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Geometric Precipices in String Cosmology

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ABSTRACT

We consider the effects of graviton multiplet fields on transitions between string gas phases. Focusing on the dilaton field, we show that it may obstruct transitions between different thermodynamic phases of the string gas, because the sign of its dimensionally reduced, T-duality invariant, part is conserved when the energy density of the universe is positive. Thus, many interesting solutions for which this sign is positive end up in a future curvature singularity. Because of this, some of the thermodynamic phases of the usual gravitating string gases behave like superselection sectors. For example, a past-regular Hagedorn phase and an expanding FRW phase dominated by string momentum modes cannot be smoothly connected in the framework of string cosmology with positive sources. The singularity separates them like a geometric precipice in the moduli space, preventing the dynamics of the theory from bridging across. Sources which simultaneously violate the positivity of energy and NEC could modify these conclusions. We provide a quantitative measure of positivity of energy and NEC violations that would be necessary for such transitions. These effects must dominate the universe at the moment of transition, altering the standard gas pictures. At present, it is not known how to construct such sources from first principles in string theory.

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1 Introduction

String theory differs dramatically from quantum field theory at short distances and high energies. At those scales the extended nature of strings comes into play, and may presumably regulate ultraviolet divergences inherent in field theory. Yet, since the string scale is high, $\ell_s{}^{-1} \gtrsim$ TeV (with the bound being saturated only in most optimistic cases), it is well neigh impossible to test string theory in low energy experiments. On the other hand, successes of modern cosmology have led to the picture of an expanding universe, which was much denser and hotter in its early age. At such high energies, full blown string dynamics may have taken place. Its effects may be important for many reasons. In the limits of string theory which are under full computational control, calculability requires the presence of additional dimensions of space, which somehow must have ended up hidden from low energy probes in the course of cosmic evolution. String theory may also have played a role in controlling and selecting cosmological initial conditions and subsequent evolution which produced a large and smooth universe, by inflation or otherwise. In the course of this evolution, some signatures of string theory may have survived the long era of cosmic dilution, and they could yield new insights in the microphysics of our world. Ultimately, one may hope that string cosmology can shed new light on cosmological singularities, where field theory formulations of matter and gravity fail completely, having no direct means to describe the universe at the densest state.

For these reasons, significant efforts have been invested in the formulation of string cosmology. Interesting models emerged, which treat string matter in thermal equilibrium [1,2,5], evolving under the influence of gravity but with specific stringy phenomena accounted for in the description of the string matter. This approximation clearly cannot fully reveal the details of the microscopic dynamics. Yet, one may at least explore the effects of long range string phenomena on some of the outstanding cosmological problems [6–18]. An intriguing idea was that at very high energies, because strings are extended objects, the string matter can undergo a Hagedorn transition, where the energy density saturates at a finite value controlled by the Hagedorn temperature, taming the violent divergences in the field theoretic models of matter [5]. If this occurs, then using the network of string dualities one might be able to describe the universe beyond the string scale by some dual picture, which admits a different, but controllable perturbative theory.

On the other hand, following the growth of the stringy Hubble volume, one may imagine that as the universe cools the Hagedorn phase should connect onto the standard expanding cosmology describing the evolution of dilute strings. The post-Hagedorn stage should initially be dominated by relativistic strings, behaving as a radiation dominated FRW universe [5]. Similarly, one can also imagine how the Hagdorn phase could be an end product of a collapsing universe controlled by string winding modes which are getting dense and transmuting into the Hagedorn phase. If these transitions can be realized consistently, one could explore their implications[1] for large universes such as our own. The question is whether such transitions are possible.

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[1]Some novel ideas for generating scale-invariant perturbations of the geometry were proposed [19]- [21], based on the proposed transitions between different stages of the string gas. However to date they have not succeeded [20]. Perhaps it is interesting to explore them further, at least to verify how robust the connection between scale-invariant perturbations and inflation is.

Our aim here is to study in more detail the effects of the graviton multiplet fields on the transitions between the string gas phases. We will focus on the dilaton field, and show that in fact it stands in the way of smoothly connecting different thermodynamic phases of the string gas. This is because there is a discrete charge characterizing the dilaton sector: the sign of the dimensionally reduced effective dilaton, $\mathrm{sgn}(\dot{\varphi}_s)$ is a conserved quantity, when the energy density of the universe is non-negative [22]. The solutions fall into branches characterized by this sign, and have a general feature that the location of the cosmological singularity strictly correlates with the sign. On the $(+)$ branch, the singularity is in the future, while on the $(-)$ branch it is in the past. The singularity may be either a pole super-inflation, or a Big Crunch, which can be reinterpreted as a pole super-inflation in the T-dual geometry. On the other hand, if the effective energy density is allowed to dip below zero temporarily, the transitions may be facilitated [23], and for conventional sources which satisfy the null energy condition (NEC) those that can complete are the $(-) \rightarrow (+)$ transitions [22]. Such solutions describe the cosmologies which develop future singularities induced by the transitions. The transitions in the opposite direction, from $(+)$ to $(-)$ branch, are impossible unless NEC is violated [22, 24]. This means, that the universes which start evolving towards a future singularity cannot evade this fate as long as NEC holds.

This implies that some of the proposed connections between the Hagedorn phase and the expanding FRW phase dominated by string momentum mode radiation [5] are impossible in the conventional framework of string cosmology, since those solutions lie on different branches. Instead, the decaying Hagedorn stage ends up as a collapsing FRW cosmology, which crunches into the future spacelike singularity. Similarly, we will see that a collapsing universe dominated by string winding modes will not condense into the long-lived Hagedorn stage evolving towards the attractor, again because its dilaton charge is different. The thermodynamic phases of the usual gravitating string gases behave like superselection sectors, characterized by whether they evolve towards or away from a singular region, which separates them like a geometric precipice in the moduli space, where the dynamics of the theory cannot bring them together.

Sources which simultaneously violate the positivity of energy and NEC could modify these conclusions [22, 24]. There are ways to mimic violations of positivity of energy in string theory, such as by inducing negative cosmological constant, allowing for Casimir energy effects or dimensionally reducing on compact spaces of positive curvature [24]. To evaluate the possibilities and impossibilities of any NEC violations, we retain the perfect fluid description and provide a quantitative measure of positivity of energy and NEC violations that could facilitate the transitions between different thermodynamic phases of string gas. These effects must be dramatic, as they need to dominate the universe at the moment of transition, altering the standard gas pictures [25]. Such dramatic NEC violations are not derived from first principles at present. Further, generically there are dangers that covariant theories which violate positivity of energy and NEC may give rise to ghosts. Perhaps it is possible to tame the ghost instabilities if Lorentz symmetry is broken [26], and induce cosmological bounces away from the singularity as in [27, 28]. On the other hand, string theory in its standard formulation is known not to have ghosts [29–31]. One must properly account for this when attempting to build cosmological scenarios which strive to employ string matter phase transitions as a method for generating cosmological perturbations. Before inventing

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mechanisms that rely on crossing between various phases of string cosmology, one must show that the bridges that need to be crossed aren't too far, but are firmly grounded in theoretical foundation.

2   A Review of Gravitating String Ensembles

Cosmological solutions notoriously involve spacelike singularities, where the curvature blows up, and the effective field theory description of geometry breaks down. In string cosmology, one hopes that the singularities of the effective description could be ameliorated by string effects. The complexities of string theory in time dependent backgrounds are a formidable hurdle to this program, having so far obstructed singularity resolutions, even casting doubt on their existence. To gain some insight in what may be going on, one may instead seek to improve the effective theory description by resorting to some fundamental features of string theory, such as dualities, which may even survive supersymmetry breaking. Indeed, in the string gas approach one should follow the effective geometry to near the singularity, and then invoke a hypothesis that as the original geometry begins to explore curvatures beyond the cutoff, one may replace it by a dual geometry, where the curvatures again turn up small and a perturbative description is restored [2,5]. This very appealing idea thus offers a way to address singularity resolutions entirely within the effective theory language. In this framework one can ask questions about the consistency of duality-related solutions with the desired asymptotic limits to see if they are at least compatible with the standard lore of string theory.

   This is the philosophy which we will pursue in what follows. Assuming that we begin in some large universe where effective theory works well, one can coarse-grain the sources of gravity that inhabit the geometry, using the rules of string thermodynamics to determine the leading-order equations of state and gravitational properties of the string ensembles. This yields the evolutionary laws for homogeneous cosmologies, which we can follow to near the strong coupling/large curvature regime, where the string apparent horizon becomes comparable to the string scale $\ell_s \sim M_s^{-1}$. At this stage, the interstitial distance between the microscopic strings becomes small and the coarse-grained description of cosmology must fail. However, if we follow the dictum of dualities and take it that as distances become smaller, there exists a dual description where the relevant degrees of freedom are getting more dilute, we can assume that it also admits a coarse-grained description and simply ask the question about how these two descriptions connect together. In this way, all we need to do is to correctly and systematically follow the (semi-)classical solutions of the theory based on the truncation of string effective actions to graviton multiplet two-derivative sector to near the string scale [5]. In the string frame the action is given by

$$S_e = \frac{1}{2\kappa_N} \int d^{N+1}x \sqrt{-g_s}\, e^{-\phi_s} \left(R_s + (\partial\phi_s)^2 - \mathcal{L}_m \right) , \tag{1}$$

where $\mathcal{L}_m$ encodes the string ensemble contributions, the effects of stringy corrections, and possibly any other sources of energy density, including the dilaton potential. The dimensional scale $\kappa_N = M_s^{2-N}$ is the string scale in $D = N + 1$ dimensions, and ϕ_s is the string frame

dilaton, related to the string coupling by $g_s^2 = e^{\langle \phi_s \rangle}$. The subscript s indicates a quantity computed in string conformal frame, and not only a string gas source. For the exploration of gravitational effects of coarse-grained string sources, it is enough to explore the theory (1) only on homogeneous backgrounds. However, to elucidate the form of the string gas source terms, following [5] one can use the anisotropic metrics

