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Measuring Social Unrest Based on Income Distribution

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Abstract

This paper* develops a social unrest measure by revising Esteban-Ray (1994, Econometrica) polarization index. For the purpose of measuring more effectively the level of social unrest that is generated by separation of income classes, the new index allows for asymmetry between the rich and the poor groups' alienation feeling against the other, and it constructs a more effective group identification function. To facilitate statistical inferences, asymptotic distribution of the proposed measure is also derived using results from U-statistics, and an easy-to-implement jackknife-based variance estimation algorithm is obtained. Since the new index is general enough to include the Esteban-Ray index and the Gini index for group data as special cases, the asymptotic results can be readily applied to these popular indices. Evidence based on the Panel Study of Income Dynamics data suggests that the level of social unrest has generally increased over the sample period of 1981-2005, particular since the late 1990's, and the increase is statistically significant.

JEL No. C13; D31; D63; I32

Keywords: Polarization, Social Unrest, Asymmetry, U-statistics, Jackknife

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1 Introduction

In development economics, the sizable private and social costs of political instability generated by conflicts between subgroups of a society have been regarded as a major constraint to economic growth and human development (e.g., Collier, 1999; Fearon and Laitin, 2003). In general, measuring the level of social unrest is of interest in social science since it is related with crime, collective action, riot, social conflict, or civil war (e.g., D'Ambrosio and Wolff, 2001; Reynal-Querol, 2002).

However, studies on developing the measures of social unrest level are somewhat limited except the polarization index by Esteban and Ray (1994).[1] Esteban and Ray (1994, p.820) writes,

> " ... why are we interested in polarization? ... the phenomenon of polarization is closely linked to the generation of tensions, to the possibilities of articulated rebellion and revolt, and to the existence of social unrest in general."

As emphasized by Esteban and Ray (1994) and Wolfson (1994), polarization is conceptually different from inequality: The latter describes overall dispersion of income distribution, whereas the former emphasizes the within-group clustering as well as the distance between different income groups so that it can describe phenomena of the disappearing middle class and formation of two segregated income classes. As such, polarization indices serve as a measure of between-group conflicts, and Esteban and Ray (1999) try to identify the type of distributions under which social conflict is most likely. There has been more studies on the measures of [bi]polarization of income distribution (e.g., Wolfson, 1994; Esteban and Ray, 1999; Duclos, Esteban and Ray, 2004; Esteban, Gradin, and Ray, 2007; Foster and Wolfson, 2010) but none of them focus on the level of social unrest.

This paper contributes to the existing literature in two-fold: It develops a generalized polarization index that measures the level of social unrest that is generated by separation of income classes more effectively than existing indices; and it analytically provides a basis for

[1] The measure of segregation by Hutchens (2004) is related.

statistical inference of the new index. First, we extend the Esteban-Ray polarization index by allowing asymmetry between the rich and the poor groups' alienation feeling against the other, and by constructing a more effective group identification function relative to existing indices. When we consider the total antagonism all individuals have in a society, it is natural to believe that, for a given statistical properties of income distribution, the poor feel a greater extent of alienation against the rich than vice versa. In addition, for a sensible representation of the degree of group-specific identity, which is the key element that distinguishes the concept of polarization from that of inequality, we consider not only the group size but also the degree of group-specific income clustering in the group identification function:[2] A person feels stronger group identity to his own group either when the size of own group is larger or when within-group income distribution is less dispersed. The current measure is general enough to include the Esteban-Ray index and the Gini index for group data as special cases.

Second, despite repeated reports in the empirical research that income distribution became more (or less) [bi]polarized between two time points (e.g., Gradin, 2000; Gradin and Rossi, 2006; Esteban, Gradin, and Ray, 2007; Hussain, 2009), few studies provide formal statistical conclusions because there is little study on the theoretical distributions of those indices. To facilitate statistical inferences, we derive the asymptotic distribution of the proposed measure using results from U-statistics, which generalizes Bishop, Formby and Zheng (1997). This result is quite useful since it can be naturally extended to existing Esteban-Ray type polarization indices and the Gini index for grouped data. Understanding that estimation of the asymptotic variance can be tedious, we also propose an easy-to-implement jackknife-based variance estimation algorithm.

The remainder of the paper is organized as follows. Section 2 develops a social unrest measure that represents the total effective antagonism in a society. Section 3 develops the asymptotic distribution of the new measure. As a special case, Section 4 deals with the polarization into two groups (i.e., bipolarization) along with jackknife estimation strategy of the asymptotic variance. As an illustration, Section 5 obtains the trajectory of the social unrest

[2] In contrast, Esteban and Rey (1994) consider only the group size effect in designing the group identification function. Esteban, Gradin, and Ray (2007) do not consider group-specific clustering effects but sum of all group clustering effects as a whole in extending Esteban and Ray's (1994) index.

level in the U.S. using the Panel Study of Income Dynamics data and compares them before and after adjusting government taxes and transfers. Section 6 concludes the paper with some remarks. All technical proofs are collected in the Appendix.

2 Measuring the Level of Social Unrest

We assume that a set of individual income data $\{y_i\}_{i=1}^n$ is a random sample from an underlying distribution $F(y)$, whose support is given by $[y_{\min}, y_{\max}]$ with $0 < y_{\min} < y_{\max} < \infty$. We consider K number of pre-specified income groups $\{A_k\}_{k=1}^K$, where $A_k = (a_{k-1}, a_k]$ for $k = 1, 2, \cdots, K$ and $2 \leq K < n$. Without loss of generality, we let $y_{\min} = a_0 < a_1 < \cdots < a_{K-1} < a_K = y_{\max}$ and define the first interval $[a_0, a_1]$ to be closed. The number of intervals, K, is given and it is assumed to be fixed (i.e., not growing with n) and small (e.g., $K = 2$ for the case of bipolarization, i.e., polarization into two groups). For each group A_k, we define the population fraction π_k and the group mean μ_k as

$$\pi_k = \int_{a_{k-1}}^{a_k} dF(y) \quad \text{and} \quad \mu_k = \frac{1}{\pi_k} \int_{a_{k-1}}^{a_k} y dF(y),$$

where we assume that $\pi_k > 0$ for all k. It follows that $\sum_{k=1}^K \pi_k = 1$ and the overall mean is given as $\mu = \int y dF(y) = \sum_{k=1}^K \pi_k \mu_k$. Note that the group means are in ascending order by construction so that $\mu_k < \mu_\ell$ if $k < \ell$.

We measure the level of social unrest by

$$\mathcal{P}(\alpha, \theta) = \frac{1}{2\mu} \sum_{k=1}^K \sum_{\ell=1}^K \pi_k \pi_\ell \phi_k(\alpha) \rho_\theta (\mu_k - \mu_\ell) \tag{1}$$

for some constants α and θ (to be discussed subsequently) that are chosen by the researcher. $\phi_k(\alpha) > 0$ represents the *within-group identity* measure of group k and $\rho_\theta(u) = 2u(\theta - \mathbb{I}\{u < 0\})$ represents the *between-group alienation* measure that depends on the mean income distance between different income groups, where $\mathbb{I}\{\cdot\}$ is the binary indicator. Similar to Esteban and Ray (1994), therefore, $\mathcal{P}(\alpha, \theta)$ combines the following two concepts: within-group *identity* and between-group *alienation*. Recall that the polarization index developed by Esteban and Ray

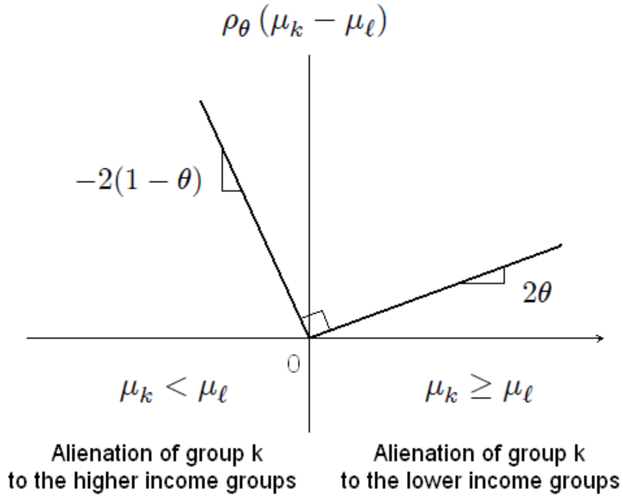

Figure 1: $\rho_\theta(\mu_k - \mu_\ell)$ describes asymmetric alienations of group k towards ℓ

(1994) is defined as $ER(\alpha) = (1/\mu)\sum_{k=1}^{K}\sum_{\ell=1}^{K}\pi_k^{1+\alpha}\pi_j|\mu_k - \mu_\ell|$.

