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Perturbative Renormalization of Lattice $\mathcal{N} = 4$ Super Yang-Mills Theory

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ABSTRACT: We consider $\mathcal{N} = 4$ super Yang-Mills theory on a four-dimensional lattice. The lattice formulation under consideration retains one exact supersymmetry at non-zero lattice spacing. We show that this feature combined with gauge invariance and the large point group symmetry of the lattice theory ensures that the only counterterms that appear at any order in perturbation theory correspond to renormalizations of existing terms in the bare lattice action. In particular we find that no mass terms are generated at any finite order of perturbation theory. We calculate these renormalizations by examining the fermion and auxiliary boson self energies at one loop and find that they all exhibit a common logarithmic divergence which can be absorbed by a single wavefunction renormalization. This finding implies that at one loop only a fine tuning of the finite parts is required to regain full supersymmetry in the continuum limit.

KEYWORDS: Lattice Quantum Field Theory, Supersymmetric Gauge Theory, Topological Field Theories, Extended Supersymmetry.

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1. Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions is both a fascinating quantum field theory in its own right and in addition plays a crucial role in the well known AdS/CFT correspondence [1, 2] where it is thought to provide a dual description of type IIB string theory in $AdS_5 \times S_5$ space.

The strong coupling, large- N limit (N being the number of colors) of this gauge theory has been extensively studied since the corresponding dual theory reduces to a weakly coupled supergravity theory, which describes the low energy limit of type IIB string theory. However it has proven difficult to go beyond this leading supergravity approximation in most situations. Having a lattice formulation of the super Yang-Mills theory would clearly be very advantageous as it would give a non-perturbative definition of the gauge theory and offer up new tools to investigate its strong coupling dynamics and thence the dynamics of the dual theory. Indeed such a lattice construction would allow for a systematic study of the classical and quantum string corrections to the supergravity solution. Unfortunately straightforward attempts to discretize the continuum theory are well known to break supersymmetry completely leading to a profusion of supersymmetry violating counterterms (four or six counterterms, depending on the gauge group, in the effective action whose couplings must be fine tuned to approach the correct continuum limit - see Ref.[3]).

However, recent formulations of supersymmetric lattice theories, which retain exact supersymmetry at non zero lattice spacing offer the hope of at least partially evading these fine tuning problems - the existence of an exact lattice supersymmetry protecting the theory from many of these dangerous counterterms [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. See also the recent reviews [15, 16] for further references. In addition, there has recently been a great deal of other work, both theoretical and numerical, focused on discrete formulations of $\mathcal{N} = 4$ SYM see Refs. [17, 18, 19, 20, 21, 22, 23]. These alternative approaches should be viewed as complementary to the lattice construction described in this paper.

In the case of $\mathcal{N} = 4$ SYM the corresponding lattice theory retains only one out of the sixteen continuum supersymmetric invariances and the question of how much fine tuning is required to take a continuum limit of this lattice theory targeting the usual $\mathcal{N} = 4$ theory has been unclear up to this point in time. This paper aims to address this issue using both general arguments valid to all orders in perturbation theory and an explicit calculation of the renormalization of the lattice theory to one loop order.

We will argue quite generally that the symmetries of the lattice theory strongly constrain the possible counterterms that can arise as a result of quantum corrections; we find that the only relevant operators that can be induced via radiative effects correspond to renormalizations of four marginal operators already present in the tree level theory. These operators correspond to kinetic terms and we show, using a topological argument based on the exact (twisted) lattice supersymmetry, that no mass terms are induced to all orders in perturbation theory.

The remaining fine tuning question then hinges on what divergences can arise in the renormalization of these four bare couplings. We proceed to calculate these divergences at one loop using lattice perturbation theory. Exact lattice supersymmetry allows us to extract these leading divergences by examining the renormalization of the three types of twisted fermion propagator and a single propagator for an auxiliary bosonic field. We show that all these exhibit a common logarithmic divergence at one loop. The appearance of a single logarithmic divergence ensures that at one loop only finite parts need to be fine tuned in order to regain full supersymmetry in the continuum limit. This is a huge advantage of this approach as compared to earlier efforts at constructing supersymmetric lattice theories

in four dimensions.

We start with a discussion of the approach to supersymmetric lattices through discretization of a topologically twisted formulation of the super Yang-Mills theory, write down the lattice theory and discuss the constraints on its renormalization implied by the lattice symmetries. We then derive the Feynman rules governing the perturbative structure of the lattice theory and write down the diagrams needed to renormalize the theory at one loop. We compute the partition function at one loop and show that it is independent of any background fields and furthermore that this is true to all orders in perturbation theory. We then evaluate the one loop diagrams, extract their leading logarithmic divergences and compute the required renormalization of the lattice theory. We conclude with a summary of our main results.

2. Twisted $\mathcal{N} = 4$, $d = 4$ Super Yang Mills

The key idea that allows us to construct a supersymmetric lattice theory that targets $\mathcal{N} = 4$ super Yang-Mills in the (naïve) continuum limit is called topological twisting - see [16] and references therein¹. This twisting process, which in flat space can be thought of as merely a change of variables, exposes a scalar, nilpotent supercharge \mathcal{Q} . It is the supersymmetry associated with this supercharge which can be implemented exactly in the lattice theory.

To understand how the twisted theory is constructed one needs to examine the relevant global symmetries of the continuum Yang-Mills theory. The Euclidean version of the $\mathcal{N} = 4$ SYM theory on \mathbb{R}^4 can be obtained by dimensionally reducing $\mathcal{N} = 1$ SYM theory on \mathbb{R}^{10} down to \mathbb{R}^4 . The ten-dimensional theory possesses an $SO(10)$ Euclidean (Lorentz) rotation group. After dimensional reduction it reduces to

$$SO(10)_E \rightarrow SO(4)_E \times SO(6)_I ,$$

where $SO(4)_E \sim SU(2) \times SU(2)$ is the four-dimensional Euclidean (Lorentz) symmetry on \mathbb{R}^4 and $SO(6)_I \sim SU(4)_R$ is the global internal R -symmetry group of the dimensionally reduced theory.

The basic idea of twisting is to decompose the fields of the theory in irreducible representations of a twisted rotation group which involves both the usual rotations and the R -symmetry². The global R -symmetry group of the dimensionally reduced theory contains a subgroup $SO(4)_R \times U(1)$. To construct the relevant twist needed for the lattice theory, we adopt as twisted rotation group $SO(4)'$ the diagonal subgroup of $SO(4)_E \times SO(4)_R$. The global symmetry is now given by

$$\begin{aligned} G' &= SO(4)' \times U(1) \\ &\sim SU(2)' \times SU(2)' \times U(1) \end{aligned}$$

¹This approach to supersymmetric lattices has been shown to be entirely equivalent to the orbifold/deconstruction formulations pioneered by Kaplan, Ünsal and collaborators [10] which predated the twisted constructions in the case of gauge theories. However for simplicity we will mostly be using the language of twisting in this paper.

²The $\mathcal{N} = 4$ theory may be twisted in three inequivalent ways [24, 25, 26] however only one of these - that due to Marcus and described in the text is ultimately compatible with discretization.

Notice that since the $U(1)$ part of the global internal symmetry group is undisturbed, it remains as a global R -symmetry of the twisted theory. The supercharges and fermions transform under the new rotation group as

$$SU(2)' \times SU(2)' \times U(1) \rightarrow (1, 1)_{\frac{1}{2}} \oplus (2, 2)_{-\frac{1}{2}} \oplus [(3, 1) \oplus (1, 3)]_{\frac{1}{2}} \oplus (2, 2)_{-\frac{1}{2}} \oplus (1, 1)_{\frac{1}{2}}, \quad (2.1)$$

or equivalently

$$SO(4)' \times U(1) \rightarrow 1_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 6_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 1_{\frac{1}{2}}. \quad (2.2)$$

As a result of this choice of embedding, the twisted theory contains supersymmetries and fermions in integer spin representations. They transform as scalars, vectors and higher rank p -form tensors:

$$\begin{array}{l} \text{Supercharges:} \quad \mathcal{Q}^{(0)} \oplus \mathcal{Q}^{(1)} \oplus \mathcal{Q}^{(2)} \oplus \mathcal{Q}^{(3)} \oplus \mathcal{Q}^{(4)} \\ \text{Fermions:} \quad \Psi^{(0)} \oplus \Psi^{(1)} \oplus \Psi^{(2)} \oplus \Psi^{(3)} \oplus \Psi^{(4)} \end{array}$$

We parametrize the fermionic content of the theory by

$$\Psi = \begin{cases} \eta & 1 \\ \psi_\mu & 4 \\ \chi_{\mu\nu} & 6 \\ \bar{\psi}_\mu \equiv \epsilon_{\mu\nu\rho\lambda} \xi_{\nu\rho\lambda} & 4 \\ \bar{\eta} \equiv \epsilon_{\mu\nu\rho\lambda} \psi_{\mu\nu\rho\lambda} & 1 \end{cases} \quad (2.3)$$

The four gauge bosons transform as $(2, 2)$ under the twisted rotation group. We label them as a vector field A_μ . Similarly, four of the six scalars of the theory are now elevated to the same footing as the gauge bosons; they also transform as $(2, 2)$ under the twisted rotation group. We label them as a vector field B_μ . The two other scalars remain as singlets under the twisted rotation group. We label them by ϕ and $\bar{\phi}$. Thus the bosons of the twisted theory transform as:

$$SU(2)' \times SU(2)' \times U(1) \rightarrow (1, 1)_1 \oplus (2, 2)_0 \oplus (2, 2)_0 \oplus (1, 1)_{-1}, \quad (2.4)$$

or equivalently

$$SO(4)' \times U(1) \rightarrow 1_1 \oplus 4_0 \oplus 4_0 \oplus 1_{-1}. \quad (2.5)$$

We parametrize the bosonic content of the theory by

$$\Phi = \begin{cases} \phi & 1 \\ A_\mu & 4 \\ B_\mu & 4 \\ \bar{\phi} & 1 \end{cases} \quad (2.6)$$

For future reference the fields of the twisted theory have the following mass dimensions:

Twisted field	A_μ	B_μ	ϕ	$\bar{\phi}$	$\chi_{\mu\nu}$	ψ_μ	$\bar{\psi}_\mu$	η	$\bar{\eta}$
Mass dimension	1	1	1	1	3/2	3/2	3/2	3/2	3/2

In this paper we are interested in $Q^{(0)}$, the scalar supercharge which is nilpotent $(Q^{(0)})^2 \cdot = 0$. It is also interesting to see that with this particular choice of twist, the resulting twisted fermions are just sufficient to saturate a single Dirac-Kähler field [9].

3. Continuum Action and Nilpotent Supersymmetry

Since the two vector fields A_μ and B_μ of the twisted theory transform the same way under the twisted rotation group we can describe the theory in a compact way if we combine the vector fields into a complex vector field \mathcal{A}_μ [26]³:

$$\mathcal{A}_\mu \equiv A_\mu + iB_\mu , \quad (3.1)$$

$$\bar{\mathcal{A}}_\mu \equiv A_\mu - iB_\mu . \quad (3.2)$$

Using these connections one can now define three covariant derivatives and field strengths:

$$D_\mu \cdot \equiv \partial_\mu + ig[A_\mu, \cdot] , \quad F_{\mu\nu} \equiv -\frac{i}{g}[D_\mu, D_\nu] , \quad (3.3)$$

$$\mathcal{D}_\mu \cdot \equiv \partial_\mu + ig[\mathcal{A}_\mu, \cdot] , \quad \mathcal{F}_{\mu\nu} \equiv -\frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] , \quad (3.4)$$

$$\bar{\mathcal{D}}_\mu \cdot \equiv \partial_\mu + ig[\bar{\mathcal{A}}_\mu, \cdot] , \quad \bar{\mathcal{F}}_{\mu\nu} \equiv -\frac{i}{g}[\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu] . \quad (3.5)$$

To make contact with the final lattice construction it is useful to assemble the complexified gauge fields and the two scalar fields into a single five-component complexified connection

$$\mathcal{A}_a = \left(\mathcal{A}_\mu \equiv A_\mu + iB_\mu, \quad \mathcal{A}_5 \equiv A_5 + iB_5 \right) , \quad a = 1, \dots, 5 ; \mu = 1, \dots, 4 \quad (3.6)$$

where the fifth component $\mathcal{A}_5 = \phi$ and $\bar{\mathcal{A}}_5 = \bar{\phi}$. Correspondingly the fermions can be packaged as five-dimensional scalar, vector and antisymmetric tensors $(\eta, \psi_a, \chi_{ab})$. The original twisted theory will then be obtained by simple dimensional reduction of a theory in five dimensions. A similar language arises in the orbifold construction of this theory where the fermions and bosons transform in representations of $SU(5)$. In addition to these fields we introduce one auxiliary bosonic scalar field d for off-shell completion of the scalar supersymmetry.

The nilpotent scalar supersymmetry \mathcal{Q} (from now on we denote the scalar supersymmetry $\mathcal{Q}^{(0)}$ by \mathcal{Q}) now acts on these fields in a simple manner

$$\mathcal{Q}\mathcal{A}_a = \psi_a \quad (3.7)$$

$$\mathcal{Q}\psi_a = 0 \quad (3.8)$$

$$\mathcal{Q}\bar{\mathcal{A}}_a = 0 \quad (3.9)$$

$$\mathcal{Q}\chi_{ab} = \bar{\mathcal{F}}_{ab} \quad (3.10)$$

$$\mathcal{Q}\eta = d \quad (3.11)$$

$$\mathcal{Q}d = 0 \quad (3.12)$$

The action of the twisted theory can now be expressed in a compact five-dimensional form, as a linear combination of \mathcal{Q} -exact and \mathcal{Q} -closed terms

$$S = \mathcal{Q}\Lambda + S_{\mathcal{Q}\text{-closed}} , \quad (3.13)$$

³Throughout this paper we will be employing a hermitian basis for the generators satisfying $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

where

$$\Lambda = \int \text{Tr} \left(\chi_{ab} \mathcal{F}_{ab} - \frac{i}{g} \eta [\overline{\mathcal{D}}_a, \mathcal{D}_a] + \frac{1}{2} \eta d \right), \quad (3.14)$$

and

$$S_{\mathcal{Q}\text{-closed}} = -\frac{1}{2} \int \text{Tr} \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab}. \quad (3.15)$$

The invariance of the \mathcal{Q} -closed term is a result of the Bianchi identity (or Jacobi identity for covariant derivatives)

$$\epsilon_{abcde} \overline{\mathcal{D}}_c \overline{\mathcal{F}}_{de} = -\frac{i}{g} \epsilon_{abcde} [\overline{\mathcal{D}}_c, [\overline{\mathcal{D}}_d, \overline{\mathcal{D}}_e]] = 0. \quad (3.16)$$

Carrying out the \mathcal{Q} -variation and subsequently eliminating the auxiliary field d using the equation of motion we can write down the action in terms of the propagating fields.

$$S = \int \text{Tr} \left(\overline{\mathcal{F}}_{ab} \mathcal{F}_{ab} + \frac{1}{2g^2} [\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab} \mathcal{D}_{[a} \psi_{b]} - \eta \overline{\mathcal{D}}_a \psi_a - \frac{1}{2} \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab} \right). \quad (3.17)$$

After a redefinition of the fields $g\eta \rightarrow \eta$, $g\psi_a \rightarrow \psi_a$, $g\chi_{ab} \rightarrow \chi_{ab}$ and $g\mathcal{A}_a \rightarrow \mathcal{A}_a$ we have

$$S = \frac{1}{g^2} \int \text{Tr} \left(-[\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b] [\mathcal{D}_a, \mathcal{D}_b] + \frac{1}{2} [\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab} \mathcal{D}_{[a} \psi_{b]} - \eta \overline{\mathcal{D}}_a \psi_a - \frac{1}{2} \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab} \right). \quad (3.18)$$

The target twisted theory in four dimensions can be obtained by dimensional reduction of this theory along the 5th direction. We write down the decomposition of five-dimensional fields into four-dimensional fields as follows

$$\mathcal{A}_a \rightarrow \mathcal{A}_\mu \oplus \phi \quad (3.19)$$

$$\mathcal{F}_{ab} \rightarrow \mathcal{F}_{\mu\nu} \oplus \mathcal{D}_\mu \phi \quad (3.20)$$

$$[\overline{\mathcal{D}}_a, \mathcal{D}_a] \rightarrow [\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] \oplus -[\overline{\phi}, \phi] \quad (3.21)$$

$$\psi_a \rightarrow \psi_\mu \oplus \overline{\eta} \quad (3.22)$$

$$\chi_{ab} \rightarrow \chi_{\mu\nu} \oplus \overline{\psi}_\mu \quad (3.23)$$

The action (3.18), after dimensional reduction, yields

$$S = \frac{1}{g^2} \int \text{Tr} \left(-[\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu] [\mathcal{D}_\mu, \mathcal{D}_\nu] + \frac{1}{2} [\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 + \frac{1}{2} [\overline{\phi}, \phi]^2 - (\overline{\mathcal{D}}_\mu \overline{\phi})(\mathcal{D}_\mu \phi) - \chi_{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]} - \overline{\psi}_\mu \mathcal{D}_\mu \overline{\eta} - i \overline{\psi} [\phi, \psi_\mu] - \eta \overline{\mathcal{D}}_\mu \psi_\mu - i \eta [\overline{\phi}, \overline{\eta}] - \chi_{\mu\nu}^* \overline{\mathcal{D}}_\mu \overline{\psi}_\nu - \frac{i}{2} \chi_{\mu\nu}^* [\overline{\phi}, \chi_{\mu\nu}] \right), \quad (3.24)$$

where the last two terms arise from the dimensional reduction of the \mathcal{Q} -closed term with χ^* , the Hodge dual of χ , defined as $\chi_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \chi_{\rho\lambda}$ and $\overline{\psi}_\mu = \frac{1}{2} \chi_{5\mu}$.