$$ds^2 = -n(t)^2 dt^2 + \sum_{i=1}^{N} e^{2\lambda_s^{(i)}(t)} \, dx_i^2 \,, \tag{2}$$

where $a_s^{(i)}(t) = e^{\lambda_s^{(i)}(t)}$ are the string frame scale factors of the toroidal directions parameterized by x^is, and $n(t)$ is the lapse function, which can be gauge fixed to unity. While such configurations are generically infested with curvature singularities [5], where some dimensions shrink to zero while others explode, away from the singularities one can use thermodynamic arguments to deduce the form of the string sources, and specifically energy density and pressures, at large scales in the gravitational equations. So, specializing for the moment to only ensembles of gravitating string gas, dimensionally reducing the theory (1) on the commuting spatial Killing vectors ∂_i of (2), and using the shifted string frame dilaton $\varphi_s = \phi_s - \sum_{i=1}^{N} \lambda_s^{(i)}$, one finds the effective $1D$ action [5],

$$S_{1D} = \int dt \, n \left\{ \frac{e^{-\varphi_s}}{M_s} \left(\sum_i \frac{(\dot{\lambda}_s^{(i)})^2}{n^2} - \frac{\dot{\varphi}_s^2}{n^2} \right) + F(\lambda_s^{(i)}, n\beta) \right\}, \tag{3}$$

where F is the comoving free energy of the string gas, which depends on the temperature β^{-1} and the scales of the comoving 'container' $\lambda_s^{(i)}$. Varying this action with respect to intensive parameters and using the standard relations between energy, pressure and free energy, $E_s = -(F + \beta \partial_\beta F)$ and $P_i = \partial_{\lambda_s^{(i)}} F = -(\partial_{\lambda_s^{(i)}} E)|_{\beta=const}$, in the gauge $n = 1$ one finds the string frame Hamiltonian constraint and equations of motion

$$\dot{\varphi}_s^2 - \sum_{i=1}^{N} \left(\dot{\lambda}_s^{(i)} \right)^2 = e^{\varphi_s} E_s \,, \tag{4}$$

$$\ddot{\lambda}_s^{(i)} - \dot{\varphi}_s \dot{\lambda}_s^{(i)} = \frac{1}{2} e^{\varphi_s} P_s^{(i)} \,, \tag{5}$$

$$\ddot{\varphi}_s - \sum_{i=1}^{N} \left(\dot{\lambda}_s^{(i)} \right)^2 = \frac{1}{2} e^{\varphi_s} E_s \,, \tag{6}$$

respectively, where we have chosen the units $M_s = 1$. The sources obey the conservation equation,

$$\dot{E}_s + \sum_{i=1}^{N} \dot{\lambda}_s^{(i)} P_i = 0 \,, \tag{7}$$

which follows from (4)-(6). We will focus on the sources of the form $E_s = \sum_{i=1}^{N} E_{s0} \, e^{\beta \lambda_s^{(i)}}$, in the limit when the distribution is isotropic, $\lambda_s^{(i)} \equiv \lambda_s$. The pressures in different directions

P_i are replaced by the mean pressure $P_s = \frac{1}{N} \sum_{i=1}^{N} P_s^{(i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial E_s}{\partial \lambda_s^{(i)}} = -\frac{\beta}{N} E_s$, such that the equation of state parameters $\gamma^{(i)} = P_s^{(i)}/E_s$ reduce in the isotropic case to $\gamma = -\beta/N$.

Once the equations of motion (4)-(6) are derived, it is convenient to replace the comoving energy E_s and comoving pressure P_s by the energy density ρ_s and pressure p_s. They are defined by $\rho_s = E_s e^{-N\lambda}$ and $p_s = P_s e^{-N\lambda}$, such that the relevant sources in the equations of motion (4)-(6) can be substituted according to

$$e^{\varphi_s} E_s = e^{\phi_s} \rho_s\,, \qquad\qquad e^{\varphi_s} P_s = e^{\phi_s} p_s\,. \tag{8}$$

This allows one to rewrite the equations (4)-(6) in terms of the original $N + 1$-dimensional dilaton variable ϕ_s as

$$\left(\frac{N^2 - N}{2}\right) \dot{\lambda}_s^2 + \frac{1}{2}\dot{\phi}_s^2 - N\dot{\lambda}_s\dot{\phi}_s = \frac{1}{2}e^{\phi_s}\rho_s\,, \tag{9}$$

$$\ddot{\lambda}_s - \dot{\phi}_s\dot{\lambda}_s + N\dot{\lambda}_s^2 = \frac{1}{2}e^{\phi_s}p_s\,, \tag{10}$$

$$\dot{\rho}_s + N\dot{\lambda}_s\left(\rho_s + p_s\right) = 0\,. \tag{11}$$

The last equation is the stress-energy conservation, which follows from (7) and (8). The dilaton equation of motion follows from stress-energy conservation, as one can verify by taking a derivative of (9) and using Eqs. (9)-(11) to solve for $\ddot{\phi}_s$. This form of the equations of motion is convenient to go beyond the gravitating string ensemble approximation and introduce more general sources of stress-energy, which may depend on the dilaton field, including any dilaton potential on the landscape. In that case, of course, the simple equation of state $p_s = \gamma\rho_s$ will be replaced by a more complicated expression, derived from the Lagrangian. This will be useful to extend our results to more general sources, particularly when formulating the conditions for the connection between different gravitating string ensembles.

2.1 Isotropic Single Fluid Cosmologies

The equations of motion (4)-(6) or (9)-(11) can be easily solved for the single fluid component, with a constant equation of state γ. These solutions serve as the attractors of the homogeneous and isotropic multi-component fluid cosmologies. Indeed, starting with a multi-component source, an expanding universe eventually ends up being dominated by the component with the least positive equation of state parameter γ. Conversely, the fluid with the most positive γ will asymptotically dominate in the contracting universe limits. When the fluids are composed of unstable states which may decay, the equation of state may eventually change, transmogrifying the dominant source of stress-energy and the universe it inhabits into a different fluid model. One may therefore try to patch together general cosmologies from such attractor geometries. The key for this quilt work is to determine the rules for connecting solutions. At the leading order, one may assume that the decay time is much faster than the Hubble time at the moment of decay, and approximate the transition by an instantaneous one. In this limit one can again ignore the microscopics and retain the long range, coarse-grained description, using Israel junction conditions to relate the field

variables at the spacelike boundary between different epochs. We will pursue this approach in what follows.

The starting point is to taxonomize the various interesting single component solutions. To count them up, one can first rewrite the system of equations (9)-(11) in the canonical form, by solving the Hamiltonian constraint (9). This yields

$$
\dot{\phi}_s = NH_s \pm \sqrt{NH_s^2 + e^{\phi_s}\rho_s} \,, \tag{12}
$$

$$
\dot{H}_s = \frac{1}{2}e^{\phi_s}p_s \pm H_s\sqrt{NH_s^2 + e^{\phi_s}\rho_s} \,, \tag{13}
$$

$$
\dot{\rho}_s = -NH_s(\rho_s + p_s) \,, \tag{14}
$$

where we have introduced the notation $H_s \equiv \dot{\lambda}_s$. We immediately note that the solutions can be classified by the sign in front of the root, which by the definition of $\varphi_s = \phi_s - N\lambda_s$ is precisely the sign of the reduced dilaton flow, i.e. $\mathrm{sgn}(\dot{\varphi}_s)$, as we can verify by comparing (4) and (12), and by the arrow of time. The sign of $\dot{\varphi}_s$ can change only if the discriminant $N\dot{\lambda}_s^2 + e^{\phi_s}\rho_s$ vanishes. Clearly, this is impossible if $\rho_s \geq 0$. We will discuss this in more detail later on. The sign of H_s can change if the universe 'bounces', which may occur in the string frame due to the fact that the dilaton kinetic term is not canonical. At any rate, this discussion shows that for any value of the source terms there are four solutions, described by the combinations of the two signs. The latter sign selection is the same one encountered in the pre-Big Bang cosmology, and so we adopt the same terminology here, as it obviously generalizes. From now on, we will refer to the $(\pm)$-branch by referring to the sign in front of the square root.

Equations (12)-(13) show that as long as $\rho_s \geq 0$, in the expanding universe on the $(+)$ branch, the string coupling $g_s = e^{\langle \phi_s \rangle/2}$ increases. We also see that on the $(-)$ branch, the string coupling increases during the contracting phase. On the other hand, the evolution of the string coupling in a $(+)$ branch contracting universe and a $(-)$ branch expanding universe is more sensitive to the sources, and depends on the time dependence of $e^{\phi_s}\rho_s$. In particular, it may occur that the $\rho_s > 0$ fluid can flip the sign of $\dot{\phi}_s$ in these cases, as can be seen by rewriting Eq. (12) as $\dot{\phi}_s = \pm\sqrt{N}\sqrt{\dot{\varphi}_s^2 - e^{\phi_s}\rho_s} + \dot{\varphi}_s$. This also shows that if $\rho_s < 0$, the coupling is fixed to grow on the $(+)$ branch contracting solutions, and decrease on the $(-)$ branch expanding solutions.

We should note that we can generate new solutions by the symmetry transformations of the effective action, which are the time-reversal $dt \to -dt$ and T-duality [4]. In fact, any two solutions which reside on the *same* branch, as defined by the sign of $\dot{\varphi}$, but differ by the sign of H_s, are T-dual images of each other. The T-duality operates on the background as the map $\lambda_s \to -\lambda_s$, $\varphi_s \to \varphi_s$. This also implies the change of sign of the equation of state parameter $\gamma = p/\rho$, which is easy to see from the fact that the T-dual of the decaying energy density increases, and vice-versa. An important thing to stress is that T-duality does not change the branch since the sign of dt, and therefore $\dot{\varphi}_s$, remains unchanged. On the other hand, the time-reversal operation changes the branch on which the solution resides, since under it the sign of $\dot{\varphi}_s$ changes. Thus starting from any one solution, we can get four different ones by the sequence of maps T-duality $\to$ time-reversal $\to$ T-duality. In the case $\gamma = 0$, or when $p_s = \rho_s = 0$ as in pre-Big Bang cosmology [6], this exhausts all the independent families of solutions, but not in general cases, as we will see in what follows.

