10-29-2010

Twisted Lattice Supersymmetry and Applications to AdS/CFT

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Twisted lattice supersymmetry and applications to AdS/CFT

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I review recent approaches to constructing supersymmetric lattice theories focusing in particular on the concept of topological twisting. The latter technique is shown to expose a nilpotent, scalar supersymmetry which can be implemented exactly in the lattice theory. Using these ideas a lattice action for $\mathcal{N}=4$ super Yang-Mills in four dimensions can be written down which is gauge invariant, free of fermion doublers and respects one out of a total of 16 continuum supersymmetries. It is shown how these exact symmetries together with the large point group symmetry of the lattice strongly constrain the possible counterterms needed to renormalize the theory and hence determine how much residual fine tuning will be needed to restore all supersymmetries in the continuum limit. We report on progress to study these renormalization effects at one loop. We go on to give examples of applications of these supersymmetric lattice theories to explore the connections between gauge theories and gravity.

The XXVIII International Symposium on Lattice Field Theory, Lattice2010
June 14-19, 2010
Villasimius, Italy

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[†]Supported in part by DOE grant DE-FG02-85ER40237

1. Introduction

In recent years there has been a resurgence of interest in the problem of formulating supersymmetric lattice theories. This has largely been driven by the realization that in certain classes of theory a discretization can be employed which preserves a subset of the continuum supersymmetry.

In this talk we will review some of these basic features concentrating on arguably the most interesting example; namely $\mathcal{N}=4$ super Yang-Mills in four dimensions. This theory, in addition to being one of the few finite four dimensional field theories, plays a crucial role in the original AdS/CFT correspondence which posits an exact equivalence between the gauge theory and a *dual* gravitational theory in five dimensional AdS space. The ability both to define and ultimately study this theory nonperturbatively is clearly very exciting since it allows us to test a variety of *holographic* conjectures connecting this theory and its dimensional reductions to a set of different gravitational theories. If one is optimistic it is possible that the existence of a non-perturbative construction of this theory may lead to new insights into the nature of quantum gravity itself.

The lattice theories I will describe have been derived using two broadly different strategies; by careful discretization of a topologically twisted form of the target continuum theory and also via a carefully chosen orbifolding procedure applied to a zero dimensional mother theory obtained by dimensional reduction of the same target theory. Remarkably these two approaches have been shown to lead to essentially the same lattice theories which leads one to suspect that the lattice constructions may be somewhat unique. Many people have contributed to one or both of these approaches; indeed there have been more than 150 papers published on the subject in the last decade. Because of this I have not tried to be comprehensive in citing most of the literature on exact lattice supersymmetry but merely refer the interested reader to the recent reviews [1, 2] for a more comprehensive bibliography.

In addition a body of recent work has attempted to study theories which cannot be treated using the methods described here – for studies of $\mathcal{N}=1$ super Yang-Mills using both Wilson and domain wall fermions see [3, 4, 5] and for treatments of the Wess Zumino model in four dimensions see [7, 8, 6]. There have also been efforts directed at constructing twisted models which preserve all continuum supersymmetries [9, 10, 11]. I will not talk about any of these other interesting topics here.

I will start with a few comments concerning the traditional difficulties of formulating a lattice theory with exact supersymmetry, explain the concept of twisting and how it allows us to circumvent most of the usual difficulties and then I will describe the most important elements in the construction of the $\mathcal{N}=4$ lattice theory. I will show how the exact symmetries of the lattice theory powerfully constrain the form of the renormalized action and hence the nature of the continuum limit. I will go on to describe the elements of an ongoing program to study the weak coupling structure of the theory using lattice perturbation theory. This work is a collaboration with Eric Dzienkowski, Joel Giedt, Anosh Joseph and Robert Wells [12].

Finally we will turn to applications of this lattice theory which hinge on exploring the possible holographic connections between strongly coupled supersymmetric gauge theories and (super)gravity. As you will see, these initial studies have shown that practical calculations, using the familiar algorithms used for lattice QCD, can indeed be used successfully to study the non-perturbative regime of these supersymmetric theories.

2. Why lattice supersymmetry is hard

The problem of formulating supersymmetric theories on lattices has a long history going back to the earliest days of lattice gauge theory. However, after initial efforts failed to produce useful supersymmetric lattice actions the topic languished for many years. Indeed a folklore developed that supersymmetry and the lattice were mutually incompatible. However, the application of new ideas and tools drawn from other areas of theoretical physics has shown this folklore to be incorrect and as this talk will show for certain classes of theory the problem has been solved.[1]

First, let me explain why discretization of supersymmetric theories resisted solution for so long. The central problem is that naive discretizations of continuum supersymmetric theories break supersymmetry completely and radiative effects lead to a profusion of *relevant* supersymmetry breaking counterterms in the renormalized lattice action. The coefficients to these counterterms must then be carefully fine tuned as the lattice spacing is sent to zero in order to arrive at a supersymmetric theory in the continuum limit. In most cases this is both unnatural and practically impossible – particularly if the theory contains scalar fields.