This action can be identified with the twisted $\mathcal{N} = 4$ SYM action in four dimensions written down by Marcus [26], up to a trivial rescaling of the fields (with a gauge parameter $\alpha = 1$ in [26]). It is important to note that in flat space, this twisted action is just a rewriting of the usual $\mathcal{N} = 4$ SYM theory in four dimensions and is physically equivalent to it.

4. Lattice Theory

Discretization of the twisted theory described in the previous section proceeds straightforwardly; complex continuum gauge fields are represented as complexified Wilson gauge links $\mathcal{U}_a(\mathbf{n})$ living on links μ_a , $a = 1 \dots 5$ of a four-dimensional lattice. Since there are five such vectors it should be clear that this lattice will have five basis vectors. To ensure that the lattice theory enjoys a maximal symmetry we would like these basis vectors to all be equivalent. This requirement means that the lattice must possess an S^5 point group symmetry (the Weyl group of $SU(5)$). The unique solution to these constraints in four dimensions is the so-called A_4^* lattice. This will hence be the underlying lattice used in our work.

A specific basis for the A_4^* lattice is given in the form of five lattice vectors

$$\hat{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \quad (4.1)$$

$$\hat{e}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \quad (4.2)$$

$$\hat{e}_3 = \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \quad (4.3)$$

$$\hat{e}_4 = \left(0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \quad (4.4)$$

$$\hat{e}_5 = \left(0, 0, 0, -\frac{4}{\sqrt{20}} \right). \quad (4.5)$$

These lattice vectors connect the center of a 4-simplex to its five corners. They are related to the $SU(5)$ weights of the $\mathbf{5}$ representation. The unit cell of the A_4^* lattice is a compound of two 4-simplices corresponding to the $\mathbf{5}$ (formed by the basis vectors \hat{e}_m) and $\bar{\mathbf{5}}$ (formed by the basis vectors $-\hat{e}_m$) representations of $SU(5)$. The basis vectors satisfy the relations

$$\sum_{m=1}^5 \hat{e}_m = 0; \quad \hat{e}_m \cdot \hat{e}_n = \left(\delta_{mn} - \frac{1}{5} \right); \quad \sum_{m=1}^5 (\hat{e}_m)_\mu (\hat{e}_m)_\nu = \delta_{\mu\nu}; \quad \mu, \nu = 1, \dots, 4. \quad (4.6)$$

Notice also that S^5 is a subgroup of the twisted rotation symmetry group $SO(4)'$ and that the lattice fields transform in reducible representations of this discrete group - for example the vector \mathcal{A}_a decomposes into a four component vector \mathcal{A}_μ and a scalar field ϕ under $SO(4)'$. Invariance of the lattice theory with respect to these discrete rotations then guarantees that the theory will inherit full invariance under twisted rotations in the continuum limit.

Proceeding in this manner it is possible to assign all the remaining fields to links on the A_4^* lattice. Since $\psi_a(\mathbf{n})$ is a superpartner of $\mathcal{U}_a(\mathbf{n})$ it must also reside on the link connecting $\mathbf{n} \rightarrow \mathbf{n} + \hat{e}_a$. Conversely the field $\mathcal{U}_a^\dagger(\mathbf{n})$ resides on the oppositely oriented link from $\mathbf{n} \rightarrow \mathbf{n} - \hat{e}_a$. The ten fermions $\chi_{ab}(\mathbf{n})$ are then chosen to reside on new fermionic links $\mathbf{n} + \hat{e}_m + \hat{e}_n \rightarrow \mathbf{n}$ while the singlet fermionic field $\eta(\mathbf{n})$ is assigned to the degenerate link consisting of a single site \mathbf{n} .

The action of the theory takes the following form

$$S = \frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d, e} \left\{ \mathcal{Q} \text{Tr} \left[-i\chi_{ab} \mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) - \eta(\mathbf{n}) \left(i\mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) - \frac{1}{2} d(\mathbf{n}) \right) \right] - \frac{1}{2} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b + \widehat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c) \right\}. \quad (4.7)$$

where the lattice field strength is given by

$$\mathcal{F}_{ab}(\mathbf{n}) \equiv -\frac{i}{g} \mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) = -\frac{i}{g} \left(\mathcal{U}_a(\mathbf{n}) \mathcal{U}_b(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a) - \mathcal{U}_b(\mathbf{n}) \mathcal{U}_a(\mathbf{n} + \widehat{\boldsymbol{\mu}}_b) \right). \quad (4.8)$$

and the covariant difference operators appearing in this expression are given by

$$\mathcal{D}_c^{(+)} f(\mathbf{n}) = \mathcal{U}_c(\mathbf{n}) f(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c) - f(\mathbf{n}) \mathcal{U}_c(\mathbf{n}), \quad (4.9)$$

$$\mathcal{D}_c^{(+)} f_d(\mathbf{n}) = \mathcal{U}_c(\mathbf{n}) f_d(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c) - f_d(\mathbf{n}) \mathcal{U}_c(\mathbf{n} + \widehat{\boldsymbol{\mu}}_d), \quad (4.10)$$

$$\mathcal{D}_c^{\dagger(-)} f_c(\mathbf{n}) = f_c(\mathbf{n}) \mathcal{U}_c^\dagger(\mathbf{n}) - \mathcal{U}_c^\dagger(\mathbf{n} - \widehat{\boldsymbol{\mu}}_c) f_c(\mathbf{n} - \widehat{\boldsymbol{\mu}}_c), \quad (4.11)$$

$$\mathcal{D}_c^{\dagger(-)} f_{ab}(\mathbf{n}) = f_{ab}(\mathbf{n}) \mathcal{U}_c^\dagger(\mathbf{n} - \widehat{\boldsymbol{\mu}}_c) - \mathcal{U}_c^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b - \widehat{\boldsymbol{\mu}}_c) f_{ab}(\mathbf{n} - \widehat{\boldsymbol{\mu}}_c). \quad (4.12)$$

Notice that these definitions reduce to the usual adjoint covariant derivative in the naïve continuum limit corresponding to $\mathcal{U}_a = I + \mathcal{A}_a(x) + \dots$ and furthermore guarantee that the resultant discrete expressions transform covariantly under lattice gauge transformation. Furthermore, this use of forward and backward difference operators guarantees that the solutions of the theory map one-to-one with the solutions of the continuum theory and hence fermion doubling problems are evaded [27]. Indeed, by introducing a lattice with half the lattice spacing one can map this Dirac-Kähler fermion action into the action for staggered fermions [28].

It is important to realize that the vectors \mathbf{n} , $\mathbf{n} + \widehat{\boldsymbol{\mu}}_a$ etc appearing in this action do not correspond to the positions in spacetime of sites and links of the original A_4^* lattice itself – instead they span an abstract hypercubic lattice whose sites and links are given by integer valued lattice vectors. (Which are related to the \mathbf{r} -charges defined in the orbifold formulation [10].) These 4-vectors $\widehat{\boldsymbol{\mu}}_a$ are defined as

$$\begin{aligned} \widehat{\boldsymbol{\mu}}_1 &= (1, 0, 0, 0) \\ \widehat{\boldsymbol{\mu}}_2 &= (0, 1, 0, 0) \\ \widehat{\boldsymbol{\mu}}_3 &= (0, 0, 1, 0) \\ \widehat{\boldsymbol{\mu}}_4 &= (0, 0, 0, 1) \\ \widehat{\boldsymbol{\mu}}_5 &= (-1, -1, -1, -1) \end{aligned} \quad (4.13)$$

The integer-valued lattice site \mathbf{n} can be related to the physical location in spacetime using the A_4^* basis vectors $\widehat{\mathbf{e}}_a$.

$$\mathbf{R} = a \sum_{\nu=1}^4 (\mu_\nu \cdot \mathbf{n}) \widehat{\mathbf{e}}_\nu = a \sum_{\nu=1}^4 n_\nu \widehat{\mathbf{e}}_\nu, \quad (4.14)$$

where a is the lattice spacing. On using the fact that $\sum_m \hat{\mathbf{e}}_m = 0$, we can show that a small lattice displacement of the form $d\mathbf{n} = \hat{\boldsymbol{\mu}}_m$ corresponds to a spacetime translation by ($a\hat{\mathbf{e}}_m$):

$$d\mathbf{R} = a \sum_{\nu=1}^4 (\mu_\nu \cdot d\mathbf{n}) \hat{\mathbf{e}}_\nu = a \sum_{\nu=1}^4 (\hat{\boldsymbol{\mu}}_\nu \cdot \hat{\boldsymbol{\mu}}_m) \hat{\mathbf{e}}_\nu = a \hat{\mathbf{e}}_m . \quad (4.15)$$

The supersymmetry transformations on the lattice fields are almost identical to their continuum counterparts:

$$\mathcal{Q}\mathcal{U}_a(\mathbf{n}) = \psi_a(\mathbf{n}) \quad (4.16)$$

$$\mathcal{Q}\psi_a(\mathbf{n}) = 0 \quad (4.17)$$

$$\mathcal{Q}\mathcal{U}_a^\dagger(\mathbf{n}) = 0 \quad (4.18)$$

$$\mathcal{Q}\chi_{ab}(\mathbf{n}) = i(\mathcal{D}_a^{(+)}\mathcal{U}_b)^\dagger(\mathbf{n}) \quad (4.19)$$

$$\mathcal{Q}\eta(\mathbf{n}) = d \quad (4.20)$$

$$\mathcal{Q}d(\mathbf{n}) = 0 \quad (4.21)$$

After the \mathcal{Q} -variation, as performed in the continuum, and integrating out the auxiliary field d , the final lattice action is

$$S = \frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d, e} \text{Tr} \left[\left(\mathcal{D}_a^{(+)}\mathcal{U}_b(\mathbf{n}) \right)^\dagger \left(\mathcal{D}_a^{(+)}\mathcal{U}_b(\mathbf{n}) \right) + \frac{1}{2} \left(\mathcal{D}_a^{\dagger(-)}\mathcal{U}_a(\mathbf{n}) \right)^2 - \chi_{ab}(\mathbf{n}) \mathcal{D}_{[a}^{(+)}\psi_{b]}(\mathbf{n}) \right. \\ \left. - \eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)}\psi_a(\mathbf{n}) - \frac{1}{2} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)}\chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \right]. \quad (4.22)$$

To see that this action targets the continuum twisted theory one needs to expand \mathcal{U}_a about the unit matrix [10]⁴

$$\mathcal{U}_a(\mathbf{n}) = \frac{1}{a} \mathbb{I}_N + i\mathcal{A}_a(\mathbf{n}) , \quad (4.23)$$

$$\mathcal{U}_a^\dagger(\mathbf{n}) = \frac{1}{a} \mathbb{I}_N - i\bar{\mathcal{A}}_a(\mathbf{n}) . \quad (4.24)$$

While the supersymmetric invariance of the \mathcal{Q} -exact term is manifest in the lattice theory it is not immediately clear that the \mathcal{Q} -closed term remains supersymmetric after discretization. Remarkably, this can be shown using a remarkable property of the discrete field strength which can be shown to satisfy an exact Bianchi identity just as for the continuum [29].

$$\epsilon_{abcde} \mathcal{D}_c^{\dagger(-)} \mathcal{F}_{ab}^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) = 0 \quad (4.25)$$

5. Renormalization - General Analysis

Power counting reveals that the continuum four-dimensional theory has an infinite number of superficially divergent Feynman diagrams occurring at all orders of perturbation theory.

⁴To leading order this is equivalent to the more conventional expression $\mathcal{U}_a(x) = \frac{1}{a} e^{ia\mathcal{A}_a(x)}$. We will see that the linear representation offers important advantages over the exponential in our later calculations.

Of course in the continuum target theory all of these potential divergences cancel between diagrams to render the quantum theory finite. However, since the lattice theory does not possess all the supersymmetries of the continuum theory, it is not clear how many of these will continue to cancel in the lattice theory.

Before we embark on a general perturbative analysis of this lattice theory it is instructive to try to ascertain what kinds of counter terms are permitted by the lattice symmetries. In the case of A_4^* lattice, these symmetries are

- a) Exact \mathcal{Q} supersymmetry.
- b) Gauge invariance
- c) S_5 point group symmetry and discrete translations.

In fact, other than exact lattice supersymmetry, the $U(N)$ lattice gauge theory also has a second fermionic symmetry, given by

$$\eta(\mathbf{n}) \rightarrow \eta(\mathbf{n}) + \epsilon \mathbb{1}_N, \quad \delta(\text{all other fields}) = 0 \quad (5.1)$$

where ϵ is an infinitesimal Grassmann parameter. Thus, we extend our list to include

- d) Fermionic shift symmetry

In practice we are primarily interested in relevant or marginal operators; that is operators whose mass dimension is less than or equal to four. We will see that the set of relevant counterterms in the lattice theory is rather short – the lattice symmetries, gauge invariance in particular, being extremely restrictive in comparison to the equivalent situation in the continuum. The argument starts by assigning canonical dimensions to the fields $[\mathcal{U}_a] = 1$, $[\Psi] = \frac{3}{2}$ and $[\mathcal{Q}] = \frac{1}{2}$ where Ψ stands for any of the twisted fermion fields $(\lambda, \psi_a, \chi_{ab})$. Invariance under \mathcal{Q} restricts the possible counterterms to be either of a \mathcal{Q} -exact form, or of \mathcal{Q} -closed form. There is only one \mathcal{Q} -closed operator permitted by the lattice symmetries and it is already present in our bare lattice action. A possible renormalization of this fermion kinetic term is hence allowed. Beyond that the exact lattice supersymmetry forces us to look at the set of \mathcal{Q} -exact counterterms.

Any such counterterm must be of the form $\mathcal{O} = \mathcal{Q}\text{Tr}(\Psi f(\mathcal{U}, \mathcal{U}^\dagger))$. There are thus no terms permitted by symmetries with dimension less than two. In addition gauge invariance tells us that each term must correspond to the trace of a closed loop on the lattice. The smallest dimension gauge invariant operator is then just $\mathcal{Q}(\text{Tr} \psi_a \mathcal{U}_a^\dagger)$. But this vanishes identically since both \mathcal{U}_a^\dagger and ψ_a are singlets under \mathcal{Q} . No dimension $\frac{7}{2}$ operators can be constructed with this structure and we are left with just dimension four counterterms. Notice, in particular that lattice symmetries permit no simple fermion bi-linear mass terms. However, gauge invariant fermion bi-linears with link field insertions are possible and their effect should be accounted for carefully.

Possible dimension four operators are, schematically,

$$L_1 = g^{-2} \mathcal{Q}\text{Tr}(\chi_{ab} \mathcal{U}_a \mathcal{U}_b)$$

$$\begin{aligned}
L_2 &= g^{-2} \mathcal{Q} \text{Tr} (\eta \mathcal{D}_a^\dagger \mathcal{U}_a) \\
L_3 &= g^{-2} \mathcal{Q} \text{Tr} (\eta \mathcal{U}_a \mathcal{U}_a^\dagger) \\
L_4 &= g^{-2} \mathcal{Q} \text{Tr} (\eta) \text{Tr} (\mathcal{U}_a \mathcal{U}_a^\dagger)
\end{aligned} \tag{5.2}$$

The first operator can be simplified on account of the antisymmetry of χ_{ab} to simply $\mathcal{Q}(\chi_{ab} \mathcal{F}_{ab})$, which again is nothing but one of the continuum \mathcal{Q} -exact terms present in the bare action. The second term in (5.2) also corresponds to one of the \mathcal{Q} -exact terms in the bare action. However the third term L_3 is a new operator not present in the bare Lagrangian and the same is true for the final double-trace operator L_4 . Both of these operators transform non-trivially under the fermionic shift symmetry, but a linear combination of the two

$$D = L_3 - \frac{1}{N} L_4 \tag{5.3}$$

is invariant under the shift symmetry with N the rank of the gauge group $U(N)$.

By these arguments it appears that the only relevant counterterms correspond to renormalizations of operators already present in the bare action together with D . This is quite remarkable. The most general form for the renormalized lattice Lagrangian is hence

$$\begin{aligned}
\mathcal{L} &= \sum_{\mathbf{n}, a, b, c, d, e} \left\{ \mathcal{Q} \text{Tr} \left[-i\alpha_1 \chi_{ab} \mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) - i\alpha_2 \eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) + \frac{\alpha_3}{2} \eta(\mathbf{n}) d(\mathbf{n}) \right] \right. \\
&\quad \left. - \frac{\alpha_4}{2} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \right\} + \mathcal{Q} \beta D,
\end{aligned} \tag{5.4}$$

where $(\alpha_i, i = 1 \dots 4)$ and β are dimensionless numbers taking values $(1, 1, 1, 1)$ and 0 respectively in the classical lattice theory. Thus it appears that at most four dimensionless ratios of these couplings might need to be tuned to approach $\mathcal{N} = 4$ Yang-Mills in the continuum limit. Furthermore, since these operators are dimension four we expect this tuning to be at worst logarithmic in the cut-off.

