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Exact Lattice Supersymmetry

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Abstract

We provide an introduction to recent lattice formulations of supersymmetric theories which are invariant under one or more real supersymmetries at nonzero lattice spacing. These include the especially interesting case of $\mathcal{N} = 4$ SYM in four dimensions. We discuss approaches based both on twisted supersymmetry and orbifold-deconstruction techniques and show their equivalence in the case of gauge theories. The presence of an exact supersymmetry reduces and in some cases eliminates the need for fine tuning to achieve a continuum limit invariant under the full supersymmetry of the target theory. We discuss open problems.

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1. Introduction

Whether or not supersymmetry is discovered to be a symmetry of nature, strongly coupled supersymmetric theories will always be a source of fascination [1, 2, 3, 4]. In these theories one can find explicit examples of many of the basic mechanisms and objects put forward in the early days of gauge theories: confinement, chiral symmetry breaking, magnetic monopoles and dyons, conformal field theories, etc. Especially intriguing are the connections between theories with sixteen supercharges and both supergravity and string theory [5, 6, 7].

Until recently, a nonperturbative lattice formulation for all but a few of these theories remained elusive despite many efforts over the years. The problem has been that discretization tends to completely break the supersymmetry, so that no characteristics of the continuum theory are present without excessive fine-tuning. In the language of the renormalization group, the lattice theory typically flows away from any supersymmetric fixed point as the cut-off is removed. Past attempts to fix this by imposing an exact supersymmetric subalgebra on the lattice action typically resulted in a loss of Poincaré invariance [8].

In the past few years, however, there have been significant advances in our understanding, which have led to the construction of a number of interesting supersymmetric lattice theories, including $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) in four dimensions, in which these fine tuning problems are under much better control.

The new development has been the construction of lattice actions which possess a subset of the supersymmetries of the continuum theory and have a Poincaré invariant continuum limit. The presence of the exact supersymmetry on the lattice provides a way to obtain the continuum limit with no fine tuning, or fine tuning much less than conventional lattice constructions (in which there is no exact supersymmetry at the cut-off scale.) In this review, we introduce some of the ideas which lead to the construction of these supersymmetric lattice theories.

Two main approaches have been proposed to formulate such supersymmetric lattice theories, which are now understood to be closely related. One is based on the idea of 'twisting' and Dirac-Kähler fermions [9, 10]. The twisting procedure is based on a decomposition of Lorentz spinor supercharges into a sum of integer spin (p-form) tensors under a diagonal subgroup of the Lorentz group and some large global symmetry of the theory, usually referred to as R-symmetry. The twisted formulation of supersymmetry goes back to Witten [11] in his seminal construction of topological field theories, but actually had been anticipated in earlier lattice work using Dirac-Kähler fields [12, 13, 14, 15, 16]. The precise connection between Dirac-Kähler fermions and topological twisting was found by Kawamoto and collaborators [17, 18, 19]. The key observation is that the zero-form supercharge that arises after twisting is a scalar which squares to zero, and constitutes a closed subalgebra of the full twisted superalgebra. It is this scalar supersymmetry that can be made manifest in the lattice action even at finite lattice spacing [20, 21, 22, 23, 24, 25, 26, 27, 28].

The second approach derives a supersymmetric lattice theory by orbifolding a certain supersymmetric matrix model. This 'mother' matrix theory is obtained by dimensional reduction of a SYM theory with a very large gauge symmetry. The projection is chosen so as to induce a lattice structure and to preserve one or more supersymmetries of the mother theory. The resulting theory is also gauge invariant and preserves a discrete subgroup of the continuum Lorentz and global symmetries [29, 30, 31, 32, 33, 34, 35, 36, 37]. The theories obtained in this fashion have a degenerate ground state (called a moduli space), where the distance from the origin of the moduli space has an interpretation as the inverse lattice spacing, as first implemented in "deconstruction" [38, 30]. The continuum limit is thus defined as a particular scaling limit out to infinity in the moduli space and the result is a supersymmetric gauge theory where full super-Poincaré symmetry is recovered[1].

Even though these two approaches seem different at first glance, they do generate similar actions and lattices. The reason behind this is that the Dirac-Kähler decomposition of the fermions is indeed encoded into the charges of the fermions encountered in the orbifold projection[39, 36, 40] – the number of non-zero components of the r-charge vector characterizing the orbifold lattice field matching the degree of the p-form component of the corresponding twisted Dirac-Kähler field. The common thread of both approaches is the exact preservation of (nilpotent) scalar supercharges on the lattice, which automatically dictates the distribution of the bosonic degrees of freedom in the lattice given the Dirac-Kähler construction. Indeed we will

[1] To be more precise the continuum limit of these constructions is actually invariant under a twisted version of the super-Poincaré group. Whether one can 'untwist' the theory to obtain a target theory with the usual super-Poincaré invariance is related to the amount of residual fine tuning needed to obtain full supersymmetry.

show that one can obtain the supersymmetric orbifold lattices by a direct discretization of an appropriately chosen twist of the target SYM theory [41, 42].

It is also useful to keep in mind the limitations of the twisting and orbifolding formalisms. These techniques only apply to a sub-class of supersymmetric gauge theories. The ability to construct a manifestly supersymmetric lattice in this formalism requires the R-symmetry group to contain $SO(d)$ – the d-dimensional (Euclidean) Lorentz symmetry group – as a subgroup. If so, one can apply the idea of twisting as shown in Fig.2. Clearly this constraint excludes the formulation of some other interesting theories, such as $\mathcal{N} = 2$ SYM (the Seiberg-Witten theory) or generic $\mathcal{N} = 1$ supersymmetric QCD theories, or theories of more phenomenological interest such as the MSSM. Lattice constructions of these interesting theories are currently open problems. It is also fair to say that much theoretical work remains to be done to understand how much fine tuning is required in order that these lattice theories inherit the full supersymmetry of the target theory in the continuum limit – perturbative calculations would be very useful in this regard as we will discuss later when describing the $\mathcal{N} = 4$ construction in detail.

The first part of the review will motivate the study of lattice supersymmetry and give an overview of some of the basic ideas: why it is difficult to build supersymmetric lattice theories and why naive discretizations of continuum supersymmetric theories lead to fine tuning problems. We argue that supersymmetry should arise as an accidental symmetry on taking the continuum limit of some suitable lattice model, and offer as an example a supersymmetric theory without scalars: the interesting $\mathcal{N} = 1$ super Yang-Mills theory in $d = 4$ dimensions. We then discuss why in supersymmetric theories with scalars, only an exact lattice supersymmetry can keep scalars massless without fine-tuning and allow for the full supersymmetry algebra to emerge as an accidental symmetry at long distances,. Tautology is avoided since the lattice model need only preserve a subset of the continuum supersymmetry to avoid or at least ameliorate fine-tuning. This naturally leads into a discussion of twisted supersymmetry and Dirac-Kähler fermions.

Before progressing to more complicated theories, we next consider supersymmetric quantum mechanics and the two dimensional Wess-Zumino and sigma models. This will allow us to illustrate the nature of the fine tuning problems that are encountered and how realization of an exact lattice supersymmetry enables the full supersymmetry to emerge in the continuum limit. The connection between twisting, Nicolai maps and topological field theories is then discussed in the context of these examples. We then turn to gauge theories, first presenting the twisted supersymmetry approach to $(2,2)$ SYM in two dimensions. We then discuss the powerful orbifold approach to lattice SYM, and show in detail how to obtain a gauge invariant lattice model invariant under one real supersymmetry for this same $(2,2)$ SYM theory. It is shown to be precisely the same as the twisted construction derived earlier. The possible supersymmetric orbifold lattices are then classified and seen to include the interesting case of $\mathcal{N}=4$ SYM. We summarize the content of this lattice model and show how it can also be generated by discretization of the Marcus twist of $\mathcal{N}=4$ SYM theory confirming, once more, the complete equivalence of the two approaches.

We emphasize that this resultant lattice action for $\mathcal{N}=4$ SYM currently offers a promising starting point for numerical simulations. Indeed, dimensional reductions of this theory are already being studied on the lattice in the context of their conjectured equivalence to string and supergravity theories [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. It is perhaps the prospect of eventually using our lattices to learn more about quantum gravity that we consider the most exciting.

2. Supersymmetry

2.1. The supersymmetry algebra

Poincaré symmetry consists of spacetime translations, generated by P_μ, and Lorentz transformations, generated by $\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}$. The algebra has the qualitative structure

$$[P,P]=0\ ,\quad [P,\Sigma]\sim P\ ,\quad [\Sigma,\Sigma]\sim\Sigma\ , \tag{1}$$

where the meaning of the three terms are (i) translations commute with each other; (ii) translations transform under the Lorentz group as a 4-vector; (iii) Lorentz transformations themselves transform as an antisymmetric tensor.

Supersymmetry is the unique extension of the Poincaré algebra consistent with the Coleman-Mandula theorem [54], where complex spinorial generators Q_α, $\bar{Q}_{\dot\alpha}$ are added with the (anti-) commutation relations:[2]

$$\{Q,Q\}=0\ ,\quad [P,Q]=0\ ,\quad [Q,\Sigma]\sim Q\ ,\quad \{Q,\bar{Q}\}\sim P\ . \tag{2}$$

[2] We have neglected the possibility of central charge terms in this simplified discussion

These terms tell us (i) Q is Grassmann; (ii) Q commutes with spacetime translations (and hence the Hamiltonian); (iii) Q transforms under Lorentz transformation as a 2-component Weyl spinor; (iv) two successive supersymmetry transformations yields a translation. From (i) and (ii) it follows that there are pairs of fermion-boson states which are degenerate (along with possible unpaired zero energy states), and from (iv) we see that in some sense a supersymmetry charge Q is a square root of the Hamiltonian (in the same sense that the Dirac operator is a square root of the Klein-Gordon operator).

2.2. Counting supercharges

The supersymmetry algebra is highly constrained, and in any given number of dimensions there are typically only a few possibilities for how many supercharges can exist. These constraints arise from the requirement that the theories not contain particles with spin greater than one (or two in the case of supergravity). This restriction on the maximal spin stems, in turn, from the requirement that the theories be renormalizable.

These different solutions are often labeled $\mathcal{N} = 1$, $\mathcal{N} = 2$, etc. What is confusing is that the number of supercharges for $\mathcal{N} = 1$ supersymmetry, for example, is different in different numbers of dimensions. Instead, when discussing supersymmetric theories in dimensions other than four, we will identify a supersymmetric theory by the spacetime dimension d, and *the number of real supercharges*, $\mathcal{Q}$. Thus $\mathcal{N} = 1$ supersymmetry in $d = 4$ has a complex pair Q, $\bar{Q}$ which are each two-component Weyl spinors, giving $\mathcal{Q} = 4$. Similarly, $\mathcal{N} = 4$ supersymmetry in $d = 4$ has $\mathcal{Q} = 16$.

2.3. R symmetries

Supersymmetric theories typically have global chiral symmetries — generically called "R-symmetries" which do not commute with the supercharges, meaning that the members of the supermultiplets transform as different multiplets under the R-symmetry. These symmetries turn out to play a crucial role in the implementation of lattice supersymmetry. The bosonic and fermionic fields furnish a representation of the R-symmetry, as well as Euclidean Lorentz symmetry $SO(d)_E$, and the same is true for the supercharges. For example, a list of the SYM theories and their Lorentz and R-symmetries (at the classical level) are given in Table 1. These symmetries are most easily determined by exploiting the fact that $\mathcal{Q} = 4, 8, 16$ SYM theories are the minimal ($\mathcal{N} = 1$) gauge theory in $d = 4, 6, 10$ dimensions respectively — in those dimensions, the theory consists of only of a gauge field and a gaugino.

Theory	Lorentz	$\mathcal{Q}=4$	$\mathcal{Q}=8$	$\mathcal{Q}=16$
$d=2$	$SO(2)$	$SO(2)\times U(1)$	$SO(4)\times SU(2)$	$SO(8)$
$d=3$	$SO(3)$	$U(1)$	$SO(3)\times SU(2)$	$SO(7)$
$d=4$	$SO(4)$	$U(1)$	$SO(2)\times SU(2)$	$SO(6)$

Table 1: *The R-symmetry groups of various SYM theories.*

The $(\mathcal{Q}=4, d=4)$ SYM theory has a $U(1)$ R-symmetry, the $(\mathcal{Q}=8, d=6)$ possesses an internal $SU(2)$ R-symmetry, while the $(\mathcal{Q}=16, d=10)$ $\mathcal{N}=1$ theory has no R-symmetry When these theories are dimensionally reduced from $d'=4,6,10$-dimensions down to d dimensions one preserves all of the supercharges while enlarging the R-symmetry by Euclidean "Lorentz" generators acting in the reduced dimensions. For example, the $\mathcal{N}=1$ theory in $d'=10$ dimensions dimensionally reduced to d dimensions has an $SO(d)_E$ Lorentz symmetry and $SO(10-d)$ R-symmetry, as shown in the last column of Table 1. There are also cases where these classical symmetries may reduce in a quantum theory due to anomalies or enhance to a larger $R-$symmetry at long distances. For a discussion of the case with $\mathcal{Q}=16$ where the latter may take place, see [55].

2.4. Why study lattice supersymmetry?

Supersymmetry is interesting in its own right. It is also potentially interesting for phenomenology, as the protection it affords scalars from additive renormalization of their masses could have something to do with the mysterious Higgs boson of the Standard Model. And it is worth studying because with the extra symmetry, many interesting results have been obtained for supersymmetric Yang-Mills (SYM) theories, including explicit examples of many mechanisms postulated in the early days of Yang-Mills theories, including spontaneous chiral symmetry breaking, confinement, magnetic monopole condensation, strong coupling - weak coupling duality, massless composite fermions, conformal field theory, and more [56, 57]. In addition, many fascinating connections have been made between between SYM theories and string theory and quantum gravity [6, 7], as well as with topology [58] .

Since there are so many interesting features of SYM theories, especially at strong coupling, it would be very desirable to be able to have a nonperturbative definition of these theories. This is important both from a mathematical viewpoint and also as a basis for numerical simulations.

3. Accidental supersymmetry and twisted supercharges

In principle, many supersymmetric theories could be studied on the lattice by choosing the right degrees of freedom, and then tuning the couplings to the critical values which yield the supersymmetric "target theory" in the infrared. However, this brute-force approach is prohibitively difficult for any but possibly the simplest theories. A more practical approach is to construct a lattice theory that respects as many of the symmetries of the target theory as possible, limiting the number of possible operators whose coefficients need to be fine-tuned. One might think that the lattice action would have to possess *all* of the symmetries of the target theory, but in fact that is not necessary due to the emergence of "accidental" symmetries. An accidental symmetry is a symmetry that emerges in the infrared (IR) limit of the lattice theory, even though it is not respected by the full lattice action. This typically occurs when the exact symmetries of the action only allow irrelevant operators which could violate the accidental symmetry transformation — such operators become unimportant in the IR, and the symmetry then emerges. A prime example of an accidental symmetry in the continuum is baryon number violation in a Grand Unified Theory (GUT) [59]: B-violation is mediated by gauge boson and scalar interactions at the GUT scale; but below the GUT scale the light degrees of freedom are such that the gauge symmetries of the standard model forbid baryon violating operators with dimension less than six, and as such, baryon number violation becomes "irrelevant" in the IR. That explains why GUTs can be consistent with the observed stringent lower bounds on the proton lifetime[3].

Accidental symmetry also explains why lattice QCD is able to recover (Euclidean) Poincaré symmetry in the continuum limit without fine-tuning, even though the lattice action only respects a discrete hypercubic subgroup of Poincaré symmetry: given the field content of QCD and both the exact hypercubic and gauge symmetries of the lattice action, the lowest dimension operators that can be added to the action which violate continuum Poincaré symmetry are dimension six, such as the discrete version of $\sum_\mu \bar{\psi}\gamma_\mu D_\mu^3 \psi$. These are irrelevant operators which become unimportant in the IR limit of the theory, and so the full Poincaré symmetry is recovered without fine-

[3] Actually, only SUSY GUTs are consistent with the limits on proton lifetime and gauge coupling unification. In SUSY GUTs, there are dimension five baryon number violating operators unless the theory is supplemented by an additional R-parity symmetry.

tuning. As a counter-example, consider Wilson fermions with the bare mass term set to zero; in this case the lattice action appears to have an exact chiral symmetry in all the relevant and marginal operators, with chiral symmetry breaking first appearing in the Wilson term, a dimension five operator of the form $\bar{\psi}D^2\psi$. Although the Wilson term is an irrelevant operator, since the exact symmetries of the lattice theory allow a dimension three fermion mass term, it will be generated radiatively, requiring $O(1/a)$ fine-tuning of the bare mass to obtain massless fermions in the IR.

What about supersymmetry? Supersymmetry certainly cannot be an exact symmetry on the lattice, since the supersymmetry algebra dictates that the anti-commutator of supercharges yield an infinitesimal translation [1], $\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta_{ij}$, and such translations do not exist on a lattice. However, as this Report documents, supersymmetry can emerge from a lattice action with little or no fine-tuning due to accidental symmetry. We first describe how this works in the four-dimensional theory that is simplest to simulate on the lattice: $\mathcal{N}=1$ SYM theory.

3.1. $\mathcal{N}=1$ *supersymmetry in* $d=4$ *and accidental susy without scalars*

The $\mathcal{N}=1$ SYM theory in $d=4$ consists of gauge bosons v_m (the "gluons", $m=1,\dots 4$) and a single Weyl fermion λ_α (the "gluinos", $\alpha=1,2$). The gluino is the supersymmetric partner of the gluon, and like it, transforms in the adjoint representation of the gauge group. Using the two-component fermion notation (see [1]), the Lagrangian for the theory is

$$\mathcal{L} = \bar{\lambda}i\bar{\sigma}^m D_m\lambda - \tfrac{1}{4}v_{mn}v^{mn} \ , \tag{3}$$

where $\bar{\sigma}^m = \{1, -\boldsymbol{\sigma}\}$ ($\boldsymbol{\sigma}$ being the three Pauli matrices and 1 being the unit matrix) and v_{mn} is the gauge field strength. This theory has only one independent coupling constant (the gauge coupling g) and is the most general Lagrangian one could write down without irrelevant operators — with the important exception that we have omitted a fermion mass term, $(m\lambda\lambda + h.c.)$. At the classical level, the theory possesses a global $U(1)$ symmetry under which $\lambda \to e^{i\alpha}\lambda$. This does not commute with supersymmetry (because there is no analogous phase rotation of the gluino's partner the gluon) and for obscure historical reasons it is therefore called an R-symmetry. Now this $U(1)$ symmetry is anomalous, and if the gauge group is $SU(N)$, only a Z_{2N} subgroup of the $U(1)$ symmetry is exact in the full quantum theory (see, for example, [60]). Note that a gluino mass term would explicitly violate this

Z_{2N} R-symmetry. It is known that gluino condensation occurs in this theory ($\langle\lambda\lambda\rangle \neq 0$), and that the global Z_{2N} symmetry is spontaneously broken to Z_2, giving rise to domain walls, where the strength of the condensate and the domain wall tension can be analytically related.

Can we investigate these properties on the lattice? After all, the theory looks simpler than QCD which has several flavors of quarks with different masses, which is routinely simulated.

The key to accidental supersymmetry for a lattice realization of this $\mathcal{N} = 1$ SYM theory is the observation that the only "bad" relevant operator allowed by gauge plus Lorentz symmetries is a gaugino mass term...and this is forbidden by the Z_{2N} chiral R-symmetry. This observation can be put to use to construct a lattice theory. Either one can use a Majorana Wilson fermion and fine-tune the gaugino mass to zero (see [61, 62, 63, 64] and references therein), or one can start with chiral lattice fermions and obtain the supersymmetric target theory without fine-tuning [4]. Luckily, the problem of how to realize chiral fermions on the lattice has already been solved: the two related techniques are to use domain wall fermions (DWF) [66], or overlap fermions[67, 68]. Formulations of $\mathcal{N} = 1$ SYM with overlap fermions were first proposed in [69]; domain wall fermion formulations of the lattice theory are found in [70, 71, 72]. Here we sketch the domain wall fermion (DWF) formulation of this theory [72], before discussing the more complicated case of how accidental supersymmetry can arise in theories with scalar fields in subsequent sections.

The DWF formulation is formulated on a (compact) five-dimensional lattice, with a massive fermion whose mass equals $+m_0$ on half the lattice and $-m_0$ on the other half. The 4-dimensional hypersurfaces where the mass changes sign are called "domain walls", and on solving the free Dirac equation, one finds two massless 4-dimensional fermion modes, one with $\gamma_5 = +1$ bound to one domain wall, and the other with $\gamma_5 = -1$ bound to the other wall, as shown in Fig. 1. (There is actually a small mass which vanishes exponentially in the fifth dimensional separation between the two domain walls, which for the purposes of this simplified discussion, we will assume is negligible - see [73, 74, 75] for a discussion of practical issues issues arising from this non-zero *residual mass*). Four dimensional gauge fields are introduced *á*

[4]This scenario whereby supersymmetry could be realized as an accidental symmetry was first proposed in [29], and specifically for lattice simulation in [65].

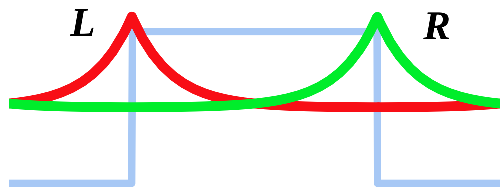

Figure 1: The profile of the domain wall fermion mass in the fifth dimension, showing the chiral zero-modes (L,R) bound to the two domain walls where the fermion mass switches sign.

la Wilson, constant in the fifth dimension, with the fermion transforming as an adjoint under the gauge group. The low energy spectrum therefore looks like a $d = 4$ theory consisting of a gauged massless adjoint Dirac fermion and gauge bosons. This $d = 4$ (Euclidean) Dirac fermion takes the form $\Psi = \begin{pmatrix}\alpha \\ \bar{\beta}\end{pmatrix}$, $\bar{\Psi} = \begin{pmatrix}\bar{\alpha}^T & \beta^T\end{pmatrix}$, where α and β are the 2-component chiral spinors stuck to the two domain walls respectively. Since the gauge fields are constant in the fifth dimension, they are insensitive to the fact that the spinors α and β reside at different places in the extra dimension. Imposition of the Majorana condition is equivalent to requiring $\Psi = R_5 C \bar{\Psi}^T$, where R_5 is the reflection in the fifth dimension which interchanges the two domain walls, and C is the $d = 4$ charge conjugation matrix. To implement a Majorana fermion in the Euclidean path integral then, we just replace $\bar{\Psi}$ everywhere by $\Psi^T R_5^T C^T$, so that the Dirac Lagrangian $\bar{\Psi} \not{D} \Psi$ becomes instead $\Psi^T R_5^T C^T \not{D} \Psi$, and the Dirac determinant $\det \not{D}$ is replaced by the Pfaffian $\mathrm{Pf}\, R_5^T C^T \not{D}$, which is real and non-negative. Simulation of this theory is computationally challenging, but recently much progress has been made with the first ab initio calculations of the chiral condensate being reported in [76, 77, 74, 75, 73].

3.2. Accidental SUSY with scalars?

In the previous section we saw that a gauged adjoint Majorana fermion in four dimensions was automatically supersymmetric provided that the relevant mass term $m\lambda\lambda$ vanished. Since this mass term violates a Z_{2N} chiral symmetry as well as supersymmetry, it follows that in a lattice theory that correctly implements the chiral symmetry, supersymmetry will automatically emerge as an accidental symmetry in the continuum limit, despite the fact that the lattice action is not supersymmetric at all.

Unfortunately, this simple reasoning does not extend readily to other supersymmetric theories, which all contain scalars as well as fermions, and possibly gauge fields. The problem is that supersymmetry is broken by the relevant operator responsible for scalar masses $m^2|\phi|^2$ (among others), which breaks the fermion-boson degeneracy. Following the example of $\mathcal{N} = 1$ SYM, we would like to identify some symmetry (*other* than supersymmetry) which is broken by a scalar mass term, and which can be implemented exactly on the lattice. Unfortunately, unlike fermions, there is no chiral symmetry which can be invoked to forbid a scalar mass; the only symmetry that can do that is a shift symmetry $\phi \to \phi + f$, and this shift symmetry is too restrictive, dictating only derivative interactions for the scalar, and hence applicable only to Goldstone bosons (furthermore sigma models are not thought to be renormalizable in four dimensions). Thus the only useful symmetry that can forbid the undesirable scalar mass term is supersymmetry itself.

We are apparently left with a paradox: implementing supersymmetry exactly on the lattice seems impossible, and so we would like it to emerge as an accidental symmetry; but in order for supersymmetry to emerge as an accidental symmetry, we are forced to suppress scalar mass renormalization, which requires implementing supersymmetry exactly on the lattice!

Perhaps we don't have to find an exact lattice implementation of *all* of the supersymmetry of the target theory, but only realize part of the supersymmetric algebra? After all, the full Poincaré group is not realized on the lattice, but only the finite subgroup generated by finite translations and rotations by $\pi/2$, yet the Poincaré group emerges as an accidental symmetry. It is natural then to ask whether there could exist a "subgroup" of supersymmetry on the lattice? But the answer is no: whereas rotations, for example, are parameterized by a bosonic angle which can be large (e.g., $\pi/2$) supersymmetric transformations are characterized by a Grassmann parameter, which is necessarily infinitesimal, just as there exist classical bosonic fields (such as the electric field) but not classical fermionic fields [5].

Instead one must ask whether it is possible to preserve a subalgebra of

[5] There are such things as "supergroups" defined with Grassmann generators, but they do not seem to be of any practical use for constructing supersymmetric lattices.

the full extended supersymmetric algebra

$$\{Q^i_\alpha, Q^j_\beta\} = 0 \ , \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0 \ , \quad \{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\} = 2P_m \sigma^m_{\alpha\dot\beta}\delta_{ij} \ , \tag{4}$$

where $i, j = 1, \ldots, \mathcal{N}$ run over different supercharges (for $\mathcal{N} = 1, 2, 4$ supersymmetry respectively in $d = 4$). A number of potential obstacles are immediately obvious:

i. The same old problem we keep returning to: how can a subalgebra of eq. (4) be chosen given that the P_m, the generator of infinitesimal translations, does not exist on the lattice? (An early attempt at lattice SUSY was to work in a Hamiltonian formulation, so that infinitesimal time translations $P_0 = H$ were maintained; however while this enabled exact lattice supersymmetry, it precluded a Lorentz invariant continuum limit [8].)

ii. How can one isolate part of the algebra without destroying the hypercubic lattice symmetry, and thereby making it impossible to recover Poincaré symmetry in the continuum, let alone supersymmetry?

iii. Less abstractly, how is it possible to implement scalars, fermions and gauge bosons in a symmetric fashion on the lattice? For example, $\mathcal{N} = 4$ SYM has one gauge field, four Weyl gauginos, and six real scalars in the same supersymmetric multiplet. If we put the gauge fields on links, surely their scalar superpartners have to be on links too! But then the scalars will transform nontrivially under lattice rotations, which suggests they can't transform as scalars (rotationally invariant objects) in the continuum limit.

iv. SYM theories have R-symmetries ($U(1)$, $U(2)$ and $SU(4)$ respectively for $\mathcal{N} = 1, 2, 4$ theories in $d = 4$; larger symmetries in lower dimensions) which are *chiral* symmetries; how are we to implement chiral fermions in a way that makes them look symmetric with their gauge and scalar partners?!

These arguments would incorrectly seem to rule out implementing accidental lattice supersymmetry with scalars. The way out of this cul-de-sac is to recognize that in Euclidean space, Q^i and $\bar{Q}^i$ are independent; calling them all q^i, a subset of the supercharges $\{q^i\}$ can be preserved that are nilpotent: $\{q^i, q^j\} = 0$ (up to a gauge transformation). Keeping such charges exact on the lattice solves the problem of not having infinitesimal P_μ generators, but does not solve the issue of Lorentz invariance since the supercharges being kept belong to incomplete spinor representations of the Lorentz group. An answer to this conundrum can be found in the peculiar formulation of staggered or Dirac-Kähler fermions, where the discrete point symmetry of the lattice is not just embedded in the full Lorentz group (or its Euclidean analogue), but in the combined (Lorentz)×(flavor) group. Under the discrete lattice symmetry, the fermions then naturally decompose as n-index antisymmetric tensors, instead of spinors. Furthermore, these antisymmetric tensor components generically include one or more scalars. If the supercharges singled out for preservation on the lattice are scalars under this lattice symmetry, it becomes plausible that Lorentz symmetry could be preserved in the continuum limit.

While providing a clue for how to realize supersymmetric lattices, the above discussion leaves obscure how to create "staggered scalars" and "staggered gauge fields" so that supersymmetric multiplets composed of (gauge boson, gaugino, gauge scalar) could appear on the lattice. As we show below, the Gordian knot is cut by formulating twisted supersymmetry, or by following the orbifold/deconstruction procedure. By means of these related techniques we are able to solve the above conundrums in a new and unanticipated way.

3.3. Twisted supersymmetry

In this section, we first briefly review the concept of twisting in extended supersymmetric gauge theories in the continuum formulation on $\mathbb{R}^d$ [11] and sketch its relation to orbifold projections of supersymmetric matrix models, which we will discuss next.

As we have discussed, extended supersymmetric gauge theories usually possess large chiral symmetries, called R-symmetries. The bosonic and fermionic fields furnish a representation of the relevant R-symmetry, as well as Euclidean Lorentz symmetry $SO(d)_E$, and the same is true for the supercharges. For example, a list of the SYM theories and their Lorentz and R-symmetries (at the classical level) are given in Table 1. For six of the theories shown in Table 1, the R-symmetry group possess an $SO(d)_R$ subgroup. Hence, the full global symmetry of the supersymmetric theory has a subgroup $SO(d)_E \times SO(d)_R \subset SO(d)_E \times G_R$. To construct the twisted theory, we identify the diagonal $SO(d)'$ subgroup in $SO(d)_E \times SO(d)_R$, and declare and treat it as the new Lorentz symmetry of the theory [6].

$$SO(d)' = \mathrm{Diag}(SO(d)_E \times SO(d)_R) \tag{5}$$

In particular, when we eventually create a lattice theory, the point group of the lattice will be a discrete subgroup of this $SO(d)'$, and *not* of $SO(d)_E$, as one might have supposed. Since the details of each such construction are slightly different, let us restrict to generalities first (later we discuss in some detail how twisting works in the context of $(2,2)$ SYM in two dimensions). In every case we will consider, fermionic fields transform as spinor representations under both $SO(d)_E$ and $SO(d)_R$. Since the product of two half-integer spins always has integer spin, *all* fermionic degrees of freedom will be in integer spin representations of $SO(d)'$, direct sums of scalars, vectors, and general p-form tensors. Let us label a p-form fermion as $\psi^{(p)}$. In all of our applications, the $\mathcal{F}$ different fermions of a target field theory in d dimensions are distributed in multiplets of $SO(d)'$ as

$$\text{fermions} \rightarrow \frac{\mathcal{F}}{2^d}(\psi^{(0)} \oplus \psi^{(1)} \oplus \ldots \psi^{(d)}) \tag{6}$$

where the multiplicative factor up front is one, two, four or eight. For a given p-form, there are $\frac{\mathcal{F}}{2^d}\binom{d}{p}$ fermions. Summing over all p, we obtain the total number of fermions in the target theory: $\frac{\mathcal{F}}{2^d}\sum_{p=0}^{d}\binom{d}{p} = \mathcal{F}$

Turning to the bosonic fields, the gauge bosons V_μ transform as $(d,1)$ under $SO(d)_E \times SO(d)_R$, while the scalars typically include a subset we can label as S_μ, transforming as $(1,d)$. Thus both S_μ and V_μ transform as d-vectors (1-forms) under the diagonal $SO(d)'$ symmetry. In theories with more than d scalars in the untwisted theory, these remnants become either 0-forms or d-forms under $SO(d)'$.

The $\mathcal{Q}$ supercharges also decompose into a sum of p-forms under the diagonal group:

$$\text{Supercharges} \rightarrow \frac{\mathcal{Q}}{2^d}(Q^{(0)} \oplus Q^{(1)} \oplus \ldots Q^{(d)}) \tag{7}$$

We may then write the supersymmetry algebra without using any spinor indices just in terms of p-forms. What is important is the fact that there

[6] We will not distinguish spin groups $Spin(n)$ from $SO(n)$ unless otherwise specified.

exists one or more spin-0 nilpotent supercharges $Q^{(0)} \equiv Q$. That means, the twisted formulation of the supersymmetry algebra contains a subalgebra

$$Q^2 \cdot = 0 \tag{8}$$

This nilpotent supercharge is then insensitive to the background geometry. In fact, if the base space of the theory is an arbitrary d-dimensional curved manifold M^d, then there exist no covariantly constant spinors. However, there may exist covariantly constant, spin-0 fields. Hence, globally, only the scalar supercharges are preserved when the theory is carried on curved spacetime. Furthermore, if M^d is flat, this transition from integer-spin, p-form supercharges to spinor supercharges is a simple change of basis, a redefinition. In flat spacetime, so long as the scalar supercharge is not declared as a BRST operator, there is no physical distinction between the twisted and untwisted theories. We can construct true topological field theories if we additionally require that the charge Q be interpreted as a BRST operator. This is discussed throughly in the context of the string theory and the theory of four manifolds in [78].

The application to topological field theories leads to the appearance of the term "topological twisting". In fact, the twisting operation can be thought of as conceptually unrelated to topological field theory or supersymmetry. As we will discuss, the well known staggered fermion formulation of lattice fermions (or Kogut-Susskind fermions) is in fact an example of twisting, with the lattice fermions living in a diagonal subspace of the flavor and real spaces as shown in Fig. 2.

The action expressed in terms of fields which form representations of the twisted Lorentz group $SO(d)'$ instead of the usual Lorentz symmetry, is called the twisted action. Typically, the twisted action can be expressed as a sum of Q-exact and Q-closed terms, where Q refers to one or more of the scalar supercharges. The arguments for this follow directly from the structure of the twisted subalgebra as we explain later.