Of course, one might have expected problems – the supersymmetry algebra is an extension of the Poincaré algebra which is explicitly broken on the lattice. Specifically, there are no infinitesimal translation generators on a discrete spacetime so that the algebra $\{Q,\overline{Q}\} = \gamma_\mu p_\mu$ is already broken at the classical level. Equivalently it is a straightforward exercise to show that a naive supersymmetry variation of a naively discretized supersymmetric theory fails to yield zero as a consequence of the failure of the Leibniz rule when applied to lattice difference operators. The basic idea of the new approaches is to maintain a particular subalgebra of the full supersymmetry algebra in the lattice theory. The hope is that this exact symmetry will constrain the effective lattice action and protect the theory from dangerous susy violating counterterms.

3. Topological twisting

The simplest way to expose a subalgebra which is compatible with discretization is to consider not the original target theory but a so-called *twisted* variant of it. The basic idea of twisting goes back to Witten in his seminal paper on topological field theory [13] but actually had been anticipated in earlier work on staggered fermions [14]. Indeed at the free level the final lattice fermion action we will construct can be mapped exactly into that of (reduced) staggered fermions.

In our context the idea of twisting is to decompose the fields of the theory in terms of representations not of the original (Euclidean) rotational symmetry $SO_{\rm rot}(D)$ but a twisted rotational symmetry which is the diagonal subgroup of this symmetry and an $SO_{\rm R}(D)$ subgroup of the R-symmetry of the theory.

$$SO(D)' = {\rm diag}(SO_{\rm Lorentz}(D) \times SO_{\rm R}(D)) \tag{3.1}$$

To be explicit consider the case where the total number of supersymmetries is $Q = 2^D$. In this case I can treat the supercharges of the twisted theory as a $2^{D/2} \times 2^{D/2}$ matrix q. This matrix can be

[1] Solved in the sense that a supersymmetric lattice theory exists which targets a given continuum supersymmetric theory. Whether the lattice theory flows to this continuum theory in the continuum limit without additional fine tuning is still an open problem in certain cases including as we will see $\mathcal{N} = 4$ YM

expanded on products of gamma matrices

$$q = QI + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_n u + \ldots \tag{3.2}$$

The 2^D antisymmetric tensor components that arise in this basis are the twisted supercharges and satisfy a corresponding supersymmetry algebra following from the original algebra

$$\begin{aligned} Q^2 &= 0 && (3.3) \\ \{Q, Q_\mu\} &= p_\mu && (3.4) \\ &\ldots && (3.5) \end{aligned}$$

The presence of the nilpotent scalar supercharge Q is most important; it is the algebra of this charge that we can hope to translate to the lattice. The second piece of the algebra expresses the fact that the momentum is the Q-variation of something which makes plausible the statement that the energy-momentum tensor and hence the entire action can be written in Q-exact form[2]. Notice that an action written in such a Q-exact form is trivially invariant under the scalar supersymmetry provided the latter remains nilpotent under discretization.

The rewriting of the supercharges in terms of twisted variables can be repeated for the fermions of the theory and yields a set of antisymmetric tensors $(\eta, \psi_\mu, \chi_{\mu\nu}, \ldots)$ which for the case of $Q = 2^D$ matches the number of components of a real Kähler-Dirac field. This repackaging of the fermions of the theory into a Kähler-Dirac field is at the heart of how the discrete theory avoids fermion doubling as was shown by Becher, Joos and Rabin in the early days of lattice gauge theory [15, 16].

It is important to recognize that the transformation to twisted variables corresponds to a simple change of variables in flat space – one more suitable to discretization. A true topological field theory only results when the scalar charge is treated as a true BRST charge and attention is restricted to states annihilated by this charge. In the language of the supersymmetric parent theory such a restriction corresponds to a projection to the vacua of the theory. It is *not* employed in these lattice constructions.

4. Twisted $\mathcal{N} = 4$ super Yang-Mills

This theory satisfies our requirements for supersymmetric latticization; its R-symmetry possesses an $SO(4)$ subgroup corresponding to rotations of its four degenerate Majorana fermions into each other which may be twisted with the Euclidean Lorentz symmetry. The twisted fermions being initially spinors under both the R symmetry and the Lorentz group transform as integer spin antisymmetric tensors after twisting. In a similar fashion, 4 of the scalars which formed a vector representation of the R symmetry transform as vectors under the twisted rotation group. Indeed, as we will see they can be packaged with the gauge fields as part of a complexified connection.