In order to see the explicit form of the D operator close to the continuum limit, we expand the action around $\mathcal{U}_m(\mathbf{n}) = \frac{1}{a} \mathbb{I}$. The result is

$$L_4 \sim \frac{1}{a} \left[\text{Tr} \eta(\mathbf{n}) \left(\sum_{m=1}^5 \psi^m(\mathbf{n}) \right) - \frac{1}{N} \text{Tr} \eta(\mathbf{n}) \text{Tr} \left(\sum_{m=1}^5 \psi^m(\mathbf{n}) \right) \right] \dots \tag{5.5}$$

where ellipsis are dictated by supersymmetry. The reader will immediately realize that $(\sum_{a=1}^5 \psi_a)$ is nothing but the S_5 (and twisted $SO(4)'$) singlet contained in the reducible representation ψ_a . Indeed, it should be clear that it is the only field that could form a fermion mass term by pairing with η .

This is about as far as we can go by just using the lattice symmetries. We now turn to a full perturbative analysis to determine how the couplings (α_i, β) evolve with cut-off.

6. Propagators and Vertices

In this section we derive the propagators and vertices of the gauge-fixed $\mathcal{N} = 4$, $d = 4$ SYM theory on A_4^* lattice. Then we write down the one loop diagrams relevant for the renormalization of the theory.

Upon rewriting the field strength and covariant derivatives in terms of the bosonic link fields $\mathcal{U}_a(\mathbf{n})$, the classical lattice action (4.22) takes the form

$$S = S_B + S_F + S_c , \quad (6.1)$$

where

$$\begin{aligned} S_B &= \frac{1}{g^2} \sum_{\mathbf{n}, a, b} \text{Tr} \left[\left(\mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) \right)^\dagger \left(\mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) \right) + \frac{1}{2} \left(\mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) \right)^2 \right] \\ &= \frac{1}{g^2} \sum_{\mathbf{n}, a, b} \text{Tr} \left[\left(\mathcal{U}_b^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_a) \mathcal{U}_a^\dagger(\mathbf{n}) - \mathcal{U}_a^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_b) \mathcal{U}_b^\dagger(\mathbf{n}) \right) \left(\mathcal{U}_a(\mathbf{n}) \mathcal{U}_b(\mathbf{n} + \hat{\boldsymbol{\mu}}_a) - \mathcal{U}_b(\mathbf{n}) \mathcal{U}_a(\mathbf{n} + \hat{\boldsymbol{\mu}}_b) \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\mathcal{U}_a(\mathbf{n}) \mathcal{U}_a^\dagger(\mathbf{n}) - \mathcal{U}_a^\dagger(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \mathcal{U}_a(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \right)^2 \right] , \end{aligned}$$

$$\begin{aligned} S_F &= -\frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d} \text{Tr} \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) \left[\chi_{ab}(\mathbf{n}) \left(\mathcal{U}_c(\mathbf{n}) \psi_d(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) - \psi_d(\mathbf{n}) \mathcal{U}_c(\mathbf{n} + \hat{\boldsymbol{\mu}}_d) \right) \right] \\ &\quad + \eta(\mathbf{n}) \left(\psi_a(\mathbf{n}) \mathcal{U}_a^\dagger(\mathbf{n}) - \mathcal{U}_a^\dagger(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \psi_a(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \right) , \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} S_c &= -\frac{1}{2g^2} \sum_{\mathbf{n}, a, b, c, d, e} \text{Tr} \epsilon_{abcde} \left(\chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \right. \\ &\quad \left. \times \left[\chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \mathcal{U}_c^\dagger(\mathbf{n}) - \mathcal{U}_c^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b) \chi_{ab}(\mathbf{n}) \right] \right) . \end{aligned} \quad (6.3)$$

To proceed further we expand the \mathcal{U}_a fields around unity as in eqn. 4.23. Notice that this expansion point is but one of an infinite number of classical vacuum solutions – the full moduli space of the lattice theory corresponds to the set of all bosonic field variables $\mathcal{U}_a(\mathbf{n})$ such that

$$\begin{aligned} 0 &= \sum_{\mathbf{n}, a, b} \text{Tr} \left[\left(\mathcal{U}_b^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_a) \mathcal{U}_a^\dagger(\mathbf{n}) - \mathcal{U}_a^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_b) \mathcal{U}_b^\dagger(\mathbf{n}) \right) \left(\mathcal{U}_a(\mathbf{n}) \mathcal{U}_b(\mathbf{n} + \hat{\boldsymbol{\mu}}_a) - \mathcal{U}_b(\mathbf{n}) \mathcal{U}_a(\mathbf{n} + \hat{\boldsymbol{\mu}}_b) \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\mathcal{U}_a(\mathbf{n}) \mathcal{U}_a^\dagger(\mathbf{n}) - \mathcal{U}_a^\dagger(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \mathcal{U}_a(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \right)^2 \right] . \end{aligned}$$

These equations possess a large class of solutions corresponding to constant diagonal matrices modulo gauge transformations. We will use this additional freedom later when we compute the one loop contribution to the effective action of the theory.

6.1 The Bosonic Propagators

As usual it is easiest to compute the Feynman diagrams in momentum space. On the A_4^* lattice a generic field $\Phi(\mathbf{x})$ has Fourier expansion

$$\Phi(\mathbf{x}) = \frac{1}{(La)^4} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} \Phi_{\mathbf{p}} \quad (6.4)$$

where $\mathbf{x} = a \sum_{a=1}^4 n_a \hat{\mathbf{e}}_a$ denotes the position on A_4^* and the momenta lie on the dual lattice given by $\mathbf{p} = \frac{2\pi}{La} \sum_{a=1}^4 m_a \hat{\mathbf{g}}_a$ (for a lattice with spacing a and length L). The dual basis vectors $\hat{\mathbf{g}}_a, a = 1 \dots 4$ satisfy

$$\hat{\mathbf{e}}_a \cdot \hat{\mathbf{g}}_b = \delta_{ab} \quad (6.5)$$

On an L^4 lattice both sets of lattice coordinates n_a, m_a take integer values in the range $-L/2 + 1, \dots, L/2$. We will assume periodic boundary conditions in all directions in this paper. Eqn. 6.4 implies that fields are automatically invariant under translations by a lattice length in any direction and a field shifted by one of the basis vectors can be expressed as⁵

$$\Phi(\mathbf{x} + \hat{\mathbf{e}}_a) = \sum_{\mathbf{p}} e^{ip_a} e^{i\mathbf{p} \cdot \mathbf{x}} \Phi_{\mathbf{p}} \quad (6.6)$$

where $p_a = \frac{2\pi}{L} m_a$. The only remaining is the question of how to deal with shifts in the lattice action associated with the additional $\hat{\mathbf{e}}_5$ vector. However, the solution is simple: since $\sum_{a=1}^5 \hat{\mathbf{e}}_a = 0$ we simply replace any $\hat{\mathbf{e}}_5$ shift encountered in the action by the equivalent shift $-\sum_{a=1}^4 \hat{\mathbf{e}}_a$. One might have worried about an apparent lack of rotational invariance associated with the naïve continuum limit of terms in the action which resemble $\sum_{a=1}^5 \sin^2 p_a$. However, putting $p_a = \mathbf{p} \cdot \hat{\mathbf{e}}_a$ and taking the naïve continuum limit this becomes

$$\sum_{a=1}^5 p_a^2 = \sum_{\mu, \nu} \sum_{a=1}^4 p_\mu p_\nu \hat{\mathbf{e}}_a^\mu \hat{\mathbf{e}}_a^\nu = \sum_{\mu} p_\mu^2 \quad (6.7)$$

which has the correct rotationally invariant form since the Greek indices refer to a Cartesian basis.

Using these ideas the bosonic action when expanded around (4.23) and (4.24) gives the following second-order term in Fourier space

$$\begin{aligned} S_B^{(2)} \approx & 2 \sum_{\mathbf{k}, a, b} \text{Tr} \left(\bar{\mathcal{A}}_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) - f_a^*(\mathbf{k}) f_b(\mathbf{k}) \right] \mathcal{A}_b(-\mathbf{k}) \right. \\ & \left. + B_a(\mathbf{k}) \left[f_a^*(\mathbf{k}) f_b(\mathbf{k}) \right] B_b(-\mathbf{k}) \right), \end{aligned} \quad (6.8)$$

where

$$f_a(\mathbf{k}) = (e^{ik_a} - 1). \quad (6.9)$$

We need to gauge-fix the bosonic action before we derive the propagators. A natural gauge-fixing choice would be an obvious generalization of Lorentz gauge-fixing [26]

$$G(\mathbf{n}) = \sum_a \left(\partial_a^{(-)} \mathcal{A}_a(\mathbf{n}) + \partial_a^{(-)} \bar{\mathcal{A}}_a(\mathbf{n}) \right). \quad (6.10)$$

This gauge-fixing choice adds the following term to the bosonic action at quadratic order

$$S_{GF} = \frac{1}{4\alpha} \sum_{\mathbf{n}} G^2(\mathbf{n}) = \frac{1}{\alpha} \sum_{\mathbf{n}, a} \text{Tr} (\partial_a^{(-)} \mathcal{A}_a(\mathbf{n}))^2, \quad (6.11)$$

⁵For simplicity we will adopt the convention that momentum sums $\sum_{\mathbf{k}}$ automatically include the $1/(La)^4$ normalization factor.

$$\begin{array}{ccc} \text{~~~~~} & & \\ \mathcal{A}_a^A(-\mathbf{k}) & \xrightarrow{\mathbf{k}} & \bar{\mathcal{A}}_b^B(\mathbf{k}) \end{array} \quad \longrightarrow \quad \delta_{ab}\delta_{AB}\frac{1}{\widehat{\mathbf{k}}^2}$$

Figure 1: The bosonic propagator.

where $\partial_a^{(-)} f(\mathbf{n}) = f(\mathbf{n}) - f(\mathbf{n} - \hat{\boldsymbol{\mu}}_a)$. On using the relation $\sum_{\mathbf{n}} (\partial_a^{(+)} f(\mathbf{n})) g(\mathbf{n}) = -\sum_{\mathbf{n}} f(\mathbf{n}) \partial_a^{(-)} g(\mathbf{n})$, the gauge-fixing term becomes

$$S_{GF} = -\frac{1}{\alpha} \sum_{\mathbf{n}, a, b} \text{Tr} A_a(\mathbf{n}) \partial_a^{(+)} \partial_b^{(-)} A_b(\mathbf{n}). \quad (6.12)$$

In momentum space it becomes

$$S_{GF} = \frac{1}{\alpha} \sum_{\mathbf{k}, a, b} \text{Tr} A_a(\mathbf{k}) f_a^*(\mathbf{k}) f_b(\mathbf{k}) A_b(-\mathbf{k}). \quad (6.13)$$

Thus the gauge-fixed bosonic action to quadratic order is

$$\begin{aligned} S_B^{(2)} + S_{GF} \approx & 2 \sum_{\mathbf{k}, a, b, c} \text{Tr} \left(A_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) - \left(1 - \frac{1}{2\alpha}\right) f_a^*(\mathbf{k}) f_b(\mathbf{k}) \right] A_b(-\mathbf{k}) \right. \\ & \left. + B_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) \right] B_b(-\mathbf{k}) \right). \end{aligned} \quad (6.14)$$

The choice $\alpha = 1/2$ makes the above expression diagonal

$$\begin{aligned} S_B^{(2)} & \approx 2 \sum_{\mathbf{k}, a, b, c} \text{Tr} \bar{\mathcal{A}}_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) \right] \mathcal{A}_b(-\mathbf{k}) \\ & = 2 \sum_{\mathbf{k}, a, b} \text{Tr} \left[\bar{\mathcal{A}}_a(\mathbf{k}) \delta_{ab} \left(4 \sum_c \sin^2 \left(\frac{k_c}{2} \right) \right) \mathcal{A}_b(-\mathbf{k}) \right]. \end{aligned} \quad (6.15)$$

Putting in the trace (using the convention $\text{Tr}(T^A T^B) = \frac{1}{2} \delta_{AB}$) the quadratic bosonic action can be written as

$$S_B^{(2)} \approx \sum_{\mathbf{k}, a, b} \bar{\mathcal{A}}_a^A(\mathbf{k}) M_{ab}^{AB}(\mathbf{k}) \mathcal{A}_b^B(-\mathbf{k}), \quad (6.16)$$

where $M_{ab}^{AB}(\mathbf{k}) = \widehat{\mathbf{k}}^2 \delta_{ab} \delta_{AB}$, with $\widehat{\mathbf{k}}^2 = 4 \sum_c \sin^2 \left(\frac{k_c}{2} \right)$. Thus only the $\mathcal{A}\bar{\mathcal{A}}$ propagator is non-zero and it is given by (See figure 1.)

$$\langle \mathcal{A}_a^A(-\mathbf{k}) \bar{\mathcal{A}}_b^B(\mathbf{k}) \rangle = \delta_{ab} \delta_{AB} \frac{1}{\widehat{\mathbf{k}}^2}. \quad (6.17)$$

6.2 The Fermionic Propagators

The fermionic part of the action is of the form

$$S_F = -\frac{1}{g^2} \sum_{abcde} \left(\chi_{ab} \mathcal{D}_{[a}^{(+)} \psi_b \right] + \eta \mathcal{D}_a^{\dagger(-)} \psi_a + \frac{1}{2} \epsilon_{abcde} \chi_{ab} \mathcal{D}_c^{\dagger(-)} \chi_{de} \Big) \quad (6.18)$$

Explicitly, we have

$$S_F = -\frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d, e} \text{Tr} \left(\chi_{ab}(\mathbf{n}) \mathcal{D}_{[a}^{(+)} \psi_{b]}(\mathbf{n}) + \eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \psi_a(\mathbf{n}) \right. \\ \left. + \frac{1}{2} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \right), \quad (6.19)$$

when expanded up to second order in the fields using (4.23) and (4.24), it becomes

$$S_F^{(2)} \approx \frac{1}{g^2} \sum_{\mathbf{k}, a, b, c, d, e} \text{Tr} \chi_{ab}(\mathbf{k}) \left[-f_a^*(\mathbf{k}) \delta_{bc} + f_b^*(\mathbf{k}) \delta_{ac} \right] \psi_c(-\mathbf{k}) + \eta(\mathbf{k}) f_c(\mathbf{k}) \psi_c(-\mathbf{k}) \\ + \frac{1}{2} \epsilon_{abcde} \chi_{de}(\mathbf{k}) e^{i(k_a + k_b)} f_c(\mathbf{k}) \chi_{ab}(-\mathbf{k}) \quad (6.20)$$

Upon restricting the sum and rescaling the field $2\chi_{ab} \rightarrow \chi_{ab}$ the fermionic action becomes

$$S_F^{(2)} \approx \frac{1}{g^2} \sum_{\mathbf{k}, a < b, c, d < e} \text{Tr} \left(\chi_{ab}(\mathbf{k}) \left[-f_a^*(\mathbf{k}) \delta_{bc} + f_b^*(\mathbf{k}) \delta_{ac} \right] \psi_c(-\mathbf{k}) + \eta(\mathbf{k}) f_c(\mathbf{k}) \psi_c(-\mathbf{k}) \right. \\ \left. + \frac{1}{2} \epsilon_{abcde} \chi_{de}(\mathbf{k}) e^{i(k_a + k_b)} f_c(\mathbf{k}) \chi_{ab}(-\mathbf{k}) \right) \quad (6.21)$$

We can then write this action in the form of a matrix product

$$S_F^{(2)} \approx \frac{1}{g^2} \sum_{\mathbf{k}} (\Psi(\mathbf{k}) \Psi(-\mathbf{k})) \begin{pmatrix} \frac{1}{4} & \\ & -M^T(\mathbf{k}) \end{pmatrix} \begin{pmatrix} 0 & M(\mathbf{k}) \\ & 0 \end{pmatrix} \begin{pmatrix} \Psi(\mathbf{k}) \\ \Psi(-\mathbf{k}) \end{pmatrix} \\ = \frac{1}{4g^2} \sum_{\mathbf{k}} \Phi(\mathbf{k}) \mathcal{M} \Phi(\mathbf{k}) \quad (6.22)$$

where $\Phi \equiv (\Psi(\mathbf{k}), \Psi(-\mathbf{k}))$ and $\Psi_i = (\eta, \psi_1, \dots, \psi_5, \chi_{12}, \dots, \chi_{15}, \dots, \chi_{45})$ and $M(\mathbf{k})$ is given in block matrix form

$$(\eta \ \psi_a \ \chi_{de})(\mathbf{k}) \begin{pmatrix} 0 & f_b(\mathbf{k}) & 0 \\ -f_a^*(\mathbf{k}) & 0 & f_g(\mathbf{k}) \delta_{ha} - f_h(\mathbf{k}) \delta_{ga} \\ 0 & -f_d^*(\mathbf{k}) \delta_{eb} + f_e^*(\mathbf{k}) \delta_{db} & \epsilon_{ghcde} q_{gh} f_c(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \eta \\ \psi_b \\ \chi_{gh} \end{pmatrix} (-\mathbf{k}).$$

where $q_{gh} = e^{i(k_g + k_h)}$. Notice that M has the properties $M^T(\mathbf{k}) = -M^*(\mathbf{k}) = -M(-\mathbf{k})$.

