


Spring 5-1-2010

Analyzing Fractals

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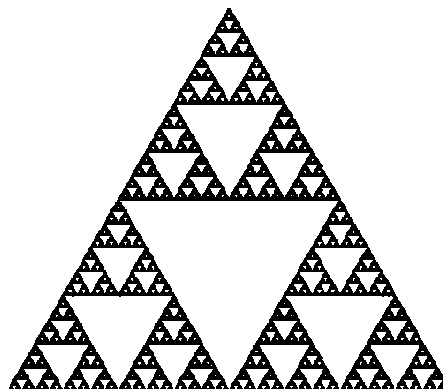
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Recommended Citation

Mesznik, Kara, "Analyzing Fractals" (2010). *Syracuse University Honors Program Capstone Projects*. 353.
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Analyzing Fractals



Fractals are pictures that are made up of smaller copies of the large picture. That is, the larger image is contracted into infinitely many smaller, self-similar images of the large picture and moved to different places using different types of transformations making up the large image. For example, the large triangle above is made up of many smaller images of itself. To understand how to produce fractals, one must first learn about spaces, sequences, and properties of functions in general. Then we apply those things to fractal geometry. We learn different theorems and how they apply to creating fractals, and finally we see an example of a classic fractal and make a fractal of our own.

I. Spaces

We begin with the concept of a *space*. A space is simply a collection of points. They can have infinitely many points, or elements, a finite number of elements, or no elements.

Definition I.1. A space X is a set. The points of the space are the elements of the set.

II. Metric Spaces

In this section we learn about *metric spaces* and certain properties they hold. When producing fractals, metric spaces are used. A *metric space* is a special type of space, in that, within the space, there is an equation called a *metric* to measure the distance between points. The *metric* is commonly notated as $d(x, y)$, which means the distance between point x and point y . This measurement equation obeys certain axioms or rules. This section covers common metrics as well as certain properties of functions. We also discuss different types of sequences.

When dealing with fractals, we are only concerned with spaces in \mathbf{R}^2 . To understand the space \mathbf{R}^2 , we need the concept of an ordered pair. Think of the normal x, y plane first introduced when learning how to graph equations in basic algebra; the horizontal line consists of x values, and the vertical line consists of y values. An ordered pair of numbers, for example $(3,4)$, is the point in \mathbf{R}^2 located three units to the right and 4 units up from the origin. The collection of all such ordered pairs is the space \mathbf{R}^2 .

Definition II.1 A metric space (X, d) is a space X together with a real-valued function $d: X \times X \rightarrow \mathbf{R}^2$, which measures the distance between pairs of point x and y in X such that d obeys certain axioms listed below.

(1) For all $x, y \in X$,

$$d(x, y) = d(y, x).$$

(2) For all $x, y \in X, x \neq y$,

$$0 < d(x, y) < \infty.$$

(3) For all $x \in X$

$$d(x, x) = 0.$$

(4) For all $x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(z, y).$$

Axiom 1 states that if the distance between two points, x and y , in the space is computed, the distance from x to y must equal the distance from y to x .

Axiom 2 states that the distance between any two points in the space must be greater than zero and less than infinity. Axiom 3 states that the distance between any point and itself is zero. Axiom 4 states when measuring the distance between two points, the distance between two points, x and z , is less than or equal to the distance from x to y plus the distance from y to z .

Two common metrics used in \mathbf{R}^2 are the *Euclidean metric* and the *Manhattan metric*.

The *Euclidean metric* measures the usual straight line distance between two points, utilizing the Pythagorean theorem ($a^2 + b^2 = c^2$) for $a = (x_1, y_1)$ and $b = (x_2, y_2)$,

$$d(a, b) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \text{ where } c = d(x, y), a = x_2 - x_1, b = y_2 - y_1.$$

Example: Let $a = (1, 3)$ and $b = (2, 0)$. The distance between these two points using the Euclidean metric is:

$$d(a, b) = \sqrt{(2-1)^2 + (0-3)^2} = \sqrt{10}.$$

The *Manhattan metric* is a different metric. This metric measures the distance only moving horizontally and vertically between the two points as if you are moving in city blocks without backtracking. That is if $a = (x_1, y_1)$ and $b = (x_2, y_2)$,

$$d(a, b) = |x_2 - x_1| + |y_2 - y_1|.$$

Example: Suppose we are in Manhattan on 14th Street and 3rd Avenue, which we will call point a , and we are trying to walk to 20th Street and 5th Avenue, which we will call point b . What is the distance from point x to point y ?

$$d(a, b) = |20 - 14| + |5 - 3| = |6| + |2| = 8.$$

This means the distance between 14th Street and 3rd Avenue and 20th Street and 5th Avenue is 8 blocks.