To be more specific, we formulate between-group alienation as

$$\rho_\theta(\mu_k - \mu_\ell) = 2(\mu_k - \mu_\ell)(\theta - \mathbb{I}\{\mu_k < \mu_\ell\}), \tag{2}$$

which is more general than $ER(\alpha)$ in the sense that we allow for *asymmetric* feelings of alienation,[3] with the degree of the asymmetry being determined by the value θ. Specifically, if we let $0 \leq \theta \leq 1/2$, then the lower income groups feel more alienated from the higher income groups than vice versa. The asymmetry gets more severe as θ goes to zero. As an extreme case, if $\theta = 0$ then the richer groups do not feel any alienation against the poorer groups (e.g., Yitzhaki, 1979). If $\theta = 1/2$ then the degree of alienation is symmetric between the groups, which corresponds to the case of $ER(\alpha)$ and the standard income inequality measures like the Gini index.

Therefore, the polarization index $\mathcal{P}(\alpha,\theta)$ reflects not only the between-group income distance (i.e., the economical aspect of the alienation) but also the asymmetric degree of feelings that each group has against the others (i.e., psychological aspect of the alienation). Figure 1

[3] For the asymmetry, Esteban and Ray (1994) briefly mentioned about such generalization but there has been no studies in that extension.

depicts $\rho_\theta(\mu_k - \mu_\ell)$, where the absolute value of the slope determines the degree of the asymmetric alienation of group k towards different income-level groups.[4] Note that the parameter of asymmetric feeling of alienation, θ, is different from the inequality aversion parameter in Atkinson's index (Atkinson, 1970) or generalized entropy index (Cowell and Kuga, 1981; Shorrocks, 1984); the latter measures the overall (and thus symmetric) inequality aversion level whereas the former measures the asymmetric inequality aversion levels in each direction.

For a more effective representation of within-group identity, we assume that the degree of group-identity is positively affected by the group size but is inversely related to within-group income dispersion. More precisely, for $\alpha > 0$, we let

$$\phi_k(\alpha) = \left(\frac{\pi_k}{\delta_k}\right)^\alpha \tag{3}$$

for some within-group income dispersion measure $\delta_k > 0$ over the interval A_k.[5] Similarly as Lee and Shin (2012), we consider the *relative* dispersion measure (e.g., σ_k/σ or G_k/G, where σ and G are the standard deviation and the Gini index of the entire population) instead of the *absolute* dispersion measure (i.e., σ_k or G_k). For example, changes in G_k also alter the overall income inequality level G so that it affects other groups' relative dispersion (i.e., other things being constant, as G_k decreases, members in other groups feel relatively less identified). In particular, we let

$$\delta_k = G_k/G \tag{4}$$

in this paper. In this specification, within-group identity gets larger either when the population share of group k increases or when the dispersion of the within-group income distribution of group k decreases. In comparison, the standard polarization index by Esteban and Ray (1994)

[4] The parameter θ represents the psychological degree of alienation of (poorer) group k toward (richer) group ℓ for all k and ℓ; the reversed direction is represented by $1 - \theta$. In general, however, this parameter could be heterogeneous across different pairs of groups (k, ℓ), depending on the relative location of k, the distance between k and ℓ and the sign of $(\ell - k)$. But we simply assume the homogeneous case in this paper to minimize the number of parameters in the index.

[5] As stated in Esteban and Ray (1994), α is a parameter that distinguishes Esteban-Ray type polarization measures from an inequality measure. $0 < \alpha \leq 1.6$ needs to be satisfied to meet the axioms of Esteban-Ray polarization concept.

assumes $\delta_k = 1$ for all k.[6] An individual feels the income class separation more as a social structural problem when the population proportion π_k of some particular income groups get larger. The income dispersion of group δ_k reflects the degree of feeling on individuals' income level clustering in group k, and an individual identifies more with her group members as within-group income levels become more similar.

Note that the polarization index $\mathcal{P}(\alpha, \theta)$ is general enough to cover the existing inequality or income polarization indices. For example, if $\theta = 1/2$ (and thus $\rho_\theta(u) = |u|$) and $\phi_k(\alpha) = \pi_k^\alpha$, then the polarization index $\mathcal{P}(\alpha, \theta)$ in (1) becomes the index developed by Esteban and Ray (1994), $ER(\alpha)$. Furthermore, if $\theta = 1/2$ and $\phi_k(\alpha) = 1$, then the index becomes the Gini index for grouped data. However, $\mathcal{P}(\alpha, \theta)$ represents the *level of social unrest*, which is implied by income distribution, more effectively than these indices, since it allows not only for asymmetric degrees of alienation among different income groups but also for a more plausible identification function explaining the within-group clustering.

Remark Though the number of groups K is arbitrarily chosen in defining the social unrest measure $\mathcal{P}(\alpha, \theta)$, we need to properly choose the cutoff points $a_1, \cdots, a_{K-1}$. In general, such a problem is solved using the K-means clustering algorithm (e.g., Hartigan and Wong, 1979) for a given number of groups. Esteban, Gradin and Ray (2007) employ Aghevli and Mehran (1981)'s method of optimal grouping for a given K. The idea is that one minimizes the sum of within-group income dispersions (e.g., the mean difference) with respect to the optimal cutoff points. Geometrically, this method corresponds to approximating the continuous Lorenz curve by piecewise linear functions and finding the optimal cutoff points that minimize the overall approximation error. Aghevli and Mehran (1981) show that the optimal cutoff point is the

[6] In $ER(\alpha)$, π_k measures the feeling of identification each individual has toward her own group members. In comparison, our new polarization index $\mathcal{P}(\alpha, \theta)$ considers not only the 'size effect' (π_k) but also the 'clustering effect' ($1/\delta_k$) in measuring the degree of within-group identity. In this regards, it can be understood that the extended polarization index by Esteban, Gradin and Ray (2007) also considers the group clustering effect implicitly, though such a point is not discussed in their paper. Note that their extended index is defined as $EGR(\alpha, \beta) = ER(\alpha) - \beta C_K$, where $\beta (> 0)$ is some arbitrary weight parameter (often $\beta = 1$) and C_K is the error in approximating the continuous Lorenz curve by K-piecewise linear functions. C_K gets smaller as within-group income distributions become more clustered around their group means. In this aspect, we can understand that $\mathcal{P}(\alpha, \theta)$ constructs the identity function of each group using its own clustering effect whereas $EGR(\alpha, \beta)$ combines all the clustering effects to consider the overall approximation.

population mean in the case of two groups.

3 Asymptotic Distribution

From (2), (3) and (4), we can readily obtain an estimator for $\mathcal{P}(\alpha,\theta)$ in (1) using proper estimators for π_k, μ_k and G_k $(k = 1, 2, \cdots, K)$ as

$$\widehat{\mathcal{P}}_n(\alpha,\theta) = \frac{1}{2\overline{y}} \sum\nolimits_{k=1}^{K} \sum\nolimits_{\ell=1}^{K} \widehat{\pi}_k \widehat{\pi}_\ell \left(\frac{\widehat{\pi}_k}{\widehat{G}_k/\widehat{G}} \right)^{\alpha} \rho_\theta \left(\overline{y}_k - \overline{y}_\ell \right), \tag{5}$$

where

$$\widehat{\pi}_k = \frac{n_k}{n} = \frac{1}{n} \sum\nolimits_{i=1}^{n} \mathbb{I}\{y_i \in A_k\}, \;\; \overline{y}_k = \frac{1}{n_k} \sum\nolimits_{i=1}^{n} y_i \mathbb{I}\{y_i \in A_k\}$$

and $\overline{y} = (1/n) \sum_{i=1}^{n} y_i$ with n_k being the number of observations in the interval A_k. Recall that $\widehat{G}_k = \left(1/(2\overline{y}_k n_k^2)\right) \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \mathbb{I}\{y_i \in A_k\} \mathbb{I}\{y_j \in A_k\}$ is the Gini coefficient for group k and $\widehat{G} = \left(1/(2\overline{y}n^2)\right) \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|$ is the standard Gini coefficient. Since the number of groups is fixed as K and independent of n, we can assume that $n_k \to \infty$ for each k as $n \to \infty$ without loss of generality. Therefore, for given (α, θ), the consistency of $\widehat{\mathcal{P}}_n(\alpha,\theta)$ to $\mathcal{P}(\alpha,\theta)$ readily follows by applying the continuous mapping theorem, since all the components in (5) are consistent to their population counterparts.