Using the property that $\sum_a \hat{\boldsymbol{\mu}}_a = 0$ we can square the matrix to obtain

$$M^2(\mathbf{k}) = -\sum_{a=1}^5 |e^{ik_a} - 1|^2 \mathbb{I}_{16} = -4 \sum_{a=1}^5 \sin^2 \left(\frac{k_a}{2} \right) \mathbb{I}_{16} = -\widehat{\mathbf{k}}^2 \mathbb{I}_{16}. \quad (6.23)$$

Thus

$$M^{-1} = -\frac{1}{\widehat{\mathbf{k}}^2} M \quad (6.24)$$

and the inverse of the full fermion matrix is

$$\mathcal{M}^{-1} = -\frac{1}{\widehat{\mathbf{k}}^2} \begin{pmatrix} 0 & -M^T(\mathbf{k}) \\ M(\mathbf{k}) & 0 \end{pmatrix}. \quad (6.25)$$

$$\begin{aligned}
\eta^A(-\mathbf{k}) \longrightarrow \begin{array}{c} \longrightarrow \\ \mathbf{k} \end{array} \longrightarrow \psi_a^B(\mathbf{k}) &\longrightarrow \delta_{AB} \frac{2}{\widehat{\mathbf{k}}^2} (e^{ik_a} - 1) \\
\psi_a^A(-\mathbf{k}) \longrightarrow \begin{array}{c} \longrightarrow \\ \mathbf{k} \end{array} \longrightarrow \chi_{bc}^B(\mathbf{k}) &\longrightarrow \delta_{AB} \frac{1}{\widehat{\mathbf{k}}^2} \left[(e^{ik_b} - 1)\delta_{ac} - (e^{ik_c} - 1)\delta_{ab} \right] \\
\chi_{ab}^A(-\mathbf{k}) \longrightarrow \begin{array}{c} \longrightarrow \\ \mathbf{k} \end{array} \longrightarrow \chi_{de}^B(\mathbf{k}) &\longrightarrow \delta_{AB} \frac{1}{2\widehat{\mathbf{k}}^2} \epsilon_{abcde} e^{i(k_d+k_e)} (e^{ik_c} - 1)
\end{aligned}$$

Figure 2: The fermionic propagators.

Then we can write the quadratic part of the fermionic action as

$$\begin{aligned}
S_F^{(2)} &= \frac{1}{4g^2} \sum_{\mathbf{k}} \text{Tr} \left[\sum_{ij} \Phi_i(\mathbf{k}) \mathcal{M}_{ij}(\mathbf{k}) \Phi_j(\mathbf{k}) \right] \\
&= \frac{1}{4g^2} \sum_{\mathbf{k}} \sum_{ij,A,B} \Phi_i^A(\mathbf{k}) \mathcal{M}_{ij}(\mathbf{k}) \Phi_j^B(\mathbf{k}) \text{Tr} (T^A T^B) \\
&= \frac{1}{8g^2} \sum_{\mathbf{k}} \sum_{ij,A,B} \Phi_i^A(\mathbf{k}) \mathcal{M}_{ij}(\mathbf{k}) \Phi_j^B(\mathbf{k}) \delta_{AB} , \tag{6.26}
\end{aligned}$$

where we have expanded the fermions as $\Phi = \Phi^A T^A$ and used $\text{Tr} (T^A T^B) = \frac{1}{2} \delta_{AB}$. Thus we write the propagators as

$$\langle \Phi_i^A(\mathbf{k}) \Phi_j^B(\mathbf{k}) \rangle = 2\mathcal{M}_{ij}^{-1}(\mathbf{k}) \delta_{AB} \tag{6.27}$$

or alternatively

$$\langle \Psi_i^A(\mathbf{k}) \Psi_j^B(-\mathbf{k}) \rangle = \frac{2}{\widehat{\mathbf{k}}^2} M_{ij}^T(\mathbf{k}) \delta_{AB} . \tag{6.28}$$

Notice that by switching the fields (with some relabeling) we have

$$\langle \Psi_i^A(-\mathbf{k}) \Psi_j^B(\mathbf{k}) \rangle = -\langle \Psi_j^B(\mathbf{k}) \Psi_i^A(-\mathbf{k}) \rangle = -\frac{2}{\widehat{\mathbf{k}}^2} M_{ji}^T(\mathbf{k}) \delta_{BA} = -\frac{2}{\widehat{\mathbf{k}}^2} M_{ij}(\mathbf{k}) \delta_{AB} . \tag{6.29}$$

For a consistency check we replace \mathbf{k} with $-\mathbf{k}$ and get

$$\langle \Psi_i^A(-\mathbf{k}) \Psi_j^B(\mathbf{k}) \rangle = \frac{2}{\widehat{\mathbf{k}}^2} M_{ij}^T(-\mathbf{k}) \delta_{AB} = -\frac{2}{\widehat{\mathbf{k}}^2} M_{ij}(\mathbf{k}) \delta_{AB} . \tag{6.30}$$

We must also undo the earlier rescaling of the χ field giving a factor of $\frac{1}{2}$ in the $\psi\chi$ propagators and a factor of $\frac{1}{4}$ in the $\chi\chi$ propagators. It is also important to note that if we switch the direction of fermion flow in the propagators then we pick up an additional minus sign.

6.3 The Vertices

Before we write down the expressions for vertices, let us further fix our conventions on the trace algebra. For the generators T^A of $U(N)$ one has

$$T^A T^B = \frac{1}{2}(d_{ABC} + if_{ABC})T^C. \quad (6.31)$$

where d_{ABC} and f_{ABC} are the symmetric and antisymmetric structure constants, respectively. This product formula is consistent with our previous trace convention $\text{Tr}(T^A T^B) = \frac{1}{2}\delta_{AB}$ and in addition yields the results

$$\begin{aligned} \text{Tr}(T^A T^B T^C) &= \text{Tr}\left(\frac{1}{2}(d_{ABD} + if_{ABD})T^D T^C\right) \\ &= \frac{1}{2}(d_{ABD} + if_{ABD})\text{Tr}[T^D T^C] \\ &= \frac{1}{2}(d_{ABD} + if_{ABD})\frac{1}{2}\delta_{DC} \\ &= \frac{1}{4}(d_{ABC} + if_{ABC}) = \frac{1}{4}\lambda_{ABC}. \end{aligned} \quad (6.32)$$

Since f_{ABC} is antisymmetric and d_{ABC} is symmetric it follows that

$$\lambda_{ACB} = \bar{\lambda}_{ABC}. \quad (6.33)$$

To extract expressions for the vertices we now return to the original gauge-fixed action for the theory given by

$$\begin{aligned} S &= \frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d, e} \text{Tr} \left[\left(\mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) \right)^\dagger \left(\mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) \right) + \frac{1}{2} \left(\mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) \right)^2 \right. \\ &\quad + 2A_a(\mathbf{n}) \partial_a^{(+)} \partial_b^{(-)} A_b(\mathbf{n}) - \left(\chi_{ab}(\mathbf{n}) \mathcal{D}_{[a}^{(+)} \psi_{b]}(\mathbf{n}) + \eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \psi_a(\mathbf{n}) \right. \\ &\quad \left. \left. + \frac{1}{2} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \right) \right]. \end{aligned} \quad (6.34)$$

The last three terms of the action give rise to vertices between varying number of \mathcal{A} 's and the fermions η , ψ_a , and χ_{ab} . There are three vertices that arise at linear order in \mathcal{A} :

- The $\psi \bar{\mathcal{A}} \eta$ vertex

$$\begin{aligned} V_{\psi \bar{\mathcal{A}} \eta} &= - \sum_{\mathbf{n}, a} \text{Tr} \left(\eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \psi_a(\mathbf{n}) \right) \\ &= - \sum_{\mathbf{n}, a} \text{Tr} \left(\eta(\mathbf{n}) \psi_a(\mathbf{n}) \mathcal{U}_a^\dagger(\mathbf{n}) - \eta(\mathbf{n}) \mathcal{U}_a^\dagger(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \psi_a(\mathbf{n} - \hat{\boldsymbol{\mu}}_a) \right) \\ &= - \sum_{\mathbf{n}, \mathbf{k}, \mathbf{q}, \mathbf{p}, a} \text{Tr} \left(e^{i(\mathbf{k} + \mathbf{q} + \mathbf{p}) \cdot \mathbf{n}} \left(\eta(\mathbf{k}) \psi_a(\mathbf{q}) (-i) \bar{\mathcal{A}}_a(\mathbf{p}) - \eta(\mathbf{k}) (-i) \bar{\mathcal{A}}_a(\mathbf{p}) e^{ip_a} \psi_a(\mathbf{q}) e^{iq_a} \right) \right) \\ &= \sum_{\mathbf{k}, \mathbf{q}, \mathbf{p}} \delta_{-\mathbf{k}, \mathbf{q} + \mathbf{p}} \eta^C(\mathbf{k}) \bar{\mathcal{A}}_b^B(\mathbf{p}) \psi_a^A(\mathbf{q}) \left(\frac{i}{4} \right) \delta_{ab} [\lambda_{ABC} - \bar{\lambda}_{ABC} e^{-i(p_a + q_a)}]. \end{aligned} \quad (6.35)$$

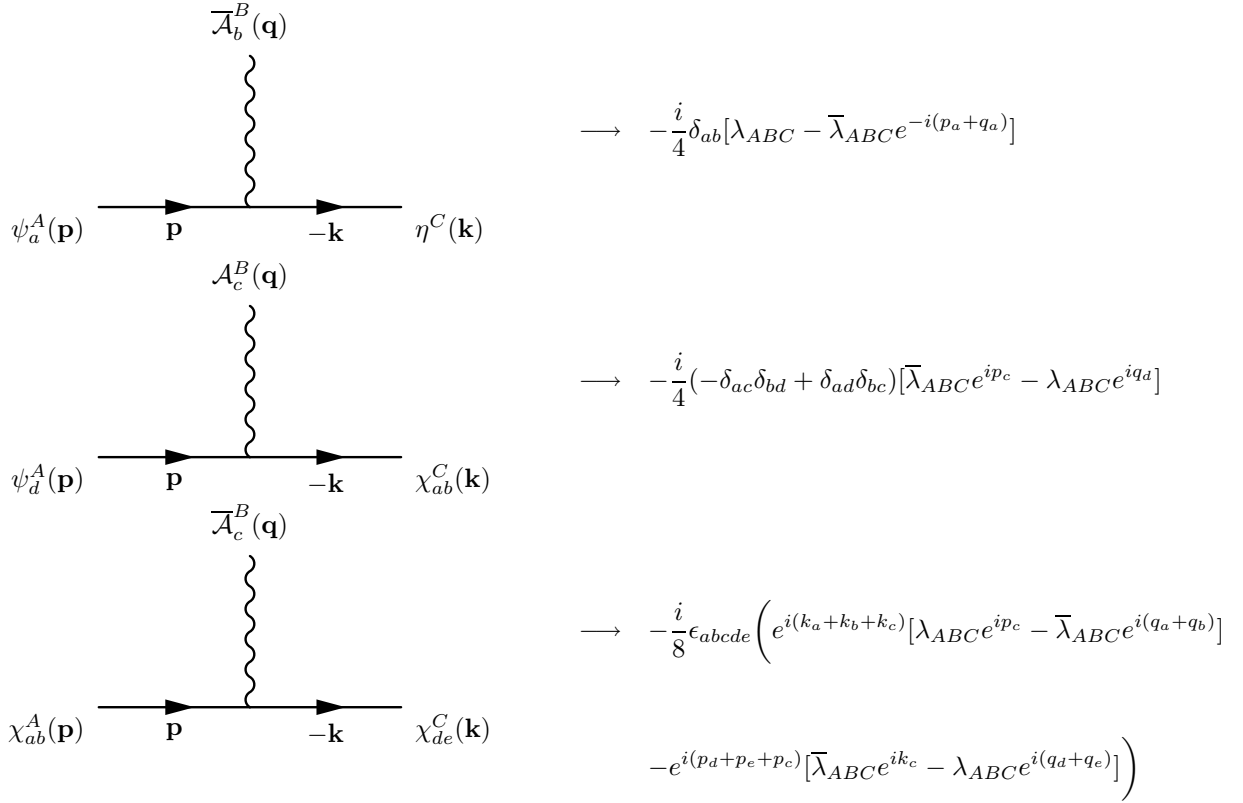


Figure 3: The vertices connecting fermions and complexified gauge fields.

Thus the Feynman diagram contribution for this vertex is (add a minus since it comes from the first order term of e^{-S})

$$V_{\eta\overline{\mathcal{A}}\psi} = -\frac{i}{4}\delta_{ab}[\lambda_{ABC} - \overline{\lambda}_{ABC}e^{-i(p_a+q_a)}]. \quad (6.36)$$

- The $\psi\mathcal{A}\chi$ vertex

$$\begin{aligned}
V_{\psi\mathcal{A}\chi} &= -\sum_{\mathbf{n},a,b} \text{Tr} \chi_{ab}(\mathbf{n}) \mathcal{D}_{[a}^{(+)} \psi_{b]}(\mathbf{n}) \\
&= \sum_{\mathbf{n},a,b} \text{Tr} \left(-\chi_{ab}(\mathbf{n}) \mathcal{D}_a^{(+)} \psi_b(\mathbf{n}) + \chi_{ab}(\mathbf{n}) \mathcal{D}_b^{(+)} \psi_a(\mathbf{n}) \right) \\
&= \sum_{\mathbf{n},a,b,c,d} (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \text{Tr} \left[\chi_{ab}(\mathbf{n}) \left(\mathcal{U}_c(\mathbf{n}) \psi_d(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) - \psi_d(\mathbf{n}) \mathcal{U}_c(\mathbf{n} + \hat{\boldsymbol{\mu}}_d) \right) \right] \\
&= \sum_{\mathbf{k},\mathbf{q},\mathbf{p},a,b,c,d} \delta_{-\mathbf{k},\mathbf{q}+\mathbf{p}} (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \chi_{ab}^C(\mathbf{k}) \mathcal{A}_c^B(\mathbf{q}) \psi_d^A(\mathbf{p}) \frac{i}{4} [\overline{\lambda}_{ABC}e^{ip_c} - \lambda_{ABC}e^{iq_d}]. \quad (6.37)
\end{aligned}$$

The vertex is given by

$$V_{\chi\mathcal{A}\psi} = -\frac{i}{4}(-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})[\overline{\lambda}_{ABC}e^{ip_c} - \lambda_{ABC}e^{iq_d}]. \quad (6.38)$$

- The $\chi\bar{\mathcal{A}}\chi$ vertex

$$\begin{aligned}
V_{\chi\bar{\mathcal{A}}\chi} &= -\frac{1}{2} \sum_{\mathbf{n},a,b,c,d,e} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \\
&= -\frac{1}{2} \sum_{\mathbf{n},a,b,c,d,e} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b + \hat{\boldsymbol{\mu}}_c) \left(\chi_{ab}(\mathbf{n} + \hat{\boldsymbol{\mu}}_c) \mathcal{U}_c^\dagger(\mathbf{n}) - \mathcal{U}_c^\dagger(\mathbf{n} + \hat{\boldsymbol{\mu}}_a + \hat{\boldsymbol{\mu}}_b) \chi_{ab}(\mathbf{n}) \right) \\
&= \frac{1}{2} \sum_{\mathbf{k},\mathbf{p},\mathbf{q},a,b,c,d,e} \delta_{-\mathbf{k},\mathbf{q}+\mathbf{p}} \epsilon_{abcde} \chi_{de}^C(\mathbf{k}) \bar{\mathcal{A}}_c^B(\mathbf{q}) \chi_{ab}^A(\mathbf{p}) \left(e^{i(k_a+k_b+k_c)} \frac{i}{4} [\lambda_{ABC} e^{ip_c} - \bar{\lambda}_{ABC} e^{i(q_a+q_b)}] \right. \\
&\quad \left. - e^{i(p_d+p_e+p_c)} \frac{i}{4} [\bar{\lambda}_{ABC} e^{ik_c} - \lambda_{ABC} e^{i(q_d+q_e)}] \right).
\end{aligned}$$

The vertex is given by (taking into account both possible contractions with external propagators)

$$V_{\chi\bar{\mathcal{A}}\chi} = -\frac{i}{8} \epsilon_{abcde} \left(e^{i(k_a+k_b+k_c)} [\lambda_{ABC} e^{ip_c} - \bar{\lambda}_{ABC} e^{i(q_a+q_b)}] - e^{i(p_d+p_e+p_c)} [\bar{\lambda}_{ABC} e^{ik_c} - \lambda_{ABC} e^{i(q_d+q_e)}] \right). \quad (6.39)$$

7. One Loop Diagrams for the Renormalized Fermion Propagators

Using these propagators and vertices it is straightforward to see that the renormalized fermion propagators receive contributions from the following four *amputated* diagrams.