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Cosmic expansion and/or acceleration in the string frame do not imply the same in the Einstein conformal frame, where the graviton kinetic terms are normalized canonically. On the contrary, when the dilaton evolves fast, the picture in the Einstein frame may be completely reversed, so that an expanding string frame universe looks like a collapsing universe crashing into a singularity. This is relevant to understand the obstructions to pasting together different cosmic epochs. To go to the Einstein frame, one performs the conformal map to the action where the graviton and dilaton terms are canonically normalized. The field redefinition which relates the Einstein frame and the string frame variables is

$$g^e_{\mu\nu} = e^{-2\phi_S/(N-1)} g^s_{\mu\nu}, \qquad \phi_e = \sqrt{\frac{2}{N-1}}\, \phi_s, \tag{15}$$

which on FRW backgrounds yields $a_e = e^{-\phi_s/(N-1)} a_s$. Further, one must be careful with the comoving time relationship as well, which is given by gauge-fixing the two lapse functions to unity. This yields $d\tau_e = e^{-\phi_s/(N-1)} dt$. Using $\phi_s = \varphi_s + N\lambda_s$, the logs of the scale factors and their time derivatives are related by

$$\lambda_e = -\frac{\lambda_s + \varphi_s}{N-1}, \qquad \frac{d\lambda_e}{d\tau_e} = -\left(\frac{\dot{\lambda}_s + \dot{\varphi}_s}{N-1}\right)\frac{dt}{d\tau_e}. \tag{16}$$

From the second of (16) we see that the sign of the Einstein frame Hubble parameter $H_e \equiv \frac{d\lambda_e}{d\tau_e}$ can be reversed relative to the sign of H_s, depending on the evolution of ϕ_s. This can be seen most easily on a case to case basis.

Let us now write down the exact single fluid isotropic solution, defined by a constant equation of state parameter $\gamma = P_s/E_s = p_s/\rho_s$ and the scaling law $E = E_0 \exp(-\gamma N\lambda)$. For this purpose it is useful to go back to the (isotropic limit of) the equations (4)-(6) and introduce a new time coordinate x defined by $E = \frac{dx}{dt}$. The equations of motion become

$$\varphi_s'^2 - N\lambda_s'^2 = E_0^{-1} e^{\varphi_s + \gamma N\lambda_s}, \tag{17}$$

$$\lambda_s'' - \gamma N\lambda_s'^2 - \varphi_s'\lambda_s' = \frac{1}{2}\gamma E_0^{-1} e^{\varphi_s + \gamma N\lambda_s}, \tag{18}$$

$$\varphi_s'' - \gamma N\varphi_s'\lambda_s' - N\lambda_s'^2 = \frac{1}{2} E_0^{-1} e^{\varphi_s + \gamma N\lambda_s}, \tag{19}$$

respectively, where the prime denotes a derivative with respect to x. The solutions of these equations are [6, 32]

$$\lambda_s = \lambda_{s0} + \frac{\gamma}{\alpha}\ln[x(x-x_*)] + \frac{1}{\alpha\sqrt{N}}\ln\left(1 - \frac{x_*}{x}\right), \tag{20}$$

$$\varphi_s = \varphi_{s0} - \frac{1}{\alpha}\ln[x(x-x_*)] - \frac{\gamma\sqrt{N}}{\alpha}\ln\left(1 - \frac{x_*}{x}\right), \tag{21}$$

$$\phi_s = \phi_{s0} - \frac{1-N\gamma}{\alpha}\ln[x(x-x_*)] - \frac{(\gamma-1)\sqrt{N}}{\alpha}\ln\left(1 - \frac{x_*}{x}\right), \tag{22}$$

where $\alpha = 1 - N\gamma^2$. The integration constant λ_{s0} is pure gauge, and can be changed arbitrarily by a rescaling of the spatial coordinates $x^k \to \zeta x^k$. The constant φ_{s0} is linked

to the comoving energy E_0 according to $e^{\varphi_{s0}} = \frac{4}{\alpha} E_0 \, e^{-\gamma N \lambda_{s0}}$. Note that in principle, we can always write the solution in the form where the integration constant x_* is positive. Indeed, to do it, it is sufficient to shift $x \to x + x_*$. In this case, the only change in the solution (20)-(22) is the sign flip of the terms $\propto \ln \left(1 - \frac{x_*}{x}\right)$. It is quite clear that the times $x = 0$ or $x = x_*$ are special moments in the evolution of the epoch described by (20)-(22), as there the scale factor $a = e^{\lambda_s}$ can vanish or diverge. To see precisely what goes on, however, it is more convenient to study separately the cosmologies of different relevant string phases.

2.2 Hagedorn Phase

In string theory, the density of states diverges at some high temperature, where the theory undergoes Hagedorn transition. Beyond this scale, the injection of additional energy into the system does not affect the temperature, which remains constant, but simply increases the entropy [33]. Essentially, what happens is that the strings become extremely wiggly as they keep absorbing the energy [34]. This saturation of the temperature implies that the dependence on the other intensive parameters vanes, and so the pressures vanish. Hence the Hagedorn phase is described by the constant comoving energy $E = E_0$ and vanishing pressure $P = 0$, giving the equation of state parameter $\gamma = 0$ [5]. This implies that the parameter α as defined in the solution (20)-(22) is $\alpha = 1$. Rewriting this solution in terms of the comoving string frame time coordinate, by using $t = x/E_0$, and setting $x_*/E_0 = t_*$, we find [5]

$$
\begin{aligned}
\lambda_s &= \lambda_{s0} + \frac{1}{\sqrt{N}} \ln \left(1 - \frac{t_*}{t}\right) , \\
\varphi_s &= \varphi_{s0} - \ln \left[t \left(t - t_*\right)\right] .
\end{aligned}
\tag{23}
$$

The unshifted dilaton is

$$
\phi_s = \phi_{s0} - 2 \ln |t| + \left(\sqrt{N} - 1\right) \ln \left(1 - \frac{t_*}{t}\right) ,
\tag{24}
$$

so that the string coupling is $g_s = \frac{g_{s0}}{|t|} \left(1 - \frac{t_*}{t}\right)^{\frac{\sqrt{N}-1}{2}}$.

First we note immediately that the special solution with $t_* = 0$ describes a static string geometry, with $\lambda_2 = \lambda_{s0} = \text{const}$ and $\varphi_s = \varphi_{s0} - 2 \ln t$, where the runaway dilaton absorbs the effects of the Hagedorn source allowing the universe to loiter forever. This special solution is *not* a universal attractor, since it is unstable: while there are configurations which flow towards it, representing either universes expanding from zero size or contracting from infinite size (as exemplified by the Class III and IV solutions described below), their time-reversals, which are perfectly allowed by the time-reversal symmetry of the field equations, describe the evolution away from this special geometry, either towards a crunch or an infinite universe.

Now, the solutions for $t_* \neq 0$ have classical curvature singularities at $t = 0$ or $t = t_*$, where the scale factor either diverges or vanishes. The interval between 0 and t_* is classically forbidden, since inside it the logarithms are imaginary. Note that the ordering 0 and t_* does not introduce extra solutions because the order can be changed by the shift $t \to t + t_*$, which sets $t_* > 0$ while flipping the sign of the log term in the equation for the scale factor.

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This means, that the solutions with $t_* < 0$ are simply T-duals of the solutions with $t_* > 0$ for a fixed sign of t. On the other hand, the time-reversed geometries are obtained by simultaneously flipping the signs of both t and t_* in the solution. The solutions (23) and (24) are perturbatively meaningful descriptions of the following classical string geometries:

- **Class I:** The universe expands from finite size, starting with $g_s \ll 1$, on the interval $t \leq 0$, with $t_* > 0$ and $\dot{\varphi}_s > 0$ (evolving away from the $t_* = 0$ repeller) until $t = 0$, where the scale factor and the string coupling diverge, and yield a curvature divergence as well. Effective field theory description will cease in the course of evolution towards this limit.

- **Class II:** T-dual of Class I, which contracts on the interval $t \leq 0$, with $t_* > 0$ and $\dot{\varphi}_s > 0$ (evolving away from the $t_* = 0$ repeller) from finite size to a Big Crunch at $t = 0$, while the dilaton flows to weak coupling $g_s \ll 1$. Geometric description ceases in this limit.

- **Class III:** The universe slowly expands for $t \geq t_* > 0$ from zero size to a constant size at $t \to \infty$, with $\dot{\varphi}_s < 0$ (approaching the $t_* = 0$ attractor) and the dilaton grows, $\dot{\phi}_s > 0$, such that the string coupling becomes strong at late times, and weakly coupled perturbative picture must eventually be replaced by a strongly coupled description. During the early epoch when $a < 1$, the geometric description fails. This solution is in fact time-reversal of the Class II solution.

- **Class IV:** T-dual of Class III, which during times $t \geq t_* > 0$ contracts from infinite size to a constant finite size at $t \to \infty$ with $\dot{\varphi}_s < 0$ (approaching the $t_* = 0$ attractor) and has a finite string coupling, that initially grows ($\dot{\phi}_s > 0$) but eventually decreases ($\dot{\phi}_s < 0$), evolving to weak coupling regime. Again, the geometric description of this solution cannot be trusted beyond $a < 1$. This solution is time-reversal of the Class I solution.