Despite the repeated reports that income distribution has become more bipolarized, few studies provide formal statistical conclusions since the asymptotic distribution results of those polarization indices are not well established. This section derives the asymptotic distribution of the generalized index estimator $\widehat{\mathcal{P}}_n(\alpha,\theta)$ to facilitate further (distribution-free) statistical inferences for the index. As discussed above, the new index is general enough to include the Esteban-Ray type indices as special cases and thus the statistical results below can be directly applied to those indices.

We let $\widehat{g} = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|$ be the standard mean difference coefficient and $\widehat{g}_{k\ell} = \widehat{g}_{\ell k} = n_k^{-1} n_\ell^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \mathbb{I}\{y_i \in A_k\} \mathbb{I}\{y_i \in A_\ell\}$ be the sub-group mean difference coefficient for all $k, \ell = 1, 2, \cdots K$. It holds that $\widehat{g} = \sum_{k=1}^{K} \sum_{\ell=1}^{K} \widehat{g}_{k\ell} \widehat{\pi}_k \widehat{\pi}_\ell$ (e.g., Dagum, 1997) and $\overline{y}=\sum_{k=1}^{K}\widehat{\pi}_k\overline{y}_k$ by construction. Then $\widehat{\mathcal{P}}_n(\alpha,\theta)$ in (5) can be rewritten as

$$
\begin{aligned}
\widehat{\mathcal{P}}_n(\alpha,\theta) &= \frac{\left(\sum_{k=1}^{K}\sum_{\ell=1}^{K}\widehat{g}_{k\ell}\widehat{\pi}_k\widehat{\pi}_\ell\right)^{\alpha}}{\left(\sum_{k=1}^{K}\widehat{\pi}_k\overline{y}_k\right)^{1+\alpha}} \\
&\quad\times\sum_{k=1}^{K}\widehat{\pi}_k^{1+\alpha}\left(\frac{\overline{y}_k}{\widehat{g}_{kk}}\right)^{\alpha}\left\{\theta\sum_{\ell<k}\widehat{\pi}_\ell\left(\overline{y}_k-\overline{y}_\ell\right)+(1-\theta)\sum_{\ell>k}\widehat{\pi}_\ell\left(\overline{y}_\ell-\overline{y}_k\right)\right\}
\end{aligned}
$$

since $\widehat{G}=\widehat{g}/2\overline{y}$ and $\widehat{G}_k=\widehat{g}_{kk}/2\overline{y}_k$ for each k. In order to derive the asymptotic distribution of $\widehat{\mathcal{P}}_n(\alpha,\theta)$, we introduce the following U-statistics for $k,\ell=1,2,\cdots,K$:

$$
\begin{aligned}
U_{0,k} &= n^{-1}\sum_{i=1}^{n}\mathbb{I}\left\{y_i\in A_k\right\}, \\
U_{1,k} &= n^{-1}\sum_{i=1}^{n}y_i\mathbb{I}\left\{y_i\in A_k\right\}, \\
U_{2,k\ell} &= n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|y_i-y_j\right|\mathbb{I}\left\{y_i\in A_k\right\}\mathbb{I}\left\{y_j\in A_\ell\right\},
\end{aligned}
$$

that are consistent estimators of $\upsilon_{0,k}$, $\upsilon_{1,k}$ and $\upsilon_{2,k\ell}$, respectively, where

$$
\begin{aligned}
\upsilon_{0,k} &= \int_{A_k}dF(y), \\
\upsilon_{1,k} &= \int_{A_k}y\,dF(y), \\
\upsilon_{2,k\ell} &= \int_{A_k}\int_{A_\ell}\left|y-x\right|dF(x)\,dF(y).
\end{aligned}
$$

Using these U-statistics, since $\widehat{\pi}_k=U_{0,k}$, $\overline{y}_k=U_{1,k}/U_{0,k}$, $\widehat{g}_{k\ell}=U_{2,k\ell}/U_{0,k}U_{0,\ell}$ for all $k,\ell=1,2,\cdots,K$, we can rewrite $\widehat{\mathcal{P}}_n(\alpha,\theta)$ as

$$
\begin{aligned}
\widehat{\mathcal{P}}_n(\alpha,\theta) &= \frac{\left(\sum_{k=1}^{K}\sum_{\ell=1}^{K}U_{2,k\ell}\right)^{\alpha}}{\left(\sum_{k=1}^{K}U_{1,k}\right)^{1+\alpha}}\sum_{k=1}^{K}\frac{1}{U_{2,kk}^{\alpha}}\left\{\theta\sum_{\ell<k}\left(U_{0,k}^{2\alpha}U_{0,\ell}U_{1,k}^{1+\alpha}-U_{0,k}^{1+2\alpha}U_{1,k}^{\alpha}U_{1,\ell}\right)\right. \\
&\quad\left.+(1-\theta)\sum_{\ell>k}\left(U_{0,k}^{1+2\alpha}U_{1,k}^{\alpha}U_{1,\ell}-U_{0,k}^{2\alpha}U_{0,\ell}U_{1,k}^{1+\alpha}\right)\right\}
\end{aligned}
$$

that is consistent for (provided $\upsilon_{2,kk} > 0$ for all k)

$$\mathcal{P}(\alpha,\theta) = \frac{\left(\sum_{k=1}^{K}\sum_{\ell=1}^{K}\upsilon_{2,k\ell}\right)^{\alpha}}{\left(\sum_{k=1}^{K}\upsilon_{1,k}\right)^{1+\alpha}}\sum_{k=1}^{K}\frac{1}{\upsilon_{2,kk}^{\alpha}}\left\{\theta\sum_{\ell<k}\left(\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{1+\alpha}-\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\upsilon_{1,\ell}\right)\right.$$
$$\left.+(1-\theta)\sum_{\ell>k}\left(\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\upsilon_{1,\ell}-\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{1+\alpha}\right)\right\}$$

as $n \to \infty$ from the Slutsky's theorem.

To derive asymptotic distribution of $\widehat{\mathcal{P}}_n(\alpha,\theta)$, we first need to obtain joint asymptotic distribution of the vector of U-statistics of $U_{0,k}$, $U_{1,k}$ and $U_{2,k\ell}$ for all $k,\ell = 1,2,\cdots,K$. For the representation purposes, however, it is suffice to obtain joint distribution of the 9×1 vector of U-statistics $\mathbf{U}^* \equiv (U_{0,k}, U_{0,\ell}, U_{1,k}, U_{1,\ell}, U_{2,kk}, U_{2,\ell\ell}, U_{2,k\ell}, U_{2,km}, U_{2,hb})'$, where $k=\ell=m=h=b$. The following lemma summarizes asymptotic distribution of $\mathbf{U}^*$ from Theorem 7.1 of Hoeffding (1948). We let $\boldsymbol{\upsilon}^* \equiv (\upsilon_{0,k}, \upsilon_{0,\ell}, \upsilon_{1,k}, \upsilon_{1,\ell}, \upsilon_{2,kk}, \upsilon_{2,\ell\ell}, \upsilon_{2,k\ell}, \upsilon_{2,km}, \upsilon_{2,hb})'$.

Lemma 1 *Let $\{y_i\}_{i=1}^n$ be i.i.d. with continuous distribution $F(y)$ and finite variance. If $\epsilon < F(a_k) < 1-\epsilon$ for all $k = 1,2,\cdots,K-1$ and for some $\epsilon \in (0,1)$, then the joint distribution of $\sqrt{n}(\mathbf{U}^* - \boldsymbol{\upsilon}^*)$ tends to the 9-variate normal distribution as $n \to \infty$ with zero mean and covariance matrix Σ^*, which is given by (A.1) in the Appendix.*

Note that Bishop, Formby and Zheng (1997) consider the joint distribution of $(U_{0,k}, U_{1,k}, U_{2,kk})$ for the particular case of $A_k = (0,z]$ with some $z > 0$; Lemma 1 extends their result. Using Lemma 1 and Theorem 7.5 of Hoeffding (1948), we obtain asymptotic distribution of $\widehat{\mathcal{P}}_n(\alpha,\theta)$ as follows. We let $\mathbf{U} \equiv (U_{0,1},\cdots,U_{0,K},U_{1,1},\cdots,U_{1,K},U_{2,11},\cdots,U_{2,KK})'$. and $\boldsymbol{\upsilon} \equiv (\upsilon_{0,1},\cdots,\upsilon_{0,K},\upsilon_{1,1},\cdots,\upsilon_{1,K},\upsilon_{2,11},\cdots,\upsilon_{2,KK})'$.

