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Anti-de Sitter space from supersymmetric gauge theory

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ABSTRACT: We construct a four dimensional Yang-Mills theory with $\mathcal{N} = 4$ twisted supersymmetry whose classical vacua correspond to four dimensional anti-de Sitter space. The theory utilizes a complex gauge field whose real part is a spin connection which is used to enforce local Lorentz invariance. The imaginary part of the connection can then be interpreted as a vierbein. The topological construction ensures that the partition function and classical vacua are independent of the background geometry. Additionally a supersymmetric and gauge invariant lattice construction is possible yielding a non-perturbative definition of the theory.

Contents

1. Introduction

The problem of constructing a theory of gravity which is consistent with quantum mechanics has a long history. The non-renormalizability of quantum general relativity can be traced to the fact that the Newton constant carries the dimensions of length squared corresponding to an action linear in the curvature. Actions containing higher powers of the curvature are usually thought to lead to violations of unitarity and can at best be thought of only as effective field theories [1].

Alternative approaches to gravity have tried to highlight the similarities to gauge theory, where in the case of gravity, the local symmetry corresponds to Lorentz invariance. In this case the corresponding gauge field is called a spin connection. It is necessary to introduce such an object when discussing fermions in curved spacetime where it is partnered by a new field the so-called vierbein which describes the local Lorentz frame and from which the usual metric tensor can be reconstructed.

Einstein's theory can be reproduced from the spin connection and vierbein provided a certain additional condition is satisfied; the vanishing of the torsion, which determines the spin connection in terms of the vierbein and its derivatives. The renormalizability problems of quantum gravity reemerge in this approach after this last condition is imposed; since then the usual Yang-Mills action for the spin connection becomes fourth order in derivatives - see eg. [2].

A further obstruction to constructing a quantum theory of gravity is the requirement of background independence – the result of quantizing the classical theory should not depend on the choice of background metric. Thus the theory should be topological in character.

In this paper we will write down a locally Lorentz invariant theory in which the spin connection and vierbein are treated as *independent fundamental fields*. They appear as the real and imaginary parts of a *complex* connection. The action contains a conventional Yang-Mills term which is quadratic in the curvature and the theory is thus renormalizable. Furthermore the complexified Yang-Mills Bianchi identity automatically yields the two gravitational Bianchi identities of Riemann-Cartan gravity. In the classical limit the torsion vanishes and the vierbein and spin connection conspire to yield anti-de Sitter space. The construction corresponds to a modification of a well known twist of $\mathcal{N} = 4$ super Yang-Mills due to Marcus and inherits that theory's topological character. Specifically, the theory possesses a $\mathcal{Q}$-exact energy-momentum tensor where the scalar nilpotent supercharge $\mathcal{Q}$ appears as a consequence of the twisting procedure. This structure ensures that the expectation value of any $\mathcal{Q}$-invariant observable is independent of any background metric used to construct the Yang-Mills theory.

We will argue later that this $\mathcal{Q}$-invariant and background independent sector of the model corresponds to a topological supergravity theory.[1] True background independence in a theory of quantum gravity would require that *all* gauge invariant observables in the theory possess vacuum expectation values which are independent of the background metric. The current construction does not seem to allow this. However, we are able to write down a lattice theory which preserves both the scalar supersymmetry and gauge invariance and which may represent a partial step in this direction. This lattice model certainly approximates the continuum theory in a flat background geometry and serves as a non-perturbative definition of the theory in such a background.

We derive the theory in a series of steps; in the next section we review the Marcus twist of $\mathcal{N} = 4$ super Yang-Mills which is the basis of the construction. In section 3 we modify the gauge field of this theory to describe a connection whose real part implements local $SO(4)$ invariance. We show that the imaginary part of the connection can be interpreted as a vierbein and the vacua of the theory correspond to the sphere S^4. In section 4 we show that implementing the same procedure for gauge group $SO(1,3)$ yields a theory whose vacua correspond to anti-de Sitter space. We note that similar arguments can be used to write down a five dimensional model whose classical vacua correspond to AdS$_5$. This forms the basis for a lattice construction of the four dimensional theory which we describe. The final section summarizes our findings and points to future work.