- The amputated $\eta\psi$ diagram. We have an $\bar{\mathcal{A}}\mathcal{A}$ propagator, a $\psi\chi$ propagator, an $\eta\bar{\mathcal{A}}\psi$ vertex, and a $\chi\mathcal{A}\psi$ vertex. Using the expressions above we have

$$\begin{aligned}
I_{\eta\psi}(\mathbf{p}) &= \sum_{\mathbf{k},\mathbf{q}} \sum_{BC} \sum_{abc} \delta_{-\mathbf{p},\mathbf{k}+\mathbf{q}} \left[\frac{1}{\widehat{\mathbf{k}}^2} [(e^{ik_b} - 1)\delta_{ac} - (e^{ik_c} - 1)\delta_{ab}] \right] \cdot \left[\frac{1}{\widehat{\mathbf{q}}^2} \right] \cdot \left[\frac{i}{4} [\lambda_{ABC} - \bar{\lambda}_{ABC} e^{i(k_a+q_a)}] \right] \\
&\quad \cdot \left[\frac{i}{4} (-\delta_{ba}\delta_{cd} + \delta_{bd}\delta_{ca}) [\bar{\lambda}_{BCD} e^{-ip_a} - \lambda_{BCD} e^{iq_d}] \right]. \quad (7.1)
\end{aligned}$$

- The first amputated $\psi\chi$ diagram. We have an $\bar{\mathcal{A}}\bar{\mathcal{A}}$ propagator, a $\chi\chi$ propagator, a $\psi\mathcal{A}\chi$ vertex, and a $\chi\bar{\mathcal{A}}\chi$ vertex.

$$\begin{aligned}
I_{\psi\chi}^1(\mathbf{p}) &= \sum_{\mathbf{k},\mathbf{q}} \sum_{bcdefm} \sum_{BC} \left[\frac{1}{2\widehat{\mathbf{k}}^2} \epsilon_{bcme} f e^{i(k_e+k_f)} (e^{ik_m} - 1) \right] \\
&\quad \cdot \left[\frac{1}{\widehat{\mathbf{q}}^2} \right] \cdot \left[-\frac{i}{4} (-\delta_{bd}\delta_{ca} + \delta_{ba}\delta_{cd}) [\bar{\lambda}_{ACB} e^{ip_d} - \lambda_{ACB} e^{-iq_a}] \right] \\
&\quad \cdot \left[\frac{i}{8} \epsilon_{efdg} h \left(e^{ik(d+g+h)} [\bar{\lambda}_{BCD} e^{-ip_d} - \lambda_{BCD} e^{i(q_g+q_h)}] \right. \right. \\
&\quad \left. \left. - e^{-ip(d+e+f)} [\lambda_{BCD} e^{ik_d} - \bar{\lambda}_{BCD} e^{i(q_e+q_f)}] \right) \right]. \quad (7.2)
\end{aligned}$$

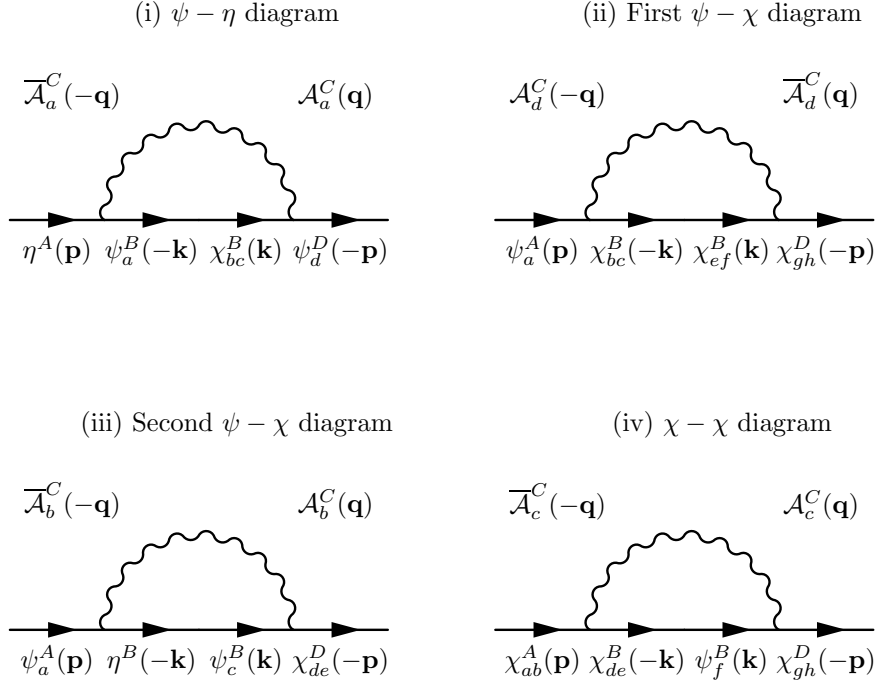


Figure 4: The one loop diagrams of fermions and complexified gauge fields.

- The second amputated $\psi\chi$ diagram has an $\bar{\mathcal{A}}\mathcal{A}$ propagator, an $\eta\psi$ propagator, a $\psi\bar{\mathcal{A}}\eta$ vertex, and a $\psi\mathcal{A}\chi$ vertex. This yields

$$\begin{aligned}
I_{\psi\chi}^2(\mathbf{p}) = & \sum_{\mathbf{k}, \mathbf{q}} \sum_{bc} \sum_{BC} \left[\frac{2}{\widehat{\mathbf{k}}^2} (e^{ik_c} - 1) \right] \cdot \left[\frac{1}{\widehat{\mathbf{q}}^2} \right] \cdot \delta_{ab} \left[-\frac{i}{4} [\lambda_{ACB} - \bar{\lambda}_{ACB} e^{-i(p_a - q_a)}] \right] \\
& \cdot \left[-\frac{i}{4} (-\delta_{db}\delta_{ec} + \delta_{dc}\delta_{eb}) [\lambda_{DCB} e^{ik_b} - \bar{\lambda}_{DCB} e^{iq_c}] \right]. \quad (7.3)
\end{aligned}$$

- The amputated $\chi\chi$ diagram. It has a $\bar{\mathcal{A}}\mathcal{A}$ propagator, a $\chi\psi$ propagator, a $\chi\bar{\mathcal{A}}\chi$ vertex, and a $\psi\mathcal{A}\chi$.

$$\begin{aligned}
I_{\chi\chi}(\mathbf{p}) = & \sum_{\mathbf{k}, \mathbf{q}} \sum_{cdef} \sum_{BC} \delta_{\mathbf{k}+\mathbf{q}-\mathbf{p}, 0} \left[\frac{1}{\widehat{\mathbf{k}}^2} [(e^{-ik_e} - 1)\delta_{fd} - (e^{-ik_d} - 1)\delta_{fe}] \right] \cdot \left[\frac{1}{\widehat{\mathbf{q}}^2} \right] \\
& \cdot \left[-\frac{i}{8} \epsilon_{abcde} \left(e^{-ik_{(a+b+c)}} [\lambda_{ACB} e^{ip_c} - \bar{\lambda}_{ACB} e^{-i(q_a + q_b)}] \right. \right. \\
& \left. \left. - e^{ip_{(c+d+e)}} [\bar{\lambda}_{ACB} e^{-ik_c} - \lambda_{ACB} e^{-i(q_d + q_e)}] \right) \right] \\
& \cdot \left[-\frac{i}{4} (-\delta_{gc}\delta_{hf} + \delta_{gf}\delta_{hc}) [\bar{\lambda}_{BCD} e^{ik_c} - \lambda_{BCD} e^{iq_f}] \right]. \quad (7.4)
\end{aligned}$$

In appendix A we show that the contributions of these diagrams all vanish in the limit $p \rightarrow 0$ indicating that mass counterterms are absent in the lattice theory at one loop. In our general argument of section 5 we argued that the only dangerous mass term involved a coupling of η and ψ_a . We now see that this term does not arise at one loop. In the next

section we will show that this feature persists to all orders and thus our general conclusion will be that *no mass counterterms are needed at any finite order of perturbation theory*.

8. Effective Action

In this section we will compute the partition function of the lattice theory in one loop order around an arbitrary classical vacuum state in which the fermions vanish and the bosonic fields correspond to constant commuting matrices. To start we expand the fields around such a constant commuting background,

$$\mathcal{U}_a(\mathbf{n}) = \mathcal{U}_a + i\mathcal{A}_a(\mathbf{n}), \quad \mathcal{U}_a^\dagger(\mathbf{n}) = \mathcal{U}_a^\dagger - i\bar{\mathcal{A}}_a(\mathbf{n}) \quad (8.1)$$

Choosing the gauge $\alpha = 1/2$, the quadratic part of the bosonic action then takes the form

$$S_B = -2 \sum_{\mathbf{n}, a, b} \text{Tr} \mathcal{A}_b(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \mathcal{D}_a^{(+)} \mathcal{A}_b(\mathbf{n}) . \quad (8.2)$$

Here the covariant derivatives depend on the constant commuting classical background $[\mathcal{U}_a, \mathcal{U}_a^\dagger] = 0$. After integration over the fluctuations in the bosonic fields one finds the bosonic contribution to the one loop partition function is given by

$$\det^{-5}(\mathcal{D}_a^{\dagger(-)} \mathcal{D}_a^{(+)}) \quad (8.3)$$

The gauge fixing functional (6.10) leads to the quadratic ghost action

$$S_G = \sum_{\mathbf{n}, a} \text{Tr} \bar{c} \mathcal{D}_a^{\dagger(-)} \mathcal{D}_a^{(+)} c . \quad (8.4)$$

The quadratic fermionic part of the action is given by the corresponding terms in (4.22), except that now the covariant derivatives depend only on the background fields.

Since the background is constant, we can pass to momentum space in which the action separates into terms for each mode \mathbf{k} . The 16×16 fermion matrix $M(\mathbf{k})$ for the mode \mathbf{k} then can be shown (using MAPLE to compute the determinant) to satisfy

$$\det M(\mathbf{k}) = \det(\mathcal{D}_a^{\dagger(-)}(\mathbf{k}) \mathcal{D}_a^{(+)}(\mathbf{k}))^8 . \quad (8.5)$$

Going back to position space, and taking into account the fact that there is a double counting of modes in the matrix form (6.22), we obtain

$$\text{Pf}(\mathcal{M}) = \det^4(\bar{\mathcal{D}}_a^{\dagger(-)} \mathcal{D}_a^{(+)}) . \quad (8.6)$$

The ghosts add another factor of $\det(\bar{\mathcal{D}}_a^{\dagger(-)} \mathcal{D}_a^{(+)})$, which is just what is needed to cancel the bosonic contribution given earlier.

In conclusion, we have shown that the one loop effective action of the lattice theory obtained by expanding about an arbitrary point in the classical moduli space is identically zero. Thus, as for the continuum, the moduli space is not lifted in this analysis and hence there can be no boson or fermion masses at one loop. Furthermore, we expect

that we can extend this analysis to all loops since the partition function of the lattice theory is a topological invariant and hence can be computed exactly in the semi-classical approximation (see Appendix C). Indeed, Matsuura uses similar arguments to show that the vacuum energy of supersymmetric lattice theories with four and eight supercharges remains zero to all orders in the coupling [30]. The calculation presented here extends this to the case of sixteen supercharges⁶. Thus we conclude that boson and scalar masses remain zero to all orders in the coupling constant. This implies that the fermions also remain massless which is consistent with our explicit one loop calculation.

At this point we have derived expressions for the amputated one loop diagrams that contribute to the renormalization of the three twisted fermion propagators. This is sufficient to calculate α_1, α_2 and α_4 that appear in the general action

$$\mathcal{L} = \frac{1}{g^2} \sum_{\mathbf{n}, a, b, c, d, e} \left\{ \mathcal{Q} \text{Tr} \left[-i\alpha_1 \chi_{ab} \mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n}) - i\alpha_2 \eta(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) + \frac{\alpha_3}{2} \eta(\mathbf{n}) d(\mathbf{n}) \right] - \frac{\alpha_4}{2} \text{Tr} \epsilon_{abcde} \chi_{de}(\mathbf{n} + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \mathcal{D}_c^{\dagger(-)} \chi_{ab}(\mathbf{n} + \hat{\mu}_c) \right\}. \quad (8.7)$$

However the coefficient α_3 requires further work. One simple way to extract it is via a computation of the renormalized auxiliary boson propagator which we turn to in the next section.

9. One Loop Diagrams for the Auxiliary Field Propagator

We have shown that the off-shell form of the bosonic action is given by

$$S_B = \sum_{\mathbf{n}, a, b} \text{Tr} \left(\mathcal{F}_{ab}^{\dagger}(\mathbf{n}) \mathcal{F}_{ab}(\mathbf{n}) - \frac{i}{g} d(\mathbf{n}) \mathcal{D}_a^{\dagger(-)} \mathcal{U}_a(\mathbf{n}) + \frac{1}{2} d^2(\mathbf{n}) \right), \quad (9.1)$$

where $\mathcal{F}_{ab}(\mathbf{n}) = -\frac{i}{g} \mathcal{D}_a^{(+)} \mathcal{U}_b(\mathbf{n})$.

In our previous computation of the fermion diagrams we integrated out the field d to give an on shell action defined just in terms of the complex gauge link fields \mathcal{U}_a and \mathcal{U}_a^{\dagger} . In this section we will not do this but instead focus on a computation of the renormalized propagator for the d field. The Feynman rules for the fermions will be identical to our previous scheme but the boson propagators will change and so we need to recompute those propagators in this off shell scheme. We proceed in the standard fashion by expanding the link field $\mathcal{U}_a(\mathbf{n})$

$$\mathcal{U}_a(\mathbf{n}) = \mathbf{1} + ig\mathcal{A}_a(\mathbf{n}), \quad \mathcal{U}_a^{\dagger}(\mathbf{n}) = \mathbf{1} - ig\bar{\mathcal{A}}_a(\mathbf{n}). \quad (9.2)$$

and using the same lattice gauge-fixing term as before

$$S_{GF}[A] = -\frac{1}{\alpha} \sum_{\mathbf{n}, a} \text{Tr} (\partial_a^{(-)} A_a(\mathbf{n}))^2, \quad (9.3)$$

⁶Notice that in this calculation we have not included any mass terms that would guarantee the stability of the initial classical vacuum state we have chosen to expand around. We have also ignored a potential sign problem associated with the replacement of a Pfaffian with a square root of a determinant. Nevertheless we expect the result to be robust; the existence of an exact supersymmetry should ensure that the object we are computing is a lattice regularized Witten index and hence independent of both coupling constant and background field.

we find the momentum space form

$$S_{GF}[A] = \frac{1}{\alpha} \sum_{\mathbf{k}, a, b} \text{Tr} A_a(\mathbf{k}) f_a^*(\mathbf{k}) f_b(\mathbf{k}) A_b(-\mathbf{k}) . \quad (9.4)$$

It is convenient in this calculation to work with the real and imaginary parts of the complex gauge field explicitly. Thus

$$\mathcal{A}_a = A_a + iB_a \quad (9.5)$$

The gauge-fixed bosonic action on the lattice to quadratic order in fields, with the choice $\alpha = \frac{1}{2}$, is then

$$\begin{aligned} S_B^{(2)} = & \sum_{\mathbf{k}, a, b} \text{Tr} 2A_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) \right] A_b(-\mathbf{k}) + 2B_a(\mathbf{k}) \left[\delta_{ab} f_c(\mathbf{k}) f_c^*(\mathbf{k}) - f_a^*(\mathbf{k}) f_b(\mathbf{k}) \right] B_b(-\mathbf{k}) \\ & - 2id(\mathbf{k}) f_a(\mathbf{k}) B_a(-\mathbf{k}) + \frac{1}{2} d(\mathbf{k}) d(-\mathbf{k}) \end{aligned} \quad (9.6)$$

We see that the $d - B_a$ system decouples from A_a to this order. Its action is given by

$$\begin{aligned} S_B^{(2)}[d, B_a] \sim & \sum_{\mathbf{k}, a, b} \text{Tr} 2B_a(\mathbf{k}) \left[\delta_{ab} f_c^*(\mathbf{k}) f_c(\mathbf{k}) - f_a^*(\mathbf{k}) f_b(\mathbf{k}) \right] B_b(-\mathbf{k}) \\ & - 2id(\mathbf{k}) f_a(\mathbf{k}) B_a(-\mathbf{k}) + \frac{1}{2} d(\mathbf{k}) d(-\mathbf{k}) \end{aligned} \quad (9.7)$$

or in matrix form

$$\begin{pmatrix} d & B_a \end{pmatrix}(\mathbf{k}) \begin{pmatrix} \frac{1}{2} & -if_b(\mathbf{k}) \\ -if_a^*(\mathbf{k}) & M_{ab}(\mathbf{k}) \end{pmatrix} \begin{pmatrix} d \\ B_b \end{pmatrix}(-\mathbf{k}) \quad (9.8)$$

where $M_{ab}(\mathbf{k}) = 2[\delta_{ab} \sum_c f_c(\mathbf{k}) f_c^*(\mathbf{k}) - f_a^*(\mathbf{k}) f_b(\mathbf{k})]$. Using standard identities for the inverse of a partitioned matrix we find

