May 2016

Lipschitz Geometry of Banach and Metric Spaces

Patrick Biermann
Syracuse University

ABSTRACT

We will investigate Lipschitz and Hölder continuous maps between a Banach space X and its dual space X^*, the space of continuous linear functionals. The existence of these maps is related to the smoothness of the norm of the space and isomorphic invariants called type and cotype will play a central role as well. The main result will be answering a question asked by W.B. Johnson about the isomorphic classification of Hilbert spaces.

Next we will find bounds on the distortion of subsets of infinite dimensional Hilbert space. This concept compares the extrinsic straight line distance inherited from the underlying space to the path metric of a subset. We will then use this concept to glean insight into the surjectivity of Bilipschitz maps.

The tight span of a finite metric space is a polyhedral complex embedded in a normed space. We will show that the position and lengths of the vertices and edges that arise in this construction are Lipschitz continuous with respect to the underlying space X.

We will study symmetric products, where it is an open problem whether the symmetric product of a space X inherits the property of being an absolute Lipschitz retract. We will propose an approach that utilizes tight spans toward answering this question.

Lipschitz Geometry of Banach and Metric Spaces

by

Patrick Biermann

M.S., Syracuse University, 2011

Dissertation

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

Syracuse University

May 2016

Contents

List of Figures

List of Symbols

$(K)_r$ r–skeleton of K

BM the space of all bounded semimetric spaces

distort(U) distortion of U

FM the space of all flat semimetric spaces

$\Lambda(X)$ best absolute Lipschitz constant of X

QM_n the space of all n–point quasimetric spaces

SM_n the space of all n–point semimetric spaces

EX tight span of X

$B(X)$ (open) unit ball of X

$C^{0,\alpha}$ α–Hölder continuous maps

d_H Hausdorff distance

d_{BM} Banach-Mazur distance

d_{GH} Gromov-Hausdorff distance

$S(X)$ unit sphere of X

X^* dual space of X

$X^{(n)}$ n-fold symmetric product of X

Chapter 1

Introduction

1.1 Metric and normed spaces

In this thesis we will address the geometry of spaces with different structures and we will devote this section to introducing these structures. We begin by considering spaces in which we have a way of measuring distances between points in a consistent manner.

Definition 1.1.1 (Quasimetric and metric spaces)**.** Let X be a space and $d\colon X \times X \to [0, \infty)$ a map. We call d a metric and consequently X a metric space if:

(i) $d(x, y) \geq 0$ for every $x, y \in X$ (non–negative)

(ii) $d(x, y) = 0$ if and only if $x = y$ (positive definite)

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetric)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality)

If d only satisfies properties (i) and (iii) then d is a quasimetric and X a quasimetric space. We denote the space of all n-point quasimetric spaces by QM_n. Furthermore we call d a semimetric if it satisfies (i), (iii) and (iv). The space of semimetric n–point spaces will be denoted by SM_n. If a finite semimetric space can be isometrically embedded into $\mathbb{R}$ we call it flat and the space of all flat n–point semimetric spaces is denoted FM_n. Finally, we call the space of all bounded semimetric spaces BM.

The most prominent metric space is the real line $\mathbb{R}$ with distance function $d(x, y) = |x-y|$. Also, every weighted graph can be equipped with the path metric, meaning the distance between two vertices is the length of the shortest path connecting them. In addition, any set X can be equipped with the discrete metric, i.e. we define $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$, showing that a space does not have to have much initial structure for us to be able to equip it with a distance function. On the other hand, the discrete metric is not particularly interesting.

We will introduce the notion of norms next, which will require the space to have a linear structure.

Definition 1.1.2 (Normed spaces)**.** Let X be a vector space and $\|\cdot\|\colon X \to [0, \infty)$ a map. We call $\|\cdot\|$ a norm if:

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and every $x \in X$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

The pair $(X, \|\cdot\|)$ is called a normed space.

Since the absolute value function is in fact a norm, the real line is our first example of a normed space. The spaces $\mathbb{R}^n$ allow a multitude of norms to be defined on them via

$$\|(x_1, \ldots, x_n)\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

for $1 \leq p < \infty$ and also

$$\|(x_1, \ldots, x_n)\|_\infty = \sup\{|x_i| \mid 1 \leq i \leq n\}$$

If we want to consider $\mathbb{R}^n$ equipped with $\|\cdot\|_p$, $1 \leq p \leq \infty$ for a particular p we will denote the space by ℓ_p^n. The spaces ℓ_p of sequences are defined analogously.

For compact subsets K of $\mathbb{R}^n$ we can consider the spaces

$$C(K) = \{f\colon K \to \mathbb{R} \mid f \text{ is continuous}\}$$

where

$$\|f - g\|_\infty = \max\{|f(x) - g(x)| \mid x \in K\}.$$

We are frequently concerned with the matter of convergence, which leads us to the next type of space.

Definition 1.1.3 (Banach space)**.** A normed space X is a Banach space, if it is complete, i.e. if every Cauchy–Sequence in X converges in X.

In studying the geometry of a space, the notion of angles is immensely useful and the next structure we will introduce will allow us to carry this concept into abstract spaces.

Definition 1.1.4 (Hilbert space)**.** We say that $\langle\cdot,\cdot\rangle\colon X \times X \to \mathbb{R}$ is an inner product if it satisfies the following:

(i) $\langle x, y\rangle = \langle y, x\rangle$ for every x, $y \in X$ (symmetry)

(ii) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$ for x, y and $z \in X$ and λ, $\mu \in \mathbb{R}$ (linearity in first component)

(iii) $\langle x, x\rangle \geq 0$ for every $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$ (positive definite)

Every inner product induces a norm via $\|x\| = \sqrt{\langle x, x\rangle}$ and that this is in fact a norm is due to the Cauchy–Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$. Then X is a Hilbert space if it is a Banach space and the norm is induced by an inner product.

One of the first Hilbert spaces to be investigated was the space $\mathbb{R}^2$ equipped with the classic Dot–product, this implies that $x \cdot y = \|x\|_2\|y\|_2 \cos\theta$, where θ is the angle between the vectors x and y. It is due to the Cauchy–Schwarz inequality and this relationship that it makes sense to define

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|_2\|y\|_2}$$

for arbitrary Hilbert spaces $\mathcal{H}$. This in turn allows us to employ the usual law of cosines in every Hilbert space. To be precise, this means that for a triangle with side length a, b and c we have

$$c^2 = a^2 + b^2 - 2ab\cos\theta \tag{1.1}$$

where θ is the angle between the sides with length a and b.

1.2 Maps between spaces

In the previous section, we addressed the different types of spaces we will investigate in the upcoming chapters and this section will be devoted to defining the classes of maps between them that we will employ to do so.

One of our main goals is to decide when two spaces X and Y are the same and there are different notions of sameness. On the one hand we could consider isometries, i.e. maps that satisfy $d_Y(f(x), f(y)) = d_X(x, y)$ and thus completely preserve the metric structure. These are very rigid. One might also consider homeomorphisms, continuous surjective maps that have a continuous inverse, which completely ignore the distance structure. Bilipschitz maps sit between these two extremes in the way that they don't exactly replicate the metric but still preserve a sufficient amount. In this sense they provide a natural notion of equivalence of metric spaces. They also have the nice property that they preserve all (sensible) notions of dimension that we have for metric spaces.

Definition 1.2.1 (Bilipschitz)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ a map. Then f is (l, L)–Bilipschitz if

$$l \leq \sup_{x,y\in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \leq L$$

We say f is L–Bilipschitz if it is (L^{-1}, L)–Bilipschitz and two spaces X and Y are bilipschitz equivalent if there is a surjective (l, L)–Bilipschitz map from X to Y.

A first relaxation of the Lipschitz condition is the following.

Definition 1.2.2 (Hölder continuity)**.** We say that a map $f\colon X \to Y$ between two Banach spaces X and Y is α–Hölder continuous ($0 < \alpha \leq 1$) if

$$\|f(x) - f(y)\|_Y \leq C\|x - y\|_X^\alpha$$

for all x, $y \in X$. The space of all α–Hölder continuous maps is denoted by $C^{0,\alpha}$.

If $\alpha = 1$ in this definition, it is more common to call f Lipschitz. This gives us another way to look at Bilipschitz maps as maps that are Lipschitz and have an inverse that is also Lipschitz.

The classic example of an α–Hölder continuous map for $0 < \alpha < 1$ is $f(x) = x^\alpha$ for $x \in [0, \infty)$ which is in $C^{0,\alpha}$. The identity map $x \mapsto x$ is one of the simplest Lipschitz functions, but there are many more. For example, it is a quick consequence of the mean value Theorem that every map with bounded derivative is Lipschitz continuous.

1.3 Summary of results

In this section we want to give a short overview of the main results of this thesis. Chapter 2 focuses on Lipschitz and Hölder continuous maps between a Banach space and its dual space X^*, the space of continuous linear functionals. We will see that the existence of these maps is related to the smoothness of the norm of the space and isomorphic invariants called type and cotype will play a central role as well. The main result will be answering a question asked by W.B. Johnson about the isomorphic classification of Hilbert spaces.

Then we will take a closer look at Hilbert space in Chapter 3. We will find bounds on the distortion of subsets of Hilbert space. This concept compares the extrinsic straight line distance inherited from the underlying space to the path metric of a subset. For example, the distortion of a circle is $\pi/2$. We will extend results from the finite dimensional case to the setting of infinite dimensional Hilbert space. Proving that the distortion of the complement of a bounded set is $\pi/2\sqrt{2}$ will necessitate a different approach than the one used to prove the same bounds in [15] in the finite dimensional case. We will then use our results about distortion to glean insight into the surjectivity of Bilipschitz maps.

Chapter 4 considers the Lipschitz continuity of the tight span of a finite metric space X. Tight spans are a concept introduced originally by Isbell in 1964 and considered for various purposes since then. Our main result is that the vertices and edges of the tight span are Lipschitz continuous with respect to the underlying space X. In order to prove these results we will establish a connection between our geometric understanding of tight spans and their algebraic definition. We will see that the choice of metric has a significant effect on these results and we will mainly contrast the Gromov–Hausdorff and Hausdorff distance.

The last chapter, Chapter 5, applies these continuity results to the study of symmetric products. In this area it is an open problem whether the symmetric product of a space X inherits the property of being an absolute Lipschitz retract, see for example Question 3.3.9 in [4]. We will introduce an approach towards this problem using the tight span construction. This will allow us to answer the question in some cases.

Chapter 2

Approximate duality maps and type

2.1 Introduction

In the study of Banach spaces the notion of duality plays a pivotal role.

Definition 2.1.1. For a Banach space X its dual space

$$X^* = \{T\colon X \to \mathbb{R} \mid T \text{ linear and bounded}\}$$

is the space of all linear functionals from X into the real number line. It is a Banach space itself equipped with the operatornorm

$$\|T\| = \sup\{\|Tx\| \,:\, \|x\| \leq 1\}.$$

We will adopt the notation that $S(X) = \{x \in X \,:\, \|x\| = 1\}$ denotes the unit sphere of X and $B(X) = \{x \in X \,:\, \|x\| < 1\}$ is the (open) unit ball of X. Then we will study (approximate) duality maps as maps between the unit sphere of X and the unit ball of its dual X^*. Before we give a formal definition of these type of maps, we want to see what motivates their study.

If we consider a Hilbert space $\mathcal{H}$, then the inner product can be used to define a map from $\mathcal{H}$ to $\mathcal{H}^*$ by considering the map $x \mapsto \langle x, \cdot \rangle$ and it is not hard to see that this map is in fact an isometry. The Riesz Representation Theorem says that this map is in fact onto, in other words a Hilbert space is isometrically isomorphic to its dual space.

An important property of duality maps is that they are designed to stand in for an inner product. For example, in a Hilbert space $\mathcal{H}$ we say an operator $T \colon \mathcal{H} \to \mathcal{H}$ is positive if it satisfies $\langle Tx, x \rangle \geq 0$ for every $x \in \mathcal{H}$ and similarly, if φ is a duality map of X then we can call an operator $T \colon X \to X$ positive if $\langle \varphi(x), Tx \rangle \geq 0$ for every $x \in X$. They also capture the geometry of X in the sense that if there is a duality map φ that is uniformly continuous on the unit ball then X is uniformly smooth. In this context, smoothness is defined as follows.

Definition 2.1.2 (Modulus of smoothness)**.** We define the modulus of smoothness of a Banach space $(X, \|\cdot\|)$ as

$$\rho_X(\tau) = \sup \left\{ \frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 \mid \|x\| = 1, \|\tau\| \leq 1 \right\}$$

We then call X uniformly smooth if $\rho_X(\tau) \in o(\tau)$ and we say that X is $\alpha + 1$ smooth if $\rho_X(\tau) \leq C\tau^{\alpha+1}$ for some constant $C > 0$.

Before we delve deeper into the use of (approximate) duality maps, we will properly define them.

Definition 2.1.3 ((Approximate) Duality map)**.** Let X be a Banach space and denote by X^* its dual space

1. We call $\varphi \colon S(X) \to \overline{B}(X^*)$ a *duality map* if φ satisfies

$$\langle \varphi(x), x \rangle = 1 \text{ for all } x \in S(X)$$

2. We call $\varphi \colon S(X) \to \overline{B}(X^*)$ an *approximate duality map* if

$$\langle \varphi(x), x \rangle \geq \eta \text{ for some } \eta \in (0, 1) \text{ and all } x \in S(X)$$

Whenever it is convenient to do so we will consider the homogeneous degree 1 extension of these maps to all of X, so we set $\hat{\varphi}(x) = \|x\|\varphi\left(\frac{x}{\|x\|}\right)$ for $x \neq 0$ and $\hat{\varphi}(0) = 0$. Frequently, we will denote them by φ as well.

If the magnitude of the lower bound η is important, we will refer to the map as a η–approximate duality map.

The first thing we should observe is that if we use the homogeneous degree 1 extension of these maps to all of X, then duality maps satisfy $\langle\varphi(x), x\rangle = \|x\|^2$ and similarly approximate duality maps $\langle\varphi(x), x\rangle \geq \eta\|x\|^2$.

We should consider how this extension impacts the Hölder continuity, but fortunately we have

Lemma 2.1.4. *Let X and Y be Banach spaces and $f\colon S(X) \to \overline{B}(Y)$ is a $C^{0,\alpha}$ map, $0 < \alpha \leq 1$, then the homogeneous degree 1 extension $\hat{f}$ of f to all of X satisfies*

$$\|\hat{f}(x) - \hat{f}(y)\| \leq K\beta(x, y)\|x - y\|^\alpha$$

for every x, y in X, with K independent of x and y.
Here, $\beta(x, y) = \|x\|^{1-\alpha} + \|y\|^{1-\alpha}$ or $\beta(x, y) = \max(\|x\|, \|y\|)^{1-\alpha}$.

Proof. Let x,$y \in X \setminus \{0\}$, we first note that the triangle inequality implies

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{2}{\|x\|}\|x - y\|$$

Then

$$\begin{aligned}\left\| \|x\| f\left(\frac{x}{\|x\|}\right) - \|y\| f\left(\frac{y}{\|y\|}\right) \right\| &\leq \|x\| C \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^{\alpha} + \left\| f\left(\frac{y}{\|y\|}\right) \right\| \|x-y\| \\ &\leq \|x\| C \frac{2^{\alpha}}{\|x\|^{\alpha}} \|x-y\|^{\alpha} + \|x-y\| \\ &= (C2^{\alpha}\|x\|^{1-\alpha} + \|x-y\|^{1-\alpha})\|x-y\|^{\alpha} \\ &\leq (C2^{\alpha}\|x\|^{1-\alpha} + 2^{1-\alpha}\max(\|x\|,\|y\|)^{1-\alpha})\|x-y\|^{\alpha} \\ &\leq K\beta(x,y)\|x-y\|^{\alpha}\end{aligned}$$

which finishes the proof. □

This establishes that Hölder continuity can be used as long as we work on bounded sets and that homogeneous degree 1 extensions preserve Lipschitz continuity.

As was mentioned earlier Hilbert spaces are isomorphic to their dual spaces via the duality map and it would be a natural question to ask whether being isomorphic to ones dual is enough to characterize Hilbert spaces. The following example demonstrates that this is a futile hope.

Example 2.1.5. Consider conjugate exponents p and q with $1 < p < 2$ then $T\colon \ell_p \oplus \ell_q \to \ell_q \oplus \ell_p$, $(x, y) \mapsto (y, x)$ is an isomorphism, but $\ell_p \oplus \ell_q$ is not a Hilbert space.

Evidently, it is not enough to be simply isomorphic to ones dual space. However, it can be easily checked that in this example we have $\langle T(x,0),(x,0)\rangle = 0$ for every $x \in \ell_p$. What if we require that $T\colon X \to X^*$ is an isomorphism that satisfies $\langle Tx, x\rangle > 0$ for every $x \in X \setminus \{0\}$ instead? This modification is not enough as the following example will show.

Example 2.1.6 (Pisier-Xu space)**.** Consider the space $A_{\frac{1}{2}2} = (v_1, \ell_\infty)_{\frac{1}{2}2}$ from [29] and let $T\colon A_{\frac{1}{2}2} \to A_{\frac{1}{2}2}$ be the identity map. Then for $x = (x_1, x_2, \ldots)$ we have

$$\langle Tx, x\rangle = \sum x_i(x_i - x_{i+1}) \geq \sum x_i^2 - \frac{1}{2}x_i^2 - \frac{1}{2}x_{i+1}^2 = \frac{1}{2}\sum x_i^2 - x_{i+1}^2 = \frac{1}{2}x_1^2$$

This shows that $\langle Tx, x\rangle \geq 0$, however closer inspection shows that the inequality is strict unless $x_i = x_{i+1}$ for all i. Thus we can see that $\langle Tx, x\rangle > 0$ for all $x \neq 0$.

Now it is in fact true that if $T\colon X \to X^*$ is a linear isomorphism that satisfies $\langle Tx, x\rangle \geq \eta\|x\|^2$ for every $x \in X$ and some $\eta > 0$ then X is isomorphic to a Hilbert space. In this case we can define an inner product on X via

$$\langle x, y\rangle_X = \langle Tx, y\rangle + \langle Ty, x\rangle$$

Let us check that this indeed defines an inner product. By definition $\langle x, y\rangle_X$ is symmetric and T being a linear operator gives linearity in the first component. In addition we have

$$\langle x, x\rangle_X = \langle Tx, x\rangle + \langle Tx, x\rangle \geq 2\eta\|x\|^2$$

which clearly establishes that our candidate for an inner product is positive definite and thus an actual inner product.

Along this same line of inquiry, W. B. Johnson has asked in [20] if a space is isomorphic to a Hilbert space whenever there exists a duality map that is a Bilipschitz equivalence. In the next section we will settle this question in the positive.

In the following we will investigate what happens if we drop the condition that this map be linear and replace it with varying conditions on its continuity.

We will begin by showing that if a space X has a Hölder continuous η–approximate duality map φ, then the Bochner spaces $L_p(X)$ on X also admit one with the same parameter and the modulus of continuity is dependent on p and the modulus of continuity of φ.

Lemma 2.1.7. *Let X be a Banach space that admits a $C^{0,\alpha}$ η–approximate duality map φ then the Bochner space $L_p(X)$ admits an η–approximate duality map $\Phi \in C^{0,\gamma}$ with $\gamma = \min(p-1, \alpha)$ for $1 < p < \infty$.*

Proof. We define $\Phi\colon L_p(X) \to L_q(X^*)$, where $1/p + 1/q = 1$ by

$$\Phi(f)(\omega) = \varphi\left(\frac{f(\omega)}{\|f(\omega)\|}\right)\|f(\omega)\|^{\frac{p}{q}}$$

as long as $f(\omega)$ isn't 0, and 0 otherwise.