2. Twisted $\mathcal{N} = 4$ gauge theory

We start from the Marcus twist of $\mathcal{N} = 4$ super Yang-Mills [4] in which the bosonic fields of the model comprise a *complex* connection $\mathcal{A}_\mu, \mu = 1 \ldots 4$ and two scalar fields $\phi, \overline{\phi}$ together

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[1]Topological $\mathcal{N} = 2$ supergravity was considered in [3]

with a set of antisymmetric tensor fields $(\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})$ representing the sixteen fermion degrees of freedom[2] [6, 7]

The action takes the form

$$S = \frac{1}{g^2}\left[\mathcal{Q}\int d^4x\ \sqrt{h}\Lambda - \frac{1}{2}\int d^4x\ \mathrm{Tr}\ \epsilon^{\rho\lambda\mu\nu}\chi_{\rho\lambda}\left(\overline{\mathcal{D}}_{[\mu}\overline{\psi}_{\nu]} + \frac{1}{2}[\overline{\phi}, \chi_{\mu\nu}]\right)\right] \tag{2.1}$$

where the $\mathcal{Q}$-exact term is given by

$$\Lambda = \mathrm{Tr}\ \left(\chi^{\mu\nu}\mathcal{F}_{\mu\nu} + \overline{\psi}^\mu \mathcal{D}_\mu\phi + \eta\left([\overline{\mathcal{D}}^\mu, \mathcal{D}_\mu] + [\overline{\phi}, \phi]\right) - \frac{1}{2}\eta d\right) \tag{2.2}$$

and indices are raised and lowered by a background metric $h_{\mu\nu}(x)$. The nilpotent supersymmetry $\mathcal{Q}$ acts on the fields as follows

$$\mathcal{Q}\,\mathcal{A}_\mu = \psi_\mu$$
$$\mathcal{Q}\,\phi = \overline{\eta}$$
$$\mathcal{Q}\,\psi_\mu = 0$$
$$\mathcal{Q}\,\overline{\eta} = 0$$
$$\mathcal{Q}\,\overline{\mathcal{A}}_\mu = 0$$
$$\mathcal{Q}\,\overline{\phi} = 0$$
$$\mathcal{Q}\,\chi_{\mu\nu} = -\overline{\mathcal{F}}_{\mu\nu}$$
$$\mathcal{Q}\,\overline{\psi}_\mu = -\overline{\mathcal{D}}_\mu\overline{\phi}$$
$$\mathcal{Q}\,\eta = d$$
$$\mathcal{Q}\,d = 0 \tag{2.3}$$

The second $\mathcal{Q}$-closed term is supersymmetric on account of the Bianchi identities

$$\epsilon^{\rho\lambda\mu\nu}\mathcal{D}_\lambda\mathcal{F}_{\mu\nu} = 0 \tag{2.4}$$
$$\epsilon^{\rho\lambda\mu\nu}[\phi, \mathcal{F}_{\mu\nu}] = 0 \tag{2.5}$$
$$\epsilon^{\rho\lambda\mu\nu}\mathcal{D}_\lambda\mathcal{D}_\mu\phi = 0 \tag{2.6}$$

The complex covariant derivatives appearing in these expressions are defined by

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu = \partial_\mu + A_\mu + iB_\mu$$
$$\overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu = \partial_\mu + A_\mu - iB_\mu \tag{2.7}$$

while in the original construction all fields take values in the adjoint of a $U(N)$ gauge group[3]. It should be noted that despite the appearance of a complexified connection and field strength the theory possesses only the usual $U(N)$ gauge invariance corresponding to the real part of the connection.

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[2]It is also the twist of $\mathcal{N} = 4$ YM used in the Geometric Langlands program [5]

[3]The generators are taken to be *anti-hermitian* matrices satisfying $\mathrm{Tr}\ (T^a T^b) = -\delta^{ab}$

Doing the $\mathcal{Q}$-variation and integrating out the field d yields

$$S = \frac{1}{g^2} \int d^4x \sqrt{h} \, (L_1 + L_2 + L_3 + L_4) \tag{2.8}$$

where

$$
\begin{aligned}
L_1 &= \operatorname{Tr} \left(-\overline{\mathcal{F}}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}^\mu, \mathcal{D}_\mu]^2) \right) \\
L_2 &= \operatorname{Tr} \left(2(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) + \frac{1}{2}[\phi, \overline{\phi}]^2 \right) \\
L_3 &= \operatorname{Tr} \left(-\chi^{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]} - \overline{\psi}^\mu \mathcal{D}_\mu \overline{\eta} - \psi^\mu \overline{\mathcal{D}}_\mu \eta - \chi^{*\mu\nu} \overline{\mathcal{D}}_{[\mu} \overline{\psi}_{\nu]} \right) \\
L_4 &= \operatorname{Tr} \left(-\overline{\psi}^\mu [\phi, \psi_\mu] - \eta [\overline{\phi}, \overline{\eta}] - \frac{1}{2} \chi^{*\mu\nu} [\overline{\phi}, \chi_{\mu\nu}] \right)
\end{aligned}
\tag{2.9}
$$