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & -if_b(\mathbf{k}) \\ -if_a^*(\mathbf{k}) & M_{ab}(\mathbf{k}) \end{pmatrix}^{-1} = \frac{1}{\sum_c f_c(\mathbf{k}) f_c^*(\mathbf{k})} \begin{pmatrix} 0 & if_b(\mathbf{k}) \\ if_a^*(\mathbf{k}) & \frac{1}{2} \mathbf{1}_5 \end{pmatrix} \quad (9.9)$$

We have $\sum_c f_c(\mathbf{k}) f_c^*(\mathbf{k}) = 4 \sum_c \sin^2 \left(\frac{\mathbf{k}_c}{2} \right)$ and as before we define $\widehat{\mathbf{k}}^2 \equiv 4 \sum_c \sin^2 \left(\frac{\mathbf{k}_c}{2} \right)$. Thus the lattice propagators are

$$\langle d^A(\mathbf{k}) d^B(-\mathbf{k}) \rangle = 0 \quad (9.10)$$

$$\langle d^A(\mathbf{k}) B_a^B(-\mathbf{k}) \rangle = i \delta_{AB} \frac{(e^{-ik_a} - 1)}{\widehat{\mathbf{k}}^2} \quad (9.11)$$

$$\langle B_a^A(\mathbf{k}) B_b^B(-\mathbf{k}) \rangle = \delta_{ab} \delta_{AB} \frac{1}{2\widehat{\mathbf{k}}^2} \quad (9.12)$$

From eqn 9.6 the propagator for the A field is also

$$\langle A_a^A(\mathbf{k}) A_b^B(-\mathbf{k}) \rangle = \delta_{ab} \delta_{AB} \frac{1}{2\widehat{\mathbf{k}}^2} . \quad (9.13)$$

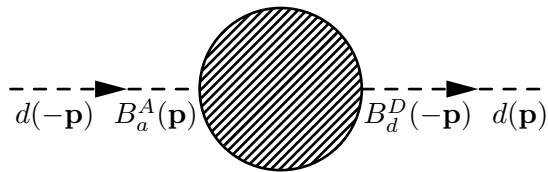


Figure 5: The generic diagram contributing to the renormalized d propagator

Notice that the field d is non-propagating at tree level. Using these propagators and those derived earlier for the fermions and ghosts we can now write down the generic Feynman diagram contributing to a renormalization of the auxiliary boson propagator. It is shown in figure 5 and represents the set of amputated diagrams possessing two external B field legs. These combine with the external $\langle dB \rangle$ propagators derived above to yield the renormalized propagator for the auxiliary field d . Notice that the vanishing of the tree level $\langle dd \rangle$ propagators ensures that no amputated diagrams with 2 d field external legs contribute. The set of all such lattice Feynman diagrams is shown below and corresponds to a subset of the B field vacuum polarization diagrams. It is important to notice that almost all these diagrams appear in the continuum off shell twisted theory; the exceptions are just the diagrams containing a BBd vertex which corresponds to the lattice vertex

$$V_{dBB} = \langle d^A(-\mathbf{k} - \mathbf{q}) B_a^B(\mathbf{k}) B_b^C(\mathbf{q}) \rangle = \frac{i}{2} \delta_{ab} (\lambda_{ABC} + \bar{\lambda}_{ABC}) (1 - e^{-i(k_a + q_a)}) . \quad (9.14)$$

Clearly this vertex vanishes as the lattice spacing is sent to zero and hence this diagram does not contribute to the divergent piece in the $\langle dd \rangle$ propagator at this order of perturbation theory.

Hence we are left with a set of diagrams which correspond to those of the equivalent continuum theory at one loop order. This fact can be exploited later to allow us to argue that the leading logarithmic divergences of the lattice theory are shared with the continuum theory. Anticipating this we will not write down explicit expressions for these amputated lattice diagrams in this section.

10. Divergence Structure of the One Loop Diagrams

At this point we have derived expressions for the amputated one loop diagrams that determine the renormalization of three fermion propagators and also the set of Feynman graphs needed to renormalize the auxiliary bosonic field propagator. In principle, this input will allow us to determine all four coefficients α_i appearing in the renormalized action eqn. 5.4. Of course the question of how much fine tuning is required to regain full supersymmetry is determined by the parts of these expressions which diverge as the lattice spacing is sent to zero. We must therefore evaluate the expressions for the one loop integrals as the lattice spacing tends to zero.

First, let us discuss the diagrams contributing to the fermion propagators. We have shown in Appendix B that the three fermion amplitudes all vanish for vanishing external

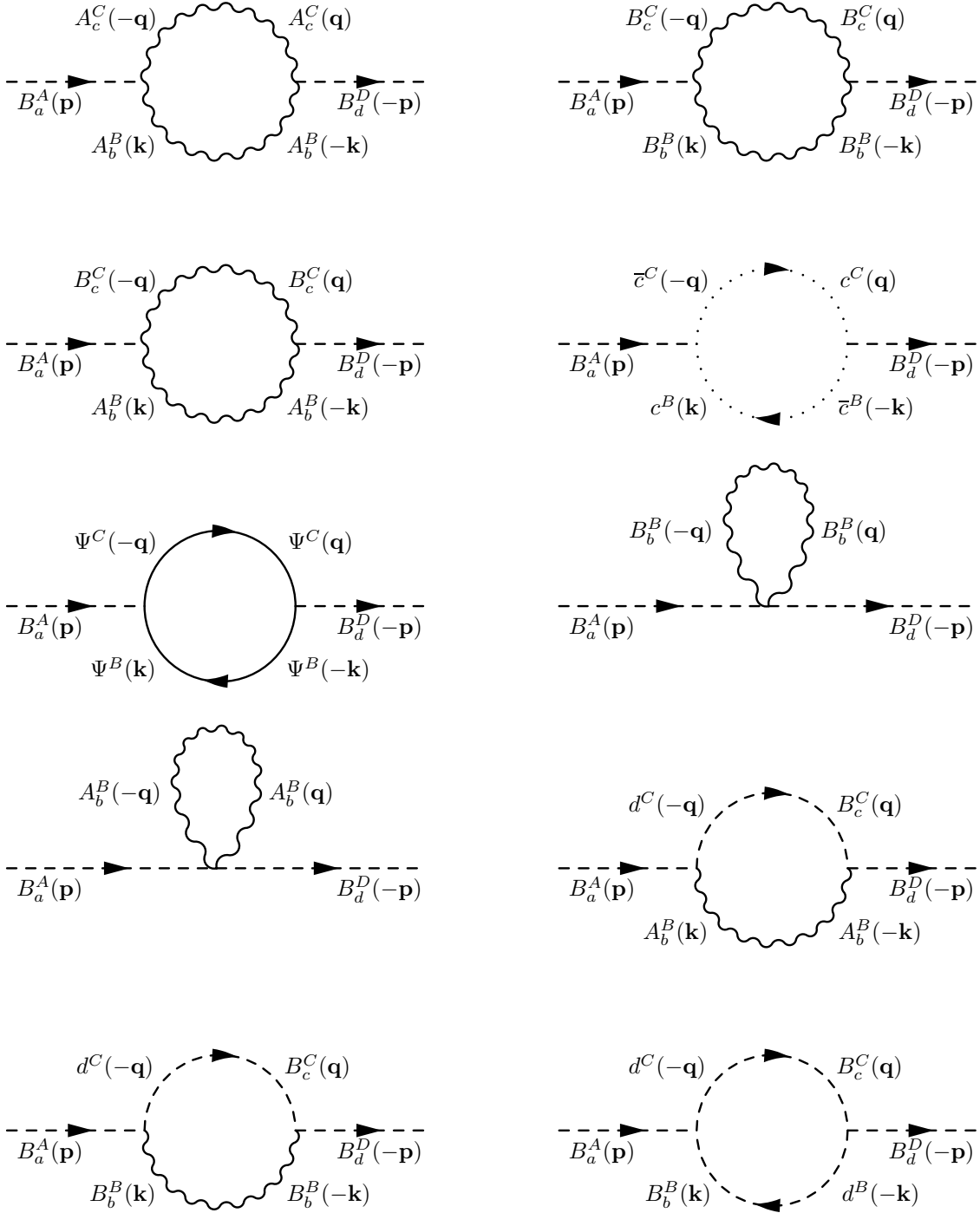


Figure 6: Set of all lattice amputated Feynman diagrams contributing the renormalized d propagator

momentum which is consistent with our effective action computation showing that no fermionic mass terms can be generated perturbatively. Reisz's power counting theorem [31]-[34] shows us that we cannot simply take the naïve continuum limit of the expressions for

the amputated one loop diagrams as they have a naïve degree of divergence of 1. However we can use a trick due to [35] and detailed in [36] to extract the leading divergences.

We split the integral $I(\mathbf{p})$ into two pieces as follows:

$$\lim_{a \rightarrow 0} I(\mathbf{p}) = \lim_{a \rightarrow 0} \left[I(\mathbf{p}) - I(\mathbf{0}) - \sum_b p_b \frac{\partial I}{\partial p_b} \Big|_{\mathbf{p}=\mathbf{0}} \right] + \lim_{a \rightarrow 0} \left[I(\mathbf{0}) + \sum_b p_b \frac{\partial I}{\partial p_b} \Big|_{\mathbf{p}=\mathbf{0}} \right] \quad (10.1)$$

The first term in square brackets can now be evaluated in the naïve continuum limit and contains no divergence. The second term contains the divergence but contains no external momenta in the integrand which simplifies its evaluation on the lattice. In addition we know that $I(\mathbf{0})$ vanishes for each of our diagrams so the calculation becomes simpler still.

We will find that the resulting expressions have logarithmic divergences of the form $\ln \mu a$ where μ is a small mass parameter used to regulate the behavior of the integrand close to the origin of momentum space and a the lattice spacing⁷.

One obvious way to proceed is simply to numerically evaluate the integral for a variety of regular masses μ and extract the logarithmic divergence and any constant contributions using a fitting procedure. However, if we are only interested in the leading log divergences there is a simpler approach detailed in the next section in which a naïve continuum limit can be taken and the expressions evaluated using, for example, dimensional regularization.

In the next section we give an example of this procedure for the amputated $\eta\psi$ diagram and show how to extract similar results for the remaining fermion self energy diagrams. We will also see that the same procedure allows us argue that the leading log divergent contribution to α_3 is also equal to its value in the continuum theory.

10.1 The Amputated Fermion Diagrams

We start with our simplified expression for $I_{\eta\psi_d}(\mathbf{p})$ given in Appendix A

$$\begin{aligned} I_{\eta\psi_d}(\mathbf{p}) = \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{BC} \left[\frac{1}{8(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} (1 - e^{i(p-q)_d}) \right] & \left[- \sum_{a \neq d} [d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a + iq_d}) \right. \\ & \left. + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a + iq_d}) \right] . \end{aligned} \quad (10.2)$$

As a first step we need to calculate the derivative of the diagram (re-inserting the lattice spacing a and the infra-red cutoff μ)

$$\begin{aligned} \frac{\partial I_{\eta\psi_d}(\mathbf{p})}{\partial p_b} \Big|_{\mathbf{p}=\mathbf{0}} &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 \mathbf{q}}{(2\pi)^4} \frac{-2a^4 \sin aq_b}{(\widehat{\mathbf{q}}^2 + \mu^2 a^2)^3} (1 - e^{-iaq_d}) f_{ABC} f_{BCD} (1 + e^{iaq_d}) \\ &+ \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 \mathbf{q}}{(2\pi)^4} \frac{-a^3}{(\widehat{\mathbf{q}}^2 + \mu^2 a^2)^2} (-ia \delta_{db} e^{-iaq_d}) f_{ABC} f_{BCD} (1 + e^{iaq_d}) \\ &+ \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 \mathbf{q}}{(2\pi)^4} \frac{-a^3}{8(\widehat{\mathbf{q}}^2 + \mu^2 a^2)^2} (1 - e^{-iaq_d}) \\ &\times \sum_{a \neq d} (d_{ABC} d_{BCD} + f_{ABC} f_{BCD}) \delta_{ab} (-ia) (1 - e^{iaq_d}) . \end{aligned} \quad (10.3)$$

⁷We will only consider the case of infinite lattice size which reduces all lattice sums in momentum space to integrals.

A further simplification now occurs; if we are only interested in the leading $\log \mu a$ coefficient we can evaluate this integral in a small q region around zero. This is because the contribution of the integrand to the $\log \mu a$ coefficient comes only from small q . Furthermore in the region $q \rightarrow 0$ the propagators and vertices inside the integral will approach their continuum counterparts and hence the logarithmic divergence can be extracted by replacing the lattice integrals by their naïve continuum limit. Note that this only works for the coefficient of the log - we must evaluate the integral numerically (and then fit) in order to extract the constant terms. This (longer) calculation is in progress and will be published in a future paper. Thus we find:

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{\partial I_{\eta\psi_d}(\mathbf{p})}{\partial p_b} \Big|_{\mathbf{p}=\mathbf{0}} &\sim \int_{-\infty}^{\infty} \frac{d^4 \mathbf{q}}{(2\pi)^4} \frac{-4iq_b q_d}{(q^2 + \mu^2)^3} f_{ABC} f_{BCD} \\ &+ \int_{-\infty}^{\infty} \frac{d^4 \mathbf{q}}{(2\pi)^4} \frac{2i}{(q^2 + \mu^2)^2} \delta_{db} f_{ABC} f_{BCD} . \end{aligned} \quad (10.4)$$

Note that we cannot just set the first term in this expression to zero as \hat{e}_d and \hat{e}_b are not orthogonal to each other, instead we have

$$\begin{aligned} \int d^d \mathbf{q} \frac{q_b q_d}{(q^2 + \mu^2)^3} &= e_b^\mu e_d^\nu \int d^d \mathbf{q} \frac{q^\mu q^\nu}{(q^2 + \mu^2)^3} \\ &= \hat{e}_b \cdot \hat{e}_d \int d^d \mathbf{q} \frac{q^2}{d(q^2 + \mu^2)^3} . \end{aligned} \quad (10.5)$$

Then $\hat{e}_b \cdot \hat{e}_d = \delta_{bd} - \frac{1}{5}$. We use dimensional regularization and the fact that $\sum_b p_b = 0$ to evaluate the resulting integrals getting

$$\begin{aligned} I_{\eta\psi_d}(\mathbf{p}) &\sim \sum_b p_b \frac{\partial I_{\eta\psi_d}(\mathbf{p})}{\partial p_b} \Big|_{\mathbf{p}=\mathbf{0}} \\ &\sim -\frac{i}{8\pi^2} p_d f_{ABC} f_{BCD} \log \mu a . \end{aligned} \quad (10.6)$$

Note that we have inserted the cutoff $\frac{1}{a}$ inside the logarithm to ensure that it is dimensionless.

Since all the Feynman graphs we need to evaluate are logarithmically divergent and in one-to-one correspondence with continuum diagrams, the resulting logarithmic divergences can all be extracted by following a similar procedure i.e. taking the naïve continuum limit of the relevant $I(\mathbf{p})$.

$$\begin{aligned} \lim_{a \rightarrow 0} I_{\psi_a \chi_{gh}}^{(1)}(\mathbf{p}) &\sim \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_m \frac{-i(p-q)_m}{2(q^2 + \mu^2)((\mathbf{p}-\mathbf{q})^2 + \mu^2)} (3\delta_{ag}\delta_{mh} - 3\delta_{ah}\delta_{mg}) f_{ABC} f_{BCD} \\ &\sim \frac{3i}{32\pi^2} f_{ABC} f_{BCD} (\delta_{ag} p_h - \delta_{ah} p_g) \log \mu a \end{aligned} \quad (10.7)$$

$$\begin{aligned} \lim_{a \rightarrow 0} I_{\psi_a \chi_{de}}^{(2)}(\mathbf{p}) &\sim \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_c \frac{-i(p-q)_c}{2(q^2 + \mu^2)((\mathbf{p}-\mathbf{q})^2 + \mu^2)} (\delta_{da}\delta_{ec} - \delta_{dc}\delta_{ea}) f_{ABC} f_{BCD} \\ &\sim \frac{i}{32\pi^2} f_{ABC} f_{BCD} (\delta_{da} p_e - \delta_{ea} p_d) \log \mu a \end{aligned} \quad (10.8)$$

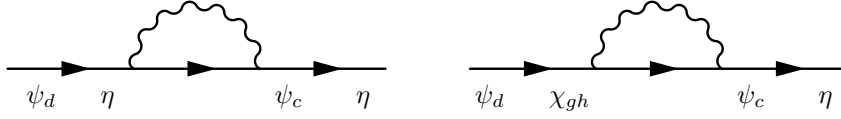


Figure 7: Full $\eta\psi$ Propagators

This obviously leads us to define $I_{\psi_a\chi_{de}}(\mathbf{p}) = I_{\psi_a\chi_{de}}^{(1)}(\mathbf{p}) + I_{\psi_a\chi_{de}}^{(2)}(\mathbf{p})$ and therefore

$$I_{\psi_a\chi_{de}}(\mathbf{p}) \sim \frac{i}{8\pi^2} f_{ABC} f_{BCD} (\delta_{da} p_e - \delta_{ea} p_d) \quad (10.9)$$

$$\begin{aligned} \lim_{a \rightarrow 0} I_{\chi_{ab}\chi_{gh}}(\mathbf{p}) &\sim \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_d \frac{i(p-q)_d}{2(q^2 + \mu^2)((\mathbf{p}-\mathbf{q})^2 + \mu^2)} \epsilon_{abdgh} f_{ABC} f_{BCD} \\ &\quad - (h \leftrightarrow g) \\ &\sim -\frac{i}{16\pi^2} f_{ABC} f_{BCD} \sum_d \epsilon_{abdgh} p_d \log \mu a \end{aligned} \quad (10.10)$$

Note that these calculations of the log terms for the other diagrams have also been verified by numerical evaluation and fitting of the resulting lattice integrals.