The theory can be written in twisted form as

$$S = \frac{1}{g^2}\left(S_{\rm exact} + S_{\rm closed}\right) \tag{4.1}$$

[2] Actually in the case of $\mathcal{N} = 4$ there is an additional Q-closed term needed

where

$$S_{\rm exact} = Q\int {\rm Tr}\,\left(\chi_{ab}\mathscr{F}_{ab} + \eta\,[\overline{\mathscr{D}}_a, \mathscr{D}_a] - \frac{1}{2}\eta d\right) \tag{4.2}$$

In this expression the indices run not over $1\ldots 4$ as one would expect but $1\ldots 5$ corresponding to a theory with a twisted $SO(5)$ invariance. Indeed the compact expression given in eqn. 4.2 is most easily understood as a naive dimensional reduction of a *five* dimensional theory. The ten bosons of $\mathscr{N}=4$ YM are thus mapped after dimensional reduction to a complex scalar $\mathscr{A}_5$ and a complex gauge field $\mathscr{A}_\mu, \mu = 1\ldots 4$ in which the remaining four scalars of $\mathscr{N}=4$ appear as the imaginary parts of the gauge field. This five component notation will prove particularly useful later when we discretize the theory on a lattice. The 16 fermions $(\eta, \psi_a, \chi_{ab})$ naturally decompose on the fields required for a 4d Kähler-Dirac field. Explicitly

$$\begin{aligned}
\mathscr{A}_a &\to \mathscr{A}_\mu \oplus \phi \\
\mathscr{F}_{ab} &\to \mathscr{F}_{\mu\nu} \oplus \mathscr{D}_\mu \phi \\
\left[\overline{\mathscr{D}}_a, \mathscr{D}_a\right] &\to \left[\overline{\mathscr{D}}_\mu, \mathscr{D}_\mu\right] \oplus \left[\overline{\phi}, \phi\right] \\
\psi_a &\to \psi_\mu \oplus \overline{\eta} \\
\chi_{ab} &\to \chi_{\mu\nu} \oplus \overline{\psi}_\mu
\end{aligned} \tag{4.3}$$

where we will employ the convention that Greek indices run from one to four and are reserved for four dimensional tensors while Roman indices refer to the original five dimensional theory. The resulting action is just the one constructed by Marcus [18] and later referred to as the GL-twist [19]. This twisted action is well known to be fully equivalent to the usual form of $\mathscr{N}=4$ in flat space.

The nilpotent transformations associated with Q are given explicitly by

$$\begin{aligned}
Q\,\mathscr{A}_a &= \psi_a \\
Q\,\psi_a &= 0 \\
Q\,\overline{\mathscr{A}}_a &= 0 \\
Q\,\chi_{ab} &= -\overline{\mathscr{F}}_{ab} \\
Q\,\eta &= d \\
Q\,d &= 0
\end{aligned}$$

The second component of the action $S_{\rm closed}$ takes the form

$$S_{\rm closed} = -\frac{1}{2}\int \varepsilon_{abcde}\chi_{ab}\overline{\mathscr{D}}_c\chi_{de} \tag{4.4}$$

and is supersymmetric on account of the (five dimensional) Bianchi identity.

Performing the Q-variation and integrating out the auxiliary field d yields

$$S = \frac{1}{g^2}\int {\rm Tr}\,\left(-\overline{\mathscr{F}}_{ab}\mathscr{F}_{ab} + \frac{1}{2}[\overline{\mathscr{D}}_a, \mathscr{D}_a]^2 - \chi_{ab}\mathscr{D}_{[a}\,\psi_{b]} - \eta\,\overline{\mathscr{D}}_a\psi_a\right) - \frac{1}{2g^2}\int \varepsilon_{abcde}\chi_{ab}\overline{\mathscr{D}}_c\chi_{de} \tag{4.5}$$

5. Discretization

The prescription for discretization is somewhat natural. Complex gauge fields are represented as complexified Wilson gauge links $\mathcal{U}_a(x) = e^{\mathcal{A}}_{\mu}(x)$ living on links of a lattice. The most natural lattice possesses *five* equivalent basis vectors satisfying $\sum_{i=1}^{5} e_i = 0$ and is called A_4^*. It has a the large point group symmetry S^5 - far larger than the usual hypercubic group. This will prove important later when we consider the renormalization of the theory.

The link fields transform in the usual way under $U(N)$ lattice gauge transformations

$$\mathcal{U}_a(x) \to G(x)\mathcal{U}_a(x)G^{\dagger}(x+a) \tag{5.1}$$

Supersymmetric invariance then implies that $\psi_a(x)$ live on the same links and transform identically. The scalar fermion $\eta(x)$ is clearly most naturally associated with a site and transforms accordingly

$$\eta(x) \to G(x)\eta(x)G^{\dagger}(x) \tag{5.2}$$

The field χ_{ab} is slightly more difficult. Naturally as a 2-form it should be associated with a plaquette. In practice we introduce additional links running from $x \to x+a+b$ and let χ_{ab} lie *with opposite orientation* along those links. This choice of orientation will be necessary to ensure gauge invariance.