As explained above, there us no more independent solutions, essentially because λ_s depends only on the ratio $\frac{t-t_*}{t}$. This degeneracy follows from $\gamma = 0$, so that all the gravitating Hagedorn phases remain connected by duality and time-reversal. The solutions are summarized in Table (1). Note that the (+) branch solutions evolve towards the singularity, and (−) branch away from it. The evolution of these solutions is represented in the $\dot{\varphi}_s, H_s$ phase plane in Figure (1), with the convention that the time flows from left to right.

The perturbative description is limited by the flow of a solution to short distances and/or strong coupling. The geometrical singularities appear particularly dangerous. Of course, once we recall the philosophy behind dualities, the singularities may be interpreted as merely an indicator of dual branches which the solution must be replaced by, once the scale factor dives below the string scale. The question then is if such dual maps relate geometries that admit asymptotic description in some supergravity. In particular, one would like to be able to have a cosmology whose late time behavior describes a radiation dominated stage, with the coupling possibly evolving away from extreme weak coupling, so that the theory may yield useful mechanisms for moduli stabilization and asymptote to a phenomenologically

Class	Branch	Expansion	Shifted Dilaton	Time	Singularity
I	$(+)$	$H_s > 0$	$\dot{\varphi}_s > 0$	$t < 0$	Future
II	$(+)$	$H_s < 0$	$\dot{\varphi}_s > 0$	$t < 0$	Future
III	$(-)$	$H_s > 0$	$\dot{\varphi}_s < 0$	$t > t_*$	Past
IV	$(-)$	$H_s < 0$	$\dot{\varphi}_s < 0$	$t > t_*$	Past

Table 1: The four Classes of the Hagedorn solution (23). In Class II, if $(\sqrt{N}+1)/2 > t/t_* > 1$ then $\dot{\phi}_s > 0$, otherwise $\dot{\phi}_s < 0$. Likewise, in Class III if $(\sqrt{N}+1)/2 < t/t_* < 1$ then $\dot{\phi}_s < 0$, otherwise $\dot{\phi}_s > 0$. Class I and II, and Class II and IV, respectively, are related by T-duality, and Class I and IV and II and III, respectively, by time-reversal.
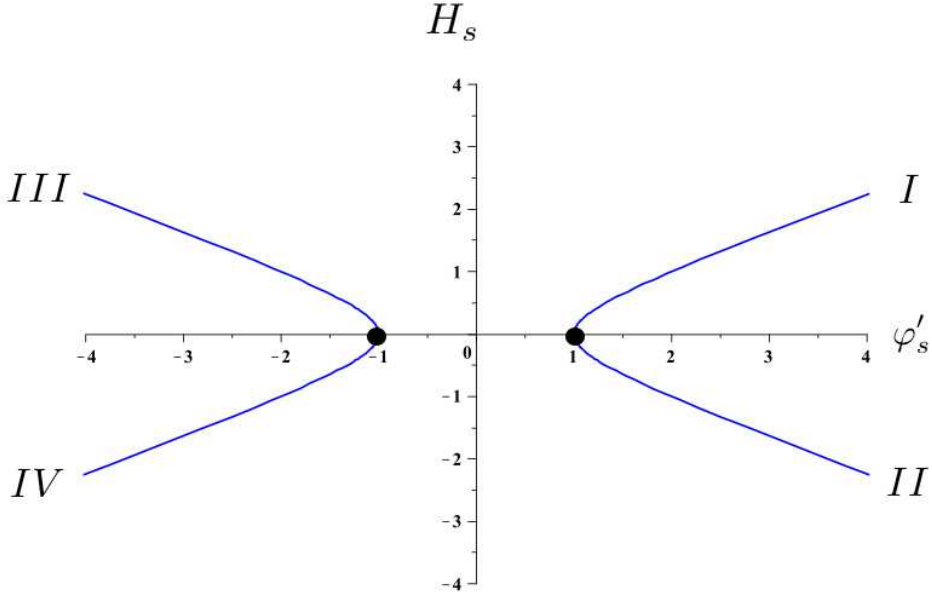

Figure 1: The (φ'_s, H_s) phase diagram of Hagedorn cosmologies. The black dots mark the Hagedorn fixed points, $(\varphi'_s, H_s) = (\pm\sqrt{E_0}, 0)$.

reasonable regime. To see if such a stage can be preceded by a Hagedorn stage, we must first explore the phase space of cosmology of excited dilute strings, to which we turn to next.

2.3 Dilute String Phase

Below the Hagedorn temperature, the ensemble of strings behaves differently. If spatial directions are compact, then the dilute strings can carry energy in two different channels: i) the momentum modes, which describe the propagation of disturbances along the string, and ii) the winding modes, which store the energy to stretch the string around the compact dimension. In the former case, the momentum mode channels are relativistic at high energies, and so their thermal ensemble behaves as a gas of radiation in the string frame, with the equation of state parameter obeying $\gamma = 1/N$, yielding $E = E_0 e^{-\lambda}$. The winding modes are the T-dual of the momentum modes, and so their equation of state parameter, by the fact that under T-duality the pressure changes sign, is $\gamma = -1/N$, such that $E = E_0 e^{\lambda}$ [4, 5]. The negative pressure reflects the fact that when the dimensions expand, the winding mode energy *increases* since it is proportional to the length of the string. Hence the energy density dilutes more slowly: although the number density drops down as is usual, with the one power of spatial volume, the energy increases and compensates some of the dilution. Of course, in the case when strings gravitate, the whole geometry needs to be consistently subjected to the T-duality transformation.

The homogeneous cosmology dominated by momentum or winding modes is still given by the general solution (20)-(22) but with the parameter α now given by $\alpha = \frac{N-1}{N}$. Thus the solution for the scale factor and the shifted dilaton formally reads

$$\lambda_s = \lambda_{s0} \pm \frac{1}{N-1} \ln\left[x(x - x_*)\right] + \frac{\sqrt{N}}{N-1} \ln\left(1 - \frac{x_*}{x}\right) , \qquad (25)$$

$$\varphi_s = \varphi_{s0} - \frac{N}{N-1} \ln\left[x(x - x_*)\right] \mp \frac{\sqrt{N}}{N-1} \ln\left(1 - \frac{x_*}{x}\right) , \qquad (26)$$

where the upper sign formally describes a momentum mode dominated universe, and the lower sign a winding mode dominated universe. The singular points of the solutions are as before 0 and x_*, and the interval between them is classically forbidden, by the condition of reality of the solution. The unshifted dilaton in a universe dominated by the momentum modes, which evolve as string radiation, is then given by

$$\phi_s^m = \phi_{s0} + \sqrt{N} \ln\left(1 - \frac{x_*}{x}\right) , \qquad (27)$$

whereas in the universe dominated by the winding modes it is

$$\phi_s^w = \phi_{s0} - \frac{4N}{N-1} \ln|x| + \sqrt{N} \frac{\sqrt{N}-1}{\sqrt{N}+1} \ln\left(1 - \frac{x_*}{x}\right) , \qquad (28)$$

as dictated by T-duality.

To see that the solutions with the opposite signs of γ are in fact T-duals of each other, we need to shift the 'time' by $x \to x + x_*$ and flip the sign of the integration constant x_*.

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With this, it becomes clear that $\lambda_s - \lambda_{s0} \to -(\lambda_s - \lambda_{s0})$ while φ_s remains unaffected. In fact, this becomes obvious if we write the expressions for the scale factor and the exponent of the shifted dilaton,

$$
\begin{aligned}
a_s &= e^{\lambda_{s0}} x^{-\frac{\sqrt{N}\mp1}{(\sqrt{N}-1)(\sqrt{N}+1)}} (x - x_*)^{\frac{\sqrt{N}\pm1}{(\sqrt{N}-1)(\sqrt{N}+1)}}, \\
e^{\varphi_s/2} &= \frac{e^{\varphi_{s0}/2}}{x^{\sqrt{N}\frac{\sqrt{N}\pm1}{(\sqrt{N}-1)(\sqrt{N}+1)}} (x - x_*)^{\sqrt{N}\frac{\sqrt{N}\mp1}{(\sqrt{N}-1)(\sqrt{N}+1)}}}.
\end{aligned} \tag{29}
$$

However, unlike before, the solutions depend on different powers of x and $x - x_*$, so that the sequence of T-duality and time-reversal cannot link all of them directly. Instead, there are two different families of linked solutions, which arise because T-duality interchanges momentum and winding modes, and flips the sign of $\gamma \neq 0$, breaking the degeneracy found in the gravitating Hagedorn gas example.

To understand the universes which these solutions describe, we need to follow their evolution for specific values of x_*. As before, there are the special solutions with $x_* = 0$, which are

$$
\lambda_s = \lambda_{s0} \pm \frac{2}{N-1} \ln|x|, \qquad \varphi_s = \varphi_{s0} - \frac{2N}{N-1} \ln|x|, \qquad \phi_s = \phi_{s0} + \frac{2N(1\pm1)}{N-1} \ln|x|. \tag{30}
$$

They are clearly T-duals of each other, representing asymptotic attractors for momentum mode-dominated and winding mode-dominate cosmologies, respectively. For general $x_* \neq 0$, there are eight distinct solutions that comprise two families interlinked by T-duality and time reversal, which are are conveniently parameterized by the branch sign and the signs of γ and x_*. They are as follows:

- **Class 1:** The momentum mode-dominated universe which expands forever as a power law, starting with $g_s \ll 1$, on the interval $x \geq x_* > 0$, with $\dot{\varphi}_s < 0$, such that asymptotically $g_s \to g_{s0} = $ const., evolving towards the $x_* = 0$ momentum mode attractor. This is a $(-)$ branch solution. Near the singularity $x = x_*$ the geometric description will cease, and the classical geometry there resembles a Big Bang.

- **Class 1$^{\text{TR}}$:** The time-reversal of the Class 1 solution, found by flipping the signs of both x and x_*. Evolves away from the $x_* = 0$ momentum mode repeller.