A key feature of this twisting process (shown in Fig.2) is that none of the degrees of freedom are spinors under $SO(d)'$. Both bosons and fermions are in integer spin representations. They are p-form tensors of $SO(d)'$. This particular form of the twisted theory is the bridge to lattice supersymmetry and orbifold lattices. A p-form continuum field may naturally be associated with a p-cell on the hypercubic lattice [9]. We will see that this is exactly what an orbifold lattice does. The orbifold projection places the fermions on

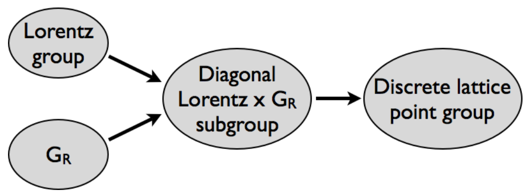

Figure 2: The lattice point group in supersymmetric lattices cannot be considered to be a subgroup of just the Lorentz group, but rather it is a subgroup of the product of the Lorentz group and and the R-symmetry group, G_R.

sites, links, faces, or more generally to p-cells. Moreover, since scalars of the Lorentz symmetry are 1-forms of the twisted theory, they can naturally be amalgamated with the gauge bosons into complex bosons – the two degrees of freedom being associated with oppositely oriented links. This complexification of gauge fields had been noted earlier in continuum twists of the $\mathcal{N} = 4$ theory [79].

We now see how twisting allows us to circumvent the first three obstacles listed in section 3.2

i. By focusing on the nilpotent supercharge, the connection between supersymmetry and infinitesimal translations is sidestepped;
ii. By identifying the lattice symmetry with a discrete subgroup of $SO(d)'$ and implementing the supercharges which are $SO(d)'$-scalars, the supersymmetric subalgebra we have selected does not interfere with obtaining a Poincaré invariant continuum limit;
iii. With fermions and bosons falling into similar $SO(d)'$ representations, it becomes possible to imagine that they could be treated similarly on the lattice as would befit members of the same supersymmetry multiplet.

In relation to the fourth point in section 3.2, we will see that the R-symmetries are not exact on the lattice, but emerge as accidental symmetries in the continuum limit, along with Poincaré invariance and full supersymmetry.

One may ask how do these orbifold projections know about the representations of the twisted group ? We will come back to this point later, after discussing several applications. The punchline is that the point group symmetry of the supersymmetric lattice is *not* a subgroup of the Euclidean Lorentz group, but in fact a discrete subgroup of the twisted rotation group $SO(d)'$. In the continuum, the orbifold lattice theory becomes the twisted version of the desired target field theory. And in flat space, the change of variables which takes the twisted form to the canonical form essentially undoes the twist.

The type of twist discussed in this section is sometimes referred as *maximal twist* as it involves the twisting of the full Lorentz symmetry group as opposed to twisting one of its subgroups. In this sense, the four dimensional $\mathcal{N} = 2$ theory can only admit a *half twisting* as its R-symmetry group is not as large as $SO(4)_E$ [11]. The other two theories, $\mathcal{N} = 1$ in $d = 4$ and $\mathcal{N} = 1$ in $d = 3$ shown in Table 1 do not admit a nontrivial twisting as there is no nontrivial homomorphism from their Euclidean rotation group to their R-symmetry group. Thus the methods described in this Report cannot be used in those cases.

After this general discussion we turn now to a pedagogical discussion of these problems and their solution in the context of simpler models – namely supersymmetric quantum mechanics and the Wess-Zumino model.

4. Supersymmetric quantum mechanics on the lattice

Supersymmetric quantum mechanics constitutes a good toy model for both understanding some of the problems encountered when trying to study supersymmetry on the lattice and some of the ways to circumvent these problems. Specifically even in this simple model we will see the issue of fine tuning arising in naive discretizations of the continuum theory and how this can be handled in low dimensions by performing perturbative lattice calculations to subtract off the dangerous radiative corrections. Furthermore, we will also see how we can make a change of variables which exposes a nilpotent supersymmetry and allows us to write down a lattice action which is explicitly invariant under this supersymmetry. This change of variables is just the twisting procedure we have already alluded to but restricted to the case of one (Euclidean) dimension. We also show that the exact supersymmetry ensures that these dangerous radiative corrections then cancel automatically and the resulting lattice theory does not suffer from fine tuning problems.

The continuum theory was first written down by Witten as a toy model for understanding supersymmetry breaking [80]. The model comprises a single commuting bosonic coordinate $\phi(t)$ and two anticommuting fermionic coordinates $\psi_1(t)$, $\psi_2(t)$. We will be working in the language of path integrals in Euclidean space which here means we treat the time coordinate t as

Euclidean. The continuum action reads

$$S = \int dt\, \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\psi_i\frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi) \tag{9}$$

where P' is an arbitrary polynomial in ϕ and P'' its derivative. The function $P(\phi)$ is often called the superpotential.

4.1. Algebra - two supersymmetries

This action is invariant under the two supersymmetries given below where ϵ_A, ϵ_B are infinitesimal Grassmann parameters.

$$\begin{aligned}
\delta_A\phi &= \psi_1\epsilon_A & \delta_B\phi &= \psi_2\epsilon_B \\
\delta_A\psi_1 &= \frac{d\phi}{dt}\epsilon_A & \delta_B\psi_1 &= -iP'\epsilon_B \\
\delta_A\psi_2 &= iP'\epsilon_A & \delta_B\psi_2 &= \frac{d\phi}{dt}\epsilon_B\ .
\end{aligned} \tag{10}$$

It is a simple exercise to verify these invariances. Simply carry out the variation of the fields and use the Grassmann property e.g. $\{\epsilon_A, \psi_i\} = 0$. In both cases the only slightly nontrivial terms encountered take, in the former case, the form

$$\delta_A S = \int dt\, i\epsilon\left(P'\frac{d\psi_2}{dt} + \frac{d\phi}{dt}P''\psi_2\right) \tag{11}$$

In this case a simple integration by parts sets the term inside the brackets equal to zero. From an operational point of view this is what ruins supersymmetry on the lattice – since this operation requires the use of the Leibniz rule which does not hold for lattice difference operators [81, 82]. Notice that $\delta_A^2 = \delta_B^2 = \frac{d}{dt}$ when acting on any field[7]. Since $H \equiv \frac{d}{dt}$ in Euclidean space this corresponds to the usual supersymmetry algebra reduced to the quantum mechanics case of one dimension.

4.2. Naive discretization

Let us now proceed to discretize this theory initially in a naive manner [83, 84]. Define the fields on lattice sites $x = na$, $n = 0 \ldots L-1$ and replace integrals by sums using periodic boundary conditions on all fields.

[7]We need to use the equations of motion to show this for $\overline{\psi}$

To eliminate fermion doubling problems it is sufficient in one dimension to replace the continuum derivative with a backward (or forward) difference operator.

$$\Delta^- f_x = f(x) - f(x-a) \tag{12}$$

Upon this replacement and carrying out supersymmetry variation, one finds a non-vanishing variation in the A case of the form (B is similar)

$$\delta_A S_L = \sum_x \; i\epsilon \left(P' \Delta^- \psi_2 + \Delta^- \phi P'' \psi_2 \right) \,. \tag{13}$$

Using lattice integration by parts we find

$$\delta_A S_L = i \sum_x \; \epsilon \psi_2 \left(-\Delta^+ P' + \Delta^- \phi P'' \right) \tag{14}$$

Since $\Delta^- \to \Delta^+ \to \frac{d}{dt}$ in the naive continuum limit it is clear that this term is O(a) and vanishes in the naive continuum limit. However it clearly does not vanishing at finite lattice spacing and thus the naive lattice action is *not* invariant under supersymmetry transformations.

As we emphasized earlier, the absence of an exact classical supersymmetry allows the quantum effective action to develop further SUSY violating interactions. This problem can be seen explicitly when we simulate this naive lattice theory. Fig. 3 shows a plot of the boson and fermion masses $m_B L$, $m_F L$ extracted from a simulation of the naively discretized action with $P(\phi) = m\phi + g\phi^3$ with $mL = 10.0$ and $gL^2 = 100.0$ shown as a function of the lattice spacing. Clearly they are not equal and indeed the problem worsens as $a \to 0$ consistent with the existence of relevant SUSY breaking counterterms. The plot also shows the expected result in the continuum limit $m_B L = m_F L = 16.87$ which can be computed straightforwardly using Hamiltonian methods.

The mismatch arises through radiative corrections and hence we are led to an analysis of loop corrections in the lattice theory in the next section.

4.3. Feynman diagrams and power counting

In practice the only diagrams we need to be concerned about are ones with a positive superficial degree of divergence. Only these will generate relevant SUSY violating interactions in the lattice effective action. Reisz's theorem [85] guarantees that all Feynman diagrams with a negative degree

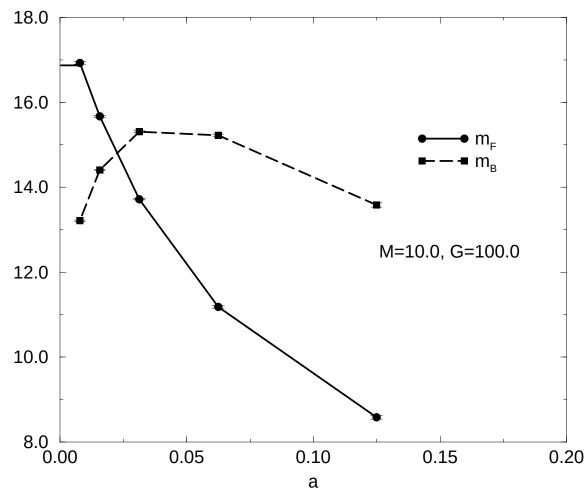

Figure 3: Boson and fermion masses vs lattice spacing for naive action

of divergence converge to their continuum counterparts as $a \to 0$ – and hence cannot contribute new supersymmetry breaking terms.

The good thing is that since this quantum mechanical model is super-renormalizable theory, there are only a finite number of such U.V sensitive diagrams and they occur at low orders in perturbation theory [83, 86, 87, 88, 89, 90]. Only these diagrams need to be examined carefully when we go to the lattice. We will see that it is possible that the contribution of such graphs in lattice perturbation theory *do not* converge to the continuum result as $a \to 0$ and hence can yield SUSY breaking effects. The reason lies with the would-be fermion doublers in the lattice description – it is possible for these high momentum states to contribute additional effects at the cut-off scale. This is similar to the classic calculation of Karsten and Smit showing how the chiral anomaly arises in lattice QCD [91].

Let us take as an example once again the potential $P' = m\phi + g\phi^3$. In this case it is a simple exercise to show that the only dangerous Feynman graph is the one-loop fermion contribution to the boson propagator (the diagram on the left in Fig. 4) [83].

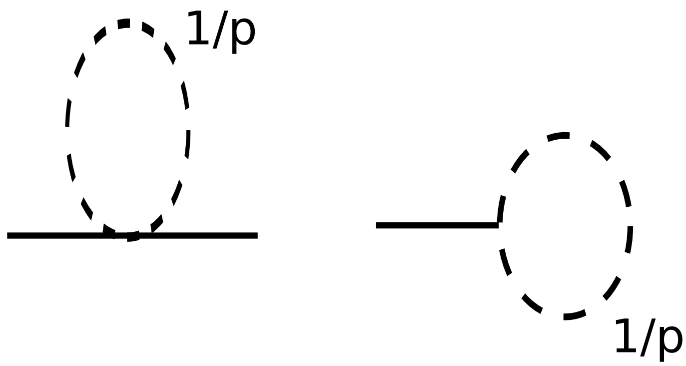

Figure 4: One loop Feynman graphs for SUSYQM

In the continuum this contributes:

$$\sigma_{\rm cont} = 6g \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{-ip+m}{p^2+m^2} \tag{15}$$

where we use π/a as the effective continuum momentum cut-off. The divergent piece of the integral is zero by the symmetry $p \to -p$ and we find

$$\Sigma_{\rm cont} = 6g\left(\frac{1}{\pi}\tan^{-1}\frac{\pi}{2ma}\right) \sim 6g\left(\frac{1}{2} + \mathcal{O}(ma)\right) \tag{16}$$

The same diagram on a lattice of size L (and using a backward difference operator) yields

$$\Sigma_{\rm latt} = \frac{6g}{L}\sum_{k=0}^{L-1} \frac{-2i\sin\left(\pi k L\right) e^{i\left(\frac{\pi k}{L}\right)} + ma}{\sin^2\left(\frac{\pi k}{L}\right) + m^2} \tag{17}$$

Notice that the phase factor breaks the $k \to -k$ symmetry. Indeed, the lattice yields *twice* the continuum result when the limit $a \to 0$ is taken *after* doing the sum. In order to understand this effect, use $\Delta^- = \Delta^S + \frac{1}{2}m_W$ where Δ^S is the symmetric difference operator having the same symmetry $k \to -k$ as the continuum, and m_W is the Wilson operator i.e difference between forward and backward difference operators or equivalently the discrete laplacian. We can understand the lack of convergence to the continuum result as resulting from the additional contribution of a doubler state with $k \sim \frac{\pi}{a}$ and mass determined by the Wilson term $m \sim \mathcal{O}(\frac{1}{a})$.

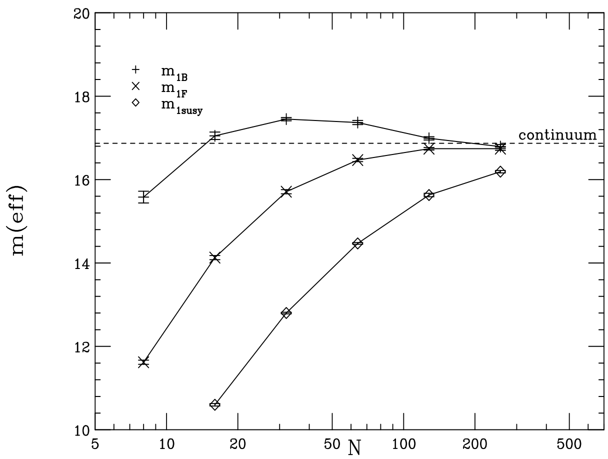

Figure 5: Counterterm corrected boson/fermion masses (From Giedt et al. [83])

This additional contribution shifts the mass squared of the boson by an additional $3g$ which breaks supersymmetry. To restore SUSY one needs only add a new boson mass counterterm to the lattice action

$$S_L \rightarrow S_L + \sum_x 3g\phi^2 \tag{18}$$

The resultant lattice theory does *not* manifest exact supersymmetry at finite lattice spacing but will nevertheless flow to the correct supersymmetric continuum theory without further fine tuning as $a \rightarrow 0$. This can be seen in Fig. 5 which shows the boson and fermion masses derived from a simulation of the naive action corrected by this one-loop counter term. The x-axis shows the number of lattice points $N = L_{\rm phys}/a$. The upper two curves correspond to the boson and fermion masses. The lower curve corresponds to the result from the exactly supersymmetric action which we discuss next. The dotted line is the expected continuum value (the parameters coincide with those used earlier for the naive lattice action)

Clearly the counterterm corrected lattice action generates boson and fermion masses which approach each other as the lattice spacing is reduced.

We now turn to a lattice discretization which preserves supersymmetry *exactly* at non-zero lattice spacing. Such a theory will necessarily contain the counterterm we have just discovered plus additional irrelevant terms which will keep the boson and fermion masses equal to all orders in the lattice spacing.

4.4. Exact lattice supersymmetry

In the last section we discussed how to diagnose the counterterms which must be added to the lattice action to avoid fine tuning. In general for $d = 1, 2, 3$ dimensions, the coefficients to these may be computed in perturbation theory. However, in the case of this supersymmetric quantum mechanics model we can do better – it is possible to find a linear combination of the supersymmetries which can be transferred intact to the lattice. To uncover this let us go back to eq. (13) which shows the variation of the lattice action under the A-type supersymmetry and recognize that this term may be rewritten in terms of a variation of another operator under the B-type supersymmetry

$$\delta_A S_L = -i\delta_B \sum_x P' \Delta^- \phi \tag{19}$$

Similarly it is not hard to show that

$$\delta_B S_L = i\delta_A \sum_x P' \Delta^- \phi \tag{20}$$

Thus the linear combination

$$(\delta_A + i\delta_B)\, S_L = -\,(\delta_A + i\delta_B)\, O \tag{21}$$

where $O = \sum_x P' \Delta^- \phi$. Notice, again, that O would vanish in a continuum theory as it would correspond to a total derivative term. Thus a lattice action invariant under supersymmetry $\delta S_{\rm exact} = 0$ can be constructed by adding O to the original naive action

$$S_L^{\rm exact} = \sum_x \frac{1}{2}(\Delta^- \phi)^2 + \frac{1}{2}P'^2 + P'\Delta^-\phi + \overline{\psi}(\Delta^- + P'')\psi \tag{22}$$

where we have defined new fermion fields

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \\ \overline{\psi} &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \end{aligned} \tag{23}$$

and the exact supersymmetry $\delta = \frac{1}{\sqrt{2}}(\delta_A + i\delta_B)$. The latter symmetry acts on the fields as follows

$$\begin{aligned} \delta x &= \psi\epsilon \\ \delta\psi &= 0 \\ \delta\overline{\psi} &= (\Delta^{-}\phi + P')\epsilon \end{aligned} \tag{24}$$

Notice that this derived supersymmetry is no longer the "square root" of a translation but instead is *nilpotent* (using the equations of motion). This fact is at the heart of how we are able to build an exact supersymmetry – the algebra of the corresponding supercharge is simply $\{Q, Q\} = 0$ and does not involve the energy or momentum. Equivalently, the invariance $\delta S_L = 0$ does not require use of the Leibniz rule.

Notice also that *two* supersymmetries were required to find such a nilpotent supercharge – the continuum theory has *extended* supersymmetry. This will be seen to be a general property of lattice models with exact supersymmetry. It is not hard to show that the other supersymmetry $\delta' = \frac{1}{\sqrt{2}}(\delta_1 - i\delta_2)$ is still broken on the lattice

$$\delta' S_L = 2\delta' O \tag{25}$$

Finally, if we examine the form of the lattice action in eq. (22) we see the bosonic piece can be rewritten

$$S_B = \sum_x \left(\Delta^{-}\phi + P'(\phi)\right)^2 \tag{26}$$

The cross term that appears after the square is expanded is the correction needed to ensure exact supersymmetry. It vanishes in the continuum as a total derivative. While it is identically zero in the continuum it constitutes a new relevant operator by lattice power counting. It generates an additional bosonic 1-loop Feynman graph which cancels the corresponding fermion loop (due to the derivative interaction) restoring supersymmetry.

4.5. Nicolai map

The partition function[8] governing the quantum theory takes the form

$$Z = \int D\phi D\psi D\overline{\psi} e^{-S} = \int D\phi \det\left(\Delta^- + P''\right) e^{-S_B} \tag{27}$$

Notice, however, a curious fact; if we imagine changing variables from ϕ to new variables $\mathcal{N} = \Delta^-\phi + P'(\phi)$ we will encounter a Jacobian which is just det $\left(\frac{\delta\mathcal{N}}{\delta\phi}\right)$. This Jacobian *cancels* the fermionic action and yields a very simple expression for the partition function [92]

$$Z = \int D\mathcal{N}\ e^{-\sum_x \mathcal{N}^2} \tag{28}$$

corresponding to a set of simple uncoupled bosonic oscillators. This change of variables is called a *Nicolai map* and the existence of a *local* Nicolai map is at the heart of why these models may be discretized in a SUSY preserving manner [92, 93].

It has immediate consequences; the detailed form of the superpotential $P(\phi)$ has disappeared in this final form of Z – hence the latter cannot depend on any coupling constants in the model – it is a *topological invariant* [22]. We may use this fact to derive an *exact* value for the expectation value of the bosonic action which holds for all interaction couplings. Replacing S_L by μS_L it is clear that Z does not change. The statement $\frac{\partial \ln Z}{\partial\mu} = 0$ then implies that $< S_L >= 0$ which in turn implies that the bosonic action S_B is given by the expectation value of the fermionic action. Since the latter is quadratic in the fermion fields a simple scaling argument shows the expectation value of the latter simply counts the number of degrees of freedom. We thus find that

$$< S_B >= \frac{1}{2} N_{\rm d.o.f} \tag{29}$$

a result which usually only applies to a free theory but here is valid as a consequence of supersymmetry at all values of the coupling. As an example of this we quote the measured value $< S_B >= 1.99985(20)$ obtained from a

[8]In the path integral formulation, we impose supersymmetry preserving periodic boundary conditions for all fields. In operator formalism, this corresponds to the partition function $Z = \mathrm{tr}[e^{-\beta H}(-1)^F]$ which, for supersymmetric theories, is just the Witten index of the theory.

Monte Carlo calculation using a lattice with $L = 4$ points and corresponding to the superpotential parameters $mL_{\rm phys} = 2.5$ and $gL^2_{\rm phys} = 6.25$. This is to be compared with the exact result expected on the basis of exact supersymmetry $< S_B >= 2$.

4.6. Ward identities

The classical invariance of the action is manifested in the quantum theory by exact relationships between different correlation functions. Consider the expectation value of some operator $\mathcal{O}$. If we make a change of variables $\phi \to \phi' = \phi + \delta\phi$ we find

$$< \mathcal{O} >= \frac{1}{Z} \int D\phi' O(\phi') e^{-S(\phi')} \tag{30}$$

If $\delta\phi$ corresponds to some symmetry $S(\phi') = S(\phi)$ and, assuming the measure is also invariant under this shift, we deduce a corresponding Ward identity

$$< \delta\mathcal{O} >= 0 \tag{31}$$

Notice, that this result can be reinterpreted in the language of canonical quantization as the usual statement that the supercharge annihilates the vacuum in the absence of spontaneous symmetry breaking.

This can be exemplified in the case of supersymmetry by choosing the operator $\mathcal{O} = \overline{\psi}_x \phi_y$ yielding the Ward identity

$$< \overline{\psi}_x \psi_y > + < (\Delta^- \phi + P')_x \phi_y >= 0 \tag{32}$$

relating the fermion 2pt function to a bosonic correlator. A numerical calculation of this Ward identity is shown in Fig. 6 for the lattice parameters $L = 16$, $mL = 10$ and $gL^2 = 100$ corresponding to a strongly interacting theory with dimensionless coupling $g/m^2 = 1$.

This result already ensures that the masses of the lowest lying fermionic and bosonic excitations must be equal – one of the most obvious predictions of a supersymmetric theory. Further evidence to this effect is shown in Fig. 7 which plots the lowest lying bosonic and fermionic masses versus lattice spacing. Evidently the masses are degenerate within rather small statistical errors. Additionally note that they appear to extrapolate rather nicely to the exact continuum answer obtained independently by Hamiltonian methods[9].

[9]Additional work on lattice implementations of this model including some high statistics, fine lattice spacing results can be found in the recent work [83, 94].

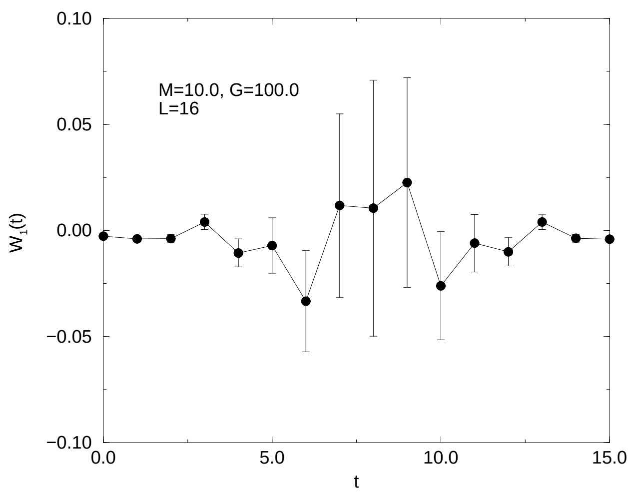

Figure 6: Ward identity for supersymmetric quantum mechanics

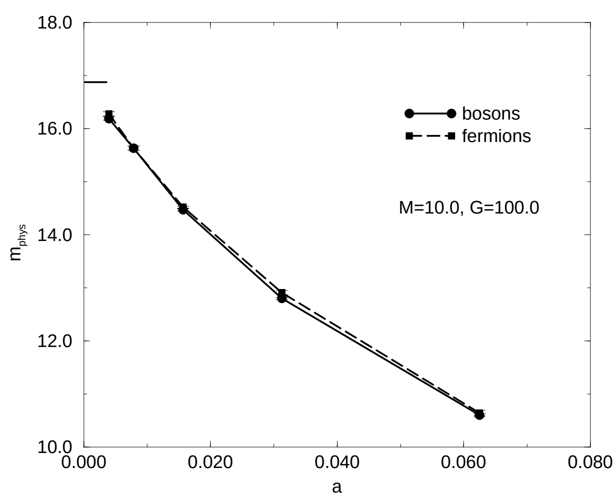

Figure 7: Boson and fermion masses vs lattice spacing for supersymmetric action

Contrast this with the result obtained for the naive action shown in Fig. 3 for the same lattice parameters (again the continuum result is shown as a tick mark on the y axis). Clearly the boson and fermion masses are very different for small lattice spacing reflecting the necessity of introducing the appropriate counter term to ensure supersymmetry as $a \to 0$.

4.7. Topological field theory form - twisting

The nilpotent character of the exact supersymmetry can be rendered true off-shell by adding an auxiliary field B [22]. The off-shell algebra reads

$$\begin{aligned} Q\phi &= \psi \\ Q\psi &= 0 \\ Q\overline{\psi} &= B \\ QB &= 0 \end{aligned} \tag{33}$$

where we absorbed the anticommuting parameter ϵ into the variation δ and introduced a corresponding fermionic variation Q which closely corresponds to the exact supercharge of the canonical formalism. It is trivial to verify that $Q^2 = 0$ now on all fields without use of the equations of motion. Using this field B the bosonic action can now be rewritten as

$$S_B = \sum_x -B(D^-\phi + P') - \frac{1}{2}B^2 \tag{34}$$

The original action is recovered after integration over B. It is easy to verify that the supersymmetry variation of the new action is still zero.

Furthermore, it is now trivial to show that entire action is nothing but the Q variation of a particular function – it is said to be Q-exact.

$$S_L = Q \sum_x \overline{\psi}(-D^-\phi - P' - \frac{1}{2}B) \tag{35}$$

In obtaining the action, one must treat the fermionic variation Q as a Grassmann and anticommute it through the other Grassmann fields.

In this form the invariance of the lattice action is manifest – it simply relies on the nilpotent property of Q and the fact that the action is the Q-variation of something. This Q-exact structure should remind one of BRST gauge fixing and it turns out that this is not a coincidence – theories with local Nicolai maps and nilpotent supercharges such as the quantum mechanics model discussed here can be obtained by a process of gauge fixing. However, unlike the usual gauge fixing procedure the gauge fixing employed in this context arises during quantization of a bosonic model with a classical topological shift symmetry.

Consider a theory comprising a bosonic field ϕ living on a lattice in Euclidean time. Take as classical action the trivial function $S(\phi) = 0$. To construct the quantum theory we must integrate over the field ϕ. But the classical theory is invariant under the topological symmetry

$$x \rightarrow x + \epsilon \tag{36}$$

where ϵ is an *arbitrary* smooth function. Thus, to quantize this model we must fix a gauge. Let us employ the gauge condition $\mathcal{N}(\phi) = 0$. The correct quantum partition function is then given by

$$Z = \int D\phi \det(\frac{\partial \mathcal{N}}{\partial \phi}) e^{-\frac{1}{2\alpha}\mathcal{N}^2(\phi)} \tag{37}$$

where we have included the usual Fadeev-Popov determinant and inserted an arbitrary gauge fixing parameter α. If we represent the determinant in terms of ghost fields ψ and $\overline{\psi}$ and choose the specific gauge fixing function $\mathcal{N} = D^-\phi + P'$ (with Feynman gauge parameter $\alpha = 1$) we recover our original quantum mechanics model! In this case the nilpotent supersymmetry we uncovered is nothing more than the usual BRST symmetry arising from quantizing the underlying topological symmetry and the Nicolai map is just the gauge fixing function! Thus, we see in our simple quantum mechanics example, that the nilpotent twisted supersymmetry we have constructed by taking linear combinations of the original supercharges has an alternative interpretation as a BRST charge arising in quantizing an underlying bosonic theory.

In BRST gauge fixing we would then go on to impose the physical state condition that

$$Q|\text{physical state} >= 0 \tag{38}$$

To construct a true topological quantum field theory this is what is done.

However, in the context of the lattice SUSY constructions described in this Report this is *not* what is done. Such a restriction would be equivalent to a projection to the vacuum states of the target supersymmetric theory – which is much too restrictive. Hence we will *not* impose this condition here and instead merely use the topologically twisted reformulations of these supersymmetric theories as simply more convenient starting points for constructing lattice actions which retain a degree of supersymmetry.

As we have seen they simply correspond to an exotic change of variables in the original theory – one that exposes a nilpotent supersymmetry explicitly.

4.8. Semiclassical exactness

The topological structure that we have exposed in this quantum mechanics model, which is at the heart of our ability to discretize it, has many important consequences. Consider a set of Q-invariant operators such as $\mathcal{O}_1(x_1), \ldots, \mathcal{O}_n(x_n)$. Most importantly, the expectation values and connected correlators of such operators in a theory with a Q-exact action $S = Q\Lambda$ may be computed *exactly* in the semi-classical limit. It is easy to see this – replacing S by μS we can write down an expression for the expectation value

$$\langle \mathcal{O} \rangle_\mu = \frac{1}{Z} \int \mathcal{O}\; e^{-\mu S} \tag{39}$$

Differentiating this expression with respect to μ leads to

$$\frac{\partial}{\partial \mu} \langle \mathcal{O} \rangle_\mu = -\langle \mathcal{O} Q\Lambda \rangle_\mu + \langle \mathcal{O} \rangle_\mu \langle Q\Lambda \rangle_\mu = 0 \tag{40}$$

where the equality just follows from recognizing that $\mathcal{O}Q\Lambda = Q(\mathcal{O}\Lambda)$ which can be recognized as a supersymmetric Ward identity. Thus expectation values of Q-invariant observables are independent of μ *in the absence of spontaneous breaking of the Q-symmetry* and hence can be computed exactly in the semiclassical limit $\mu \to \infty$. In this limit we need only do 1-loop calculations around the classical vacua[10]. Generalizations of this argument allow one to show that Q-invariant observables in the continuum twisted theories are independent of any background metric and, in the case where they are not Q-exact, possess expectation values that correspond to topological invariants of the background space.

4.9. Supersymmetry breaking

In this section we discuss, in the context of our lattice quantum mechanics model, a mechanism by which Q-supersymmetry may be spontaneously

[10]This is the basis of the argument used by Matsuura to show that the vacuum energy of certain supersymmetric orbifold theories remains zero to all orders in the coupling [95]

broken. This offers a prototype for the kind of dynamical susy breaking mechanism we would ultimately like to examine in lattice simulations of more realistic models.

The partition function for the system with periodic boundary conditions on all fields is a topological invariant called the Witten index. We may evaluate it easily from the Nicolai map formulation. We may deform the map

$$\mathcal{N}(\phi) = \Delta^{-}\phi + P'(\phi) \tag{41}$$

in such a way as keep only the highest power of ϕ in the potential $\mathcal{N}(\phi) \to \phi^n$. Consider the expression for Z in this limit (here S contains both the bosonic action S_B and the effective interaction coming from the fermions $\ln\det(\frac{\partial\mathcal{N}}{\partial\phi})$)

$$Z = \int_{-\infty}^{\infty} D\phi e^{-S} = \int_{-\infty}^{0} D\phi e^{-S} + \int_{0}^{\infty} D\phi e^{-S} \tag{42}$$

In the case where n is odd the map $\phi \to \mathcal{N} = \phi^n$ is single valued and the previous expression is just equivalent to

$$Z = \int_{-\infty}^{\infty} D\mathcal{N} e^{-\mathcal{N}^2} \tag{43}$$

But for n even the map is not single valued and the limits on the first integral change leading to the result

$$Z = \int_{\infty}^{0} D\mathcal{N} e^{-\mathcal{N}^2} + \int_{0}^{\infty} D\mathcal{N} e^{-\mathcal{N}^2} = 0 \tag{44}$$

Thus the Witten index is zero for superpotentials where the highest power of ϕ in $P'(\phi)$ is even. A non-zero Witten index implies that supersymmetry is unbroken. A vanishing Witten index allows for supersymmetry breaking. To see this recall that the Witten index is simply the difference between the number of fermionic and bosonic vacua. If this is non-zero supersymmetry *cannot* break since any vacuum state that is lifted to positive energy necessarily occurs with a superpartner state of opposite statistics which is not possible if W is non-zero. However, if $W = 0$ supersymmetry breaking can (and often does) occur.

However, powerful non-renormalization theorems guarantee that supersymmetry cannot break in any finite order of perturbation theory [96]. If it is to occur it must proceed through a non-perturbative mechanism - for example instantons. In the quantum mechanics model these correspond to non-trivial field configurations satisfying

$$\frac{d\phi}{dt} + P'(\phi) = 0 \tag{45}$$

Such non-trivial field configurations can occur when the asymptotic values of ϕ tend to two different classical vacua as $t \to \pm\infty$. The instantons are then just kink solutions.

When $P'(\phi) = 0$ has only one solution they cannot occur but it is easy to construct examples where there are two solutions eg. $P'(\phi) = (\phi^2 - a^2)$ which clearly corresponds to a theory with vanishing Witten index $W = 0$. The instanton solution is then

$$\phi = a \tanh \frac{1}{2}(t - c) \tag{46}$$

where c is an arbitrary constant corresponding to the center of the instanton. For the bosonic action based on $\mathcal{N}^2$ the action of this configuration is zero. Furthermore, associated with the center coordinate c is an exact bosonic zero mode of the form

$$\phi_0 = a \mathrm{sech}^2 \frac{1}{2}(t - c) \tag{47}$$

which is exponentially localized on the instanton. Supersymmetry then dictates that there is a corresponding fermionic zero mode. One can understand the vanishing of W as a consequence of the Grassmann integration over this fermionic zero mode. Indeed, to get nonzero expectation values we need observables that absorb this zero mode. They are of the form

$$\mathcal{O} = f'(\phi)\psi \tag{48}$$

But notice that this observable is itself the Q-variation of something. Hence, if instantons condense in the vacuum of this theory we will find $< Q\mathcal{O} > \neq 0$ signaling supersymmetry breaking.