where χ^* is the Hodge dual of χ, $\chi^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\chi_{\rho\lambda}$. The terms appearing in L_1 can then be written

$$
\begin{aligned}
\overline{\mathcal{F}}^{\mu\nu} \mathcal{F}_{\mu\nu} &= (F_{\mu\nu} - [B_\mu, B_\nu])(F^{\mu\nu} - [B^\mu, B^\nu]) + (D_{[\mu} B_{\nu]})(D^{[\mu} B^{\nu]}) \\
\frac{1}{2} \left[\overline{\mathcal{D}}^\mu, \mathcal{D}_\mu \right]^2 &= -2 (D_\mu B^\mu)^2
\end{aligned}
\tag{2.10}
$$

where $F_{\mu\nu}$ and D_μ denote the usual field strength and covariant derivative depending on the real part of the connection A_μ. The classical vacua of this theory correspond to solutions of the equations

$$
\begin{aligned}
F_{\mu\nu} - [B_\mu, B_\nu] &= 0 \\
D_{[\mu} B_{\nu]} &= 0 \\
D_\mu B_\mu + [\phi_1, \phi_2] &= 0 \\
D_\mu \phi_1 - [B_\mu, \phi_2] &= 0 \\
D_\mu \phi_2 - [\phi_1, B_\mu] &= 0
\end{aligned}
\tag{2.11}
$$

where $\phi = \phi_1 + i\phi_2$. In [4] this moduli space is argued to correspond to the space of flat *complexified* connections modulo gauge transformations[4]. The topological character of the theory guarantees that any $\mathcal{Q}$-invariant observable such as the partition function can be evaluated *exactly* by considering only gaussian fluctuations about such vacuum configurations. Indeed, it is easy to see from eqn. 2.1 that the energy momentum tensor of this theory is $\mathcal{Q}$-exact rendering the expectation values of such topological observables independent of smooth deformations of the background metric $h_{\mu\nu}(x)$.

Returning to the bosonic action and integrating by parts we find that the term linear in $F_{\mu\nu}$ cancels and the contribution of L_1 reads

$$L_1 = \operatorname{Tr} \left(F_{\mu\nu} F^{\mu\nu} + 2B^\mu D^\nu D_\nu B_\mu - [B_\mu, B_\nu][B^\mu, B^\nu] - 2R^{\mu\nu} B_\mu B_\nu \right) \tag{2.12}$$

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[4]This assumes that the vacuum value of the scalars is zero which is clearly one solution to these equations

where $R_{\mu\nu}$ is the background Ricci tensor. Thus at strong coupling the bosonic theory possesses the usual Yang-Mills field strength for a real $U(N)$ gauge field together with four massless vectors arising from the imaginary part of the connection and an additional two scalar fields. The fermionic sector comprises a Kähler-Dirac field arising from twisting the four Majorana fermions of $\mathcal{N} = 4$ Yang-Mills. In flat space the theory is fully equivalent to the usual $\mathcal{N} = 4$ theory as this twisting operation can be regarded merely as an exotic change of variables – the massless vectors can combined with the two scalars to yield the usual six scalars of $\mathcal{N} = 4$ Yang-Mills and the Kähler-Dirac action is equivalent to the Dirac action for four degenerate Majorana spinors. In this case we can lift the restriction to the topological subsector and consider the set of all gauge invariant operators as spanning the space of physical observables.

3. A generalized spin connection

The appearance of a complex connection naturally lead one to try to find a relationship to gravity [8]. Since the Yang-Mills theory contains no symmetric tensor that could play the role of a metric we will instead focus on alternative formulations of gravity which utilize the language of vielbeins and spin connections. As the latter function as the gauge field for local Lorentz transformations we are led to examine a generalized complex spin connection of the form

$$\mathcal{A}_\mu(x) = \omega_\mu{}^{ab}(x)\sigma^{ab} + \frac{i}{l_G}e_\mu{}^a(x)\gamma^a \tag{3.1}$$

where the indices $a, b = 1 \ldots 4$ correspond to indices in a flat Euclidean tangent space and we employ the definition

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{3.2}$$

With the Euclidean gamma matrices taken *anti-hermitian* the corresponding tangent space metric is just

$$\frac{1}{2}\{\gamma^a, \gamma^b\} = \eta^{ab} = -\delta^{ab} \tag{3.3}$$

The set of matrices (σ^{ab}, γ^c) obey the additional algebra

$$[\sigma^{ab}, \gamma^c] = \delta^{ca}\gamma^b - \delta^{cb}\gamma^a \tag{3.4}$$

It is not hard to show that this set of 10 matrices can be taken as a basis for the algebra $so(5)$. However, by construction, the action will only be locally invariant under the real part of this connection corresponding to the group $SO(4)$. Notice that we have also inserted an explicit length scale l_G into the definition so that the function e_μ^a is dimensionless.