- **Class 2:** Another momentum mode-dominated universe, which however resides on the interval $x < 0$, with $x_* > 0$, and which starts out as a collapsing universe but undergoes a string frame bounce, and runs off to a super-inflating pole singularity at $x = 0$. This is a $(+)$ branch solution, $\dot{\varphi}_s > 0$, starting with some $g_{s0} = $ const, evolving away from the $x_* = 0$ momentum mode repeller towards strong coupling $g_s \gg 1$. Near the pole, both the geometric description and perturbation theory in g_s break down. This solution cannot be related to the Class 1 case neither by T-duality nor time-reversal, which is clearly from the existence of the string frame bounce.

- **Class 2$^{\text{TR}}$:** The time-reversal of the Class 2 solution. Again, the signs of both x and x_* are flipped. Evolves towards the $x_* = 0$ momentum mode attractor.

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- **Class 3:** T-dual of the Class 1 solution, describing a winding mode-dominated universe, which is collapsing from infinite size at $x = x_* > 0$ to zero as $x \to \infty$, with $\dot{\varphi}_s < 0$. This is a $(-)$ branch solution. In the course of evolution, the string coupling changes from $g_s \gg 1$ to $g_s \ll 1$. Initially, the theory is given by strongly coupled winding mode gas, which evolves towards weakly coupled dense gas, described by the $x_* = 0$ winding mode attractor, where the geometric picture breaks down.

- **Class 3$^{\text{TR}}$:** The time-reversal of the Class 3 solution. Evolves away from the $x_* = 0$ winding mode repeller.

- **Class 4:** T-dual of Class 2, a winding mode-dominated universe defined on the domain $x < 0$ (with $x_* > 0$) which initially slowly expands out of zero size, evolving away from the $x_* = 0$ winding mode repeller. It reaches a maximum size due to the resistance from the winding modes, and starts to collapse again. In the course of evolution, $\dot{\varphi}_s > 0$ so this is a $(+)$ branch solution. The string coupling varies from $g_s \ll 1$ to $g_s \gg 1$, so that near the crunch both the geometric description and g_s expansion fail.

- **Class 4$^{\text{TR}}$:** The time-reversal of the Class 4 solution. Evolves towards the $x_* = 0$ winding mode attractor.

Rewriting these solutions in terms of the comoving time t is cumbersome. However in the asymptotic limits $|x| \to \pm\infty$ it is straightforward to extract the leading order form of the solution as a function of t. In fact these limits are exactly the attractors (30). Using $dx/dt = E_0 e^{-\lambda_s}$, and the fact that (25) converges to $\lambda_s \to \pm\frac{1}{N-1} \ln x^2$ when $|x| \to \infty$, we find that $|x| \sim |t|^{(N-1)/(N+1)}$ when momentum modes dominate. Similarly, when winding modes dominate, $|x| \sim |t|^{(N-1)/(N-3)}$ for $N \neq 3$, and $|x| \sim e^{E_0|t|}$ for $N = 3$, as is straightforward to see. Therefore the momentum mode solutions in the limit $x \to \pm\infty$ approach

$$a_s \to a_{s0}|t|^{\frac{2}{N+1}}, \qquad \varphi_s \to \varphi_{s0} - \frac{2N}{N+1}\ln|t|, \qquad \phi_s \to \phi_{s0}, \tag{31}$$

which is precisely the form of an FRW radiation-dominated cosmology in $N+1$ dimensions. The fact that the string dilaton ϕ_s approaches a constant in this limit despite the absence of a stabilizing potential is a well known consequence of the conformal symmetry of the sources which has this attractor behavior [35]. On the other hand, as $x \to \pm\infty$ the winding mode solutions converge to

$$a_s \to \frac{a_{s0}}{|t|^{\frac{2}{N-3}}}, \qquad \varphi_s \to \varphi_{s0} - \frac{2N}{N-3}\ln|t|, \qquad \phi_s \to \phi_{s0} - \frac{4N}{N-3}\ln|t|, \tag{32}$$

for $N \neq 3$, which changes to

$$a_s \to a_{s0}e^{-E_0|t|}, \qquad \varphi_s \to \varphi_{s0} - 3E_0\ln|t|, \qquad \phi_s \to \phi_{s0} - 6E_0|t|, \tag{33}$$

when $N = 3$. We therefore see that the winding mode geometries describe an inflating or deflating asymptotic cosmology in the string frame, depending on the direction of the flow of t.
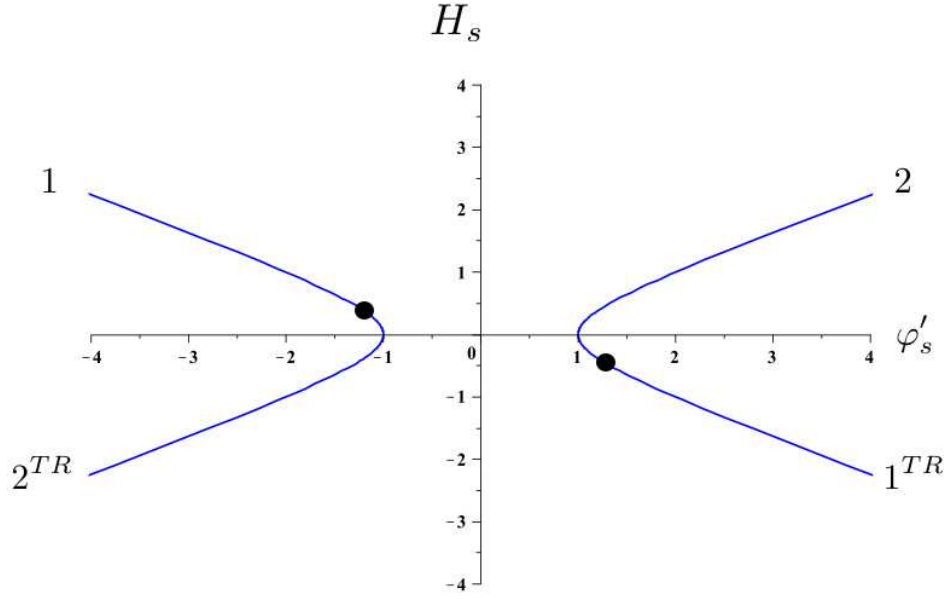
Figure 2: The (φ'_s, H_s) phase diagram of momentum mode-dominated cosmologies. The black dots mark the momentum mode fixed point cosmologies, which in the τ variable are $(\varphi'_s, H_s) = (\mp\sqrt{3E_0/2}, \pm\sqrt{E_0/6})$.
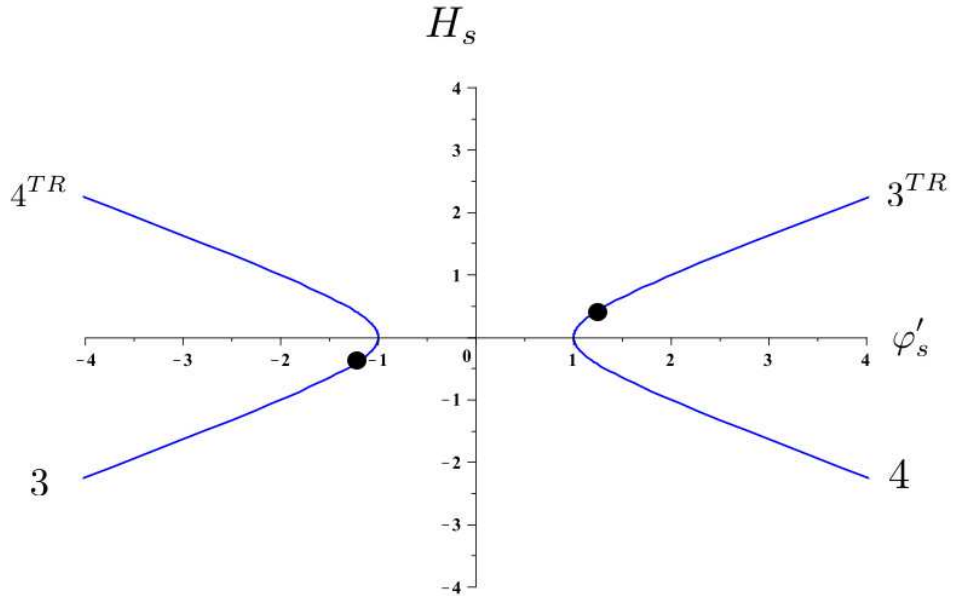

Figure 3: The (φ'_s, H_s) phase diagram of winding mode-dominated cosmologies. The black dots mark the winding mode fixed points, $(\varphi'_s, H_s) = (\pm\sqrt{3E_0/2}), \pm\sqrt{E_0/6})$.

These solutions are depicted in Figures (2) (momentum mode-dominated cosmologies) and (3) (winding mode-dominated cosmologies), where we use the time variable τ, defined by the relation $d\tau = \exp(\frac{\varphi_s - \gamma N\lambda_s}{2})\,dt$ (see below), and assume that the time flows again from left to right. The two families of solutions mutually linked by time-reversal and T-duality are also summarized in Tables (2) and (3). Again, we see that the $(+)$ branch solutions run to a singularity, while the $(-)$ branch ones evolve away from it. We can now turn to determining how and when these solutions may be matched onto the Hagedorn gas cosmologies of the previous section.

Class	Branch	Expansion	Shifted Dilaton	Time	Singularity
1	$(-)$	$H_s > 0$	$\dot{\varphi}_s < 0$	$x \geq x_* > 0$	Past
1^{TR}	$(+)$	$H_s < 0$	$\dot{\varphi}_s > 0$	$x \leq x_* < 0$	Future
3	$(-)$	$H_s < 0$	$\dot{\varphi}_s < 0$	$x \geq x_* > 0$	Past
3^{TR}	$(+)$	$H_s > 0$	$\dot{\varphi}_s > 0$	$x \leq x_* < 0$	Future

Table 2: Smooth family, without string frame bounce. Classes 1 and 3 are T-duals of each other, as are their time-reversed solutions, describing momentum mode-dominated and winding mode-dominated cosmologies, respectively.