It is instructive to see how one might see this breaking within a Monte Carlo simulation. On the lattice an isolated instanton cannot be realized because of the periodic boundary conditions on the bosonic field. The lowest energy configuration then consists of an instanton anti-instanton pair. If the pair are widely separated this is an approximate solution of the classical equations of motion with a classical action $S_{\mathrm{I}\bar{\mathrm{I}}} = \frac{4}{3}a^3$ Associated with this configuration there will be a low lying, localized *and hence normalizable* fermion mode which is a superpartner to the approximate bosonic zero mode corresponding to motion of the instanton center. In the thermodynamic limit this mode can induce supersymmetry breaking. The cleanest way to see this is to consider the system at non-zero temperature by employing antiperiodic boundary conditions for the fermions. These boundary conditions break supersymmetry explicitly and the question of whether supersymmetry breaks spontaneously is then reduced to a computation of the expectation value of the energy (written in Q-exact form) in the thermodynamic limit as the temperature is sent to zero. Numerical simulations of this theory indicate that the energy, extrapolated to first to zero lattice spacing, and subsequently to zero temperature, is indeed non-zero [97, 98].

4.10. Generalizations

4.10.1. Supersymmetric Yang Mills Quantum Mechanics

These models arise as dimensional reductions of $\mathcal{N} = 1$ SYM theory in $d = 4, 6, 10$ dimensions down to one (Euclidean) dimension. The $d = 10$ theory with $\mathcal{Q} = 16$ supercharges and N colors is especially interesting as it is conjectured to be dual to type IIa string theory containing N D0-branes. The type IIA string theory reduces to a supergravity theory for low energies compared to the string scale $(\alpha')^{-1/2}$. In this limit the thermal theory contains black holes with N units of D0-charge. Their energy, E, is a function of their Hawking temperature, T. Defining $\lambda = N g_s \alpha'^{-3/2}$ where g_s is the string coupling, we may write a dimensionless energy and temperature $\epsilon = E\lambda^{-1/3}$ and $t = T\lambda^{-1/3}$. One finds provided we take N large and $t \ll 1$ the black hole is weakly curved on string scales and the quantum string corrections are suppressed. The energy of this black hole can be precisely computed by standard methods [7] giving,

$$\epsilon = c\, N^2 t^{14/5} \qquad c = \left(\frac{2^{21}3^{12}5^2}{7^{19}}\pi^{14}\right)^{1/5} \simeq 7.41. \tag{49}$$

Duality posits that the thermodynamics of this black hole should be reproduced by the dual Yang-Mills quantum mechanics at the same temperature with $g_s\alpha'^{-3/2} = g^2_{YM}$ so that λ is to be identified with the 't Hooft coupling.

A one dimensional lattice action with $\mathcal{Q} = 8$ exact supersymmetries has been constructed for the target $\mathcal{Q} = 16$ matrix quantum mechanics [34]. However, as our previous analysis shows, the renormalization of supersymmetric quantum mechanics is sufficiently simple that a naive latticization may also be employed. The only diagram exhibiting a superficial degree of divergence $D \geq 0$ corresponds to the one loop fermion contribution to the tadpole diagram for the field ψ which has $d = 0$. However an easy calculation shows that the log divergent part of this amplitude contains a factor

$$f_{abc}\delta_{ab} \tag{50}$$

which vanishes on account of the antisymmetry of the structure constants f_{abc}. Thus *any* naive discretization of this theory (in which there are no doubles) will flow automatically to the supersymmetric theory in the continuum limit as was explicitly shown in [99].

A suitable lattice action takes the form $S = S_B - \ln \mathrm{Pf}(\mathcal{O})$ where

$$S_B = \frac{NL^3}{\lambda R^3} \sum_{a=0}^{L-1} \mathrm{Tr}\left[\frac{1}{2}(D_- X_i) - [X_{i,a}, X_{j,a}]^2\right] \tag{51}$$

and the fermion operator $\mathcal{O}$ is defined by

$$\mathcal{O}_{ab} = \begin{pmatrix} 0 & (D_-)_{ab} \\ -(D_-)_{ba} & 0 \end{pmatrix} - \gamma^i \left[X_{i,a}, \cdot\right] \mathrm{Id}_{ab} \tag{52}$$

and we have rescaled the continuum fields $X_{i,a}$ and $\Psi_{i,\alpha}$ by powers of the lattice spacing R/L to render them dimensionless (L is the number of lattice points). The covariant derivatives are given by $(D_- W_i)_a = W_{i,a} - U^\dagger_{a-1} W_{i,a-1} U_{a-1}$ and we have introduced a Wilson gauge link field U_a. Notice that the fermionic operator is free of doublers and is manifestly antisymmetric in this basis.

Studies of the 16 supercharge model using this action have have been initiated in [45, 44]. Work using a gauge fixed approach momentum space approach have been reported in [46, 49, 50, 47, 48, 51, 52]. The results from the two methods are in agreement and lend strong support to the conjectured duality of strong coupling Yang-Mills quantum mechanics and type II string theory in AdS space. A comparison between the Yang-Mills system and its gravity dual is shown in Fig. 8

The plots shows the mean Yang-Mills energy (in units of $\lambda^{\frac{1}{3}}$) versus dimensionless temperature $t = T/\lambda^{\frac{1}{3}}$ for a number of colors N in the range $N = 3 - 12$ obtained from a Monte Carlo simulation of the model [44]. The solid line shows the corresponding result obtained for the charge N black hole solution of type IIa supergravity as given in earlier in eq. (49). The dashed line gives the high temperature result. It should be noted that the agreement seen in the figure was obtained without any fitting of the data.

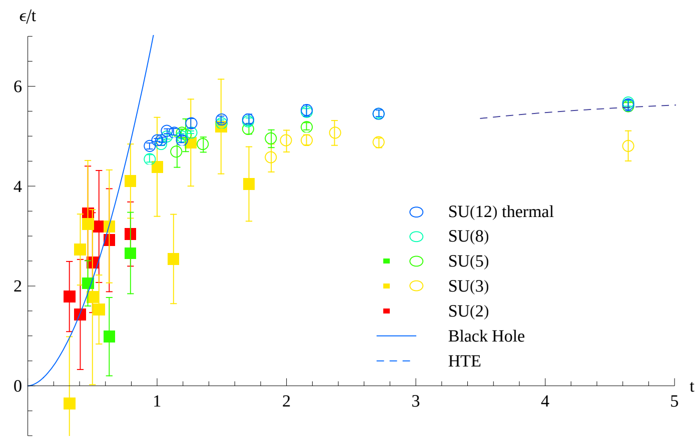

Figure 8: Mean energy vs temperature for SYMQM (points). Solid line corresponds to the prediction from the black hole solution in supergravity

4.10.2. Sigma models

The simple quantum mechanics model may be generalized by replacing the field ϕ by a set of fields ϕ^i to be considered as coordinates in some target space with metric g_{ij}. The fermions and auxiliary field B also pick up target space indices and the nilpotent symmetry is naturally generalized to [100]

$$\begin{aligned} Q\phi^i &= \psi^i \\ Q\psi^i &= 0 \\ Q\overline{\psi}_i &= B_i\overline{\psi}_j\Gamma^j_{ik}\psi_k \\ QB_i &= B_j\Gamma^j_{ik}\psi_k - \frac{1}{2}\overline{\psi}_j R^j_{ilk}\psi_l\psi_k \end{aligned} \tag{53}$$

The quantities Γ^i_{jk} and R^j_{jkl} are the usual Riemann connection and curvature. It is straightforward to show that $Q^2 = 0$ on all fields. As before the supersymmetric lattice action takes the form

$$S = \beta Q \sum_x \overline{\psi}_i \left(\mathcal{N}^i - \frac{1}{2}g^{ij}B_j\right) \tag{54}$$

Carrying out the variation and integrating over the field B yields

$$S = \beta \sum_x \left(\frac{1}{2}\mathcal{N}^i\mathcal{N}_i - \overline{\psi}_i\nabla_k N^i\psi_k + \frac{1}{4}R_{jklm}\overline{\psi}^j\overline{\psi}^l\psi^m\psi^k\right) \tag{55}$$

which is invariant under the on-shell SUSY

$$\begin{aligned}
Q\phi^i &= \psi^i \\
Q\psi^i &= 0 \\
Q\overline{\psi}_i &= \mathcal{N}_i - \overline{\psi}_j \Gamma^j_{ik} \psi_k
\end{aligned} \tag{56}$$

The curved space lattice Nicolai map is simply

$$\mathcal{N}^i = \Delta^+ \phi^i \tag{57}$$

which allows us to rewrite the fermion kinetic term in the form

$$\overline{\psi}_i \left(\Delta^+ \psi^i + \Gamma^i_{kj} \Delta^+ \phi^k \psi^j \right) \tag{58}$$

Notice that while the lattice action is now manifestly supersymmetric (and free of doubles) it is no longer general coordinate invariant in the target space – the term $\Delta^+ \phi^k$ is *not* a target space vector unlike its continuum cousin involving derivatives. However, one might hope that this symmetry along with the other supersymmetry is restored in the continuum limit (large β) where the fields vary slowly over the lattice. There is some evidence for this in the analogous two dimensional sigma models [101]. It would be very interesting to investigate this in more detail in this simpler one dimensional case.

4.11. Summary

There are several conclusions we can draw from the discussion of these quantum mechanical models. The first is that low dimension supersymmetric theories can sometimes be handled using discretizations that break all the continuum supersymmetries – the super-renormalizable nature of the theories ensures that only a finite number of dangerous Feynman diagrams exist and they occur in low orders of perturbation theory. These diagrams can be computed using lattice perturbation theory and counter-terms can be used to subtract off the susy breaking effects.

This has been verified in great detail in the work of the Jena group who have conducted high precision simulations of the non-gauge quantum mechanics theory with very convincing results [94]. But we have seen that things are even easier with exact supersymmetry – linear combinations of the original supersymmetries can be found which are nilpotent and a lattice action constructed that is exactly invariant under one of these new supersymmetries. This is the simplest example of the more general twisting procedure discussed in the introduction to this Report. In one dimension there is no notion of Lorentz symmetry and the procedure reduces to the simple observation that from the two real supersymmetries obeying $Q_1^2 = Q_2^2 = H$ one can construct two complex conjugate symmetries $Q = Q_1 + iQ_2, \overline{Q} = Q_1 - iQ_2$, which are nilpotent. One of these can then be chosen to survive the transition to the lattice.

We now turn to the simplest quantum field theory where a similar construction is possible and where Lorentz symmetry plays a less trivial role – the two dimensional Wess-Zumino model.

5. Two dimensional theories without gauge symmetry

5.1. Wess-Zumino Model

Having discussed the case of quantum mechanics the next task is to see how to generalize these ideas to field theory. The simplest place to start is in two dimensions and perhaps the simplest example of a lattice theory which exhibits an exact supersymmetry is gotten by lifting the (non-gauge) quantum mechanics to two dimensions. This will lead to a Wess-Zumino model [23, 102, 25, 103, 104, 105, 106]. (An alternative is to fine-tune a finite number of counter-terms as in the 2+1 dimensional Wess-Zumino model [107].) We saw that for quantum mechanics a minimum of two supersymmetries was necessary to build an exact lattice supersymmetry so we are led to consider the $\mathcal{N} = 2$ Wess-Zumino model with continuum action

$$\begin{aligned} S_{\rm WZ} &= \int d^2x\, \partial_\mu\phi\partial_\mu\overline{\phi} + W'(\phi)W'(\overline{\phi}) + \overline{\psi}\gamma_\mu\partial_\mu\psi + \\ &+ \overline{\psi}\left(\frac{1}{2}\left(1+\gamma_5\right)W''(\phi) + \frac{1}{2}\left(1-\gamma_5\right)W''(\overline{\phi})\right)\psi \end{aligned} \tag{59}$$

which contains a complex scalar field ϕ with analytic superpotential $W(\phi)$, coupled to a Dirac fermion ψ. It will turn out to be easier to rewrite the fermion action. Choosing a chiral basis

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{60}$$

allows us to rewrite the fermion operator as

$$M_{\mathrm{F}} = \begin{pmatrix} iW''(\overline{\phi}) & \partial_{\overline{z}} \\ \partial_z & -iW''(\phi) \end{pmatrix} \tag{61}$$

To build a lattice action which preserves supersymmetry we will require a local Nicolai map as for quantum mechanics [13]. This means that the fermion operator should be realizable as a Jacobian representing the change of variables from $\phi \to \mathcal{N}(\phi)$. Analogy with one dimension together with the above form of the fermion operator suggests the form

$$\mathcal{N}(\phi) = \partial_{\overline{z}}\phi + iW'(\overline{\phi}) \tag{62}$$

Indeed in the continuum the bosonic action derived from $\mathcal{N}\overline{\mathcal{N}}$ differs from the continuum one given by eq. (59) by a cross term $\int dz d\overline{z} \partial_z \overline{\phi} W'(\overline{\phi})$+h.c, similar to that encountered in supersymmetric quantum mechanics. Again in the continuum this term is a total derivative and can be discarded. Discretizing this Nicolai map then leads to a lattice action S_L which is invariant under the following supersymmetry [23, 13, 25]

$$S_L = \sum_x \mathcal{N}\overline{\mathcal{N}} + \overline{\omega}\left(D^s_{\overline{z}}\lambda + iW''_L(\overline{\phi})\omega\right) + \overline{\lambda}\left(D^s_z\omega - iW''_L(\overline{\phi})\lambda\right) \tag{63}$$

where

$$\begin{aligned} Q\phi &= \lambda \\ Q\overline{\phi} &= \omega \\ Q\lambda &= 0 \\ Q\omega &= 0 \\ Q\overline{\omega} &= \overline{\mathcal{N}} \\ Q\overline{\lambda} &= \mathcal{N} \end{aligned} \tag{64}$$

where $\psi = \begin{pmatrix} \omega \\ \lambda \end{pmatrix}$, $\overline{\psi} = \begin{pmatrix} \overline{\omega} \\ \overline{\lambda} \end{pmatrix}$. In this expression we have replaced the continuum derivative by a *symmetric* difference operator. To remove the would be doublers from this expression one can add a Wilson mass term to the superpotential

$$W'_L(\phi) = W'(\phi) + \frac{1}{2} D^+_z D^-_{\overline{z}}\phi \tag{65}$$

L	$< S_B >$	$\frac{1}{2}N_{dof}$
4	31.93(6)	32
8	127.97(7)	128
16	512.0(3)	512
32	2046(3)	2048

Table 2: Mean bosonic action $< S_B >$ versus exact SUSY value, equal to half the number of degrees of freedom.

Again it is easy to see that $Q^2 = 0$ using the equations of motion. By introducing fields $B, \overline{B}$ we can again write the action in Q-exact form and render the symmetry Q nilpotent off-shell [22].

$$S_L = Q \sum_x \left[\overline{\omega} \left(\mathcal{N} + \frac{1}{2} B \right) + \overline{\lambda} \left(\overline{\mathcal{N}} + \frac{1}{2} \overline{B} \right) \right] \tag{66}$$

where the action of the supersymmetry Q contains the new elements

$$\begin{aligned} Q\overline{\omega} &= \overline{B} \\ Q\overline{\lambda} &= B \\ QB &= 0 \\ Q\overline{B} &= 0 \end{aligned} \tag{67}$$

The existence of an exact supersymmetry results in a set of exact lattice Ward identities constraining the form of the quantum theory. Perhaps the simplest of these identities is given by $< S_L >=< Q\Lambda >= 0$ leading to a prediction for the mean bosonic action for *any* superpotential. This is illustrated in table 2 which shows the expectation value of the bosonic action as a function of the number of lattice sites.

Again, the restoration of full supersymmetry occurs without fine tuning as a consequence of exact supersymmetry [23, 25, 106, 94, 108]. Figure 9 shows the bosonic and fermionic contributions to a particular Q-supersymmetric Ward identity resulting from the Q-variation of the operator $O = \phi_x \overline{\lambda}_y$ in the theory with $W(\phi) = m\phi + g\phi^2$. The parameters correspond to $mL = 10.0$ and $gL = 3.0$ on a lattice with $L = 8 \times 8$ sites. Clearly the two curves add to zero as would be expected for a theory which realizes exact supersymmetry.

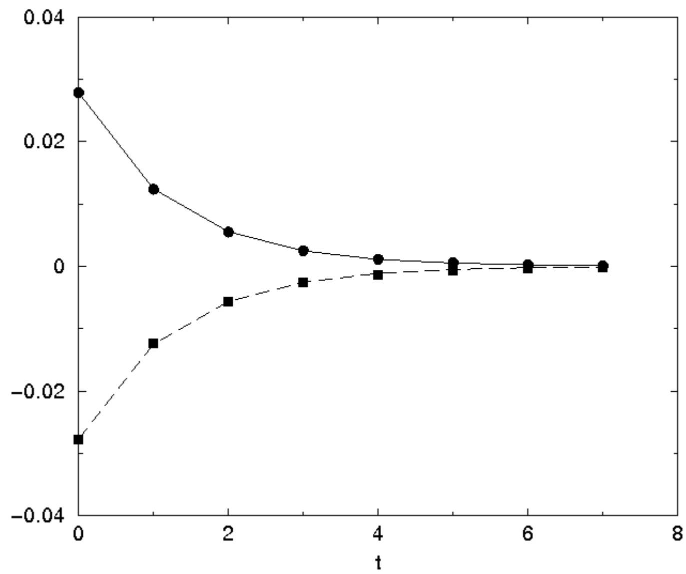

Figure 9: Real parts of bosonic $< \phi_x \mathcal{N}_y >$ and fermionic contributions $< \lambda_x \overline{\lambda}_y >$ to Q-ward identity for Wess-Zumino model with superpotential parameters $mL = 10.0$ and $gL = 3.0$

5.2. Sigma models

The one dimensional sigma model may be lifted to two dimensions if sufficient constraints are placed on the target space. In practice this means that the manifold should be Kähler. In the case of a curved target space we must add the Wilson term in the form of a *twisted* mass term rather than a superpotential. We refer to [101] for details. This breaks the Q symmetry softly although numerical results still favor the restoration of full supersymmetry without fine tuning in the continuum limit [101].

5.3. Twisting in two dimensions

The process of exposing a linear combination of supercharges that is nilpotent can be understood in a systematic way. This will furnish an explicit example of the twisting process discussed in the introduction.

Consider a two dimensional theory with 4 (real) supercharges ($\mathcal{N}=2$ supersymmetry). Such a theory contains 2 degenerate Majorana spinors which transform into each other under an internal $SO(2)_I$ symmetry. The spinors also transform under an independent $SO(2)_E$ (Euclidean) Lorentz symmetry. Following on from our general discussion of twisting in the introduction it is possible to decompose the fields of the theory under the *diagonal subgroup* corresponding to making equal rotations in the base and internal spaces

$$SO(2)' = \mathrm{Diag}\,(SO(2)_I \times SO(2)_L) \tag{68}$$

In practice this means that the supercharges q^i_α (i corresponds to internal space, α to rotations) are to be treated as a 2×2 matrix [19, 18]. It is then natural to expand the supercharge matrix on products of two dimensional Dirac gamma matrices (Pauli matrices) ($\mu = 1, 2$)

$$q = QI + Q_\mu\gamma_\mu + Q_{12}\gamma_1\gamma_2 \tag{69}$$

In this process we see that all the supercharges are decomposed in terms of geometrical quantities – scalars, vectors and antisymmetric tensors. Furthermore the original supersymmetry algebra becomes

$$\begin{aligned}
\{Q, Q\} &= \{Q_{12}, Q_{12}\} = \{Q, Q_{12}\} = \{Q_\mu, Q_\nu\} = 0 \\
\{Q, Q_\mu\} &= p_\mu \\
\{Q_{12}, Q_\mu\} &= \epsilon_{\mu\nu}p_\nu
\end{aligned} \tag{70}$$

showing that indeed the scalar component is nilpotent as required. Actually we see that the momentum is Q-exact which makes it plausible that the entire energy-momentum tensor is nilpotent. Thus it should be no surprise that supersymmetric theories reformulated in this *twisted* basis often have Q-exact actions as we have already seen [21][11].

5.4. Dirac-Kähler fermions in two dimensions

If the supercharges are tensorial we would expect the same to be true for the fermions themselves. Taking the standard free fermion action for a theory with 2 degenerate Majorana species and replacing the fermions ψ^i_α by matrices we find that it can be trivially rewritten as [19, 109]

$$S_{\rm F} = {\rm Tr}\left[\Psi^\dagger\gamma_\mu\partial_\mu\Psi\right] \tag{71}$$

Expanding the matrices into (real) components $(\frac{1}{2}\eta, \psi_\mu, \chi_{12})$ and doing the trace yields

$$S_{\rm F} = \frac{1}{2}\eta\partial_\mu\psi_\mu + \chi_{12}\left(\partial_1\psi_2 - \partial_2\psi_1\right) \tag{72}$$

This geometrical rewriting of the fermionic action yields the so-called Kähler-Dirac action which is most naturally rewritten using the language of differential forms as

$$\Psi.\left(d - d^\dagger\right)\Psi \tag{73}$$

where the Dirac-Kähler field Ψ is now just the set of components $(\frac{1}{2}\eta, \psi_\mu, \chi_{12})$ and d is the exterior derivative whose action on general rank p antisymmetric tensors (forms) $\omega_{[\mu_1\ldots\mu_p]}$ yields a rank $p{+}1$ tensor with components $\omega_{[\mu_1\ldots\mu_p\mu_{p+1}]}$ and the square bracket notation indicates complete antisymmetrization between all indices. The dot notation just indicates that corresponding tensor components are multiplied and integrated over space. The operator $d^\dagger$ is the corresponding adjoint operator mapping rank p tensors to rank $p-1$.

This recasting of the action in geometrical terms not only yields a nilpotent supersymmetry but allows us to discretize the action *without* inducing fermion doubles [9]. The prescription is simple.

- Replace a continuum derivative by a *forward difference* operator if it derives from the exterior derivative, a curl-like operation on the component fields.

[11] Two known exceptions are the $(\mathcal{Q} = 8, d = 3)$ and $(\mathcal{Q} = 16, d = 4)$ target SYM theories. These theories, in their twisted and lattice regularized forms, contain a Q-closed operator in their action [33, 34].

- Replace a continuum derivative by the *backward* difference if it comes from the adjoint operator $d^\dagger$, which implements a divergence-like operation on the component fields.

We can see this explicitly if we take the preceding fermion action and write it in the form

$$(\frac{1}{2}\eta, \chi_{12}). \left(\begin{array}{cc} \Delta_1^- & \Delta_2^- \\ \Delta_2^+ & -\Delta_1^+ \end{array} \right) \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \tag{74}$$

It is clear that the determinant of the matrix operator is equal to the usual double free bosonic determinant precluding the existence of additional zeros of the fermion operator. Thus, discretizations based on Dirac-Kähler fermions do not require additional ad hoc Wilson mass terms to be added. This turns out to be particularly useful for models with massless fermions such as the extended supersymmetric Yang Mills theories we consider later.

Finally, notice that the action for free Dirac-Kähler fermions can be mapped into the action for staggered quarks. In two dimensions simply introduce a lattice with half the original lattice spacing. Place the link fields $\psi_1(x)$, $\psi_2(x)$ and the plaquette field $\chi_{12}(x)$ on the sites of this new lattice with the scalar $\eta(x)$ being placed at the original site x. It is straightforward to verify that all the backward and forward differences now become symmetric differences acting now on these new site fields on the doubled lattice. The usual staggered phases arise as a consequence of the antisymmetrization of the derivatives.

Indeed this latter construction offers yet another way to see that Dirac-Kähler fields have no doubling – they are equivalent at the free field level (all that matters for doubling) to staggered fermions. However, unlike the usual situation in QCD the supersymmetric theories we are studying automatically contain the correct number of degenerate fermion flavors represented by the full staggered fermion determinant and the usual rooting problem that plagues staggered fermion formulations of QCD is avoided.

Finally, it is possible to recast our previous Wess Zumino construction in the language of these Dirac-Kähler fields. Consider just the kinetic term and let

$$\begin{aligned} \omega &= \frac{1}{2}\eta + i\chi_{12} \\ \lambda &= \psi_1 + i\psi_2 \end{aligned} \tag{75}$$

Then the expression

$$\omega^\dagger D_{\overline{z}}\lambda \tag{76}$$

yields the previous Dirac-Kähler action in two dimensions. The scalar field and its complex conjugate are formed from the bosonic partners (C, A_μ, B_{12}) in the same way.

Actually the appearance of these complex field combinations can be understood from within a Dirac-Kähler perspective. From a given Dirac-Kähler field $\Psi = (\frac{1}{2}\eta, \psi_\mu, \chi_{12})$ we can construct a dual field $\tilde{\Psi}$ with dual tensor components. This duality operation takes a rank p tensor and replaces it by a $(d-p)$ rank tensor, and is a realization of Hodge duality on the lattice. Schematically

$$f_p \xrightarrow{*} f_{d-p} \tag{77}$$

where the dual components are given by

$$\tilde{f}_{\mu_1 \cdots \mu_{d-p}} = \epsilon_{\mu_1 \cdots \mu_d} f_{\mu_{d-p+1} \cdots \mu_d} \tag{78}$$

Notice that two applications of the duality operation yields *minus* the identity. Thus a projector onto self-dual Dirac-Kähler fields would take the form

$$P^+ = \frac{1}{2}\left(I + i*\right) \tag{79}$$

with a corresponding projector equipped with a minus sign for projection on anti-self dual fields. If we decompose the original Dirac-Kähler field on its self-dual and anti-self-dual parts we can verify that the coontinuum Dirac-Kähler action separates into two independent parts. This allows us to restrict attention to say the self-dual component. This is what happens in the Wess-Zumino and sigma model cases.

6. Two dimensional gauge field theories – twisted $\mathcal{N} = (2, 2)$ SYM

6.1. Continuum formulation

In section 5.3 we argued that any two dimensional supersymmetric theory with four (or integer multiples of four) supercharges can be reformulated in twisted variables. Furthermore we have an uncovered explicit examples of this in the case of the $\mathcal{N} = 2$ Wess-Zumino model and sigma models. However, while the existence of a twisted formulation of a given continuum field theory is certainly a necessary condition for constructing a lattice model with exact supersymmetry, it does not guarantee that one exists. In general it is necessary for any lattice theory to satisfy additional constraints. Perhaps the most important of these is seen when we try to implement the procedure for gauge theories. In this case the requirement that the discretization procedure maintaining exact gauge introduces additional difficulties which we have not considered up to this point. In this section we examine this issue in some detail concentrating on perhaps the simplest canonical case of

$\mathcal{N}=(2,2)$ SYM target theory in $d=2$ dimensions.[12] The first Euclidean lattice formulation for this theory was given in Ref.[32] using the orbifold approach, which will be discussed in §7.2. Lattice formulations based on the concept of twisting were then proposed in [26, 109]. These latter approaches start from a particular twist of the continuum theory given by the action

$$S=\beta Q\mathrm{Tr}\int d^2x\left(\frac{1}{4}\eta[\phi,\overline{\phi}]+2\chi_{12}F_{12}+\chi_{12}B_{12}+\psi_\mu D_\mu\overline{\phi}\right) \tag{80}$$

Here all fields $f(x)$ are in the adjoint representation of $SU(N)$ with $f(x)=\sum_{a=1}^{N^2-1}f^a(x)T^a$ with *antihermitian* generators T^a satisfying $\mathrm{Tr}T^aT^b=-\delta^{ab}$ [13]. The covariant derivatives act as

$$D_\mu f=\partial_\mu f+[A_\mu,f] \tag{81}$$

while the action of Q on the twisted fields is given by

$$\begin{aligned}
QA_\mu &= \psi_\mu\\
Q\psi_\mu &= -D_\mu\phi\\
Q\overline{\phi} &= \eta\\
Q\eta &= [\phi,\overline{\phi}]\\
QB_{12} &= [\phi,\chi_{12}]\\
Q\chi_{12} &= B_{12}\\
Q\phi &= 0
\end{aligned} \tag{82}$$

Notice that $Q^2=\delta^\phi$ an infinitesimal gauge transformation on the fields. Carrying out the Q-variation on eq. (80) and subsequently integrating over the field B_{12} leads to the action

$$\begin{aligned}
S &= \beta\mathrm{Tr}\int d^2x\left(\frac{1}{4}[\phi,\overline{\phi}]^2-\frac{1}{4}\eta[\phi,\eta]-F_{12}^2-D_\mu\phi D_\mu\overline{\phi}\right.\\
&- \left.\chi_{12}[\phi,\chi_{12}]-2\chi_{12}\left(D_1\psi_2-D_2\psi_1\right)-\psi_\mu D_\mu\eta+\psi_\mu[\overline{\phi},\psi_\mu]\right)
\end{aligned} \tag{83}$$

[12]In two dimensions supercharges can be specified as "left-handed" or "right-handed", and this theory has two of each, so it is often called $(2,2)$ SYM.

[13] In §6 and §6.4, we use anti-hermitian generators with $\mathrm{Tr}T^aT^b=-\delta^{ab}$. In the rest, our Lie algebra generators and their normalization convention is $\mathrm{Tr}T^aT^b=+\delta^{ab}$.

The bosonic sector of this action is precisely the usual Yang-Mills action while the fermionic sector constitutes, as expected, a Dirac-Kähler representation of the usual spinorial action [21].

It is worth pointing out that the twisted theory possesses an additional $U(1)$ symmetry inherited from the remaining R-symmetry of the model which is given by

$$\begin{aligned} \psi_\mu &\to e^{i\alpha}\psi_\mu \\ \chi_{12} &\to e^{-i\alpha}\chi_{12} \\ \eta &\to e^{-i\alpha}\eta \\ \phi &\to e^{2i\alpha}\phi \\ \overline{\phi} &\to e^{-2i\alpha}\overline{\phi} \end{aligned} \tag{84}$$

Two different discretization schemes have been proposed to generate a lattice model from this continuum theory [110, 110, 109] and preliminary simulations have already been done [111, 112, 97, 113]. However, the lattice formulation described in [109] and [114] suffers from a doubling of degrees of freedom with respect to the continuum theory which has been discussed in [40, 115]. The lattice formulation introduced by Sugino in [110] is also problematic since the vacuum state turns out to be infinitely degenerate at least in dimensions greater than two.

Both sets of problems can be evaded using a alternative twist based on the strictly nilpotent supercharge $Q + iQ_{12}$ introduced later [41, 42]. As we will see the resultant lattice actions then reproduce *precisely* the corresponding orbifold actions [32], which we discuss in §7.2. Before describing this alternative twist we show how to construct the action of the additional (non-scalar) twisted supersymmetries on the twisted fields. This is an important issue as it allows us to construct Ward identities for all the broken supersymmetries in the discretized theory. The question of whether the full supersymmetry of the continuum target theory is regained in the continuum limit can then be examined by examining whether these Ward identities are satisfied as the lattice spacing is sent to zero. Preliminary work in this direction is reported in [116, 112].

6.2. Additional twisted supersymmetries

It is straightforward to construct the additional twisted supersymmetry transformations of the component fields [112]. As we have described in section 5.4 the fermion kinetic term can be written in the matrix form

$$S_F = \int d^2x \mathrm{Tr}\Psi^\dagger\gamma.D\Psi \tag{85}$$

where Ψ corresponds to the matrix form of the Dirac-Kähler field

$$\Psi = \frac{\eta}{2}I + \psi_\mu\gamma_\mu + \chi_{12}\gamma_1\gamma_2 \tag{86}$$

This term is clearly invariant under $\Psi \rightarrow \Psi\Gamma^i, i = 1\ldots 4$ and Γ^i corresponds to one of the set $(I, \gamma_1, \gamma_2, \gamma_1\gamma_2)$ Consider first the case $\Gamma^4 = \gamma_1\gamma_2$. In terms of the component fields the transformation $\Psi \rightarrow \Psi\Gamma^4$ effects a duality map

$$\begin{aligned}
\frac{\eta}{2} &\rightarrow -\chi_{12} \\
\chi_{12} &\rightarrow \frac{\eta}{2} \\
\psi_\mu &\rightarrow -\epsilon_{\mu\nu}\psi_\nu
\end{aligned} \tag{87}$$

Such an operation clearly leaves the Yukawa terms invariant and trivially all bosonic terms. It is thus a symmetry of the continuum action. By combining such a transformation with the original action of the scalar supercharge one derives an additional supersymmetry of the theory – that corresponding to the twisted supercharge Q_{12}. Explicitly this supersymmetry will transform the component fields of the continuum theory in the following way

$$\begin{aligned}
Q_{12}A_\mu &= -\epsilon_{\mu\nu}\psi_\nu \\
Q_{12}\psi_\mu &= -\epsilon_{\mu\nu}D_\nu\phi \\
Q_{12}\chi_{12} &= -\frac{1}{2}[\phi, \overline{\phi}] \\
Q_{12}B_{12} &= [\phi, \frac{\eta}{2}] \\
Q_{12}\overline{\phi} &= -2\chi_{12} \\
Q_{12}\frac{\eta}{2} &= B_{12} \\
Q_{12}\phi &= 0
\end{aligned} \tag{88}$$

From the Q and Q_{12} transformations it is straightforward to verify the following algebra holds

$$\begin{aligned}
\{Q, Q\} &= \{Q_{12}, Q_{12}\} = \delta_\phi \\
\{Q, Q_{12}\} &= 0
\end{aligned} \tag{89}$$

where δ_ϕ denotes an infinitesimal gauge transformation with parameter ϕ. This allows us to construct strictly nilpotent symmetries $\hat{Q}_\pm = Q \pm iQ_{12}$ in the continuum theory corresponding to using the (anti)self-dual components of the original Dirac-Kähler field.