To see that $e_\mu{}^a$ can be identified as a vierbein let us examine the variation of $e_\mu = \gamma^a e_\mu{}^a$ under an infinitesimal (adjoint) gauge transformation $\delta e_\mu = [\lambda, e_\mu]$. Using the algebra given in eqn. 3.4 we find

$$\delta e_\mu{}^a = \lambda^{ab}e_\mu{}^b \tag{3.5}$$

Thus $e_\mu{}^a$ transforms correctly as a fundamental field under local Lorentz transformations. In a similar way the covariant derivative $D_\mu e_\nu = \partial_\mu e_\nu + [\omega_\mu, e_\nu]$ leads to the appropriate covariant derivative of the vierbein e_μ^a

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_\nu{}^b \qquad (3.6)$$

The gauge transformation of the spin connection is the usual one

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + \omega_\mu{}^{ac} \lambda^{cb} + \omega_\mu{}^{bc} \lambda^{ac} \qquad (3.7)$$

while the curvature of the spin connection is given by the Yang-Mills field strength

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^{cb} - \omega_\nu{}^{ac} \omega_\mu{}^{cb} \qquad (3.8)$$

In a similar fashion each fermion field can be decomposed into a real part which is antisymmetric tensor in the frame indices and transforms as an adjoint under gauge transformations and an imaginary part which is vector in the local frame and which gauge transforms in a similar fashion to the vierbein eg.

$$\psi_\mu = \psi_\mu{}^{ab}(x) \sigma^{ab} + i \psi_\mu{}^a(x) \gamma^a \qquad (3.9)$$

Complexified covariant derivatives acting on such fields also inherit this same structure.

The interpretation of this Yang-Mills system as a gravitational theory is further strengthened by examining the first of the three complex Yang-Mills Bianchi identities given in eqn. 2.6. It is straightforward to show that this leads to the usual two Bianchi identities associated with a gravitational theory

$$\epsilon^{\rho\lambda\mu\nu} D_\lambda R_{\mu\nu} = 0$$
$$\epsilon^{\rho\lambda\mu\nu} [e_\lambda, R_{\mu\nu}] + \epsilon^{\rho\lambda\mu\nu} D_\lambda T_{\mu\nu} = 0 \qquad (3.10)$$

where the covariant derivatives appearing in the above expressions contain only the spin connection and we have introduced the torsion $T_{\mu\nu} = D_{[\mu} e_{\nu]}$.

Thus the original Yang-Mills model modified to incorporate the generalized spin connection given in eqn. 3.1 contains all the ingredients of a gravitational theory. The theory contains a spin connection and a vierbein which satisfy the gravitational Bianchi identities and an action that is invariant under both local (Euclidean) Lorentz rotations and general coordinate transformations with respect to the background metric $h_{\mu\nu}$. Furthermore, the topological nature of the twisted construction ensures that the partition function and any $\mathcal{Q}$-invariant observables are independent of the choice of this background metric. The twisted theory is necessarily supersymmetric which ensures that the theory possesses additional twisted anticommuting fields which serve as superpartners for both the spin connection and vierbein.

Further evidence supporting this gravitational interpretation can be seen by examining the moduli space of the theory. The first two vacuum equations given in eqn. 2.11 become

now[5]

$$R_{\mu\nu}{}^{ab} - \frac{2}{l_G^2}\left(e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a\right) = 0$$
$$T_{\mu\nu}{}^a = D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \qquad (3.11)$$

The second relation sets the torsion $T_{\mu\nu}{}^a$ to zero. This together with the metricity condition $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ is sufficient to ensure that the (ω_μ, e_μ) system can reconstruct a Riemannian geometry. Actually we must be careful here; the interpretation of $e_\mu{}^a$ as a vierbein strictly requires that the inverse matrix $\hat{e}^\mu{}_a$ exist with

$$\hat{e}^\mu{}_a e_\mu{}^b = \delta_a^b \qquad (3.12)$$

If this is the case we can recast the first of these equations as

$$R = \hat{e}^\mu{}_a \hat{e}^\nu{}_b R_{\mu\nu}{}^{ab} = \frac{24}{l_G^2} \qquad (3.13)$$