To complete the discretization we need to describe how continuum derivatives are to be replaced by difference operators. A natural technology for accomplishing this in the case of adjoint fields was developed many years ago and yields expressions for the derivative operator applied to arbitrary lattice p-forms [17]. In the case discussed here we need just three derivatives given by the expressions

$$\mathcal{D}_a^{(+)} f_a = \mathcal{U}_a(x) f_b(x+a) - f_b(x)\mathcal{U}_a(x+b) \tag{5.3}$$

$$\overline{\mathcal{D}}_a^{(-)} f_a = f_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-a) f_a(x-a) \tag{5.4}$$

$$\overline{\mathcal{D}}_c^{(-)} f_{ab} = f_{ab}(x)\overline{\mathcal{U}}_c(x-c) - \overline{\mathcal{U}}_c(x+a+b-c) f_{ab}(x-c) \tag{5.5}$$

The lattice field strength is given by the gauged forward difference $\mathcal{F}_{ab} = \mathcal{D}_a^{(+)}\mathcal{U}_b$ and is automatically antisymmetric in its indices. Furthermore it transforms like a lattice 2-form and yields a gauge invariant loop on the lattice when contracted with χ_{ab}. Similarly the covariant backward difference appearing in $\overline{\mathcal{D}}_a\mathcal{U}_a$ transforms as a 0-form or site field and hence can be contracted with the site field η. The expression in the third line of eqn. 5.5 is carefully chosen so that the Q-closed term is gauge invariant on the lattice – the epsilon tensor forces all the indices to be distinct in eqn. 4.4 and since the basis vectors sum to zero the expression can be seen to correspond yet again to a closed loop. Furthermore the lattice field strength satisfies a Bianchi identity just as for the continuum so the term is also Q-supersymmetric.

This use of forward and backward difference operators guarantees that the solutions of the theory map one-to-one with the solutions of the continuum theory and hence fermion doubling problems are evaded [15]. Indeed, by introducing a lattice with half the lattice spacing one can map this Kähler-Dirac fermion action into the action for staggered fermions. Notice that, unlike the case of QCD, there is no rooting problem in this supersymmetric construction since the additional fermion degeneracy is already required by the continuum theory.

To summarize; the lattice theory we have constructed by discretization of a twisted reformulation of the target SYM theory is gauge invariant, Q-supersymmetric and free of fermion doubling problems. However, many questions remain to be answered; is the theory rotationally invariant in the continuum limit, does it flow to the target theory with full supersymmetry as the lattice spacing is reduced and does the classical moduli space survive under quantum correction. We now summarize what is known about these issues by examining the theory using perturbative methods.

6. Renormalization of twisted lattice theory

Before we embark on a general perturbative analysis of this lattice theory it is instructive to try to ascertain what kinds of counter terms are permitted by the lattice symmetries. In the case of A_4^* lattice, these symmetries are exact Q supersymmetry, gauge invariance and the S^5 point symmetry of the A_4^* lattice. In fact, the $U(N)$ lattice gauge theory also has a second fermionic symmetry, given by

$$\eta \to \eta + \varepsilon \mathbb{I}_N, \qquad \delta(\text{all other fields}) = 0 \tag{6.1}$$

where ε is an infinitesimal Grassmann parameter. This acts as a further constraint on the structure of the renormalized lattice action.

In practice we are primarily interested in relevant or marginal operators; that is operators whose mass dimension is less than or equal to four. We will see that the set of relevant counterterms in the lattice theory is rather short – the lattice symmetries, gauge invariance in particular, being extremely restrictive in comparison to the equivalent situation in the continuum. The argument starts by assigning canonical dimensions to the fields $[\mathcal{U}_a] = [\overline{\mathcal{U}}_a] = 1$, $[\Psi] = \frac{3}{2}$ and $[Q] = \frac{1}{2}$ where Ψ stands for any of the twisted fermion fields $(\lambda, \psi^m, \xi_{mn})$. Invariance under Q restricts the possible counterterms to be either of a Q-exact form, or of Q-closed form. There is only one Q-closed operator permitted by the lattice symmetries and it is already present in our bare lattice action. A possible renormalization of this fermion kinetic term is hence allowed. Beyond that the exact lattice supersymmetry forces us to look at the set of Q-exact counterterms.

Any such counterterm must be of the form $\mathcal{O} = Q\mathrm{Tr}\,(\Psi f(\mathcal{U}, \overline{\mathcal{U}}))$. There are thus no terms permitted by symmetries with dimension less than two. In addition gauge invariance tells us that each term must correspond to the trace of a closed loop on the lattice. The smallest dimension gauge invariant operator is then just $Q(\mathrm{Tr}\,\psi_a \overline{\mathcal{U}}_a)$. But this vanishes identically since both $\overline{\mathcal{U}}_a$ and ψ_a are singlets under Q. No dimension $\frac{7}{2}$ operators can be constructed with this structure and we are left with just dimension four counterterms. Notice, in particular that lattice symmetries permit no simple fermion bilinear mass terms. However, gauge invariant fermion bi-linears with link field insertions are possible and their effect should be accounted for carefully. Possible dimension four operators are, schematically,

$$\begin{aligned}
L_1 &= Q\mathrm{Tr}\,(\chi_{ab}\mathcal{U}_a\mathcal{U}_b) \\
L_2 &= Q\mathrm{Tr}\,(\eta \overline{\mathcal{D}}_a \mathcal{U}_a) \\
L_3 &= Q\mathrm{Tr}\,(\eta \mathcal{U}_a \overline{\mathcal{U}}_a) \\
L_4 &= Q\mathrm{Tr}\,(\eta)\mathrm{Tr}\,(\mathcal{U}_a \overline{\mathcal{U}}_a)
\end{aligned} \tag{6.2}$$

The first operator can be simplified on account of the antisymmetry of χ_{ab} to simply $Q(\chi_{ab}\mathcal{F}_{ab})$, which again is nothing but one of the continuum Q-exact terms present in the bare action. The second term in (6.2) also corresponds to one of the Q-exact terms in the bare action. However the third term L_3 is a new operator not present in the bare Lagrangian and the same is true for the final double-trace operator L_4. Both of these operators transform non-trivially under the fermionic shift symmetry, but a linear combination of the two

$$D = L_3 - \frac{1}{N} L_4 \tag{6.3}$$

is invariant under the shift symmetry with N the number of colors of the gauge group $U(N)$.