In the same way we can try to build an additional supersymmetry by combining the invariance of the fermion kinetic term under $\Psi \to \Psi\Gamma^1$ with the existing scalar supersymmetry. This effects the following transformation of fermion fields:

$$\begin{aligned} \frac{\eta}{2} &\to \psi_1 \\ \chi_{12} &\to -\psi_2 \\ \psi_1 &\to \frac{\eta}{2} \\ \psi_2 &\to -\chi_{12} \end{aligned} \qquad (90)$$

However, the Yukawas and bosonic terms are only invariant under such a transformation if we simultaneously make the transformation $\phi \to -\overline{\phi}$. The resultant explicit action of Q_1 and Q_2 on the component fields is given by

$$\begin{aligned} Q_1 A_1 &= \frac{\eta}{2} & Q_2 A_1 &= \chi_{12} \\ Q_1 A_2 &= -\chi_{12} & Q_2 A_2 &= \frac{\eta}{2} \\ Q_1 \psi_1 &= -\frac{1}{2}[\phi, \overline{\phi}] & Q_2 \psi_1 &= B_{12} \\ Q_1 \psi_2 &= -B_{12} & Q_2 \psi_2 &= -\frac{1}{2}[\phi, \overline{\phi}] \\ Q_1 \chi_{12} &= -D_2 \overline{\phi} & Q_2 \chi_{12} &= D_1 \overline{\phi} \\ Q_1 B_{12} &= [\overline{\phi}, \psi_2] & Q_2 B_{12} &= -[\overline{\phi}, \psi_1] \\ Q_1 \overline{\phi} &= 0 & Q_2 \overline{\phi} &= 0 \\ Q_1 \frac{\eta}{2} &= D_1 \overline{\phi} & Q_2 \frac{\eta}{2} &= D_2 \overline{\phi} \\ Q_1 \phi &= -2\psi_1 & Q_2 \phi &= -2\psi_2 \end{aligned} \qquad (91)$$

Again, we can verify the following algebra holds

$$\begin{aligned} \{Q_1, Q_1\} &= \{Q_2, Q_2\} = \delta_{-\overline{\phi}} \\ \{Q_1, Q_2\} &= 0 \end{aligned} \qquad (92)$$

with $\delta_{-\overline{\phi}}$ a corresponding gauge transformation with parameter $-\overline{\phi}$. This allows us to construct yet another pair of nilpotent supercharges in the continuum theory $\overline{Q}_{\pm} = Q_1 \pm iQ_2$.

It is interesting to check also the anticommutators of these new charges $\hat{Q}_{\pm}$ and $\overline{Q}_{\pm}$. It is a straightforward exercise to verify the following algebra holds *on-shell*

$$\begin{aligned}
\{\hat{Q}_+, \overline{Q}_-\} &= \{\hat{Q}_-, \overline{Q}_+\} = 0 \\
\{\hat{Q}_+, \overline{Q}_+\} &= 4(D_1 + iD_2) \\
\{\hat{Q}_-, \overline{Q}_-\} &= 4(D_1 - iD_2)
\end{aligned} \tag{93}$$

As an example consider $\{\hat{Q}_+, \overline{Q}_+\}\psi_1$

$$\{\hat{Q}_+, \overline{Q}_+\} = \{Q, Q_1\} - \{Q_{12}, Q_2\} + i\left(\{Q_{12}, Q_1\} + \{Q, Q_2\}\right) \tag{94}$$

Using the component transformations listed above the relevant anticommutators are

$$\begin{aligned}
\{Q, Q_1\}\psi_1 &= 2D_1\psi_1 \\
\{Q, Q_2\}\psi_1 &= 2D_1\psi_2 + 2[\phi, \chi_{12}] \\
\{Q_{12}, Q_1\}\psi_1 &= 2D_2\psi_1 \\
\{Q_{12}, Q_2\}\psi_1 &= 2D_2\psi_2 + [\phi, \eta]
\end{aligned} \tag{95}$$

Thus we find

$$\{\hat{Q}_+, \overline{Q}_+\}\psi_1 = 2D_1\psi_1 - 2D_2\psi_2 - [\phi, \eta] + i(2D_1\psi_2 + 2D_2\psi_1 + 2[\phi, \chi_{12}]) \tag{96}$$

Using the equations of motion

$$\begin{aligned}
-2D_1\psi_1 - 2D_2\psi_2 - [\phi, \eta] &= 0 \\
-2D_2\psi_1 + 2D_1\psi_2 + 2[\phi, \chi_{12}] &= 0
\end{aligned} \tag{97}$$

we can easily verify the second line of eq. (93). Notice that from these new charges $\hat{Q}_{\pm}$, $\overline{Q}_{\pm}$ we can build spinorial supercharges of the form

$$\begin{pmatrix} \hat{Q}_+ \\ \overline{Q}_- \end{pmatrix} \tag{98}$$

in which case the algebra given in eq. (93) represents the usual supersymmetry algebra in a chiral basis (up to a gauge transformation).

6.3. Self-dual twist

As we have seen it is possible to derive an additional *strictly nilpotent* supersymmetry in the gauge theory case by combining the scalar charge Q with its dual pseudoscalar charge Q_{12} in the form $\mathcal{Q} = Q - iQ_{12}$. Furthermore, using the transformations derived in the previous section we can easily show that

$$\begin{aligned} \mathcal{Q}\mathcal{A} = \mathcal{Q}\left(A_1 + iA_2\right) &= 2\left(\psi_1 + i\psi_2\right) \\ \mathcal{Q}\left(\psi_1 + i\psi_2\right) &= 0 \\ \mathcal{Q}\overline{\mathcal{A}} = \mathcal{Q}\left(A_1 - iA_2\right) &= 0 \end{aligned} \tag{99}$$

This new supercharge is associated to a alternative twist of the Yang-Mills theory which we may call the *self-dual twist*. It will turn out that this twist is intimately connected to the orbifold lattice constructions we discuss next. The key observation is that the original 4 on-shell bosonic degrees of freedom can be realized in terms of the *complex* gauge fields $\mathcal{A}_\mu$ and $\overline{\mathcal{A}}_\mu$ together with a new set of twisted supersymmetry transformations

$$\begin{aligned} \mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\ \mathcal{Q}\,\psi_\mu &= 0 \\ \mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\ \mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\ \mathcal{Q}\,\eta &= d \\ \mathcal{Q}\,d &= 0 \end{aligned} \tag{100}$$

Notice that this supersymmetry implies that the fermions are to be treated as complex which is natural in a Euclidean theory. As in previous constructions the twisted action in two dimensions can be written in $\mathcal{Q}$-exact form $S = \beta\mathcal{Q}\,\Lambda$ where Λ now is given by the expression

$$\Lambda = \int \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right) \tag{101}$$

and we have introduced the complexified covariant derivatives (we again employ an antihermitian basis for the generators of $U(N)$)

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + \mathcal{A}_\mu = \partial_\mu + A_\mu + iB_\mu \\ \overline{\mathcal{D}}_\mu &= \partial_\mu + \overline{\mathcal{A}}_\mu = \partial_\mu + A_\mu - iB_\mu \end{aligned} \tag{102}$$

Doing the $\mathcal{Q}$-variation and integrating out the field d yields

$$S = \int \mathrm{Tr}\ \left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu \right) \tag{103}$$

The bosonic terms can be written

$$\begin{aligned} \overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} &= (F_{\mu\nu} - [B_\mu, B_\nu])^2 + \left(D_{[\mu}B_{\nu]}\right)^2 \\ \frac{1}{2}\left[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu\right]^2 &= -2\left(D_\mu B_\mu\right)^2 \end{aligned} \tag{104}$$

where $F_{\mu\nu}$ and D_μ denote the usual field strength and covariant derivative depending on the real part of the connection A_μ. After integrating by parts the term linear in $F_{\mu\nu}$ cancels and the final bosonic action reads[14]

$$S_B = \int \mathrm{Tr}\ \left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2\right) \tag{105}$$

Notice that the imaginary parts of the gauge field have transformed into the two scalars of the SYM theory! This is further confirmed by looking at the fermionic part of the action which can be rewritten in 2×2 block form as

$$\begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{106}$$

which is easily recognized as the dimensional reduction of $\mathcal{N} = 1$ SYM theory in four dimensions in which a chiral representation is employed for the fermions. As usual the scalar fields B_μ arise from the gauge fields in the reduced directions.

6.4. Lattice theory for $(2, 2)$ *SYM*

In this section we show how to discretize this self-dual twist of the two dimensional Yang-Mills model with $\mathcal{Q} = 4$ supercharges. To do this we employ the geometrical discretization scheme proposed in [109]. In general continuum p-form fields are mapped to lattice fields defined on p-subsimplices of a general simplicial lattice. In the case of hypercubic lattices this assignment is equivalent to placing a p-form with indices $\mu_1 \ldots \mu_p$ on the link connecting

[14] The bosonic action is real positive definite on account of the antihermitian basis that we have chosen.

$\mathbf{x}$ with $(\mathbf{x}+\boldsymbol{\mu}_1+\ldots+\boldsymbol{\mu}_p)$ where $\boldsymbol{\mu}_i, i=1\ldots p$ corresponds to a unit vector in the lattice. Actually this is not quite the full story; each link has two possible orientations and we must also specify which orientation is to be used for a given field. A positively oriented field corresponds to one in which the link vector has positive components with respect to this coordinate basis.

Continuum derivatives on such a hypercubic lattice are represented by lattice difference operators acting on these link fields. Specifically, covariant derivatives appearing in curl-like operations and *acting on positively oriented fields* are replaced by a lattice gauge covariant forward difference operator whose action on lattice scalar and vector fields is given by

$$\begin{aligned} \mathcal{D}^{(+)}_\mu f(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x})f(\mathbf{x}+\boldsymbol{\mu}) - f(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}) \\ \mathcal{D}^{(+)}_\mu f_\nu(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x})f_\nu(\mathbf{x}+\boldsymbol{\mu}) - f_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\nu}) \end{aligned} \tag{107}$$

where $\mathbf{x}$ denotes a two dimensional lattice vector and $\boldsymbol{\mu}=(1,0)$, $\boldsymbol{\nu}=(0,1)$ unit vectors in the two coordinate directions. Here, we have replaced the continuum *complex* gauge fields $\mathcal{A}_\mu$ by non-unitary link fields $\mathcal{U}_\mu = e^{i\mathcal{A}_\mu}$. The backward difference operator $\overline{\mathcal{D}}^-_\mu$ replaces the continuum covariant derivative in divergence-like operations and its action on (positively oriented) lattice vector fields can be gotten by requiring that it to be the adjoint to $\mathcal{D}^+_\mu$. Specifically its action on lattice vectors is

$$\overline{\mathcal{D}}^{(-)}_\mu f_\mu(\mathbf{x}) = f_\mu(\mathbf{x})\overline{\mathcal{U}}_\mu(\mathbf{x}) - \overline{\mathcal{U}}_\mu(\mathbf{x}-\boldsymbol{\mu})f_\mu(\mathbf{x}-\boldsymbol{\mu}) \tag{108}$$

The nilpotent scalar supersymmetry now acts on the lattice fields as

$$\begin{aligned} \mathcal{Q}\,\mathcal{U}_\mu &= \psi_\mu \\ \mathcal{Q}\,\psi_\mu &= 0 \\ \mathcal{Q}\,\overline{\mathcal{U}}_\mu &= 0 \\ \mathcal{Q}\,\chi_{\mu\nu} &= \mathcal{F}^{L\dagger}_{\mu\nu} \\ \mathcal{Q}\,\eta &= d \\ \mathcal{Q}\,d &= 0 \end{aligned} \tag{109}$$

Here we written the lattice field strength as

$$\mathcal{F}^L_{\mu\nu} = \mathcal{D}^{(+)}_\mu \mathcal{U}_\nu(\mathbf{x}) = \mathcal{U}_\mu(\mathbf{x})\mathcal{U}_\nu(\mathbf{x}+\boldsymbol{\mu}) - \mathcal{U}_\nu(\mathbf{x})\mathcal{U}_\mu(\mathbf{x}+\boldsymbol{\nu}) \tag{110}$$

which reduces to the continuum (complex) field strength in the naive continuum limit and is automatically antisymmetric in the indices (μ,ν).

Notice that this supersymmetry transformation implies that the fermion fields ψ_μ have the same orientation as their superpartners the gauge links $\mathcal{U}_\mu$ and run from $\mathbf{x}$ to $(\mathbf{x}+\boldsymbol{\mu})$. However, the field $\chi_{\mu\nu}$ must have the same orientation as $\mathcal{F}^{L\dagger}_{\mu\nu}$ and hence is to be assigned to the negatively oriented link running from $(x+\boldsymbol{\mu}+\boldsymbol{\nu})$ down to $\mathbf{x}$ i.e parallel to the vector $(-1,-1)$. This link choice also follows naturally from the matrix representation of the Dirac-Kähler field Ψ

$$\Psi = \eta I + \psi_\mu\gamma_\mu + \chi_{12}\gamma_1\gamma_2 \tag{111}$$

which associates the field χ_{12} with the lattice vector $\boldsymbol{\mu}_1+\boldsymbol{\mu}_2=\boldsymbol{\mu}+\boldsymbol{\nu}$. We will see that the negative orientation is crucial for allowing us to write down gauge invariant expressions for the fermion kinetic term. Finally, it should be clear that the scalar fields η and d can be taken to transform simply as site fields.

These link mappings and orientations are conveniently summarized by giving the gauge transformation properties of the lattice fields

$$\begin{aligned}
\eta(\mathbf{x}) &\rightarrow G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x})\\
\psi_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\psi_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\boldsymbol{\mu})\\
\chi_{\mu\nu}(\mathbf{x}) &\rightarrow G(\mathbf{x}+\boldsymbol{\mu}+\boldsymbol{\nu})\chi_{\mu\nu}G^\dagger(\mathbf{x})\\
\mathcal{U}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x})\\
\overline{\mathcal{U}}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x}+\boldsymbol{\mu})\overline{\mathcal{U}}_\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned} \tag{112}$$

We will see shortly that this decomposition of the fermionic degrees of freedom over the lattice is identical to that encountered in the orbifolding approach to lattice supersymmetry [32]. Furthermore, the above $\mathcal{Q}$-variations and field assignments are equivalent to the formulation described in [28] *provided* that we set the fermionic shift parameter a in that formulation to zero and consider only the corresponding scalar supersymmetry.

The lattice gauge fermion now takes the form

$$\Lambda = \sum_{\mathbf{x}} \mathrm{Tr}\,\left(\chi_{\mu\nu}\mathcal{D}^{(+)}_\mu\mathcal{U}_\nu + \eta\overline{\mathcal{D}}^{(-)}_\mu\mathcal{U}_\mu - \frac{1}{2}\eta d\right) \tag{113}$$

It is easy to see that in the naive continuum limit the lattice divergence $\overline{\mathcal{D}}^{(-)}_\mu\mathcal{U}_\mu$ equals $[\overline{\mathcal{D}}_\mu,\mathcal{D}_\mu]$. Notice that with the previous choice of orientation for the various fermionic link fields this gauge fermion is automatically invariant under lattice gauge transformations. There is no need for the doubling of degrees of freedom encountered in [109, 117]. Those constructions utilize the twist described earlier in section 6.1 in which the nature of the gauge fermion and the scalar supercharge led to the presence of explicit Yukawa interactions in the theory. These, in turn, required the lattice theory to contain fermion link fields of both orientations and hence led to a doubling of degrees of freedom with respect to the continuum theory. For the self-dual twist the Yukawa interactions are embedded into the complexified covariant derivatives and successive components of the Dirac-Kähler field representing the fermions can be chosen with alternating orientations leading to a Dirac-Kähler action which is automatically gauge invariant without these extra degrees of freedom.

Acting with the $\mathcal{Q}$-transformation shown above and again integrating out the auxiliary field d we derive the gauge and $\mathcal{Q}$-invariant lattice action

$$S = \sum_{\mathbf{x}} \mathrm{Tr}\, \left(\mathcal{F}^{L\dagger}_{\mu\nu}\mathcal{F}^{L}_{\mu\nu} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{\mu}\mathcal{U}_{\mu}\right)^{2} - \chi_{\mu\nu}\mathcal{D}^{(+)}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}^{(-)}_{\mu}\psi_{\mu} \right) \qquad (114)$$

But this is precisely the orbifold action arising in [32] with the modified deconstruction step described in [118] and [40] which we will describe in detail in the next section. The two approaches are thus entirely equivalent.

We can use this geometrical formulation to show very easily that the lattice theory exhibits no fermion doubling problems. The simplest way to do this is merely to notice that the lattice action at zero coupling $\mathcal{U} \to I$ conforms to the canonical form required for no doubling by the theorem of Rabin [9]. Explicitly, discretization of continuum actions written in terms of p-forms will not encounter doubling problems if continuum derivatives acting in curl-like operations are replaced by forward differences in the lattice theory while continuum derivatives appearing in divergence-like operations are represented by backward differences on the lattice. More precisely the continuum exterior derivative d is mapped to a forward difference while its adjoint $d^{\dagger}$ is represented by a backward difference.

An alternative way to see this is to examine the the form of the fermion operator arising in this construction.

$$\left(\chi_{12} \quad \frac{\eta}{2} \right) \begin{pmatrix} -\mathcal{D}^{(+)}_{2} & \mathcal{D}^{(+)}_{1} \\ \mathcal{D}^{(-)}_{1} & \mathcal{D}^{(-)} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad (115)$$

Clearly the determinant of this operator in the free limit is nothing more than the usual determinant encountered for scalars in two dimensions and hence possesses no extraneous zeroes that survive the continuum limit.

κ	κS_B	exact
1.0	4.39(2)	4.5
10.0	4.49(1)	4.5
100.0	4.49(1)	4.5
1000.0	4.52(2)	4.5

Table 3: Bosonic action versus exact SUSY value for the $\mathcal{Q} = 4$, $SU(2)$ matrix model.

Numerical simulations of this and related models are just beginning. They rely on using the RHMC algorithm [119] to handle the non-local Pfaffian that arises after integration over the twisted fermions. Table 3 shows results from a simulation of the zero dimensional $SU(2)$ matrix system which arises by dimensional reduction of the $\mathcal{Q} = 4$ twisted action we have described. As for quantum mechanics, exact supersymmetry allows us to predict a value for the expectation value of the bosonic action κS_B which is independent of coupling κ and this result is strongly borne out by the Monte Carlo data[15].

One of the most interesting questions that arises in these models concerns the nature of the vacuum. Classically the models possess a continuous infinity of vacua corresponding to taking the complex bosonic link fields to be diagonal matrices which are constant over the lattice. Integration over these flat directions may then lead to IR divergences. This issue has already been examined via numerical simulations (see fig. 10) where it is found that contrary to naive expectation the eigenvalues of the scalar fields (imaginary parts of the complex link field) remain localized close to the origin in field space [122]. These preliminary simulations are encouraging as they show that these new lattice actions may indeed be very useful starting points for numerical explorations of strongly coupled supersymmetric systems.

Our discussion thus far has taken us from simple quantum mechanics models to genuine field theories with non-abelian gauge symmetry. In all cases the approach we have followed is to rewrite the theory in terms of so-called twisted variables which naturally exposes a scalar supercharge which,

[15]In this case the Pfaffian phase is exactly zero and so poses no problem in the simulation. In two dimensions large phase fluctuations are encountered for the model with $\mathcal{Q} = 4$ supersymmetries [120, 121, 122]. These phase fluctuations seem much smaller for the model with $\mathcal{Q} = 16$ supersymmetries and can be handled by standard re-weighting techniques.

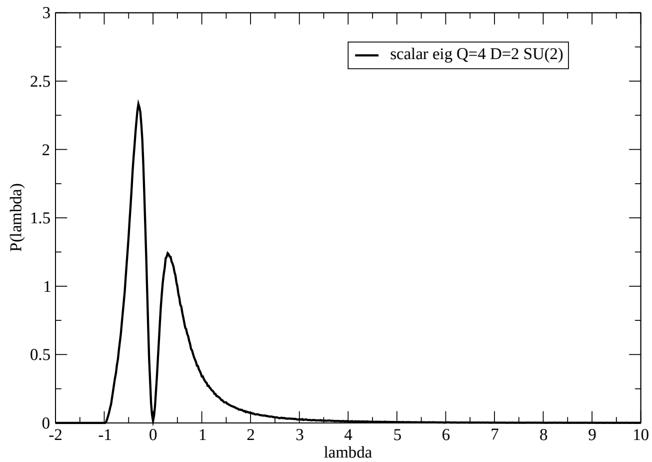

Figure 10: Probability distribution for the eigenvalues of scalar fields in $SU(2)$ theory with $\mathcal{Q} = 4$ supersymmetries in two dimensions

with care, may then be transfered to the lattice. The twisted constructions are elegant and physically well motivated but the precise discretization prescription has been arrived at in a somewhat ad hoc manner. We will now turn to the orbifold constructions for gauge theories and show how these discretization rules re-emerge in an essentially unique way, being determined only by the global symmetry of the continuum theory and the requirement of one or more exact supersymmetries.

7. Supersymmetric lattices from orbifold projection

In this section we turn to an alternative construction of supersymmetric lattice actions based on the ideas of deconstruction and orbifolding. On the face of it this seems quite independent of the discretized twisted constructions we have discussed in the last couple of sections. Nevertheless we will see on closer analysis that both approaches are intimately connected and lead to similar lattice actions in the case of Yang-Mills theories. As we will show, the orbifold approach is a very powerful way to generate all the known SYM lattices, and in fact was how they were first derived for spatial lattices [31] and Euclidean spacetime lattices [32, 33, 34, 35].

7.1. Deconstruction:The AHCG model

The starting point is the deconstruction method of Arkani-Hamed, Cohen and Georgi (AHCG). In reference [38] the authors were not concerned with latticizing supersymmetry; instead they wanted a precise field theoretic way to examine claims about the phenomenology of certain field theories in five dimensions. In order to avoid ill-defined problems with renormalization in five dimensions, they constructed a theory with four continuous dimensions, and a latticized fifth dimension. This can be viewed as a $d = 4$ field theory with many "flavors" of fields, corresponding to the discrete values of the fifth coordinate. A diagram of the theory of interest is given in Fig. 11; it is an $\mathcal{N} = 1$ supersymmetric field theory in $d = 4$ with gauge group $U(k)^N$ with a single gauge coupling g, where each $U(k)$ factor appears as a node in the picture. The n^{th} node has a vector multiplet associated with it — a gauge field $v_m^{(n)}$ and a gaugino $\lambda^{(n)}$. In addition there are matter fields in the form of chiral supermultiplets Φ_n which appear in the figure as directed links between nodes n and $(n+1)$; they transform as bifundamentals $(\Box, \overline{\Box})$ under the $U(k)\times U(k)$ gauge symmetry associated with those two nodes, and are neutral under the rest of the gauge symmetry; they represent the scalar and fermion component fields $(\phi^{(n)}, \psi^{(n)})$. All the interactions in this model are supersymmetric gauge interactions (which include certain Yukawa and ϕ^4 couplings). Note that since all the fields transform as either adjoints of $U(k)$ or bifundamentals of $U(k) \times U(k)$, they can all be represented as $k \times k$ matrices with non-zero trace.

So far, this model doesn't look at all like a lattice for a 5D theory; although there are interactions between nearest neighbors the fifth direction, there are no bilinear "hopping terms" corresponding to kinetic energy operators for motion in this extra dimension. However, the authors noted that the theory has a "flat direction" corresponding to

$$\langle \phi^{(n)} \rangle = \frac{1}{a\sqrt{2}}\,\mathbf{1}_k \tag{116}$$

where $\mathbf{1}_k$ represents that $k \times k$ unit matrix, and a is a length scale. By flat direction, we mean that the theory has a degenerate ground state, where the vacuum energy is unaffected by the simultaneous shift of all the scalar link fields $\phi^{(n)}$ as in eq. (116). Furthermore, as we will elaborate on below, AHCG noted that the parameter a behaves like a lattice spacing, and that in the

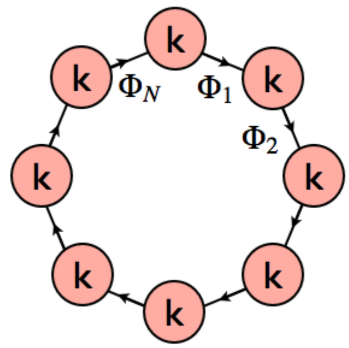

Figure 11: A diagram for the AHCG deconstruction model, which is a $d = 4$, $\mathcal{N} = 1$ supersymmetric gauge theory. Each node corresponds to an independent $U(k)$ gauge symmetry, with the associated vector supermultiplet V_n. The links represent chiral superfields Φ_n which transform as bifundamentals under the gauge symmetries of the nodes they connect.

limit

$$N \to \infty \ , \quad a \to 0 \ , \quad g \to 0 \ , \quad aN \equiv L_5 \text{ (fixed)} \ , \quad g^2/a \equiv g_5^2 \text{ (fixed)} \ , \tag{117}$$

the model of Fig. 11 has two amazing properties:

- it possesses $d = 5$ Poincaré invariance;
- it possesses $\mathcal{Q} = 8$ supercharges, even though the $d = 4$ model in Fig. 11 only respected $\mathcal{Q} = 4$ exact supersymmetries.

This is exactly the type of phenomenon we were looking for! Both Poincaré symmetry and supersymmetry are enhanced in the continuum limit without any fine tuning of the theory.

We now sketch out how the 5D kinetic terms emerge in the AHCG model in the $a \to 0$ limit, and then discuss how to generalize their procedure to generate true lattices where every spacetime dimension is discretized, a method called "orbifolding".

7.2. Continuum limit of the AHCG model

The Lagrangian for the AHCG model possesses four types of terms:

1. The Yang-Mills action for the gauge fields $v_m^{(n)}$;

2. Gauge interactions for the adjoint gauginos $\lambda^{(n)}$ and the bifundamental matter fields $\phi^{(n)}$ and $\psi^{(n)}$, the latter involving both $v_m^{(n)}$ and $v_m^{(n-1)}$;
3. Yukawa interactions for the form $\sum_n \mathrm{Tr}\, \lambda^{(n)} \left(\psi^{(n)}\bar{\phi}^{(n)} - \bar{\phi}^{(n-1)}\psi^{(n-1)}\right)$;
4. A ϕ^4 interaction (called the "D-term") proportional to $\sum_n \mathrm{Tr}\, \left(\phi^{(n+1)}\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\phi^{(n)}\right)^2$

It is easy to see then that indeed eq. (116) is a flat direction of the theory, since the D-term vanishes if each field $\phi^{(n)}$ equals the same diagonal matrix. To see how the continuum limit emerges, we expand the ϕ fields about their vacuum value as

$$\phi^{(n)}(x) = \frac{1}{a\sqrt{2}}\,\mathbf{1}_k + \frac{s^{(n)}(x) + iv_5^{(n)}(x)}{\sqrt{2}} \tag{118}$$

where s and v_5 are hermitean matrices. Then, for example, the ($d = 4$) kinetic term for ϕ in the AHCG action is

$$\begin{aligned}
&\frac{1}{g^2}\sum_n \int d^4x\, \mathrm{Tr}\, |D_\mu \phi^{(n)}|^2 = \frac{1}{g^2}\sum_n \int d^4x\, \mathrm{Tr}\, |\partial_\mu \phi^{(n)} + iv_\mu^{(n)}\phi^{(n)} - i\phi^{(n)}v_\mu^{(n+1)}|^2 \\
=\; &\frac{1}{2g^2}\sum_n \int d^4x\, \mathrm{Tr}\, \Big|\left(\partial_\mu s^{(n)} + iv_\mu^{(n)}s^{(n)} - is^{(n)}v_\mu^{(n+1)}\right) \\
&\quad +i\left(\partial_\mu v_5^{(n)} + iv_\mu^{(n)}v_5^{(n)} - iv_5^{(n)}v_\mu^{(n+1)}\right) + i\left(v_\mu^{(n)} - v_\mu^{(n+1)}\right)/a\Big|^2 \\
&\xrightarrow[a\to 0]{} \frac{1}{2g_5^2}\int d^5x\, \mathrm{Tr}\, (D_\mu s)^2 - \mathrm{Tr}\, v_{\mu 5}v^{\mu 5}\ ,
\end{aligned} \tag{119}$$

where v_{mn} is the $d = 5$ gauge field strength. Note that the 5D kinetic term for the gauge field has emerged in this limit.

The scalar "D-term" in the AHCG model provided the 5D kinetic term for the field s in the same limit:

$$\frac{1}{2g^2}\sum_n \int d^4x\, \mathrm{Tr}\, \left(\phi^{(n+1)}\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\phi^{(n)}\right) \xrightarrow[a\to 0]{} \frac{1}{2g_5^2}\int d^5x\, \mathrm{Tr}\, (D_5 s)^2\ . \tag{120}$$

Note that this $5D$ term is normalized the same way as the $(D_\mu s)^2$ term in the previous equation, as required by 5D Lorentz invariance.

In the AHCG model, the two Weyl fermions—the gaugino λ and the matter field ψ—combine to form one, 4-component, $d = 5$ fermion

$$\Psi = \begin{pmatrix}\lambda \\ \bar{\psi}\end{pmatrix}\ , \qquad \bar{\Psi} = (\psi \quad \lambda) \tag{121}$$

in the γ-matrix basis

$$\gamma_\mu = \begin{pmatrix} & \bar{\sigma}_\mu \\ \sigma_\mu & \end{pmatrix} \ , \qquad \gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{122}$$

The fifth dimensional part of the fermion kinetic term (and the $\Psi - s$ interaction) arises from the Yukawa interaction in the $d = 4$ theory:

$$\begin{aligned} &\frac{1}{g^2} \sum_n \int d^4x \, i\sqrt{2}\mathrm{Tr} \ \lambda^{(n)} \left(\psi^{(n)} \bar{\phi}^{(n)} - \bar{\phi}^{(n-1)} \psi^{(n-1)} \right) + \text{h.c.} \\ &\xrightarrow[a\to 0]{} \frac{1}{2g_5^2} \int d^5x \ \mathrm{Tr} \ \left(\bar{\Psi} i\gamma_5 D_5 \Psi - \bar{\Psi}\gamma_5 [s, \Psi] \right) \ . \end{aligned} \tag{123}$$

It is easy to figure out the limit of the remaining terms. The conclusion is that a 5D supersymmetric gauge theory emerges in the continuum limit, consisting of the scalar s arising as the real part of the link scalar ϕ, the fermion $\Psi, \bar{\Psi}$ arising both from the gauginos λ living at the sites of Fig. 11, as well as the link fermions ψ; and the 5D gauge field consisting of the four components of v_μ living on the sites, and v_5 arising as the imaginary part of the link scalar ϕ. It is fascinating to see how these 5D multiplets form by combining both site and link variables. Most importantly for our purposes, recall the claim that this 5D gauge theory possesses $\mathcal{Q} = 8$ supersymmetries, which has somehow emerged in the $a \to 0$ limit from the original $\mathcal{Q} = 4$ theory, without any fine tuning.

The mechanism by which enhanced supersymmetry emerges in the continuum limit of the AHCG model [38] is what has been long sought for in a lattice theory — but it is itself still a theory in four continuous dimensions and not on a lattice. To construct a true supersymmetric lattice, we must "reverse engineer" the AHCG model to find general principles for how it is constructed, and then apply those principles to constructing true spacetime lattices.

7.3. The AHCG model via orbifolding

A simple procedure exists for producing the theory represented by Fig. 11 with N sites and a $U(k)^N$ gauge symmetry. The idea is to start with a "mother theory" which has the following properties:

- it is a $d = 4$ field theory like the AHCG model;
- it possesses the huge gauge group $U(Nk)$;

- it respects the number of supersymmetries of the *target* theory, namely $\mathcal{Q} = 8$.

In other words, it is a $d = 4$, $\mathcal{Q} = 8$ gauge theory with gauge group $U(Nk)$; such a theory is known as an $\mathcal{N} = 2$ SYM theory.

What we will then do is project out a Z_N symmetry (which means: identify a Z_N symmetry in the theory, and set to zero all fields which aren't neutral under that symmetry). This projection (called an orbifold projection) breaks the gauge symmetry from $U(Nk) \to U(k)^N$, and it breaks half the supersymmetries of the theory, from $\mathcal{Q} = 8$ to $\mathcal{Q} = 4$. That leaves us with the AHCG model.

To see how this works, consider the field content of an $\mathcal{N} = 2$ SYM theory. The gauge multiplet consists of a gauge field v_μ, two Weyl gauginos $\lambda^{(1,2)}$, and a complex scalar ϕ. It is also useful to decode the structure of the $\mathcal{N} = 2$ supersymmetry in terms of $\mathcal{N} = 1$ supersymmetry multiplet as we eventually want to know which supersymmetries survive the projection. The $\mathcal{N} = 2$ matter content diamond shown in Fig. 12 can be decomposed in terms of $\mathcal{N} = 1$ multiplets $V = (v_m, \lambda^{(1)})$, $\Phi = (\phi, \lambda^{(2)})$ and $\mathcal{N} = 1'$ multiplets $V' = (v_m, \lambda^{(2)})$, $\Phi' = (\phi, \lambda^{(1)})$ as shown below:

$$\begin{tikzcd}
& v_m \arrow[dl, leftrightarrow, "\mathcal{N}=1"'] \arrow[dr, leftrightarrow, dotted, "\mathcal{N}=1'"] & \\
\lambda^{(1)} \arrow[dr, leftrightarrow, dotted, "\mathcal{N}=1'"'] & & \lambda^{(2)} \arrow[dl, leftrightarrow, "\mathcal{N}=1"] \\
& \phi &
\end{tikzcd} \tag{124}$$

Note the similarity between this multiplet and the field content appearing in Fig. 11. Each of the fields transforms as the adjoint representation of the gauge group, which in our case is $U(Nk)$; that means we can represent the fields as $Nk \times Nk$ matrices, acted upon by the gauge transformation U as $\phi \to U\phi U^\dagger$ (except for the gauge field, which has the usual inhomogeneous transformation).

But how to define the Z_N symmetry which tells some fields to become site variables and others to become link variables in the AHCG model? The $\mathcal{N} = 2$ SYM action possesses an $SU(2) \times U(1)$ R-symmetry, under which the fields transform as shown in Fig. 12. We can find a symmetry which distinguishes between fields destined to become site variables (v_m and $\lambda^{(1)}$)

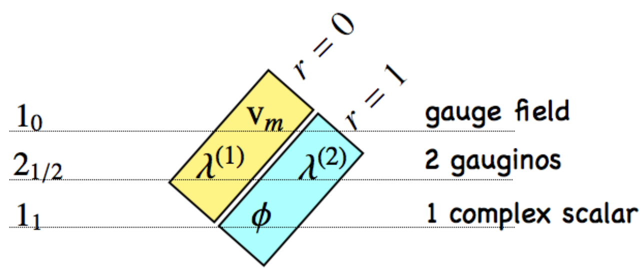

Figure 12: The fields for $\mathcal{N}=2$ SYM theory, along with their $SU(2)\times U(1)$ R-symmetry quantum numbers. The charge $r=(Y-T_3)$ distinguishes which fields become site variables in the AHCG model and which become link variables ($r=0$ and $r=1$ respectively).

