10-20-2010

Three Dimensional Lattice Gravity as Supersymmetric Yang-Mills Theory

Simon Catterall
Syracuse University

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Simon Catterall[*,†]
Department of Physics, Syracuse University, Syracuse NY 13244
E-mail: smcatterall@gmail.com

We argue that a certain twisted supersymmetric Yang-Mills theory in three dimensions with gauge group $SU(2)$ possesses a set of topological observables whose expectation values can be computed in a related Chern Simons theory. This Chern Simons theory has been proposed as a definition of three dimensional Euclidean quantum gravity. Since the YM theory admits a discretization which preserves the values of topological observables we conjecture that it can be used as a non-perturbative definition of the quantum gravity theory.

The XXVIII International Symposium on Lattice Field Theory, Lattice2010
June 14-19, 2010
Villasimius, Italy

[*] Speaker.
[†] Supported in part by DOE grant DE-FG02-85ER40237

1. Introduction

In recent years there has been a resurgence of interest in the problem of formulating supersymmetric lattice theories. This has largely been driven by the realization that in certain classes of theory a discretization can be derived which preserves a subset of the continuum supersymmetry - see for example the recent review [1] and references therein.

For our purposes in this paper the key idea that lies behind these new constructions is that of *topological twisting* – one discretizes not the original supersymmetric theory but a theory formulated in terms a new set of *twisted* variables. This twisting procedure has the merit of exposing a nilpotent, scalar supercharge which is compatible with an underlying lattice structure - the associated fermionic symmetry holds exactly on the lattice and leads to a number of remarkable properties; the vacuum energy is zero to all orders in the lattice spacing, the boson-fermion spectrum is degenerate and the lattice theory contains cut-off independent observables whose expectation values can be computed exactly and which are independent of the coupling constant. These latter observables correspond to a topological sector of the continuum theory; a set of observables whose expectation values are independent of smooth deformations in the background space. This independence on the background metric is also common to certain Chern Simons theories and specifically can be used to write down background independent formulations of topological quantum gravity [2, 3, 4].

The three dimensional Chern Simons gravitational theory has been well studied and has been shown to correspond to the usual Einstein-Hilbert theory [5]. In this talk we show that the relevant gravitational observables appearing in this three dimensional Chern Simons theory can be computed in an associated twisted Yang-Mills theory in three dimensions. Furthermore, the lattice theory that targets this continuum twisted theory can be discretized and hence can serve as a non-perturbative definition of the quantum gravity theory.

In this talk we will first review the Chern Simons formulation of (Euclidean) quantum gravity and then go on to show that the relevant topological observables can be computed in an equivalent continuum twisted Yang-Mills theory with $\mathcal{N}=4$ supersymmetry. We then go to review the construction of a lattice theory which targets this twisted YM theory in the naive continuum limit.

2. Chern-Simons formulation of three dimensional gravity

It has been known for a long time that three dimensional gravity can be reformulated in the language of gauge theory [2, 3]. The construction employs a Chern-Simons action and in Euclidean space the relevant local symmetry corresponds to the group $SO(3,1) \sim SL(2,C)$ - the complexified $SU(2)$ group. Notice that this includes both local Lorentz rotations and translations. Furthermore the resulting theory is topological and global gravitational observables are determined by integrals over the moduli space of flat $SL(2,C)$ connections. We show these properties explicitly below.

Consider the following three dimensional Chern-Simons action

$$\int d^3x \varepsilon_{\mu\nu\lambda} \hat{\mathrm{Tr}} \left(A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu \left[A_\nu, A_\lambda\right] \right) \tag{2.1}$$

Furthermore, assume that the gauge field A_μ takes values in the adjoint representation of the group $SO(3,1)$. A convenient representation for the six generators of the Lie algebra of this group is then given by commutators of the (Lorentzian) Dirac matrices $\gamma^{AB} = \frac{1}{4}\left[\gamma^A, \gamma^B\right]$ where $(\gamma^a)^\dagger = \gamma^a$ $a = 1\ldots3$ and $\left(\gamma^4\right)^\dagger = -\gamma^4$ This yields an expression for the gauge field of the form

$$A_\mu = \sum_{A<B} A_\mu^{AB}\gamma^{AB} \quad A,B = 1\ldots4 \tag{2.2}$$

Finally, the group indices are contracted using the invariant tensor ε_{ABCD} corresponding to a trace of the form

$$\hat{\mathrm{Tr}}(X) = \mathrm{Tr}(\gamma_5 X) \tag{2.3}$$

To see explicitly that the resulting theory is just three dimensional gravity we decompose the gauge field and field strength in terms of an $SO(3)$ subgroup