By these arguments it appears that the only relevant counterterms correspond to renormalizations of marginal operators already present in the bare action together with D. The effective lattice action taking the form

$$S = \sum_x Q\left(\alpha_1 \chi_{ab} F_{ab} + \alpha_2 \psi_a \overline{\mathcal{D}}_a \eta + \frac{\alpha_3}{2} \eta d\right) + \alpha_4 S_{\rm closed} + \alpha_5 D \tag{6.4}$$

This implies that a maximum of four couplings might require log tuning to achieve a continuum limit invariant under full supersymmetry with only the dangerous mass term potentially needing power law tuning with the lattice spacing. In fact we will now argue that the the dangerous mass terms in D are in fact absent to all orders in perturbation theory reducing any "fine" tuning to just four independent logarithmic terms. The absence of mass terms will be shown in the next section using by a general background field calculation of the effective potential in the lattice theory.

7. Effective action

Here we expand about a constant commuting background corresponding to a generic point in the moduli space space and a solution of the classical equations of motion.

$$\mathcal{U}_a(n) = \mathcal{U}_a + i\mathcal{A}_a(n), \quad \overline{\mathcal{U}}_a(n) = \overline{\mathcal{U}}_a - i\overline{\mathcal{A}}_a(n) \tag{7.1}$$

with $[\mathcal{U}_a, \overline{\mathcal{U}}_b] = 0$. We further add a Lorentz gauge fixing term to the action of the form

$$\frac{1}{4\alpha} \mathrm{Tr}\, (\overline{\mathcal{D}}_a \mathcal{A}_a + \mathcal{D}_a \overline{\mathcal{A}}_a)^2 \tag{7.2}$$

Choosing the gauge $\alpha = 1$, we can show that the quadratic part of the bosonic action then takes the form

$$S_B = -2 \sum_n \mathrm{Tr}\, \mathcal{A}_b \overline{\mathcal{D}}_a^- \mathcal{D}_a^+ \mathcal{A}_b \tag{7.3}$$

Here the covariant derivatives depend only on the constant commuting classical background fields $\mathcal{U}_a, \overline{\mathcal{U}}_a$. This gauge fixing functional leads to the quadratic ghost action

$$S_G = \sum_n \mathrm{Tr}\, \overline{c} \overline{\mathcal{D}}_a^- \mathcal{D}_a^+ c \tag{7.4}$$

The quadratic fermionic part of the action is given by the corresponding expression in eqn. 4.2, except that now the covariant derivatives now depend only on the background fields.

Since the background is constant, we can pass to momentum space in which the action separates into terms for each mode k. The 16×16 fermion matrix $M(k)$ (see next section for details) for the mode k then can be shown (using Maple to compute the determinant) to satisfy

$$\det M(k) = \det{}^8(\overline{\mathscr{D}}^-_a(k)\mathscr{D}^+_a(k)) \tag{7.5}$$

Going back to position space we obtain

$$\mathrm{Pf}M = \det{}^4(\overline{\mathscr{D}}^-_a\mathscr{D}^+_a) \tag{7.6}$$

The ghosts add another factor of $\det(\overline{\mathscr{D}}^-_a\mathscr{D}^+_a)$ which together with the twisted fermions is just what is needed to cancel the factor of $\det^5(\overline{\mathscr{D}}^-_a\mathscr{D}^+_a)$ which comes from the bosons in the denominator. In conclusion, for a constant, commuting background, the one-loop effective action is zero - bosonic and fermionic fluctuations cancel exactly. Thus the moduli space is not lifted in this analysis and hence there can be no boson or fermion masses at one loop. We have verified this in the next section by explicit computation of the fermion self-energies which all vanish as the external momentum goes to zero.

Furthermore, we expect that we can extend this result to all loops since the full effective action of the lattice is related to the partition function of the theory Z. The latter should be a topological invariant since it is just the Witten index for the theory if periodic boundary conditions are used for all fields. This fact implies that Z that it can be computed exactly at one loop. Indeed, Matsuura uses similar arguments to show that the vacuum energy of supersymmetric orbifold theories with four and eight supercharges remains zero to all orders in the coupling [20]. Thus we conclude that boson and scalar masses remain zero to all orders in the coupling constant.

To proceed further we need to setup a perturbative calculation for the remaining coefficients $\alpha_i, i = 1\ldots 4$.