Thus the moduli space corresponds to four dimensional Euclidean metrics with constant positive curvature; that is the sphere S^4 with radius l_G. Actually this conclusion is strictly valid only for vacuum solutions in which the scalars $\phi, \overline{\phi}$ have vanishing expectation values which may not exhaust all possible classical vacua of the system [6]. Finally, notice that sphere S^4 is nothing more than the coset space $SO(5)/SO(4)$ - that is the space invariant under global $SO(5)$ transformations modulo local $SO(4)$ rotations.

In general a metric is determined from the vielbeins via the relation

$$g_{\mu\nu} = -e_\mu{}^a e_\nu{}^a = -\mathrm{Tr}\,(e_\mu e_\nu) \qquad (3.14)$$

where the unconventional minus sign reflects our use of the frame metric $(-1,-1,-1,-1)$. Thus the moduli space equations imply that this latter Lorentz invariant operator develops an expectation value in the classical limit corresponding to the appearance of an emergent metric. Notice that while the vacuum energy of the system is always zero because of supersymmetry the classical limit of the theory nonetheless leads to a space with non-zero curvature.

It is likely that at strong coupling the expectation value of $g_{\mu\nu}$ is zero and the theory is best described in terms of a strongly coupled Yang-Mills system of propagating vierbeins and spin connections together with their superpartners as in eqn. 2.12[7].

---

[5]It is not clear what is the correct gravitational interpretation of the 3rd equation $D^\mu e_\mu = 0$. Its primary role appears to act as a gauge fixing term to fix the local translational invariance corresponding to the imaginary part of the connection. It also may play a role in selecting out only homogeneous solutions to the vacuum equations

[6]Note, though, that this model does not possess the continuum of flat directions found in the usual $U(N)$ theories

[7]Notice that the the theory as written contains no analog of the Einstein-Hilbert term which is linear in the curvature. This fact decouples the magnitude of any effective Newton constant arising from eg supersymmetry breaking from the scale l_G. Such a term could be added by augmenting the original action by a topological term built from the complex curvature $\mathcal{R}_{\mu\nu}$ of the form $\int d^4x\,\epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\mathcal{R}_{\mu\nu}{}^{ab}\mathcal{R}_{\rho\lambda}{}^{cd}$

4. Lorentzian signature

The generalization to background metrics and tangent spaces with Lorentz signature is straightforward – the gamma matrices are taken Lorentzian and the algebra eqn. 3.4 replaced with

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$$
$$\left[\sigma^{ab}, \gamma^c\right] = \eta^{ca}\gamma^b - \eta^{cb}\gamma^a \tag{4.1}$$

where $\eta^{ab} = \frac{1}{2}\{\gamma^a, \gamma^b\}$ corresponds to the Lorentz metric $(+1, -1, -1, -1)$. The algebra of the generalized spin connection is now isomorphic to the algebra of the group $SO(2,3)$ - the isometries of four dimensional anti-de Sitter space. However, as before, only the Lorentz subgroup $SO(1,3)$ is gauged in the theory. This generalized connection is essentially the same as that used in [9]. The theory described here differs from those earlier constructions by virtue of the twisted supersymmetry which allows us to contract indices using a background metric while maintaining background independence of operators which are invariant under the twisted supersymmetry.

To create expressions invariant under this Lorentz symmetry we need to raise and lower tangent space indices with the metric η_{ab}. For example, the covariant derivative of the vierbein and curvature become

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b$$
$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b \tag{4.2}$$

while the metric is now given by

$$g_{\mu\nu} = e_\mu{}^a e_{a\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b \tag{4.3}$$

The action can be rendered locally Lorentz invariant by ensuring that all tangent space indices are contracted using η_{ab} in the usual way. Again, the torsion vanishes on the moduli space and the spin connection and vierbein combine to yield a constant curvature space. In this case the metric describes the coset space $SO(2,3)/SO(1,3)$ or four dimensional anti-de Sitter space with curvature $\frac{24}{l_G^2}$.