An important property of functions involved with fractals is *continuity*. The easiest way to understand the definition of a continuous function is by looking at the graph of a function from \mathbf{R}^1 to \mathbf{R}^1 . The graph of a *continuous function* is unbroken. This means that the graph can be drawn without ever having to lift the pencil off the paper.

Definition II.2 A function $f: X_1 \rightarrow X_2$ from a metric space (X_1, d_1) into a metric space (X_2, d_2) is continuous if, for each $\varepsilon > 0$ and $x, y \in X_1$, there is a $\delta > 0$ so that

$$d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$$

To understand this more easily, let's say we drew a circle around the original point and the point that the function is being graphed onto. The original point has a radius δ and the second point has a radius ε . In a continuous function, if you start with a point anywhere within the circle of radius δ , your point will be graphed into the circle of radius ε because the size of the circle of radius ε grows and shrinks based on the size of the circle of radius δ .

Fractals depend upon *sequences* in a metric space. A *sequence* is an ordered set of points. A sequence, for example, of the positive squared numbers is denoted $\{n^2\}_{n=1}^{\infty}$. This means that for each term $n=1, 2, 3, \dots$ and continuing until infinity, each term is squared:

$$1^2, 2^2, 3^2, \dots$$

$$\text{So } \{n^2\}_{n=1}^{\infty} = 1, 4, 9, 16, \dots$$

This sequence is known as an *infinite sequence* because it continues indefinitely as opposed to having a finite number of terms. Sequences can have sequences within them, which are known as *subsequences*. One subsequence of the previous example is the list of positive odd squares:

$$1, 9, 25, 49, \dots$$

When dealing with metric spaces, we are only interested in infinite sequences.

III. Cauchy Sequences, Limit Points, Closed Sets, and Complete Metric Spaces

In this section, we learn more about metric spaces, more specifically, the types of sequences in them, and certain properties about them. We learn about *Cauchy* sequences, which are very important to fractals. Also we go over *limits* and *convergence* and how these two concepts relate to Cauchy sequences. We also learn about what it means for a metric space to be *complete* and what it means for it to be *closed*.

Cauchy sequences are important in the analysis of fractals. In a *Cauchy* sequence, the elements become arbitrarily close together as the sequence progresses. This means that as the sequence progresses, the distance between any two elements becomes smaller.

Definition III.1 A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric space (X, d) is called a Cauchy sequence if, for any given number $\varepsilon > 0$, there is an integer $N > 0$ so that, for all $n, m > N$

$$d(x_n, x_m) < \varepsilon$$

This means that for any number greater than zero chosen, which we denote ε , you can find an integer N , also greater than zero such that the distance between any two elements further along in the sequence than N is less than ε .

Example Show $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is Cauchy.

In light of Definition III.1, given any $\varepsilon > 0$, we must find an integer $N > 0$ such that for all $n, m > N$, $d(x_n, x_m) < \varepsilon$. Let's take $N > \frac{2}{\varepsilon}$ so $m, n > \frac{2}{\varepsilon}$. Using axiom (4) from Definition II.1, for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$. So plugging it into our equation, $d(x_n, x_m) \leq d(x_n, 0) + d(0, x_m)$. So

$d(x_n, x_m) \leq |x_n - 0| + |x_m - 0|$. So $x_n + x_m$. Because $n, m > \frac{2}{\varepsilon}$, it is implied that

$\frac{1}{n}, \frac{1}{m} < \frac{\varepsilon}{2}$. So $\frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. Therefore, $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$ is Cauchy.