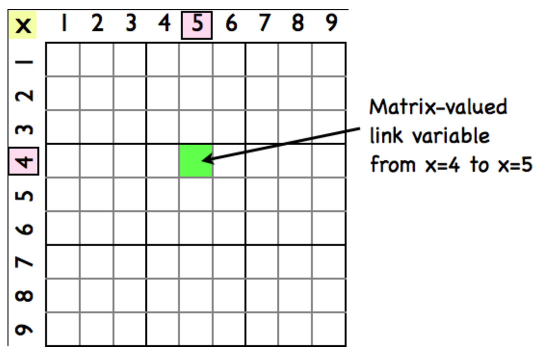

Figure 13: Illustration of how a $9k\times 9k$ matrix can represent a 1D lattice with $N=9$ sites. The highlighted $k\times k$ block represents a $k\times k$ matrix-valued field residing on the directed link from site $x=4$ to site $x=5$.

and link variables ($\lambda^{(2)}$ and ϕ) by defining a $U(1)$ charge r which lives in the $SU(2)\times U(1)$ R-symmetry: $r=Y-T_3$, where Y is the $U(1)$ charge and T_3 is the third $SU(2)$ generator. Then as shown in Fig. 12, site variables have $r=0$ and link variables have $r=1$.

Each of the different types of fields of the AHCG model — each of the N "flavors" of $k\times k$ matrices — can be represented as a single sparse $Nk\times Nk$ matrix, as illustrated in Fig. 13. We think of the big $Nk\times Nk$ matrix as being made of N^2 $k\times k$ blocks, each labeled by a row number n_i and a column number n_f; then that block can be thought of as living on a 1D lattice as a link running from site n_i to site n_f. Thus for the site variables ($r=0$) we want to have an $Nk\times Nk$ matrix with only diagonal $k\times k$ blocks surviving; the link variables ($r=1$) in Fig. 11 should become sparse $Nk\times Nk$ matrices with nonzero blocks only appearing one row above the diagonal.

We can attain the desired result by defining a Z_N symmetry which combines the r symmetry with a particular $U(Nk)$ transformation:

$$Z_N : \Phi \to \hat{\gamma}\Phi \equiv \omega^r \Omega \Phi \Omega^\dagger \ , \qquad \Omega = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^N \end{pmatrix} \ , \qquad \omega = e^{2\pi i/N} \ , \tag{125}$$

where r is the particular r-charge for that field Φ, and each entry in Ω is proportional to a $k \times k$ unit matrix. We then define the orbifold projection operator $\hat{P}\Phi = \frac{1}{N}\sum_{i=p}^{N} \hat{\gamma}^p \Phi$ which annihilates any sub–block in the matrix Φ which is not invariant (this follows from the fact that $[\omega + \omega^2 + \ldots + \omega^N] = 0$). Note that this projection does not commute with the full $U(Nk)$ gauge symmetry of the mother theory and leaves intact only the $U(k)^N$ subgroup which commutes with Ω. The result of this projection is shown in Fig. 14 and can be depicted as in Fig.11, a segment of which is shown below:

$$\xrightarrow{\Phi_{n-2}} V_{n-1} \xrightarrow{\Phi_{n-1}} V_n \xrightarrow{\Phi_n} V_{n+1} \xrightarrow{\Phi_{n+1}} \tag{126}$$

Note that evidently $\hat{P}$ also breaks the $\mathcal{N} = 2$ supersymmetry, since it treats the different members of the gauge multiplet differently. It does, however, preserve an $\mathcal{N} = 1$ supersymmetry, with $V_n = (A_{\mu,n}, \lambda_n^{(1)})$ being an $\mathcal{N} = 1$ vector supermultiplet, and $\Phi_n = (\phi_n, \lambda_n^{(2)}) \equiv (\phi_{n,n+1}, \lambda_{n,n+1}^{(2)})$ forming an $\mathcal{N} = 1$ chiral matter multiplet. The vector multiplets V_n transform as adjoint under the gauge group factor G_n and chiral multiplets Φ_n transform as bi-fundamental $(\Box, \overline{\Box})$ under $G_n \times G_{n+1}$. Thus, in the quiver, the $\mathcal{N} = 1'$ is explicitly violated since the gauge rotation properties of used-to-be $\mathcal{N} = 1'$ multiplet no longer matches as shown below:

$$\begin{tikzcd} & v_{m,n} \arrow[dl, leftrightarrow, "\mathcal{N}=1"'] \arrow[dr, leftrightarrow, dotted, "\text{nothing}"] & \\ \lambda_n^{(1)} \arrow[dr, leftrightarrow, dotted, "\text{nothing}"'] & & \lambda_{n,n+1}^{(2)} \\ & \phi_{n,n+1} \arrow[ur, leftrightarrow, "\mathcal{N}=1"'] & \end{tikzcd} \tag{127}$$

The action of a global supersymmetry transformation of an adjoint cannot produce a bi-fundamental.

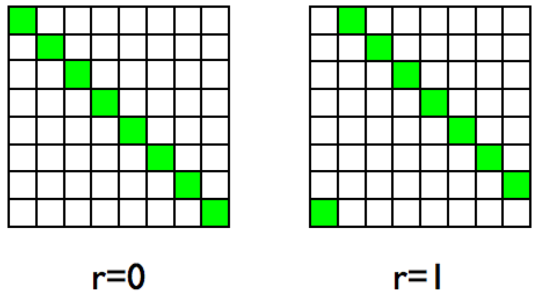

Figure 14: The result of the Z_N orbifold projection: For the fields v_m and $\lambda^{(1)}$ with $r = 0$, only the diagonal $k \times k$ blocks survive, and these can be interpreted as site variables, transforming as adjoints under the unbroken $U(k)^N$ gauge symmetry. The $\lambda^{(2)}$ and ϕ fields with $r = 1$ have only the superdiagonal blocks survive; these transform as bifundamentals under the $U(k)^N$ gauge symmetry, and represent the link variables in Fig 11 (with $\lambda^{(2)} \equiv \psi$).

The punchline: by plugging the sparse matrices obtained after projection back into the $\mathcal{N} = 2$ action, one recovers the full action of the AHCG model! (Also see [123, 124])

It is straightforward now to generalize our orbifold projection prescription in order to construct true lattices, of varying dimensions. For example, to produce a $d = 2$ lattice, we need to start with a mother theory with a $U(N^2k)$ gauge symmetry, and project out a $Z_N \times Z_N$ symmetry. The idea is that we take the $N^2k \times N^2k$ matrices in the mother theory, divide them into N^2 $NK \times Nk$ blocks, and then subdivide those into N^2 $k \times k$ sub-blocks. The location of each $k \times k$ sub-block can then be specified by four integers; the interpretation is that this is a link variable going from one site on a 2D lattice (specified by two integers) to another (specified by another two integers); see Fig. 15.

We now have a method for generating supersymmetric lattice actions:

i. Start with a mother theory which is an SYM with the same number of supercharges $\mathcal{Q}$ as the target theory in the continuum;

ii. This mother theory should be formulated in *zero* dimensions (in other words: it is a matrix model, not a field theory), since we don't want any continuous dimensions, unlike the AHCG model which was formulated in $d = 4$;

iii. For a target theory with d continuous dimensions, make the gauge group of the mother theory $U(N^dk)$, identify the appropriate Z_N^d symmetry

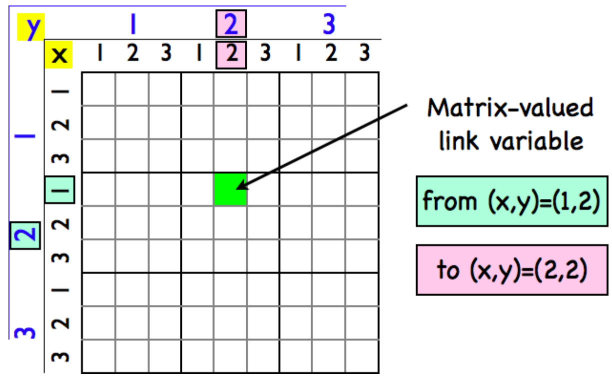

Figure 15: Representing a two-dimensional 3×3 lattice by a sparse 9×9 matrix: each sub-block can be identified as a site or link variable on the 2D lattice.

that resides partly in the gauge group and partly in the R-symmetry group of the mother theory, and project it out;

iv. Travel out along the flat direction in the degenerate vacua as in eq. (117), in order to recover the continuum limit of the target theory.

Oddly enough, this diabolical recipe really works! And in fact, it has shown that all the different constructions of lattice SYM theories in the literature can be shown to be equivalent to ones obtained through orbifold projection [40]. As with all pacts with the devil there is a price: item (i) and item (iii) above are not in general compatible, since a theory with a small number of supercharges will have a small R-symmetry which will not contain a Z_N^d subgroup for large d. Equivalently, since each dimension requires a Z_N projection which breaks half of the remaining supercharges of the mother theory (and since we want the lattice theory to possess at least one unbroken supercharge) we require $\mathcal{Q} \geq 2^d$. Thus to go to higher dimension d, one needs to consider highly supersymmetric theories with large $\mathcal{Q}$. For $d = 4$, the only supersymmetric lattice that can be constructed via this method must have $\mathcal{Q} \geq 16$, leaving $\mathcal{N} = 4$ SYM theory as the only possibility.

7.4. Orbifold Lattice Theory for $\mathcal{N} = (2, 2)$ SYM

We now briefly describe the construction of the four supercharge theory in two dimensions which was previously discussed from within the twisted approach. The action for this theory is easy to write down: start with the familiar $\mathcal{N} = 1$ SYM theory in $d = 4$ dimensions (a gauge theory with a massless Weyl adjoint fermion), and erase two of the space dimensions. The gaugino becomes a 2-component Dirac fermion ψ (since γ matrices in $d=2$ are just Pauli matrices, Dirac spinors have only two components). The four component gauge boson in $d=4$ becomes a two component gauge boson plus one complex scalar field s. The gluon and gaugino interactions in the $d=4$ action become 2D gauge interactions, plus Yukawa and s^4 interactions. The result is the action (in Euclidean spacetime)

$$\mathcal{L} = \frac{1}{g_2^2}\,\mathrm{Tr}\left(\left|D_m s\right|^2 + \bar\psi\, iD_m\gamma_m\psi + \tfrac{1}{4}v_{mn}v_{mn} + i\sqrt{2}(\bar\psi_L[s,\psi_R] + \bar\psi_R[s^\dagger,\psi_L]) + \tfrac{1}{2}[s^\dagger, s\,]^2\right) \tag{128}$$

where $m,n=1,2$, ψ_R and ψ_L are the right- and left-chiral components of a two-component Dirac field ψ, $D_m = \partial_m + i[v_m,\,\cdot\,]$ is the covariant derivative, and $v_{mn} = -i[D_m, D_n]$ is the field strength. All fields are rank-k matrices transforming as the adjoint representation of $U(k)$. This is the target theory for which we want to construct a lattice.

To construct a lattice for this target theory, we need to start with a matrix theory with a $U(N^2k)$ gauge symmetry with $\mathcal{Q}=4$ supersymmetries. The way to obtain the $\mathcal{Q}=4$ matrix theory is simple: Start with the same $\mathcal{N}=1$ SYM theory in $d=4$, which we know has $\mathcal{Q}=4$ supersymmetries...and then erase *all* spacetime coordinates from the action (and therefore, all derivatives). The result is a very simple action which will serve as our mother theory:

$$S = \frac{1}{g^2}\left(\frac{1}{4}\mathrm{Tr}\; v_{mn}v_{mn} + \mathrm{Tr}\;\bar\psi\,\bar\sigma_m[v_m,\psi]\right)\ , \tag{129}$$

where $m,n=0,\ldots,3$, ψ and $\bar\psi$ are independent complex two-component spinors, v_m is the 4-vector of constant gauge potentials, and

$$v_{mn} = i[v_m, v_n]\ , \quad \sigma_m = \{1,\, -i\boldsymbol{\sigma}\}\ , \quad \bar\sigma_m = \{1,\, i\boldsymbol{\sigma}\}\ , \tag{130}$$

This mother theory is invariant under four independent supersymmetries, characterized by the transformations

$$\delta v_m = -i\bar\psi\,\bar\sigma_m\kappa + i\bar\kappa\,\bar\sigma_m\psi\ , \quad \delta\psi = -iv_{mn}\sigma_{mn}\kappa\ , \quad \delta\bar\psi = iv_{mn}\,\bar\kappa\,\bar\sigma_{mn}\ , \tag{131}$$

where

$$\sigma_{mn} \equiv \tfrac{i}{4}\left(\sigma_m\bar\sigma_n - \sigma_n\bar\sigma_m\right)\ , \qquad \bar\sigma_{mn} \equiv \tfrac{i}{4}\left(\bar\sigma_m\sigma_n - \bar\sigma_n\sigma_m\right)\ . \tag{132}$$

where κ and $\bar{\kappa}$ are independent two-component Grassmann parameters.

The R-symmetry of the mother theory is $SO(4)\times U(1) = SU(2)\times SU(2)\times U(1)$. This result is not very mysterious: the $U(1)$ factor is just the $U(1)$ R-symmetry associated with the $d = 4$ $\mathcal{N} = 1$ SYM theory we started with to derive the mother theory. The $SO(4) = SU(2) \times SU(2)$ factor is nothing but what remains of the (Euclidean) Lorentz symmetry that remains even after all spacetime coordinates are removed from the $d = 4$ theory. Therefore v_m transforms as a 4-vector $= (2, 2)$ under this $SU(2) \times SU(2)$, while ψ transforms as a $(2, 1)$ and $\bar{\psi}$ as a $(1, 2)$.

The "daughter theory" we will derive from this mother theory by orbifolding will be a two-dimensional lattice with N^2 sites and a $U(k)$ symmetry associated with each site (the conventional way to realize a $U(k)$ gauge symmetry on a lattice). To obtain this daughter theory we must identify the correct $Z_N \times Z_N$ symmetry to project out. The trick is to define two independent analogues of the r-charge from the previous section — We will call them $\vec{r} = \{r_1, r_2\}$. This vector $\vec{r}$ is interpreted as the directed link in the unit cell on which a given variable resides. For example, $\vec{r} = \{0, 0\}$, $\vec{r} = \{1, 0\}$ and $\vec{r} = \{1, 1\}$ are interpreted as a site, an x-link, and a diagonal link respectively. As such, we need to define the $Z_N \times Z_N$ symmetry so that $\vec{r}$ components only take on the values $0, \pm 1$. Furthermore, one can show that the number of unbroken supercharges on the lattice equals the number of fermions with $\vec{r} = \{0, 0\}$ (e.g., living on the sites), and so we want to choose the $Z_N \times Z_N$ symmetry to maximize this number. With a little work, it is possible to show that a suitable choice yields the charge assignments displayed in Table 4 [32], where we have written the fermion components as

$$\psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \qquad \bar{\psi} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \end{pmatrix} \tag{133}$$

This choice is unique up to uninteresting permutations. We can then use these r-charges to define a $Z_N \times Z_N$ projection which creates the lattice shown in Fig. 16. Note that the placement of each degree of freedom in this figure is simply determined by the corresponding $\vec{r}$ charge appearing in Table 4.

We won't give any of the details here, but it is not too difficult to construct the lattice action by substituting the orbifold projected matrices back into the action of the mother theory, eq. (130). One then follows the path of

bosons	r_1	r_2
$z_1 = \frac{v_0 - i v_3}{\sqrt{2}}$	1	0
$\bar{z}_1 = \frac{v_0 + i v_3}{\sqrt{2}}$	-1	0
$z_2 = -i\frac{v_1 - i v_2}{\sqrt{2}}$	0	1
$\bar{z}_2 = i\frac{v_1 + i v_2}{\sqrt{2}}$	0	-1

fermions	r_1	r_2
λ_1	0	0
λ_2	-1	-1
$\bar{\lambda}_1$	1	0
$\bar{\lambda}_2$	0	1

Table 4: *Assignment of the $Z_N \times Z_N$ charges for the variables of the mother theory eq.* (129)*; see [32] for details.*

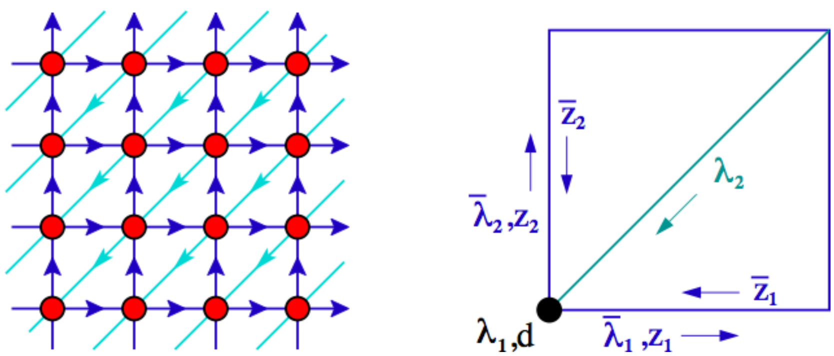

Figure 16: The lattice structure and the unit cell for the target theory of eq. (128), $(2, 2)$ SYM in two dimensions. The "d" variable is an auxiliary field you can ignore; it proves to be convenient when developing a superfield formulation for the lattice theory.

deconstruction, expanding the boson fields as

$$z_i = \frac{1}{a\sqrt{2}}\mathbf{1}_k + \frac{s_i + i v_i}{\sqrt{2}} \tag{134}$$

and taking the continuum limit $a \to 0$ with $g^2 a^2 = g_2^2$ kept fixed. Amazingly enough, one finds the target theory eq. (128) in this limit, with the identification

$$s = \frac{s_1 + i s_2}{\sqrt{2}}\,, \qquad \psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\,, \qquad \bar{\psi} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \end{pmatrix}\,, \qquad v_m = \begin{pmatrix} v_1 & v_2 \end{pmatrix}\,. \tag{135}$$

So what about the list of obstructions mentioned at the end of §4? How does this theory get around them? Well, for one thing, the conundrum of satisfying $\{Q, \bar{Q}\} = \gamma.P$ when there is no P operator on the lattice is circumvented by the fact that we have a Q charge on our lattice, but no corresponding $\bar{Q}$! This follows because our construction leads to a single site fermion λ_1 (which has

$\vec{r} = \{0,0\}$) but no corresponding site variable $\bar{\lambda}_1$. This is one of the funny things about supersymmetry in Euclidean spacetime: it is possible to have a theory respecting a single supercharge, which is impossible in Minkowski space. This feature is related to the strange property of fermions continued to Euclidean space, that $\bar{\psi}$ is not related to the hermitean conjugate of ψ, i.e, $\bar{\psi} \neq \psi^\dagger \gamma_0$.

Other questions raised in §4 remain to be answered: for example, the target theory has a chiral $U(1)$ R-symmetry which is exact up to anomalies; how does this symmetry arise in the lattice theory? Did we invent a new type of lattice chiral fermion? Also, we are claiming that the scalar s in the target theory is represented on the lattice by s_1 and s_2 (the real parts of $z_{1,2}$) which are link variables; this means that even though s_1 and s_2 transform into each other non trivially under lattice rotations, they must be invariant under rotations in the continuum! Isn't this absurd, since the continuum rotations contain lattice rotations as a subgroup, and an object transforming non trivially under the latter must transform non trivially under the former?

To understand what is going on, let us first focus on the quadratic part of the boson action, which looks like:

$$\begin{aligned}
&\frac{1}{2g^2a^2}\sum_{\mathbf{n}} \mathrm{Tr}\ \Big[\left(s_{1,\mathbf{n}-\hat{\mathbf{x}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}-\hat{\mathbf{y}}} - s_{2,\mathbf{n}}\right)^2 \\
&\qquad\qquad + \left| \left(s_{1,\mathbf{n}+\hat{\mathbf{y}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}} - s_{2,\mathbf{n}+\hat{\mathbf{x}}}\right) - i\left(v_{1,\mathbf{n}+\hat{\mathbf{y}}} - v_{1,\mathbf{n}} - v_{2,\mathbf{n}+\hat{\mathbf{x}}} + v_{2,\mathbf{n}}\right)\right|^2 \Big] \\
&= \frac{1}{2g^2}\sum_{\mathbf{n}} \mathrm{Tr}\ \left[\sum_{\hat{\boldsymbol{\mu}}}\sum_{i=1,2} \left(\frac{s_{i,\mathbf{n}} - s_{i,\mathbf{n}-\hat{\boldsymbol{\mu}}}}{a}\right)^2 + \left(\frac{v_{1,\mathbf{n}+\hat{\mathbf{y}}} - v_{1,\mathbf{n}}}{a} - \frac{v_{2,\mathbf{n}+\hat{\mathbf{x}}} - v_{2,\mathbf{n}}}{a}\right)^2\right] ,
\end{aligned} \tag{136}$$

When we take the continuum limit, we get

$$\frac{1}{g_2^2}\int d^2x\, \tfrac{1}{2}\mathrm{Tr}\ \left[(\partial_1 s_1 + \partial_2 s_2)^2 + (\partial_2 s_1 - \partial_1 s_2)^2 + (\partial_2 v_1 - \partial_1 v_2)^2\right] . \tag{137}$$

Note that the first two terms make (s_1, s_2) look like a vector (as you would expect from link variables!) rather than components of scalar: the first term looks like $(\vec{\nabla}\cdot\vec{s})^2$, while the second term looks like $(\vec{\nabla}\times\vec{s})^2$; neither term looks like the scalar kinetic term $[(\partial_m s_1)^2 + (\partial_m s_2)^2]$...yet amazingly enough, when you add the two terms and integrate by parts, that is *exactly* what you get! Not only do we get the correct $SO(2)$ Euclidean "Lorentz" invariance with s_i

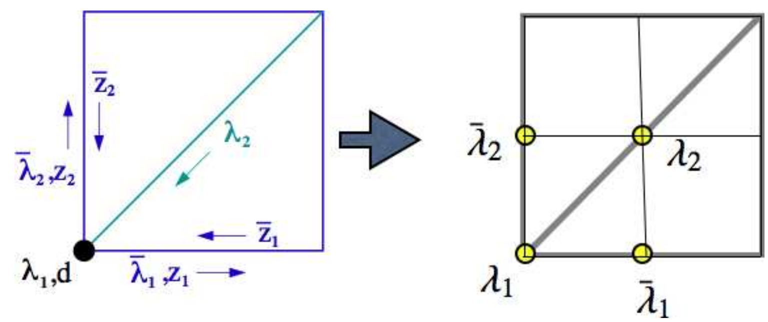

Figure 17: The fermions mapped onto a lattice with half the spacing can be recognized as reduced staggered fermions.

being invariant, but we get an *independent* internal $SO(2)$ symmetry where the s_i rotate into each other while the derivatives ∂_m remain unchanged. The latter $SO(2) = U(1)$ is just the R-symmetry's action on the scalar s!

If we turn to the quadratic part of the fermion action, we find something more familiar. If one takes our rather unconventional lattice, and superimpose upon it a lattice with spacing $a/2$, the fermions can all be mapped onto sites of this finer lattice, as shown in Fig. 17. Examining the lattice action for these fermions in the coordinates of this sublattice, one discovers that the fermions are none other than "reduced staggered fermions" as discussed in [125]. Again, you might wonder how a collection of fermions scattered over different parts of the lattice could reassemble themselves into a continuum spinor; it seems as mysterious as how our link bosons became a complex scalar. Understanding these features goes a long way toward explaining how the obstacles facing lattice supersymmetry have been circumvented by this orbifold projection technique.

7.5. Fine tuning

We will finish up this section with a brief discussion about quantum corrections in our lattice theory for $(2, 2)$ SYM. Recall that the goal of a supersymmetric lattice action was to prevent unwanted relevant or marginal operators from being radiatively generated which could spoil supersymmetry in the the continuum limit. The single exact lattice supercharge is enough to protect the lattice theory from unwanted radiatively induced operators which could spoil the supersymmetric continuum limit of the lattice theory, just as we hoped. To show this we can construct the *Symanzik action* for the theory: we expand the z variables about the flat direction $\langle z \rangle = \mathbf{1}_k/a\sqrt{2}$, expand the action for smooth fields in powers of $1/a$, include all operators allowed by the exact symmetries of the lattice, and then consider radiative corrections to the coefficients of these operators, paying special attention to relevant and marginal operators which violate the full $\mathcal{Q} = 4$ supersymmetry of the target theory, and whose coefficients by definition do not vanish in the $a \to 0$ limit. The key is to identify all the operators allowed by the exact lattice symmetries, which include the single supersymmetry. This is most easily done by constructing superfields: we introduce a Grassmann coordinate θ, which has mass dimension 1/2 (where spacetime coordinates x have mass dimension -1), and define the exact lattice supercharge to be $Q = \partial_\theta$. With this definition of Q, and knowing the action of Q on the lattice variables, it is possible to construct superfields as is done in the more familiar $d = 4$, $\mathcal{N} = 1$ supersymmetry [1]. One finds the following superfields on the unit cell at site $\mathbf{n}$:

$$\begin{aligned}
Z_1(\mathbf{n}) &= z_1(\mathbf{n}) + \sqrt{2}\theta\bar{\lambda}_1(\mathbf{n}) \ , \\
Z_2(\mathbf{n}) &= z_2(\mathbf{n}) + \sqrt{2}\theta\bar{\lambda}_2(\mathbf{n}) \ , \\
\Xi(\mathbf{n}) &= \lambda_2(\mathbf{n}) + 2\left[\bar{z}_1(\mathbf{n}+\hat{\mathbf{y}})\bar{z}_2(\mathbf{n}) - \bar{z}_2(\mathbf{n}+\hat{\mathbf{x}})\bar{z}_1(\mathbf{n})\right]\theta \ , \\
\Lambda(\mathbf{n}) &= \lambda_1(\mathbf{n}) - \left[\bar{z}_1(\mathbf{n}-\hat{\mathbf{x}})z_1(\mathbf{n}-\hat{\mathbf{x}}) - z_1(\mathbf{n})\bar{z}_1(\mathbf{n})\right. \\
&\qquad \left. +\bar{z}_2(\mathbf{n}-\hat{\mathbf{y}})z_2(\mathbf{n}-\hat{\mathbf{y}}) - z_2(\mathbf{n})\bar{z}_2(\mathbf{n}) + id(\mathbf{n})\right]\theta \ .
\end{aligned} \tag{138}$$

Since $Q = \partial_\theta$, the most general supersymmetric action can be written as

$$\frac{1}{g_2^2}\int d\theta \int d^2x \sum_{\mathcal{O}} C_{\mathcal{O}}\mathcal{O}(x,\theta) \tag{139}$$

where the $\mathcal{O}$ are local Grassmann operators. This expression is obviously annihilated by Q since it doesn't depend on θ. Since the action has to be dimensionless, if $\mathcal{O}$ has mass dimension p, it is easy to check that the operator coefficient C_O must have dimension $(7/2 - p)$. Now, since the action has a $1/g_2^2$ out front (where g_2 has mass dimension 1), radiative corrections to $C_{\mathcal{O}}$ at ℓ loops will be of the form

$$\delta C_{\mathcal{O}} \sim c_\ell a^{(p-7/2)}(g_2^2a^2)^\ell \ , \tag{140}$$

where the c_ℓ are dimensionless coefficients and can only depend on a logarithmically. Since we only care about operator coefficients which do not vanish as $a \to 0$, we need only consider operators and loops satisfying $p \le (7/2 - 2\ell)$. At $\ell = 0$ (tree level) we claim our lattice action gives the correct target theory in the continuum limit. At $\ell = 1$ we need to consider $p \le 3/2$; it turns out we cannot construct operators with $p \le 1/2$ so that's it! It is then a quick job to convince oneself that there are no bad operators $\mathcal{O}$ with $p = 3/2$ which one can construct. Therefore, we can prove that the supersymmetric lattice does what it was supposed to do: it allows one to realize the supersymmetric target theory without fine-tuning. The exact supersymmetry of the lattice was crucial for this to be possible. For example, in a non-supersymmetric lattice formulation, scalar mass terms are permitted and needs to be fine-tuned, which is forbidden in a lattice theory with exact supersymmetry. We refer interested readers to ref. [32] for details of the argument. The analysis for this theory was simplified by the fact that it is "super-renormalizable", namely that each loop correction introduced positive powers of a. In §8.4, we briefly discuss renormalization of a $d = 4$ theory, in which divergent contributions may arise at arbitrary loop order.

7.6. Other supersymmetric lattices

So what are the supersymmetric lattices we have constructed to date? SYM theories exist in $d \ge 2$ with $\mathcal{Q} = 2, 4, 8, 16$. Since each dimension requires projecting out a Z_N factor, and each projection costs one half of the remaining supersymmetries of the mother theory, and we want at least one unbroken supercharge on the lattice, we can only consider SYM theories with $\mathcal{Q} \ge 2^d$. That constrains us to

$$\begin{aligned} \mathcal{Q} = 4: \quad d &= 2 \\ \mathcal{Q} = 8: \quad d &= 2, 3 \\ \mathcal{Q} = 16: \quad d &= 2, 3, 4 \ , \end{aligned} \tag{141}$$

and all of these lattice have been constructed. As we have discussed the $\mathcal{Q} = 16$ theories are especially interesting and have especially symmetric lattices, shown in Fig. 18.

The $d = 1$ lattices for $\mathcal{Q} = 16$ SYM give an alternative to the naive lattice actions we introduced before for simulating $\mathcal{Q} = 16$ quantum mechanics. In addition to pure SYM theories, a lattice for $(2, 2)$ SYM has also been constructed with certain classes of matter fields [35], which we discuss next.

7.7. Addition of matter to $(2, 2)$ theories

The supersymmetric lattice construction of the last section can be generalized to include charged matter fields interacting via a superpotential [35].

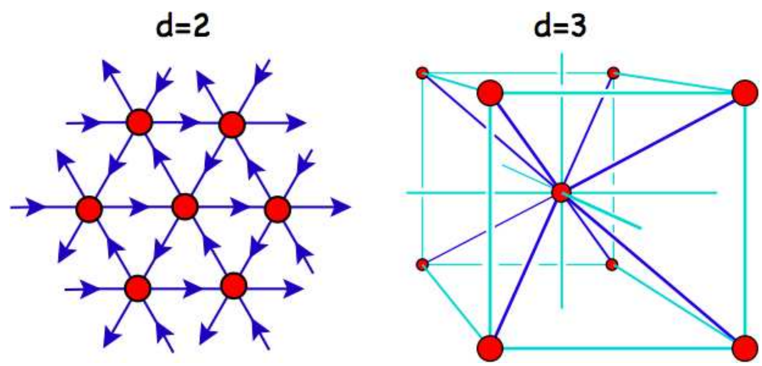

Figure 18: The lattices for $\mathcal{Q} = 16$ supersymmetry in $d = 2$ and $d = 3$ dimensions [34]. In $d = 4$ the lattice for $\mathcal{N} = 4$ SYM has the A_4^* lattice structure.

In [126] Giedt extended this construction to $(4, 4)$ theories with matter. Such theories are lower dimensional counterparts of super QCD in $d = 4$ dimensions. More recently orbifold/twisted constructions have been obtained with matter in the fundamental representation [127, 128]. These are very interesting as they open up the possibility of defining phenomenologically more realistic models with exact supersymmetry.

8. $\mathcal{N} = 4$ SYM in four dimensions

In this section we discuss perhaps one of the most interesting applications of these ideas – the construction of a lattice model invariant under a single exact supersymmetry whose naive continuum limit is $\mathcal{N} = 4$ SYM theory in four dimensions. This gauge theory is thought to be dual to type IIB string theory in $AdS^5 \times S^5$ space. In the large 't Hooft coupling limit, it is conjectured to be describe the supergravity limit of that string theory. The lattice theory constitutes the only known example of a supersymmetric lattice model in four dimensions[16]. We first summarize the construction of the supersymmetric orbifold action for this model, then go on to re-derive it by discretization of an appropriate twist of the continuum theory.

[16]Recently a paper has appeared which shows how a non-supersymmetric formulation of $\mathcal{N} = 4$ SYM can be constructed with domain wall fermions which has significantly reduced fine-tuning compared to what one might expect, and which may be numerically tractable [129]; see also [130]. A alternative regularization for $\mathcal{N} = 4$ SYM in the planar limit was given in [53]

8.1. Orbifold action

The strategy to obtain the supersymmetric lattice action for the $\mathcal{N} = 4$ target theory is four-dimensional generalization of the one given for $\mathcal{N} = (2,2)$ model in two dimensions. As discussed in §7.2, to build the $\mathcal{N} = (2,2)$ model we orbifolded the $\mathcal{Q} = 4$ mother matrix theory. This mother theory possesses an $SO(4)\times U(1)$ R-symmetry group with a maximal abelian $U(1)^3$ subgroup which allowed us to build a lattice theory in two dimensions.

To obtain the target $\mathcal{N} = 4$ target theory in four dimensions, we start with $\mathcal{Q} = 16$ matrix model. The matrix model may be obtained by dimensionally reducing the $d = 10$ dimensional $\mathcal{N} = 1$ SYM theory down to $d = 0$ dimensions. The reduced model possesses $SO(10)$ R-symmetry inherited from the Lorentz symmetry of the $d = 10$ dimensional theory prior to reduction. The field content of the mother theory is ten bosonic and sixteen Grassmann odd fermionic matrices transforming as **10** and **16** representation of the R-symmetry, and in the adjoint representation of the gauge group.