$$\begin{aligned} A_\mu &= \sum_{a<b} \omega_\mu^{ab}\gamma^{ab} + \frac{1}{l}e_\mu^a\gamma^{4a} \quad a,b = 1\ldots3 \\ F_{\mu\nu}^{ab} &= \sum_{a<b}\left(R_{\mu\nu}^{ab} + \frac{1}{l^2}e^a_{[\mu}\,e^b_{\nu]}\right)\gamma^{ab} \\ F_{\mu\nu}^a &= \sum_a D_{[\mu}\,e^a_{\nu]} \end{aligned} \tag{2.4}$$

The covariant derivative appearing in the field strength contains just the $SO(3)$ gauge field ω_μ and we have introduced a explicit length scale l into the definition of the gauge fields e_μ. After substituting into the Chern-Simons action one recognizes that it corresponds to three dimensional Einstein-Hilbert gravity including a cosmological constant and written in the first order tetrad-Palatini formalism [2].

$$S_{\mathrm{EH}} = \frac{1}{l}\int \varepsilon^{\mu\nu\lambda}\varepsilon_{abc}\left(e_\mu^a R_{\mu\nu}^{bc} - \frac{1}{3l^2}e_\mu^a e_\nu^b e_\lambda^c\right) \tag{2.5}$$

with ω_μ and e_μ corresponding to the spin connection and dreibein and $1/l^2$ playing the role of a cosmological constant.

To see that this theory is classically equivalent to the usual metric theory of gravity one merely has to notice that the classical equations of the Chern-Simons theory require that the $SO(3,1)$ field strength vanish $F_{\mu\nu}^{AB} = 0$. One consequence of this is that the torsion $T = F^{4a} = D_{[\mu}\,e_{\nu]}$ must vanish. This condition allows one to solve for the spin connection as a function of the dreibein $\omega = \omega(e)$ and ensures that the curvature appearing in eqn. 2.5 is indeed nothing more than the usual Riemann curvature associated to a torsion free connection determined by the metric $g_{\mu\nu} = e_\mu^a e_\nu^a$.

Indeed the condition $F_{\mu\nu} = 0$ also requires that the $SO(3)$ curvature $R_{\mu\nu}$ must take the constant value $-1/l^2$. Such a solution corresponds (at least locally) to hyperbolic three space $H^3 \sim SO(3,1)/SO(3)$ which is the correct solution to a metric theory of three dimensional Euclidean quantum gravity in the presence of a cosmological constant.

Finally one can show that the theory restricted to this space of flat connections is also invariant under diffeomorphisms [2] at least on shell. This result follows from the fact that one can express a general coordinate transformation on A_μ with parameter $-\xi^\nu$ as a gauge transformation with parameter $\xi^\mu A_\mu$ plus a term which vanishes on flat connections.

$$\delta A_\mu^\xi = -D_\mu(\xi^\nu A_\nu) - \xi^\nu F_{\mu\nu} \tag{2.6}$$

The classical solutions of this Chern Simons theory thus correspond to the space of flat $SL(2,C)$ (or equivalently $SO(3,1)$) connections up to $SL(2,C)$ gauge transformations. One might ask how such a situation can correspond to gravity since this solution does not seem to allow for the propagation of any local degrees of freedom. In fact, it is well known that three dimensional gravity possesses no gravitons so this result should not be too surprising [5].

However, notice that there may still be non-local observables in such a theory corresponding to the expectation value of Wilson lines of the $SL(2,C)$ group which cannot be shrunk to a point.

Since the expectation values of such Wilson lines cannot depend on any background metric introduced to construct the Chern Simons theory they are explicitly topological in nature. It is important to stress that *physical spacetime* is not determined by the choice of background metric but instead determined dynamically by the values of the dreibeins.

We now turn to a different theory based on a twisted supersymmetric Yang-Mills which will turn out to possess the same vacuum state and the same set of topological observables whose vacuum expectation values will also be background independent.

3. Twisted $\mathcal{N}=4$ gauge theory in three dimensions

The twist of $\mathcal{N}=4$ super Yang-Mills that we are interested in can be most succinctly written in the form where

$$S = \frac{1}{g^2} Q \int d^3x \sqrt{h} \left(\chi^{\mu\nu} \mathcal{F}_{\mu\nu} + \eta \left[\overline{\mathcal{D}}^{\mu}, \mathcal{D}_{\mu} \right] + \frac{1}{2} \eta d + B_{\mu\nu\lambda} \overline{\mathcal{D}}^{\lambda} \chi^{\mu\nu} \right) \tag{3.1}$$

The fermions comprise a multiplet of p-form fields $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\lambda})$[1] where in three dimensions $p = 0 \ldots 3$. This multiplet of twisted fermions corresponds to a single Kähler-Dirac field and here possesses eight single component fields as expected for a theory with $\mathcal{N}=4$ supersymmetry in three dimensions.