Convergence is another important term used in the analysis of sequences. A *convergent* sequence is one in which all of the terms are heading towards a certain point.

Definition III.2 A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric space (X, d) is said to converge to a point $x \in X$ if, for any given number $\varepsilon > 0$, there is an integer $N > 0$ such that for all $n > N$,

$$d(x_n, x) < \varepsilon$$

In this case the point $x \in X$, to which the sequence converges, is called the *limit* of the sequence, and we use the notation

$$x = \lim_{n \rightarrow \infty} x_n.$$

Suppose a sequence converges to a specific point x . Then for any positive number ε that you choose, no matter how small, you can find an integer N (probably large) such that the distance between any term n in the sequence (which is greater than the N chosen) and point x is less than ε . The point of convergence is also known as the *limit* of the sequence.

Example One convergent sequence is $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. This sequence converges to zero. As n increases towards infinity, the terms in the sequence continuously decrease, approaching, but never equaling zero.

Theorem III.1 *If a sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) converges to a point $x \in X$, then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

Proof: To show that a sequence is Cauchy we must show that points in the sequence get closer and closer together as the sequence progresses. Let $\varepsilon > 0$. Now must find a large N such that for $n, m > N$, $d(x_n, x_m) < \varepsilon$. We know the distance between x_n or x_m (for a large n, m) and x is small since x is the limit of the sequence. So let's take n, m big enough so that $d(x_n, x) < \frac{\varepsilon}{2}$ and $d(x_m, x) < \frac{\varepsilon}{2}$. By Definition II.1 axiom (4) $d(x, y) \leq d(x, z) + d(z, y)$. So $d(x_n, x_m) < d(x_m, x) + d(x_n, x)$. Then $d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $n, m > N$. Therefore the sequence is Cauchy. This completes the proof.

Definition III.3 *A metric space (X, d) is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit $x \in X$.*

Every Cauchy sequence in a complete metric space converges or has a limit.

Definition III.4 Let $S \subset X$ be a subset of a metric space (X, d) . A point $x \in X$ is called a limit point of S if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points $x_n \in S$ such that

$$x = \lim_{n \rightarrow \infty} x_n.$$

In this definition, S is any subset of the metric space, meaning that everything in S is also in the metric space. If there is a sequence of points in S that approaches a specific point x , that point x is called a limit point of S .

Definition III.5 Let $S \subset X$ be a subset of a metric space (X, d) . The closure of S is defined to be everything in space $S \cup \{\text{limit points of } S\}$. S is closed if it contains all of its limit points.

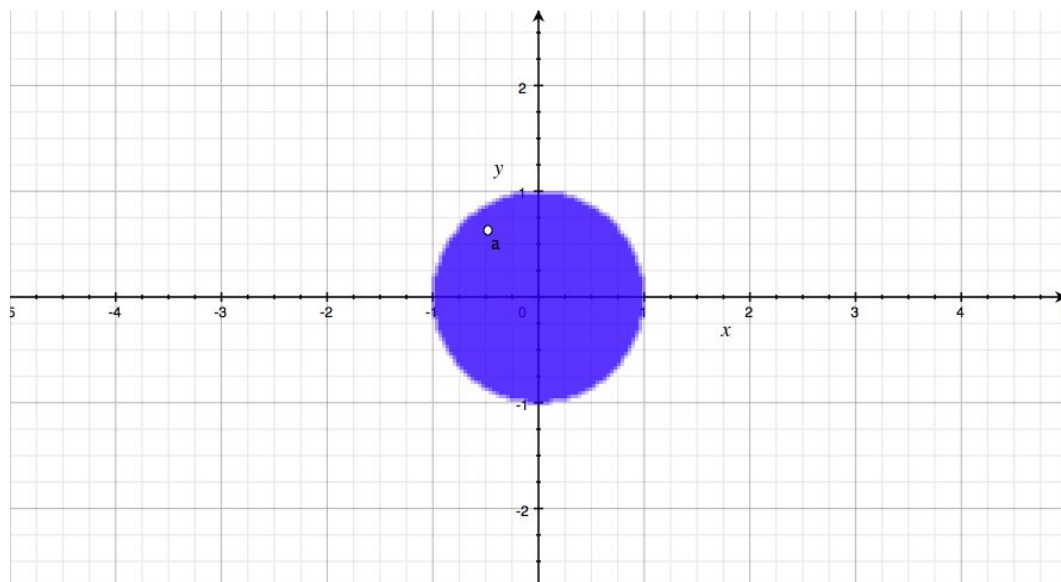
Once again let's say that S is a subset of the metric space. Then S is defined as closed if all of its limit points are contained within the space S .