To proceed, it is more convenient to decompose the variables of the mother theory under the $SU(5) \times U(1)$ subgroup of $SO(10)$.

$$
\begin{aligned}
&\text{bosons} : \mathbf{10} \rightarrow \mathbf{5} \oplus \overline{\mathbf{5}} = z^m \oplus \overline{z}_m \\
&\text{fermions} : \mathbf{16} \rightarrow \mathbf{1} \oplus \mathbf{5} \oplus \overline{\mathbf{10}} = \lambda \oplus \psi^m \oplus \xi_{mn}.
\end{aligned} \tag{142}
$$

Written in terms of this $SU(5) \times U(1)$ decomposition, the action of the type IIB matrix theory becomes

$$
\begin{aligned}
S \;=\; \frac{1}{g^2}\mathrm{Tr}\,\Big[&\sum_{m,n}\left(\frac{1}{2}[\overline{z}_m, z^m][\overline{z}_n, z^n] + [z^m, z^n][\overline{z}_n, \overline{z}_m]\right) \\
&+\sqrt{2}\left(\lambda[\overline{z}_m, \psi^m] - \xi_{mn}[z^m, \psi^n] + \frac{1}{8}\epsilon^{mnpqr}\xi_{mn}[\overline{z}_p, \xi_{qr}]\right)\Big]
\end{aligned} \tag{143}
$$

Below, we employ the $U(1)^5$ abelian subgroup of the R-symmetry group to generate the four dimensional lattice with one exact supersymmetry. As usual, the starting point is the mother theory with $U(N^d k)$ gauge group with $d = 4$. An orbifold projection by $(Z_N)^4$ symmetry generates the four dimensional lattice. The ten bosonic and sixteen fermionic lattice fields, their charges under $U(1)^5$ and their associated $\mathbf{r}$-charges (which determines the position and orientation of each lattice field on the unit cell) are given in Table 5. In Table 5, we also define five $\boldsymbol{\mu}_m$ vectors which will be used to specify the $\mathbf{r}$-charges directly in terms of $SU(5)$ tensor indices. For the further details of this procedure, we refer to [34] for details.

	$2Q_0$	$2q_1$	$2q_2$	$2q_3$	$2q_4$	$2q_5$	$\mathbf{r}$
z_1	2	2	0	0	0	0	$\{1,0,0,0\} = \boldsymbol{\mu}_1$
z_2	2	0	2	0	0	0	$\{0,1,0,0\} = \boldsymbol{\mu}_2$
z_3	2	0	0	2	0	0	$\{0,0,1,0\} = \boldsymbol{\mu}_3$
z_4	2	0	0	0	2	0	$\{0,0,0,1\} = \boldsymbol{\mu}_4$
z_5	2	0	0	0	0	2	$\{-1,-1,-1,-1\} = \boldsymbol{\mu}_5$
$\bar{z}_1$	-2	-2	0	0	0	0	$\{-1,0,0,0\} = -\boldsymbol{\mu}_1$
$\bar{z}_2$	-2	0	-2	0	0	0	$\{0,-1,0,0\} = -\boldsymbol{\mu}_2$
$\bar{z}_3$	-2	0	0	-2	0	0	$\{0,0,-1,0\} = -\boldsymbol{\mu}_3$
$\bar{z}_4$	-2	0	0	0	-2	0	$\{0,0,0,-1\} = -\boldsymbol{\mu}_4$
$\bar{z}_5$	-2	0	0	0	0	-2	$\{1,1,1,1\} = -\boldsymbol{\mu}_5$
λ	5	1	1	1	1	1	$\{0,0,0,0\} = \mathbf{0}$
ψ_1	-3	1	-1	-1	-1	-1	$\{1,0,0,0\} = \boldsymbol{\mu}_1$
ψ_2	-3	-1	1	-1	-1	-1	$\{0,1,0,0\} = \boldsymbol{\mu}_2$
ψ_3	-3	-1	-1	1	-1	-1	$\{0,0,1,0\} = \boldsymbol{\mu}_3$
ψ_4	-3	-1	-1	-1	1	-1	$\{0,0,0,1\} = \boldsymbol{\mu}_4$
ψ_5	-3	-1	-1	-1	-1	1	$\{-1,-1,-1,-1\} = \boldsymbol{\mu}_5$
ξ_{12}	1	-1	-1	1	1	1	$\{-1,-1,0,0\} = -\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$
ξ_{13}	1	-1	1	-1	1	1	$\{-1,0,-1,0\} = -\boldsymbol{\mu}_1 - \boldsymbol{\mu}_3$
ξ_{14}	1	-1	1	1	-1	1	$\{-1,0,0,-1\} = -\boldsymbol{\mu}_1 - \boldsymbol{\mu}_4$
ξ_{23}	1	1	-1	-1	1	1	$\{0,-1,-1,0\} = -\boldsymbol{\mu}_2 - \boldsymbol{\mu}_3$
ξ_{24}	1	1	-1	1	-1	1	$\{0,-1,0,-1\} = -\boldsymbol{\mu}_2 - \boldsymbol{\mu}_4$
ξ_{34}	1	1	1	-1	-1	1	$\{0,0,-1,-1\} = -\boldsymbol{\mu}_3 - \boldsymbol{\mu}_4$
ξ_{15}	1	-1	1	1	1	-1	$\{0,1,1,1\} = -\boldsymbol{\mu}_1 - \boldsymbol{\mu}_5$
ξ_{25}	1	1	-1	1	1	-1	$\{1,0,1,1\} = -\boldsymbol{\mu}_2 - \boldsymbol{\mu}_5$
ξ_{35}	1	1	1	-1	1	-1	$\{1,1,0,1\} = -\boldsymbol{\mu}_3 - \boldsymbol{\mu}_5$
ξ_{45}	1	1	1	1	-1	-1	$\{1,1,1,0\} = -\boldsymbol{\mu}_4 - \boldsymbol{\mu}_5$

Table 5: The Q_0, q_m and $r_\mu = (q_\mu - q_5)$ charges of the bosonic variables v and fermionic variables ω of the $\mathcal{Q} = 16$ mother theory under the $SO(10) \supset SU(5)$ decomposition $v = 10 \to 5 \oplus \bar{5} = z^m \oplus \bar{z}_m$, and $\omega = 16 \to 1 \oplus 5 \oplus \overline{10} = \lambda \oplus \psi^m \oplus \xi_{mn}$.

The action of the lattice gauge theory that results from the orbifold projection may be written in component form as [34]

$$
\begin{aligned}
S = \frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\, \Bigg[& \tfrac{1}{2}\Big(\sum_{m=1}^{5} \left(\bar{z}_m(\mathbf{n}-\boldsymbol{\mu}_m) z^m(\mathbf{n}-\boldsymbol{\mu}_m) - z^m(\mathbf{n})\bar{z}_m(\mathbf{n})\right)\Big)^2 \\
& + \sum_{m,n=1}^{5} \Big| z^m(\mathbf{n}) z^n(\mathbf{n}+\boldsymbol{\mu}_m) - z^n(\mathbf{n}) z^m(\mathbf{n}+\boldsymbol{\mu}_n)\Big|^2 \\
& - \sqrt{2}\Big(\Delta_{\mathbf{n}}(\lambda, \bar{z}_m, \psi^m) + \Delta_{\mathbf{n}}(\xi_{mn}, z^m, \psi^n) + \frac{1}{8}\epsilon^{mnpqr}\Delta_{\mathbf{n}}(\xi_{mn}, \bar{z}_p, \xi_{qr})\Big)\Bigg]
\end{aligned}
\tag{144}
$$

We have introduced the labeling convention that $z^m(\mathbf{n})$, $\psi^m(\mathbf{n})$ and $\bar{z}_m(\mathbf{n})$ live on the same link, running between site $\mathbf{n}$ and site $(\mathbf{n}+\boldsymbol{\mu}_m)$; similarly $\xi_{mn}(\mathbf{n})$ lives on the link between sites $\mathbf{n}$ and $(\mathbf{n}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n)$, while $\lambda(\mathbf{n})$ resides at the site $\mathbf{n}$. The site vector $\mathbf{n}$, a four-vector with integer-valued components, should be distinguished from $SU(5)$ indices n.

We have introduced the triangular plaquette function $\Delta_{\mathbf{n}}$ defined as:

$$
\begin{aligned}
\Delta_{\mathbf{n}}(\lambda, \bar{z}_m, \psi^m) =& -\lambda(\mathbf{n})\Big(\bar{z}_m(\mathbf{n}-\boldsymbol{\mu}_m)\psi^m(\mathbf{n}-\boldsymbol{\mu}_m) - \psi^m(\mathbf{n})\bar{z}_m(\mathbf{n})\Big)\ , \\
\Delta_{\mathbf{n}}(\xi_{mn}, z^m, \psi^n) =& \xi_{mn}(\mathbf{n})\Big(z^m(\mathbf{n})\psi^n(\mathbf{n}+\boldsymbol{\mu}_m) - \psi^n(\mathbf{n}) z^m(\mathbf{n}+\boldsymbol{\mu}_n)\Big)\ , \\
\Delta_{\mathbf{n}}(\xi_{mn}, \bar{z}_p, \xi_{qr}) =& -\xi_{mn}(\mathbf{n})\Big(\bar{z}_p(\mathbf{n}-\boldsymbol{\mu}_p)\xi_{qr}(\mathbf{n}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n) \\
& - \xi_{qr}(\mathbf{n}-\boldsymbol{\mu}_q-\boldsymbol{\mu}_r)\bar{z}_p(\mathbf{n}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n)\Big)
\end{aligned}
\tag{145}
$$

Note that Δ corresponds to the signed sum of two terms, each of which is a string of three variables along a closed and oriented path on the lattice, with the sign determined by the orientation of the path. As discussed in § 2, there is a $U(k)$ gauge symmetry associated with each site, with $\lambda(\mathbf{n})$ transforming as an adjoint, while the oriented link variables transform as bifundamentals under the two $U(k)$ groups associated with the originating and destination sites of the link. A string of variables along any closed path on the lattice, such as we see in the definition of Δ, is gauge invariant. In the continuum limit, the Δ terms will form the gaugino hopping terms and Yukawa couplings of the $\mathcal{Q} = 16$ SYM theory.

It is now simple to write down the action for the lattice theory that results from the orbifold projection, in a form which is manifestly $\mathcal{Q} = 1$ supersymmetric.

After orbifold projection, there are superfields associated with each lattice site $\mathbf{n}$, where $\mathbf{n}$ is a four component vector of integers, each component ranging from 1 to N:

$$\begin{aligned}
\mathbf{Z}^m(\mathbf{n}) &= z^m(\mathbf{n}) + \sqrt{2}\theta\psi^m(\mathbf{n}) \\
\mathbf{\Lambda}(\mathbf{n}) &= \lambda(\mathbf{n}) - \theta i d(\mathbf{n}) \\
\mathbf{\Xi}_{mn}(\mathbf{n}) &= \xi_{mn}(\mathbf{n}) - 2\theta\left[\,\overline{z}_m(\mathbf{n}+\boldsymbol{\mu}_n)\overline{z}_n(\mathbf{n}) - \overline{z}_n(\mathbf{n}+\boldsymbol{\mu}_m)\overline{z}_m(\mathbf{n})\right]
\end{aligned} \tag{146}$$

In addition there is the singlet field $\overline{z}_m(\mathbf{n})$. In the above expressions, subscripts and superscripts $m, n = 1, \ldots, 5$ and repeated indices are summed over. Note that the superfields are not entirely local, and that in the continuum they will depend on derivatives of fields as well as the fields themselves.

The lattice action we obtained may be written in manifestly $\mathcal{Q} = 1$ supersymmetric form as

$$\begin{aligned}
S &= \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{n}}\int d\theta\,\Big(-\frac{1}{2}\mathbf{\Lambda}(\mathbf{n})\partial_\theta\mathbf{\Lambda}(\mathbf{n}) - \mathbf{\Lambda}(\mathbf{n})\Big[\overline{z}_m(\mathbf{n}-\boldsymbol{\mu}_m)\mathbf{Z}^m(\mathbf{n}-\boldsymbol{\mu}_m) - \mathbf{Z}^m(\mathbf{n})\overline{z}_m(\mathbf{n})\Big] \\
&\qquad + \frac{1}{2}\mathbf{\Xi}_{mn}(\mathbf{n})\Big[\mathbf{Z}^m(\mathbf{n})\mathbf{Z}^n(\mathbf{n}+\boldsymbol{\mu}_m) - \mathbf{Z}^n(\mathbf{n})\mathbf{Z}^m(\mathbf{n}+\boldsymbol{\mu}_n)\Big]\Big) \\
&\quad + \frac{\sqrt{2}}{8}\epsilon^{mnpqr}\mathbf{\Xi}_{mn}(\mathbf{n})\Big[\overline{z}_p(\mathbf{n}-\boldsymbol{\mu}_p)\mathbf{\Xi}_{qr}(\mathbf{n}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n) - \mathbf{\Xi}_{qr}(\mathbf{n}-\boldsymbol{\mu}_q-\boldsymbol{\mu}_r)\overline{z}_p(\mathbf{n}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n)\Big]
\end{aligned} \tag{147}$$

The auxiliary field $d(\mathbf{n})$ has no hopping term, and after eliminating it by the equations of motion on can show that the above action in terms of superfields is equivalent to the lattice action given in component form in eq. (144). Formulating the action in this supersymmetric facilitates the analysis of allowed operators and the continuum limit of the lattice theory.

The lattice defined by the orbifold projection cannot be directly considered to be a spacetime lattice, as all terms in the lattice action are trilinear and conventional hopping terms are absent. To generate a spacetime lattice and take the continuum limit one must follow the example of deconstruction [38] and follow a particular trajectory out to infinity in the moduli space of the theory, interpreting the distance from the origin of moduli space as the inverse lattice spacing.

As can be seen in eq. (144), the moduli space in the present theory corresponds to all values for the bosonic z variables such that

$$0 = \sum_{\mathbf{n}} \mathrm{Tr}\, \Bigg[\tfrac{1}{2} \Big(\sum_m \left(\bar{z}_m(\mathbf{n} - \boldsymbol{\mu}_m) z^m(\mathbf{n} - \boldsymbol{\mu}_m) - z^m(\mathbf{n}) \bar{z}_m(\mathbf{n}) \right) \Big)^2 \\ + \sum_{m,n} \Big| z^m(\mathbf{n}) z^n(\mathbf{n} + \boldsymbol{\mu}_m) - z^n(\mathbf{n}) z^m(\mathbf{n} + \boldsymbol{\mu}_n) \Big|^2 \Bigg]. \qquad (148)$$

8.1.1. A hypercubic lattice

There are clearly a large class of solutions to these equations. One possibility is

$$\begin{aligned} z^m(\mathbf{n}) &= \bar{z}_m(\mathbf{n}) = \frac{1}{a\sqrt{2}} \mathbf{1}_k, \qquad m = 1, \ldots, 4, \\ z^5(\mathbf{n}) &= \bar{z}_5(\mathbf{n}) = 0\ , \end{aligned} \qquad (149)$$

where a is the length scale associated with the lattice spacing, interpreted as the physical length (up to a factor of 4/5) of the links on which z_m and $\bar{z}^m$ variables reside, for $m = 1, \ldots, 4$. Such a lattice can be interpreted as a hypercubic lattice of length a on an edge, since the $\mathbf{r}$ charges for these variables correspond to Cartesian unit vectors, as seen in Table 5. In this case, the physical location of site $\mathbf{n}$ is simply the four-vector $\mathbf{R} = a\mathbf{n}$. Various fields of the $SU(5)$ multiplets distribute to the hypercubic lattice as follows:

$$\begin{aligned} &\lambda \to \lambda, \qquad 0-\text{cell} \\ &\psi^m \to (\psi^\mu, \psi^5) = (\psi^\mu, \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \psi_{\mu\nu\rho\sigma}), \qquad (0-\text{cell}, 4-\text{cell}) \\ &\xi_{mn} \to (\xi_{\mu\nu}, \xi_{\mu 5}) = (\xi_{\mu\nu}, \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \xi^{\nu\rho\sigma}), \qquad (2-\text{cell}, 3-\text{cell}) \end{aligned} \qquad (150)$$

In other words, the fermions are totally anti-symmetric p-cell variables, which one would naturally associate with the p-form representation of $SO(4)$ of the continuum. Thus the fermionic content of the hypercubic lattice construction provides an explicit realization of Dirac-Kähler fermions. The distributions of bosons such as $z^m \to (z^\mu, z^5) = (z^\mu, z^5) = (z^\mu, \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} z_{\mu\nu\rho\sigma})$ as well as their orientations are dictated by the fermions because of exact supersymmetry. The symmetry of the hypercubic lattice action is S_4, much smaller than the hypercubic group, since the fields are oriented.

8.1.2. The A_4^* lattice and point group symmetry

Instead of the above trajectory, we can examine the most symmetric solution, in the hopes that the greater the symmetry of the spacetime lattice,

classes:	(1)	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
sizes:	1	10	20	30	24	15	20
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	-1	-1	1	0	1	-1
χ_6	5	1	-1	-1	0	1	1
χ_7	6	0	0	0	1	-2	0

Table 6: *The character table of S_5, the point symmetry group of A_4^* lattice. The even permutations are spacetime rotations, the odd permutations involves parity operations and hence improper rotations.*

the fewer relevant or marginal operators will exist. A solution which treats all five z^m symmetrically (and thus preserves an S_5 permutation symmetry) is to have the five links on which they reside correspond to the vectors connecting the center of a 4-simplex to its corners. The lattice generated by such vectors is known to mathematicians as A_4^*[17]. We thus expand about the symmetric point:

$$z^m(\mathbf{n}) \quad = \bar{z}_m(\mathbf{n}) = \frac{1}{a\sqrt{2}}\mathbf{1}_k, \qquad m = 1, \ldots, 5 \; . \tag{151}$$

Once again a is interpreted as the spacetime length of the link that each z^m resides upon.

The physical point group symmetry of the lattice is isomorphic to permutation group S_5, the Weyl group of $SU(5)$, corresponding to the permutations of the group indices of $SU(5)$. The character table and conjugacy classes of S_5 are given in Table 6. The group has $5! = 120$ elements and seven conjugacy classes shown in Table 6. The symmetry of the lattice action is composed of the elements of S_5. It is easy to show that **even** permutations with determinant one (which can be read off from the χ_2(or sign) representation) are pure rotational symmetries of the action. We see from Table 6 that the odd permutations have determinant minus one, and are not proper rotations. In

[17]The A_4 lattice is generated by the simple roots of $SU(5) = A_4$; then A_4^* is the dual lattice, generated by the fundamental weights of $SU(5)$, or equivalently, by the weights of the defining representation of $SU(5)$. Lower dimension analogues are A_2^*, the triangular lattice, and A_3^*, the body-centered cubic lattice.

fact, the odd permutations accompanied by

$$\lambda \to \lambda,\ (\psi^m, z^m, \overline{z}_m) \to -(\psi^m, z^m, \overline{z}_m),\ \xi_{mn} \to \xi_{mn} \tag{152}$$

generate additional symmetries of the action. Notice that the symmetry of the action is not the full symmetry of the A_4^* lattice, as reflection symmetries which exchange z_m and $\overline{z}_m$ are not symmetries of the action.

The point group symmetry combined with gauge invariance of the lattice action and exact supersymmetry is very powerful in constraining the possible fine tuning required in the continuum limit. Representation theory of S_5 is also useful in identifying the precise relation between the Marcus's twist and A_4^* lattice formulation of $\mathcal{N} = 4$ SYM. In particular, we will show that the S_5 symmetry of the A_4^* lattice lives in the twisted Lorentz group, the diagonal sum of the R-symmetry and Lorentz symmetry of the original theory as shown in Fig.2.

To relate the lattice site $\mathbf{n}$ with a physical location in spacetime, we introduce a specific basis, in the form of five, four-dimensional lattice vectors

$$\begin{aligned}
\mathbf{e_1} &= (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}) \\
\mathbf{e_2} &= (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}) \\
\mathbf{e_3} &= (0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}) \\
\mathbf{e_4} &= (0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}}) \\
\mathbf{e_5} &= (0, 0, 0, -\frac{4}{\sqrt{20}}).
\end{aligned} \tag{153}$$

These vectors satisfy the relations

$$\sum_{m=1}^{5} \mathbf{e}_m = 0\ , \qquad \mathbf{e}_m \cdot \mathbf{e}_n = \left(\delta_{mn} - \frac{1}{5}\right)\ , \qquad \sum_{m=1}^{5} (\mathbf{e}_m)_\mu (\mathbf{e}_m)_\nu = \delta_{\mu\nu}\ . \tag{154}$$

The lattice vectors eq. (153) are simply related to the $SU(5)$ weights of the $\mathbf{5}$ representation, and the 5×5 matrix $\mathbf{e}_m \cdot \mathbf{e}_n$ can be recognized as the Gram matrix for A_4^* [131]. The site $\mathbf{n}$ on our lattice is then defined to be at the spacetime location

$$\mathbf{R} = a \sum_{\nu=1}^{4} (\boldsymbol{\mu}_\nu \cdot \mathbf{n})\, \mathbf{e}_\nu = a \sum_{\nu=1}^{4} n_\nu\, \mathbf{e}_\nu\ , \tag{155}$$

where a is the lattice spacing introduced in eq. (151), and the vectors $\boldsymbol{\mu}_\nu$ (which have integer components) were defined in Table 5. By making use of the fact that $\sum_m \mathbf{e}_m = 0$, it is easy to show that a small lattice displacement of the form $d\mathbf{n} = \boldsymbol{\mu}_m$ corresponds to a spacetime translation by $(a\,\mathbf{e}_m)$:

$$d\mathbf{R} = a\sum_{\nu=1}^{4}(\boldsymbol{\mu}_\nu \cdot d\mathbf{n})\,\mathbf{e}_\nu = a\sum_{\nu=1}^{4}(\boldsymbol{\mu}_\nu \cdot \boldsymbol{\mu}_m)\,\mathbf{e}_\nu = a\,\mathbf{e}_m\ . \tag{156}$$

Thus from the last column in Table 5 one can read off the physical location of each of the variables. For example, at the site $\mathbf{n} = \mathbf{0}$, $z^1(\mathbf{0})$ lies on the link directed from $\mathbf{R} = \mathbf{0}$ to $\mathbf{R} = a\,\mathbf{e}_1$, while $\xi_{45}(\mathbf{0})$ lies on the link directed from the site $\mathbf{R} = a\,(\mathbf{e}_4 + \mathbf{e}_5)$ to the site $\mathbf{R} = \mathbf{0}$. From the relation eq. (156) we see that each of the five links occupied by the five z^m variables has length $|a\,\mathbf{e}_m| = \sqrt{\frac{4}{5}}\,a$, unlike the case of the hypercubic lattice mentioned above, where z^5 resided on a link twice as long as the links occupied by the other four z^m variables.

8.2. Twisted construction

8.2.1. Continuum theory – Marcus twist

There are three inequivalent twists of the $\mathcal{N} = 4$ SYM theory in four dimensions [132, 79]. Two of those do not emerge from the lattice construction due to reasons to be explained later. The one we will consider and which corresponds to the orbifold lattice construction is due to Marcus. In addition to its application in lattice supersymmetry [41, 39], it also plays an important role in the geometric Langlands program [133]. Here, we briefly outline this twist.

The $\mathcal{N} = 4$ SYM theory in $d = 4$ dimensions possesses a global Euclidean Lorentz symmetry $SO(4)_E \sim SU(2)\times SU(2)$ and a global R-symmetry group $SO(6) \sim SU(4)$. The R-symmetry contains a subgroup $SO(4)_R \times U(1)$. To construct the twisted theory, we take the diagonal sum of $SO(4)_E \times SO(4)_R$ and declare it the new rotation group. Since the $U(1)$ part of the symmetry group is undisturbed, it remains as a global R-symmetry of the twisted theory. Under the global $G = \big(SU(2)\times SU(2)\big)_E \times \big(SU(2)\times SU(2)\big)_R$ symmetry, the fermions transform as $(2,1,2,1) \oplus (2,1,1,2) \oplus (1,2,1,2) \oplus (1,2,2,1)$. The same fields, under $G' = SU(2)' \times SU(2)' \times U(1)$ (or under

$SO(4)' \times U(1)$) transform as[18]

$$\begin{array}{ll} \text{fermions} & \rightarrow (1,1)_{\frac{1}{2}} \oplus (2,2)_{-\frac{1}{2}} \oplus [(3,1)\oplus(1,3)]_{\frac{1}{2}} \oplus (2,2)_{-\frac{1}{2}} \oplus (1,1)_{\frac{1}{2}} \\ & \rightarrow 1_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 6_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 1_{\frac{1}{2}} \ . \end{array} \tag{157}$$

The magic of this particular embedding is clear. There are now two spin zero fermions, while the remaining fermions are now in integer spin representations of the twisted Lorentz symmetry $SO(4)'$. They transform as scalars, vectors, and higher rank p-form tensors. We parameterize these Grassmann valued tensors, accordingly, $(\lambda, \psi^\mu, \xi_{\mu\nu}, \xi^{\mu\nu\rho}, \psi_{\mu\nu\rho\sigma})$.

The gauge boson V_μ which transforms as $(2,2,1,1)$ under the group G becomes $(2,2)$ under G'. Similarly, four of the scalars S_μ which originally transformed as $(1,1,2,2)$ are now elevated to the same footing as the gauge boson and transform as $(2,2)$ under the twisted rotation group. The resulting theory is most compactly described using a complex vector field [19]

$$z^\mu = (S^\mu + iV^\mu)/\sqrt{2}, \qquad \overline{z}_\mu = (S_\mu - iV_\mu)/\sqrt{2} \qquad \mu = 1,\ldots,4 \tag{158}$$

Since there are two types of vector fields, there are indeed two types of complexified gauge covariant derivative appearing in the formulation. These are holomorphic and antiholomorphic in character

$$\mathcal{D}^\mu \cdot = \partial^\mu \cdot + \sqrt{2}[z^\mu, \cdot], \qquad \overline{\mathcal{D}}_\mu \cdot = -\partial_\mu \cdot + \sqrt{2}[\overline{z}_\mu, \cdot\,]\,, \tag{159}$$

In fact only three combinations of the covariant derivatives (similar to the F-term and D-term in $\mathcal{N} = 1$ gauge theories) appear in the formulation. These are

$$\begin{array}{ll} \mathcal{F}^{\mu\nu} & = -i[\mathcal{D}^\mu, \mathcal{D}^\nu] = F_{\mu\nu} - i[S_\mu, S_\nu] - i(D_\mu S_\nu - D_\nu S_\mu) \\ \overline{\mathcal{F}}_{\mu\nu} & = -i[\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu] = F_{\mu\nu} - i[S_\mu, S_\nu] + i(D_\mu S_\nu - D_\nu S_\mu) \\ (-id) & = \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \cdots = -D_\mu S_\mu + \cdots \end{array} \tag{160}$$

where $D_\mu \cdot = \partial_\mu \cdot + i[V_\mu, \cdot\,]$ is the usual covariant derivative and $F_{\mu\nu} = -i[D_\mu, D_\nu]$ is the nonabelian field strength. The field strength $\mathcal{F}^{\mu\nu}(x)$ is

[18] Twice of the $U(1)$ charge is usually called the ghost number in the topological counterpart of this theory.

[19] The indices $\mu, \nu, \rho, \sigma \ldots$ are $SO(4)'$ or 4-dimensional hypercubic indices and summed over $1, \ldots 4$. The indices $m, n, \ldots$ are indices for permutation group S_5 (for A_4^* lattices) and are summed over $1, \ldots, 5$.

holomorphic; depending only on the complexified vector field z^μ and not on $\bar{z}_\mu$. Likewise, $\overline{\mathcal{F}}_{\mu\nu}$ is anti-holomorphic. The $(-id)$ will come out of the solutions of equations of motion for auxiliary field d and ellipses stand for possible scalar contributions. These combinations arise naturally from **all** of the orbifold lattice constructions in any dimensions and is one of the reasons for considering this class of twist (we saw this already in our discussion of the self-dual twist of the $(2,2)$ YM theory in two dimensions).

Finally, the two other scalars remains as scalars under the twisted rotation group. Since one of the scalars is the superpartner (as will be seen below) of the four form fermion, we label them as $(z_{\mu\nu\rho\sigma}, \bar{z}^{\mu\nu\rho\sigma})$. To summarize, the bosons transform under G' as

$$\text{bosons} \rightarrow z_{\mu\nu\rho\sigma} \oplus z^\mu \oplus \bar{z}_\mu \oplus \bar{z}^{\mu\nu\rho\sigma} \;\; \rightarrow [(1,1)_1 \oplus (2,2)_0 + (2,2)_0 + (1,1)_{-1}] \tag{161}$$

As can be seen easily from the decomposition of the fermions, there are two Lorentz singlet supercharges $(1,1)$ under the twisted Lorentz group and either of these (or their linear combinations) can be used to write down the Lagrangian of the four dimensional theory in "topological" form. Below, we use the scalar supercharge associated with λ. This produces the transformations given by [79].

The continuum off-shell supersymmetry transformations are given by

$$\begin{array}{l}
Q\lambda = -id, \qquad Qd = 0 \\
Qz^\mu = \sqrt{2}\;\psi^\mu, \qquad Q\psi^\mu = 0 \\
Q\bar{z}_\mu = 0 \\
Q\xi_{\mu\nu} = -i\overline{\mathcal{F}}_{\mu\nu} \\
Q\xi^{\nu\rho\sigma} = \sqrt{2}\,\overline{\mathcal{D}}_\mu \bar{z}^{\mu\nu\rho\sigma} \\
Qz_{\mu\nu\rho\sigma} = \sqrt{2}\psi_{\mu\nu\rho\sigma}, \qquad Q\psi_{\mu\nu\rho\sigma} = 0 \\
Q\bar{z}^{\mu\nu\rho\sigma} = 0
\end{array} \tag{162}$$

where d is an auxiliary field introduced for the off-shell completion of the supersymmetry algebra. Clearly, the scalar supercharge is nilpotent

$$Q^2 \cdot = 0. \tag{163}$$

owing to the anti-holomorphy of $\overline{\mathcal{F}}_{\mu\nu}$ etc. The fact that the subalgebra $(Q^2 = 0)$ does not produce any spacetime translations makes it possible to carry it easily onto the lattice. This exact nilpotent property, in contrast to nilpotency only up to gauge transformation, has a technical advantage - it admits a superfield formulation of the target supersymmetric field theory.

The twisted Lagrangian may be written as a sum of Q-exact and Q-closed terms:

$$g^2\mathcal{L} = \quad \mathcal{L}_{exact} + \mathcal{L}_{closed} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = Q\widetilde{\mathcal{L}}_{exact} + \mathcal{L}_{closed}, \quad (164)$$

where g is coupling constant and $\widetilde{\mathcal{L}}_{exact} = \widetilde{\mathcal{L}}_{e,1} + \widetilde{\mathcal{L}}_{e,2}$ is given by

$$\begin{aligned} \widetilde{\mathcal{L}}_{e,1} &= \mathrm{Tr}\left(\lambda(\tfrac{1}{2}id + \tfrac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \tfrac{1}{24}[\overline{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}])\right) \\ \widetilde{\mathcal{L}}_{e,2} &= \mathrm{Tr}\left(\tfrac{i}{4}\xi_{\mu\nu}\mathcal{F}^{\mu\nu} + \tfrac{1}{12\sqrt{2}}\xi^{\nu\rho\sigma}\mathcal{D}^\mu z_{\mu\nu\rho\sigma}\right) \end{aligned} \quad (165)$$

and $\mathcal{L}_{closed}$ is given by

$$\mathcal{L}_{closed} = \mathcal{L}_3 = \mathrm{Tr}\ \tfrac{1}{2}\xi_{\mu\nu}\overline{\mathcal{D}}_\rho\xi^{\mu\nu\rho} + \tfrac{\sqrt{2}}{8}\ \xi_{\mu\nu}[\overline{z}^{\mu\nu\rho\sigma}, \xi_{\rho\sigma}] \quad (166)$$

By using the transformation properties of fields and the equation of motion of the auxiliary field d

$$(-id) = \tfrac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \tfrac{1}{24}[\overline{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}]\ , \quad (167)$$

we obtain the Lagrangian expressed in terms of propagating degrees of freedom: [20]

$$\begin{aligned} \mathcal{L}_1 &= \mathrm{Tr}\left(\tfrac{1}{2}(\tfrac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \tfrac{1}{24}[\overline{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}])^2 + \lambda(\overline{\mathcal{D}}_\mu\psi^\mu + \tfrac{1}{24}[\overline{z}^{\mu\nu\rho\sigma}, \psi_{\mu\nu\rho\sigma}])\right) \\ \mathcal{L}_2 &= \mathrm{Tr}\left(\tfrac{1}{4}\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}^{\mu\nu} + \xi_{\mu\nu}\mathcal{D}^\mu\psi^\nu + \tfrac{1}{12}|\mathcal{D}^\mu z_{\mu\nu\rho\sigma}|^2 + \tfrac{1}{12}\xi^{\nu\rho\sigma}\mathcal{D}^\mu\psi_{\mu\nu\rho\sigma} + \tfrac{1}{6\sqrt{2}}\xi^{\nu\rho\sigma}[\psi^\mu, z_{\mu\nu\rho\sigma}]\right) \\ \mathcal{L}_3 &= \mathrm{Tr}\left(\tfrac{1}{2}\xi_{\mu\nu}\overline{\mathcal{D}}_\rho\xi^{\mu\nu\rho} + \tfrac{\sqrt{2}}{8}\ \xi_{\mu\nu}[\overline{z}^{\mu\nu\rho\sigma}, \xi_{\rho\sigma}]\right)\ . \end{aligned} \quad (168)$$

The Q-invariance of the $\mathcal{L}_{exact}$ is obvious and follows from supersymmetry algebra $Q^2 = 0$. To show the invariance of Q-closed term requires the use of the Bianchi (or Jacobi identity for covariant derivatives) identity

$$\epsilon^{\sigma\mu\nu\rho}\overline{\mathcal{D}}_\mu\overline{\mathcal{F}}_{\nu\rho} = \epsilon^{\sigma\mu\nu\rho}[\overline{\mathcal{D}}_\mu, [\overline{\mathcal{D}}_\nu, \overline{\mathcal{D}}_\rho]] = 0 \quad (169)$$

and a similar identity involving scalars. The action is expressed in terms of the twisted Lorentz multiplets, and the $SO(4)' \times U(1)$ symmetry is manifest.

[20] Notice that the splitting of the exact terms in Lagrangian into $\mathcal{L}_1$ and $\mathcal{L}_2$ is not identical to the one used by Marcus. The reason for the above splitting lies in the symmetries of the cut-off theory (A_d^* lattice theory) that will be discussed later.

The Lagrangian eq. (168) arises in the classical continuum limit of the hypercubic lattice and the A_4^* lattice action. In the former, a lattice p-cell field is identified with a continuum p-form under twisted $SO(4)'$. In the latter case, the matching of the fields can be deduced by the representation theory of S_5 as we will see.