Theorem 2 *Under the same conditions as Lemma 1, we have $\sqrt{n}(\widehat{\mathcal{P}}_n(\alpha,\theta) - \mathcal{P}(\alpha,\theta)) \to_d \mathcal{N}(0, V_{\mathcal{P}}(\alpha,\theta))$ as $n\to\infty$, where $V_{\mathcal{P}}(\alpha,\theta) = [\nabla\mathcal{P}(\alpha,\theta)]'\Sigma[\nabla\mathcal{P}(\alpha,\theta)]$, $\nabla\mathcal{P}(\alpha,\theta)$ is the $(2K+K(K+1)/2)\times 1$ vector of partial derivatives of $\mathcal{P}(\alpha,\theta)$ with respect to $\boldsymbol{\upsilon}$, and Σ is the $(2K+K(K+1)/2)\times(2K+K(K+1)/2)$ matrix of asymptotic variance matrix of $\mathbf{U}$ whose elements can be obtained from Σ^* in Lemma 1. The specific form of $\nabla\mathcal{P}(\alpha,\theta)$ is given in the Appendix.*

Distributions of various inequality measures and polarization measure can be obtained from Theorem 2. For example, for general number of K groups, $\mathcal{P}(\alpha, 1/2)$ with $\phi_k(\alpha) = \pi_k^\alpha$ becomes the polarization index $ER(\alpha)$ developed by Esteban and Ray (1994);[7] $\mathcal{P}(0, 1/2)$ becomes the Gini index for grouped data. Furthermore, when $K = 2$ (and thus two income groups are given as $[0, z]$ and (z, ∞) for some constant $z > 0$), if we let $U_{0,1} = n^{-1} \sum_{i=1}^n \mathbb{I}\{y_i \leq z\}$, $U_{1,1} = n^{-1} \sum_{i=1}^n y_i \mathbb{I}\{y_i \leq z\}$ and $U_{2,1} = n^{-2} \sum_{i=1}^n \sum_{j=1}^n |y_i - y_j| \mathbb{I}\{y_i \leq z\} \mathbb{I}\{y_j \leq z\}$, we can also obtain asymptotic distributions of several poverty indices similarly as Bishop, Formby and Zheng (1997), Xu (2007), and Barrett and Donald (2009).

4 The Case of Bipolarization

In many studies, polarization into two groups (i.e., bipolarization) is considered important because of the following reasons. First, the income polarization literature starts with the historical event of the disappearing middle class resulting in formation of two income groups, which has been observed in many developed countries such as the United States and the United Kingdom. In fact, another commonly used polarization index by Wolfson (1994) deals only with the bipolarization case. Second, conceptually speaking, the issue of polarization becomes less important as the number of groups increases. In extreme, when all individuals in the population form their own groups, neither group size nor within-group homogeneity matters in the concept and the measurement of polarization, and only inequality concept remains. As a special example of the general polarization index, this section focuses on bipolarization (i.e., $K = 2$), for which the closed form expressions are also obtained of the main results in the previous section.

First note that, when $K = 2$, $\mathcal{P}(\alpha, \theta)$ in (1) can be rewritten as

$$\mathcal{B}(\alpha, \theta) = \frac{\mu_2 - \mu_1}{\mu} \pi_1 \pi_2 \left[(1 - \theta) \left(\frac{\pi_1}{G_1/G} \right)^\alpha + \theta \left(\frac{\pi_2}{G_2/G} \right)^\alpha \right]$$

[7] While we consider the case with fixed K but large n asymptotics above, Duclos, Esteban and Ray (2004) derive the asymptotic properties of $ER(\alpha)$ under the large K asymptotics.

for $\mu_1 < \mu_2$ by construction, whose consistent estimator can be obtained as

$$\widehat{\mathcal{B}}_n(\alpha,\theta) = \left(1 - \frac{\overline{y}_1}{\overline{y}}\right)\widehat{\pi}_1\widehat{G}^{\alpha}\left[(1-\theta)\left(\frac{\widehat{\pi}_1}{\widehat{G}_1}\right)^{\alpha} + \theta\left(\frac{1-\widehat{\pi}_1}{\widehat{G}_2}\right)^{\alpha}\right]. \tag{6}$$

Here, we use a new notation $\mathcal{B}(\alpha,\theta)$ to highlight that we consider $K=2$ case. Similarly as the previous section, we can rewrite $\widehat{\mathcal{B}}_n(\alpha,\theta)$ as

$$\widehat{\mathcal{B}}_n(\alpha,\theta) = \frac{U_5U_1 - U_2U_4}{(U_2+U_5)^{1+\alpha}}(U_3 + 2U_7 + U_6)^{\alpha}\left\{(1-\theta)\left(\frac{U_2U_1^2}{U_3}\right)^{\alpha} + \theta\left(\frac{U_5U_4^2}{U_6}\right)^{\alpha}\right\}$$

using the U-statistics

$$\begin{aligned}
U_1 &= n^{-1}\textstyle\sum_{i=1}^n \mathbb{I}\{y_i \le y^*\} \\
U_2 &= n^{-1}\textstyle\sum_{i=1}^n y_i\mathbb{I}\{y_i \le y^*\} \\
U_3 &= n^{-2}\textstyle\sum_{i=1}^n\sum_{j=1}^n |y_i - y_j|\,\mathbb{I}\{y_i \le y^*\}\,\mathbb{I}\{y_j \le y^*\} \\
U_4 &= n^{-1}\textstyle\sum_{i=1}^n \mathbb{I}\{y_i > y^*\} \\
U_5 &= n^{-1}\textstyle\sum_{i=1}^n y_i\mathbb{I}\{y_i > y^*\} \\
U_6 &= n^{-2}\textstyle\sum_{i=1}^n\sum_{j=1}^n |y_i - y_j|\,\mathbb{I}\{y_i > y^*\}\,\mathbb{I}\{y_j > y^*\} \\
U_7 &= n^{-2}\textstyle\sum_{i=1}^n\sum_{j=1}^n |y_i - y_j|\,\mathbb{I}\{y_i > y^*\}\,\mathbb{I}\{y_j \le y^*\},
\end{aligned}$$

where we consider two income groups $[y_{\min}, y^*]$ and $(y^*, y_{\max}]$ for some cutoff point y^*. Denoting

$$\begin{aligned}
\upsilon_1 &= \textstyle\int_{-\infty}^{y^*} dF(y) = F(y^*) \\
\upsilon_2 &= \textstyle\int_{-\infty}^{y^*} y\,dF(y) \\
\upsilon_3 &= \textstyle\int_{-\infty}^{y^*}\int_{-\infty}^{y^*} |y-x|\,dF(y)\,dF(x) \\
\upsilon_4 &= \textstyle\int_{y^*}^{\infty} dF(y) = 1 - F(y^*) \\
\upsilon_5 &= \textstyle\int_{y^*}^{\infty} y\,dF(y) \\
\upsilon_6 &= \textstyle\int_{y^*}^{\infty}\int_{y^*}^{\infty} |y-x|\,dF(y)\,dF(x) \\
\upsilon_7 &= \textstyle\int_{-\infty}^{y^*}\int_{y^*}^{\infty} |y-x|\,dF(y)\,dF(x),
\end{aligned}$$

$U_1, U_2, \cdots, U_7$ are consistent estimators of $\upsilon_1, \upsilon_2, \cdots, \upsilon_7$, respectively. Theorem 7.1 of Hoeffding (1948) gives that the joint distribution of $\sqrt{n}(U_m - \upsilon_m)$ for $m = 1, 2, \cdots, 7$ tends to the