Class	Branch	Expansion	Shifted Dilaton	Time	Singularity
2	$(+)$	$H_s < 0 \rightarrow H_s > 0$	$\dot{\varphi}_s > 0$	$x < 0,\ x_* > 0$	Future
2^{TR}	$(-)$	$H_s > 0 \rightarrow H_s < 0$	$\dot{\varphi}_s < 0$	$x > 0,\ x_* < 0$	Past
4	$(+)$	$H_s > 0 \rightarrow H_s < 0$	$\dot{\varphi}_s > 0$	$x < 0,\ x_* > 0$	Future
4^{TR}	$(-)$	$H_s < 0 \rightarrow H_s > 0$	$\dot{\varphi}_s < 0$	$x > 0,\ x_* < 0$	Past

Table 3: A family with the string frame bounce. Classes 2 and 4 are T-duals of each other, as are their time-reversed solutions, again describing momentum mode-dominated and winding mode-dominated cosmologies, respectively.

3 Branch Changes and Energy Conditions

The transitions between different phases of string are normally precipitated by the change in internal energy. As the dilute string gas is heated to the Hagedorn temperature, it undergoes the Hagedorn transition, and starts to absorb energy by increasing its entropy while keeping a constant temperature. Conversely, if the internal energy is dissipated, by decreasing entropy, the Hagedorn gas will reach the transition point and begin to cool. An obvious arena for such transitions is an expanding universe, where the expansion dials the internal energy of the matter up and down. Thus in string theory, it is expected that in cosmological backgrounds the gas of string will scan through various allowed thermodynamic phases as the universe evolves. This has led [2, 5] to propose models of early universe which may even provide a way around the cosmological singularity, not directly by a bounce but rather by a dual

relationship between collapsing and expanding universes. In these works, typically one seeks to have cosmologies which look like a patchwork of different thermodynamic phases of the string, that evolve into each other dynamically.

In the presence of string gravity, however, the transitions between different string phases become subtler, complicated by the presence of the dilaton. The dilaton may significantly affect the evolution of the background geometry inhabited by the string gas. If one treats it too cavalierly, one may not notice that the dilaton can obstruct transitions between some of the gravitating thermodynamic string ensembles. One may argue that just before such transitions the dilaton is stabilized by dynamical effects [14,19]. But if the theory is to retain its duality symmetries at the fundamental level, the dilaton stabilization and decoupling should only occur at low energies, below which the dualities are obscured by the infra-red effects. Hence one expects that any phase transitions at high energies, in a very early universe, should be subjected to the selection rules dictated by the full, T-duality symmetric dilaton gravity which governs the geometry. Let us now determine these rules in a precise manner. We will show that as long as the energy density inhabiting the universe is non-negative, branch changing is impossible, and the transitions can only occur between string phases on the same geometric branch. Further, if positivity of energy is not maintained, branches may change, but unless the sources also violate NEC, which in homogeneous and isotropic cosmologies requires $p + \rho < 0$, it is impossible to go from a $(+)$ branch solution to a $(-)$ branch one. Since $(+)$ solutions have future singularities, this result is really a singularity theorem in a different guise: it implies that the solutions cannot be relieved of strong curvatures unless NEC is violated.

Indeed, when discussing the equations (12)-(14), we already observed the first part of this statement, that the solutions on opposite branches, as characterized by $\mathtt{sgn}(\dot{\varphi})$, cannot link up unless the discriminant $N\dot{\lambda}_s^2 + e^{\phi_s}\rho_s$ vanishes. Since the first term is positive definite, this implies that if the energy density ρ_s is also positive definite, a transition between different branches is *impossible*. Hence to have any chance for changing branches, the energy density must dip below zero, which of course by itself still does not guarantee that the branch change will actually occur.

Although very simple, this statement of positivity of energy excluding branch changing is already quite strong: it already excludes some of the phase transitions described in the literature. For example, [5] describe a universe which expands away from the self-dual point on the Hagedorn phase, and eventually below the Hagedorn transition it transits to a radiation phase, or possibly to an expanding winding mode phase. Near the self-dual point, the universe just hovers, with the background asymptotically evolving out of the attractor solution, given by the $t_* = 0$ limit of (23). Prior to the self-dual point, the universe may have evolved as a time-reversed cosmology of either of the two post self-dual point options. However, the inspection of the Hagedorn phase solutions, in Table (1), immediately shows that the Hagedorn cosmology that slowly flows out of the fixed point at the self-dual radius is the Class I geometry, which is a $(+)$ branch solution. On the other hand, the radiation cosmology which has no future singularities but evolves towards a conventional FRW universe with a frozen dilaton is the Class 1 geometry, a $(-)$ branch solution, as we see from Table (2). Thus if the energy density dominating the universe is always positive definite, the decay of the Hagedorn gas in a cosmology evolving from the attractor *will not* yield an asymptotic

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radiation-dominated FRW universe. Alternatively, the Hagedorn gas stage is really evolving towards the strong curvature regime, whereas the radiation FRW universe is not. To change from one to another, the evolution to large curvature must be averted, and that is impossible with the positive definite energy density.

If we turn to winding mode cosmologies, we can see by a similar argument as above that the Class I Hagedorn gas universe, which flows out of the attractor, can only connect to the Class 4 geometry, which is also a $(+)$ branch solution (see Table (3)). This transition is possible early on while the winding mode universe is still expanding. However, the expansion ceases in this universe and it collapses back to a future singularity. Once it passes below the self-dual radius, we may replace it by its T-dual description, however this is the Class 2 momentum-mode dominated cosmology which experiences a pole singularity in the future as opposed to asymptoting to the conventional radiation-dominated FRW universe. The bottomline is that the strong curvature singularity in the future is unavoidable. Similar obstructions can be inferred for the time-reversed configurations describing the universe before the self-dual point.

Thus as long as the energy density is non-negative, we can only sew together the solutions on the *same* branch [22]. For instance, requiring that the late universe is a radiation-dominated FRW universe of Class 1 confines one to the $(-)$ branch solutions. If the evolution is then followed backward in time, it is plausible that this universe can be preceded by the Hagedorn cosmology given by the Class III solution, which is on the $(-)$ branch too. However, in the far past, this solution does *not* asymptote to the Hagedorn attractor configuration, but instead has a Big Big singularity in the string frame. It is conceivable that the small universe region near this singularity can be resolved as the T-dual geometry of Class IV, which in turn could have evolved from the Class 3 collapsing winding mode-dominated world. In this chain of events, one might be able to 'relegate' the problem of cosmological singularity, reinterpreting it as the problem of finding the correct dual description of a small universe. On the other hand, in contrast to the discussion of the cosmic phase quilt that was discussed in [5], the universe of this example does *not* spend an infinite time lingering around the Hagedorn attractor. In fact, the Hagedorn attractors are completely excised, now being in the future of the transition to FRW radiation, or in the past of the winding mode condensation. One could now imagine that the local perturbations in the collapsing T-dual geometry, if such is consistently constructed, can be computed, and that they could map onto long distance perturbations in the small universe near the singularity. This would resolve their short distance dynamics. However this would be completely different from the ideas for generation of perturbations in string cosmology which have been pursued so far. Nevertheless any applications of such ideas to phenomenology must await the development of the framework that might enable us to compute how the local perturbations in one universe T-dualize to large scale perturbations in another. The present techniques only allow a precise construction of T-dual geometries when there are isometries, which are of course all generically broken by the perturbations.

To understand the dynamics of the solutions with non-negative energy density, it is convenient to go to the phase space description. The phase space evolution of solutions can be illustrated in simple cases of Hagedorn, momentum mode or winding mode universes, with $\gamma = 0, \pm 1/N$, respectively, in Figures (4)-(6), where we now also plot the flow fields which
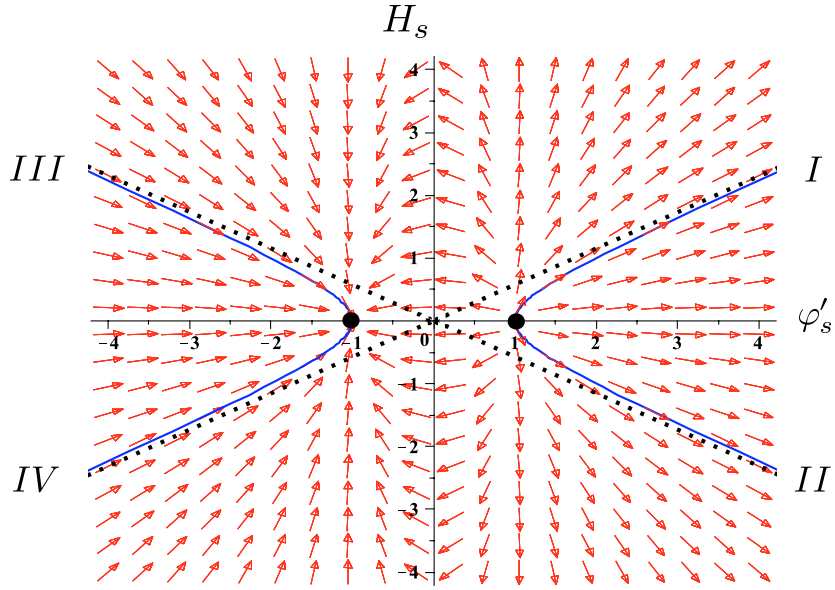
Figure 4: The (φ'_s, H_s) phase diagram of Hagedorn cosmologies, with phase space flow and the limiting envelopes (dotted lines) describing the case $E_0 = 0$.
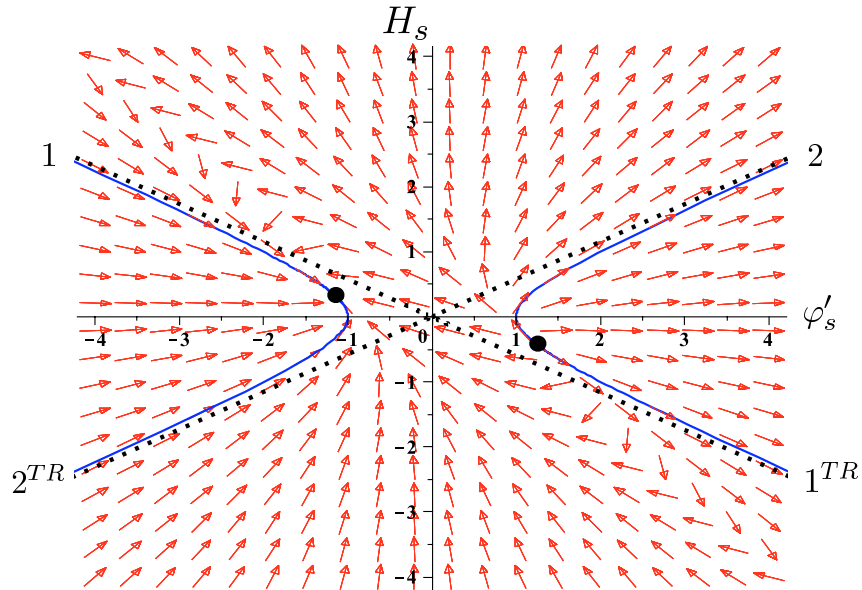