Up to trivial rescalings this is the action (with gauge parameter $\alpha = 1$) of twisted $\mathcal{N} = 4$ Yang-Mills in four dimensions written down by Marcus [79]. This twisted action is well known to be fully equivalent to the usual form of $\mathcal{N} = 4$ in flat space.

8.2.2. A shortcut derivation of the Marcus twist

There is a slick way to obtain the twisted theory described in the previous section. The idea is to amalgamate the four complexified gauge fields eq. (158) and the extra scalar into a single five-component "gauge connection".

$$\left(z^\mu = \frac{S_\mu + iV_\mu}{\sqrt{2}},\ z^5 = \frac{S_5 + iS_6}{\sqrt{2}} \right) \to z^m, \qquad m = 1, \ldots 5 \tag{170}$$

The theory may then be realized as a five dimensional $\mathcal{Q} = 16$ theory dimensionally reduced to $d = 4$ dimensions. In five dimensions, $z^5 = \frac{S_5 + iV_5}{\sqrt{2}}$ and we may identify S_6 with V_5 upon dimensional reduction. Paralleling the four supercharge theory we introduce an additional auxiliary bosonic scalar field d and a set of five dimensional antisymmetric tensor fields to represent the fermions $\Psi = (\lambda, \psi^m, \xi_{mn})$. This latter field content corresponds to considering just one of the two Dirac-Kähler fields used to represent the 32 fields of the five dimensional theory. Again, a nilpotent symmetry relates these fields

$$\begin{aligned} \mathcal{Q}\, z^m &= \sqrt{2}\psi^m \\ \mathcal{Q}\, \psi^m &= 0 \\ \mathcal{Q}\, \overline{z}_m &= 0 \\ \mathcal{Q}\, \xi_{mn} &= -i\overline{\mathcal{F}}_{mn} \\ \mathcal{Q}\, \lambda &= id \\ \mathcal{Q}\, d &= 0 \end{aligned} \tag{171}$$

and remarkably we may extract the Marcus theory from the same $\mathcal{Q}$-exact action that was employed in §.6.3 for $(2, 2)$ Yang-Mills in two dimensions $S = \beta \mathcal{Q} \Lambda$ with

$$\Lambda = \int \mathrm{Tr} \left(\lambda (\tfrac{1}{2} id + \tfrac{1}{2} [\overline{\mathcal{D}}_m, \mathcal{D}^m]) + \tfrac{i}{4} \xi_{mn} \mathcal{F}^{mn} \right) \tag{172}$$

where we have again employed complexified covariant derivatives. Carrying out the $\mathcal{Q}$-variation and subsequently integrating out the auxiliary field as for the $\mathcal{Q}=4$ theory leads to the action

$$S_{\rm exact} = \int \mathrm{Tr}\,\left(\tfrac{1}{4}\overline{\mathcal{F}}_{mn}\mathcal{F}^{mn} + \tfrac{1}{8}([\overline{\mathcal{D}}_m,\mathcal{D}^m])^2 + \lambda\overline{\mathcal{D}}_m\psi^m + \xi_{mn}\mathcal{D}^m\psi^n\right) \quad (173)$$

Actually in this theory there is another fermionic term one can write down which is also invariant under this supersymmetry:

$$S_{\rm closed} = \frac{1}{8}\epsilon^{mnpqr}\xi_{mn}\overline{\mathcal{D}}_p\xi_{qr} \quad (174)$$

The invariance of this term is just a result of the Bianchi identity $\epsilon_{mnpqr}\mathcal{D}_p\overline{\mathcal{F}}_{qr} = 0$. The final action we will employ is the sum of the $\mathcal{Q}$-exact piece and this $\mathcal{Q}$-closed term and reproduces, after dimensional reduction, the four dimensional $\mathcal{Q}$-closed term we have already discussed. The coefficient in front of this term is determined by the requirement that the theory reproduce the Marcus twist of $\mathcal{N}=4$ Yang-Mills.

Splitting $\mathcal{F}^{mn}\to\mathcal{F}^{\mu\nu}\oplus\mathcal{D}_\mu z^5$, $\left[\overline{\mathcal{D}}_m,\mathcal{D}^m\right]\to\left[\overline{\mathcal{D}}_\mu,\mathcal{D}^\mu\right]\oplus[\overline{z}_5,z^5]$ and using eq. (160) and eq. (150) gives the $\mathcal{N}=4$ SYM action in the twisted form shown in eq. (168).

8.2.3. Lattice theory

The discretization scheme that is employed is precisely the same as the $\mathcal{N}=(2,2)$ target theory in $d=2$ dimensions as described in §.6.4. Specifically the continuum gauge field is exponentiated into a non-unitary link field with

$$\mathcal{U}^\mu = \frac{1}{\sqrt{2}a}e^{a(S_{\mu,\mathbf{n}}+iV_{\mu,\mathbf{n}})},\ \ \mathcal{U}^5 = \frac{1}{\sqrt{2}a}e^{a(S_{5,\mathbf{n}}+iS_{6,\mathbf{n}})} \quad (175)$$

as described in the continuum in eq. (170) [21]. The $\mathcal{Q}$-supersymmetry is essentially the same as in the continuum and remains nilpotent

$$\begin{aligned}
\mathcal{Q}\,\mathcal{U}^m &= \sqrt{2}\psi^m \\
\mathcal{Q}\,\overline{\mathcal{U}}_m &= 0 \\
\mathcal{Q}\,\psi^m &= 0 \\
\mathcal{Q}\,\xi_{mn} &= -2\left(\mathcal{F}^L_{mn}\right)^\dagger \\
\mathcal{Q}\,\lambda &= id \\
\mathcal{Q}\,d &= 0
\end{aligned} \tag{176}$$

where the lattice field strength $\mathcal{F}^L_{ab}$ is given by eq. (110) as before.

As for the $(2,2)$ twisted SYM model the twisted fermions are to be placed on p-cells in the lattice. However, there is one remaining wrinkle in this mapping; for each p-cell (with $1 \leq p \leq 4$) field associated with hypercubic lattice, we may have two possible orientations. This orientation is physical and determines the gauge rotation properties of the fields. We need to give a prescription to go from Marcus's twist to the lattice. As we will see, exact supersymmetry also plays an important role here.

For the moment let us base our discretization scheme around a hypercubic lattice. Then the gauge links $\mathcal{U}^\mu(\mathbf{x}) \equiv \mathcal{U}^m, m = 1\ldots 4$ should live on elementary coordinate directions in the unit hypercube, running from $\mathbf{x} \to \mathbf{x} + \boldsymbol{\mu}$. We will adopt the notation that these four basis vectors are labeled $\boldsymbol{\mu}_a, a = 1\ldots 4$. This assignment then implies that the superpartners of the gauge links $\psi^\mu(\mathbf{x})$ should also live on the same links and be oriented identically. Evidently, $\overline{\mathcal{U}}_m(\mathbf{x})$ is oriented oppositely, running from $\mathbf{x}+\boldsymbol{\mu} \to \mathbf{x}$. By eq.(110), the complexified field strength runs from $\mathbf{x} \to \mathbf{x}+\boldsymbol{\mu}+\boldsymbol{\nu}$, hence $\left(\mathcal{F}^L_{mn}\right)^\dagger$ and by exact supersymmetry $\xi_{\mu\nu}$ runs oppositely. The reader may have a feeling that, in this way, we are essentially re-constructing Table 5, and indeed, this is true.

However, a priori, the assignment of ψ^5 is not immediately obvious – a naive assignment to a site field would result in *two* fermionic 0-forms which is not what is expected for a four dimensional Dirac-Kähler field. Dirac-Kähler

[21] The definition of eq. (175) is rescaled relative to the discussion in §. 6.4 by a factor of $\sqrt{2}$. With this modification, the small field expansion of non-unitary link field is $\mathcal{U}^\mu = \frac{1}{\sqrt{2}a} + \frac{S_{\mu,\mathbf{n}}+iV_{\mu,\mathbf{n}}}{\sqrt{2}}$, and same as the one used in deconstruction/orbifold approach as in §. 7 and eq. (118).

decomposition demands a 4-form, associated with the chiral matrix of the four dimensional theory $\Gamma^5 = \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. This motivates assigning the lattice field to the body diagonal of the unit hypercube, a 4-cell. The ability to construct gauge invariant expressions involving the 4-cell fields (such as the last term in $\mathcal{L}_3$ in eq. (168)) demands that ψ^5 and z^5 fields to be oriented along the vector $\boldsymbol{\mu}_5 = (-1,-1,-1,-1)$. Notice that this assignment also ensures that $\sum_{m=1}^{5}\boldsymbol{\mu}_m = \mathbf{0}$ which will be seen to be crucial for constructing gauge invariant quantities.

To summarize the p-cell *and* orientation assignments of lattice fields, we write down their lattice gauge transformations:

$$\begin{array}{rcl}
\lambda(\mathbf{x}) & \to & G(\mathbf{x})\lambda(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi^m(\mathbf{x}) & \to & G(\mathbf{x})\psi^m(\mathbf{x})G^\dagger(\mathbf{x}+\boldsymbol{\mu}_m) \\
\xi_{mn}(\mathbf{x}) & \to & G(\mathbf{x}+\boldsymbol{\mu}_m+\boldsymbol{\mu}_n)\xi_{mn}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}^m(\mathbf{x}) & \to & G(\mathbf{x})\mathcal{U}^m(\mathbf{x})G^\dagger(\mathbf{x}+\boldsymbol{\mu}_m) \\
\overline{\mathcal{U}}_m(x) & \to & G(\mathbf{x}+\boldsymbol{\mu}_m)\overline{\mathcal{U}}_m(\mathbf{x})G^\dagger(\mathbf{x})
\end{array} \tag{177}$$

Notice also that these link choices and orientations match exactly the **r**-charge assignments of the orbifold action for the sixteen supercharge theory in four dimensions given in Table 5. As for two dimensions, successive components of the resultant fermionic Dirac-Kähler field alternate in orientation which will be the key to writing down gauge invariant fermion kinetic terms. Switching back to the four component anti-symmetric index notation, the set of four Majorana fermions required for $\mathcal{N} = 4$ YM are now compactly expressed in matrix form

$$(\Psi^{\mathrm{Maj}})_{\Upsilon I} = \left(\lambda 1 + \psi^\mu\gamma_\mu + \xi_{\mu\nu}\gamma^{[\mu\nu]} + \xi^{\mu\nu\rho}\gamma_{[\mu\nu\rho]} + \psi_{\mu\nu\rho\sigma}\gamma^{[\mu\nu\rho\sigma]}\right)_{\Upsilon I}, \Upsilon, I = 1,\ldots,4 \tag{178}$$

where upper index means oriented along the unit vectors and lower index means oppositely oriented. Thus, 1 and 3-form fermions $(\psi^{(1)},\psi^{(3)})$ are positively oriented and 0, 2 and 4-forms $(\psi^{(0)},\psi^{(2)},\psi^{(4)},)$ are negatively oriented This property is crucial (for any supersymmetric (orbifold) lattice in any dimension) both for gauge invariance and an absence of fermion doubling in any supersymmetric (orbifold) lattice theory.

Using the prescription of §6.4, and eq. (107) and eq. (110) produces precisely the supersymmetric lattice action for the $\mathcal{N} = 4$ SYM target theory given in eq. (144), modulo the replacement $z^m(\mathbf{n}) \to \mathcal{U}^m(\mathbf{n})$. The $\mathcal{Q}$-exact part becomes $S_{\text{exact}} = \mathcal{Q}\Lambda$ where

$$\Lambda = \sum_x \text{Tr}\, \left(-\tfrac{1}{2}\xi_{mn}\mathcal{F}^L_{mn} - \lambda\overline{\mathcal{D}}^{(-)}_m\mathcal{U}_m + \tfrac{1}{2}\lambda(id)\right) \tag{179}$$

which after $\mathcal{Q}$-variation and elimination of the auxiliary d yields

$$S = \sum_{\mathbf{x}} \text{Tr}\, \left(\mathcal{F}^{L\dagger}_{mn}\mathcal{F}^L_{mn} + \tfrac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_m\mathcal{U}_m\right)^2 - \sqrt{2}\left(\lambda\overline{\mathcal{D}}^{(-)}_m\psi_m + \xi_{mn}\mathcal{D}^{(+)}_{[m}\psi_{n]}\right)\right) \tag{180}$$

where the lattice field strength is given by the same expression as in §6.4. The third triangular fermion plaquette term arising in the orbifold action is now seen to be a discretized version of the $\mathcal{Q}$-closed term

$$S_{\text{closed}} = -\frac{\sqrt{2}}{8}\sum_{\mathbf{x}} \text{Tr}\, \epsilon_{mnpqr}\xi_{qr}(\mathbf{x} + \boldsymbol{\mu}_m + \boldsymbol{\mu}_n + \boldsymbol{\mu}_p)\overline{\mathcal{D}}^{(-)}_p\xi_{mn}(\mathbf{x} + \boldsymbol{\mu}_p) \tag{181}$$

where

$$\overline{\mathcal{D}}^{(-)}_p\xi_{mn}(\mathbf{x}) = \xi_{mn}(\mathbf{x})\overline{\mathcal{U}}_p(\mathbf{x} - \boldsymbol{\mu}_p) - \overline{\mathcal{U}}_p(\mathbf{x} + \boldsymbol{\mu}_m + \boldsymbol{\mu}_n - \boldsymbol{\mu}_p)\xi_{mn}(\mathbf{x} - \boldsymbol{\mu}_p) \tag{182}$$

Notice that the ϵ-tensor forces all indices to be distinct and the gauge invariance of this result follows from the fact that $\sum_{m=1}^{5}\boldsymbol{\mu}_m = \mathbf{0}$. In the continuum the invariance of this term under $\mathcal{Q}$-transformations requires use of the Bianchi identity. Remarkably, the lattice difference operator satisfies a similar identity (see [114] for the four dimensional result)

$$\epsilon_{mnpqr}D^{(+)}_p\mathcal{F}^L_{qr} = 0 \tag{183}$$

Furthermore, since the bosonic and fermionic link fields entering each lattice site in a hypercubic lattice construction are the same as the A_4^* lattice, we can obtain both hypercubic and A_4^* lattices from the twisted construction as well.

Preliminary simulations of this model have already been performed with encouraging results [122]. Table 7 shows the mean bosonic action for lattices with volume $2^4, 3^4$ at fixed 't Hooft coupling $\lambda = 0.5$ (the data corresponds to 6000 and 1000 configurations for linear size $L = 2$ and $L = 3$ respectively). As for the $\mathcal{N} = (2,2)$ model this observable can be computed exactly using a $\mathcal{Q}$-Ward identity yielding the exact result quoted in the last column. For

L	$< S_B >$	S_B^{exact}
2	211.2(2)	216.0
3	1075.0(35)	1093.5

Table 7: Observables for $SU(2)$ $\mathcal{Q} = 16$ model in $D = 4$ at $\lambda = 0.5$

L	$< S_B >$	S_B^{exact}
2	211.5(5)	216.0
3	1080.5(45)	1093.5

Table 8: Observables for $SU(2)$ $\mathcal{Q} = 16$ model in $D = 4$ at $\lambda = 0.25$

comparison, table 8 shows the same quantities for 't Hooft coupling $\lambda = 0.25$. Notice that as we approach weak coupling and smaller lattice spacings the bosonic action moves towards its exact supersymmetric value as expected.[22] Finally the scalar eigenvalue distribution is shown in figure 19 and looks qualitatively similar to what was seen for $(2, 2)$ YM with the important caveat that the tail of the distribution is much more rapidly damped in the $\mathcal{Q} = 16$ supercharge case. This is similar to what had been observed before in simulations of the corresponding matrix models [134].

To conclude, we have shown how to derive the supersymmetric orbifold lattice corresponding to $\mathcal{N} = 4$ SYM [34] by geometrical discretization of the continuum twisted SYM theory. This connection is not unexpected – it was shown earlier in [39] that the naive continuum limit of the $\mathcal{Q} = 16$ orbifold theory in four dimensions corresponded to the Marcus twist of $\mathcal{N} = 4$ and more recent work by Damgaard et al. [40] and Takimi [115] have exhibited the strong connections between discretizations of the twisted theory and orbifold theories. In this section we have completed this connection – the two approaches are in fact identical provided one chooses the exact lattice supersymmetry carefully - we must use the *self-dual twist* introduced earlier and employ the geometric discretization proposed in [109]. Additionally, as

[22] The small breaking of susy seen in this data is associated with the truncation $U(2) \to SU(2)$ employed in the simulations. This was necessary to avoid a vacuum instability problem. For further details on this and the issue of the Pfaffian phase we refer the reader to [122].

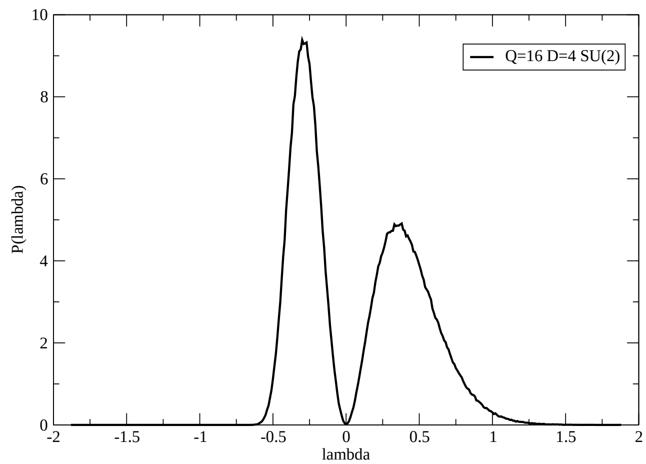

Figure 19: Probability distribution of scalar eigenvalues for $\mathcal{Q} = 16$, $D = 4$, $L = 2$ and $SU(2)$

was pointed out by Damgaard et al. [135] this lattice theory is essentially equivalent to the one proposed by d'Adda and collaborators [28] *provided* that the fermionic shift parameter employed in that model is chosen to be zero and we restrict our attention solely to the corresponding scalar supercharge.

This connection between the twisting and orbifolding methods is most clearly exhibited by recasting the usual Marcus twist of $\mathcal{N} = 4$ Yang-Mills as the dimensional reduction of a very simple five dimensional theory. The $\mathcal{Q}$-exact part of the action is then essentially identical to the two dimensional theory with $(2, 2)$ supersymmetry with the primary difference between the two theories arising because of the appearance of a new $\mathcal{Q}$-closed term which was not possible in two dimensions. Nevertheless discretization proceeds along the same lines, the one subtlety being the lattice link assignment of the fifth component of the complex gauge field after dimensional reduction. The key requirement governing discretization is that successive components of the Dirac-Kähler field representing the fermions have opposite orientations. This allows the fermionic action to be gauge invariant without any additional doubling of degrees of freedom. It seems likely that all the orbifold actions in various dimensions can be obtained in this manner.

8.2.4. *Absence of fermion doubling*

There are two independent ways to demonstrate the absence of unwanted doublers in our formulation. One is, to calculate the spectrum of scalars (which is technically simpler) and show that the bosonic action is doubler-free. Then, by exact supersymmetry, the fermionic spectrum is as well doubler free. This was the point of view taken in Appendix B of [34].

There is also an elegant way, which is made manifest by the geometric approach and which makes it easier to understand why this lattice theory does not suffer from doubling problems. This argument does not rely on supersymmetry, hence it is also useful for doubler-free formulations of four-flavor non-supersymmetric theories.

We will analyze this question in the context of the hypercubic lattice discretization. Clearly most of the fermionic kinetic terms manifestly satisfy the double free discretization prescription given by Rabin [9]. This prescription is; use forward lattice difference $(\mathcal{D}^\mu)^{(+)}$ whenever the continuum derivative acts as a gauged exterior derivative and use the backward difference $\overline{\mathcal{D}}^{(-)}_\mu$ whenever the continuum derivative appears as an adjoint of the exterior derivative. Most of the terms appearing in this action manifestly satisfy the requirements for this theorem.

The only subtleties arise when one or more tensor indices of the fields equal $m = 5$. Expressions involving these fields are not located wholly in the positively oriented unit hypercube and must be translated into the hypercube before they can examined from the perspective of this prescription. This has the effect of changing a forward difference to a backward difference operator after which it is easily seen that the term satisfies the requirements of this theorem. For more details, see [41].

8.3. *A_4^* lattice and Dirac-Kähler fermions*

We have seen that the $\mathcal{Q} = 1$ hypercubic supersymmetric lattice provides a realization of Dirac-Kähler fermions and a natural latticization of Marcus's twist. This discussion also makes it clear that the hypercubic lattice resides in the diagonal sub-space of the R-symmetry and original Lorentz symmetry. In this section, we wish to identify the relation between Dirac-Kähler fermions and the fermions of the A_4^* lattice. Recall that the A_4^* lattice is the maximally symmetric lattice realization of $\mathcal{N} = 4$ SYM theory in four dimensions. In A_4^*, the fermions are distributed as single site fermion λ, five link fermions ψ^m and an additional ten face fermions ξ_{mn}. The symmetry of A_4^* makes it clear that all 5 link fermions are on equal footing and all 10 face fermions are on equal standing as well. However, we know that in the continuum, under the twisted rotation group $SO(4)'$, the fermions must fill in antisymmetric tensor representations as shown in eq. (157). In particular, it is evident that the 5 link and 10 face fermions of the A_4^* lattice must be reducible. In order to see this explicitly, we need to decompose the lattice fields in terms of irreducible representation of S_5, as shown in Table 6.

The symmetry operations and characters of the S_5 point group symmetry are given in Table 6. By choosing a representative from each symmetry conjugacy class, we can calculate the character of the corresponding group element. In Table 9, we show how a particular representative from each conjugacy class acts on the fermion link fields and calculate the character

Operation	$(\psi^1, \psi^2, \psi^3, \psi^4, \psi^5)$	$\chi(g_{(rep)})$
(1)	$(\psi^1, \psi^2, \psi^3, \psi^4, \psi^5)$	5
(12)	$-(\psi^2, \psi^1, \psi^3, \psi^4, \psi^5)$	-3
(123)	$(\psi^3, \psi^1, \psi^2, \psi^4, \psi^5)$	2
(1234)	$-(\psi^4, \psi^1, \psi^2, \psi^3, \psi^5)$	-1
(12345)	$(\psi^5, \psi^1, \psi^2, \psi^3, \psi^4)$	0
(12)(34)	$(\psi^2, \psi^1, \psi^4, \psi^3, \psi^5)$	1
(12)(345)	$-(\psi^2, \psi^1, \psi^5, \psi^3, \psi^4)$	0

Table 9: *A representative of each conjugacy class and their action on the site and link fields are shown in the table. The five link fermions ψ^m transform in the same way with z^m. The transformation of ten fermions ξ_{mn} can be deduced from the antisymmetric product representation of $\bar{z}_m$ with itself.*

$\chi(g) = \text{Tr}\,(O(g))$, where g is a representative of each class and O is a matrix representation of the operation. Since the character is a class function, it is independent of representative. For the fermion fields, we obtain

$$\chi(\psi^m) = (5, -3, 2, -1, 0, 1, 0) \tag{184}$$

Note that the odd permutations are accompanied by the transformation eq. (152), since the combined operation is a symmetry of the action. Inspecting the character table of S_5, we see that this is not an irreducible representation. It is a linear combination of two irreducible representations,

$$\chi(\psi^m) = \chi_4 \oplus \chi_2, \tag{185}$$

a four-dimensional pseudo-vector and a singlet pseudo-scalar. We can also relate this representation theory argument to the detailed calculation given in [34]. Recall that under a group operation (see Table 9), $\psi^m \to O^{mn}(g)\psi^n$. The fact that the group action on the link field is reducible means there is a similarity transformation which takes all of the $O(g)$ into a block diagonal form. In this case, two blocks have sizes 1×1 and 4×4. Now, let us introduce the orthogonal matrix $\mathcal{E}$ that block-diagonalizes $O(g)$ for all $g \in S_5$. It is, not surprisingly, related to the basis vectors $\mathbf{e}_m$ of the A_4^* lattice.

$$\mathcal{E}_{m\mu} = (\mathbf{e}_m)_\mu, \qquad \mathcal{E}_{m5} = \frac{1}{\sqrt{5}}\,. \tag{186}$$

The matrix $\mathcal{E}_{mn}$ then forms a bridge between the irreducible representation of S_5 and the representations of the twisted Lorentz group $SO(4)'$. Thus, we obtain the following relations dictated by symmetry arguments:

$$\psi^\mu = \mathcal{E}_{m\mu}\psi^m, \qquad \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\psi^{\mu\nu\rho\sigma} = \mathcal{E}_{m5}\psi^m = \frac{1}{\sqrt{5}}\sum_{m=1}^{5}\psi^m \tag{187}$$

Obviously, we could have easily guessed the form of the singlet.

Performing the same exercise for all lattice fields, we obtain

$$\begin{aligned}
&\chi(\lambda) = \chi(d) = (1,1,1,1,1) \sim \chi_1 \\
&\chi(z^m) = \chi(\psi^m) = \chi(\overline{z}_m) = (5,-3,2,-1,0,1,0) \sim \chi_4 \oplus \chi_2 \\
&\chi(\xi_{mn}) = \chi([\overline{z}_m, \overline{z}_n]) = (10,2,1,0,0,-2,-1) \sim \chi_7 \oplus \chi_3\,.
\end{aligned} \tag{188}$$

This means that the sixteen fermions appearing in the unit cell of the A_4^* lattice branch into

$$\text{fermions} \to \chi_1 \oplus \chi_3 \oplus \chi_7 \oplus \chi_4 \oplus \chi_2 \tag{189}$$

irreducible representations of S_5, which is nothing but eq. (157). We may also write expressions relating the A_4^* lattice fields to the continuum twisted Dirac-Kähler fermions, by using the eq. (186). It is

$$\xi_{\mu\nu} = \xi_{mn}\mathcal{E}_{m\mu}\mathcal{E}_{n\nu}, \qquad \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\xi^{\nu\rho\sigma} = \xi_{mn}\mathcal{E}_{m\mu}\mathcal{E}_{n5}\,. \tag{190}$$

This completes our discussion of the relation between the A_4^* fermions and fermions in the twisted theory eq. (168).

Retrospectively, these relations are not surprising. We already knew that the fermions on the hypercubic lattice are Dirac-Kähler fermions. We may deform the hypercubic lattice into a A_4^* lattice while remaining within the moduli space of our orbifolded matrix model eq. (147). The number of bosonic and fermionic fields leaving and entering each lattice site is equal in these two lattice constructions. Obviously, the bosons work in a similar manner, which follows from exact supersymmetry.

In the continuum, the point group symmetry S_5 of the lattice action enhances to the twisted rotation group $SO(4)'$.

$$S_5 \subset SO(4)' \tag{191}$$

without any fine-tuning, thanks to the microscopic symmetries. In the renormalization discussion of §8.4, we will show that there are no relevant or marginal twisted $SO(4)'$ violating operators. Hence, in the continuum, we are guaranteed to get a $\mathcal{Q} = 1$, twisted Lorentz symmetry invariant gauge theory without any fine tuning. (In this sense, the enhancement of S_5 into twisted Lorentz symmetry is analogous to the pure YM theory on lattice, where hypercubic symmetry enhances to Lorentz symmetry.) Unfortunately, this does **not** imply that we can immediately undo the twist and obtain the $\mathcal{Q} = 16$ target theory. In particular, there are a few relevant or marginal operators which respect $SO(4)'$, gauge symmetry and $\mathcal{Q} = 1$, but not $SO(4)_E$. This means, some amount of fine tuning may be necessary in order attain the desired $\mathcal{Q} = 16$ target theory in the continuum limit. We examine these issues in more detail in the next section.

8.4. Renormalization

The immediate question that arises for this discretization of $\mathcal{N} = 4$ super Yang Mills theory is how much residual fine tuning will be required to ensure the restoration of full supersymmetry in the continuum limit. Clearly the existence of one exact supersymmetry improves the situation over any naive discretization but it is not immediately clear what additional counter terms will be needed to realize the full supersymmetry of the continuum theory.

Unlike the case of $d \leq 3$ dimensions power counting reveals that the continuum four dimensional theory has an infinite number of superficially divergent Feynman diagrams occurring at all orders of perturbation theory. Of course in the continuum target theory all of these potential divergences cancel between diagrams to render the quantum theory finite. However, since the lattice theory does not possess all the supersymmetries of the continuum theory, it is not clear how many of these will continue to cancel in the lattice theory.

As a first step to understanding the structure of the effective action that arises in this lattice theory as a result of radiative corrections one can attempt to write down the structure of all possible counterterms which are consistent with the exact lattice symmetries. In the case of A_4^* lattice, these symmetries are

a) Exact $\mathcal{Q} = 1$ supersymmetry.

b) Gauge invariance

c) S_5 point group symmetry and discrete translations.

In fact, other than exact lattice supersymmetry, the $U(k)$ lattice gauge theory also has a second fermionic symmetry, given by

$$\lambda(\mathbf{n}) \to \lambda(\mathbf{n}) + \epsilon \mathbf{1_k}, \qquad \delta(\text{all other fields}) = 0 \tag{192}$$

where ϵ is an infinitesimal Grassmann parameter. Thus, we extend our list to include

d) Fermionic shift symmetry

In practice we are primarily interested in relevant or marginal operators; that is operators whose mass dimension is less than or equal to four. We will see that the set of relevant counterterms in the lattice theory is rather short – the lattice symmetries, gauge invariance in particular, being extremely restrictive in comparison to the equivalent situation in the continuum. The argument starts by assigning canonical dimensions to the fields $[\mathcal{U}_a] = [\overline{\mathcal{U}}_a] = 1$, $[\Psi] = \frac{3}{2}$ and $[\mathcal{Q}] = \frac{1}{2}$ where Ψ stands for any of the twisted fermion fields $(\lambda, \psi^m, \xi_{mn})$. Invariance under $\mathcal{Q}$ restricts the possible counterterms to be either of a $\mathcal{Q}$-exact form, or of $\mathcal{Q}$-closed form. There is only one $\mathcal{Q}$-closed operator permitted by lattice symmetries, and it corresponds to the continuum term $\mathcal{L}_3$ in eq. (168). Thus, we need to look to the set of $\mathcal{Q}$-exact counterterms. Any such counterterm must be of the form $\mathcal{O} = \mathcal{Q}\mathrm{Tr}\,(\Psi f(\mathcal{U}, \overline{\mathcal{U}}))$. There are thus no terms permitted by symmetries and with dimension less than two. In addition gauge invariance tells us that each term must correspond to the traces of a closed loop on the lattice. The smallest dimension gauge invariant operator is then just $\mathcal{Q}(\mathrm{Tr}\ \psi^m \overline{\mathcal{U}}_m)$. But this vanishes identically since both $\overline{\mathcal{U}}_m$ and ψ_m are singlets under $\mathcal{Q}$. No dimension $\frac{7}{2}$ operators can be constructed with this structure and we are left with just dimension four counterterms. Notice, in particular that lattice symmetries permit no simple fermion bilinear mass terms. However, gauge invariant fermion bi-linears with link field insertions are possible and their effect should be accounted for carefully.

Possible dimension four operators are, schematically,

$$\begin{aligned}&\mathcal{Q}\mathrm{Tr}\ (\xi_{mn}\mathcal{U}^m\mathcal{U}^n)\\&\mathcal{Q}\mathrm{Tr}\ (\lambda\mathcal{U}^m\overline{\mathcal{U}}_m)\\&\mathcal{Q}\mathrm{Tr}\ (\lambda)\mathrm{Tr}\ (\mathcal{U}^m\overline{\mathcal{U}}_m),\end{aligned} \tag{193}$$

The first operator can be simplified on account of the antisymmetry of ξ_{mn} to simply $\mathcal{Q}(\xi_{mn}\mathcal{F}^{mn})$, which is nothing but the continuum term $\mathcal{L}_2$ in eq. (168).

The second term in eq. (193) requires more care. There are two operators of this type permitted by lattice symmetries, not including the fermionic shift symmetry. These are

$$\mathcal{L}_{1,\mp} = \mathcal{Q}\mathrm{Tr}\ \lambda(\mathbf{n})\Big(\overline{\mathcal{U}}_m(\mathbf{n}-\boldsymbol{\mu}_m)\mathcal{U}^m(\mathbf{n}-\boldsymbol{\mu}_m) \mp \mathcal{U}^m(\mathbf{n})\overline{\mathcal{U}}_m(\mathbf{n})\Big) \tag{194}$$

where both anti-commutator and commutator structure are allowed. Clearly, the operator with the relative minus sign is $\mathcal{L}_1$ in eq. (168), but the one involving the anti-commutator is a new operator not present in the bare Lagrangian. The only operator of the third type is a double-trace operator

$$\mathcal{L}_{1,+}^{\mathrm{d.t.}} = \mathcal{Q}\mathrm{Tr}\ \lambda(\mathbf{n})\mathrm{Tr}\ \Big(\overline{\mathcal{U}}_m(\mathbf{n}-\boldsymbol{\mu}_m)\mathcal{U}^m(\mathbf{n}-\boldsymbol{\mu}_m) + \mathcal{U}^m(\mathbf{n})\overline{\mathcal{U}}_m(\mathbf{n})\Big) \tag{195}$$

Note that both $\mathcal{L}_{1,+}$ and $\mathcal{L}_{1,+}^{\mathrm{d.t.}}$ transform non-trivially under the fermionic shift symmetry, but a linear combination of the two

$$\mathcal{L}_4 = \mathcal{L}_{1,+} - \frac{1}{k}\mathcal{L}_{1,+}^{\mathrm{d.t.}} \tag{196}$$

is invariant under the shift symmetry with k the rank of the gauge group $U(k)$.

By these arguments it appears that the only relevant counterterms correspond to renormalizations of operators already present in the bare action together with $\mathcal{L}_4$. This is quite remarkable. The most general form for the renormalized lattice Lagrangian is hence

$$g^2\mathcal{L} = \mathcal{L}_1 + \alpha\mathcal{L}_2 + \beta\mathcal{L}_3 + \gamma\mathcal{L}_4 \tag{197}$$

where α, β, γ are dimensionless numbers taking the value $(1, 1, 0)$ in the classical lattice theory and g^2 is a renormalized coupling constant. Thus it appears that at most three couplings might need to be fine tuned to approach $\mathcal{N} = 4$ Yang-Mills in the continuum limit.