First we check that the parameter η is preserved.

$$\begin{aligned}
\langle\Phi(f), f\rangle &= \int\left\langle\varphi\left(\frac{f(\omega)}{\|f(\omega)\|}\right)\|f(\omega)\|^{\frac{p}{q}}, f(\omega)\right\rangle \mathrm{d}\omega \\
&= \int \|f(\omega)\|^{\frac{p}{q}+1}\left\langle\varphi\left(\frac{f(\omega)}{\|f(\omega)\|}\right), \frac{f(\omega)}{\|f(\omega)\|}\right\rangle \mathrm{d}\omega \\
&\geq \int \eta\|f(\omega)\|^{\frac{p}{q}+1}\mathrm{d}\omega = \eta
\end{aligned}$$

Let f, $g \in S(L_p(X))$ and consider the following pointwise inequality

$$\begin{aligned}
\|\Phi(f) - \Phi(g)\|_{X^*} &\leq \left\|\varphi\left(\frac{f}{\|f\|}\right)\right\|\left|\|f\|^{\frac{p}{q}} - \|g\|^{\frac{p}{q}}\right| + \|g\|^{\frac{p}{q}}\left\|\varphi\left(\frac{f}{\|f\|}\right) - \varphi\left(\frac{g}{\|g\|}\right)\right\| \\
&\leq \|f-g\|^{\frac{p}{q}} + K\|g\|^{\frac{p}{q}}\left\|\frac{f}{\|f\|} - \frac{g}{\|g\|}\right\|^{\alpha} \\
&\leq \left(\|f-g\|^{\frac{p}{q}-\gamma} + K2^{\alpha-\gamma}\|g\|^{\frac{p}{q}-\gamma}\right)\|f-g\|^{\gamma} \\
&\leq \tilde{K}\max(\|f\|, \|g\|)^{\frac{p}{q}-\gamma}\|f-g\|^{\gamma}
\end{aligned}$$

Here, all instances of f and g are to be considered evaluated at ω.

Then we have the following

$$\begin{aligned}
\|\Phi(f) - \Phi(g)\|_q^q &\leq \int \tilde{K}^q\|f-g\|^{\gamma q}\max(\|f\|, \|g\|)^{p-\gamma q}\mathrm{d}\omega \\
&\leq K\left(\int\|f-g\|^p\mathrm{d}\omega\right)^{\frac{\gamma q}{p}}\left(\int\max(\|f\|, \|g\|)^p\mathrm{d}\omega\right)^{\frac{p-\gamma q}{p}} \\
&\leq K2^{p-\gamma q}\|f-g\|_p^{\gamma q}
\end{aligned}$$

and finally

$$\|\Phi(f) - \Phi(g)\|_q \leq C\|f-g\|_p^{\gamma}$$

□

2.2 Approximate duality maps and Rademacher–type

Type and cotype are two numbers associated to a Banach space that are invariant under isomorphisms and it turns out that approximate duality maps are a good tool to investigate these.

Before we can properly talk about them we first define the following.

Definition 2.2.1 (Average)**.** For n elements $x_1, \ldots, x_n$ of a Banach space and $0 < p < \infty$, we define

$$\operatorname*{Avg}_{\pm} \left\| \sum \pm x_i \right\|^p = \frac{1}{2^n} \sum_{\epsilon \in \{-1,1\}^n} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\|^p$$

as the average.

Definition 2.2.2 (Type and cotype)**.** Let X be a Banach space, then X has type p (or Rademacher type) if

$$(\operatorname*{Avg}_{\pm} \| \sum \pm x_i \|^p)^{\frac{1}{p}} \leq T_p (\sum \|x_i\|^p)^{\frac{1}{p}}$$

for all finite sums, where T_p is a constant independent of n and we denote by $p_X = \sup\{p : X \text{ has type } p\}$ the supremal type of X.

Similarly, X has cotype q if

$$(\operatorname*{Avg}_{\pm} \| \sum \pm x_i \|^q)^{\frac{1}{q}} \geq C_q (\sum \|x_i\|^q)^{\frac{1}{q}}$$

for all finite sums, where C_q is a constant independent of n and we denote by $q_X = \inf\{q : X \text{ has cotype } q\}$ the infimal cotype of X.

It is known that $p_X \in [1, 2]$, $q_X \in [2, \infty]$ and that a space X does not necessarily have type p_X or cotype q_X. The left sides of the inequalities for type and cotype are equivalent for all p and q as the following result by Kahane (for a proof see [24]) shows.

Theorem 2.2.3 (Kahane's Inequality)**.** *Let X be a Banach space and $0 < p < q < \infty$, then* *there is a constant K_{pq} independent of n, such that*

$$\left(\operatorname*{Avg}_{\pm}\left\|\sum \pm x_i\right\|^q\right)^{\frac{1}{q}} \leq \left(\operatorname*{Avg}_{\pm}\left\|\sum \pm x_i\right\|^p\right)^{\frac{1}{p}} \leq K_{pq}\left(\operatorname*{Avg}_{\pm}\left\|\sum \pm x_i\right\|^q\right)^{\frac{1}{q}}$$

We will now state and prove that the existence of certain maps will provide bounds on the supremal type and infimal cotype of X or even show that a space X has certain type. One major theorem that we will use to establish these bounds will be the following theorem due to Maurey and Pisier [25], which we will state after introducing the necessary definitions.

Definition 2.2.4 (Banach-Mazur distance)**.** The Banach-Mazur distance between two Banach spaces X and Y is defined as

$$d_{BM}(X,Y) = \inf\{\|T\|\|T^{-1}\| \,:\, T\colon X \to Y \text{ isomorphism}\}$$

Definition 2.2.5 (Finitely represented)**.** A Banach space Y is finitely represented in another Banach space X if for every $\epsilon > 0$ and every finite dimensional subspace Y_1 of Y there is a finite dimensional subspace X_1 of X such that $d_{BM}(Y_1, X_1) < 1 + \epsilon$. If instead for every ϵ the last inequality only holds only for some, we say that Y is crudely represented in X.

The next Theorem will be very helpful in taking a look at the case of a uniformly continuous approximate duality map and a proof can be found in [3]

Theorem 2.2.6 (Maurey–Pisier, '76)**.** *If X is an infinite dimensional Banach space then ℓ_p is finitely represented in X for $p = p_X$ and $p = q_X$.*

Lemma 2.2.7. *Let X be a Banach space with dim $X = \infty$.*
If X admits a uniformly continuous approximate duality map $\hat{\varphi}$ then $p_X > 1$.

Proof. Suppose that ℓ_1 is finitely represented in X and let

$$T : \ell_1^n \longrightarrow T(\ell_1^n)$$

be an isomorphism such that $\|T\| = 1$ and $\|T^{-1}\| < 1 + \epsilon$. We will now use the map $\hat{\varphi}$ to define an approximate duality map φ from $S(\ell_1^n)$ to $\overline{B}(\ell_\infty^n)$.

Consider

$$\begin{tikzcd} X \arrow[r, "\hat{\varphi}"] & X^* \arrow[d, "T^*"] \\ S(\ell_1^n) \arrow[u, "T"] \arrow[r, "\varphi"] & \overline{B}(\ell_\infty^n) \end{tikzcd}$$

Figure 1

where $\varphi = T^* \circ \hat{\varphi} \circ T$ makes the diagram in Figure 1 commute. It remains to be shown that φ is still a uniformly continuous approximate duality map. If we denote by $\omega_{\hat{\varphi}}$ the modulus of continuity of $\hat{\varphi}$, we get

$$\|\varphi(x) - \varphi(y)\|_\infty \leq \|T^*\| \, \omega_{\hat{\varphi}}(\|T(x-y)\|_X) \leq \omega_{\hat{\varphi}}(\|x-y\|_1)$$

To show that φ is an approximate duality map, we first note that

$$\frac{\|x\|}{1+\epsilon} \leq \|Tx\| \leq \|x\|$$

Then $\langle \varphi(x), x \rangle = \langle T^*(\hat{\varphi}(Tx)), x \rangle = \langle \hat{\varphi}(Tx), Tx \rangle$ and considering the homogeneous degree 1 extension we arrive at

$$\|Tx\|^2 \left\langle \hat{\varphi}\left(\frac{Tx}{\|Tx\|}\right), \frac{Tx}{\|Tx\|} \right\rangle \geq \eta \|Tx\|^2 \geq \frac{\eta}{(1+\epsilon)^2}\|x\|^2 = \hat{\eta}\|x\|^2$$

We will now consider the reflections ρ_i about the hyperplanes $\{x_i = 0\}$ in ℓ_1^n and denote by G the group generated by these reflections. Then define

$$\tilde{\varphi} := \frac{1}{|G|} \sum_{g \in G} g \circ \varphi \circ g$$

This new map is symmetric in the following sense $\tilde{\varphi} \circ g = g \circ \tilde{\varphi}$ $\forall g \in G$ and it is still a uniformly continuous $\hat{\eta}$-approximate duality map with the same modulus of continuity as φ.

We should now consider the midpoint $m = (1/n, \ldots, 1/n)$ of one face of $S(\ell_1^n)$ and the midpoints $m_i = \rho_i(m)$ of adjacent faces. Since they differ only in one coordinate their distance is $\|m - m_i\|_1 = \frac{2}{n}$.

By the symmetry of $\tilde{\varphi}$ the images of those midpoints are completely determined by $\tilde{\varphi}(m) = (y_1, \ldots, y_n)$. We get that $\frac{1}{n}\sum y_i = \langle \tilde{\varphi}(m), m\rangle \geq \hat{\eta}$, so for at least one y_i we have $y_i \geq \hat{\eta}$. We conclude that

$$2\hat{\eta} \leq 2|y_i| = \|\tilde{\varphi}(m) - \tilde{\varphi}(m_i)\|_\infty \leq \omega_\varphi(\|m - m_i\|_1) = \omega_\varphi(2/n)$$

where the right side tends to 0 as $n \to \infty$. Since $\hat{\eta}$ did not depend on n this is a contradiction and thus ℓ_1 can not be finitely represented in X. $\square$

We can see that this proof works even if we just assume that ℓ_1 is crudely represented in X. Now we will state and prove a lemma that connects Hölder continuity of an approximate duality map directly with Rademacher type.

Lemma 2.2.8. *Let X be a Banach space with $\dim X = \infty$ and $0 < \alpha \leq 1$. If X admits a $C^{0,\alpha}$ approximate duality map φ then X has type $\alpha + 1$.*

Proof. Let $x_1, \ldots, x_n$ be elements of X, throughout the proof we will adopt the convention that $\epsilon x = \sum_{i=1}^n \epsilon_i x_i$ and denote by ϵ^{j-} the vector $(\epsilon_1, \ldots, -1, \ldots, \epsilon_n)$, where the j–th component is -1.

Using the lower bound from approximate duality and linearity we obtain

$$\left(\frac{1}{2^n}\sum_{\epsilon\in\{-1,1\}^n}\|\epsilon x\|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{\eta 2^n}\sum_{\epsilon\in\{-1,1\}^n}\sum_{j=1}^n \epsilon_j\langle\varphi(\epsilon x), x_j\rangle\right)^{\frac{1}{2}}$$

We first can write

$$\sum_{\epsilon\in\{-1,1\}^n}\sum_{j=1}^{n}\epsilon_j\langle\varphi(\epsilon x),x_j\rangle=\sum_{j=1}^{n}\left(\sum_{\epsilon_j=1}\langle\varphi(\epsilon x),x_j\rangle-\sum_{\epsilon_j=-1}\langle\varphi(\epsilon x),x_j\rangle\right)$$
$$=\sum_{j=1}^{n}\sum_{\epsilon_j=1}\langle\varphi(\epsilon x)-\varphi(\epsilon^{j-}x),x_j\rangle$$

And combining this with

$$\langle\varphi(\epsilon x)-\varphi(\epsilon^{j-}x),x_j\rangle\le\|\varphi(\epsilon x)-\varphi(\epsilon^{j-}x\|\|x_j\|\overset{2.1.4}{\le}K(\|\epsilon x\|^{1-\alpha}+\|\epsilon^{j-}x\|^{1-\alpha})\|\epsilon x-\epsilon^{j-}x\|^{\alpha}\|x_j\|$$

we have

$$\sum_{\epsilon\in\{-1,1\}^n}\sum_{j=1}^{n}\epsilon_j\langle\varphi(\epsilon x),x_j\rangle\le\sum_{j=1}^{n}\sum_{\epsilon_j=1}K(\|\epsilon x\|^{1-\alpha}+\|\epsilon^{j-}x\|^{1-\alpha})\|x_j\|^{\alpha+1}$$
$$=K\sum_{j=1}^{n}\|x_j\|^{\alpha+1}\sum_{\epsilon\in\{-1,1\}^n}\|\epsilon x\|^{1-\alpha}$$
$$=2^nK\operatorname*{Avg}_{\pm}\left\|\sum\pm x_i\right\|^{1-\alpha}\sum_{j=1}^{n}\|x_j\|^{\alpha+1}$$

Employing Kahane's inequality 2.2.3 we obtain the following from the above

$$\left(\operatorname*{Avg}_{\pm}\left\|\sum\pm x_i\right\|^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\le\sqrt{\frac{K}{\eta}}\left(\sum_{j=1}^{n}\|x_j\|^{\alpha+1}\operatorname*{Avg}_{\pm}\left\|\sum\pm x_i\right\|^{1-\alpha}\right)^{\frac{1}{2}}$$

Which is equivalent to

$$\left(\operatorname*{Avg}_{\pm}\left\|\sum\pm x_i\right\|^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\le\tilde{K}\left(\sum_{j=1}^{n}\|x_j\|^{\alpha+1}\right)^{\frac{1}{\alpha+1}}$$

and thus X has type $\alpha+1$. $\square$

We are almost ready to answer W. B. Johnson's question and the last missing piece is the following theorem due to Kwapien ([22]).

Theorem 2.2.9 (Kwapien, '72)**.** *A Banach space X is isomorphic to a Hilbert space if and only if X has type 2 and cotype 2.*

Theorem 2.2.10. *A Banach space $(X, \|\cdot\|)$ is isomorphic to a Hilbert space if and only if it admits an η–approximate duality map that is a Bilipschitz equivalence.*

Proof. First, assume that X is isomorphic to a Hilbert space $\mathcal{H}$ via an isomorphism T. Then there is an adjoint of this map from $\mathcal{H} \to X^*$. Furthermore, let $id_{\mathcal{H}}$ denote the identity map on $\mathcal{H}$, then this defines a duality map for $\mathcal{H}$ since $\mathcal{H}^* = \mathcal{H}$. We can then define $\varphi \colon S(X) \to B(X^*)$ as

$$\varphi = \frac{1}{\|T\|^2} T^* \circ \underset{\mathcal{H}}{\mathrm{id}} \circ T$$

which is Bilipschitz since all maps involved are bounded and linear. We then have, for $x \in S(X)$

$$\begin{aligned}\langle \varphi(x), y \rangle &= \frac{1}{\|T\|^2} \langle (\underset{\mathcal{H}}{\mathrm{id}} \circ T)(x), Ty \rangle \\ &= \frac{1}{\|T\|^2} \langle Tx, Ty \rangle_{\mathcal{H}} \leq \frac{1}{\|T\|^2} \|Tx\|_{\mathcal{H}} \|Ty\|_{\mathcal{H}} \\ &\leq \|y\|\end{aligned}$$

and also

$$\langle \varphi(x), x \rangle = \frac{1}{\|T\|^2} \langle Tx, Tx \rangle_{\mathcal{H}} = \frac{1}{\|T\|^2} \|Tx\|_{\mathcal{H}}^2 \geq \frac{1}{\|T\|^2 \|T^{-1}\|^2}$$

which shows that φ is in fact an approximate duality map.

In order to show the other implication, let $\varphi \colon S(X) \to S(X^*)$ be an η–approximate duality map that is a Bilipschitz equivalence. Then 2.2.8 shows that X has type 2. In addition $\varphi^{-1} \colon S(X^*) \to S(X) \subset S(X^{**})$ is still an η–approximate duality map which by assumption is Lipschitz and thus we conclude that X^* has type 2. In [19] it is shown that this in turn implies that X has cotype 2. But then Kwapien's result 2.2.9 finishes the proof. □

We should put our results in context. In combination with known results we have the following chain of implications

$$(\alpha+1)\text{–smooth} \implies \exists\, C^{0,\alpha} \text{ approximate duality map} \implies \text{type } (\alpha+1) \quad . \tag{2.1}$$

In fact, we know that the duality map of a space X being α–Hölder is equivalent to that space having modulus of smoothness $(\alpha+1)$ and this implies Rademacher type $(\alpha+1)$. There are however examples in [29] that there are spaces with type $(\alpha+1)$ that are not $(\alpha+1)$ smooth. This shows that in (2.1) not both arrows can be reversed. It remains unknown whether $(\alpha+1)$–smoothness or having type $(\alpha+1)$ is equivalent to the existence of a Hölder continuous approximate duality map.

Chapter 3

Distortion of subsets of Hilbert spaces

3.1 Introduction

In Banach space theory, a driving question of the field is the Lipschitz classification of Banach spaces and a recent survey by Benyamini [2] shows that although substantial progress has been made in that area there is still one big question that remains open. It is unclear what kind of infinite–dimensional Banach spaces X satisfy that the unit ball $B(X)$ is Lipschitz equivalent to the unit sphere $S(X)$ in the sense that there is a bijective Lipschitz map from $B(X)$ to $S(X)$ that has a Lipschitz inverse. The space constructed by Gowers and Maurey in [14] provides an example that the answer can not be "every Banach space". It is an open problem whether being a Hilbert space is enough or if there are any natural obstructions.

We will investigate whether a continuous map f from an infinite–dimensional Hilbert space $\mathcal{H}$ into itself that is expanding and whose image has nonempty interior is actually onto, which is a question Louis Nirenberg asked in his book *Topics in Nonlinear Functional Analysis* ([28]) from 1974. Morel and Steinlein found a counterexample to this question in 1984 ([27]) if $\mathcal{H}$ is replaced by the Banach space ℓ^1.

These two problems are closely related, since Benyamini and Lindenstrauss show in [3] that a Lipschitz equivalence between $B(\mathcal{H})$ and $S(\mathcal{H})$ can be used to construct a counterexample to Nirenberg's question, in other words, if one of the problems can be solved with a positive answer it must mean that the answer to the other problem is negative.

In the case that $\mathcal{H}$ is finite dimensional it can be seen that Nirenberg's problem has in fact a positive answer which is a direct consequence of Brouwer's invariance of domain theorem. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an expansive map, i.e. $\|f(x)-f(y)\| \geq \|x-y\|$, with nonempty interior, then it is injective and thus Brouwer's theorem implies that $f(\mathbb{R}^n)$ is open. On the other hand, completeness shows that $f(\mathbb{R}^n)$ is closed and thus $f(\mathbb{R}^n) = \mathbb{R}^n$.

The following examples will show that neither expansiveness nor the image having open interior alone are sufficient to impose surjectivity.

Example 3.1.1. Consider $\mathcal{H} = \ell^2$ and let $R\colon \ell^2 \to \ell^2$ be the right shift operator, i.e. $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$. This map is clearly expanding, it is even an isometry. On the other hand it is easy to see that the right shift is by no means a surjection.

The next example will demonstrate that the image of a map having nonempty interior alone is not enough to imply surjectivity.

Example 3.1.2. Let $\mathcal{H}$ be any Hilbert space, then the projection P onto the closed unit ball is continuous. Obviously $P(\mathcal{H}) = \overline{B}(\mathcal{H}) \neq \mathcal{H}$.

Before we try to answer this question partially, we will first take a small excursion into the geometry of Hilbert space.

3.2 Distortion and quasiconvexity

In order to answer some of the questions raised in the preceding section, we will need the concept of distortion. In colloquial terms, the distortion describes the ratio of distances inside a given set in relation to the distances in the ambient space. Often the distortion of a set is also referred to as the constant of quasiconvexity.

Definition 3.2.1 (Distortion)**.** Let $(X, \|\cdot\|)$ be a Banach space and $U \subset X$. The intrinsic distance $d(x, y)$ between two points x, $y \in U$ is the infimum of lenghts of curves in U connecting x to y. We then define

$$\operatorname{distort}(U) = \sup_{x,y \in U} \frac{d(x, y)}{\|x - y\|}$$

as the distortion of U.

Our results are inspired by results from M. Gromov and P. Pansu, who have proved that a compact set in $\mathbb{R}^n$ whose distortion is less than $\frac{\pi}{2}$ has to be simply connected and if the distortion of a compact set is less than $\frac{\pi}{2\sqrt{2}}$ then the set has to be contractible [15]. The approach used to prove these results is heavily reliant on the Hilbert space in question being finite dimensional. We will be able to prove very similar results by utilizing only basic properties of Hilbert space that are not dependent on finite dimension.

Let B be a bounded convex subset of $\mathcal{H}$, can we estimate the distortion of $\mathbb{R}^n \setminus B$ from below? This is a difficult question to answer even in finite dimensions, for example, it is known that the distortion of $\mathbb{R}^2 \setminus B$ is bounded below by $\pi/2$ and this bound is sharp as a disk realizes this bound. On the other hand, even if $n = 3$ the best known bound is $\pi/2\sqrt{2}$ but it is unclear if this is sharp. Noam Elkies in [13] provided an example that there are closed and bounded convex subsets B of $\mathbb{R}^3$ such that $\operatorname{distort}(\mathbb{R}^3 \setminus B) < \pi/2$ and thus the bound in $\mathbb{R}^3$ can not be improved to the bound from the 2–dimensional case.