8. One loop lattice perturbation theory

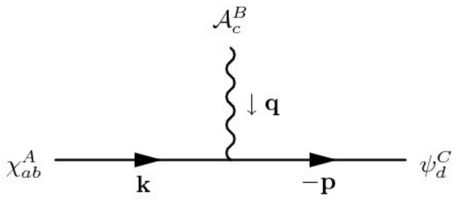
Three of the four coefficients α_1, α_2 and α_4 can be computed by examining the fermion self energy diagrams. The basic ingredients required for the perturbative computation of these quantum corrections are the propagators for the fermion and boson fields and the vertices. These are derived in the usual way by substituting the expression for the $\mathscr{U}$ fields given in eqn. 7.1 into the lattice action and identifying the quadratic pieces with the propagators and the vertices from the interactions. If we adopt the gauge fixing described in the previous section only a single non-zero boson propagator is required which takes the form in momentum space

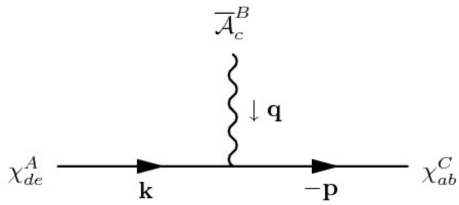
$$<\overline{\mathscr{A}}^C_a(k)\mathscr{A}^D_b(-k)>=\frac{1}{\hat{k}^2}\delta_{ab}\delta^{CD} \tag{8.1}$$

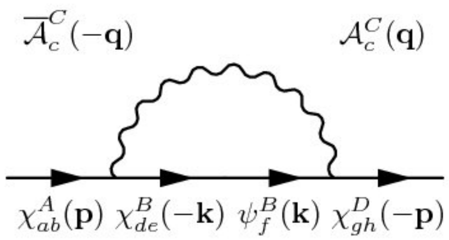
The fermion propagator takes the form of a 16×16 block matrix which acts on the twisted fields $\Psi_i = (\eta, \psi_a, \chi_{ab}), i = 1\ldots 16$. In momentum space it is

$$\langle\Psi^A_i(-k)\Psi^B_j(k)\rangle = \frac{1}{2\hat{k}^2}M_{ij}(k)\delta_{AB} \tag{8.2}$$

where the matrix M is the discrete Kähler-Dirac operator and $\widehat{k}^2 = \sum_a \sin^2 \frac{k_a}{2}$. Notice that the square of the fermion propagator is nothing more than the boson propagator as a consequence of the supersymmetric structure of the lattice action. The vertices correspond to terms involving the fields $\psi\eta$, $\psi\chi$ and $\chi\chi$. It can be easily be shown that only four Feynman diagrams are needed at one loop to find $\alpha_1, \alpha_2, \alpha_4$. An example of a typical contribution to the one loop renormalization of the $\chi - \chi$ propagator is shown below:







The corresponding expression that follows by applying the Feynman rules to this diagram is somewhat complicated and we refer the reader to [12] for its precise form but it can be shown to possess the following properties;

- It vanishes as $p \to 0$ as expected from the general arguments given in the last section concerning the absence of mass terms.

- It contains a logarithmic divergence $Ag^2 \log \mu a$ which can be obtained by fitting the infinite lattice integral for the amputated diagram as a function of the IR regular mass μ.

It can shown that all the fermion self-energy one loop terms possess the same properties.

In renormalized perturbation theory the log divergences can then be absorbed using wavefunction renormalization counterterms with the finite parts of the counterterms determined as usual by choosing suitable normalization conditions on the two point functions. The coefficients of the log divergences yield weak coupling estimates for the anomalous dimensions of the twisted fermion fields and at the same time give the required tuning as $a \to 0$ of the bare parameters $\alpha_1, \alpha_2, \alpha_4$ required to to ensure that full supersymmetry is restored in the lattice theory at weak coupling. A complete understanding of this issue requires in addition a computation of the renormalization of α_3 which can be obtained via the vacuum polarization of the complex gluons.

This program sketched out in this section is near completion and we refer the interested reader to [12] for further details. We expect this one loop calculation will serve as an important guide for numerical explorations of the theory at strong coupling.

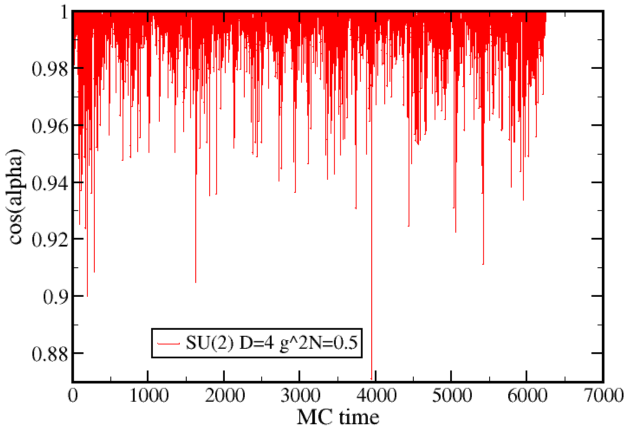


Figure 1: Cosine of pfaffian phase vs Monte Carlo time

9. Numerical simulations

We have conducted preliminary investigations of the theory on small lattices and confirmed that Monte Carlo simulations of the full theory are practicable; Figure 1. illustrates that the observed Pfaffian phase is small in the parameter regions of interest at least for the small lattices that have been examined so far[3][21]. Furthermore, the flat directions of the theory corresponding to constant commuting fields do not seem to cause problems of convergence; the scalar fields of the lattice theory do not appear to evolve to ever increasing values as the simulation progresses at least for zero temperature but remain somewhat localized close to the origin in field space as can be seen in Figure 2.