Figure 5: The (φ'_s, H_s) phase diagram of momentum mode cosmologies, with phase space flow and the limiting envelopes (dotted lines) describing the case $E_0 = 0$.
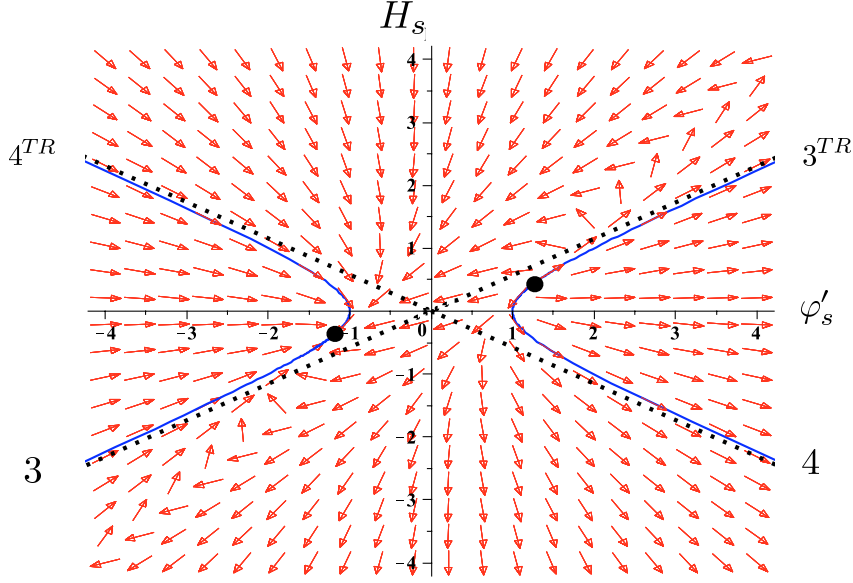
Figure 6: The (φ'_s, H_s) phase diagram of winding mode cosmologies, with phase space flow and the limiting envelopes (dotted lines) describing the case $E_0 = 0$.

indicate the direction of evolution for nearby trajectories. In the plots, we start with the Eqs. (12)-(14), rescale the time variable to $d\tau = e^{(\varphi_s - \gamma N \lambda_s)/2} dt$ and introduce the variables $l = \lambda'$ and $f = \varphi'$ to recast these equations as a first-order system,

$$f' - \frac{1}{2}f(\gamma N l + f) = -\frac{1}{2}E_0\,, \qquad l' - \frac{1}{2}l(\gamma N l + f) = \frac{\gamma}{2}E_0\,, \qquad f^2 - N l^2 = E_0\,. \qquad (34)$$

This system has fixed points

$$(f, l) = \left(\pm sgn(\gamma)\sqrt{\frac{E_0}{1 - \gamma^2 N}}, \pm \sqrt{\frac{\gamma^2 E_0}{(1 - \gamma^2 N)}}, \right)\,, \qquad (35)$$

which are precisely the relevant attractor/repeller solutions that we discussed previously. The flow fields are sensitive to the location of the fixed points, as controlled by the initial comoving energy E_0. However, to understand the dynamics we must keep in mind that the flows are constrained by the last of Eqs. (34) to follow a specific trajectory in phase space as proscribed by the Hamiltonian constrain from the gravitational sector, which in simpler parlance is just the Friedman equation. An arbitrary trajectory in phase space for a fixed energy density and pressure, that may be a solution of the first two of (34) is *not* a solution of the full gravity equations unless the constraint is satisfied explicitly. So, the flow fields in the Figures (4)-(6) are to be understood merely as an indicator for how the system will evolve if it is perturbed by a source that is asymptotically subleading to the dominant sources parameterized by E_0, which would deform the system trajectory away from the depicted solid curve, with the deformation becoming small near the fixed points. Conversely, if we wish

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to follow a randomly chosen trajectory in the phase plane, which is not one of the depicted solid lines, we *must* provide the correct sources in order to ensure that the trajectory is a solution of the full gravitational system. But in such cases, the flow lines will be different than those depicted, particularly if the new sources turn out to be dominant.

As a consequence of this discussion, we see immediately that if we want to follow a trajectory that would take us to the coordinate origin $(f, l) = (0, 0)$, we must dial the value of E_0 to zero as time goes on. This of course means that the fluid dominating the universe will be neither Hagedorn, nor the momentum mode, nor the winding mode gas. Further, as long as the energy density remains positive, the system will have to follow a trajectory which is composed piecewise of the infinitesimal portions of the depicted hypebolas, because of the last of Eqs. (34). This then shows that for such evolution the coordinate origin will be an accumulation point of the fixed points on the one-parameter family of hyperbolas enclosed between the dashed straight lines in the Figures (4)-(6), and so, it itself will be a fixed point. Therefore, it will take an infinite time to reach it. As a result, the universe represented by such solutions will be future geodesically complete, and its passage through the origin will be impossible. This shows that the idea of the loitering universe dynamics of [14] cannot be realized with positive sources. The transition from one side of the phase diagram to another will never complete, and the universe on one side, approaching the fixed point, will be left hanging there forever.

One may also ask how robust are these conclusions, since quantum corrections may violate positivity of energy, and perhaps relax some of the restrictions on the dynamics. In fact one does expect additional contributions to the effective Lagrangian, that could arise from other stringy matter, higher order α' or g_s corrections, spatial curvature, and various possible contributions to the effective dilaton potential, and so this question is relevant. If the effective energy density in the course of evolution were negative for a while, then a change from $(-)$ branch solutions to $(+)$ branch ones may occur, redirecting the flow of solutions from weak to strong curvature regime [22]. However the transitions from $(+)$ to $(-)$ branch, that could avert the future singularity once the cosmology starts evolving towards it, cannot happen if the sources which dominate the universe *do not* violate NEC during the transition from different string phases. To see this, we can generalize Eqs (12)-(14) to nominally include additional contributions to the dynamics in the effective Lagrangian $\mathcal{L}_m = \mathcal{L}_m(\phi, g_{\mu\nu}, \dots)$ (denoted by $\dots$), arising from any other stringy matter, higher order α' and g_s corrections, spatial curvature, and various possible corrections to the dilaton potential. Restricting the analysis to the homogeneous and isotropic perfect fluid cosmologies as before, with the string ensemble sources $T_{\mu\nu} = \mathrm{diag}\,(\rho_s, p_s, \dots, p_s)$, now including the contributions from the corrections to $\mathcal{L}_m$ as well as the string ensemble terms, we find [24]

$$\dot{\varphi}_s = \pm\sqrt{NH_s^2 + e^{\phi_s}\rho_s}\,, \tag{36}$$

$$\dot{H}_s = \pm H_s\sqrt{NH_s^2 + e^{\phi_s}\rho_s} + \frac{1}{2}e^{\phi_s}\left(p_s + \Delta_\phi\mathcal{L}_m\right)\,, \tag{37}$$

$$\dot{\rho}_s = -NH_s\left(\rho_s + p_s\right) - \dot{\phi}_s\Delta_\phi\mathcal{L}_m\,, \tag{38}$$

$$\ddot{\varphi}_s = NH_s^2 + \frac{1}{2}e^{\phi_s}\left(\rho_s - \Delta_\phi\mathcal{L}_m\right)\,, \tag{39}$$

where $\Delta_\phi\mathcal{L}_m$ is the variational derivative of the effective Lagrangian with respect to the

dilaton including any couplings of the dilaton to the general sources which are absent in the string ensemble description (3). Here clearly the upper sign refers to the (+) branch configurations, and the lower sign to the (−) branch. The Eq. (39) follows by taking a derivative of (36) and straightforwardly manipulating the terms.

As noted before, a branch change may not occur unless the discriminant, or the 'egg' function [23]

$$\mathsf{e} = NH_s^2 + e^{\phi_s}\rho_s \,, \tag{40}$$

vanishes during some stage of the evolution, which requires $\rho_s < 0$. Now, suppose that ρ_s does dip below zero for some range of variables and parameters. From Eqs. (36)-(39) it is clear that the phase space trajectories can only explore the region of $\rho_s < 0$ down to the contour $\mathsf{e} = 0$. The region $\mathsf{e} < 0$ is excluded by demanding the reality of the solutions. Now, the region where $\rho_s < 0$ should be compact, in order to match it asymptotically to the string gas cosmologies discussed above, for which $\rho_s \geq 0$. Thus, the contour $\mathsf{e} = 0$ should be bounded, with its interior excluded, just like the 'egg' region discussed in the pre-Big Bang attempts to induce branch changing [23, 24]. The mechanism of branch-changing can then be viewed as a collision between the system trajectory in the phase space and the 'egg' contour $\mathsf{e} = 0$, that needs to stand in its way. A branch changing would then require the following ingredients:

- Collision with the 'egg'; branch changing without collisions is impossible, since the sign of $\dot{\varphi}_s$ will not change unless $e = 0$ somewhere along the trajectory.