7-variate normal distribution as $n \to \infty$ with zero mean and covariance matrix $\Sigma_{\mathcal{B}}$ given by

$$\begin{pmatrix} \upsilon_1(1-\upsilon_1) & \upsilon_2(1-\upsilon_1) & 2\upsilon_3(1-\upsilon_1) & -\upsilon_1\upsilon_4 & -\upsilon_1\upsilon_5 & -\upsilon_1\upsilon_6 & -\upsilon_1\upsilon_7 \\ & \xi_2 & 2\xi_{2,3} & -\upsilon_2\upsilon_4 & -\upsilon_2\upsilon_5 & -\upsilon_2\upsilon_6 & -\upsilon_2\upsilon_7 \\ & & 4\xi_3 & -\upsilon_3\upsilon_4 & -\upsilon_3\upsilon_5 & -\upsilon_3\upsilon_6 & -\upsilon_3\upsilon_7 \\ & & & \upsilon_4(1-\upsilon_4) & \upsilon_5(1-\upsilon_4) & 2\upsilon_6(1-\upsilon_4) & \upsilon_7(1-\upsilon_4) \\ & & & & \xi_5 & 2\xi_{5,6} & 2\upsilon_7(1-\upsilon_5) \\ & & & & & 4\xi_6 & 4\xi_{6,7} \\ & & & & & & 4\xi_7 \end{pmatrix},$$

where

$$\begin{aligned}
\xi_2 &= \textstyle\int_{-\infty}^{y^*} y^2 dF(y) - \upsilon_2^2 \\
\xi_3 &= \textstyle\int_{-\infty}^{y^*} \left\{\int_{-\infty}^{y^*} |y-x|\, dF(x)\right\}^2 dF(y) - \upsilon_3^2 \\
\xi_{2,3} &= \textstyle\int_{-\infty}^{y^*}\int_{-\infty}^{y^*} y\,|y-x|\, dF(y)\, dF(x) - \upsilon_2\upsilon_3 \\
\xi_5 &= \textstyle\int_{y^*}^{\infty} y^2 dF(y) - \upsilon_5^2 \\
\xi_6 &= \textstyle\int_{y^*}^{\infty} \left\{\int_{y^*}^{\infty} |y-x|\, dF(x)\right\}^2 dF(y) - \upsilon_6^2 \\
\xi_7 &= \textstyle\int_{y^*}^{\infty} \left\{\int_{-\infty}^{y^*} (y-x) dF(x)\right\}^2 dF(y) - \upsilon_7^2 \\
\xi_{5,6} &= \textstyle\int_{y^*}^{\infty}\int_{y^*}^{\infty} y\,|y-x|\, dF(y)\, dF(x) - \upsilon_5\upsilon_6 \\
\xi_{6,7} &= \textstyle\int_{y^*}^{\infty} \left\{\int_{y^*}^{\infty} |y-x|\, dF(y)\right\}\left\{\int_{-\infty}^{y^*} (y-x) dF(y)\right\} dF(x) - \upsilon_6\upsilon_7.
\end{aligned}$$

Using this result, Theorem 7.5 of Hoeffding (1948) gives asymptotic distribution of $\widehat{\mathcal{B}}_n(\alpha,\theta)$ as follows.

Corollary 3 *Let $\{y_i\}_{i=1}^n$ be i.i.d. with continuous distribution $F(y)$ and finite variance. If $\epsilon < F(y^*) < 1-\epsilon$ for some $\epsilon \in (0,1)$, we have $\sqrt{n}(\widehat{\mathcal{B}}_n(\alpha,\theta) - \mathcal{B}(\alpha,\theta)) \to_d \mathcal{N}(0, V_{\mathcal{B}}(\alpha,\theta))$ as*

$n \to \infty$, where $V_{\mathcal{B}}(\alpha,\theta) = [\nabla\mathcal{B}(\alpha,\theta)]'\Sigma_{\mathcal{B}}[\nabla\mathcal{B}(\alpha,\theta)]$ and the 7×1 vector $\nabla\mathcal{B}(\alpha,\theta)$ is given by

$$\frac{\zeta g^{\alpha}}{\mu^{1+\alpha}} \begin{pmatrix} \upsilon_5/\zeta + 2\alpha/\upsilon_1 & \upsilon_5/\zeta \\ -\upsilon_4/\zeta - (1+\alpha)/\mu + \alpha/\upsilon_2 & -\upsilon_4/\zeta - (1+\alpha)/\mu \\ \alpha/\zeta - \alpha/\upsilon_3 & \alpha/\zeta \\ -\upsilon_2/\zeta & -\upsilon_2/\zeta + 2\alpha/\upsilon_4 \\ \upsilon_1/\zeta - (1+\alpha)/\mu & \upsilon_1/\zeta - (1+\alpha)/\mu + \alpha/\upsilon_5 \\ \alpha/\zeta & \alpha/\zeta - \alpha/\upsilon_6 \\ 2\alpha/\zeta & 2\alpha/\zeta \end{pmatrix} \begin{pmatrix} (1-\theta)\left(\upsilon_2\upsilon_1^2/\upsilon_3\right)^{\alpha} \\ \theta\left(\upsilon_5\upsilon_4^2/\upsilon_6\right)^{\alpha} \end{pmatrix}$$

with $\zeta = (\mu_2 - \mu_1)\pi_1\pi_2$.

In practice, we mostly need to obtain the standard error of $\widehat{\mathcal{B}}_n(\alpha,\theta)$ for further statistical inference. For example, when we want to compare the levels of social unrest between two different groups or to test for changes in the level of social unrest over time, the standard error is a key ingredient for constructing any test statistics. The asymptotic variance $V_{\mathcal{B}}(\alpha,\theta)$ can be consistently estimated using the sample counterparts of υ_m's for $m = 1, 2, \cdots, 7$ (i.e., their U-statistics, $U_1, U_2, \cdots, U_7$), but the calculation is quite complicated even for the bipolarization case as appears in Corollary 3. To facilitate the variance estimation of $\widehat{\mathcal{B}}_n(\alpha,\theta)$, we propose a subsampling method, specifically the jackknife variance estimation (e.g., Yitzhaki, 1991; Karagiannis and Kovacevic, 2000). The procedure is summarized in detail as follows.[8]

1. We sort the original income data in *ascending* order and denote them as $\{y_i\}_{i=1}^n$; therefore, the index of y_i also represents its rank r_i.

   (a) We calculate the sample mean $\overline{y} = (1/n)\sum_{i=1}^n y_i$.

   (b) We define $L = \sum_{i=1}^n r_i y_i$ and $H_i = \sum_{j=i+1}^n y_j$ for $i = 1, 2, \cdots, n$ with $H_n = 0$.

[8] Note that bootstrap variance estimation of the Gini coefficient is still computationally demanding especially when n is large like the conventional income data. This is still the case for $\widehat{\mathcal{B}}_n(\alpha,\theta)$ since we need to calculate the Gini coefficients $\widehat{G}$ and $\widehat{G}_k$ in each iteration step. On the other hand, the jackknife variance estimator can be obtained much faster than the bootstrap variance estimator. In this paper, we use the algorithm suggested by Karagiannis and Kovacevic (2000). In comparison, Xu (2000) proposes the iterated-bootstrap method for inference for (generalized) Gini indices.

(c) Then the Gini coefficient can be obtained as $\widehat{G} = (2L) / (\overline{y}n^2) - (n+1)/n$.

2. We group the data in two using a given cutoff point y^* (e.g., the sample mean $\overline{y}$), and let $A_1 = \{y_i | y_i < y^*\}$ and $A_2 = \{y_i | y_i \geq y^*\}$.

   (a) Since the original data is already sorted in step 1, the data in each group is also properly ordered. For each group $k = 1, 2$, we let n_k be the number of observations in group k and $\{y_{k,i}\}_{i=1}^{n_k}$ be the sorted income data in group k. We also denote $r_{k,i}$ as the rank of $y_{k,i}$'s in group k.

   (b) We calculate the group sample proportion $\widehat{\pi}_k = n_k/n$ and the group sample mean $\overline{y}_k = (1/n_k) \sum_{i=1}^{n_k} y_{k,i}$. We also define $L_k = \sum_{i=1}^{n_k} r_{k,i} y_{k,i}$ and $H_{k,i} = \sum_{j=i+1}^{n_k} y_{k,j}$ for $i = 1, 2, \cdots, n_k$ with $H_{k,n_k} = 0$.

   (c) Then the Gini coefficient of group k can be obtained as $\widehat{G}_k = (2L_k) / (\overline{y}_k n_k^2) - (n_k + 1)/n_k$.

   (d) Using values obtained in steps 1 and 2, we calculate $\widehat{\mathcal{B}}_n(\alpha, \theta)$ as in (6) for given α and θ.

3. From the entire sample, we omit the i-th observation y_i. (We do not change the groups A_1 and A_2 in Step 2 even after omitting one observation.)