We will give bounds in the infinite dimensional case under reasonable assumptions. We need one more definition.

Definition 3.2.2 (Inradius)**.** Let $(X, \|\cdot\|)$ be a Banach space, and $U \subset X$, then we call

$$R_U = \sup_{x \in U} \{r \colon B(x, r) \subset U\}$$

the inradius of U.

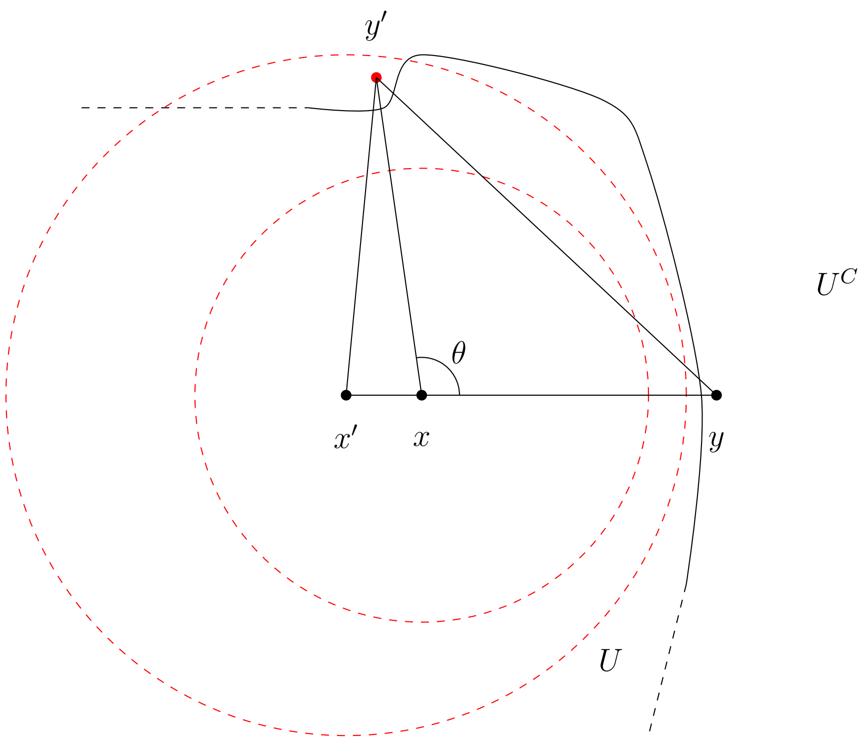

Figure 2: Illustration of Theorem 3.2.3

The first Theorem we will prove explores how Hilbert space gets distorted if we cut out an open set that is not too big in the sense that it does not allow balls of arbitrary size inside of it.

Theorem 3.2.3. *Let $\mathcal{H}$ be a Hilbert space and $U \subset \mathcal{H}$ open such that $0 < R_U < \infty$, then*

$$\operatorname{distort}\left(U^C\right) \geq \frac{\pi}{2\sqrt{2}}$$

The main idea in the proof is to show that we can find points in the complement of U such that the angle to the center of an almost maximal ball contained in U is obtuse, as can be seen in the included figure.

Proof. Let $\epsilon > 0$ and pick x such that $B(x, R_U - \epsilon) \subset U$. Then there is $y \in U^C$ such that $\|x - y\| < R_U + \epsilon$. If we choose $x' = x + 2\sqrt{\epsilon R_U}\frac{x-y}{\|x-y\|}$ we can find $y' \in B(x', R_U + \epsilon) \setminus B(x, R_U - \epsilon)$, with $y' \in U^C$. Using the law of cosines (1.1) we will show that the angle between $y - x$ and $y' - x$, we call it θ, is obtuse.

$$\begin{aligned}\cos\theta &= \frac{\|x' - y'\|^2 - \|x - x'\|^2 - \|x - y'\|^2}{2\|x - x'\|\|x - y'\|} \\ &\leq \frac{(R_U + \epsilon)^2 - 4\epsilon R_U - (R_U - \epsilon)^2}{2\|x - x'\|\|x - y'\|} \leq 0\end{aligned}$$

Since the nearest point projection onto a convex and closed set is a contraction in Hilbert space, we have that

$$d(y, y') \geq d(P(y), P(y')) = (R_U - \epsilon)\theta = (R_U + \mathcal{O}(\epsilon))\theta$$

where P denotes the nearest point projection onto $B(x, R_U - \epsilon)$.

$$\begin{aligned}\|y - y'\| &= \sqrt{\|x - y\|^2 + \|x - y'\|^2 - 2\|x - y\|\|x - y'\|\cos\theta} \\ &\leq \sqrt{(R_U + \epsilon)^2 + (4\epsilon R_U + R_U + \epsilon)^2 - 2(R_U + \epsilon)(4\epsilon R_U + R_U + \epsilon)\cos\theta} \\ &= \sqrt{2(R_U + \mathcal{O}(\epsilon))^2(1 - \cos\theta)} \\ &= (R + \mathcal{O}(\epsilon))\sqrt{2}\sqrt{1 - \cos\theta} = 2(R + \mathcal{O}(\epsilon))\sin\frac{\theta}{2}\end{aligned}$$

This results in

$$\operatorname{distort}(U^C) \geq \frac{(R + \mathcal{O}(\epsilon))\theta}{2(R + \mathcal{O}(\epsilon))\sin\frac{\theta}{2}}$$

for every $\epsilon > 0$. Then letting $\epsilon \to 0$ leads to

$$\operatorname{distort}(U^C) \geq \frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}} \geq \frac{\pi}{2\sqrt{2}}$$

The last inequality is due to the fact that $\dfrac{x}{\sin x}$ is monotonically increasing on $[0, \pi]$ and θ is bounded below by $\dfrac{\pi}{2}$. $\square$

In a very similar fashion an analogous result for cone shaped regions can be proved.

Theorem 3.2.4. *Let $\mathcal{H}$ be a Hilbert space and $U \subset \mathcal{H}$ open such that there exists $0 < \alpha < 1$*

for which $\mathrm{dist}(x, U^C) \leq \alpha\|x\|$ *for every* $x \in \mathcal{H}$, *then*

$$\mathrm{distort}(U^C) \geq \frac{\beta}{2\sin\frac{\beta}{2}},$$

where $\beta = \arccos\alpha$.

Proof. Let $\epsilon > 0$ and then pick $x \in U$ such that $B(x, \alpha\|x\|) \subset U$. Without loss of generality we can assume that $\|x\| = 1$. Then we can find $y \in U^C$ no farther away from x than $\alpha + \epsilon$ and as in the proof of Theorem 3.2.3 we choose $x' = x + \frac{d}{\|x-y\|}(x - y)$ and locate a $y' \in B(x', (1+d)\alpha) \setminus B(x, \alpha)$ with $y' \in U^C$. Let us denote by θ the angle between $y - x$ and $y' - x$, as in the proof of Theorem 3.2.3 we employ the law of cosines (1.1) to find a suitable bound.

$$\begin{aligned}
\cos\theta &= \frac{\|x'-y'\|^2 - \|x-x'\|^2 - \|x-y'\|^2}{2\|x-x'\|\|x-y'\|} \\
&\leq \frac{((1+d)\alpha+\epsilon)^2 - d^2 - (\alpha-\epsilon)^2}{2\|x-x'\|\|x-y'\|} \\
&\leq \frac{((1+d)\alpha+\epsilon)^2 - d^2 - (\alpha-\epsilon)^2}{2d(\alpha-\epsilon)} \\
&= \frac{2\alpha^2 d + d^2(\alpha^2-1) + \mathcal{O}(\epsilon)}{2d\alpha - \mathcal{O}(\epsilon)}
\end{aligned}$$

Choosing $d = \sqrt{\epsilon}$ and letting $\epsilon \to 0$ we obtain $\cos\theta \leq \alpha$.

We see that the intrinsic distance between y and y' is bounded from below by $(\alpha + \mathcal{O}(\epsilon))\theta$ by the same technique employed in the proof of Theorem 3.2.3 and we can bound the regular distance by the following.

$$\begin{aligned}
\|y - y'\| &= \sqrt{\|y-x\|^2 + \|x-y'\|^2 - 2\cos\theta\|y-x\|\|x-y'\|} \\
&\leq \sqrt{(\alpha+\epsilon)^2 + (d + (1+d)\alpha + \epsilon)^2 - 2\cos\theta(\alpha-\epsilon)^2} \\
&= \sqrt{2(\alpha+\mathcal{O}(\epsilon))^2 - 2\cos\theta(\alpha+\mathcal{O}(\epsilon))^2} = (\alpha+\mathcal{O}(\epsilon))\sqrt{2}\sqrt{1-\cos\theta} \\
&= (\alpha+\mathcal{O}(\epsilon))2\sin\frac{\theta}{2}
\end{aligned}$$

Finally, using the monotonicity of $x/\sin x$ on $[0, \pi]$ and combining this with the bound on $\cos\theta$ we have

$$\operatorname{distort}(U^C) \geq \frac{\theta}{2\sin\frac{\theta}{2}} \geq \frac{\beta}{2\sin\frac{\beta}{2}} \qquad \square$$

3.3 Surjectivity and distortion

In this section we will employ the results about the distortion of subsets of Hilbert space to obtain conclusions on the surjectivity of Bilipschitz maps. We begin by taking a look at the distortion of the Bilipschitz image of a space X, since this will be fundamental to our results.

Lemma 3.3.1. *If $f\colon X \to Y$ is a (l,L)-Bilipschitz map, then* $\operatorname{distort}(f(X)) \leq \frac{L}{l}\operatorname{distort}(X)$.

Proof. Let $f(x)$ and $f(y)$ be in $f(X)$ and consider a curve γ in X that connects x and y. Then $f(\gamma)$ defines a curve that connects $f(x)$ and $f(y)$ and since f is L-Lipschitz we conclude that $\operatorname{length}(f(\gamma)) \leq L\operatorname{length}(\gamma)$. This means that $d(f(x), f(y)) \leq Ld(x, y)$ at the same time we have that $\|f(x) - f(y)\| \geq l\|x - y\|$ and thus

$$\operatorname{distort}(f(X)) \leq \frac{L}{l}\operatorname{distort}(X) \qquad \square$$

A first application of this Lemma will give us one condition for when a Bilipschitz map from a Hilbert space into itself is actually onto, as long as it has finite distance from the identity map.

Theorem 3.3.2. *Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \mathcal{H}$ a map such that $\|f(x) - x\| < R < \infty$ for every $x \in \mathcal{H}$. If f is also (l,L)-Bilipschitz with $\frac{L}{l} < \frac{\pi}{2\sqrt{2}}$ then f is onto.*

Proof. Suppose that f is not onto, then $U = \mathcal{H} \setminus f(\mathcal{H})$ is open and non-empty. Since $\operatorname{dist}(x, f(\mathcal{H})) \leq \|x - f(x)\| < R$ for every $x \in U$ we can apply Theorem 3.2.3 and conclude that $\operatorname{distort}(U^C) \geq \frac{\pi}{2\sqrt{2}}$. This is a contradiction to Lemma 3.3.1 and thus f is onto. $\square$

Our next result will focus on maps that satisfy the assumptions of 3.2.4.

Proposition 3.3.3. *Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \mathcal{H}$ a map such that there is $\alpha < 1$ for which $\|f(x) - x\| \leq \alpha\|x\|$ for every $x \in \mathcal{H}$. If f is also (l,L)-Bilipschitz with $\frac{L}{l} < \frac{\beta}{2\sin\frac{\beta}{2}}$, with $\beta = \arccos\alpha$, then f is onto.*

Proof. If we assume that f is not onto we see that $\mathcal{H} \setminus f(\mathcal{H})$ is open and non-empty. By our assumptions for the displacement of $f(x)$ the conditions of Theorem 3.2.4 are satisfied and thus $\operatorname{distort}(f(\mathcal{H})) \geq \frac{\beta}{2\sin\frac{\beta}{2}}$ which contradicts Lemma 3.3.1. □

The next result will give us some more information about when Bilipschitz approximate duality maps are surjective.

Proposition 3.3.4. *Let $\mathcal{H}$ be a Hilbert space and $\varphi\colon \mathcal{H} \to \mathcal{H}$ an η-approximate duality map that is also (l, L)-Bilipschitz. If $\frac{L}{l} < \frac{\beta}{2\sin\frac{\beta}{2}}$, where $\beta = \arcsin\eta$, then φ is onto.*

Proof. Suppose φ is not onto. For $x \in \mathcal{H}$ we denote by θ the angle between $\varphi(x)$ and x. We observe that $\|x\| = \|\varphi(x)\|$ and since φ is an η–approximate duality map we can conclude that $\cos\theta = \frac{\langle\varphi(x),x\rangle}{\|x\|^2} > \eta$. Approximate duality maps are homogeneous of degree 1 and thus we know that the whole line segment L from 0 to $\varphi(x)$ is contained in $\varphi(\mathcal{H})$. We then have that $\operatorname{dist}(x, \varphi(\mathcal{H})) \leq \operatorname{dist}(x, L) = \sin\theta\|x\| < \sqrt{1-\eta^2}\|x\|$. Since $\arccos\sqrt{1-\eta^2} = \arcsin\eta$ this shows that φ satisfies the conditions of Theorem 3.2.4 whose assertion contradicts Lemma 3.3.1. □

It remains unknown whether Bilipschitz approximate duality maps are always onto.

Chapter 4

Lipschitz geometry of tight spans

4.1 Introduction and the general construction

Our main goal in this chapter will be to investigate the Lipschitz geometry of the tight span of a (finite) metric or normed space X. Tight spans were first introduced in 1964 by Isbell in [18], where it was also noted that they have a polyhedral structure if the metric space is finite. We will take a closer look at the Lipschitz geometry by paying special attention to this structure, in particular, we will investigate how vertices of the tight span behave under changes of the underlying metric space and derive stability results from there.

One area where tight spans see use is in phylogenetics, which is the study of evolutionary history. The goal is building a phylogenetic tree and once a number of different species are considered and one has decided on a measure of distance between them one has automatically build a finite metric space. In considering the tight span a tree–like structure can be found. As we will soon see the tight span of a space is rarely a tree and a part of phylogenetic combinatorics deals with the ramifications of this fact in the form of studying tree–like metrics and studying how far a given metric is away from being tree–like.

In addition, it is known (see for example [11]) that the 1–skeleton of the tight span of a finite metric space (X, d) is a realization of the metric space in the form of an edge–weighted graph, i.e the 1–skeleton is a subset of ℓ_∞^n such that the distance in the path–metric between vertices v_x and v_y corresponding to original points x, y of X is precisely $d(x, y)$. Moreover, there is a close relationship between tight spans and optimal realizations as shown in [17].

More recently the construction has been used by Chrobak and Larmore for the *Equipoise* algorithm, which is a 11–competitive algorithm for the 3–server problem (see [8]).

In this chapter we will focus on finite metric spaces spaces where all points have a label, i.e. $X = \{x_0, \dots, x_n\}$, and thus we can write the metric in a few different ways. One way of stating the metric would be as a symmetric matrix $D = (d_{ij})_{i=0,\dots,n}^{j=0,\dots,n}$, where $d_{ij} = d(x_i, x_j)$ and another way would be writing the entries of this matrix in a vector d, leaving out the diagonal entries if it is convenient to do so. This gives us a natural correspondence between an $(n+1)$-point metric space and elements of the normed space ℓ_∞^m, where $m = \binom{n+1}{2}$.

In view of this correspondence, given two metric spaces $X = \{x_0, x_1, \dots, x_n\}$ and $Y = \{y_0, y_1, \dots, y_n\}$ with said vectors of pairwise distances being denoted x and y respectively, we can say that $d(X, Y) = \|x - y\|_\infty$, giving us a natural way to talk about the distance between two finite metric spaces.

We are going to introduce two other ways of measuring the distance between metric spaces, the Hausdorff distance and the Gromov–Hausdorff distance.

Definition 4.1.1 (Hausdorff distance)**.** Let X be a metric space, then for two subsets A and B we call

$$d_H(A, B) = \inf\{\delta \mid A \subset B(\delta), B \subset A(\delta)\}$$

the Hausdorff distance between A and B. Here $A(\delta)$ denotes the δ-neighborhood of A.

It is apparent from the definition, that the Hausdorff distance can only be used if the two metric spaces in question already naturally sit inside another one. We will see that this is the case in the context we will be working in.

Now let us introduce the Gromov–Hausdorff distance.

Definition 4.1.2 (Gromov–Hausdorff distance)**.** Let X and Y be metric spaces, then we call

$$d_{GH}(X,Y) = \inf\{d_H(f(X), g(Y)) \mid f\colon X \to Z, g\colon Y \to Z\},$$

where f and g are isometries, the Gromov–Hausdorff distance between X and Y.

In order to state a more useful, equivalent definition we first need the concept of correspondence.

Definition 4.1.3 (Correspondence)**.** Let X and Y be sets. We call a relation $R \subset X \times Y$ that satisfies

(i) for every $x \in X$ there exists a $y \in Y$ such that $(x, y) \in R$ (right complete)

(ii) for every $y \in Y$ there exists a $x \in X$ such that $(x, y) \in R$ (left complete)

a correspondence between X and Y.

Then for a correspondence R between X and Y, we define

$$\mathrm{AD}(R) = \sup\{|d_X(x_1, x_2) - d_Y(y_1, y_2)| \mid x_1 R y_1, x_2 R y_2\}$$

and it is a well known fact (see for example Theorem 7.3.25 in [6]) that

$$d_{GH}(X,Y) = \frac{1}{2} \inf_R \mathrm{AD}(R)$$

We are now almost ready to define the tight span of a metric space. This is a construction originally invented by Isbell to study and create injective metric spaces. When talking about metric spaces, we say that a space E is injective if every 1–Lipschitz mapping from $A \subset X$ into E can be extended to a 1–Lipschitz map from X into E. In a later chapter we will talk about absolute Lipschitz retracts and want to point out at this point that injective metric spaces and 1–absolute Lipschitz retracts are precisely the same.

Definition 4.1.4 (Tight span)**.** A mapping $e\colon X \to E$ of metric spaces is called the tight span of X if E is injective, e is an isometric embedding and no injective proper subspace of

E contains $e(X)$. In the literature the tight span is sometimes also referred to as an injective envelope or injective hull. We will denote the tight span of X by EX.

Our next goal is to give insight into how Isbell constructed tight spans of metric spaces and indicate how this illustrates the polyhedral structure mentioned earlier. A fundamental part of the tight span construction are so called extremal functions.

Definition 4.1.5 (Extremal function)**.** We define an extremal function on a metric space X as a non-negative function

$$f\colon X \to \mathbb{R}$$

that is pointwise minimal subject to

$$f(x) + f(y) \geq d(x, y)$$

and we denote the space of all extremal functions on X by ϵX.

If $X = \{x_1, \ldots, x_n\}$ is a finite metric space with pairwise distances d_{ij} we can naturally identify extremal functions with elements $z \in \ell_\infty^n$ that satisfy

$$z_i + z_j \geq d_{ij}$$

and are minimal componentwise.

In his paper [18] Isbell proved the following relationship between extremal functions on a space X and its tight span EX.

Theorem 4.1.6 (Isbell)**.** *The map $e\colon X \to \epsilon X$ defined by $e(x)(y) = d(x, y)$ is a tight span and all tight spans are equivalent. Furthermore, the tight span of a finite space is a polyhedral complex.*

The preceding theorem together with the identification of extremal functions with elements of ℓ_∞^n shows that the tight span of an n-point metric space can be viewed as a subset of ℓ_∞^n and as such we are justified in using the Hausdorff distance to measure the distance between the tight spans of n-point metric spaces.

In addition, we can interpret extremal functions as minimal solutions to the matrix inequality $Az \geq d$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \tag{4.1}$$

$z = (z_1, z_2, \ldots, z_n)^T$, $d = (d_{12}, d_{13}, \ldots, d_{23}, \ldots, d_{n-1,n})^T$ and A is the incidence matrix of a complete graph on n vertices. We can see the polyhedral structure of the tight span if we enforce $z_i + z_j = d_{ij}$ for some indices, i.e. if we force equality for some rows of of this matrix inequality. Our next immediate goal will be to explore this relationship. It will prove fruitful to use graphs and subgraphs to describe this precisely and it is not hard to see that a good description of an n–point metric space $X = \{x_1, \ldots, x_n\}$ is a complete weigthed graph on n vertices, where the weight of an edge between vertices i and j is exactly the distance $d(x_i, x_j)$. We call such a graph the associated graph and denote it by Γ_X.

Definition 4.1.7 (Spanning subgraph)**.** For a given graph Γ on n vertices we call a subset S of the edges a spanning subgraph of Γ if every vertex is incident to at least one edge in S. Equivalently, a spanning subgraph can be represented as a set of two point subsets of $\{1, \ldots, n\}$ that covers the whole set.The set of all spanning subgraphs will be denoted by $\mathcal{S}$.