IV. Compact Sets, Bounded Sets, and Boundaries

In this section we learn about different properties of metric spaces. We cover the two different ways of defining a metric space as *compact*. We also go over what it means for a metric space to be *bounded* and how to define *interior* and *boundary points* of metric spaces.

Definition IV.1 Let $S \subset X$ be a subset of a metric space (X, d) . S is bounded if there is a point $a \in X$ and a number $R > 0$ so that

$$d(a, x) < R \text{ for all } x \in S.$$



To put it more simply, a space is bounded if a circle can be drawn around the space containing all of the points in the space. In the picture above, we have a metric space X in which a is the only element. This space is bounded because a circle can be drawn around the space containing point a .

A *boundary point* of a metric space is defined as a point that has the property that when even the smallest possible circle is drawn around it, at least one point in the circle is in the space and one point in the circle is outside of the space.

The notation $B(x, \varepsilon)$ is used to denote a ball centered at x and with a radius of ε

Definition IV.2 Let $S \subset X$ be a subset of a metric space (X, d) . A point $x \in X$ is a boundary point of S if for every number $\varepsilon > 0$, $B(x, \varepsilon)$ contains a point in X not in S and a point in S . The set of all boundary points of S is called the boundary of S and is denoted as ∂S .

Interior points of metric spaces are points within the metric space that are not boundary points. For a point to be interior, there must exist at least one circle that can be drawn around the point such that the circle is contained within the space.

Definition IV.3 Let $S \subset X$ be a subset of a metric space (X, d) . A point $x \in S$ is called an interior point of S if there is a number $\varepsilon > 0$ such that $B(x, \varepsilon) \subset S$. The set of interior points of S is called the interior of S and is denoted S^0 .

When producing fractals, we are interested in metric spaces where the elements in the space are only *compact sets*. In a *compact set*, every infinite sequence in the space must have a subsequence that has a limit in the space.

Definition IV.4 Let $S \subset X$ be a subset of a metric space (X, d) . S is compact if every infinite sequence $\{x_n\}_{n=1}^{\infty}$ in S contains a subsequence having a limit in S .

Also, all compact sets are closed and bounded.

Theorem IV.1 Let (X, d) be a complete metric space. Let $S \subset X$. Then S is compact if and only if it is closed and bounded.

Proof (\Leftarrow): Suppose S is closed and bounded. Because S is bounded, we can draw a ball around S enclosing all of its points. Because S is closed, we can assume that all of the points *and* limit points are enclosed in that ball. We can also assume that every Cauchy sequence in S converges to a point in S . Suppose we drew a square around the ball with a side length equal to L (the radius of the ball). Recall that to show that S is compact, we must show that every sequence $\{x_n\}$, where $x_n \in S$, has a convergent subsequence x_{n_i} with limit x_{n_i} in S . Let $\{x_n\}$ be a sequence of points in S . Now we must find a convergent subsequence. Divide the square into 4 sub-squares: $D_1, D_2, D_3,$ and D_4 . Suppose D_2 contains the most points of the sequence. In this case, let's suppose it contains infinitely many points. Choose one point from D_2 : x_{i_1} . Now divide D_2 into 4 sub-squares: $D_{21}, D_{22}, D_{23},$ and D_{24} . Suppose D_{23} has the most points out of $D_{21}, D_{22}, D_{23},$ and D_{24} . Because D_2 contained an infinite number of points, the sub-square with the largest number of points (in this case D_{23}) also has an infinite number of points. Choose one point from D_{23} :