   (a) Using $(n-1)$-number of observations, we obtain the new sample mean and the Gini coefficient as

$$\overline{y}_{(-i)} = \frac{1}{n-1}(n\overline{y} - y_i) \quad \text{and} \quad \widehat{G}_{(-i)} = \frac{2}{\overline{y}_{(-i)}(n-1)^2}(L - r_i y_i - H_i) - \frac{n}{n-1}.$$

(b) We let

$$\widehat{\pi}_{1,(-i)} = \begin{cases} (n_1 - 1) / (n - 1) & \text{if } y_i \in A_1 \\ n_1 / (n - 1) & \text{if } y_i \in A_2 \end{cases}$$

$$\overline{y}_{1,(-i)} = \begin{cases} (n_1 \overline{y}_1 - y_i) / (n_1 - 1) & \text{if } y_i \in A_1 \\ \overline{y}_1 & \text{if } y_i \in A_2 \end{cases}$$

$$\overline{y}_{2,(-i)} = \begin{cases} \overline{y}_2 & \text{if } y_i \in A_1 \\ (n_2 \overline{y}_2 - y_i) / (n_2 - 1) & \text{if } y_i \in A_2 \end{cases}.$$

Then the Gini coefficients of group 1 and 2 can be obtained as

$$\widehat{G}_{1,(-i)} = \begin{cases} \frac{2}{\overline{y}_{1,(-i)}(n_1-1)^2} (L_1 - r_{1,i} y_i - H_{1,i}) - \frac{n_1}{n_1 - 1} & \text{if } y_i \in A_1 \\ \widehat{G}_1 & \text{if } y_i \in A_2 \end{cases},$$

$$\widehat{G}_{2,(-i)} = \begin{cases} \widehat{G}_2 & \text{if } y_i \in A_1 \\ \frac{2}{\overline{y}_{2,(-i)}(n_2-1)^2} (L_2 - r_{2,i} y_i - H_{2,i}) - \frac{n_2}{n_2 - 1} & \text{if } y_i \in A_2 \end{cases}.$$

(c) Using values obtained in step 3 above, we get $\widehat{\mathcal{B}}_{n,(-i)} (\alpha, \theta)$ as

$$\widehat{\mathcal{B}}_{n,(-i)} (\alpha, \theta) = \left(1 - \frac{\overline{y}_{1,(-i)}}{\overline{y}_{(-i)}}\right) \widehat{\pi}_{1,(-i)} \left[(1 - \theta) \left(\frac{\widehat{\pi}_{1,(-i)}}{\widehat{G}_{1,(-i)} / \widehat{G}_{(-i)}}\right)^{\alpha} + \theta \left(\frac{1 - \widehat{\pi}_{1,(-i)}}{\widehat{G}_{2,(-i)} / \widehat{G}_{(-i)}}\right)^{\alpha}\right].$$

4. We iterate step 3 from $i = 1$ to n and recursively calculate

$$V_i = V_{i-1} + \frac{n-1}{n} \left(\widehat{\mathcal{B}}_{n,(-i)} (\alpha, \theta) - \widehat{\mathcal{B}}_n (\alpha, \theta)\right)^2$$

with $V_0 = 0$. Then, V_n is the jackknife variance estimate of $\widehat{\mathcal{B}}_n (\alpha, \theta)$.

5 Empirical Illustration

Using the Panel Study of Income Dynamics (PSID) data, this section illustrates how the level of social unrest, as measured by the new index for the case of two income groups, has evolved over the survey period from 1981 to 2005. The PSID is a longitudinal survey administered by the Survey Research Center (SRC) of the University of Michigan every year from 1968 through 1997 and every other year afterwards. We exploit the SRC's random sample, excluding the so-called 'poverty' sample. Due to unavailability of information on government taxes/transfers, we exclude the pre-1981 period from the sample. For consistency of the survey frequency, we exploit surveys only for odd years. (Note that, however, since each year's survey contains the total family income for the previous year, our sample period in fact runs every other year from 1980 through 2004.) We analyze two income variables at the family level: the total family income and the total family income adjusted by government actions. The former is defined by the sum of family labor earnings, family asset income, family private transfers, and family private retirement income. The latter is defined by subtracting household taxes from the total family income and adding public transfer income and social security pensions.

Figures 2 through 4 display how the level of social unrest has evolved over the sample period for different values of θ. In each figure, the line connecting rectangular data points shows the level of social unrest represented by the total family income, and the line connecting circular data points displays the level of social unrest computed by the adjusted family income. Each series of the social unrest level is accompanied by a pair of dotted lines, which represent a (pointwise) confidence interval at the 95% level. For all figures, α is set to be 1.6.[9] A value of θ smaller than 0.5 implies that the poor feel more alienated against the rich than the rich do against the poor. The degree of this asymmetry gets stronger, as we move from Figures 2 to 4, this is, a greater weight is placed on the poor group.

Several conclusions emerge from comparison of these figures. First, in all cases and for both income variables, the level of social unrest has generally increased over the sample period, particular since the late 1990's. And the change is statistically significant. Second, for both income variables, as we change the value of θ from 0.5 to 0.0, so placing a greater weight on the poor group, there is a tendency of observing a larger increase in the measured level of social unrest. Third, for each value of θ, the level of social unrest measured by the adjusted family income is generally lower than that by the total family income, implying that the government

[9] The current findings are quite robust with respect to different values of α in a qualitative sense.

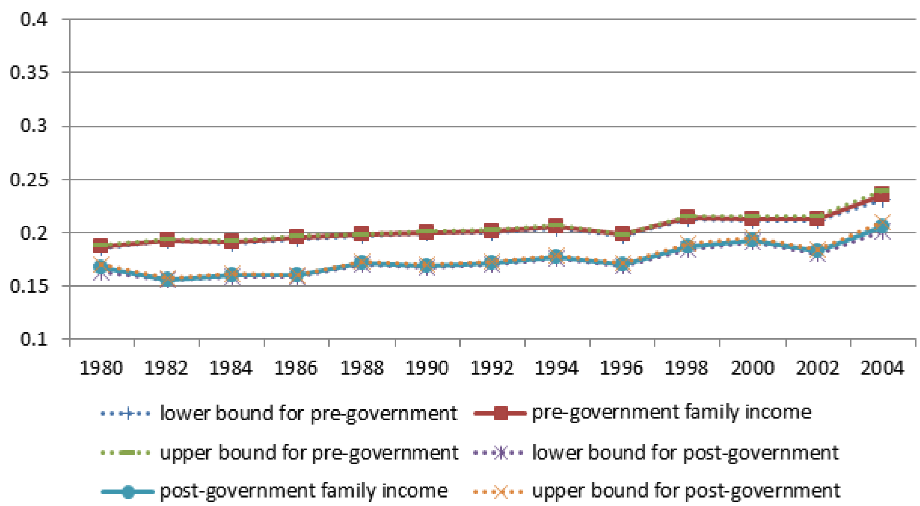

Figure 2: Evolution of the Level of Social Unrest: $(\theta, \alpha) = (0.5, 1.6)$. Data source: Panel Study of Income Dynamics. The horizontal axis represents survey years. The 95% confidence level is adopted in deriving (pointwise) confidence intervals.

taxes/transfers program has been generally effective in mitigating the level of social unrest. (cf. Bishop, Formby and Zheng, 1998) Fourth, as we place a greater weight to the poor group, the government actions become less effective in reducing the level of social unrest for the late 1980s to 2000 period, not for the other period.

6 Concluding Remarks

For more effective representation of the level of social unrest, existing Esteban-Ray type indices are revised in a way that the new index allows for asymmetric feeling of alienation between different income groups and includes a more sensible group identification function relative to existing ones. Furthermore, to facilitate statistical inferences, asymptotic distribution of the new index is derived using results from U-statistics, and an easy-to-implement jackknife-based variance estimation algorithm is obtained. The new index is general enough to include the index by Esteban and Ray (1994) and the Gini index for group data as special cases.

One fundamental feature underlying all existing polarization indices including the new one developed in this paper is that even the richer groups contribute to the index. In case of

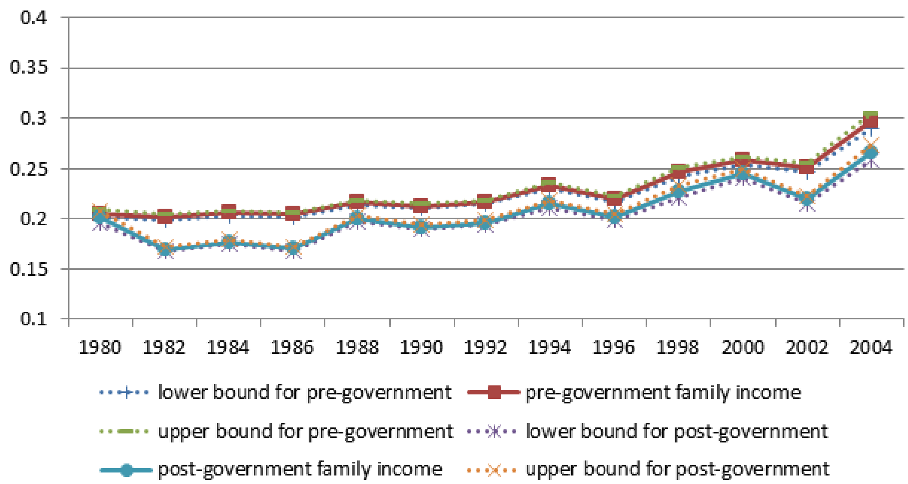

Figure 3: Evolution of the Level of Social Unrest: $(\theta, \alpha) = (0.25, 1.6)$.