We can associate an incidence matrix $M(S)$ to every spanning subgraph S. This is a binary $|S| \times n$ matrix with constant row sum 2, where every row corresponds to an edge and every column corresponds to a vertex. For every edge we put a 1 in the columns corresponding to the vertices to which the edge is incident. It is clear that this matrix is not unique since the rows can be permuted freely.

This definition allows us to formalize our insight into the structure of tight spans.

Definition 4.1.8 (Cells and combinatorial structure)**.** For a finite metric space $X = \{x_1, \ldots, x_n\}$ with pairwise distances d_{ij} we call

$$\underset{X}{\operatorname{cell}}(S) = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+ \mid z_i + z_j = d_{ij}, \text{ whenever } \{i, j\} \in S \text{ and } z_i + z_j \geq d_{ij} \forall\, i, j\}$$

a cell of X with respect to the spanning subgraph S of Γ_X.

We say two finite metric spaces X and Y have the same combinatorial structure (or are combinatorially equivalent) if

$$\dim \underset{X}{\operatorname{cell}}(S) = \dim \underset{Y}{\operatorname{cell}}(S) \text{ for every } S \in \mathcal{S}$$

In combination with Theorem 4.1.6 and the definition of extremal functions we can see that the combinatorial structure as outlined above describes the polyhedral structure of the tight span and the incidence matrices $M(S)$ correspond to submatrices of the matrix (4.1). If, for a finite metric space (X, d), $\dim \operatorname{cell}_X(S) = 0$ we are dealing with a vertex of the tight span.

We should look at two concrete examples of constructing the tight span of a finite metric space space.

Example 4.1.9 (Tight span of a 3–point metric space)**.** Consider a 3–point metric space $X = \{x_1, x_2, x_3\}$ with pairwise distances denoted by d_{ij}. Then the vertices of EX corresponding to the points of X are (with a slight abuse of notation) $x_1 = (0, d_{12}, d_{13})$, $x_2 = (d_{12}, 0, d_{23})$ and $x_3 = (d_{13}, d_{23}, 0)$ and the only other vertex z of the tight span has to correspond to the subgraph $S = \{(1, 2), (1, 3), (2, 3)\}$, since that is the only possible subgraph whose incidence matrix can have rank 3 and consequently $\dim \operatorname{cell}_X(S) = 0$. This yields $z = {}^1\!/_2(d_{12} + d_{13} -$

$d_{23}, d_{12} + d_{23} - d_{13}, d_{13} + d_{23} - d_{12})$ and we should note that all components of z are non–negative because of the triangle inequality. In particular, we observe that if one component of z is 0, then z coincides with the vertex corresponding to one of the elements of X and closer inspection reveals that $\|x_i - z\|_\infty = z_i$, for $i = 1, 2, 3$. We can use these results for a nice representation of the space as

d	x_1	x_2	x_3
x_1	0	$z_1 + z_2$	$z_1 + z_3$
x_2	$z_1 + z_2$	0	$z_2 + z_3$
x_3	$z_1 + z_3$	$z_2 + z_3$	0

and furthermore we can visualize these results in the following figure, which highlights the polyhedral structure.

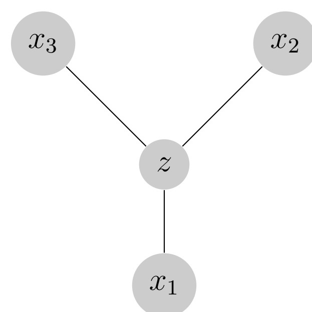

Figure 3: EX of 4.1.9

This illustrates clearly that the only combinatorial structures possible are a tripod or a line segment.

In the next example we want to take a quick look at the tight span of a 4–point metric space.

Example 4.1.10 (Tight span of a 4–point metric space)**.** In [10] it is shown that given a general 4–point metric space $X = \{x_1, x_2, x_3, x_4\}$, we can find nonnegative numbers a, b, c, d, e and f such that the metric, after relabeling if necessary, can be represented as

d	x_1	x_2	x_3	x_4
x_1	0	$a+b+f$	$a+c+e+f$	$a+d+e$
x_2	$a+b+f$	0	$b+c+e$	$b+d+e+f$
x_3	$a+c+e+f$	$b+c+e$	0	$c+d+f$
x_4	$a+d+e$	$b+d+e+f$	$c+d+f$	0

These numbers occur in the tight span in the following way.

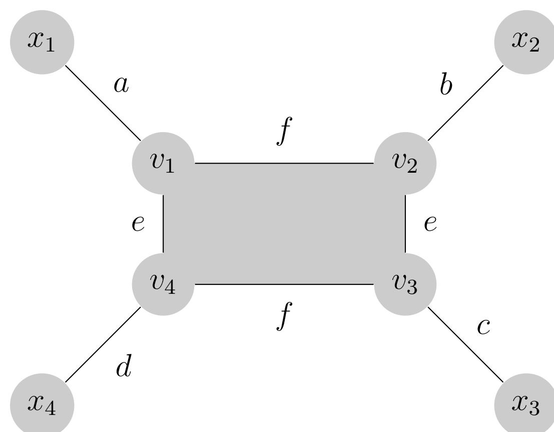

Figure 4: EX of 4.1.10

This representation is possible, since even though the tight span of a 4–point metric space naturally sits in ℓ_1^4, the highest dimensional cells are of dimension 2 ([16]). This is the smallest example where we can see that that there are multiple different combinatorial structures possible. The tight span could be a square (in case $a = b = c = d = 0$), a cross (if $e = f = 0$), have the shape of an H (if either $e = 0$ or $f = 0$) or even look like a horse (if a, b, c or $d = 0$).

4.2 Lipschitz continuity of vertices in the same combinatorial type

This section is dedicated to taking a closer look at how sensitive the position of vertices of the tight span is under pertubation of the underlying metric. We will see that this requires a better understanding of the matrices $M(S)$ corresponding to spanning subgraphs S which have full rank. We need to establish results about the general structure of these matrices to facilitate this investigation. This will be done in two steps.

Proposition 4.2.1. *After possibly relabeling the vertices of a graph on n vertices, the matrix $M(S)$ of a spanning subgraph S with $|S| = n$ and m connected components can be put into the form*

$$M(S) = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & C_m \end{pmatrix}$$

where C_i is the matrix of a connected component of the spanning subgraph. The matrix is invertible if and only if every C_i is an invertible, square matrix.

Proof. Let n_i $(i = 1, \ldots, m)$ denote the number of vertices in the i-th connected component, then we can write $\{1, \ldots, n\} = \bigcup_{i=1,\ldots,m} A_i$, where $|A_i| = n_i$ and the elements of A_i correspond to the vertices in the i-th connected component. It is clear that if $\{i, j\} \in S$ then both i and j are elements of the same A_k for some k and we call n'_k the number of such pairs in A_k. Now labeling the vertices in A_k by $\{n'_1 + \ldots + n'_{k-1} + 1, \ldots, n'_1 + \ldots + n'_k\}$ and have the $n'_1 + \ldots + n'_{k-1} + 1$ to $n'_1 + \ldots + n'_k$ rows corresponds to pairs $\{i, j\}$ where one (and thus both) of the indices is an element of A_k we achieve the desired form of our matrix.

Since the resulting matrix is block diagonal we can apply a well known result from linear algebra that block diagonal matrices are invertible if and only if every block is square and invertible to finish the proof. $\square$

The next step will be to investigate the structure of the matrices of the connected components as the behavior of them will determine the behavior of $M(S)$ as a whole. Fortunately we have the following result.

Proposition 4.2.2. *Let S be a connected covering of a graph on n vertices with $|S| = n$. Then the incidence matrix $M(S)$ can be arragend into the form*

$$M(S) = \begin{pmatrix} A & 0 \\ R_1 & R_2 \end{pmatrix}$$

where $A = (a_{ij})$ is an $m \times m$ matrix of the form

$$a_{ij} = \begin{cases} 1 & , \textit{ if } i = j \\ 1 & , \textit{ if } i + 1 = j \\ 1 & , \textit{ if } i = m, j = 1 \\ 0 & \textit{otherwise} \end{cases}$$

where m is the size of the largest cycle in S, and the matrix $R = (R_1|R_2)$ is a binary matrix with constant row sum 2 and R_2 is a lower triangular matrix with unit diagonal.

Proof. A graph on n vertices with n edges can't be a tree (see for example [7, Theorem 7.7]) and as such has to contain at least one cycle. We can label the vertices in the largest cycle 1 to m in such a way that $\{1, 2\}, \{2, 3\}, \ldots, \{m-1, m\}, \{1, m\} \in S$ and assign the remaining $n - m$ numbers to the other vertices. We can further permute the rows in such a way that the i-th row ($i = 1, \ldots, m-1$) corresponds to $\{i, i+1\}$ and the m-th row corresponds to $\{1, m\}$. After this process we have achieved a matrix of the form

$$\begin{pmatrix} A & 0 \\ R_1 & R_2 \end{pmatrix}$$

and since S is connected there is an edge represented by $\{i, j\}$ with $i \in \{1, \ldots, m\}$ and $j > m$ which connects some vertex j to the cycle (or more generally to the first m connected vertices). After permuting the row corresponding to this edge into the $(m + 1)$ row and switching the labels of the vertices j and $(m+1)$ the first row of R_1 is precisely e_i (size m) and the first row of R_2 is exactly e_1 (size $n-m$). The remaining $n-m-1$ vertices still have to be connected to the first $m+1$ ones so we can repeat this argument to achieve the desired form of the matrix. □

The next Lemma will show that if we want A to be invertible the cycles have to be of odd length and furthermore shows how the inverse has to look explicitly. The invertibility result was already proved by Cyriel van Nuffelen in [31] in 1976, but it is included in this proof for completeness sake.

Lemma 4.2.3. *The $m \times m$ matrix $A = (a_{ij})$ with*

$$a_{ij} = \begin{cases} 1 & , \textit{if } i = j \\ 1 & , \textit{if } i+1 = j \\ 1 & , \textit{if } i = m, j = 1 \\ 0 & \textit{otherwise} \end{cases}$$

is invertible with inverse $A^{-1} = (b_{ij})$ where

$$b_{ij} = \begin{cases} \frac{1}{2}(-1)^{i+j} & , \textit{if } i \leq j \\ -\frac{1}{2}(-1)^{i+j} & , \textit{if } i > j \end{cases}$$

if m is odd and singular otherwise.

Proof. If m is even we can compute the determinant of A by expanding after the first column and we immediately see that in that case $\det(A) = 0$. Now let m be odd. The matrix AA^{-1} consists of rows where, every row is made up of the sum of two consecutive rows (mod m)

of A^{-1}. First let us consider the i-th row for $1 \leq i < m$, then

$$b_{kj} + b_{(k+1)j} = \begin{cases} \frac{1}{2}((-1)^{i+j} + (-1)^{i+1+j}) = 0 & \text{, if } i < j \\ \frac{1}{2}((-1)^{i+j} + (-1)^{i+2+j}) = 1 & \text{, if } i = j \\ -\frac{1}{2}((-1)^{i+j} + (-1)^{i+1+j}) = 0 & \text{, if } i > j \end{cases}$$

and in the m-th row we will have

$$b_{mj} + b_{1j} = \begin{cases} \frac{1}{2}((-1)^{2m} + (-1)^{m+1}) = 1 & \text{, if } m = j \\ \frac{1}{2}((-1)^{m+1+j} + (-1)^{1+j}) = 0 & \text{, if } m > j \end{cases}$$

which works since m is odd. □

This looks a little unappealing, so let us take a look at the inverse if the matrix A in Lemme 4.2.3 is of dimension 5×5.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

We can see that we have a band of 1's on the diagonal and the first subdiagonal. In each row the entries outside of this band are alternating in sign, but all of magnitude $1/2$.

Lemma 4.2.4. *If the matrix $M(S)$ representing a spanning subgraph of S has an inverse, then*

$$\|M(S)^{-1}\| \leq n - \frac{3}{2}$$

and this bound is sharp.

Proof. Let m be the number of connected components of the spanning subgraph S and let

$M(S)$ be invertible. Then by Proposition 4.2.1 $M(S)$ is of the form

$$M(S) = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & C_m \end{pmatrix}$$

and as such the inverse of this block-diagonal matrix has the form

$$M(S)^{-1} = \begin{pmatrix} C_1^{-1} & 0 & \cdots & 0 \\ 0 & C_2^{-1} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & C_m^{-1} \end{pmatrix}$$

which shows that $\|M(S)^{-1}\| = \max(\|C_1^{-1}\|, \ldots, \|C_m^{-1}\|)$. This shows that for the norm of the inverse it will be enough to investigate the norm of the inverse of the connected components of S.

Now by Proposition 4.2.2 we can assume that each of these C_i has the form

$$C = \begin{pmatrix} A & 0 \\ R_1 & R_2 \end{pmatrix}$$

Since this is a block triangular matrix we know that the inverse has to be of the form

$$C^{-1} = \begin{pmatrix} A^{-1} & 0 \\ D_1 & D_2 \end{pmatrix}$$

and we will show that the entries of D_1 and D_2 are either 1 or $\frac{1}{2}$ in absolute value. If we denote the rows of A^{-1} by a_i and the rows of $(D_1 \mid D_2)$ by a_{k+i} we can write

$$CC^{-1} = C \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{pmatrix}$$

by the structure of C we have that $e_{k+1} = a_j + a_{k+1}$ for some $j \leq k$, i.e. $a_{k+1} = e_{k+1} - a_j$ a vector where the first $k+1$ entries are nonzero and the rest is 0. Since this is the matrix of a connected component, we have for $m > k+1$ that $a_m = e_m - a_j$ for some $j < m$.
We have shown that the entries in every row of C^{-1} are either 0, $\frac{1}{2}$ or 1 in absolute value and that the number of entries of size $\frac{1}{2}$ is exactly the size of A, which is at least 3. This means that $\|C^{-1}\| \leq n - \frac{k}{2} \leq n - \frac{3}{2}$.
If we consider the matrix

$$M(S) = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \ddots & 0 \\ 1 & 0 & 1 & 0 & \ddots & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

we can see that $\|M(S)^{-1}\| = n - \frac{3}{2}$ so the bound is sharp. $\square$

If the vector d denotes the pairwise distances in the space X, then we understand the notation

$$z = M(S)^{-1} d$$

as the point z in ℓ_∞^n that is a result of multiplying the matrix $M(S)^{-1}$ with a subvector of d corresponding to the pairs $\{i, j\}$ in S. This in particular shows that if S is a spanning subgraph corresponding to a vertex in the tight span and if $\hat{d}$ is a metric with the same combinatorial structure then $\|M(S)^{-1}d - M(S)^{-1}\hat{d}\|_\infty \le (n - 3/2)\,\|d - \hat{d}\|_\infty$, i.e. the position of vertices is a Lipschitz function of the metrics as long as the metrics give rise to tight spans with the same combinatorial type.

4.3 Global continuity of vertices

For the investigation of the global continuity of vertices, the following Theorem from page 66 of [30] is needed.

Theorem 4.3.1 (Schneider). *Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of convex subsets of ℓ_∞^n converging to the convex subset K. Then for $r = 0, \ldots, n-1$,*

$$\overline{(K)_r} \subset \liminf_{i\to\infty} (K_i)_r$$

Here $x \in \liminf_{i\to\infty} A_i$ if every neighborhood of x meets A_i for almost all i.

In the Theorem and throughout the rest of the thesis, the notation $(K)_r$ describes the r–skeleton of K.

By its definition, points $z = (z_1, \ldots, z_n)$ of a tight span are minimal and thus have to satisfy $z_i + z_j = d_{ij}$ for some pairs $\{i, j\}$. This leads to the following definition.

Definition 4.3.2 (Equation set). For every $x = (x_1, \ldots, x_n) \in EX$ we call the set of two point subsets of $\{1, \ldots, n\}$, such that $x_i + x_j = d_{ij}$ the equation set of x, denoted by $\mathrm{eq}(x)$.

Just as for spanning subgraphs we can associate an incidence matrix to $\mathrm{eq}(x)$ and so when we write $\mathrm{rk}(\mathrm{eq}(x)) = k$, we mean that the rank of the associated incidence matrix is k.

If the rank $\mathrm{rk}(\mathrm{eq}(x)) = k$ for $x \in EX$ then x is an interior point of an $(n-k)$–dimensional hyperplane contained in EX.

The language we have used so far lends itself to an algebraic point of view of vertices via matrices. Now we want to express this in terms of $\mathrm{eq}(x)$ and extend it to describe edges as well.

Definition 4.3.3 (Algebraic vertex and edge)**.** Let $d = (d_{ij}) \in \ell_\infty^{\binom{n}{2}}$, $H_{ij}^+(d) = \{z \in \ell_\infty^n \colon z_i + z_j \geq d_{ij}\}$ and $H^+(d) = \bigcap_{i,j} H_{ij}^+(d)$. We say that an element $z = (z_1, \ldots, z_n) \in H^+(d)$ is an algebraic vertex if z satisfies n linear independent equations $z_i + z_j = d_{ij}$. An algebraic edge is a segment $[a, b] \subset H^+(d)$ such that a, b are algebraic vertices of $H^+(d)$ and $\mathrm{rk}(\mathrm{eq}(z)) = n-1$ for every $z \in (a, b)$.

The relationship between EX of a finite metric space (X, d) and $H^+(d)$ from the previous definition is explored in [11], where it is shown that the faces/vertices of EX are precisely the bounded faces of $H^+(d)$.

The fact that EX has a polyhedral structure implies that a rather geometric point of view can be taken during this investigation and the next definition formalizes this.

Definition 4.3.4 (Geometric vertex and edge)**.** Let $H^+ \subset \mathbb{R}^n$ be convex. We call a convex subset $F \in H^+$ a face if for $x, y \in H^+$ such that $\frac{x+y}{2} \in F$ we have that $x, y \in F$. A geometric vertex is a 0 dimensional face and a geometric edge is a 1 dimensional face.

Our next immediate goal is to show that these definitions of geometric and algebraic vertices/edges are equivalent, but to do so we need two Lemmas, one that addresses the equation set of points on the line segment $[a.b]$ between two elements of the tight span and one that refines this result to the special case when $[a, b]$ is an algebraic edge.

Lemma 4.3.5. *Let X be a metric space, then for a, $b \in EX$ we have*

$$\mathrm{eq}\left(\frac{a+b}{2}\right) = \mathrm{eq}(a) \cap \mathrm{eq}(b).$$

Proof. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ then

$$\begin{aligned} d_{ij} &= \frac{a_i + b_i}{2} + \frac{a_j + b_j}{2} \\ &= \frac{a_i + a_j}{2} + \frac{b_i + b_j}{2} \\ &\geq \frac{d_{ij}}{2} + \frac{d_{ij}}{2} = d_{ij} \end{aligned}$$

with equality if and only if $a_i + a_j = b_i + b_j = d_{ij}$ □

Lemma 4.3.6. *Two vertices a and b in EX are connected by an edge if and only if*

$$\operatorname{rk}(\operatorname{eq}(a) \cap \operatorname{eq}(b)) = n - 1$$

Proof. If $[a, b]$ is an algebraic edge, then $\operatorname{rk}\operatorname{eq}(z) = n-1$ for every $z \in (a, b)$. But by Lemma 4.3.5 we have $\operatorname{eq}(a) \cap \operatorname{eq}(b) = \operatorname{eq}\left(\frac{a+b}{2}\right)$ so their ranks are the same.
Now suppose that $\operatorname{rk}(\operatorname{eq}(a)\cap\operatorname{eq}(b)) = n-1$, then it is clear that every $z = (z_1, \dots, z_n) \in (a, b)$ satisfies $\operatorname{rk}\operatorname{eq}(z) = n - 1$ and we need to show that the subgraph S associated with the equation set of z is spanning. Suppose on the contrary that $1 \notin \bigcup_{A\in\operatorname{eq}(z)} A$, then $\operatorname{eq}(a)\cap\operatorname{eq}(b)$ completely determines $z_2, \dots, z_n$ and moreover $z_i = a_i = b_i$ for $i = 2, \dots, n$. Thus every $z \in (a, b)$ is given by $(\lambda a_1 + (1 - \lambda)b_1, a_2, \dots, a_n)$, where without loss of generality $a_1 > b_1$. We know that $\{1, i\} \in \operatorname{eq}(a)$ for some i and as such satisfies $a_1 + a_j = d_{1j}$, but then z satisfies $z_1 + z_j = \lambda a_1 + (1 - \lambda)b_1 + a_j < a_1 + a_j = d_{1j}$, which is a contradiction. □

We are now ready to show that the algebraic and geometric definitions are in fact equivalent.