5. Five dimensional AdS

As was shown in [7] the Marcus twist of $\mathcal{N} = 4$ Yang-Mills is most succinctly written as the dimensional reduction of a five dimensional gauge theory comprising a complex gauge field $\mathcal{A}_\mu, \mu = 1\ldots 5$ and a multiplet of twisted fermions $(\eta, \psi_\mu, \chi_{\mu\nu})$. The five dimensional action takes the form

$$S = \frac{1}{g_5^2}\left(\mathcal{Q}\int d^5x \sqrt{h}\Lambda - \frac{1}{4}\int d^5x \ \text{Tr} \ \epsilon^{\mu\nu\rho\lambda\delta}\chi_{\mu\nu}\overline{\mathcal{D}}_\rho\chi_{\lambda\delta}\right) \tag{5.1}$$

where the $\mathcal{Q}$-exact term is given by

$$\Lambda = \int \mathrm{Tr} \left(\chi^{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}^{\mu}, \mathcal{D}_{\mu}] - \frac{1}{2}\eta d \right) \tag{5.2}$$

where $\mathcal{F}_{\mu\nu}$ is the complex Yang-Mills curvature associated to the five dimensional complex connection $\mathcal{A}_{\mu}$ and we have introduced now a five dimensional background metric $h_{\mu\nu}(x)$. As before the nilpotent supersymmetry acts on the fields as follows

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_{\mu} &= \psi_{\mu} \\
\mathcal{Q}\,\psi_{\mu} &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_{\mu} &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned} \tag{5.3}$$

The second $\mathcal{Q}$-closed term is supersymmetric on account of the generalized Bianchi identity

$$\epsilon^{\mu\nu\rho\lambda\delta}\mathcal{D}_{\rho}\mathcal{F}_{\lambda\delta} = 0 \tag{5.4}$$

The definitions of the complexified covariant derivatives follow those of the four dimensional theory.

The transition to a theory of gravity is performed as before; we expand the connections not on the Lie algebra of $U(N)$ but that of the *five dimensional* anti-de Sitter algebra $SO(2,4)$[8]. Again this algebra separates into a piece corresponding to the five dimensional Lorentz group $SO(1,4)$ which is gauged and a translational component representing the 5-bein. Again, the complexified Bianchi identities yield the two Bianchi identities of a gravitational theory. And once again the expectation values of topological observables are independent of smooth deformations of the background metric $h_{\mu\nu}$.

The classical vacua are now given simply by the conditions

$$\begin{aligned}
R_{\mu\nu}{}^{ab} - \frac{2}{l_G^2}\left(e_{\mu}{}^a e_{\nu}{}^b - e_{\mu}{}^b e_{\nu}{}^a\right) &= 0 \\
T_{\mu\nu}{}^a = D_{\mu}e_{\nu}{}^a - D_{\nu}e_{\mu}{}^a &= 0 \\
D^{\mu}e_{\mu}{}^a &= 0
\end{aligned} \tag{5.5}$$

where $R_{\mu\nu}$ is the curvature associated to the spin connection ω_{μ} and e_{μ} is the 5-bein. As for four dimensions these equations can be interpreted as yielding five dimensional anti-de Sitter space with radius l_G.

The fermions appearing in this construction bear some resemblance to those of five dimensional $\mathcal{N}=4$ supergravity [10]. For example, the theory contains a fermionic superpartner

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[8]As for four dimensions we may also use the algebra $so(6)$ which will lead to a S^5 vacuum state

of the generalized spin connection ψ_μ whose component fields $\psi_\mu{}^{ab}, \psi_\mu{}^a$ can be thought of as resulting from the four degenerate gravitinos expected in $\mathcal{N} = 4$ supergravity after twisting the frame indices with the associated flavor symmetry. Similarly, the fermions η^a, η^{ab} can be mapped into the four spinors of such a theory after the same twisting of tangent space indices with $\mathcal{N} = 4$ flavor indices. It is also interesting to observe that the *global* background $\mathcal{Q}$ supersymmetry transforms the 5-bein into the twisted gravitino as required by *local* supersymmetry [2].

6. Lattice construction

The five dimensional construction described above is the basis for recent efforts to derive lattice actions for $\mathcal{N} = 4$ super Yang-Mills theory which retain an exact supersymmetry at non zero lattice spacing. For completeness we summarize this construction here specializing where necessary to the gravitational case. For more details the reader is referred to [7, 11, 12].

Clearly to make contact with a twist of $\mathcal{N} = 4$ in four dimensions we must dimensionally reduce this theory along the 5th direction. This will yield the complex scalar $\phi = A_5 + iB_5$ and its superpartner $\overline{\eta}$ that appeared in the continuum construction given in section 2. Similarly the 10 five dimensional fermions $\chi_{\mu\nu}, \mu = 1 \ldots 5$ naturally decompose into the 6 fields of a 2-form in four dimensions and the vector $\overline{\psi}_\mu, \mu = 1 \ldots 4$. However the lattice action is most simply expressed in language of the five dimensional theory and we will hence use that notation in what follows - thus all indices will be taken as varying in the range $\mu, \nu = 1 \ldots 5$.