10.2 The Auxiliary Field Diagram

Since the amputated divergent diagrams for the lattice d propagator are log divergent we can extract the sum of these logarithmic divergences using the same tricks we used for the fermions; evaluating the diagram in the naïve continuum limit. The sum of all these diagrams, contracted with external dB propagators, will then yield a log divergent term of the form

$$C_{dd} = c f_{ACB} f_{DCB} \ln(\mu a) \quad (10.11)$$

where c is a constant to be determined by explicitly evaluating the diagrams. However, we will argue in the next section that there that it is not necessary to evaluate these diagrams, even in the continuum, to determine α_3 – the requirement that the continuum theory preserve full supersymmetry will automatically determine α_3 in terms of the other α_i corresponding to the fermion propagator renormalization.

10.3 From Amputated Diagrams to Renormalized Propagators

The leading logarithmic divergences appearing in the renormalized propagators are obtained by combining the (divergent parts of) the individual amputated diagrams we have just computed. In principle several of the amputated fermion diagrams can appear as internal bubbles when correcting a given fermion propagator. As an example consider the $\psi\eta$ diagram shown in Figure 7. Naïvely we see that three of our amputated diagrams contribute to the renormalization of this propagator. However we find that (at least in the case of the log divergences) the Lorentz structure of the propagators and integrals means that only the $\eta\psi$ amputated diagram contributes to the renormalization of the $\eta\psi$ propagator.

We demonstrate this through explicit calculation. Denoting the full diagrams by C and noting that as we are dealing with only the divergent part we can approximate the lattice propagators by their continuum analogues we find

$$\begin{aligned}
C_{\psi a \eta} &= \frac{2ip_d}{p^2} \sum_c I_{\eta\psi_c}(\mathbf{p}) \frac{2ip_c}{p^2} + \sum_{c,g,h} \frac{ip_g\delta_{dh} - ip_h\delta_{dg}}{p^2} I_{\chi_{gh}\psi_c}(\mathbf{p}) \frac{2ip_c}{p^2} \\
&\sim -\frac{2ip_d}{p^2} \frac{i}{8\pi^2} f_{ABC} f_{BCD} \sum_c p_c \frac{2ip_c}{p^2} \log \mu a \\
&\sim \frac{1}{4\pi^2} f_{ABC} f_{BCD} \frac{2ip_d}{p^2} \log \mu a .
\end{aligned} \tag{10.12}$$

The second term disappears as

$$\begin{aligned}
\sum_{c,g,h} (p_g\delta_{dh} - p_h\delta_{dg})(\delta_{cg}p_h - \delta_{ch}p_g)p_c &= \sum_c (p_c p_d - p^2\delta_{cd} - p^2\delta_{dc} + p_c p_d)p_c \\
&= 0 .
\end{aligned} \tag{10.13}$$

We can similarly show that only $I_{\psi\chi}$ contributes to $C_{\psi\chi}$ and $I_{\chi\chi}$ to $C_{\chi\chi}$. Note, however, that this analysis strictly only applies to the logarithmically divergent piece in C .

$$\begin{aligned}
C_{\psi_a\chi_{de}} &= \frac{i}{8\pi^2} f_{ABC} f_{BCD} \sum_{g,h,c} \frac{ip_g\delta_{ah} - ip_h\delta_{ag}}{p^2} (\delta_{gc}p_h - \delta_{hc}p_g) \frac{ip_d\delta_{ce} - ip_e\delta_{cd}}{p^2} \log \mu a \\
&= \frac{1}{4\pi^2} f_{ABC} f_{BCD} \frac{ip_d\delta_{ae} - ip_e\delta_{ad}}{p^2} \log \mu a .
\end{aligned} \tag{10.14}$$

In calculating $C_{\chi\chi}$ we must take into account that the internal propagator in $I_{\chi\chi}$ can be a $\psi\chi$ or $\chi\psi$. This contributes another factor of 2 to $C_{\chi\chi}$.

$$\begin{aligned}
C_{\chi_{ab}\chi_{de}} &= -\frac{i}{8\pi^2} f_{ABC} f_{BCD} \log \mu a \sum_{c,f,g,i,h,j,k} \epsilon_{abcfg} \frac{ip_c}{2p^2} \epsilon_{fghij} p_i \epsilon_{hjkde} \frac{ip_k}{2p^2} \\
&= -\frac{i}{2\pi^2} f_{ABC} f_{BCD} \log \mu a \sum_{c,i,k} \frac{ip_c}{2p^2} p_i \frac{ip_k}{2p^2} (\delta_{ai}\epsilon_{bckde} + \delta_{bi}\epsilon_{cakde} + \delta_{ci}\epsilon_{abkde}) \\
&= \frac{1}{4\pi^2} f_{ABC} f_{BCD} \log \mu a \sum_k \frac{ip_k}{2p^2} \epsilon_{abkde} .
\end{aligned} \tag{10.15}$$

The coefficients α_i are now determined by the coefficient of the propagator in the renormalized propagator amplitudes C . Explicitly we find

$$\alpha_i = 1 + b_i \ln \mu a \quad i = 1, 2, 4 \tag{10.16}$$

where $b_i = b = \frac{g^2 N}{4\pi^2}$. Note that we have used $f_{ABC} f_{BCD} = N\delta_{AD}$. This is required as the colour structure of any counterterms must match the tree propagators. However this is strictly only true for $SU(N)$ as $f_{ABC} f_{BCD} = N(\delta_{AD} - \delta_{A0}\delta_{D0})$ for $U(N)$. This does not matter in the continuum as the $U(1)$ trace piece simply decouples from the rest of the system and can be ignored. When doing lattice simulations we might imagine achieving a similar result by giving the $U(1)$ mode a large mass of the order of the cut-off which

will serve to decouple it from the $SU(N)$ modes at finite lattice spacing. The breaking of supersymmetry in this sector may then be removed by sending this $U(1)$ mass to zero *after* taking the continuum limit.

While naïvely one might have expected the coefficients b_i to be all different our results indicate that in fact the log divergent parts of b_i and hence α_i are actually all equal. This fact can be understood quite simply; to untwist the continuum theory into a theory with four Majorana spinors requires that the continuum twisted fermions exhibit a common wavefunction renormalization. This just follows from the fact that the individual components of the spinors mix the different twisted fermions together. To achieve this requires that the corresponding renormalization constants of the kinetic terms α_i should all be equal – just as we find. Furthermore, since the leading log behavior of the lattice theory is the same as the continuum we should expect that the log divergent part of the lattice couplings behave in the same way. Thus a single wavefunction renormalization of the twisted lattice fermions is all that is needed to render the renormalized theory finite. The common anomalous dimension of the fermions in this twisted scheme is then given by $\gamma = \frac{g^2 N}{8\pi^2}$.

In the case of the $\langle dd \rangle$ propagator the leading log divergent contribution can be computed from the naïve continuum limit of the corresponding continuum expression for the sum of the BB bubble diagrams given in diagram 6. Combined with the fact that the tree level $\langle dB \rangle$ propagators required on the outside of these BB amputated diagrams are the same as the continuum to $\mathcal{O}(a)$ we find that the log divergence in the mass renormalisation of the d field must be the same on the lattice as in the continuum. Using this fact we can argue that the log divergent part of α_3 must actually be equal to that of the fermions eg α_1 . This follows from the fact that the bosonic action for general α_i can be rewritten as

$$\alpha_1 \left(\overline{\mathcal{F}}_{ab} \mathcal{F}_{ab} \right) + \frac{\alpha_2^2}{\alpha_3} \left(\frac{1}{2} [\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 \right) \quad (10.17)$$

Only for $\alpha_3 = \alpha_2 = \alpha_1$ can this renormalized bosonic action be untwisted to yield the conventional gauge field plus scalar action in the continuum limit. But since the continuum twisted theory possesses full supersymmetry this must be true. And our general arguments then tell us the log divergence of α_3 on the lattice must satisfy the same property.

To summarise; we find that the log divergent parts of the coefficients $\alpha_i, i = 1 \dots 4$ must all be equal to one loop order in the lattice theory. This implies that a common wavefunction renormalization of both twisted fermions and bosons is sufficient to render the renormalized theory finite at one loop with all fields acquiring an anomalous dimension (in this scheme) given by $\gamma = \frac{g^2 N}{8\pi^2}$. Physically, the equality of the couplings $\alpha_i, i = 1 \dots 4$ means that *no* logarithmic fine tuning is required at weak coupling for the lattice theory to exhibit full supersymmetry as the lattice spacing is sent to zero.

11. Conclusion and Discussion

We have examined a recently developed lattice construction for $\mathcal{N} = 4, d = 4$ super Yang-Mills theory using perturbation theory. We argue that the exact symmetries of the

classical lattice theory; namely gauge invariance, a single exact supersymmetry \mathcal{Q} and the (large) point group symmetry of the lattice strongly constrain the possible counter terms induced by quantum corrections. Indeed with one exception the only relevant counterterms correspond to renormalizations of existing terms in the action. We furthermore show by a computation of the effective action that the one new operator which cannot be excluded in the general analysis actually makes no appearance to all orders in perturbation theory⁸

The renormalized action can then be written in terms of 4 coupling constants α_i which take the value unity in the classical lattice action. We evaluate the renormalization of these couplings at one loop using lattice perturbation theory. Three of the couplings can be computed by examining the renormalization of the three twisted fermion propagators. The final coupling is most easily read off from a one loop contribution to the propagator for a bosonic auxiliary field. The relevant propagators and vertices are derived and the amputated one loop diagrams constructed. All these diagrams possess identical logarithmic divergences of the form $\ln \mu a$ where a is the lattice spacing and μ a mass scale introduced to regulate the small momentum behavior of the integrands. This divergence can be absorbed by a common wavefunction renormalization Z of the twisted fermions and bosons.

The simplest way to understand this rather surprising result is to realize that the coefficient of the logarithmic divergence of some one loop diagram in the lattice theory can be extracted by taking a naïve continuum limit of the diagram, since the log divergence comes from the small loop momentum region of the integral. Provided that the lattice diagrams correspond one-to-one with equivalent continuum diagrams,⁹ and that all lattice propagators and vertices reduce to their continuum counterparts for small momenta, this means that the log divergences of the lattice theory are equal to the same divergences in the continuum theory¹⁰. Furthermore, since the twisted continuum theory is equivalent to the usual $\mathcal{N} = 4$ theory in flat space it must possess the full $\mathcal{Q} = 16$ supersymmetry. This fact ensures that all divergences present in the twisted continuum fermion self energies must be equal - which is indeed what we find. And this structure is necessarily inherited by the log divergent parts of the lattice theory at one loop. This is what leads to our main result; that only a one time tuning of the finite parts of the wavefunction renormalization needs to be performed at one loop in order to restore the full supersymmetry.

This similarity between the divergence structure of the lattice theory and the continuum theory is strongly suggestive that the beta function of the lattice theory will also

⁸The calculation we have done is somewhat formal as it ignores possible instabilities associated with the flat directions and specifically the $U(1)$ trace mode of the scalars. It is possible that regulating these directions by eg introducing a mass term for the $U(1)$ mode might modify our conclusions since supersymmetry is broken by such terms. Notice though that at large N the dangerous double trace mass operator is suppressed and hence the result should certainly hold in that limit

⁹Note that this is true in the present, twisted construction and that it is not true in other, more naïve constructions. There, one would have $\mathcal{O}(a)$ multigluon vertices that correct the fermion self energy through a tadpole. This lattice diagram does not occur in the perturbation theory we have described above.

¹⁰Clearly both continuum and lattice perturbation theories must employ the same scheme for this to be true. In this light we would note that it is possible by addition of sufficient auxiliary fields to arrange for $Z = 1$ for all fields in continuum $\mathcal{N} = 4$ SYM. However such a superfield approach is not possible for our twisted construction. This does not spoil the finiteness of the theory which depends only on the vanishing of the beta function which is true in both twisted and supergraph schemes.

vanish at weak coupling. We leave a proof of this (and a computation of the finite parts of the diagrams) to a later paper and here merely give heuristic arguments as to why this result may indeed hold. First, note that the calculation of the beta function requires the evaluation of one loop vertex diagrams in the lattice theory. Preliminary calculations suggest that the set of relevant lattice vertex diagrams correspond one to one to continuum vertex diagrams and remain only logarithmically divergent. They may thus be evaluated in the continuum theory. The coefficient of this log divergence is then combined with the wavefunction renormalizations determined above to yield the one loop beta function in the usual manner. However we already know the result of this computation for the continuum theory; the beta function vanishes. We hence expect a similar result to hold at one loop in the lattice theory. Thus for weak coupling we expect the lattice theory to possess a line of fixed points parametrized by the bare coupling constant just as for the continuum theory. However, our calculations do not reveal whether this feature survives in the lattice theory to strong coupling. At two or more loops the divergences of the lattice Feynman diagrams will not be equal to the those of the continuum theory and hence we cannot use the latter to infer the divergence structure of the lattice theory. To understand how to take the continuum limit in this regime will then require a mixture of two loop and numerical calculations.

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A. Simplification of the One Loop Diagrams

This section shows how the diagrams in the paper are simplified. For the easy evaluation of the diagram we use the following identities.

$$\sum_{B,C} \lambda_{ABC} \lambda_{BCD} = \sum_{B,C} d_{ABC} d_{BCD} - f_{ABC} f_{BCD} . \quad (\text{A.1})$$

$$\bar{\lambda}_{ABC} = \lambda_{ACB} . \quad (\text{A.2})$$

These relations imply

$$\begin{aligned} (A\lambda_{ABC} - B\bar{\lambda}_{ABC})(C\lambda_{BCD} - D\bar{\lambda}_{BCD}) &= d_{ABC}d_{BCD}(AC - AD - BC + BD) \\ &\quad - f_{ABC}f_{BCD}(AC + AD + BC + BD) . \end{aligned} \quad (\text{A.3})$$

Starting with $I_{\eta\psi}(\mathbf{p})$:

$$\begin{aligned}
I_{\eta\psi_d}(\mathbf{p}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{a,b,c} \sum_{BC} \left[\frac{1}{16(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} \right] [\lambda_{ABC} - \bar{\lambda}_{ABC} e^{ip_a}] [[\bar{\lambda}_{BCD} e^{-ip_a} - \lambda_{BCD} e^{iq_d}]] \\
&\quad \left((1 - e^{i(p-q)b}) \delta_{ca} (\delta_{ba} \delta_{cd} - \delta_{bd} \delta_{ca}) - (1 - e^{i(p-q)c}) (\delta_{ba} \delta_{cd} - \delta_{bd} \delta_{ca}) \delta_{ba} \right) \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{a,b} \sum_{BC} \left[\frac{1}{8(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} \right] \\
&\quad [d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a+iq_d}) + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a+iq_d})] \\
&\quad (1 - e^{i(p-q)b}) (\delta_{ba} \delta_{ad} - \delta_{bd} \delta_{aa})
\end{aligned}$$

That is

$$\begin{aligned}
I_{\eta\psi_d}(\mathbf{p}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_a \sum_{BC} \left[\frac{1}{8(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} \right] \left[[d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a+iq_d}) \right. \\
&\quad + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a+iq_d})] (1 - e^{i(p-q)a}) \delta_{ad} \\
&\quad - [d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a+iq_d}) \\
&\quad + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a+iq_d})] (1 - e^{i(p-q)d}) \left. \right] \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{BC} \left[\frac{1}{8(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} (1 - e^{i(p-q)d}) \right] \left[[d_{ABC} d_{BCD} (e^{-ip_d} - e^{iq_d} - 1 + e^{i(p+q)d}) \right. \\
&\quad + f_{ABC} f_{BCD} (e^{-ip_d} + e^{iq_d} + 1 + e^{i(p+q)d})] \\
&\quad - \sum_a [d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a+iq_d}) \\
&\quad + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a+iq_d})] \left. \right] \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{BC} \left[\frac{1}{8(\widehat{\mathbf{p}-\mathbf{q}})^2 \widehat{\mathbf{q}}^2} (1 - e^{i(p-q)d}) \right] \left[- \sum_{a \neq d} [d_{ABC} d_{BCD} (e^{-ip_a} - e^{iq_d} - 1 + e^{ip_a+iq_d}) \right. \\
&\quad + f_{ABC} f_{BCD} (e^{-ip_a} + e^{iq_d} + 1 + e^{ip_a+iq_d})] \left. \right] \tag{A.4}
\end{aligned}$$