Proposition 4.3.7. *Let $d \in \ell_\infty^{\binom{n}{2}}$, then $z \in \ell_\infty^n$ is an algebraic vertex of $H^+(d)$ if and only if it is a geometric vertex of $H^+(d)$ and a segment $[a, b] \subset H^+(d)$ is an algebraic edge if and only if it is a geometric edge.*

Proof. We will start by showing that an algebraic vertex Z of $H^+(d)$ is always a geometric vertex. Suppose that there exist x, $y \in H^+(d)$ such that $z = \lambda x + (1 - \lambda)y$ then by the definition of an algebraic vertex there exist n equations such that

$$\begin{aligned} d_{ij} = z_i + z_j &= \lambda x_i + (1-\lambda)y_i + \lambda x_j + (1-\lambda)y_j \\ &= \lambda(x_i + x_j) + (1-\lambda)(y_i + y_j) \\ &\geq \lambda d_{ij} + (1-\lambda)d_{ij} \\ &= d_{ij} \end{aligned}$$

whenever $(i,j) \in \mathrm{eq}(z)$. Thus all the inequalities above are actually equalities, in particular $x_i + x_j = y_i + y_j = d_{ij}$. But the equations were linearly independent and thus the solution z is unique. That means $x = y = z$, which finishes this part of the proof. On the other hand if z is not an algebraic vertex, then $\mathrm{rk}(z) < n$ and thus z is an interior point of a hyperplane of dimension at least 1. But this is a direct contradiction to the definition of a geometric vertex. Thus we have shown that algebraic and geometric vertices are equivalent.

Now take $z \in H^+(d)$ such that $\mathrm{rk}(\mathrm{eq}(z)) = n-1$ and let $L = \{z' \in H^+(d) : \ \mathrm{eq}(z') = \mathrm{eq}(z)\}$ be the open line segment containing z. Then $\overline{L}$ is a geometric edge. If it is not, then $L \subset D$, where D is an open 2-dimensional polygon, which in turn implies that every equation in $\mathrm{eq}(z)$ defines a hyperplane that contains D. This contradicts $\mathrm{rk}(\mathrm{eq}(z)) = n-1$. Since every algebraic edge arises in this way this shows that algebraic edges are always geometric edges.

If $L \subset H^+(d)$ is a geometric edge then by Lemma 4.3.5 we have that $\mathrm{rk}(\mathrm{eq}(z)) \leq n-1$ for every interior point z of L. If however $\mathrm{rk}(\mathrm{eq}(z)) < n-1$ for some $z \in L$ we have that $\mathring{L}$ is contained in some 2-dimensional polygon, which would contradict L being a geometric edge. We have shown that every geometric edge is also an algebraic edge. □

This equivalence empowers us to show how vertices and edges of EX behave under taking limits, here the most important result is that if we consider a sequence of metric spaces with the same combinatorial structure, then their limit does not gain vertices and all vertices of the limit come from the limit of vertices.

Proposition 4.3.8. *Let* $(X_i, d^{(i)})$ *be a sequence of* n*-point metric spaces converging to an* n*-point metric space* (X, d)*, such that all* X_i *have the same combinatorial structure. For every spanning subgraph* S *with* $\operatorname{rk}(M(S)) = n$ *such that there exists* $v_S^i \in EX_i$ *we have that* $v = \lim_{i\to\infty} v_S^i$ *exists and* $\operatorname{eq}(v_S^i) \subset \operatorname{eq}(v_S)$*. In addition, every vertex of* EX *arises in this way.*

If $[a^{(i)}, b^{(i)}]$ *is an edge in* EX_i*, then* $a = \lim_{i\to\infty} a^{(i)}$ *and* $b = \lim_{i\to\infty} b^{(i)}$ *exist and* $[a, b]$ *is an edge of* EX *that is possibly degenerate. Also, all edges of* EX *arise in this way.*

Proof. Since all of the X_i have the same combinatorial structure, there is an integer N and spanning subgraphs $S_1, \ldots, S_N$ with rank n incidence matrix such that the vertices of EX_i are precisely the $v_{S_j}^i$, $j = 1, \ldots, N$. Due to the algebraic definition of the vertices via spanning subgraphs, i.e. $v_{S_j}^i = M(S_j)^{-1} d^{(i)}$, it is clear that $v_{S_j} = \lim_{i\to\infty} v_{S_j}^i$ exists as a vertex of X for every j. At the same time we have by Theorem 4.3.1 that the vertices of EX are a subset of $\{v_{S_j} : j = 1, \ldots, N\}$ and thus the two sets are equal.

Now let $[a^{(i)}, b^{(i)}]$ be an edge in EX_i, then the previous part of the proof shows that $a = \lim_{i\to\infty} a^{(i)}$ and $b = \lim_{i\to\infty} b^{(i)}$ exist and are vertices of EX. Furthermore, every $z^{(i)} \in (a^{(i)}, b^{(i)})$ satisfies that

$$\operatorname{rk}\operatorname{eq}(z^{(i)}) = \operatorname{rk}(\operatorname{eq}(a^{(i)}) \cap \operatorname{eq}(b^{(i)})) = n - 1$$

and as $i \to \infty$ we know that $\operatorname{rk}\operatorname{eq}(z) \geq n - 1$. If $\operatorname{rk}\operatorname{eq}(z) = n - 1$ we know that z is on an edge between a and b by Lemma 4.3.6 and if $\operatorname{rk}\operatorname{eq}(z) = n$ we have that $a = b = z$ and the edge is degenerate. But by the same argument as above, Theorem 4.3.1 shows that no new edges exist in the limit. This finishes the proof. $\square$

The previous Proposition shows that we can't gain vertices in the tight span by passing to a limit as long as all other spaces have the same combinatorial structure and the only thing that could possibly happen is that multiple distinct vertices of X_i collapse to the same vertex of X.

4.4 Lipschitz continuity of the 1–skeleton with respect to the Gromov–Hausdorff distance

In this section we will investigate the Lipschitz–stability of the 1–skeleton of tight spans with respect to the Gromov–Hausdorff distance, but before we start we need to get some preliminary results out of the way. The first of them deals with how we can express the length of an edge,

Lemma 4.4.1. *The length of an edge $[a, b]$ in the 1-skeleton $(EX)_1$ of a finite metric space (X, x) is given by*

$$\operatorname{length}([a, b]) = \|(M(S_2)^{-1} - M(S_1)^{-1})x\|_\infty$$

where $a = M(S_1)^{-1}x$ and $b = M(S_2)^{-1}x$. Consequently, every subsegment of the form $[a, c]$ or $[c, b]$ has length $\lambda \operatorname{length}([a, b])$ for some $0 < \lambda < 1$.

If we denote by $d_X(a, b)$ the distance between a and b in the 1-skeleton equipped with the path-metric, this demonstrates that $d_X(a, b) = \|(M(S_2)^{-1} - M(S_1)^{-1})x\|_\infty$.

Proof. Considering that $(EX)_1 \subset \ell_\infty^n$ for some n this is trivial. The second part becomes evident once we realize that $c \in (a, b)$ can be written as $(1 - \lambda)a + \lambda b$. □

We now address the difference in length between corresponding edges of two spaces with the same combinatorial type.

Lemma 4.4.2. *Let X and Y be two n–point metric spaces with the same combinatorial type. Then, using spanning subgraphs, there is a natural correspondence between vertices (and thus edges) of the 1-skeletons. Let $[a, b]$ be an edge in $(EX)_1$ and $[\hat{a}, \hat{b}]$ the corresponding edge in $(EY)_1$. We then have*

$$|\operatorname{length}([a, b]) - \operatorname{length}([\hat{a}, \hat{b}])| \leq (2n - 3)d(X, Y)$$

Proof. We denote by x the vector describing the pairwise distances in X and analogously those of y by Y. If $\text{length}([a,b]) = \|(M(S_2)^{-1} - M(S_1)^{-1})x\|_\infty$ we have

$$\begin{aligned}
|\,\text{length}([a,b]) - \text{length}([\hat{a},\hat{b}])| &= \big|\|(M(S_2)^{-1} - M(S_1)^{-1})x\|_\infty - \|(M(S_2)^{-1} - M(S_1)^{-1})y\|_\infty\big| \\
&\le \left\|M(S_1)^{-1}x - M(S_1)^{-1}y + M(S_2)^{-1}y - M(S_2)^{-1}x\right\|_\infty \\
&\le \left\|M(S_1)^{-1}x - M(S_1)^{-1}y\right\|_\infty + \left\|M(S_2)^{-1}y - M(S_2)^{-1}x\right\|_\infty \\
&\le \|M(S_1)\|\|x-y\|_\infty + \|M(S_2)\|\|x-y\|_\infty
\end{aligned}$$

Now we can employ Lemma 4.2.4 and get

$$\begin{aligned}
\|M(S_1)\|\|x-y\|_\infty + \|M(S_2)\|\|x-y\|_\infty &\le 2(n - 3/2)\|x-y\|_\infty \\
&= (2n-3)\|x-y\|_\infty
\end{aligned}$$

And since we consider finite labeled metric spaces we have that $d(X,Y) = \|x-y\|_\infty$. $\square$

Closer inspection of this proof reveals that the result is not dependent on $[a,b]$ being an edge, but rather holds for every line segment $[a.b]$, where a and b are vertices of the tight span EX. This allows us to prove the following.

Theorem 4.4.3. *Let X and Y be two n-point metric spaces with the same combinatorial structure. Then*

$$d_{GH}((EX)_1, (EY)_1) \le \frac{(n2^{n-3}+2)(2n-3)}{2} d(X,Y)$$

where both 1-skeletons are equipped with the path metric.

Proof. We denote the vector of pairwise distances between points of X by x and analogously those of Y by y. Then $d(X,Y) = \|x-y\|_\infty$.

We define a correspondence between $(EX)_1$ and $(EY)_1$ in the following way. A vertex v in $(EX)_0$ corresponds to a spanning subgraph S via $v = M(S)^{-1}x$ and so we define

$$vRw \Leftrightarrow v = M(S)^{-1}x, \text{ and } w = M(S)^{-1}y$$

where $v \in (EX)_0$ and $w \in (EY)_0$. Similarly we can identify a point c in the 1-skeleton of X that is not a vertex via $c = (\lambda M(S_1)^{-1} + (1-\lambda)M(S_2)^{-1})x$ for some $0 < \lambda < 1$. We define

$$cRd \Leftrightarrow c = (\lambda M(S_1)^{-1} + (1-\lambda)M(S_2)^{-1})x, \text{ and } d = (\lambda M(S_1)^{-1} + (1-\lambda)M(S_2)^{-1})y$$

where $c \in (EX)_1 \setminus (EX)_0$ and $d \in (EY)_1 \setminus (EY)_0$. Since the spaces share the same combinatorial structure this defines a correspondence between their 1-skeletons.

Now given two elements x_1 and x_2 in the 1-skeleton of X we can find a sequence of consecutive vertices $v_0, \ldots, v_m$ such that they lie on a path from x_1 to x_2 realizing the path distance, i.e.

$$d(x_1, x_2) = d(x_1, v_0) + \sum_{k=1}^{m} d(v_{k-1}, v_k) + d(v_m, x_2)$$

If $|d(x_1, x_2) - d(y_1, y_2)| = d(y_1, y_2) - d(x_1, x_2)$ we can use this sequence of vertices to write

$$d(y_1, y_2) - d(x_1, x_2) = d(y_1, y_2) - d(x_1, v_0) - \sum_{k=1}^{m} d(v_{k-1}, v_k) + d(v_m, x_2)$$

Since the two tight spans have the same combinatorial structure, we can use the sequence of corresponding vertices $\hat{v}_0, \ldots, \hat{v}_m$ to find an upper bound on $d(y_1, y_2)$ and so we have

$$d(y_1, y_2) - d(x_1, x_2) \leq d(y_1, \hat{v}_0) + \sum_{k=1}^{m} d(\hat{v}_{k-1}, \hat{v}_k) + d(\hat{v}_m, x_2) - d(x_1, v_0) - \sum_{k=1}^{m} d(v_{k-1}, v_k) + d(v_m, x_2)$$

Using the triangle inequality, we can look at expressions of the form $|d(y_1, \hat{v}_0) - d(x_1, v_0)|$, $|d(\hat{v}_{k-1}, \hat{v}_k) - d(v_{k-1}, v_k)|$ and $|d(\hat{v}_m, y_2) - d(v_m, x_2)|$, which by Lemma 4.4.2 can all be bounded above by $(2n-3)\|x - y\|_\infty$ and as such we have shown

$$d(y_1, y_2) - d(x_1, x_2) \leq (m+2)(2n-3)\|x - y\|_\infty$$

If $|d(x_1, x_2) - d(y_1, y_2)| = d(x_1, x_2) - d(y_1, y_2)$ we can pick a sequence of consecutive vertices $w_0, \ldots, w_M$ and repeat the above argument to prove

$$d(x_1, x_2) - d(y_1, y_2) \leq (M+2)(2n-3)\|x-y\|_\infty$$

Using the total number of edges, which according to [16] is bounded above by $n2^{n-3}$, as an upper bound for m and M we arrive at

$$d_{GH}((EX)_1, (EY)_1) \leq \frac{(n2^{n-3}+2)(2n-3)}{2}\|x-y\|_\infty \qquad \square$$

The above Proposition is heavily reliant on the fact that the spaces X and Y share the same combinatorial structure, as the following example demonstrates.

Example 4.4.4. Consider the following sequence of 4-point metric spaces (X_ϵ, d_ϵ)

d_ϵ	x_1^ϵ	x_2^ϵ	x_3^ϵ	x_4^ϵ
x_1^ϵ	0	3	$4+\epsilon$	$1+\epsilon$
x_2^ϵ	3	0	$1+\epsilon$	$2+\epsilon$
x_3^ϵ	$4+\epsilon$	$1+\epsilon$	0	3
x_4^ϵ	$1+\epsilon$	$2+\epsilon$	3	0

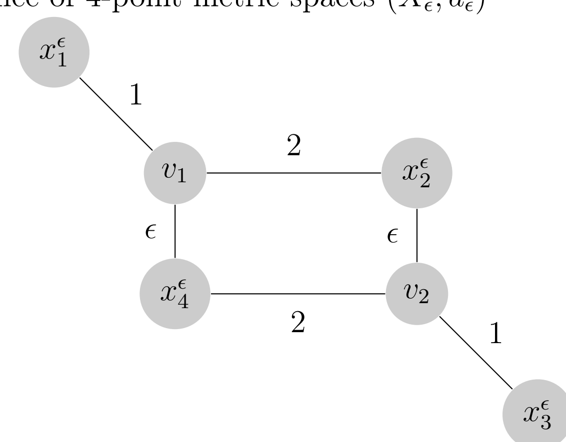

Figure 5: EX_ϵ of 4.4.4

and their limit (X, d).

d	x_1	x_2	x_3	x_4
x_1	0	3	4	1
x_2	3	0	1	2
x_3	4	1	0	3
x_4	1	2	3	0

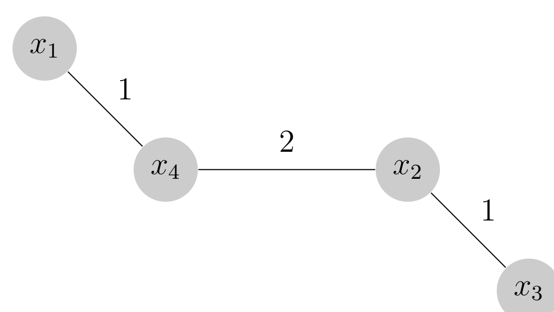

Figure 6: EX of 4.4.4

It is immediate that $d(X^\epsilon, X) = \epsilon$. Consider the set

$$S = \left\{ x_1^\epsilon, x_2^\epsilon, x_3^\epsilon, x_4^\epsilon, \frac{1}{2}(x_2^\epsilon + v_1), \frac{1}{2}(x_4^\epsilon + v_2) \right\} \subset EX_\epsilon$$

which is 1–separated, i.e. for any 2 points in this set the distance between them is at least 1. The tight span of X however is a line segment of length 4, which means the cardinality of any $(1 - \delta)$–separated subset can be at most 5 as long as $\delta < 1/5$. This implies that $d_{GH}(EX_\epsilon, EX) \geq 1/5$.

4.5 Lipschitz continuity of the 1–skeleton with respect to the Hausdorff distance

We have seen that there is a natural limit to how far we can push our results with respect to the Gromov–Hausdorff distance, so in this section we will take a closer look at stability results with a more specific metric. Unlike the Gromov–Hausdorff distance, the use of the Hausdorff distance in this section is very reliant on the fact that the tight span of an n–point metric space can be considered as a subset of ℓ_∞^n. The first result we will prove is the analogue of 4.4.3 with respect to the Hausdorff distance and we will see that we can obtain a (much) better bound on the Lipschitz constant.

Theorem 4.5.1. *Let X and Y be two n-point metric spaces with the same combinatorial structure, then*

$$d_H((EX)_1, (EY)_1) \leq \left(n - \frac{3}{2}\right) d(X, Y)$$

Proof. We denote the vector of pairwise distances between points of X by x and analogously those of Y by y. Let c be a point in the 1-skeleton of X, then there are spanning subgraphs S_1 and S_2 such that $c \in [M(S_1)^{-1}x, M(S_2)^{-1}x]$. Thus there is some constant $0 \leq \lambda \leq 1$ such that $c = \lambda M(S_1)^{-1}x + (1 - \lambda)M(S_2)^{-1}x$. If $\hat{c}$ is the point in $(EX)_1$ that corresponds to c

we have

$$\begin{aligned}
\|c-\hat{c}\|_\infty &= \|\lambda M(S_1)^{-1}x + (1-\lambda)M(S_2)^{-1}x - \lambda M(S_1)^{-1}x + (1-\lambda)M(S_2)^{-1}x\|_\infty \\
&\leq \lambda\|M(S_1)^{-1}\|\|x-y\|_\infty + (1-\lambda)\|M(S_2)^{-1}\|\|x-y\|_\infty \\
&\leq \left(n-\frac{3}{2}\right)\|x-y\|_\infty
\end{aligned}$$

We have shown that the $\left(n-\frac{3}{2}\right)$ neighborhood of $(EX)_1$ contains $(EY)_1$ and vice versa. The result follows. □

This argument can be extended to show that the same holds for the whole tight span, not just the 1-skeleton and unlike in 4.4.3 we can even drop the assumption that the spaces have to have the same combinatorial structure. Before we can prove this more general result, we should take a look at the next well–known Proposition.

Proposition 4.5.2. *Let* $X = \bigcup_{k=1,\ldots,n} E_k$*, where* E_k *is closed and* X *is a convex subset of a normed space. Let further* $f\colon X \to Y$*, with* Y *a normed space, such that the restriction of* f *to each* E_k *is L-Lipschitz, then* f *is L-Lipschitz on all of* X*.*

Proof. We will prove the Proposition by induction on n. If $n = 1$ there is nothing to be shown since $X = E_1$. Now let $X = \bigcup_{k=1,\ldots,n+1} E_k$ and $a,\ b \in X$. Since X is convex the whole line segment $[a,b]$ is contained in X and we note that for $c \in [a,b]$ we have $\|a-c\|_X + \|c-b\|_X = \|a-b\|_X$. Without loss of generality we may assume that $a \in E_1$ and we choose $c = \sup E_1 \cap [a,b]$. Clearly $c \in E_1$ since E_1 is closed. We then have

$$\|f(a)-f(b)\|_Y \leq \|f(a)-f(c)\|_Y + \|f(c)-f(b)\|_Y \leq L\|a-c\|_X + \|f(c)-f(b)\|_Y$$

and at the same time we can apply the induction hypothesis to $\|f(c)-f(b)\|_Y$, since $[c,b]$ is covered by the remaining n sets, $E_2,\ldots,E_{n+1}$. Thus we arrive at

$$\|f(a)-f(b)\|_Y \leq L\|a-c\|_X + L\|c-b\|_X = L\|a-b\|_X$$

which concludes the proof. □

We will need a slight variation of this statement.