In order to see the explicit form of the $\mathcal{L}_4$ operator close to the continuum limit, we expand the action around $\overline{\mathcal{U}} = \frac{1}{\sqrt{2}a}$. The result is

$$\mathcal{L}_4 \sim \frac{1}{a}\left[\mathrm{Tr}\ \lambda(\sum_{m=1}^{5}\psi^m) - \frac{1}{k}\mathrm{Tr}\ \lambda \mathrm{Tr}\ (\sum_{m=1}^{5}\psi^m) + \right]\ldots \tag{198}$$

where ellipsis are dictated by supersymmetry. The reader will immediately realize that $(\sum_{m=1}^{5}\psi^m) = \mathcal{E}_{5m}\psi^m$ is nothing but the S_5 (and twisted $SO(4)'$) singlet identified in §8.3. Indeed, it is the only field that could form a fermion mass term by pairing with λ.

As we remarked earlier twisted Lorentz invariance $SO(4)' = \mathrm{Diag}(SO(4)_E \times SO(4)_R)$ emerges in the continuum without any fine tuning due to microscopic symmetries of the lattice action, and gauge symmetry. Furthermore, the issue of the restoration of (untwisted) rotational invariance $SO(4)_E$, non-abelian R-symmetry invariance $SO(6)_R$, and full supersymmetry can now be formulated in terms of the magnitudes of the dimensionless coefficients α, β, γ. For example, the theory with $(\alpha, \beta, \gamma) = (1, 1, 0)$ is the Marcus twist of $\mathcal{N} = 4$ with full supersymmetry. The classical lattice theory is also defined with these initial conditions. However, it is currently not known whether the lattice theory flows to the desired target theory or not as the lattice spacing is sent to zero. A one loop calculation of α, β, γ has yet to be done but clearly is of the utmost interest in this regard. The theories for which $\gamma = 0$ also enjoy a global $U(1)_R$ symmetry, the so-called ghost number symmetry in the topological field theory literature. This $U(1)_R$ is the $SO(2)$ subgroup of the R-symmetry prior to twisting $SO(6)_R \supset SO(4)_R \times SO(2)_R$. The charges under $U(1)_R$ are given in eq. (157) and eq. (161), and apparently, the $\mathcal{L}_4$ operator explicitly violates it. The physical reason for the appearance of this mass operator in the continuum is then the absence of a continuous global chiral symmetry in A_4^* lattice formulation. In this sense, this is similar to the appearance of a Wilson mass term, in the continuum limit of a lattice theory without exact $U(1)$ chiral symmetry.

Finally, the class of theories for which $(\alpha, \beta, \gamma) \neq (1, 1, 0)$ correspond $\mathcal{N} = 1/4$ deformations of $\mathcal{N} = 4$ SYM theory and their physical interpretation is currently not known.

9. General aspects of supersymmetric lattices

9.1. Supersymmetric lattices and topological field theory

While having a non-perturbative definition of a supersymmetric gauge theory is important in its own right, it is also expected that lattice supersymmetry may lead to new insights and understanding in supersymmetric gauge dynamics. It may also offer a new non-perturbative window into general problems in quantum gravity and string theory via the AdS/CFT correspondence.

In the previous sections of this Report we have seen that supersymmetric lattices always lead to twisted supersymmetric theories in the continuum limit. These twisted theories are not topological, however, if desired, one can make them topological by declaring the scalar supercharge Q to be a true BRST operator. In this case the space of physical states of the theory is truncated to include only those $|\Omega\rangle$ annihilated by Q i.e $Q|\Omega\rangle = 0$, modulo those which can be written as $|\Omega'\rangle \sim Q|\Omega''\rangle$. In this sense, there is an intimate connection between topological field theories and supersymmetric lattices. The utility of topological field theory in the derivation of certain exact dualities of $\mathcal{N} = 4$ SYM within the restricted Hilbert spaces of the associated topological theory, as well as in the theory of 4-manifolds is well known [11, 78, 132, 133]. One of our hopes is that the lattice construction of the supersymmetric theories will shed light into the dynamics and dualities in these gauge theories beyond their topological subsectors.

A common thread in both topological field theory and lattice supersymmetry is the existence of a nilpotent scalar supercharge Q. However, although all supersymmetric lattices are associated with topological field theories, the converse statement is not true. Given a supersymmetric twist with a scalar supercharge, we are not guaranteed to have a working supersymmetric lattice formulation. Below, we examine this in connection with the twists of the $\mathcal{N} = 4$ SYM theory in $d = 4$.

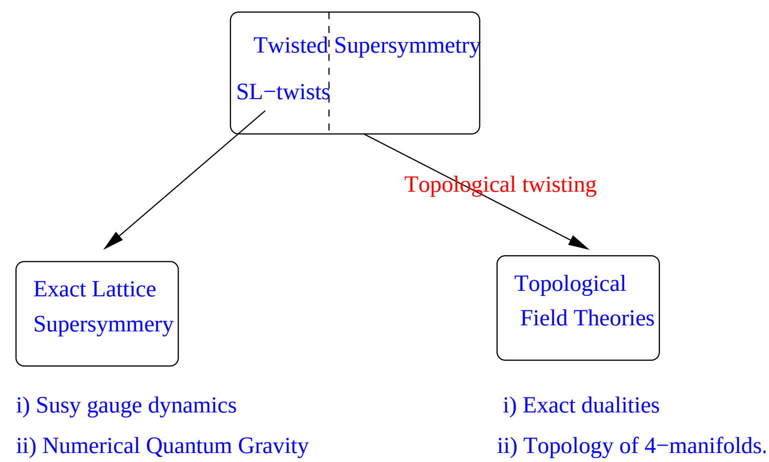

Figure 20: Classification for twisted supersymmetry, and interrelations between general twists, supersymmetric lattice (SL) twists and topological twists. SL-twists only form a special subset of all possible twists. Few selected applications to physics and mathematics are also shown.

9.1.1. Three twists of $\mathcal{N}=4$ SYM in $d=4$

The $\mathcal{N}=4$ theory on $\mathbb{R}^4$ has three inequivalent twists – to be described below – all of which admit a nilpotent scalar supercharge, with $Q^2\cdot=0$. All these twists are consistent with not having infinitesimal translation generators P_μ on the lattice, and in this sense, provide a solution to the problems quoted in §3.2. However, only one of these twists arises naturally in the context of supersymmetric orbifold lattices, and has a natural mapping into a lattice theory within the twisted/geometric approach. In this section we will explain what distinguishes these three twists and why only one admits a supersymmetric lattice construction. This will also shed light on the question of why the $\mathcal{N}=2$ theory in $d=4$, which also admits twisting and a nilpotent scalar supercharge, cannot be latticized in any simple way by the techniques described in this work.

Recall that the $\mathcal{N}=4$ theory on $\mathbb{R}^4$ can be obtained as the dimensional reduction of the $\mathcal{N}=1$ gauge theory on $\mathbb{R}^{10}$ down to $\mathbb{R}^4$. The ten dimensional theory possesses an $SO(10)$ Euclidean Lorentz rotation group. Upon reduction, the $SO(10)$ group decomposes into

$$SO(10)\longrightarrow\left(\begin{array}{c|c} SO(4) & \\ \hline & SO(6)\end{array}\right) \qquad (199)$$

where $SO(4) \sim SU(2)_L \times SU(2)_R$ is the four dimensional Lorentz symmetry action on $\mathbb{R}^4$ and $SO(6)_{\mathcal{R}} \sim SU(4)_{\mathcal{R}}$ is the internal $\mathcal{R}$-symmetry group. The **16** dimensional positive chirality spinor of $SO(10)$ decomposes as

$$Q_{\alpha,I} \oplus \overline{Q}_{\dot{\alpha},I} \sim (\mathbf{2},\mathbf{1},\mathbf{4}) \oplus (\mathbf{1},\mathbf{2},\bar{\mathbf{4}}) \in SU(2)_L \times SU(2)_R \times SU(4)_{\mathcal{R}} \tag{200}$$

The twisting procedure corresponds to a choice of a non-trivial $[SU(2) \times SU(2)]'$ embedding into $SU(2) \times SU(2) \times SU(4)_{\mathcal{R}}$.

The $\mathcal{N} = 4$ SYM theory in $d = 4$ has three inequivalent twists, i.e, three inequivalent embeddings of an $SU(2) \times SU(2)$ into $SU(4)_{\mathcal{R}}$ symmetry, each of which results in one or two scalar supersymmetries for which

$$Q^2 \cdot = 0 \qquad \text{(up to gauge rotations)} \tag{201}$$

These twists were first discussed in [132, 79] in the context of topological $\mathcal{N} = 4$ SYM theory.

However, only a subclass of these twisted theories may be defined on a lattice consistently [136]. We may refer to this class as *supersymmetric lattice twists* or *SL-twists* for short. For example, Marcus's twist is in SL-twist category, but not the other two. While the existence of a nilpotent scalar supersymmetry $Q^2 = 0$ is sufficient to formulate a topologically twisted version of a supersymmetric gauge theory on curved space, it is not sufficient to allow a lattice construction due to the other strictures of the latter.

The three independent twists of $\mathcal{N} = 4$ SYM are most easily described by providing the decomposition of the **4** of $SU(4)$ in eq. (200) under an $SU(2) \times SU(2)$ symmetry

$$\begin{aligned} &i)\ \ (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}), \qquad (\mathrm{SL-twist}) \\ &ii)\ \ (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}) \\ &iii)\ \ (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})\ . \end{aligned} \tag{202}$$

Under the twisted rotation group

$$[SU(2)_L \times SU(2)_R]' \times (G_a) \subset [SU(2)_L \times SU(2)_R] \times SU(4)_{\mathcal{R}} \tag{203}$$

where G_a $(a = i, ii, iii)$ is the global $\mathcal{R}$-symmetry of the twisted theory, the supercharges (and fermions) transform as

$$i)\ \ \text{fermions} \qquad \rightarrow (\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2}) \oplus [(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})] \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1})$$

$$\rightarrow \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{4} \oplus \mathbf{1} \qquad (\mathrm{SL-twist})$$

$$\begin{aligned} ii) \text{ fermions} \quad & \rightarrow 2 \times \Big[(\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathbf{3},\mathbf{1})\Big] \\ iii) \text{ fermions} \quad & \rightarrow \Big[(\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathbf{3},\mathbf{1})\Big] \oplus 2 \times \Big[(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\Big] \end{aligned} \tag{204}$$

Notice that we have dropped the transformation properties under G_a which are not important for our purposes. The gauge boson, which is a $SU(4)$ singlet, transforms as $(\mathbf{2},\mathbf{2})$. The scalars are a singlet under the Lorentz symmetry and transform in the $\mathbf{6} = \mathbf{4} \wedge \mathbf{4}$, anti-symmetric representation of $SU(4)$. Therefore, eq. (202) uniquely fixes the decomposition of the $\mathbf{6}$ under the twisted rotation group, for example,

$$i) \;\; [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] \wedge [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] = (\mathbf{2},\mathbf{2}) \oplus 2(\mathbf{1},\mathbf{1}), \tag{205}$$

and similarly,

$$ii) \;\; 3(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \qquad iii) \;\; 2(\mathbf{1},\mathbf{2}) \oplus 2(1,1) \tag{206}$$

As we stated above, all three of these twists support the existence of at least one nilpotent scalar supercharge $Q \sim (\mathbf{1},\mathbf{1})$, with $Q^2 = 0$, modulo gauge rotations. Indeed, the first two twists have two such nilpotent charges. One would naively expect that, since $Q^2 = 0$ does not interfere with any translation, it should be possible to implement all these twisted theories on the lattice. This intuition is not completely correct as we shall now argue.

First, note that all three twists have a copy of the twist of $\mathcal{N} = 2$ SYM theory in $d = 4$ [11] where eight supercharges decompose as $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathbf{3},\mathbf{1})$. This structure exists in a $L' \leftrightarrow R'$ symmetric manner in the first twist and asymmetric manner for the last two. This means that, in case i), instead of self-dual two-forms, we can just think of two-forms, without a self-duality condition. In lattice gauge theory, the implementation of the self-duality condition in a manifestly gauge covariant fashion is problematic. For example, in continuum, we will have $Q\psi^{\mu\nu,+} = F^{\mu\nu,+} \equiv F^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ where both of $\psi^{\mu\nu,+}$ and $F^{\mu\nu,+}$ are in the self-dual $(\mathbf{3},\mathbf{1})$ representation [11]. [23] It is not clear how to implement the self-dual field strength in a gauge covariant

[23]This equation has another utility. In the topological field theory context, the fixed points of the $\mathcal{N} = 2$ supersymmetric action are described by BPST-instantons. A useful complex generalization of the instanton equation in the $\mathcal{N} = 4$ SYM theory was obtained

way on the lattice and hence, the meaning of the left hand side (a self-dual Grassmann) is also unclear. This means, a gauge covariant implementation of the twists *ii*) and *iii*) in a lattice formulation is unlikely. Furthermore, the *iii*) case also involves double-valued representations of scalars and spinors, which are again in double-valued spinor representations of the lattice point group symmetry and do not have a natural mapping to a lattice, unlike the p-form to p-cell mapping that has been used in the constructions described in this Report.

The conclusion of these arguments seems to be that supersymmetric lattices always correspond to twists which do not involve any self-dual field-strengths and in which all the fields live in single-valued integer spin representations of the twisted rotation group. Furthermore, the spinors (and supercharges) must decompose into p-form integer spins:

$$Q_{\alpha,I} \oplus \overline{Q}_{\dot{\alpha},I} \longrightarrow Q^{(0)} \oplus Q^{(1)} \oplus Q^{(2)} \oplus Q^{(3)} \oplus Q^{(4)} \tag{207}$$

as we in for example the SL-twist of $\mathcal{N} = 4$ YM. Another way of stating this is that supersymmetric lattice theories must always contain a sufficient number of fermions to saturate one or more single Dirac-Kähler fields.

9.2. Matrix model regularizations

In this section we briefly discuss an alternative to the lattice constructions we have been describing but one which shares many of the same features - the matrix model regularization of supersymmetric gauge theories. This approach is independent of the orbifolding/deconstruction and twisting approaches. The main utility of this approach is that it can be used to construct a manifestly supersymmetric matrix regularization for certain twisted supersymmetric gauge theories formulated on curved backgrounds, such as S^2 or $S^2 \times \mathbb{R}$, which are not accessible by the techniques described so far.

It is well-known that global scalar supersymmetry may be carried to curved spaces if a twisted version of the supersymmetry algebra is used [11]. On curved space, there are no covariantly constant spinors, hence global supersymmetry cannot be achieved in any naive way. On the other hand,

---

by studying the fixed points of $\widetilde{Q} = Q + *Q^{(4)}$. The fixed points of the $\widetilde{Q}$-action yield $\overline{\mathcal{F}}^{(2)} + *\mathcal{F}^{(2)} = 0$, or in components, $\overline{\mathcal{F}}_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma} = 0$. This equation was derived first in the context of lattice supersymmetry [39] and later in the study of dualities in $\mathcal{N} = 4$ SYM [133].

covariantly constant scalars exist in curved spaces thanks to the twisting procedure. Indeed, a mass deformation of the type IIB matrix model provides a matrix model regularization for a twisted theory on a curved background – a two-sphere S^2. A remarkable feature of this construction is that both the regularized theory and continuum theory respect the same set of scalar supersymmetries. Instead of discussing an example on a curved background (see [136] for such an example), which necessitates introducing additional notation, we will highlight the main points of the matrix model regularization by employing the already established notation of §.8.

9.2.1. A deformed $\mathcal{Q}=1$ matrix model for $\mathcal{N}=4$ SYM in $d=4$

The type IIB matrix model possesses $\mathcal{Q}=16$ supersymmetries and a $SO(10)_R$ global R-symmetry and $U(N^2k)$ gauge symmetry. We first construct a $\mathcal{Q}=1$ supersymmetry and $U(1)^5$ global symmetry preserving deformation of the type IIB matrix model. This model serves as a nonperturbative regularization for $\mathcal{N}=4$ SYM theory in four Euclidean dimensions. As opposed to the orbifold projections where one starts with $U(N^4k)$ and projects out by Z_N^4 to obtain a $U(k)$ lattice gauge theory on an N^4 lattice, the deformed $U(N^2k)$ matrix model is itself a rewriting of a $U(k)$ gauge theory on N^4 lattice, without any projections. As a consequence, the latter formulation is not precisely local, however, this non-locality can be pushed to the cut-off scale by a judicious choice of the deformation parameter.

The deformed matrix model action with $\mathcal{Q}=1$ exact supersymmetry is given by

$$\begin{aligned}
S^{\mathrm{DMM}} \;&=\; \frac{\mathrm{Tr}}{g^2}\Bigg[\int d\theta\left(-\frac{1}{2}\mathbf{\Lambda}\partial_\theta\mathbf{\Lambda}-\mathbf{\Lambda}[\overline{z}_m,\mathbf{Z}_m]+\frac{1}{2}\mathbf{\Xi}_{mn}\mathbf{E}^{mn}\right)\\
&+\;\frac{\sqrt{2}}{8}\epsilon^{mnpqr}\mathbf{\Xi}_{mn}(e^{-i(\Phi_{pq}+\Phi_{pr})/2}\overline{z}_p\mathbf{\Xi}_{qr}-e^{+i(\Phi_{pq}+\Phi_{pr})/2}\mathbf{\Xi}_{qr}\overline{z}_p)\Bigg] \qquad (208)
\end{aligned}$$

where the $\mathcal{Q}=1$ supersymmetric matrix multiplets are

$$\begin{aligned}
&\mathbf{\Lambda}=\lambda-i\theta d\ ,\\
&\mathbf{z}^m=z^m+\sqrt{2}\,\theta\,\psi^m, \qquad \overline{z}_m, \qquad m=1,\ldots,5 \qquad (209)\\
&\mathbf{\Xi}_{mn}=\xi_{mn}-2\theta\ \overline{E}_{mn}\,.
\end{aligned}$$

The $\overline{z}_m$ is a supersymmetry singlet, and hence a multiplet on its own right. The fermi multiplet $\mathbf{\Xi}_{mn}$ is anti-symmetric in its indices. The holomorphic $\mathbf{E}^{mn}$ functions are the analogs of the derivative of the superpotential

$\epsilon^{mnp}\frac{\partial W(\mathbf{Z})}{\partial \mathbf{Z}^p}$ and given by

$$
\begin{aligned}
\mathbf{E}_{mn}(\mathbf{Z}) \quad &= e^{-i\Phi_{mn}/2}\mathbf{Z}^m\mathbf{Z}^n - e^{+i\Phi_{mn}/2}\mathbf{Z}^n\mathbf{Z}^m, \\
\overline{E}_{mn}(\overline{z}) \quad &= e^{-i\Phi_{mn}/2}\overline{z}_m\overline{z}_n - e^{+i\Phi_{mn}/2}\overline{z}_n\overline{z}_m \ .
\end{aligned}
\tag{210}
$$

The result eq. (208) is the $\mathcal{Q} = 1$ supersymmetry preserving deformed matrix model formulation of the target $\mathcal{N} = 4$ SYM theory. For $[\Phi_{mn}] = 0$, it is simply a rewriting of the $\mathcal{Q} = 16$ theory in terms of $\mathcal{Q} = 1$ superfields.

We choose the gauge group of the deformed matrix model as $U(N^2k)$ and a convenient choice of deformation (flux) matrix with a local continuum limit is

$$
[\Phi_{mn}] = \left[\begin{array}{cc|cc|c}
 & +\frac{2\pi}{N} & & & -\frac{2\pi}{N} \\
-\frac{2\pi}{N} & & & & +\frac{2\pi}{N} \\
\hline
 & & & +\frac{2\pi}{N} & -\frac{2\pi}{N} \\
 & & -\frac{2\pi}{N} & & +\frac{2\pi}{N} \\
\hline
+\frac{2\pi}{N} & -\frac{2\pi}{N} & +\frac{2\pi}{N} & -\frac{2\pi}{N} &
\end{array}\right]
\tag{211}
$$

With this choice of the flux matrix, we may use the background solution to form a basis for a lattice theory on an N^4 lattice. (For details, see [137].) Splitting the background and fluctuations of the matrix field in eq. (208) formally as

$$
U(N^2k) \longrightarrow \underbrace{U(N^2)}_{T^4 \text{ background}} \otimes \underbrace{U(k)}_{\text{gauge fluctuations}}
\tag{212}
$$

we obtain the $\mathcal{Q} = 1$ lattice gauge theory action of eq. (144) *except* with a modified (non-local) $\star$ product of lattice superfields. The exact $\mathcal{Q} = 1$ supersymmetry of the deformed model is same as the exact lattice supersymmetry of the lattice formulation. The $\star$-product (which is more commonly known as Moyal $\star$-product) is encoded into a kernel $K(\mathbf{j} - \mathbf{n},\ \mathbf{k} - \mathbf{n})$

$$
\begin{aligned}
\Psi_1(\mathbf{n}) \star \Psi_2(\mathbf{n}) = \quad & \sum_{\mathbf{j},\mathbf{k}} \Psi_1(\mathbf{j})\ K(\mathbf{j} - \mathbf{n},\ \mathbf{k} - \mathbf{n})\Psi_2(\mathbf{k}) \\
\equiv \quad & \sum_{\mathbf{j},\mathbf{k}} \Psi_1(\mathbf{j}) \left(\frac{1}{N^4} e^{-\frac{4\pi i}{N^2\theta'}\ (\mathbf{j}-\mathbf{n})\wedge(\mathbf{k}-\mathbf{n})}\right) \Psi_2(\mathbf{k})
\end{aligned}
\tag{213}
$$

In this formula $\theta' = 2/N$ is a dimensionless non-commutativity parameter on the lattice, and $\wedge$ is the usual skew-product.

The resulting model corresponds to a $U(k)$ lattice gauge theory on a N^4 lattice. The hypercubic lattice and A_4^* lattice are special points in its moduli space and were examined in §.8.1.1 and §.8.1.2. Distinct from the discussion in §.8.1.2, there is now a dimensionful length scale which measures the non-locality of the kernel in eq. (213). Restoring the dimensions, it is equal to

$$\Theta = \frac{N^2 a^2 \theta'}{4\pi} \tag{214}$$

The length scale associated with the non-locality of the $\star$-product is,

$$\ell_\star \sim \sqrt{\Theta} \sim Na\sqrt{\theta'} \sim \sqrt{N}a, \tag{215}$$

Compared to the box size, which is $L = Na$, we have

$$\frac{\ell_\star}{L} \sim \sqrt{\theta'} \sim \frac{1}{\sqrt{N}} \to 0\ . \tag{216}$$

This means, in the continuum limit where we take $N \to \infty$, the non-locality of the matrix model action tends to zero relative to the size of the box.

A few remarks are in order: The deformed matrix model is a natural generalization of the β-deformed $\mathcal{N} = 4$ SYM theory in $d = 4$, which is used to deconstruct slightly fuzzy theories in six dimensions [138]. By tuning θ' to be $O(1)$ in N counting, we may also achieve a non-commutative $\mathcal{N} = 4$ SYM theory on T^4 or $\mathbb{R}^4$ as in the supersymmetric examples of Refs.[139, 140]. In Refs. [139, 140], such supersymmetric non-commutative theories were obtained by using orbifolds with discrete torsion, which is just a way of saying that the orbifold projection matrices used in generating various dimensions commute with each other only up to a phase, which substitutes for the deformation matrix in eq. (211). The lack of a need to orbifold the matrix model at all, provided the model was suitably deformed, was recognized later in [137].

There has been also some recent interesting progress in the non-supersymmetric version of the deformed matrix model, which is known as the twisted Eguchi-Kawai (TEK) model. Along the same lines as above, a $U(N^2k)$ TEK model, at the classical level, produces a slightly non-commutative $U(k)$ Yang-Mills theory on four dimensional N^4 lattice. Recently, [141, 142] showed that, there is an quantum mechanical instability in the bosonic TEK model, and the relation to the lattice theory is spoiled. Ref. [141] argued that, in supersymmetric theories, or supersymmetric theories with softly broken supersymmetry, the analog of the instability that takes place in the pure TEK model is cured. Thus, the deformed matrix model shown in eq. (208) with appropriate choice of flux yields a non-perturbatively stable $d = 4$ non-commutative supersymmetric gauge theory according to the criteria of Ref. [141].

10. Lattice Supergravity?

We have seen that it is possible to construct globally supersymmetric lattices and that they have a lot of interesting mathematical structure. For example, the series of well prescribed mathematical steps described in this review could have been used to discover staggered fermions (if the methods hadn't come along 30 years too late!). One might wonder though whether the power of the analytical approach used here could be harnessed to create a lattice for *local* supersymmetry, known as supergravity. It would be pretty nifty if we could construct a lattice theory for quantum gravity by walking down a straight and narrow algebraic path without having to worry about the meaning of geometry and spacetime! In this section we briefly outline such an attempt which was *not* successful, in hope that it might inspire the reader towards something better [24].

Consider $(2,2)$ supergravity in $d = 2$ dimensions. It's action is derived from $\mathcal{N} = 1$ supergravity in $d = 4$ dimensions by erasing two spacetime dimensions. The particle content of the theory is a graviton and a spin $\frac{3}{2}$ gravitino; the action for the graviton is the usual Hilbert action, and for the gravitino, the Rarita-Schwinger action. The theory also has lots of auxiliary fields required to make the theory manifestly supersymmetric off-shell. The idea we will follow will be to invent "staggered" gravitinos on the lattice. We will then introduce staggered vierbeins, and try to realize one exact supercharge on the lattice, and then hope that the action has enough Lorentz symmetry and supersymmetry to have the desired continuum limit.

10.1. Staggered gravitinos

Consider spin 3/2 Majorana fermions in four dimensions. These are self-conjugate Dirac spinors ψ_m where m is a 4-vector index. The Rarita-Schwinger action is given by

$$\epsilon_{mnpq}\psi_m^T C\gamma_n\gamma_5\partial_p\psi_q \ . \tag{217}$$

[24] The material in this section is unpublished work by D.B. Kaplan and Michael Endres.

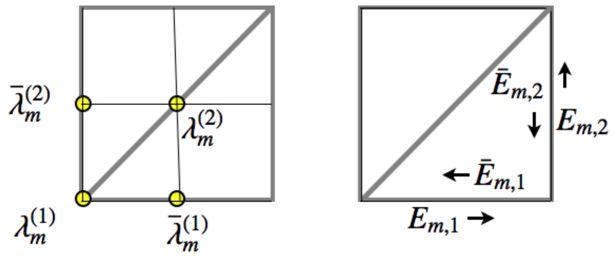

Figure 21: Lattice assignments for the gravitino and vierbein for $(2,2)$ supergravity; $m = 1,2$ is a curved space index.

This possesses a gauge symmetry $\psi_m \to \psi_m + \partial_m \chi$, where χ is an arbitrary Dirac spinor. Following the derivation for staggered fermions, we construct a naive latticization of this action:

$$\frac{1}{2a}\epsilon_{mnpq}\psi_m^T(\mathbf{n})C\gamma_n\gamma_5\left[\psi_q(\mathbf{n}+\hat{p}) - \psi_q(\mathbf{n}-\hat{p})\right] \ . \tag{218}$$

This lattice action also possesses a gauge symmetry, $\psi_m(\mathbf{n}) \to \psi_m(\mathbf{n}) + (\chi(\mathbf{n}+\hat{m}) - \chi(\mathbf{n}-\hat{m}))/(2a)$. We now substitute

$$\psi_m(\mathbf{n}) = \gamma_m\left(\gamma_1^{n_1}\cdots\gamma_4^{n_4}\right)\lambda(\mathbf{n}) \tag{219}$$

which is easily shown to eliminate the Dirac structure in the action, leaving us with four identical copies of the action for each spinor component of λ_m. We can therefore choose λ_m to be a one-component fermion (with a four-vector index). The lattice then has one of these four-vector fermions at each site and a simple action involving lattice derivatives with signs that encode the spin 3/2 structure.

In General Relativity the vector index on the gravitino lives in curved spacetime, while the spinor index lives in the tangent space; the way the two talk to each other is through the vierbein e_m^a, where m is a curved space index and a is a tangent space index; the vierbein is related to the metric by $e_{am}e_n^a = g_{mn}$ and to Lorentz symmetry by $e_{am}e_b^m = \eta_{ab}$, where η is the usual flat (Minkowski or Euclidean) space metric. The ease with which one can construct staggered spin 3/2 fermions is encouraging, but the fact that the curved space index does not play any structural role on the lattice is disturbing, even though the action couples the curved space index to the index of lattice derivative operators.

Ignoring gathering confusion, one can try to construct a lattice theory for $(2,2)$ supergravity in $d = 2$. The gravitino is readily latticized following

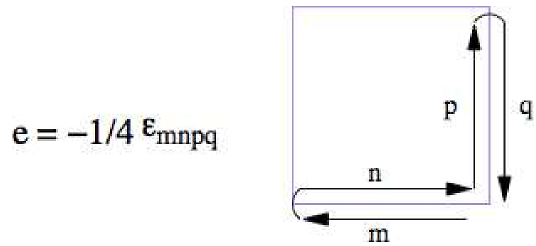

Figure 22: A picture of the lattice operator equal to $e = \det e_m^a$ in the continuum: A directed product of the E fields defined of Fig. 21, where the letters represent the curved space indices of the E variable, which are contracted by the ϵ tensor.

the staggering procedure, and the lattice assignments are shown in Fig. 21. Pushing on, one can latticize the gravitino's supersymmetric partner, the vierbein. Using the structure of our $(2,2)$ lattice construction with matter fields [35] as a guide, as well as the supersymmetry transformations between vierbeins and gravitinos in $(2,2)$ supergravity, one can define

$$e_m^a \sigma_{a\alpha\dot{\beta}} \equiv \begin{pmatrix} E_{m,1} & E_{m,2} \\ -\bar{E}_{m,2} & \bar{E}_{m,1} \end{pmatrix} \tag{220}$$

and assign the E fields lattice positions shown in Fig. 21. A heartening result is that various objects needed in the supergravity action, such as $e \equiv \det e_m^a$ and $(e_m^a)^{-1}$ are easily constructed as local lattice operators. For example, the determinant e is represented as a "staple" as shown in Fig. 22.

Nevertheless, it seems difficult to understand how to formulate the lattice covariant derivative in this theory. At this time it is an open and compelling question: can lattice supersymmetry give us new insights into lattice supergravity, and therefore about quantum gravity in general ?

11. Conclusions, prospects and open problems

In this report we have discussed some of the problems facing efforts to discretize supersymmetric theories. In general one faces fine tuning problems when one tries to do this as the classical symmetry is generally entirely broken under discretization. However, we stress that in dimensions less than four this is not necessarily disastrous – such theories possess only a finite set of U.V divergent diagrams which occur at low orders in perturbation theory. In principle such diagrams can be calculated in lattice perturbation theory and appropriate counterterms constructed, which when added to the lattice action, ensure that the resulting theory flows automatically to the supersymmetric fixed point in the continuum limit. In general non-supersymmetric discretizations may offer computational benefits such as positive definite determinants over lattice actions with exact supersymmetry.

That said, we have spent the bulk of this review discussing new ideas on how to put supersymmetric theories on the lattice in a way which guarantees a subset of the full supersymmetry is preserved at non-zero lattice spacing [25]. The approach only works for theories with a number of supercharges which is an integer multiple of 2^d if d is the dimension of (Euclidean) spacetime. This includes quantum mechanics, the two dimensional Wess-Zumino model, sigma models and a large class of SYM theories, including the important case of $\mathcal{N} = 4$ SYM in four dimensions.

Two constructions have been described; direct discretization of a twisted form of the theory and a construction based on orbifolding a matrix theory. The former technique can be used for theories both with and without gauge symmetry, the latter is a powerful technique for deriving lattices for supersymmetric Yang-Mills theories. Remarkably the two approaches can be explicitly connected in the case of gauge theories and in that case have been shown to be precisely equivalent [41, 42, 39]. In general the fermions and supercharges of these theories can be embedded into one or more Dirac-Kähler fields containing integer spin fields. The mapping of these fields onto the lattice is then very natural. The scalar components of these Dirac-Kähler fields map to site fields and correspond to supersymmetries that can be preserved on the lattice. Furthermore, it has been known for a long time that Dirac-Kähler fields can be mapped into staggered fermion fields at the level of free field theory which is one way of seeing that these lattice supersymmetric models do not exhibit fermion doubling.

In four dimensions there is a unique theory that can be treated this way – $\mathcal{N} = 4$ SYM. The resulting lattice action, derived either from orbifolding or twisting, is invariant under both lattice gauge transformations and a single scalar supersymmetry and is free of fermion doubles. Understanding the renormalization structure of this lattice theory is a pressing issue since it governs whether the lattice theory requires additional fine tuning in order

[25] Approaches have also been pioneered based on twisting which claim to preserve *all* supercharges on the lattice – see [143, 144, 145, 146, 147, 148]. These approaches have been examined in [82, 149, 136]. Large discrete chiral and space-time symmetries of these lattice theories is emphasized in [136].

for it to yield the correct continuum limit. One and two loop calculations of the counterterms in this model are crucial in thsi respect and await the interested researcher.

All these approaches potentially suffer from a complex fermion effective action and it is an open question how well current Monte Carlo algorithms can handle the resulting system. Initial results, particularly for thermal systems are quite encouraging but much more work needs to be done [122].

The lattices described in this report represent only a small fraction of the continuum supersymmetric theories one would like to study, and it would be interesting to see if somehow the techniques could be extended to include, for example, supersymmetric QCD in four dimensions. The extensions to systems with fermions in the fundamental representation are very interesting in this regard [127, 128]. Since numbers of quark flavors other than four cannot be represented by staggered fermions, it would also be interesting to see if one could somehow implement domain wall fermions in lattice supersymmetry and escape the flavor tyranny of staggered/Dirac-Kähler fermions.

Lattice supersymmetry has seen a resurgence of activity in recent years. After years in the desert, it is delightful to contemplate the intricate structure of the supersymmetric lattices described here and how they evade all the challenging obstacles outlined earlier. We still have some hope that these lattices will not only eventually be useful for numerical studies of extended SYM theories, but also that their reach might be extended to shed light on both phenomenologically more realistic supersymmetric theories and perhaps some restricted class of lattice supergravity theories.

12. Acknowledgments

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