x_{i_2} . As we continue dividing the square with infinite points into four sub-squares and choosing one point, we end up with a subsequence of points x_{i_n} . We now must prove that $\{x_{i_n}\}$ is Cauchy. Let $\varepsilon > 0$. Find some large N such that for $i_n, i_m \geq N$, $d(x_{i_n}, x_{i_m}) < \varepsilon$ where x_{i_n} and x_{i_m} are elements in D . The furthest distance that two points can be from each other is the distance of the diagonal of the box. The original square has a width of L . Using the Pythagorean theorem, $a^2 + b^2 = c^2$, the diagonal of the square is $\sqrt{2}L$. Each of the four initial sub-squares, D_1, D_2, D_3 , and D_4 , has a width of $\frac{L}{2}$. Every subsequent set of 4 sub-squares therefore has a width of $\frac{L}{2^N}$. This means that the length of every subsequent diagonal is $\frac{\sqrt{2}L}{2^N}$. Now pick N such that $\varepsilon > \frac{\sqrt{2}L}{2^N}$. We are able to do this because as N gets larger, the size of the diagonal gets smaller approaching 0. Because the largest distance between any two points in the subsequence is $\frac{\sqrt{2}L}{2^N}$ and $\varepsilon > \frac{\sqrt{2}L}{2^N}$, $d(x_{i_n}, x_{i_m}) < \varepsilon$. Since S is closed the limit of x_{i_n} must be in S . Therefore S is compact. This completes the proof.

V. The Metric Space $(\mathcal{H}(X), h)$: The Space Where Fractals Live

In this section, we learn about the metric space used for producing fractals $(\mathcal{H}(X), h)$. In the space $(\mathcal{H}(X), h)$, each element is a compact set of points.

We also go over the distance function used in $\mathcal{H}(X)$ for making fractals, the *Hausdorff distance*.

Definition V.1 Let (X, d) be a complete metric space. Then $\mathcal{H}(X)$ denotes the space whose points are the compact, non- empty subsets of X .

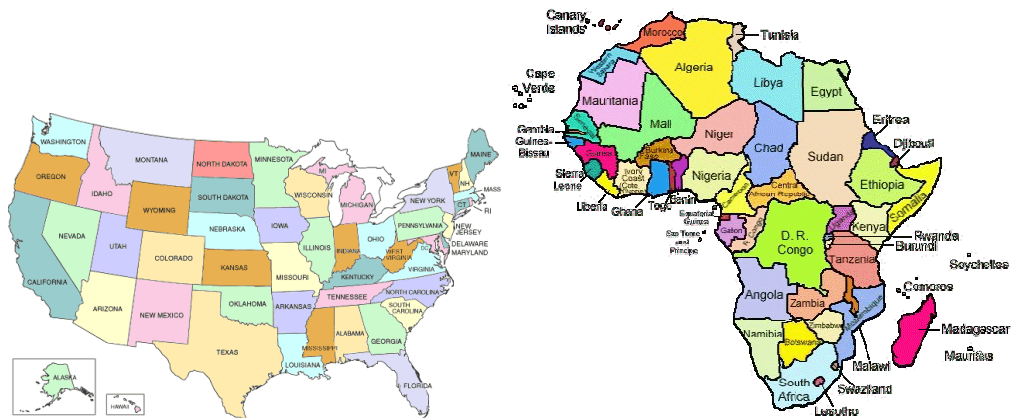
Suppose compact set A and compact set B are both in $\mathcal{H}(X)$. The distance from compact set A to compact set B is found by measuring the distance from the furthest point in A from B to the closest point in B from A .

Definition V.2 Let (X, d) be a complete metric space. Let $A, B \in \mathcal{H}(X)$. Define

$$d(A, B) = \max \{d(x, B) : x \in A\}.$$

$d(A, B)$ is called the distance from the set $A \in \mathcal{H}(X)$ to the set $B \in \mathcal{H}(X)$.

Example: Let's say we have a space where the maps of the United States and Africa are two compact sets in the space. We will call the map of the United States A and the map of Africa B .



To find the distance from the United States to Africa, we measure from the furthest point in A from B to the closest point in B from A . We would likely be measuring from some point in California to some point in Liberia.