Corollary 4.5.3. *Let* $X = \bigcup_{k=1,\ldots,n} \overline{E}_k$ *be convex subset of a normed space. Let further* $f\colon X \to Y$*,* Y *normed, be a continuous map such that the restriction of* f *to each* E_k *is* L*-Lipschitz. Then* f *is* L*-Lipschitz on* X*.*

Proof. Let a, $b \in \overline{E}_k$ for some k and let a_n be a sequence in E_k converging to a and b_n a sequence in E_k converging to b, then we have

$$\|f(a) - f(b)\|_Y = \lim_{n\to\infty} \|f(a_n) - f(b_n)\|_Y \leq \lim_{n\to\infty} L\|a_n - b_n\|_X = L\|a - b\|_X$$

This shows that f is in fact L-Lipschitz on $\overline{E}_k$ and thus the statement follows from 4.5.2. □

Theorem 4.5.4. *Let* X *and* Y *be two* n*-point metric spaces, then the map* $X \mapsto EX$ *is* $\left(n - \frac{3}{2}\right)$*-Lipschitz, or in other words*

$$d_H(EX, EY) \leq \left(n - \frac{3}{2}\right) d(X, Y)$$

Proof. We will prove this statement in three steps, first we will show that the statement is true within the same combinatorial structure, then we will show that $X \mapsto EX$ is continuous everywhere and finally use 4.5.3 to finish the proof.

Step 1: Consider two spaces X and Y whose tight spans have the same combinatorial structure. Let $c \in EX$, then given the polyhedral structure of the tight span there is some k-cell that contains c. This implies that c can be written as a convex combination of the m vertices $v_1, \ldots, v_m$ of that cell, i.e. $c = \sum_{i=1}^m \lambda_i v_i$. Each vertex v_i can be identified with a spanning subgraph S_i and using x to denote the vector of the pairwise distances in X we have $c = \sum_{i=1}^m \lambda_i M(S_i)^{-1} x$. Since both spaces have the same combinatorial structure, we can find a point $\hat{c} = \sum_{i=1}^m \lambda_i M(S_i)^{-1} y \in EY$, where y represents the vector of pairwise distances in Y. We then have

$$\begin{aligned}
\|c-\hat{c}\|_\infty &= \left\| \sum_{i=1}^m \lambda_i M(S_i)^{-1} x - \sum_{i=1}^m \lambda_i M(S_i)^{-1} y \right\|_\infty \\
&= \left\| \sum_{i=1}^m \lambda_i M(S_i)^{-1} (x-y) \right\|_\infty \leq \sum_{i=1}^m \lambda_i \left\| M(S_i)^{-1} \right\| \|x-y\|_\infty \\
&\leq \sum_{i=1}^m \lambda_i \left(n - \frac{3}{2}\right) \|x-y\|_\infty \\
&= \left(n - \frac{3}{2}\right) \|x-y\|_\infty
\end{aligned}$$

This argument shows that the $(n - 3/2)$ neighborhood of EX contains EY. We can repeat this argument and pick an element of EY to start with which gives us the reverse inclusion and thus $d_H(EX, EY) \leq (n - \frac{3}{2})d(X, Y)$.

Step 2: To show global continuity, let $\epsilon > 0$ and cover $(EX)_r$ by $B_\epsilon(x)$. We can use compactness to select finitely many $x_1, \ldots, x_m$ such that the union of their epsilon balls still covers $(EX)_r$. Then by Theorem 4.3.1 there is $N_r(x_i)$ such that $B_\epsilon(x_i) \cap (EX^j)_r \neq \emptyset$, whenever $j \geq N(x_i)$. Let $N_r = \max N_r(x_i)$, then we have that $(EX)_r \subset ((EX^j)_r)_\epsilon$, for $j \geq N_r$. In [16] it is shown that the largest cells the tight span of an n-point metric space can contain are of dimension $\left\lfloor \frac{n}{2} \right\rfloor$ and so if we choose $N = \max N_r$ the preceding argument shows that $(EX) \subset ((EX^j))_\epsilon$ for every $j \geq N$.

We want to show that there is an N_r such that $(EX^j)_r \subset ((EX)_r)_\epsilon$ for every $j \geq N_r$. Let us assume that on the contrary there exists a sequence $x_1, x_2, \ldots$ such that $x_j \in (EX^j)_r$ but $\operatorname{dist}(x_j, (EX)_r) \geq \epsilon$. Each x_j satisfies $(n-r)$ equations that are still spanning and there are only finitely many different ways of picking $(n-r)$ equations out of the $\binom{n}{2}$ possible ones. Thus we can select a subsequence x_{j_k} such that all x_{j_k} satisfy the same equations, i.e. $\operatorname{eq}(x_{j_k}) = \operatorname{eq}(x_{j_l})$ for all k and l. Then we can select a converging subsequence $x_{j_{k_l}} \to x$, but this means that x satisfies the same equations and thus $x \in (EX)_r$ which is a contradiction.

Step 3: We can write the space of all n-point semimetric spaces as $\mathrm{SM}_n = \bigcup_{k=1}^{m} C_k$, where m is the number of possible different combinatorial structures of the tight spans of n-point semimetric spaces and $C_k = \{X \colon X \text{ has combinatorial structure } k\}$. Then the previous computations show that $X \mapsto EX$ is $\left(n - \frac{3}{2}\right)$–Lipschitz on each C_k and continuous on SM_n. By 4.5.3 $X \mapsto EX$ is $\left(n - \frac{3}{2}\right)$–Lipschitz everywhere. $\square$

Chapter 5

Absolute Lipschitz retracts of symmetric products

5.1 Introduction

This chapter will investigate the space $X^{(n)}$ as defined below.

Definition 5.1.1 (Symmetric product)**.** For a metric space (X, d) we call

$$X^{(n)} = {}^{X}\!/_{S_n}$$

the n-fold symmetric product of X, where S_n is the symmetric group on n elements. Equipped with the quotient metric, this is a metric space.

The study of these spaces is motivated through the theory of Dirichlet–energy minimizing $X^{(n)}$–valued maps, which were first introduced by Almgren in his big regularity paper [1] and this is still a current area of research as evidenced by [4]. In this investigation two maps introduced by Almgren play an important role; more concretely the map $\xi\colon X^{(n)} \to \mathbb{R}^M$ for some (possibly quite large) M depending on n and $\dim X$, and the Lipschitz retraction $\rho\colon \mathbb{R}^M \to \xi(X^{(n)})$. Unfortunately, most of this work focuses mainly on the existence of these maps and not on how to explicitly construct them. We will introduce a more constructive approach and apply this to the study of absolute Lipschitz retracts of $X^{(2)}$.

Definition 5.1.2 (Absolute Lipschitz Retract)**.** Let Y be a metric space and $X \subset Y$. A Lipschitz map $r\colon Y \to X$ is a Lipschitz retraction if it is the identity on X. If such a Lipschitz retraction exists, we say that X is a Lipschitz retraction of Y and if X is a Lipschitz retraction of every metric space containing it we say that X is an absolute Lipschitz retract.

Before we look at some examples, let us first state the following proposition which gives a characterization of absolute Lipschitz retracts in terms of extensions. A proof can be found in [3].

Proposition 5.1.3. *Let X be a metric space. Then the following are equivalent.*

(i) X is an absolute Lipschitz retract.

(ii) For every metric space Y and every subset $Z \subset Y$, every Lipschitz function $f\colon Z \to X$ can be extended to a Lipschitz function $F\colon Y \to X$.

(iii) For every metric space Y containing X and for every metric space Z, every Lipschitz function $f\colon X \to Z$ can be extended to a Lipschitz function $F\colon Y \to X$.

Now let us consider a few examples for this property to see that some of the most famous spaces satisfy it.

The real line $\mathbb{R}$ is an absolute 1–Lipschitz retract and in fact if (X, d) is a metric space and $U \subset X$ an explicit extension of a L-Lipschitz map $f\colon U \to \mathbb{R}$ is given by

$$\hat{f}(x) = \inf_{y \in U}\{f(x) + Ld(x, y)\} \tag{5.1}$$

The space c_0 of sequences converging to 0 and $C(K)$, the space of continuous functions on a compact set K are both absolute 2–Lipschitz retracts, where the result about c_0 is proved in [3] and Kalton proved the latter result in 2007 ([21]). On the other hand it is known (see [5]) that no infinite dimensional Hilbert space can be an absolute Lipschitz retract.

In the study of absolute Lipschitz retracts, the space ℓ_∞ plays a central and very special role since it is a 1–Lipschitz retract itself (which can be shown using (5.1)) and every other separable space X can be isometrically embedded into ℓ_∞ via the map $x \mapsto \Phi(x)$, where $\Phi(x)(y) = d(x,y) - d(x_0,y)$ for some arbitrarily chosen basepoint x_0 ([3]). This, together with the previous Proposition, shows that in order to prove that a certain space X is an absolute Lipschitz retract it is enough to show it is a Lipschitz retract of ℓ_∞ as the next diagram illustrates.

$$\begin{tikzcd} Z \arrow[r] \arrow[d, phantom, "\cap"] & X \arrow[r, shift right, hook, "\Phi"'] & \ell_\infty \arrow[l, shift right, "r"'] \\ Y \arrow[ur, dashed] \arrow[urr] & & \end{tikzcd}$$

Figure 7: A way to find absolute Lipschitz retracts

In the following we will closer investigate the 2-fold symmetric product of a space by decomposing it into two pieces. Before we prove the decomposition result, let us first introduce the following definition.

Definition 5.1.4. For a normed space X its projective cone $(X)_\pm$ is the space X where antipodal points are identified. The projective cone becomes a metric space if we equip it with the quotient metric, which can be written as

$$d([x],[y]) = \min\{\|x-y\|, \|x+y\|\}.$$

The projective cone of a space will play a central role in the decomposition of the symmetric product, which we will take a look at next.

Proposition 5.1.5. *Let $(X, \|\cdot\|)$ be a normed space. Then*

$$X^{(2)} \underset{BL}{\cong} X \oplus_\infty (X)_\pm$$

Proof. We begin the proof with a few considerations. The point $(x, y) \in X^2$ can be uniquely written as $(a, a) + (b, -b)$ for some $a, b \in X$, so we would hope that something similar works in $X^{(2)}$. Ideally we would want to identify the point $\{x, y\} \in X^{(2)}$ with $a \oplus b$, but the main problem here is that in $X^{(2)}$ the points $\{x, y\}$ and $\{y, x\}$ are indistinguishable. However we notice that $(x, y) = (a + b, a - b)$ and $(y, x) = (a + (-b), a - (-b))$ and as such if we identify b and $-b$ we will solve this problem.

Now we are in the position to define the following map:

$$\Phi\colon X^{(2)} \longrightarrow X \oplus_\infty X_\pm$$

with

$$\Phi(\{x, y\}) = a \oplus [b]$$

where $(x, y) = (a + b, a - b)$ and $\Phi(\{x\}) = a \oplus [0]$. Clearly this map is bijective.

In order to show that this map is in fact Bilipschitz, we will use the following two matrices

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

since $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. Let $\{x, y\}$ and $\{x', y'\}$ be in $X^{(2)}$ with $a \oplus [b]$ and $a' \oplus [b']$ their images under Φ respectively. Without loss of generality we can choose representatives b and b' such that $d([b], [b']) = |b - b'|$ and we have (after relabeling x' and y' if necessary) $d_H(\{x, y\}, \{x', y'\}) = \min(\max(|x - x'|, |y - y'|), \max(|x - y'|, y - x'|)) = \max(|x - x'|, |y - y'|)$

Then we have

$$
\begin{aligned}
d_H(\{x,y\},\{x',y'\}) &= \max(|x-x'|,|y-y'|) \\
&= \left\| A^{-1}\begin{pmatrix} a-a' \\ b-b' \end{pmatrix} \right\|_\infty \leq 2d(a\oplus[b], a'\oplus[b'])
\end{aligned}
$$

On the other hand, we also have

$$
\left\| A^{-1}\begin{pmatrix} a-a' \\ b-b' \end{pmatrix} \right\|_\infty \geq d(a\oplus[b], a'\oplus[b'])
$$

which shows that Φ is $(1,2)$-Bilipschitz, which is exactly what we wanted. □

5.2 The space $(\ell_\infty^n)_\pm$

In this section we will investigate a universal way of embedding the projective cone of ℓ_∞^n into ℓ_∞^m for some $m > n$. Then we will use the fact that ℓ_∞ is a 1-absolute Lipschitz retract to get an estimate on how good of an absolute Lipschitz retract $(\ell_\infty^n)_\pm$ can be. We first introduce the following definition.

Definition 5.2.1. For a metric space X, we denote by

$$
\Lambda(X) = \inf\{C\colon X \text{ is a } C\text{–absolute Lipschitz retract}\}
$$

the optimal absolute Lipschitz retract constant of X. For brevity we will refer to $\Lambda(X)$ as the Lipschitz constant of X.

Before we will try to construct an absolute Lipschitz retraction, we will take a look at the best constant we could hope for and the next Proposition will help us do that.

Proposition 5.2.2. *Let $\{B(x_i,r_i)\}_{i\in I}$ be a family of balls in a metric space (X,d) such that $d(x_i,x_j) \leq r_i + r_j$ for every i and j and $\bigcap B(x_i, Lr_i) = \emptyset$. Then $\Lambda(X) \geq L$.*

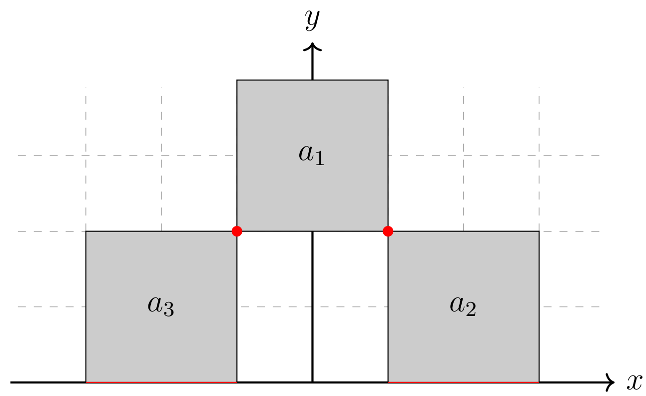

Figure 8: $\Lambda((\ell_\infty)_\pm) \geq 2$

Proof. Consider the discrete space $Z = \{z_i | i \in I\}$ and define a metric on Z by letting $d(z_i, z_j) = r_i + r_j$. Then the map $f\colon Z \to X$ defined by $f(z_i) = x_i$ is 1-Lipschitz. Now form the space $\tilde{Z} = Z \cup \{w\}$ and extend the metric by defining $d(w, z_i) = r_i$ for every i. If f has a $\tilde{L}$-Lipschitz extension to $\tilde{Z}$ we have $d(f(w), x_i) \leq \tilde{L} r_i$ for every i and thus $f(w) \in \bigcap B(x_i, \tilde{L} r_i)$. By our assumptions this must mean $\tilde{L} \geq L$ and thus $\Lambda(X) \geq L$. $\square$

For the projective cone of ℓ_∞ we can show that $\Lambda((\ell_\infty)_\pm \geq 2$ by considering the three balls $B_1(a_1)$, $B_1(a_2)$ and $B_1(a_3)$, where $a_1 = (0, 3, \ldots)$, $a_2 = (2, 1, 0, \ldots)$ and $a_3 = (-2, 1, 0, \ldots)$. Then $\|a_1 - a_2\| = \|a_1 - a_3\| - \|a_2 - a_3\| = 2$, but $\bigcap_{i=1}^3 B_r(a_i) = \emptyset$ for every $r < 2$.

Theorem 5.2.3. *Let $X = (\ell_\infty^n)_\pm$. Then $\varphi\colon X \to \ell_\infty^{n^2}$ defined as*

$$\varphi([x]) = \left(\left| \frac{x_i + x_j}{2} \right| , 1 \leq i \leq j \leq n \, , \left| \frac{x_i - x_j}{2} \right| , 1 \leq i < j \leq n \right),$$

where $[x] = [(x_1, \ldots, x_n)]$is a $\left(\frac{1}{3}, 1\right)$-Bilipschitz map .

Proof. We first note that the map φ as defined above does not depend on the choice of a representative of $[x]$. We will start by showing $d([x], [y]) \leq 3\|\varphi([x]) - \varphi([y])\|_\infty$, which is equivalent to showing

$$\left|\left|\frac{x_i \pm x_j}{2}\right| - \left|\frac{y_i \pm y_j}{2}\right|\right| \leq 1 \;\forall\, i, j \Rightarrow |x_i - y_i| \leq 3 \;\forall\, i \text{ or } |x_j + y_j| \leq 3 \;\forall\, j$$

Suppose on the contrary that there exist indices i and j such that $|x_i - y_i| > 3$ and $|x_j + y_j| > 3$, where without loss of generality we may say that $i = 1$ and $j = 2$. Notice that $||x_1| - |y_1|| \leq 1$ and $||x_2| - |y_2|| \leq 1$ by assumption. If we analyze these 4 inequalities closer we see that $|x_1 - y_1| > 3$ and $||x_1| - |y_1|| \leq 1$ can only be true at the same time when x_1 and y_1 have opposite signs, while $|x_2 + y_2| > 3$ and $||x_2| - |y_2|| \leq 1$ can only be true at the same time if x_2 and y_2 have the same sign. Without loss of generality we will choose the representatives in a way that $x_1 < 0$, $x_2 > 0$, $y_1 > 0$ and $y_2 > 0$. We need to address two cases. If $|x_1| > |x_2|$ we have

$$\begin{aligned} 2 &\geq ||x_1 + x_2| - |y_1 + y_2|| = |-x_1 - x_2 - y_1 - y_2| \geq |x_2 + y_2| - |x_1 + y_1| \\ &> 3 - |x_1 + y_1| = 3 - ||y_1| - |x_1|| \geq 2 \end{aligned}$$

On the other hand if $|x_1| \leq |x_2|$ we get

$$\begin{aligned} 2 &\geq ||x_1 + x_2| - |y_1 + y_2|| = |x_1 + x_2 - y_1 - y_2| \geq |x_1 - y_1| - |x_2 - y_2| \\ &> 3 - |x_2 - y_2| = 3 - ||x_2| - |y_2|| \geq 2 \end{aligned}$$

Thus we have reached a contradiction in either case and as such our original statement must have been true.

It remains to be shown that $\|\varphi([x]) - \varphi([y])\|_\infty \leq d([x], [y])$. We see immediately that $\varphi([x])$ contains $|x_i|$ for every i and thus $\|\varphi([x]) - \varphi([y])\|_\infty \leq \max(|\,|x_i| - |y_i|\,|,\; i = 1, \ldots, n)$. Applying the triangle inequality to the right hand side of this equation, we see that $\|\varphi([x]) - \varphi([y])\|_\infty \leq \|x - y\|_\infty$ and this finishes the proof since we can choose representative for $[x]$ and $[y]$ in a way such that $d([x], [y]) = \|x - y\|_\infty$. $\square$

$$\begin{tikzcd}
(\ell_\infty^n)_\pm \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \ell_\infty^{n^2} \\
& \mathrm{QM}_{n+1} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Figure 9: Factoring through QM_{n+1}

This theorem not only allows us to identify $(\ell_\infty^n)_\pm$ with a subset of ℓ_∞^n in a convenient way it also establishes a close relationship with quasimetric spaces on $n+1$ points QM_{n+1}, which the following Lemma demonstrates.

Lemma 5.2.4. *We let φ be the map from 5.2.3 and define*

$$p\colon (\ell_\infty^n)_\pm \longrightarrow QM_{n+1}$$

by

$$p([x_1,\dots,x_n]) = (d_{ij}) \text{ where } d_{0j} = |x_j|, 1 \le j \le n \text{ and } d_{ij} = |x_i - x_j| \text{ when } 1 \le i < j \le n$$

We define another map

$$\iota\colon QM_{n+1} \longrightarrow \ell_\infty^{n^2}$$

by

$$\iota((d_{ij})) = \begin{pmatrix} d_{01} & \dots & \max(d_{0i}, d_{0j}) - \frac{1}{2}d_{ij} \\ \vdots & \ddots & \vdots \\ \frac{1}{2}d_{ij} & \dots & d_{0n} \end{pmatrix}$$

Then the following diagram commutes:

Furthermore, p is a 2-Lipschitz map and ι is $\frac{3}{2}$-Lipschitz.

Proof. We need to show that $\iota(p([x])) = \varphi([x])$. If we investigate the diagonal entries of $\iota(p([x]))$ we see immediately that they coincide with the diagonal entries of $\varphi([x])$ and the same is true for all entries below the diagonal. Once we realize that $\frac{|x_i+x_j|}{2} + \frac{|x_i-x_j|}{2} = \max(|x_i|, |x_j|)$ it becomes evident that the entries above the diagonal of $\iota(p([x]))$ are also identical to those same entries of $\varphi([x])$. □

In essence the function φ samples from $(\ell_\infty^n)_\pm$ in a Bilipschitz way. This process is closely related to the part of harmonic analysis called phaseless reconstruction. In this area one considers a frame $\mathcal{F} = \{f_1, \ldots, f_m\}$, which is a set of $m \geq n$ vectors that span n-dimensional Hilbert space $\mathcal{H}$. One then tries to reconstruct a vector x from the magnitude of its so called frame coefficents $\{|\langle x, f_k\rangle|,\ 1 \leq k \leq m\}$. In this case the frame we use is

$$\left\{e_j,\ 1 \leq j \leq n,\ \frac{|e_i \pm e_j|}{2},\ 1 \leq i < j \leq n\right\}$$

but for the remainder we will use the frame

$$\{e_j,\ 1 \leq j \leq n,\ |e_i - e_j|,\ 1 \leq i < j \leq n\}$$

which samples the size of every entry and the pairwise distances between them. Since we sample entries we deal with the metric structure of a line, meaning the map f from the previous Lemma maps the projective cone $(\ell_\infty^n)_\pm$ into quasimetric spaces that can be isometrically embedded into $\mathbb{R}$. In higher dimensions, this will be important since we will try to find a way of retracting QM_{n+1} onto FM_{n+1}.