The imaginary parts of the complex gauge field $\mathcal{A}_\mu$ $\mu = 1 \ldots 3$ appearing in this construction yield the three scalar fields of the conventional (untwisted) theory. Fields d and $B_{\mu\nu\lambda}$ are auxiliaries introduced to render the scalar nilpotent supersymmetry Q nilpotent off shell. The latter acts on the twisted fields as follows

$$\begin{aligned}
Q\mathcal{A}_\mu &= \psi_\mu \\
Q\overline{\mathcal{A}}^\mu &= 0 \\
Q\psi_\mu &= 0 \\
Q\chi^{\mu\nu} &= \overline{\mathcal{F}}^{\mu\nu} \\
Q\eta &= d \\
Qd &= 0 \\
QB_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \qquad (3.2) \\
Q\theta_{\mu\nu\lambda} &= 0 \qquad (3.3)
\end{aligned}$$

[1] It is common in the continuum literature to replace the 2 and 3 form fields in these expressions by their Hodge duals; a second vector $\hat{\psi}_\mu$ and scalar $\hat{\eta}$ see, for example [6]

where the background metric $h_{\mu\nu}$ is used to raise and lower indices in the usual manner $\chi^{\mu\nu} = h^{\mu\alpha}h^{\nu\beta}\chi_{\alpha\beta}$ and is a Q-singlet. The topological character of the theory follows from the Q-exact structure of S[2]

The complex covariant derivatives appearing in these expressions are defined by

$$\begin{aligned}\mathscr{D}_\mu &= \partial_\mu + \mathscr{A}_\mu = \partial_\mu + A_\mu + iB_\mu \\ \overline{\mathscr{D}}^\mu &= \partial^\mu + \overline{\mathscr{A}}^\mu = \partial^\mu + A^\mu - iB^\mu\end{aligned} \tag{3.4}$$

while all fields take values in the adjoint representation of $SU(N)$[3]. It should be noted that despite the appearance of a complexified connection and field strength the theory possesses only the usual $SU(N)$ gauge invariance corresponding to the real part of the gauge field. In our current application we will need to consider only the case where the group is $SU(2)$ although the topological character of this theory holds for any value of N.

The structure of this twisted theory is similar to that of the Marcus twist of $\mathscr{N}=4$ super Yang-Mills in four dimensions [9, 10, 11] which plays an important role in the Geometric-Langlands program [12].

Doing the Q-variation, integrating out the field d and using the Bianchi identity

$$\varepsilon_{\mu\nu\lambda}\overline{\mathscr{D}}^\lambda\overline{\mathscr{F}}^{\mu\nu} = 0 \tag{3.5}$$

yields

$$S = \frac{1}{g^2}\int d^3x\,\sqrt{h}\,(L_1 + L_2) \tag{3.6}$$

where

$$\begin{aligned}L_1 &= \mathrm{Tr}\left(-\overline{\mathscr{F}}^{\mu\nu}\mathscr{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathscr{D}}^\mu, \mathscr{D}_\mu]^2)\right) \\ L_2 &= \mathrm{Tr}\left(-\chi^{\mu\nu}\mathscr{D}_{[\mu}\,\psi_{\nu]} - \psi_\mu\overline{\mathscr{D}}^\mu\eta - \theta_{\mu\nu\lambda}\overline{\mathscr{D}}^{[\lambda}\,\chi^{\mu\nu]}\right)\end{aligned} \tag{3.7}$$

The terms appearing in L_1 can then be written

$$\begin{aligned}\overline{\mathscr{F}}^{\mu\nu}\mathscr{F}_{\mu\nu} &= (F_{\mu\nu} - [B_\mu, B_\nu])(F^{\mu\nu} - [B^\mu, B^\nu]) + (D_{[\mu}\,B_{\nu]})(D^{[\mu}\,B^{\nu]}) \\ \frac{1}{2}\left[\overline{\mathscr{D}}^\mu, \mathscr{D}_\mu\right]^2 &= -2\left(D^\mu B_\mu\right)^2\end{aligned} \tag{3.8}$$

where $F_{\mu\nu}$ and D_μ denote the usual field strength and covariant derivative depending on the real part of the connection A_μ. The classical vacua of this theory correspond to solutions of the equations

$$\begin{aligned}F_{\mu\nu} - [B_\mu, B_\nu] &= 0 \\ D_{[\mu}\,B_{\nu]} &= 0 \\ D^\mu B_\mu &= 0\end{aligned} \tag{3.9}$$