The *Hausdorff distance* is the maximum distance between compact set A and compact set B . In the above example, to find this distance we would need to also measure the distance from Africa to the United States. To do this we would find the furthest point in B from A and measure it to the closest point in A from B . The maximum of the two measurements taken is the Hausdorff distance.

Definition V.3 Let (X, d) be a complete metric space. Then the Hausdorff distance between two points A and B in $\mathcal{H}(X)$ is defined by

$$h(A, B) = \max\{d(A, B) \text{ and } d(B, A)\}.$$

VI. The Completeness of the Space of Fractals

This section is rather short, but very important. Here we learn a theorem explaining the relationship between the completeness of a metric space (X, d) and the completeness of that metric space $(\mathcal{H}(X), h)$.

If you have a complete metric space, then the set of all compact sets along with the Hausdorff distance between the points in these sets is also a

complete metric space. Because $\mathcal{H}(X)$ is complete, this means that any Cauchy sequence in $\mathcal{H}(X)$ has a limit.

Theorem VI.1 The Completeness of the Space of Fractals. *Let (X, d) be a complete metric space. Then $(\mathcal{H}(X), h)$ is a complete metric space. Moreover, if $\{A_n\}_{n=1}^{\infty} \in \mathcal{H}(X)$ is a Cauchy sequence, then*

$$A = \lim_{n \rightarrow \infty} A_n \in \mathcal{H}(X)$$

can be characterized as follows:

$$A = \{x \in X: \text{there is a Cauchy sequence } \{x_n \in A_n\} \text{ that converges to } x\}.$$

VII. The Contraction Mapping Theorem

In this section, we learn about *contraction mappings* and the *Contraction Mapping Theorem*, which are used in fractals to contract the large picture into a smaller image. We also cover the definition of the *fixed point*, which could be considered the limit of the fractal picture.

Definition VII.1 *A transformation $f: X \rightarrow X$ on a metric space (X, d) is called contractive or a contraction mapping if there is a constant $0 \leq s < 1$ such that*

$$d(f(x), f(y)) \leq s \cdot d(x, y) \text{ for all } x, y \in X.$$

Any such number s is called a contractivity factor for f .

Example: Suppose we want to contract the graph of the function $y = \frac{1}{3}x^2$ in

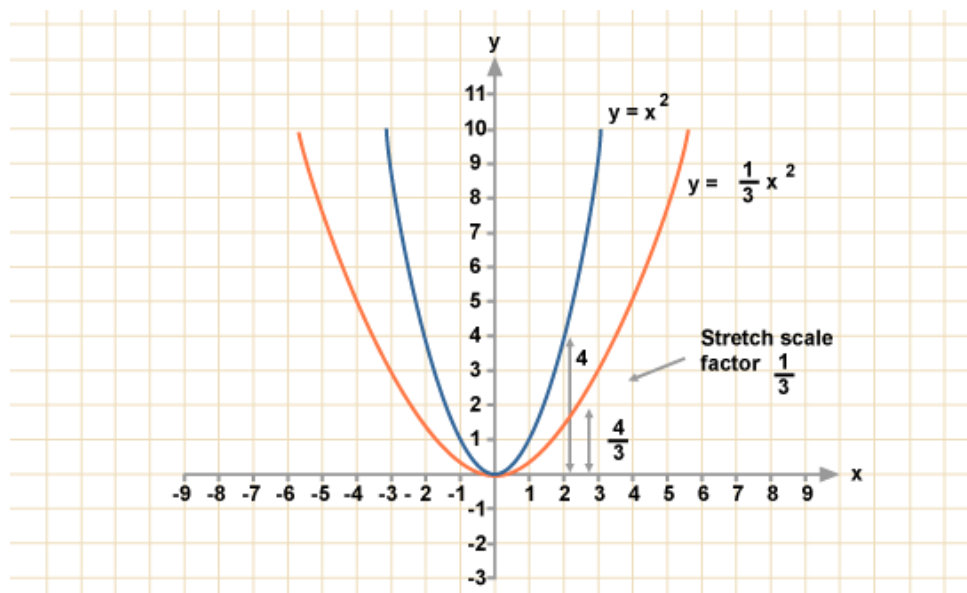
\mathbf{R}^2 by $\frac{1}{3}$ in the x direction and $\frac{1}{3}$ in the y direction. To do this set $\tilde{x} = \frac{1}{3}x$ and