Now $I_{\psi_a \chi_{gh}}^{(1)}$:

$$\begin{aligned}
I_{\psi_a \chi_{gh}}^{(1)}(\mathbf{p}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{b,c,d,e,m,f} \sum_{B,C} \frac{(-1)}{64 \widehat{\mathbf{q}}^2 (\widehat{\mathbf{p}-\mathbf{q}})^2} \epsilon_{bcme} \epsilon_{fghdef} e^{i(p-q)(e+f)} \\
&\quad (e^{i(p-q)m} - 1) (\delta_{bd} \delta_{ca} - \delta_{ba} \delta_{cd}) \\
&\quad \times (\lambda_{ABC} e^{ip_d} - \bar{\lambda}_{ABC} e^{-iq_a}) \left(e^{-ip(d+e+f)} (\bar{\lambda}_{BCD} e^{iq(e+f)} - \lambda_{BCD} e^{i(p-q)d}) \right. \\
&\quad \left. - e^{i(p-q)(d+g+h)} (\lambda_{BCD} e^{iq(g+h)} - \bar{\lambda}_{BCD} e^{-ip_d}) \right)
\end{aligned}$$

That is

$$\begin{aligned}
I_{\psi_a \chi_{gh}}^{(1)}(\mathbf{p}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{d,e,m,f} \sum_{B,C} \frac{1}{16 \widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} \epsilon_{adme} \epsilon_{ghdef} (e^{i(p-q)_m} - 1) \\
&\quad \times \left(-d_{ABC} d_{BCD} (e^{i(p_d+q_{(g+h)})} - 1 - e^{iq_{(g+h-a)}} + e^{-i(p_d+q_a)}) \right. \\
&\quad \left. + f_{ABC} f_{BCD} (e^{i(p_d+q_{(g+h)})} + 1 + e^{iq_{(g+h-a)}} + e^{-i(p_d+q_a)}) \right) \\
&= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{d,m} \sum_{B,C} \frac{1}{8 \widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} (e^{i(p-q)_m} - 1) \\
&\quad \times (\delta_{ag} \delta_{mh} + \delta_{ah} \delta_{md} \delta_{dg} + \delta_{ad} \delta_{mg} \delta_{dh} - \delta_{ah} \delta_{mg} - \delta_{ag} \delta_{md} \delta_{dh} - \delta_{ad} \delta_{mh} \delta_{dg}) \\
&\quad \times \left(d_{ABC} d_{BCD} (e^{i(p_d+q_{(g+h)})} - 1 - e^{iq_{(g+h-a)}} + e^{-i(p_d+q_a)}) \right. \\
&\quad \left. - f_{ABC} f_{BCD} (e^{i(p_d+q_{(g+h)})} + 1 + e^{iq_{(g+h-a)}} + e^{-i(p_d+q_a)}) \right) \tag{A.5}
\end{aligned}$$

Looking at the second $\psi\chi$ diagram we have:

$$\begin{aligned}
I_{\psi_a \chi_{de}}^{(2)}(\mathbf{p}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{b,c,B,C} \frac{1}{8 \widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} (e^{i(p-q)_c} - 1) \delta_{ab} (\delta_{db} \delta_{ec} - \delta_{dc} \delta_{eb}) \\
&\quad \times (\lambda_{ABC} e^{-i(p-q)_a} - \bar{\lambda}_{ABC}) (\lambda_{BCD} e^{iq_c} - \bar{\lambda}_{BCD} e^{i(p-q)_b}) \\
&= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{c,B,C} \frac{1}{8 \widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} (e^{i(p-q)_c} - 1) (\delta_{da} \delta_{ec} - \delta_{dc} \delta_{ea}) \\
&\quad \times \left(d_{ABC} d_{BCD} (e^{i(q_c - (p-q)_a)} - e^{iq_c} - 1 + e^{i(p-q)_a}) \right. \\
&\quad \left. - f_{ABC} f_{BCD} (e^{i(q_c - (p-q)_a)} + e^{iq_c} + 1 + e^{i(p-q)_a}) \right) \tag{A.6}
\end{aligned}$$

Now looking at $I_{\chi_{ab} \chi_{gh}}$:

$$\begin{aligned}
I_{\chi_{ab} \chi_{gh}}(\mathbf{p}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{c,d,e,f,B,C} \frac{1}{32 \widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} \epsilon_{abcde} (\delta_{gc} \delta_{hf} - \delta_{gf} \delta_{hc}) \\
&\quad \times \left((e^{-i(p-q)_d} - 1) \delta_{ef} - (e^{-i(p-q)_e} - 1) \delta_{df} \right) \left(\lambda_{BCD} e^{iq_f} - \bar{\lambda}_{BCD} e^{i(p-q)_c} \right) \\
&\quad \times \left(e^{-ik_{(a+b+c)}} (\bar{\lambda}_{ABC} e^{ip_c} - \lambda_{ABC} e^{-iq_{(a+b)}}) \right. \\
&\quad \left. - e^{ip_{(c+d+e)}} (\lambda_{ABC} e^{-i(p-q)_c} - \bar{\lambda}_{ABC} e^{-iq_{(d+e)}}) \right)
\end{aligned}$$

That is

$$\begin{aligned}
I_{\chi_{ab}\chi_{gh}}(\mathbf{p}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{d,e,B,C} \frac{-1}{16\widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} e^{ip(g+d+e)} \epsilon_{abgde} \\
&\quad \times \left((e^{-i(p-q)d} - 1)\delta_{eh} - (e^{-i(p-q)e} - 1)\delta_{dh} \right) \\
&\quad \times \left(\lambda_{ABC} e^{-i(p-q)g} - \bar{\lambda}_{ABC} e^{-iq(d+e)} \right) \left(\lambda_{BCD} e^{iqh} - \bar{\lambda}_{BCD} e^{i(p-q)g} \right) \\
&\quad - (h \leftrightarrow g) \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{d,B,C} \frac{1}{8\widehat{\mathbf{q}}^2(\widehat{\mathbf{p}-\mathbf{q}})} e^{ip(g+d+h)} \epsilon_{abdgh} (e^{-i(p-q)d} - 1) \\
&\quad \times (d_{ABC} d_{BCD} (e^{-i(p_g-q(g+h))} - 1 - e^{-iqd} + e^{i(p_g-q(g+d+h))}) \\
&\quad - f_{ABC} f_{BCD} (e^{-i(p_g-q(g+h))} + 1 + e^{-iqd} + e^{i(p_g-q(g+d+h))})) \\
&\quad - (h \leftrightarrow g)
\end{aligned} \tag{A.7}$$

(Note that we also need to take into account the diagram where the internal $\psi\chi$ is flipped. It is the same as what we have but with $a \leftrightarrow g$, $b \leftrightarrow h$ and $\mathbf{p} \leftrightarrow -\mathbf{p}$. We may for convenience take $\mathbf{q} \leftrightarrow -\mathbf{q}$. We pick up an additional minus sign in the $f_{ABC}f_{BCD}$ term due to the differing order of the group factors.)

B. Vanishing of One Loop Fermion Propagator at Zero Momentum

Starting with the first diagram and using the simplified forms of the integrals derived in Appendix A (assuming an IR regulator) we have:

$$\begin{aligned}
I_{\eta\psi_d}(\mathbf{0}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{BC} \left[\frac{1}{8\widehat{\mathbf{q}}^2\widehat{\mathbf{q}}^2} (1 - e^{-iqd}) \right] \left[-2 \sum_{a \neq d} f_{ABC} f_{BCD} (1 + e^{iqd}) \right] \\
&= 0
\end{aligned} \tag{B.1}$$

as $(1 - e^{-iqd})(1 + e^{iqd}) = 2i \sin qd$ and then the integrand is the combination of an odd and an even function. Next we calculate:

$$\begin{aligned}
I_{\psi_a\chi_{gh}}^{(1)}(\mathbf{0}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{d,m} \sum_{B,C} \frac{1}{8\widehat{\mathbf{q}}^2\widehat{\mathbf{q}}^2} (e^{-iqm} - 1) \\
&\quad \times (\delta_{ag}\delta_{mh} + \delta_{ah}\delta_{md}\delta_{dg} + \delta_{ad}\delta_{mg}\delta_{dh} - \delta_{ah}\delta_{mg} - \delta_{ag}\delta_{md}\delta_{dh} - \delta_{ad}\delta_{mh}\delta_{dg}) \\
&\quad \times (d_{ABC} d_{BCD} (e^{iq(g+h)} - 1 - e^{iq(g+h-a)} + e^{-iq_a}) \\
&\quad - f_{ABC} f_{BCD} (e^{iq(g+h)} + 1 + e^{iq(g+h-a)} + e^{-iq_a})) \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_m \sum_{B,C} \frac{1}{8\widehat{\mathbf{q}}^2\widehat{\mathbf{q}}^2} (e^{-iqm} - 1) (\delta_{ah}\delta_{mg} - \delta_{ag}\delta_{mh}) \\
&\quad \times (d_{ABC} d_{BCD} (e^{iq(g+h)} - 1 - e^{iq(g+h-a)} + e^{-iq_a}) \\
&\quad - f_{ABC} f_{BCD} (e^{iq(g+h)} + 1 + e^{iq(g+h-a)} + e^{-iq_a}))
\end{aligned} \tag{B.2}$$

Then we can use the fact that if $a \neq g$ and $a \neq h$ then the expression disappears. If $a = g = h$ again the expression disappears. So assuming $a = h$ and $a \neq g$ we get:

$$\begin{aligned}
I_{\psi_a \chi_{gh}}^{(1)}(\mathbf{0}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{B,C} \frac{1}{8\widehat{\mathbf{q}}^2 \widehat{\mathbf{q}}^2} (e^{-iq_g} - 1) (d_{ABC} d_{BCD} (e^{iq_{(g+a)}} - 1 - e^{iq_g} + e^{-iq_a}) \\
&\quad - f_{ABC} f_{BCD} (e^{iq_{(g+a)}} + 1 + e^{iq_g} + e^{-iq_a})) \\
&= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{B,C} \frac{2i}{8\widehat{\mathbf{q}}^2 \widehat{\mathbf{q}}^2} (d_{ABC} d_{BCD} (\sin q_a + \sin q_g - \sin q_{(a+g)}) \\
&\quad - f_{ABC} f_{BCD} (\sin q_a - \sin q_g - \sin q_{(a+g)})) \\
&= 0
\end{aligned} \tag{B.3}$$

which vanishes term by term. We then move onto the second $\psi\chi$ diagram.

$$\begin{aligned}
I_{\psi_a \chi_{de}}^{(2)}(\mathbf{0}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{c,B,C} \frac{1}{8\widehat{\mathbf{q}}^2 \widehat{\mathbf{q}}^2} (e^{-iq_c} - 1) (\delta_{da} \delta_{ec} - \delta_{dc} \delta_{ea}) \\
&\quad \times \left(d_{ABC} d_{BCD} (e^{iq_{(c+a)}} - e^{iq_c} - 1 + e^{-iq_a}) \right. \\
&\quad \left. - f_{ABC} f_{BCD} (e^{iq_{(c+a)}} + e^{iq_c} + 1 + e^{-iq_a}) \right)
\end{aligned} \tag{B.4}$$

In a similar way to the previous diagram if $a \neq d$ and $a \neq e$ then the diagram vanishes. If $a = d = e$ it also vanishes so we only need to deal with the case $a = d, a \neq e$:

$$\begin{aligned}
I_{\psi_a \chi_{de}}^{(2)}(\mathbf{0}) &= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{B,C} \frac{1}{8\widehat{\mathbf{q}}^2 \widehat{\mathbf{q}}^2} (e^{-iq_e} - 1) \left(d_{ABC} d_{BCD} (e^{iq_{(e+a)}} - e^{iq_e} - 1 + e^{-iq_a}) \right. \\
&\quad \left. - f_{ABC} f_{BCD} (e^{iq_{(e+a)}} + e^{iq_e} + 1 + e^{-iq_a}) \right) \\
&= \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{B,C} \frac{2i}{8\widehat{\mathbf{q}}^2 \widehat{\mathbf{q}}^2} (e^{-iq_e} - 1) \left(d_{ABC} d_{BCD} (\sin q_a + \sin q_{(a+e)} + \sin q_e) \right. \\
&\quad \left. - f_{ABC} f_{BCD} (\sin q_a + \sin q_{(a+e)} - \sin q_e) \right) \\
&= 0
\end{aligned} \tag{B.5}$$

which again vanishes term by term. Finally we show that $I_{\chi\chi}(\mathbf{0}) = 0$.

$$\begin{aligned}
I_{\chi_{ab}\chi_{gh}}(\mathbf{0}) &= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{d,B,C} \frac{1}{8\widehat{\mathbf{q}}^2\widehat{\mathbf{q}}^2} \epsilon_{abdgh} (e^{iq_d} - 1) \\
&\quad \times (d_{ABC}d_{BCD}(e^{iq_{(g+h)}} - 1 - e^{-iq_d} + e^{-iq_{(g+d+h)}}) \\
&\quad - f_{ABC}f_{BCD}(e^{iq_{(g+h)}} + 1 + e^{-iq_d} + e^{-iq_{(g+d+h)}})) \\
&\quad - (h \leftrightarrow g) \\
&= \int \frac{d^4\mathbf{q}}{(2\pi)^4} \sum_{d,B,C} \frac{2i}{8\widehat{\mathbf{q}}^2\widehat{\mathbf{q}}^2} \epsilon_{abdgh} \\
&\quad \times (d_{ABC}d_{BCD}(\sin q_{(d+g+h)} - \sin q_d - \sin q_{(g+h)}) \\
&\quad - f_{ABC}f_{BCD}(\sin q_{(d+g+h)} + \sin q_d - \sin q_{(g+h)})) \\
&= 0.
\end{aligned} \tag{B.6}$$

C. Coupling Constant Independence in $\mathcal{N} = 4$ SYM

The continuum twisted $\mathcal{N} = 4$ SYM discussed in this paper possesses a privileged set of operators whose expectation values can be shown to be independent of the background metric and hence topological. The condition for this to be true is that the operator be annihilated by the charge \mathcal{Q} . In addition the expectation values of these operators can be shown to be independent of the coupling constant. As we will see this property remains true in the lattice theory and provides powerful constraints on the renormalization of such operators. To see this result consider the twisted lattice action which is the sum of \mathcal{Q} -exact and \mathcal{Q} -closed terms. The coupling constant dependence of the \mathcal{Q} -closed term can be removed, without disturbing the \mathcal{Q} BRST transformation, by rescaling the fields in appropriate ways. We show this below.

The twisted action is

$$\begin{aligned}
S &= \frac{1}{g^2} S_{exact} + \frac{1}{g^2} S_{closed} \\
&= \int \text{Tr} \left\{ \frac{1}{g^2} (\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2} [\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a) \right. \\
&\quad \left. - \frac{1}{g^2} \left(\frac{1}{2} \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de} \right) \right\}.
\end{aligned} \tag{C.1}$$

A simple rescaling of the fields

$$\chi_{ab} \rightarrow \chi_{ab}/g, \quad \psi_a \rightarrow g\psi_a, \quad \eta \rightarrow \eta/g, \tag{C.2}$$

gives the action

$$\begin{aligned}
S &= \frac{1}{g^2} \int \text{Tr} \left(\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2} [\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a \right) \\
&\quad - \frac{1}{2} \int \text{Tr} \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de} \\
&= \frac{1}{g^2} S_{exact} + S_{closed}
\end{aligned} \tag{C.3}$$

Calling $\beta = \frac{1}{g^2}$ and writing the action as $S = \mathcal{Q}\Lambda + S_{\text{closed}}$ the expression for the expectation value of a \mathcal{Q} -invariant operator \mathcal{O} becomes

$$\langle \mathcal{O} \rangle_\beta = \frac{1}{Z} \int \mathcal{O} e^{-(\beta \mathcal{Q}\Lambda + S_{\text{closed}})}, \quad Z = \int e^{-(\beta \mathcal{Q}\Lambda + S_{\text{closed}})}. \quad (\text{C.4})$$

Differentiating this expression with respect to β leads to

$$\begin{aligned} \frac{\partial}{\partial \beta} \langle \mathcal{O} \rangle_\beta &= \langle \mathcal{Q}\Lambda \rangle_\beta \langle \mathcal{O} \rangle_\beta - \langle \mathcal{O} \mathcal{Q}\Lambda \rangle_\beta \\ &= \langle \mathcal{Q}\Lambda \rangle_\beta \langle \mathcal{O} \rangle_\beta - \langle \mathcal{Q}(\mathcal{O}\Lambda) \rangle_\beta \\ &= 0, \end{aligned} \quad (\text{C.5})$$

where we have used the fact that as long as the BRST symmetry is not broken spontaneously, the expectation value of the \mathcal{Q} variation of some operator vanishes. Thus expectation values of \mathcal{Q} -invariant observables are independent of β and hence can be computed exactly in the semi-classical limit $\beta \rightarrow \infty$. In this limit we need only do one loop calculations around the classical vacua.

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