A parallel code has now been constructed which uses a multiple time step RHMC algorithm to handle the integration over the twisted fermions. The code uses the MDP libraries within the FermiQCD package to provide an interface to the MPI libraries that handle communication between individual computation nodes. The plot in Figure 3. below shows the performance of this code as a function of the number of cores for a simulation of the $SU(2)$ theory on a $8^3 \times 16$ lattice.

This illustrates that simulations of $\mathcal{N} = 4$ super Yang-Mills in four dimensions using this Q-exact action are practicable and have already started. We now turn to recent applications of this numerical work to study conjectured dualities between dimensionally reduced versions of this theory and various supergravity theories. These provide concrete examples of how simulations of Yang-Mills theories regulated using these methods can provide insight into the properties of black holes.

[3]Similar studies have provided evidence that the related two dimensional lattice model with 4 supercharges also does not possess a sign problem in the continuum limit [22]

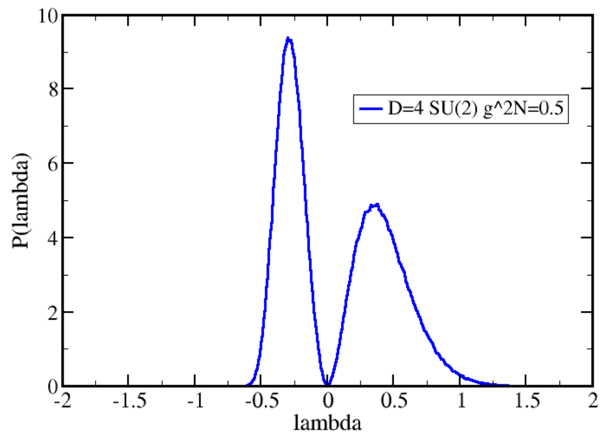


Figure 2: Probability distribution of scalar eigenvalues λ of $U_a^\dagger U_a - 1$

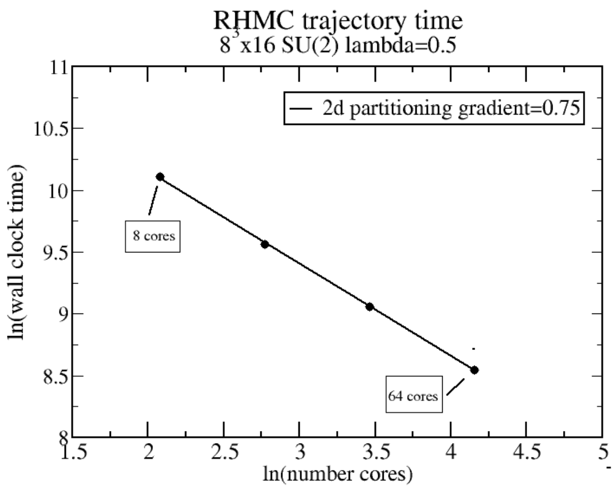


Figure 3: Wall clock time for $\mathcal{N} = 4$ SYM model on $8^3 \times 16$ vs number of cores

10. Applications: black holes from Yang-Mills theory

In recent years there has been a great deal of interest in dualities between supersymmetric Yang-Mills theories and gravitational theories. For example, the original AdSCFT correspondence has been generalized to allow for the description of the low energy dynamics of Dp-branes in various supergravity theories at finite temperature and the thermal behavior of $(p+1)$-dimensional Yang-Mills theories at strong coupling and large N [23].

The existence of supersymmetric lattice actions for these Yang-Mills theories allows us to test these conjectured correspondences in some detail and indeed precise computations in the Yang-

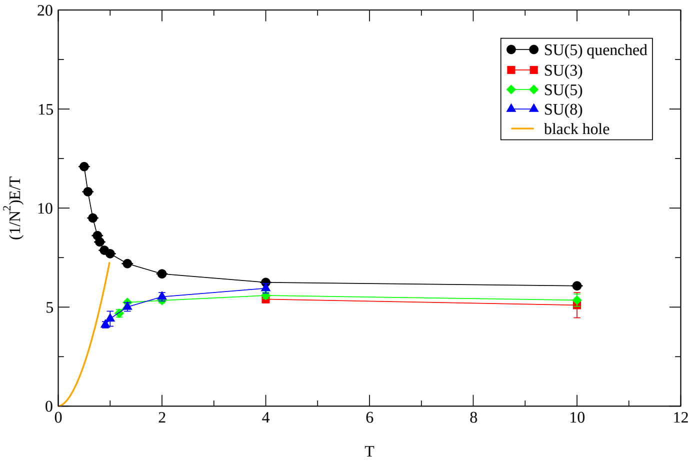


Figure 4: Comparing $E_{\rm YM}$ with $E_{\rm BH}$ for $Q = 16$ SYMQM

Mills theories can in principle yield information on the stringy corrections to the supergravity description. With this in mind in mind we have begun the study of two Yang-Mills theories both of which are derived by dimensional reduction of the $\mathcal{N} = 4$ lattice theory.

