperature results are computed for $N = 2, 3, 5$ for 5 points by simulating the periodic theory, and reweighting with the appropriate combination of the thermal and periodic Pfaffians, as described in the main text. The low temperature black hole prediction is shown. *Bottom:* A plot of the Polyakov loop observable P for the same cases.

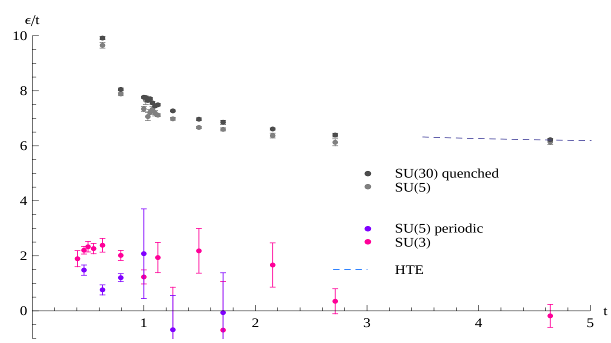

FIG. 4: For comparison with figure 3, ϵ/t verses t is shown for the quenched theory for $N = 5, 12, 30$ and periodic theory with Pfaffian reweighting for $N = 3, 5$, using 5 point lattices.

for larger t this does not occur. As argued above we believe this is an artifact of our lattice formulation and has nothing to do with continuum physics. The points plotted in the figure are taken only from simulations where the scalar distribution remained bounded for hundreds of physical RHMC times (the observed instability sets in very quickly in RHMC time, so the change in behavior is easy to identify). The plot shows that these lattice spacing effects are small and hence for the remainder of our results we show only data from $L = 5$ point lattices.

In the lower plot of figure 2 we show the mean cosine of the Pfaffian phase for the thermal $SU(5)$ theory with $L = 5$ lattice sites. As expected this phase becomes more important at lower temperatures but the actual value is close to one over the range of temperatures where we can directly simulate the thermal theory. Indeed the effects of reweighting are negligible in this temperature regime. Hence for the data we present later for direct simulation of the thermal theory we use the phase quenched approximation. Since the Pfaffian is very costly to compute this allows us to work at larger N.

We now turn to the main results of this letter. In figure 3 we plot the energy and Polyakov loop for various N and $L = 5$ point lattices. For high temperature we have used direct simulation of the thermal theory (phase quenched as discussed above), and we are able to obtain results up to $N = 12$. At low temperatures we obtain results by a reweighting of the periodic simulations as discussed above. The results from both methods agree in the regime where they overlap $t \sim 1$.

At very high temperatures the curves approach a constant corresponding to the result from classical equipartition assuming N^2 deconfined gluonic states (the fermions are lifted out of the dynamics by their thermal mass in this limit). In contrast for low temperatures the energy approaches zero signaling the presence of a supersymmetric vacuum at vanishing temperature.

As seen before [6], we see that t'Hooft scaling sets in for small N, with $N = 3$ already giving results close to an extrapolated large N result. The high temperature asymptotics computed in [19] are also plotted and agree with our data. Our results also appear consistent with those found recently using non-lattice methods [9].

There are two important physical observations. Firstly the curves appear to interpolate from high to low temperature smoothly – there is no obviously discontinuous behavior. This is to be contrasted with the quenched version of this theory which has a large N confinement/deconfinement phase transition at $t \simeq 0.9$ [20, 21]. Since the intermediate temperature range $t \sim 1$ is dual to the regime where the thermal D0-branes have a radius comparable to the string scale, we are probing the Horowitz-Polchinski correspondence regime, and seeing apparently smooth behavior there.

Secondly, the low temperature behavior of the theory appears consistent with the prediction from supergravity, also shown in the plots. This is to be contrasted with the quenched energy curve shown for comparison in figure 4 which departs strongly from the black hole prediction at low temperature. In this figure we also show the periodic theory which shows the degree of supersymmetry breaking for these lattices is small.

It would be very interesting to extend these calculations to 2 and 3 dimensional Yang-Mills systems which are thought to be dual to D1 and D2 brane systems using recent lattice formulations retaining exact supersymmetry [22].

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