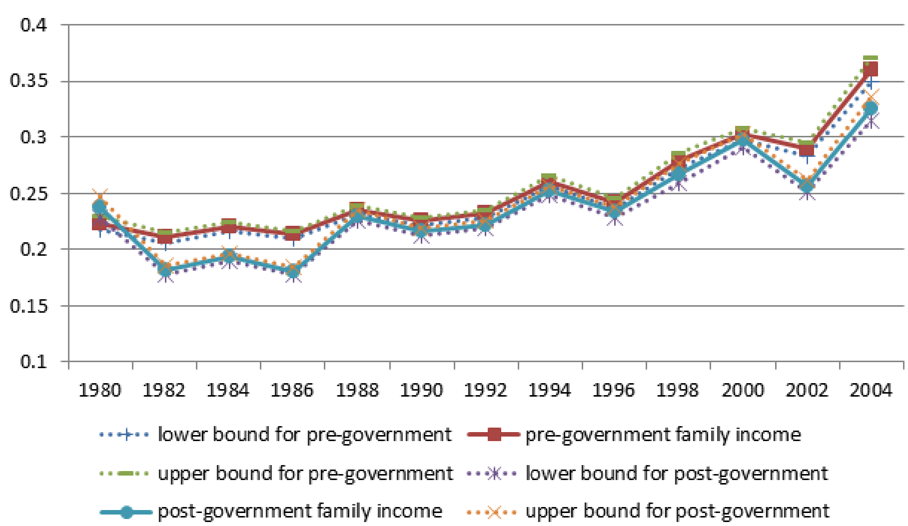

Figure 4: Evolution of the Level of Social Unrest: $(\theta, \alpha) = (0, 1.6)$.

bipolarization, more specifically, either when the between-group income distance gets longer or when the within-group income distribution becomes less dispersed among the rich, the level of contribution of the rich toward the index value increases. This feature is based on the presumption that the rich and the poor are antagonistic against each other. While such assumption is essential for the purpose of explaining between-group conflicts or 'collective crimes' from the view point of an individual's economic crime incentive, the rich would feel less crime incentives as they expect that their life-time income will be more secured following bipolarization of the current income distribution (e.g., Lee and Shin, 2012; Lee, Shin and Shin, 2013). For a better representation of the level of social unrest, future research could be directed to further generalize the current index by considering both the collective and the individual crime incentives simultaneously.

Appendix: Mathematical Proofs

Proof of Lemma 1 From Theorem 7.1 of Hoeffding (1948), the asymptotic variance Σ^* can be obtained as

$$\Sigma^* = \begin{pmatrix} \Sigma_{11}^* & \Sigma_{12}^* \\ \Sigma_{12}^{*\prime} & \Sigma_{22}^* \end{pmatrix}, \tag{A.1}$$

where $\underset{(4\times 4)}{\Sigma_{11}^*}$, $\underset{(4\times 5)}{\Sigma_{12}^*}$ and $\underset{(5\times 5)}{\Sigma_{22}^*}$ are given as

$$\begin{pmatrix} \upsilon_{0,k}(1-\upsilon_{0,k}) & -\upsilon_{0,k}\upsilon_{0,\ell} & \upsilon_{1,k}(1-\upsilon_{0,k}) & -\upsilon_{0,k}\upsilon_{1,\ell} \\ & \upsilon_{0,\ell}(1-\upsilon_{0,\ell}) & -\upsilon_{0,\ell}\upsilon_{1,k} & \upsilon_{1,\ell}(1-\upsilon_{0,\ell}) \\ & & \xi_{1,k}-\upsilon_{1,k}^2 & -\upsilon_{1,k}\upsilon_{1,\ell} \\ & & & \xi_{1,\ell}-\upsilon_{1,\ell}^2 \end{pmatrix},$$

$$\begin{pmatrix} 2(1-\upsilon_{0,k})\upsilon_{2,kk} & -2\upsilon_{0,k}\upsilon_{2,\ell\ell} & (1-2\upsilon_{0,k})\,\upsilon_{2,k\ell} & (1-2\upsilon_{0,k})\,\upsilon_{2,km} & -2\upsilon_{0,k}\upsilon_{2,hb} \\ -2\upsilon_{0,\ell}\upsilon_{2,kk} & 2\upsilon_{2,\ell\ell}(1-\upsilon_{0,\ell}) & (1-2\upsilon_{0,\ell})\,\upsilon_{2,k\ell} & -2\upsilon_{0,\ell}\upsilon_{2,km} & -2\upsilon_{0,\ell}\upsilon_{2,hb} \\ 2\left(\xi_{2,kk}-\upsilon_{1,k}\upsilon_{2,kk}\right) & -2\upsilon_{1,k}\upsilon_{2,\ell\ell} & \xi_{2,k\ell}-2\upsilon_{1,k}\upsilon_{2,k\ell} & \xi_{2,km}-2\upsilon_{1,k}\upsilon_{2,km} & -2\upsilon_{1,k}\upsilon_{2,hb} \\ -2\upsilon_{1,\ell}\upsilon_{2,kk} & 2\left(\xi_{2,\ell\ell}-\upsilon_{1,\ell}\upsilon_{2,\ell\ell}\right) & \xi_{2,\ell k}-2\upsilon_{1,\ell}\upsilon_{2,k\ell} & -2\upsilon_{1,\ell}\upsilon_{2,km} & -2\upsilon_{1,\ell}\upsilon_{2,hb} \end{pmatrix},$$

$$\begin{pmatrix} 4\left(\xi_{3,kkk}-\upsilon_{2,kk}^2\right) & -4\upsilon_{2,\ell\ell}\upsilon_{2,kk} & 2\left(\xi_{3,kk\ell}-2\upsilon_{2,kk}\upsilon_{2,k\ell}\right) & 2\left(\xi_{3,kkm}-2\upsilon_{2,kk}\upsilon_{2,km}\right) & -4\upsilon_{2,kk}\upsilon_{2,hb} \\ & 4\left(\xi_{3,\ell\ell\ell}-\upsilon_{2,\ell\ell}^2\right) & 2\left(\xi_{3,\ell\ell k}-2\upsilon_{2,\ell\ell}\upsilon_{2,k\ell}\right) & -4\upsilon_{2,\ell\ell}\upsilon_{2,km} & -4\upsilon_{2,\ell\ell}\upsilon_{2,hb} \\ & & \xi_{3,\ell kk}+\xi_{3,k\ell\ell}-4\upsilon_{2,k\ell}^2 & \xi_{3,k\ell m}-4\upsilon_{2,k\ell}\upsilon_{2,km} & -4\upsilon_{2,k\ell}\upsilon_{2,hb} \\ & & & \xi_{3,mkk}+\xi_{3,kmm}-4\upsilon_{2,km}^2 & -4\upsilon_{2,km}\upsilon_{2,hb} \\ & & & & \xi_{3,bhh}+\xi_{3,hbb}-4\upsilon_{2,hb}^2 \end{pmatrix},$$

respectively, with

$$\begin{aligned} \xi_{1,k} &= \textstyle\int_{A_k} y^2 dF(y) \\ \xi_{2,k\ell} &= \textstyle\int_{A_k}\int_{A_\ell} y\,|y-x|\,dF(x)\,dF(y) \\ \xi_{3,k\ell m} &= \textstyle\int_{A_k}\left\{\int_{A_\ell}|y-x|\,dF(x)\right\}\left\{\int_{A_m}|y-x|\,dF(x)\right\}dF(y)\,. \end{aligned}$$