The same moduli space arises in the study of the Marcus twist of four dimensional $\mathscr{N}=4$ Yang-Mills where it is argued to correspond to the space of flat *complexified* connections modulo *complex*

[2]Notice that this construction differs slightly from the one discussed in [8] in that one of the fermion terms is here trivially rewritten as a Q-exact rather than Q-closed form. This does not affect any of the subsequent arguments

[3]The generators are taken to be *anti-hermitian* matrices satisfying Tr $(T^aT^b) = -\delta^{ab}$

gauge transformations. A simple way to understand this is to recognize that the additional term $D^\mu B_\mu = 0$ appearing in the vacuum equations 3.9 resembles a partial gauge fixing of a theory with a complexified gauge invariance and associated gauge fields A_μ and B_μ down to a theory possessing just the usual $SU(N)$ (here $N = 2$) gauge invariance implemented via the gauge field A_μ.

More specifically, Marcus showed in [9] that the solutions of eqns. 3.9 modulo $SU(2)$ gauge transformations are in one to one correspondence with the space of flat complexified $SU(2) = SL(2,C)$ connections modulo complexified gauge transformations. These arguments should hold in the three dimensional case too.

The topological character of the theory then guarantees that any Q-invariant observable such as the partition function can be evaluated *exactly* by considering only Gaussian fluctuations about such vacuum configurations. Furthermore, it is easy to see from eqn. 3.1 that the energy momentum tensor of this theory is Q-exact rendering the expectation values of such topological observables independent of smooth deformations of the background metric $h_{\mu\nu}(x)$.

Most importantly notice that the fact that the complexified connection $\overline{\mathscr{A}}_\mu$ is a Q-singlet allows us to trivially construct a class of topological observables corresponding to the trace of an associated Wilson loop.

$$O(\gamma) = \mathscr{P} e^{\int_\gamma \overline{\mathscr{A}}^\mu . dx_\mu} \tag{3.10}$$

If this loop is non-contractible (for example if it winds around a cycle of a torus) we obtain the same non-trivial global topological observable we encountered in the Chern-Simons construction of three dimensional Euclidean gravity.

We can cement this connection between the twisted Yang-Mills theory and the gravitational theory by identifying the imaginary parts of the $SL(2,C)$ connection - the field B_μ occurring in the Yang-Mills theory - with the matrix valued field e_μ occurring the tetrad-Palatini action. Notice that the fact that the field B_μ transforms in the adjoint representation of the $SU(2)$ gauge group translates in the gravitational theory to the statement that the dreibein e^a_μ transforms as a *vector* under local Lorentz transformations just as it should.

These considerations together with the equivalence of the topological sectors of these two theories leads us to conjecture that the twisted two color Yang-Mills gives an alternative representation of the gravity theory. Furthermore, this alternative representation has some advantages – the path integral is now well defined and indeed may be given a non-perturbative definition as the appropriate limit of a gauge and supersymmetric invariant lattice model to which we now turn.

4. Lattice theory

The twisted theory described in the previous section may be discretized using the techniques developed in [11, 13, 14]. The resultant lattice theories have the merit of preserving both gauge invariance and the scalar component of the twisted supersymmetry. Here we show how to derive this lattice theory by direct discretization of the continuum twisted theory. We will start by assuming that the continuum theory is formulated in flat (Euclidean) space with metric $h_{\mu\nu} = \delta_{\mu\nu}$.

In the case of topological observables the choice of metric is unimportant and hence the lattice theory we construct will yield expectation values for topological operators which depend only on the topology of the lattice and not on the coupling, lattice spacing or the fact that we started by discretization of a theory in a flat background.