The transition to the lattice from the continuum theory requires a number of steps. The first, and most important, is to replace the continuum generalized spin connection $\mathcal{A}_\mu(x)$ which now takes its value in the four dimensional anti-de Sitter algebra by a set of appropriate complexified Wilson link fields which are the exponentials of these fields $\mathcal{U}_\mu(x) = e^{\mathcal{A}_\mu(x)}, \mu = 1 \ldots 4$. These lattice fields are taken to be associated with unit length vectors in the coordinate directions $\boldsymbol{\mu}$ in an abstract four dimensional hypercubic lattice. In contrast the continuum scalar field $\mathcal{A}_5$ takes its values only in the algebra and is placed on the body diagonal $\boldsymbol{\mu}_5 = (-1, -1, -1, -1)$ of the unit hypercube[9]. Notice that the sum of these five basis vectors is zero. This is very important for gauge invariance of the lattice model. By supersymmetry the fermion fields $\psi_\mu, \mu = 1 \ldots 5$ lie on the same link as their bosonic superpartners. In contrast the scalar fermion η is associated with the sites of the lattice and the tensor fermions $\chi_{\mu\nu}, \mu = 1 \ldots 5$ with the link running from $\boldsymbol{\mu} + \boldsymbol{\nu}$ down to the origin of the hypercube (remember that the set of vectors $\boldsymbol{\mu}, \boldsymbol{\nu}$ runs over the unit vectors in the coordinate directions together with the body diagonal $\boldsymbol{\mu}_5$). These are the same link assignments that occur in the lattice constructions of sixteen supercharge $U(N)$ theories [7, 11, 12].

The construction then posits that all link fields transform as bifundamental fields under

---

[9]A more symmetrical treatment of the scalar field is also possible and leads to an A_4^* lattice - see [7, 11]

gauge transformations

$$
\begin{aligned}
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G(\mathbf{x}+\boldsymbol{\mu}) \\
\chi_{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\boldsymbol{\mu}+\boldsymbol{\nu})\chi_{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\boldsymbol{\mu}) \\
\overline{\mathcal{U}}_\mu(\mathbf{x}) &\to G(\mathbf{x}+\boldsymbol{\mu})\overline{\mathcal{U}}_\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned}
\tag{6.1}
$$

The supersymmetric and gauge invariant lattice action which corresponds to eqn. 2.1 then takes a very similar form to its continuum counterpart in eqn. 5.1

$$
S = \frac{1}{g^2}\left(\mathcal{Q}\sum_x \Lambda - \frac{1}{4}\sum_x \ \mathrm{Tr} \ \epsilon^{\mu\nu\rho\lambda\delta}\chi_{\mu\nu}\overline{\mathcal{D}}_\rho^{(-)}\chi_{\lambda\delta} \right)
\tag{6.2}
$$

where the $\mathcal{Q}$-exact term is given by

$$
\Lambda = \mathrm{Tr} \ \left(\chi_{\mu\nu}\mathcal{R}_{\mu\nu} + \eta\overline{\mathcal{D}}_\mu^{(-)}\mathcal{U}_\mu - \frac{1}{2}\eta d \right)
\tag{6.3}
$$

and the form of the supersymmetry transformations are the same as the continuum theory

$$
\begin{aligned}
\mathcal{Q}\,\mathcal{U}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{U}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= \mathcal{R}_{\mu\nu}^\dagger \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}
\tag{6.4}
$$

The lattice field strength is given by

$$
\mathcal{R}_{\mu\nu} = \mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})\mathcal{U}_\nu(\mathbf{x}+\boldsymbol{\mu}) - \mathcal{U}_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\nu})
\tag{6.5}
$$

This lattice action is invariant under local $SO(1,3)$ gauge transformations. It is also free of fermion doubling problems – see the discussion in [7]. The covariant lattice difference operators appearing in these expressions are defined by

$$
\mathcal{D}_\mu^{(+)}f_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})f_\nu(\mathbf{x}+\boldsymbol{\mu}) - f_\nu(\mathbf{x})U_\mu(\mathbf{x}+\boldsymbol{\mu})
\tag{6.6}
$$
$$
\mathcal{D}_\mu^{(-)}f_\mu(\mathbf{x}) = f_\mu(\mathbf{x})\overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(x-\boldsymbol{\mu})f_\mu(\mathbf{x}-\boldsymbol{\mu})
\tag{6.7}
$$