The first of these is thermal Yang-Mills quantum mechanics which is thought to be dual to a system of $D0$ branes in type iia string theory at non zero temperature. For low temperature and large N the system should be well described by certain black hole solutions in iia supergravity and standard Bekenstein-Hawking arguments can be used to compute the mean energy and entropy of these black holes. We have checked this correspondence by simulation of the supersymmetric Yang-Mills model and find good agreement; Figure 4. shows a plot of the mean energy in the Yang-Mills system for several values of N versus the (dimensionless) temperature. The data points correspond to the Monte Carlo data while the solid curve is the prediction for the corresponding black hole solution. Technical issues related to an IR divergence of the thermal partition function make it difficult to simulate the Yang-Mills system at very low temperatures; nevertheless the agreement is quite impressive. Notice that the data points corresponding to the quenched theory rapidly diverge from the black hole curve at low temperature and contrast with the results of the dynamical fermion simulations which turn over to approach zero at low temperature as expected for a supersymmetric system. These results are discussed in more detail in [24, 25] and are in good agreement with other non lattice methods which have also targeted this system [26, 27, 28, 29].

The case of SYM in two dimensions is perhaps even more interesting; in this case the phase diagram of the Yang-Mills theory compactified on a torus is labeled by two parameters; the extent of the lattice in the (Euclidean) time direction and the spatial extent. These two directions are distinguished by the boundary conditions on the fermions; antiperiodic in the temporal direction, periodic in the spatial direction. In the supergravity dual they correspond to the temperature and the size of a compactified spatial direction. Two solutions of the supergravity equations are now allowed corresponding to a spherically symmetric black hole solution and a new type of solution with cylindical symmetry - the black string. Which solution dominates the free energy depends on the size of the black hole horizon in comparison to the size of the compactified dimension.

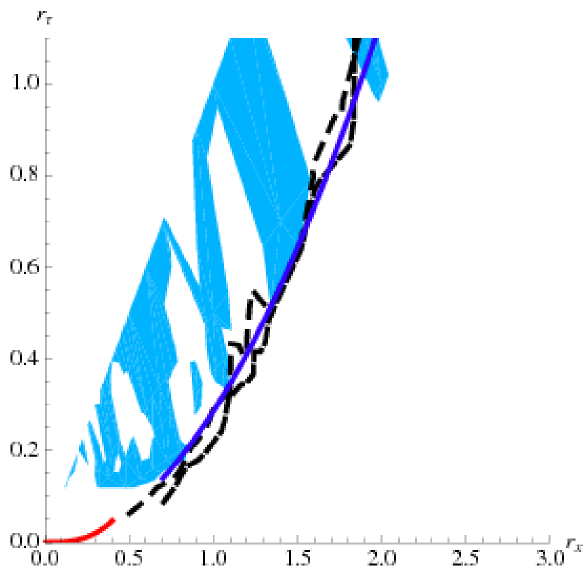


Figure 5: Transition line in P_s for $SU(3), SU(4)$ with fit to supergravity prediction in blue

If the event horizon wraps the spatial circle the black string solution is preferred. The transition between these different types of spacetime is referred to as the Gregory-LaFlamme instability [30]. In the dual Yang-Mills system it can be seen as a thermal phase transition distinguished by the spatial Polyakov line. We have investigated the phase boundary between confined and deconfined behavior of this line as the aspect ratio of the lattice and temperature are varied for several N [31].

Figure 5. shows a contour plot of our results in the parameter space corresponding to the temporal r_τ and spatial extents r_x rendered dimensionless by rescaling with the 't Hooft coupling. The solid curves correspond to results obtained from the supergravity solution in blue and the dimensionally reduced model obtained at high temperature (red). It can be seen that the numerically obtained phase boundary (defined as the point where $P_{\text{spatial}} = 0.5$) shown by the two dashed lines for $SU(3)$ and $SU(4)$ corresponds rather well to both analytic limits and in particular, confirms the expected parametric dependence of $r_\tau \sim c r_x^2$ given by supergravity. The coefficient c which was previously unknown from the gravity side taking the approximate value $c \sim 3.5$.

11. Summary

In this talk I have surveyed some of the recent theoretical work focused on constructing supersymmetric lattice theories and described some initial perturbative and numerical studies of these models with particular emphasis on applications to studies of gauge gravity duality.

Recent developments that I have not had time to cover include theoretical constructions to couple these theories to fermions in the fundamental representation [32, 33], Q-exact constructions of mass deformed YM theories [34, 35, 36] and connections of twisted YM to gravity [37]. With these continuing theoretical developments and the start of serious numerical studies I expect the next few years will prove to be an exciting time for lattice supersymmetry.

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