The transition to the lattice from the continuum theory requires a number of steps. The first, and most important, is to replace the continuum complex gauge field $\mathcal{A}_\mu(x)$ at every point by an appropriate complexified Wilson link $\mathcal{U}_\mu(\mathbf{x}) = e^{\mathcal{A}_\mu(\mathbf{x})}, \mu = 1\ldots 3$. These lattice fields are taken to be associated with unit length vectors in the coordinate directions μ in an abstract three dimensional hypercubic lattice. By supersymmetry the fermion fields $\psi_\mu(\mathbf{x}), \mu = 1\ldots 3$ lie on the same oriented link as their bosonic superpartners running from $\mathbf{x} \to \mathbf{x}+\mu$. In contrast the scalar fermion $\eta(\mathbf{x})$ is associated with the site $\mathbf{x}$ of the lattice and the tensor fermions $\chi^{\mu\nu}(\mathbf{x}), \mu < \nu = 1\ldots 3$ with a set of diagonal face links running from $\mathbf{x}+\mu+\mu \to \mathbf{x}$. The final 3 form field $\theta_{\mu\nu\lambda}(\mathbf{x})$ is then naturally placed on the body diagonal running from $\mathbf{x} \to \mathbf{x}+\mu+\mu+\lambda$. The construction then posits that all link fields transform as bifundamental fields under gauge transformations

$$
\begin{aligned}
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G(\mathbf{x}+\mu) \\
\chi^{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\mu+\mu)\chi^{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu) \\
\overline{\mathcal{U}}^\mu(\mathbf{x}) &\to G(\mathbf{x}+\mu)\overline{\mathcal{U}}^\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned}
\tag{4.1}
$$

Notice that we can keep track of the orientation of the lattice field by following its continuum index structure – upper index fields are placed on negatively orientated links, lower index fields live on positively oriented links.

The action of the scalar supersymmetry on these fields is given by the continuum expression in eqn. 3.3 with the one modification that the continuum field $\mathcal{A}_\mu(x)$ is replaced with the Wilson link $\mathcal{U}_\mu(x)$ and the lattice field strength being defined as $\mathcal{F}_{\mu\nu} = \mathcal{D}^{(+)}_\mu U_\nu$. The supersymmetric and gauge invariant lattice action which corresponds to eqn. 3.7 then takes a very similar form to its continuum counterpart

$$
\begin{aligned}
S_1 &= Q\sum_{\mathbf{x}}\left(\chi^{\mu\nu}\mathcal{F}_{\mu\nu} + \eta\left[\overline{\mathcal{D}}^{(-)\mu}\mathcal{U}_\mu\right] + \frac{1}{2}\eta d\right) \\
S_2 &= \sum_{\mathbf{x}}\theta_{\mu\nu\lambda}\overline{\mathcal{D}}^{(+)\lambda}\chi^{\mu\nu}
\end{aligned}
\tag{4.2}
$$

The covariant difference operators appearing in these expressions are defined by

$$
\begin{aligned}
\mathcal{D}^{(+)}_\mu f_\nu(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x})f_\nu(\mathbf{x}+\mu) - f_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\mu) \\
\overline{\mathcal{D}}^{(-)\mu} f_\mu(\mathbf{x}) &= f_\mu(\mathbf{x})\overline{\mathcal{U}}^\mu(\mathbf{x}) - \overline{\mathcal{U}}^\mu(\mathbf{x}-\mu)f_\mu(\mathbf{x}-\mu)
\end{aligned}
\tag{4.3}
$$

These expressions are determined by the twin requirements that they reduce to the corresponding continuum results for the adjoint covariant derivative in the naive continuum limit $\mathcal{U}_\mu \to 1 + \mathcal{A}_\mu$ and that they transform under gauge transformations like the corresponding lattice link field carrying the same indices. This allows the terms in the action to correspond to gauge invariant closed loops on the lattice. The action can also be shown to be free of fermion doubling problems – see the discussion in [11].

As in the continuum, the presence of an exact Q-symmetry allows the definition of a class of supersymmetric Wilson loop corresponding to the trace of the product of $\overline{\mathcal{U}}_\mu$ links around a closed loop in the lattice.

$$O = \prod_{t=1}^{T} \overline{\mathscr{U}}^t(\mathbf{x}) \tag{4.4}$$

In principle other non-contractible loops can also be defined corresponding to knot-like structures as in the continuum. The vacuum expectation value of these operators can be computed exactly by restriction to the moduli space of theory and can probe only topological features of the theory.

To summarize; we have constructed a supersymmetric lattice theory based on the group $SU(2)$ which possesses a topological subsector which can be identified with the set of global gravitational observables in a Chern-Simons reformulation of 3d Euclidean quantum gravity. The lattice theory is well defined and we conjecture that the lattice theory may constitute a non-perturbative definition of the gravitational theory. It is an open question as to whether any similar correspondences between Yang-Mills theories and topological Chern Simons theories can be made in higher dimensions. The author would like to thank Poul Damgaard for useful discussions.

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