Notice that $\mathcal{D}^{(+)}$ acts like an exterior derivative and promotes a lattice p-form to a $(p+1)$-form as evidenced by its gauge transformation property. Similarly $\mathcal{D}^{(-)}$ maps a lattice p-form to a $(p-1)$-form corresponding to the adjoint of the exterior derivative. Finally, the derivative appearing in the $\mathcal{Q}$-closed term is given by

$$
\overline{\mathcal{D}}_\rho^{(-)}\chi_{\mu\nu}(\mathbf{x}) = \chi_{\mu\nu}(\mathbf{x})\overline{\mathcal{U}}_\rho(\mathbf{x}-\boldsymbol{\rho}) - \overline{\mathcal{U}}_\rho(\mathbf{x}+\boldsymbol{\mu}+\boldsymbol{\nu}-\boldsymbol{\rho})\chi_{\mu\nu}(\mathbf{x}-\boldsymbol{\rho})
\tag{6.8}
$$

It is straightforward to show that with this definition the complex Yang-Mills Bianchi identity is satisfied *exactly* in the lattice theory [7].

Notice that this lattice theory bears some similarity to spin foam formulations of loop quantum gravity [13]. What is different is that the underlying lattice structure is fixed corresponding to an implicit choice of a flat background metric and the Wilson loop variables take values in the anti-de Sitter group with only the Lorentz subgroup being gauged.

7. Discussion

In this paper we have constructed a topological gravity theory of Yang-Mills type by modifying the Marcus twist of $\mathcal{N} = 4$. The key to the construction is to embed the spin connection and vierbein into a complexified connection whose generators taken together span the anti-de Sitter algebra. However only the Lorentz subalgebra is gauged. The complexified Yang-Mills Bianchi identities then yield the usual two Bianchi identities of the gravitational theory and the classical vacua are shown to correspond to anti-de Sitter space. It is only in this classical limit that a metric tensor can be clearly identified.

Away from the classical limit the torsion is non-zero, the spin connection and vierbein are independent, interacting fields and the metric tensor constructed from the vierbeins most probably has a vanishing expectation value.

The topological construction ensures that the partition function of the model is independent of the background metric and indeed may be computed exactly in the semi-classical approximation. Furthermore, the twisted supersymmetry guarantees that the vacuum energy is zero while the curvature of the space, being a parameter of the moduli space, should suffer no quantum corrections.

The appearance in this model of a moduli space corresponding to flat connections is similar to previous constructions of two dimensional topological supergravity [14]. It is also likely that the theory described here is related to a supersymmetric extension of BF-theory such as that proposed in [15, 16, 17]. In these constructions the topological terms are supplemented by additional cubic interactions which ensure the theory reduces to General Relativity in an appropriate limit. Further evidence in favor of this comes from the work of Blau and Thompson who showed that the twisted $\mathcal{N} = 4$ theory considered here with $U(N)$ gauge group could be obtained as a deformation of super BF theory [18].

While $\mathcal{Q}$-invariant observables in this continuum theory are background independent this will not be true of general gauge invariant operators. This appears to be a barrier preventing a non-topological gravity interpretation of the theory. One way to proceed is to simply give up on the requirement of general background independence and consider the Yang-Mills theory as defined on a flat base space. The gravitational interpretation of the theory then rests upon the observation that the bosonic fields of the model satisfy the gravitational Bianchi identities and that the vacua can be interpreted as a classical spacetime. It would be very interesting to see whether small fluctuations of the coupled vierbein/spin connection system around this vacuum state could be interpreted in terms of fluctuations of an emergent metric

tensor and, in particular, how to reconcile such a picture with the Weinberg-Witten theorem [19]. Notice that the latter theorem, which prohibits the appearance of a composite massless graviton, requires the existence of a Poincaré covariant stress energy tensor. In the theory described here any gravitational fluctuations occur relative to an AdS spacetime and may yield an emergent graviton with an effective mass determined by the scale l_G thereby evading the theorem.

In the case of a flat background we have also written down a lattice theory which, in the naive continuum limit, approximates this continuum theory arbitrarily well. This lattice theory is both gauge invariant and preserves the scalar supersymmetry and it can be thought of as yielding a non-perturbative formulation of the theory suitable for numerical simulations [20].

Acknowledgments

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