But first we will take a look at the case $n = 2$, which allows for a more direct approach.

5.3 The case $n = 2$

Proposition 5.3.1. *The map* $\varphi\colon (\ell_\infty^2)_\pm \to \ell_\infty^3$ *defined as*

$$\varphi([x]) = (|x_1|, |x_2|, \operatorname{sgn}(x_1 x_2) \min\{|x_1|, |x_2|\}),$$

where $[x] = [x_1, x_2]$ *is* 2*-Lipschitz.*

The image of this map

$$\operatorname{Im}\varphi = \{(x, y, \delta) : x, y \geq 0 \ , \ \delta = \pm \min\{x, y\}\}$$

is a cone in ℓ^3_∞.

Proof. Once we realize that $\operatorname{sgn}(x_1 x_2)\min\{|x_1|, |x_2|\} = \frac{|x_1+x_2|}{2} - \frac{|x_1-x_2|}{2}$ the result follows from 5.2.3. □

Proposition 5.3.2. *The map* $\psi\colon \ell^3_\infty \to (\ell^2_\infty)_\pm$ *defined by*

$$\begin{pmatrix} x \\ y \\ \delta \end{pmatrix} \longmapsto \left[\begin{pmatrix} (x-y)^+ + |\delta| \\ (y-x)^+ \operatorname{sgn}(\delta) + \delta \end{pmatrix}\right],$$

where $\operatorname{sgn}(0) = 1$*, is a* 3*-Lipschitz map that satisfies that* $\psi \circ \varphi = \mathrm{id}_{(\ell^2_\infty)_\pm}$.

Proof. Let us begin by showing that $\psi \circ \varphi = \mathrm{id}_{(\ell^2_\infty)_\pm}$, so let $[x, y]$ be an arbitrary element of $(\ell^2_\infty)_\pm$ and note that $[x, y] = [|x|, |y| \operatorname{sgn}(xy)]$.

Then

$$\begin{aligned} (\psi \circ \varphi)([x, y]) &= \psi(|x|, |y|, \min(|x|, |y|) \operatorname{sgn}(xy)) \\ &= [(|x| - |y|)^+ + \min(|x|, |y|), (|y| - |x|)^+ \operatorname{sgn}(xy) + \min(|x|, |y|) \operatorname{sgn}(xy)] \end{aligned}$$

In the case where $|x| \geq |y|$ the last line reduces to $[|x|, |y| \operatorname{sgn}(xy)]$ and we can see that in the case $|x| < |y|$ the result is exactly the same and as we have previously noted $[|x|, |y| \operatorname{sgn}(xy)] = [x, y]$.

It remains to be shown that the map ψ is indeed 3-Lipschitz. First we should note that

$$[(x-y)^+ + |\delta|, (y-x)^+ \operatorname{sgn}(\delta) + \delta] = [(x-y)^+ \operatorname{sgn}(\delta) + \delta, (y-x)^+ + |\delta|] \tag{5.2}$$

in $(\ell^2_\infty)_\pm$. This will justify us looking at either side when we investigate the map ψ. If $x \geq y$ we see that $\psi(x,y,\delta) = [x-y+|\delta|, \delta]$ which is clearly 3-Lipschitz, while if $x \leq y$ we will use the right side of the identity to get $\psi(x,y,\delta) = [\delta, y-x+|\delta|]$ □

The last part of this Proposition will be crucial in showing that $(\ell^2_\infty)_\pm$ is an absolute Lipschitz retract.

Proposition 5.3.3. *The space $(\ell^2_\infty)_\pm$ is a 6-absolute lipschitz retract.*

Proof. Let Y be a metric space, $Z \subset Y$ and $f\colon Z \to (\ell^2_\infty)_\pm$ a L-Lipschitz function. Let φ and ψ be the maps from 5.3.1 and 5.3.2 respectively. Then $\varphi \circ f\colon Z \to \ell^3_\infty$ is $2L$-Lipschitz and since ℓ^3_∞ is a 1-ALR we can extend this to a map $F\colon Y \to \ell^3_\infty$ which is also $2L$-Lipschitz. The map $\psi \circ F\colon Y \to (\ell^2_\infty)_\pm$ is $6L$ Lipschitz and for $z \in Z$ we have

$$(\psi \circ F)(z) = \psi(F(z)) = \psi(\varphi(f(z)) = (\psi \circ \varphi)(f(z)) = f(z)$$

This shows that $\psi \circ F$ extends f and we have shown that $(\ell^2_\infty)_\pm$ is a 6 absolute Lipschitz retract. □

In particular, this establishes that $2 \leq \Lambda((\ell^2_\infty)_\pm) \leq 6$ and it is not known whether the upper bound can be lowered further. For now, in combination with the decomposition from 5.1.5 we have shown the following.

Theorem 5.3.4. *The symmetric product $(\ell^2_\infty)^{(2)}$ is a 12–absolute Lipschitz retract.*

Our next goal is to settle this problem for arbitrary $n \geq 2$, where giving an explicit map to reconstruct is a lot harder. As we mentioned earlier it seems to important to decide when a space has the metric structure of the real line and this section is devoted to measuring the distance of a quasimetric space to a flat space.

5.4 Width of metric spaces

Our goal is to introduce a measurement of how far a metric space is away from being 'flat' and it seems natural to try and define this as the shortest distance to a subset of the real line.

Definition 5.4.1. Let X be a metric space. We call

$$\mathrm{width}(X) = \inf\{d_{GH}(X, Y) \mid Y \subset \mathbb{R}\}$$

the width of X. We observe that in particular width is a 1–Lipschitz map with respect to the Gromov–Hausdorff distance.

This definition is closely related to the definition of the diameter of a metric space. In fact, if we consider an arbitrary metric space (X, d) and the one point space $\{p\}$, then the only correspondence R between X and p is the one that identifies every point of X with p. Clearly

$$d_{GH}(X, \{p\}) = \frac{1}{2}\,\mathrm{AD}(R) = \frac{1}{2} \sup_{x_1, x_2 \in X} |d(x_1, x_2)| = \frac{1}{2}\,\mathrm{diam}(X)$$

Lemma 5.4.2. *Let X be a finite metric space. Then*

$$\begin{aligned}\mathrm{width}(X) &= \inf\{d_{GH}(X, Y) \mid Y \subset \mathbb{R}\} \\ &= \inf\{d_{GH}(X, Y) \mid Y \subset \mathbb{R}, |Y| \leq |X|\}\end{aligned}$$

Proof. For $Y \subset \mathbb{R}$ we label the pairwise distances by y_{ij} and similarly we label the pairwise distances in X by x_{ij}. We can arrange all of these into a matrix with $\binom{|X|}{2}$ rows and $\binom{|Y|}{2}$ columns with entries $|x_{ij} - y_{kl}|$. A correspondence has to choose at least one entry from each row and each column. If we pick entries from each row in a way that not two are in the same column we select exactly $\binom{|X|}{2}$ entries and it is an immediate observation that selecting entries from any leftover columns can not improve the infimum. As such these columns are redundant and it is enough to only consider subsets Y with $|Y| \leq |X|$. $\square$

We can come up with a very straightforward way to measure whether a triangle is degenerate in the sense that it is actually a line segment, which allows us to detect the flatness of a 3-point metric space and we can generalize that definition naturally to all bounded metric spaces.

Definition 5.4.3 (Gromov Product and Excess)**.** Let X be a metric space and x, y, $z \in X$. We call

$$(x, y)_z = \frac{1}{2}(d(x, y) + d(y, z) - d(x, z))$$

the Gromov product of x and z at y and

$$\operatorname{exc}(x, y, z) = \min_{\sigma \in S_3}(\sigma(x), \sigma(y))_{\sigma(z)}$$

the excess of the triangle defined by x, y and z. We can then define the (possibly infinite) excess of the space X as

$$\operatorname{exc}(X) = \sup\{\operatorname{exc}(x, y, z) \mid x, y, z \in X\}$$

In fact, if $\operatorname{exc}(X) = 0$ for a 3-point space $X = \{x, y, z\}$, then we can relabel the points to have $d(x, y) + d(y, z) = d(x, z)$ and define an isometric embedding f into the real line via $f(x) = 0$, $f(y) = d(x, y)$ and $f(z) = d(x, z)$. The next Lemma will establish a general relationship between $\operatorname{exc}(X)$ and $\operatorname{width}(X)$.

Lemma 5.4.4. *The map*

$$\operatorname{exc}\colon BM \to [0, \infty)$$

is 3*-Lipschitz. In particular this implies* $\operatorname{width}(X) \geq \frac{1}{3}\operatorname{exc}(X)$.

Proof. Let X and Y be bounded metric spaces and $\epsilon > 0$. We can pick a correspondence R such that

$$|xx' - yy'| \leq 2(d_{GH}(X, Y) + \epsilon)$$

whenever xRy and $x'Ry'$.

We will show that

$$|\operatorname{exc}(y_1, y_2, y_3) - \operatorname{exc}(x_1, x_2, x_3)| \leq 3(d_{GH}(X, Y) + \epsilon) \tag{5.3}$$

for any x_1, x_2, x_3 and y_1, y_2, y_3 such that x_iRy_i. We note that excess is unaffected by relabeling and as such we can assume that either $\operatorname{exc}(x_1, x_2, x_3) = x_1x_2 + x_2x_3 - x_1x_3$ or $\operatorname{exc}(y_1, y_2, y_3) = y_1y_2 + y_2y_3 - y_1y_3$. Then

$$\begin{aligned}
\operatorname{exc}(x_1, x_2, x_3) - \operatorname{exc}(y_1, y_2, y_3) &= \operatorname{exc}(x_1, x_2, x_3) - \frac{1}{2}(y_1y_2 + y_2y_3 - y_1y_3) \\
&\leq \frac{1}{2}(x_1x_2 + x_2x_3 - x_1x_3) - \frac{1}{2}(y_1y_2 + y_2y_3 - y_1y_3) \\
&\leq 3(d_{GH}(X, Y) + \epsilon)
\end{aligned}$$

We can repeat this argument to achieve $\operatorname{exc}(y_1, y_2, y_3) - \operatorname{exc}(x_1, x_2, x_3) \leq 3(d_{GH}(X, Y) + \epsilon)$ which shows (5.3).

Finally we will pick x_1,x_2,$x_3 \in X$ such that $\operatorname{exc}(x_1, x_2, x_3) \geq \operatorname{exc}(X) - \epsilon$ and y_1, y_2, y_3 such that x_iRy_i, then

$$\begin{aligned}
\operatorname{exc}(Y) \geq \operatorname{exc}(y_1, y_2, y_3) &\geq \operatorname{exc}(x_1, x_2, x_3) - 3d_{GH}(X, Y) - 3\epsilon \\
&\geq \operatorname{exc}(X) - 3d_{GH}(X, Y) - 4\epsilon
\end{aligned}$$

by symmetry we also have $\operatorname{exc}(X) \geq \operatorname{exc}(Y) - 3d_{GH}(X, Y) - 4\epsilon$ and since $\epsilon > 0$ was arbitrary this shows that exc is 3-Lipschitz.

The lower bound on the width of a space follows after the observation that $\operatorname{width}(Y) = 0$ if $Y \subset \mathbb{R}$, since

$$\operatorname{width}(X) = \inf \{d_{GH}(X, Y) \mid Y \subset \mathbb{R}\} \geq \inf \left\{ \frac{1}{3} |\operatorname{exc}(X) - \operatorname{exc}(Y)| \mid Y \subset \mathbb{R} \right\} = \frac{1}{3} \operatorname{exc}(X)$$

□

This lower bound on the width is sharp and we will prove next that every 3-point metric space attains it.

Example 5.4.5. (Width of a 3-point space) Let $X = \{x_1, x_2, x_3\}$ be a 3-point metric space. Then

$$\mathrm{width}(X) = \frac{1}{3}\,\mathrm{exc}(x_1, x_2, x_3)$$

Proof. We denote the pairwise distances in X by x_{ij} and we can assume that $\mathrm{exc}(x_1, x_2, x_3) = {}^1\!/_2(x_{12} + x_{23} - x_{13})$ without loss of generality, relabeling the points if necessary. Now let $\delta = \frac{2}{3}\,\mathrm{exc}(x_1, x_2, x_3)$ and take $Y = \{y_1, y_2, y_3\} \subset \mathbb{R}$ such that $y_{12} = x_{12} - \delta$, $y_{23} = x_{23} - \delta$ and $y_{13} = y_{12} + y_{23}$. Here y_{ij} denotes the distance between y_i and y_j. We fix a relation R such that

$$\mathrm{AD}(R) = \max(|x_{12} - y_{12}|, |x_{23} - y_{23}|, |x_{13} - y_{13}|) = \max(\delta, \delta, \delta) = \delta$$

This shows that $d_{GH}(X, Y) \leq \frac{1}{2}\delta = \frac{1}{3}\,\mathrm{exc}(x_1, x_2, x_3)$ and consequently $\mathrm{width}(X) \leq \frac{1}{3}\,\mathrm{exc}(x_1, x_2, x_3)$. Since by lemma 5.4.4 we know that $\mathrm{width}(X) \geq \frac{1}{3}\,\mathrm{exc}(x_1, x_2, x_3)$ this concludes the proof. $\square$

We have shown that the bound we found earlier works great for 3-point spaces and it is tempting to think that we could use excess, which is much easier to compute or bound, as a way of shrinking or modifying our space in an attempt to flatten it. Unfortunately the next example will demonstrate that even for 4-point spaces excess is an inadequate estimate for the width of a space.

Example 5.4.6. We consider the 4-point space given by

d	x_1	x_2	x_3	x_4
x_1	0	1	2	1
x_2	1	0	1	2
x_3	2	1	0	1
x_4	1	2	1	0

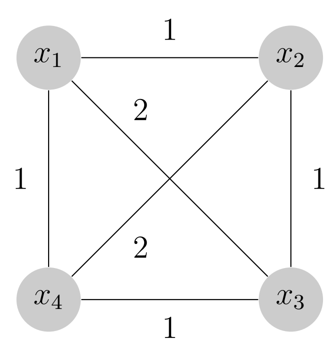

Figure 10: 4–cycle

Then a quick computation shows that $\mathrm{exc}(X) = 0$. However, it is clear that these 4 points can not be isometrically embedded into a line.

In the case of 3–point spaces we see that the excess of the space X and its tight span EX coincide by looking at Example 4.1.9, but in general the excess of a tight span is only bounded below by the excess of the underlying space. This inspires the hope that we can use the excess of the tight span of a space X instead to find an upper bound for the width of X. In particular, we should note that if the tight span of X is flat, then X itself had to be flat and $\mathrm{exc}(EX) = 0$.

Conjecture 5.4.7. There is some constant C, possibly dependent on n, such that for an n-point metric space X we have

$$\mathrm{width}(X) \leq C\,\mathrm{exc}(EX)$$

We will prove the following proposition as supporting evidence for our conjecture.

Proposition 5.4.8. *Let X be a 4–point space, then*

$$\mathrm{width}(X) \leq \frac{19}{12}\,\mathrm{exc}(EX).$$

Proof. We will start with considering the general 4–point metric space X from Example 4.1.10.

d	x_1	x_2	x_3	x_4
x_1	0	$a+b+f$	$a+c+e+f$	$a+d+e$
x_2	$a+b+f$	0	$b+c+e$	$b+d+e+f$
x_3	$a+c+e+f$	$b+c+e$	0	$c+d+f$
x_4	$a+d+e$	$b+d+e+f$	$c+d+f$	0

with $e \leq f$ and $b \leq \min\{a, c, d\}$ which gives this respective tight span EX .

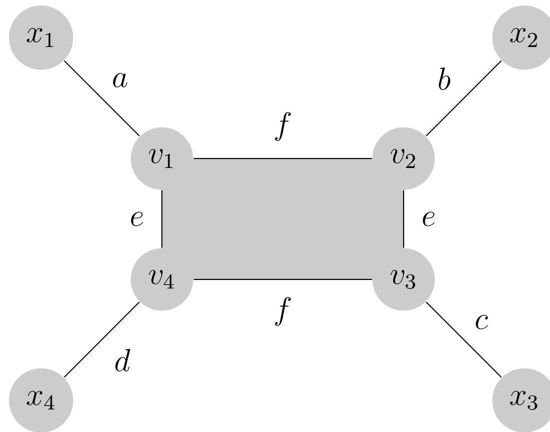

Figure 11: Tight span of a general 4–point metric space

Our first goal will be to estimate the excess of the tight span and we will do so in two steps. First consider the rectangle with side lengths e and f equipped with the ℓ_1 metric. We choose $a_1 = \frac{1}{3}v_4 + \frac{2}{3}v_3$, $a_2 = \frac{1}{3}v_4 + \frac{2}{3}v_1$ and $a_3 = v_2$ which implies that $\|a_1 - a_2\| = \frac{2}{3}(e+f)$, $\|a_1 - a_3\| = e + \frac{1}{3}f$ and $\|a_2 - a_3\| = \frac{1}{3}e + f$. A consequence is that $\operatorname{exc}(a_1, a_2, a_3) = \frac{2e}{3}$ and thus the excess of the rectangle is at least $\frac{2e}{3}$. In addition, we can consider the four 3–point configurations involving x_1, x_2, x_3 and x_4 and see that the excess of them is at least b. This allows us to conclude that

$$\operatorname{exc}(EX) \geq \max\left\{\frac{2e}{3}, b\right\} \tag{5.4}$$

From here, we consider a similar space X_1

d	x_1	x_2	x_3	x_4
x_1	0	$a+b+f$	$a+c+f$	$a+d$
x_2	$a+b+f$	0	$b+c$	$b+d+f$
x_3	$a+c+f$	$b+c$	0	$c+d+f$
x_4	$a+d$	$b+d+f$	$c+d+f$	0

with tight span EX_1.

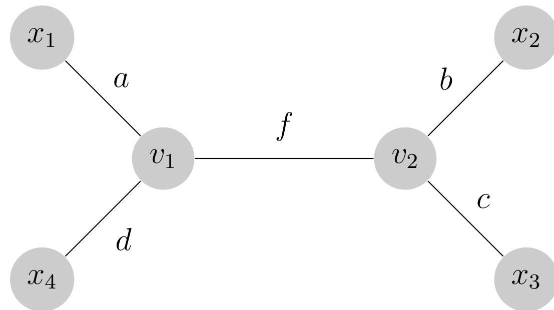

Figure 12: Tight span EX_1 after the first reduction

We can see, using the obvious correspondence, that $d_{GH}(X, X_1) \leq \frac{1}{2}e$. Modifying X_1 slightly, we arrive at the space X_2.

d	x_1	x_2	x_3	x_4
x_1	0	$a+f$	$a+c+f$	$a+d$
x_2	$a+f$	0	c	$d+f$
x_3	$a+c+f$	c	0	$c+d+f$
x_4	$a+d$	$d+f$	$c+d+f$	0

with tight span EX_2.