Though most of the terms are standard, deriving covariance terms involving $U_{2,k\ell}$ needs some care. For example, the leading term of the asymptotic variance of $U_{2,k\ell}$ can be obtained from

$$
\begin{aligned}
& \int \left(\int_{A_k} |x-y| \, \mathbb{I}\{y \in A_\ell\} \, dF(x) - \upsilon_{2,k\ell} \right)^2 dF(y) \\
& + 2 \int \left(\int_{A_k} |x-y| \, \mathbb{I}\{y \in A_\ell\} \, dF(x) - \upsilon_{2,k\ell} \right) \left(\int_{A_\ell} |x-y| \, \mathbb{I}\{y \in A_k\} \, dF(x) - \upsilon_{2,\ell k} \right) dF(y) \\
& + \int \left(\int_{A_\ell} |x-y| \, \mathbb{I}\{y \in A_k\} \, dF(x) - \upsilon_{2,\ell k} \right)^2 dF(y) \\
= \; & \int_{A_\ell} \left(\int_{A_k} |x-y| \, dF(x) \right)^2 dF(y) + \int_{A_k} \left(\int_{A_\ell} |x-y| \, dF(x) \right)^2 dF(y) - 4\upsilon_{2,\ell k}^2
\end{aligned}
$$

since $\upsilon_{2,k\ell} = \upsilon_{2,\ell k}$ and $\int \left(\int_{A_k} |x-y| \, \mathbb{I}\{y \in A_\ell\} \, dF(x) \right) \left(\int_{A_\ell} |x-y| \, \mathbb{I}\{y \in A_k\} \, dF(x) \right) dF(y) = 0$ for $\mathbb{I}\{y \in A_\ell\} \mathbb{I}\{y \in A_k\} = 0$ with $k = \ell$. The other terms can be obtained similarly. $\square$

Proof of Theorem 2 The result follows directly using Lemma 1 and Theorem 7.5 of Hoeffding (1948). In this proof, we summarize the elements of $\nabla \mathcal{P}(\alpha, \theta)$. We first note that for

$$
Q(x_1, \cdots, x_K) = \sum_{k=1}^{K} \left\{ c_1 \sum_{\ell<k} f_1(x_1, x_2) + c_2 \sum_{\ell>k} f_2(x_1, x_2) \right\}
$$

with continuously differentiable bivariate functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$, we have

$$
\frac{\partial}{\partial x_k} Q(x_1, \cdots, x_K) = \sum_{\ell<k} \{c_1 f_{1,1}(x_1, x_2) + c_2 f_{2,2}(x_1, x_2)\} + \sum_{\ell>k} \{c_2 f_{2,1}(x_1, x_2) + c_1 f_{1,2}(x_1, x_2)\}
$$

where $f_{m,h}(x_1, x_2) = \partial f_m(x_1, x_2)/\partial x_h$ for $m, h = 1, 2$. Using this result, for each $k = 1, 2, \cdots, K$, we obtain that

$$
\begin{aligned}
& \frac{\partial \mathcal{P}(\alpha, \theta)}{\partial \upsilon_{0,k}} \\
= \; & \frac{C(K,\alpha)}{\upsilon_{2,kk}^{\alpha}} \sum_{\ell<k} \left\{ \theta \left[2\alpha \upsilon_{0,k}^{2\alpha-1} \upsilon_{0,\ell} \upsilon_{1,k}^{1+\alpha} - (1+2\alpha) \upsilon_{0,k}^{2\alpha} \upsilon_{1,k}^{\alpha} \upsilon_{1,\ell} \right] + (1-\theta) \left[\upsilon_{0,k}^{1+2\alpha} \upsilon_{1,k}^{\alpha} \upsilon_{1,\ell} - \upsilon_{0,k}^{2\alpha} \upsilon_{1,k}^{1+\alpha} \right] \right\} \\
& + \frac{C(K,\alpha)}{\upsilon_{2,kk}^{\alpha}} \sum_{\ell>k} \left\{ (1-\theta) \left[(1+2\alpha) \upsilon_{0,k}^{2\alpha} \upsilon_{1,k}^{\alpha} \upsilon_{1,\ell} - 2\alpha \upsilon_{0,k}^{2\alpha-1} \upsilon_{0,\ell} \upsilon_{1,k}^{1+\alpha} \right] + \theta \left[\upsilon_{0,k}^{2\alpha} \upsilon_{1,k}^{1+\alpha} - \upsilon_{0,k}^{1+2\alpha} \upsilon_{1,k}^{\alpha} \upsilon_{1,\ell} \right] \right\} \\
= \; & C(K,\alpha) \left(\frac{\upsilon_{0,k}^2 \upsilon_{1,k}}{\upsilon_{2,kk}} \right)^{\alpha} \left\{ \theta \left[\sum_{\ell<k} \gamma_{0,1}(k,\ell) - \sum_{\ell>k} \gamma_{0,2}(k,\ell) \right] + (1-\theta) \left[\sum_{\ell<k} \gamma_{0,2}(k,\ell) - \sum_{\ell>k} \gamma_{0,1}(k,\ell) \right] \right\}
\end{aligned}
$$

where $C(K,\alpha) = (\sum_{k=1}^{K}\sum_{\ell=1}^{K} \upsilon_{2,k\ell})^{\alpha}/(\sum_{k=1}^{K}\upsilon_{1,k})^{1+\alpha}$, $\gamma_{0,1}(k,\ell) = 2\alpha\upsilon_{0,k}^{-1}\upsilon_{0,\ell}\upsilon_{1,k} - (1+2\alpha)\upsilon_{1,\ell}$ and $\gamma_{0,2}(k,\ell) = \upsilon_{0,k}\upsilon_{1,\ell} - \upsilon_{1,k}$. Similarly, for each $k = 1, 2, \cdots, K$,

$$\begin{aligned}
&\frac{\partial \mathcal{P}(\alpha,\theta)}{\partial \upsilon_{1,k}} \\
= \; & -\frac{(1+\alpha)\mathcal{P}(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}} \\
&+\frac{C(K,\alpha)}{\upsilon_{2,kk}^{\alpha}}\sum_{\ell<k}\left\{\theta\left[(1+\alpha)\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{\alpha} - \alpha\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha-1}\upsilon_{1,\ell}\right] + (1-\theta)\left[\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\right]\right\} \\
&+\frac{C(K,\alpha)}{\upsilon_{2,kk}^{\alpha}}\sum_{\ell>k}\left\{(1-\theta)\left[\alpha\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha-1}\upsilon_{1,\ell} - (1+\alpha)\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{\alpha}\right] + \theta\left[-\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\right]\right\} \\
= \; & -\frac{(1+\alpha)\mathcal{P}(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}} \\
&+C(K,\alpha)\left(\frac{\upsilon_{0,k}^{2}\upsilon_{1,k}}{\upsilon_{2,kk}}\right)^{\alpha}\left\{\theta\left[\sum_{\ell<k}\gamma_{1,1}(k,\ell) - \sum_{\ell>k}\gamma_{1,2}(k,\ell)\right] + (1-\theta)\left[\sum_{\ell<k}\gamma_{1,2}(k,\ell) - \sum_{\ell>k}\gamma_{1,1}(k,\ell)\right]\right\},
\end{aligned}$$

where $\gamma_{1,1}(k,\ell) = (1+\alpha)\upsilon_{0,\ell} - \alpha\upsilon_{0,k}\upsilon_{1,k}^{-1}\upsilon_{1,\ell}$ and $\gamma_{1,2}(k,\ell) = \upsilon_{0,k}$. The derivatives with respect to $\upsilon_{2,k\ell}$ can be readily obtained as

$$\frac{\partial \mathcal{P}(\alpha,\theta)}{\partial \upsilon_{2,kk}} = \frac{(1+\alpha)\mathcal{P}(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}} - \frac{\alpha C(K,\alpha)D(k,\alpha,\theta)}{\upsilon_{2,kk}^{\alpha+1}} \quad \text{and} \quad \frac{\partial \mathcal{P}(\alpha,\theta)}{\partial \upsilon_{2,k\ell}} = \frac{\alpha\mathcal{P}(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}}$$

for each $k, \ell = 1, 2, \cdots, K$, where

$$D(k,\alpha,\theta) = \theta\sum\nolimits_{\ell<k}\left(\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{1+\alpha} - \upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\upsilon_{1,\ell}\right) + (1-\theta)\sum\nolimits_{\ell>k}\left(\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,k}^{\alpha}\upsilon_{1,\ell} - \upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}^{1+\alpha}\right).$$

□

Proof of Corollary 3 The partial derivative vector is obtained as $\nabla\mathcal{B}(\alpha,\theta)$, where $\upsilon_3 + 2\upsilon_7 + \upsilon_6 = g$ and $\upsilon_2 + \upsilon_5 = \mu$ by construction and by letting $\zeta \equiv \upsilon_5\upsilon_1 - \upsilon_2\upsilon_4 = (\mu_2 - \mu_1)\pi_1\pi_2$. Then the result follows immediately from the result above using Theorem 7.5 of Hoeffding (1948). Note that $\upsilon_1, \upsilon_4 > 0$ since it is assumed that $\epsilon < F(y^*) < 1 - \epsilon$ for some $\epsilon \in (0,1)$, and thus $\upsilon_m > 0$ for all $m = 1, 2, \cdots, 7$. □

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