- At the moment of collision, $\dot{\varphi}_s \propto \sqrt{\mathsf{e}} = 0$. Since the trajectory cannot enter inside the 'egg' and will not stop there, as $\ddot{\varphi}_s \neq 0$, it will ricochet away, grazing the 'egg'. By continuity, the sign of $\dot{\varphi}_s$, and consequently the branch, will change.

- Subsequently the trajectory should fly away from the 'egg', such that further collisions that can change the branch again will not occur. Alternatively, for the branch change to occur there could be only an odd number of collisions.

So to have a branch changing, we must generate an 'egg' region in the way of a trajectory which asymptotically approaches those of Figures (3)-(6). Then we must tune the initial conditions such that a trajectory will glance off the 'egg', and make sure that it can flow away after an odd number of hits. To check when these requirements may be realized, write the first derivative of $\dot{\varphi}_s = \pm\sqrt{\mathsf{e}}$ using (39), $\pm\frac{d}{dt}\left(\sqrt{\mathsf{e}}\right) = \mathsf{e} + \frac{1}{2}e^{\phi_s}\rho_s - \frac{1}{2}e^{\phi_s}\Delta_\phi\mathcal{L}_m$, and using (37) eliminate $\Delta_\phi\mathcal{L}_m$ to find

$$\pm\frac{d}{dt}\left(\sqrt{\mathsf{e}}\right) = -\dot{H}_s + H_s\dot{\phi}_s + \frac{1}{2}e^{\phi_s}\left(\rho_s + p_s\right) \,. \tag{41}$$

Suppose that a branch change does occur, so that the trajectories arriving to the surface of the 'egg' at say the instant t_h, when $e = 0$, and departing away from the region of the 'egg' at say t_e when $\rho_s = 0$ and so $\mathsf{e} = NH_s^2$, and beyond which again $\rho_s \geq 0$, are on different branches. Integrating Eq. (41) between these instants yields [22]

$$\int_{t_h}^{t_e} dt\left[\frac{d}{dt}\left(\pm\sqrt{\mathsf{e}} + H_s\right) - H_s\,\dot{\phi}_s\right] = \frac{1}{2}\int_{t_h}^{t_e} dt\, e^{\phi_s}\left(\rho_s + p_s\right) \,. \tag{42}$$

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The first term is a total derivative and so is given by the difference of the quantity $\pm\sqrt{e} + H_s$ at the limits of integration. The second quantity is $\int_{t_h}^{t_e} dt\, H_s\dot{\phi}_s = \int_{t_h}^{t_e} d\phi_s H_s$, and so it is given as the line integral of $H_s(\phi_s)$ over the phase space trajectory. Because the flow of the phase space trajectories around the 'egg' is clockwise [22], which can be glimpsed from the flow lines on the Figures (4)-(6), the relation $\dot{\phi}_s = \dot{\varphi}_s + NH_s$, and the fact that around the 'egg' H_s cannot change the sign since it obeys $NH_s^2 = \mathsf{e} + e_s^\phi|\rho_s| \geq e_s^\phi|\rho_s| > 0$, this integral is *equal* to the area $\mathcal{A}$ between the projection of the phase space trajectory onto the (ϕ_s, H_s) hyperplane and hence it is positive definite. Then integrating and using that $\mathsf{e}(t_e) = NH_s^2(t_e)$ and $e(t_h) = 0$, we get

$$\left(\pm\sqrt{N} - 1\right) H_s(t_e) + H_s(t_h) + \mathcal{A} = -\frac{1}{2}\int_{t_h}^{t_e} dt\, e^{\phi_s}\left(\rho_s + p_s\right) . \tag{43}$$

Note that this formula immediately applies for any odd number of hits between t_h and t_e, thanks to the additivity of definite integrals.

Now, if the incoming trajectory is a $(-)$ branch expanding universe with $H_s > 0$, keeping the lower sign in the first term of Eq. (43) we can rewrite this equation as

$$\left(\sqrt{N} + 1\right) H_s(t_e) = H_s(t_h) + \mathcal{A} + \frac{1}{2}\int_{t_h}^{t_e} dt\, e^{\phi_s}\left(\rho_s + p_s\right) , \tag{44}$$

which in principle allows a branch change $(-) \to (+)$ to occur even when the NEC is always satisfied, $p_s + \rho_s \geq 0$. For such a transition, what is required is that prior to the transition, the expansion of the universe on the $(-)$ branch, as characterized by $H_s(t_h)$, is slowed down. This is accomplished by the effective string frame energy density ρ_s dipping below zero, but then NEC need not be violated. Such transitions can be realized by even such mundane terms as the positive spatial curvature of the universe, as is familiar in the examples considered in [22]. However, such transitions are not desired. On the contrary, if a transition from $(-)$ branch to a $(+)$ branch occurs, it then implies that the universe will end up in the future singularity, unless another transition back to $(-)$ branch happens.

Those transitions, which are also required in order to realize the cosmological scenarios discussed in [5, 14], are much harder to accomplish. Indeed, going back to Eq. (43), to describe the $(+) \to (-)$ transition we now must keep the upper sign in the first term, as that describes an initially $(+)$ branch solution. Then we can rewrite the resulting equation for an expanding universe as

$$\left(\sqrt{N} - 1\right) H_s(t_e) + H_s(t_h) + \mathcal{A} = -\frac{1}{2}\int_{t_h}^{t_e} dt\, e^{\phi_s}\left(\rho_s + p_s\right) . \tag{45}$$

Since the universe should be expanding before and after the transition, the left hand side of this equation is positive definite, and unless NEC is violated, we see that no solutions can satisfy it. Thus, if any solutions that are consistent with Eq. (45) were to exist, NEC *must* be violated in an overwhelming way - during the transition, the universe must be dominated by the sources which do not respect NEC.

It is very difficult to imagine such sources emerging in string theory. It is true that Casimir energy violates NEC, however in cosmological applications such contributions are

by and large subleading[2] to the dominant sources that control the evolution [36]. One exception, where the NEC violating effects are studied and found to significantly affect the background is the eternal inflation [38]. In eternal inflation, quantum fluctuations drive the scalar field up the potential, which violates NEC, and induces an inflationary landscape which populates the vacua of the theory. This happens without violating the positivity of energy (while keeping the quantum matter of the universe in equilibrium with gravity, as opposed to just in equilibrium) in contrast to what branch changing processes would need to do. On the other hand, the recent explorations of the string landscape have shown that it is possible to find inflation in string theory [39], and so such dynamics may occur even if the universe starts out dominated by some string gas. If this happens, however, inflation will generically completely wipe out the traces of the early initial conditions in the universe, reshaping the universe in accord with inflationary dynamics. This would then solve cosmological problems without the need to resort to the gas phase.

A possible method to induce NEC violation is to introduce ghosts in the theory, which of course typically lead to a host of problems with both the microscopic and the large scale behavior of physical systems, involving uncontrollable instabilities. An interesting suggestion that a ghost may condense [26], and its instabilities tamed in cosmological applications has been made in [27, 28], however string theory has been proven to be ghost-free [29, 30]. Furthermore, the ghost condensate model [26] is known to have a very low cutoff, which is necessary to keep the bad behavior under control. To go beyond the cutoff and explore energy scales such as those encountered in string cosmology, one should need to find a UV completion of the theory and use it at the scales above the cutoff. The investigation of the possible completions however has turned up arguments against such completions based on locality and causality [31]. Thus unless it is demonstrated that the ghost condensate can be consistently embedded in string theory, the interesting condensed ghost bounces of [27] do not really provide the mechanism to generate branch changes in string gas cosmologies. Thus for now, it appears that the phenomenological applications of (+) branch solutions, that evolve towards future singularities, invariably leads one into the string swampland [40].

4 Summary

To summarize, in this paper we have considered the obstructions to phase transitions between different thermodynamic phases of the string gas in the early universe. These arise because the string dilaton and gravity react to the background sources by generating the flow of the dilaton, or the string coupling. If the energy density of the sources is non-negative, the sign of the time derivative of the dimensionally reduced dilaton $\varphi_s = \phi_s - N\lambda_s$ is conserved, behaving like a discrete charge, and separating the solutions into two superselection sectors, or (+) and (−) branches. The (+) branch solutions are problematic since they evolve to a future spacelike singularity, which looks either like a Big Crunch or a pole inflation in the string frame. The singularity separates different branches of string gas solutions like a geometric precipice in the moduli space, and prevents them from joining together. The flow cannot

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[2]The corrections to the Hagedorn gas equation of state which have been analyzed in [37] indeed do not violate NEC.

be altered without negative energy and NEC violations. This excludes the early universe cosmologies based on the transitions of the string Hagedorn gas evolving away from the fixed point to the FRW radiation cosmology [5], and the loitering gas models of [14], unless new sources are introduced which violate both positivity of energy and NEC. We provide quantitative criteria which such sources must satisfy to yield a branch change, but recall that the known mechanisms which could in principle satisfy these criteria [27, 28] do not at present come out of string theory in a natural way. It does remain possible to build composite cosmologies using the solutions that reside on the same branch, and we have seen an example of a model patched together from winding mode, Hagedorn and momentum mode cosmologies on the $(-)$ branch. Those solutions behave differently from the ones discussed in [5, 19] because they do not have a future singularity. The past singularity could be 'resolved' by using T-duality, instead of bounces induced by negative energy and NEC violations, and is closer in spirit to the original ideas of how T-duality might help in cosmological applications. These solutions do not yet have interesting cosmological applications, but it would be interesting to consider what happens with cosmological perturbations in universes that are T-dual to each other.

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