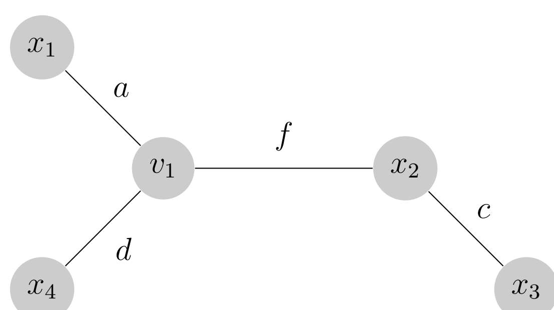

Figure 13: Tight span EX_2 after the second reduction

Once again the obvious correspondence yields that $d_{GH}(X_1, X_2) \leq \frac{b}{2}$. Since X_2 is isometrically embedded in EX_2, which in turn is isometrically embedded in EX we have that $\mathrm{exc}(X_2) \le \mathrm{exc}(EX_2) \le \mathrm{exc}(EX)$ We now employ that width is a 1–Lipschitz map to arrive at

$$\begin{aligned}\mathrm{width}(X) &\le \mathrm{width}(X_2) + d_{GH}(X, X_2)\\ &\le \frac{1}{3}\,\mathrm{exc}(X_2) + d_{GH}(X, X_1) + d_{GH}(X_1, X_2)\\ &\le \frac{1}{3}\,\mathrm{exc}(X) + \frac{1}{2}e + \frac{b}{2}\\ &\le \frac{1}{3}\,\mathrm{exc}(EX) + \frac{1}{2}e + \frac{b}{2}\end{aligned}$$

Equation (5.4) can then be used to get

$$\mathrm{width}(X) \le \frac{1}{3}\,\mathrm{exc}(EX) + \frac{1}{2}\,\mathrm{exc}(EX) + \frac{3}{4}\,\mathrm{exc}(EX) = \frac{19}{12}\,\mathrm{exc}(EX)$$

which is exactly what we wanted. □

Remark 5.4.9. By looking at EX_2 of Proposition 5.4.8 we can see that it is a tripod and as such we have that $\mathrm{exc}(EX_2) \ge \min\{a, d, c+f\}$. Combining this with the estimates we have obtained in the previous Proposition, we obtain

$$\mathrm{exc}(EX) \ge \max\left\{\frac{2e}{3}, b, \min\{a, d, c+f\}\right\}$$

And as such if we reduce all edges in the tight span by $3/2\,\mathrm{exc}(EX)$ we have collapsed the tight span to a straight line.

Our observations in this section have shown that width and excess of tight spans seem to be related, which lead us to the following remark.

Remark 5.4.10. Let X be a finite metric space, then

$$\mathrm{width}(X) = 0 \iff \mathrm{exc}(EX) = 0$$

We can sketch a proof of this in the following way. If $\mathrm{width}(X) = 0$ then X can be isometrically embedded into $\mathbb{R}$, which is an injective space. This implies that EX can also be embedded into $\mathbb{R}$ and thus $\mathrm{exc}(EX) = 0$.

If the excess of the tight span is 0 however, this means that the tight span can not have any cells of dimension 2 or higher. This establishes that EX is a graph. Furthermore, the tight span can not contain any tripods, as those have positive excess. The only option left is a graph that contains no cycles and every vertex has degree 2, but those can be isometrically embedded into $\mathbb{R}$.

5.5 The case $n = 3$

We can now use the methods introduced in the previous section to take a look at $(\ell_\infty^3)_\pm$.

Before we start with going into this special case, we would like to outline our general approach with the following diagram.

$$(\ell_\infty^n)_\pm \xrightarrow{p} \ell_\infty^{\binom{n+1}{2}} \xrightarrow{M} \mathrm{QM}_{n+1} \xrightarrow{r} \mathrm{SM}_{n+1} \xrightarrow{\psi} \mathrm{FM}_{n+1} \xrightarrow{R} (\ell_\infty^n)_\pm$$

Figure 14: General approach to bound $\Lambda((\ell_\infty^n)_\pm)$

The map p is from 5.2.4 and the map M in this diagram is taking the positive part of each component, which allows us to interpret the components as distances between $n + 1$ points. In addition, M leaves the image of p unchanged. Since all of the maps involved are retractions, this can be used to show that $(\ell_\infty^n)_\pm$ is an absolute Lipschitz retract as illustrated in Figure 7. We will see (or already have seen), that the maps p, M, r and R are Lipschitz with Lipschitz constants independent of n and the main difficulty will arise in constructing ψ. First, we show that there is a retraction from quasimetric spaces onto semimetric spaces.

Proposition 5.5.1. *There is a 4–Lipschitz retraction $r\colon QM_n \longrightarrow SM_n$*

Proof. We need to introduce some notation first. For a quasimetric space Y with quasimetric

$\rho = (\rho_{ij})$ we let $\text{defect}(\rho) = \max\{(\rho_{ij} - \rho_{ik} - \rho_{kj})^+ \mid i, j, k \in \{1, \dots, n\}\}$ the maximum amount by which ρ fails the triangle inequality, which is a 3-Lipschitz map. Now we define

$$r\colon \mathrm{QM}_n \longrightarrow \mathrm{SM}_n$$

by

$$\rho_{ij} \mapsto d_{ij} = \rho_{ij} + \text{defect}(\rho)$$

which is a 4-Lipschitz map independent of n. Since a semimetric space (X, d) satisfies the triangle inequality we have that $\text{defect}(d) = 0$ and as such r leaves semimetric spaces fixed. It remains to be checked that d_{ij} satisfy the triangle inequality and indeed for any indices i, j, k we have

$$\begin{aligned} d_{ik} + d_{kj} - d_{ij} &= \rho_{ik} + \rho_{kj} - \rho_{ij} + \text{defect}(\rho) \\ &= \text{defect}(\rho) - (\rho_{ij} - \rho_{ik} - \rho_{kj}) \geq 0 \end{aligned}$$

□

Furthermore, we have that r restricted to $p((\ell^3_\infty)_\pm)$ is the identity, since $p((\ell^3_\infty)_\pm)$ does satisfy the triangle inequality.

Lemma 5.5.2 (Flattening)**.** *There is a* 180*–Lipschitz retraction* $\psi\colon SM_4 \to FM_4$*.*

Proof. We start by considering a general 4–point space $X = \{x_0, x_1, x_2, x_3\}$. Then there is some permutation $\sigma \in S_4$ such that the space can be represented as

d	$x_{\sigma(0)}$	$x_{\sigma(1)}$	$x_{\sigma(2)}$	$x_{\sigma(3)}$
$x_{\sigma(0)}$	0	$a+b+f$	$a+c+e+f$	$a+d+e$
$x_{\sigma(1)}$	$a+b+f$	0	$b+c+e$	$b+d+e+f$
$x_{\sigma(2)}$	$a+c+e+f$	$b+c+e$	0	$c+d+f$
$x_{\sigma(3)}$	$a+d+e$	$b+d+e+f$	$c+d+f$	0

We can then create a new space by shortening all of the quantities a, b, c, d, e and f by $\delta = \frac{3}{2}\,\text{exc}(EX)$ and truncating at 0, for example $\tilde{a} = (a - \delta)^+$. It is crucial to note that

Remark 5.4.9 shows that the resulting space is indeed flat. The numbers $a, \ldots, f$ represent the length of edges in the tight span of X and thus are 5–Lipschitz by Lemma 4.4.2. In addition we know from Lemma 5.4.4 that exc is 3–Lipschitz and in [23] it is shown that $X \mapsto EX$ is a 2–Lipschitz map.
In order to put these results together, we have to note that at most 4 edges participate in the distances $d(x_{\sigma(i)}, x_{\sigma(j)})$. Then we have that the map from X to $\tilde{X}$ with $\tilde{X} \in \mathrm{FM}_4$ is 180–Lipschitz.

We note that so far this result only holds within the same combinatorial type σ, however it is easy to see that this construction is consistent on overlaps and thus by Corollary 4.5.3 actually 180–Lipschitz on SM_4.

It remains to be shown that this map is actually a retraction, but Remark 5.4.10 shows that the excess of a flat space is 0 and thus these spaces remain unchanged. □

We now turn our view to the last undescribed map from Figure 14.

Definition 5.5.3 (Restoration map)**.** We can write the space FM_{n+1}, where every $X \in \mathrm{FM}_{n+1}$ has labeled vertices $x_0, \ldots, x_n$ as the union of n pieces $A_1, \ldots, A_n$, with $A_j = \{X : d(x_0, x_j) = \max\{d(x_0, x_i) \colon i = 1, \ldots, n\}\}$. For $X \in A_j$ we define

$$R_j \colon A_j \to (\ell_\infty^n)_\pm$$

by

$$(R_j(X))_i = d(x_0, x_j) - d(x_i, x_j)$$

then we call the map

$$R \colon \mathrm{FM}_{n+1} \to (\ell_\infty^n)_\pm$$

defined as $R(X) = R_j(X)$ if $X \in A_j$ the restoration map.

The first goal is to show that this map is well–defined and Lipschitz.

Theorem 5.5.4. *The restoration map $R\colon FM_{n+1} \to (\ell_\infty^n)_\pm$ from 5.5.3 is well–defined and 2-Lipschitz.*

Proof. Let $X \in \mathrm{FM}_{n+1}$ with labeled vertices $x_0, \dots, x_n$. In view of 5.5.3 it needs to be shown that $R_k(X) = R_l(X)$ if $X \in A_k \cap A_l$. We have to distinguish between two cases, either $d(x_l, x_k) = 0$ or $d(x_l, x_k) = 2d(x_0, x_l) = 2d(x_0, x_k)$. First let $d(x_l, x_k) = 0$, then

$$(R_k(X))_i = d(x_0, x_k) - d(x_i, x_k) = d(x_0, x_l) - d(x_i, x_l) = (R_l(X))_i \;\forall\, i$$

If $d(x_l, x_k) = 2d(x_0, x_l)$ we have

$$(R_k(X))_l = d(x_0, x_k) - d(x_l, x_k) = d(x_0, x_l) - 2d(x_0, x_l) = -d(x_0, x_l) = -(R_l(X))_l$$

and

$$(R_k(X))_k = d(x_0, x_k) = 2d(x_0, x_k) - d(x_0, x_k) = d(x_k, x_l) - d(x_0, x_l) = -(R_l(X))_k$$

then for $i \notin \{j, k\}$ we notice that $d(x_k, x_l) = d(x_l, x_i) + d(x_k, x_i)$ and get

$$\begin{aligned}(R_k(X))_i &= d(x_0, x_k) - d(x_i, x_k) = d(x_0, x_l) - (d(x_k, x_l) - d(x_l, x_i)) \\ &= d(x_0, x_l) - 2d(x_0, x_l) + d(x_l, x_i) = d(x_l, x_i) - d(x_0, x_l) \\ &= -(R_l(X))_i\end{aligned}$$

We have shown that $(R_k(X))_i = -(R_l(X))_i$ for every i, so $R_k(X) = R_l(X)$ in $(\ell_\infty^n)_\pm$ and thus R is well-defined. In addition the map is clearly 2-Lipschitz by definition. □

The last thing we need to show is that for $x \in (\ell_\infty^n)_\pm$ we have $R(p([x])) = [x]$. Using the notation of the definition of the restoration map, we have that $p([x_1, \dots, x_n]) \in A_j$ for some

j and we can choose a representative such that $x_j \geq 0$. But then

$$R_j(p([x]))_i = d_{0j} - d_{ij} = |x_j| - |x_j - x_i| = x_j - x_j + x_i = x_i$$

which is exactly what we wanted.

These computations now allow us to prove

Theorem 5.5.5. *$(\ell_\infty^3)_\pm$ is a 2880–absolute Lipschitz retract.*

Proof. Let $Z \subset Y$ and $f\colon Z \to (\ell_\infty^3)_\pm$ be Lipschitz. Then the following diagram shows us how to extend it to a map $F\colon Y \to (\ell_\infty^3)_\pm$.

$$\begin{tikzcd}
 & & \mathrm{FM}_4 \arrow[dl, "2"', "R"] & \mathrm{SM}_4 \arrow[l, "180"', "\psi"] \\
Z \arrow[r, "L", "f"'] \arrow[dd, "\iota"'] & (\ell_\infty^3)_\pm \arrow[dr, "2", "p"'] & & \\
 & & \ell_\infty^6 \arrow[r, "1", "M"'] & \mathrm{QM}_4 \arrow[uu, "r", "4"'] \\
Y \arrow[uur, dashed, "F"'] \arrow[urr, "2L", "\tilde{F}"'] & & &
\end{tikzcd}$$

Figure 15: Diagram for the $n = 3$ case

Using all of our previous considerations we have that F, defined in this way, is an extension of f with Lipschitz constant $2880L$. $\square$

As we have pointed out in the beginning of this section, all maps in this diagram are independent of n except for the flattening map introduced in Lemma 5.5.2. If we can remove this dependence, that would show that $(\ell_\infty^n)^{(2)}$ is an absolute Lipschitz retract with Lipschitz constant independent of n.

Bibliography

[1] F. J. Almgren Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, vol. 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2; With a preface by Jean E. Taylor and Vladimir Scheffer.

[2] Y. Benyamini, *Introduction to the uniform classification of Banach spaces*, Advanced courses of mathematical analysis I, World Sci. Publ., Hackensack, NJ, 2004, pp. 1–29.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[4] P. Bouafia, T. de Pauw, and J. Goblet, *Existence of p harmonic multiple valued maps into a separable Hilbert space*, Annales de l'institut Fourier **65** (2015), 763-833.

[5] A. Brudnyi and Y. Brudnyi, *Methods of geometric analysis in extension and trace problems. Volume 1*, Monographs in Mathematics, vol. 102, Birkhäuser/Springer Basel AG, Basel, 2012.

[6] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.

[7] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw–Hill, 2005.

[8] M. Chrobak and L. L. Larmore, *Generosity helps or an* 11*-competitive algorithm for three servers*, J. Algorithms **16** (1994), no. 2, 234–263.

[9] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

[10] A. W. M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Adv. in Math. **74** (1989), 321–402.

[11] A. W. M. Dress, K. T. Huber, J. Koolen, V. Moulton, and A. Spillner, *Basic phylogenetic combinatorics*, Cambridge University Press, Cambridge, 2012.

[12] A. W. M. Dress, V. Moulton, and W. Terhalle, *T-theory. An overview*, Sém. Lothar. Combin. **34** (1995), Art. B34b, approx. 23 pp.

[13] N. D. Elkies, *Convex surface on which any two points a, b can be joined by a curve of length* $(\pi/2-\epsilon)|a-b|$, Mathematics Stack Exchange. URL: http://math.stackexchange.com/q/896139 (version: 2014-08-13).

[14] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), no. 4, 851–874.

[15] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original; With appendices by M. Katz, P. Pansu and S. Semmes.

[16] S. Herrmann and M. Joswig, *Bounds on the f-vectors of tight spans*, Contrib. Discrete Math. **2** (2007), no. 2, 161–184 (electronic).

[17] S. Herrmann, V. Moulton, and A. Spillner, *Searching for realizations of finite metric spaces in tight spans*, Discrete Optim. **10** (2013), no. 4, 310–319.

[18] J. R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76.

[19] W. B. Johnson and J. Lindenstrauss (eds.), *Handbook of the geometry of Banach spaces. Vol. I*, North-Holland Publishing Co., Amsterdam, 2001.

[20] S. Ivanov, *A characterization of Hilbert spaces?*, MathOverflow. URL: http://mathoverflow.net/q/130443 (version: 2013-05-12).

[21] N. J. Kalton, *Extending Lipschitz maps into $C(K)$-spaces*, Israel J. Math. **162** (2007), 275–315.

[22] S. Kwapień, *Isomorphic characterizations of Hilbert spaces by orthogonal series with vector valued coefficients*, Séminare Maurey-Schwartz Année 1972–1973: Espaces L^p et applications radonifiantes, Exp. No. 8, Centre de Math., École Polytech., Paris, 1973.

[23] U. Lang, M. Pavón, and R. Züst, *Metric stability of trees and tight spans*, Arch. Math. (Basel) **101** (2013), no. 1, 91–100.

[24] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Classics in Mathematics, Springer-Verlag, Berlin, 2011. Isoperimetry and processes; Reprint of the 1991 edition.

[25] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), no. 1, 45–90 (French).

[26] V. D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, vol. 1200, Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[27] J.-M. Morel and H. Steinlein, *On a problem of Nirenberg concerning expanding maps*, J. Funct. Anal. **59** (1984), no. 1, 145–150.

[28] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Lecture Notes in Mathematics, vol. 6, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2001.

[29] G. Pisier and Q. H. Xu, *Random series in the real interpolation spaces between the spaces v_p*, Geometrical aspects of functional analysis (1985/86), Lecture Notes in Math., vol. 1267, Springer, Berlin, 1987, pp. 185–209.

[30] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, 2014.

[31] C. Van Nuffelen, *On the incidence matrix of a graph*, IEEE Trans. Circuits and Systems **9** (1976), 572.

Patrick Biermann

Education

2011–2016 **Ph.D. in Mathematics**, *Syracuse University*, Syracuse, *current GAP 3.92*.
Dissertation: "Lipschitz Geometry of Banach and Metric spaces"
Advisor: Leonid Kovalev
Anticipated graduation date, May 2016

2010–2011 **MS, Mathematics**, *Syracuse University*, Syracuse, *GPA 3.83*.

2006–2010 **Vordiplom in Mathematics**, *Universität Ulm*, Ulm, *1.0 (equivalent to a GPA 4.0)*.
Left Ulm to go study abroad at Syracuse University

1996–2005 **Abitur**, *Mauritius Gymnasium Büren*, Büren, *1.3 (equivalent to a GPA 3.7)*.
High school

Experience

Vocational

2010-2016 **Teaching Assistant**, *Syracuse University*, Syracuse.
Teaching classes and leading recitations.
Detailed Breakdown:
- Recitation leader: MAT 121, MAT 284, MAT 397
- Classes taught as primary instructor: MAT 285, MAT 295, MAT 397
- Currently teaching: MAT 296

2010-2010 **Intern**, *Towers Watson*, Cologne.
Intern at Towers Watson with a main focus on insurance consulting for 3 months.
Detailed Achievements:
- In charge of an internal training about the changes to private health insurance due to a new law
- Maintained a database of insurance companies and their balance sheets
- Analyzed a survey on the state of unit-linked life insurance

2008–2010 **Teaching Assistant**, *Universiät Ulm*, Ulm.
Leading recitations and review sessions

2009-2009 **Intern**, *Landesbank Baden-Württemberg*, Stuttgart.
Intern in the Equity Derivatives Trading group for 3 months.
Detailed Achievements:
- Developed an Excel tool using VBA to evaluate stock options in an automated manner
- provided ad-hoc support to the traders with Excel tools

2005–2006 **Obergefreiter**, *Federal Republic of Germany*, Augustdorf.
Drafted into military service with the Panzergrenadierbattalion 212.

2004–2005 **Student**, *Universität Paderborn*, Paderborn.
Student at the University of Paderborn while in high school.

1022 Westcott Street – 13210 Syracuse, NY – USA
+1 (315) 278 7362 • *pbierman@syr.edu*

Miscellaneous

2013-2016 **Senator**, *Syracuse University*, Syracuse.
Elected senator of the mathematics department to represent the graduate students in the Graduate Student Organization (GSO). Current member of the finance committee.

2014–2015 **MGO Vice President**, *Syracuse University*, Syracuse.
Vice president of the mathematics graduate student organization.
Detailed Achievements:
- Co-organized the 40th Annual New York Regional Graduate Mathematics Conference at Syracuse University

2012–2014 **MGO President**, *Syracuse University*, Syracuse.
President of the mathematics graduate student organization, representing the mathematics and mathematics education graduate students to the department.
Detailed Achievements:
- Main organizer of the 38th Annual New York Regional Graduate Mathematics Conference at Syracuse University
- Main organizer of the 39th Annual New York Regional Graduate Mathematics Conference at Syracuse University
- Organized two annual picnics to further interaction between faculty and graduate students

2006–2010 **Student representative**, *Universität Ulm*, Ulm.
Part of student government as a representative for mathematics majors at the University of Ulm.

2001–2002 **Mediator**, *Mauritius Gymnasium Büren*, Büren.
Student volunteer, mediating conflict between students.

Talks

2015 **Lipschitz Geometry of Symmetric Products**, *Syracuse University*, Syracuse.
Analysis Seminar

2015 **The distortion of subsets of Hilbert space**, *Syracuse University*, Syracuse.
40th Annual New York Regional Graduate Mathematics Conference

2014 **Duality Maps in Banach Spaces**, *Syracuse University*, Syracuse.
Analysis Seminar

2013 **Why is MAT 183 useful?**, *Syracuse University*, Syracuse.
MGO Colloquium

2011 **A short introduction to the mapping degree**, *Syracuse University*, Syracuse.
MGO Colloquium

Awards

2010 **Fulbright Scholar**.

2008 **Gebührenstipendium**, *Universität Ulm*, Ulm.
Awarded for being in the top 4% of the mathematics program

Languages

German fluent *native language*

English fluent

1022 Westcott Street – 13210 Syracuse, NY – USA
+1 (315) 278 7362 • *pbierman@syr.edu*

French beginner

Computer skills

Programming Languages: C, C++, VBA, SQL, Java, LaTex

Programs: Microsoft Excel, MATLAB

Interests

Skiing

Board gaming

Memberships

American Mathematical Society

Phi Beta Delta International Honor Society

1022 Westcott Street – 13210 Syracuse, NY – USA

+1 (315) 278 7362 • *pbierman@syr.edu*