$\tilde{y} = \frac{1}{3}y$. Solving for x and y , $x = 3\tilde{x}$ and $y = 3\tilde{y}$. So plugging the new

variables into the equation, $(3\tilde{y}) = \frac{1}{3}(3\tilde{x})^2$. Then $(3\tilde{y}) = \frac{1}{3}(9\tilde{x}^2)$. So $3\tilde{y} = 3\tilde{x}^2$.

This means that the contraction of $y = \frac{1}{3}x^2$ by $\frac{1}{3}$ in both the x and y

directions is $\tilde{y} = \tilde{x}^2$.



In the above example the point $(0,0)$ does not move after the contraction is applied. This point is known as a *fixed point*. In any contraction that is performed on a metric space, there is always one *fixed point*. In fact, all the other points in the space that are being contracted move around that one point and converge to that point when the limit is taken.

Theorem VII.2 [The Contraction Mapping Theorem] *Let $f: X \rightarrow X$ be a contraction mapping on a complete metric space (X, d) . Then f possesses exactly one fixed point $x_f \in X$ and moreover for any point $x \in X$, the sequence $\{f^{on}(x) : n = 0, 1, 2, \dots\}$ converges to x_f . That is,*

$$\lim_{n \rightarrow \infty} f^{on}(x) = x_f, \text{ for each } x \in X.$$

VIII. Contraction Mappings on the Space of Fractals

In this section, we apply the Contraction Mapping Theorem to the space of fractals. We review properties that exist between a metric space X and $\mathcal{H}(X)$. We also learn about iterated function systems and attractors of iterated function systems, which are very important when producing fractals, and will be talked about in more detail in the Algorithm section.

If w is a contraction mapping on the metric space (X, d) , then w is also a contraction mapping on $\mathcal{H}(X)$ with the same contractivity factor.

Lemma VIII.1 *Let $w: X \rightarrow X$ be a contraction mapping on the metric space (X, d) with contractivity factor s . Then $w: \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ defined by*

$$w(B) = \{w(x) : x \in B\} \text{ for all } B \in \mathcal{H}(X)$$

is a contraction mapping on $(\mathcal{H}(X), h(d))$ with contractivity factor s .

To make fractals, you need to apply multiple contraction mappings, which often have different contractivity factors to the space $\mathcal{H}(X)$. The union of these different contraction mappings on the same space is what makes a fractal picture.

Lemma VIII.2 *Let (X, d) be a metric space. Let $\{w_n : n = 1, 2, \dots, N\}$ be contraction mappings on $(\mathcal{H}(X), h)$. Let the contractivity factor for w_n be denoted by s_n for each n . For each $B \in \mathcal{H}(X)$, define $W : \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ by*

$$\begin{aligned} W(B) &= w_1(B) \cup w_2(B) \cup \dots \cup w_n(B) \\ &= \bigcup_{n=1}^N w_n(B). \end{aligned}$$

Then W is a contraction mapping with contractivity factor

$$s = \max\{s_n : n = 1, 2, \dots, N\}.$$

Definition VIII.1 *An iterated function system, or IFS, consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_n : X \rightarrow X$ with respective contractivity factors s_n , for $n = 1, 2, \dots, N$. The notation for the IFS just announced is $\{X; w_n, n = 1, 2, \dots, N\}$ and its contractivity factor is $s = \max\{s_n : n = 1, 2, \dots, N\}$.*

An iterated function system of different contractions on a picture is what is used to generate a fractal picture.

Theorem VIII.1 *Let $\{X; w_n, n = 1, 2, \dots, N\}$ be an iterated function system with contractivity factor s . Then the transformation $W: \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ defined for all $B \in \mathcal{H}(X)$ by,*

$$W(B) = \bigcup_{n=1}^N w_n(B)$$

is a contraction mapping metric space $(\mathcal{H}(X), h(d))$ with contractivity factor s .

That is for all $B, C \in \mathcal{H}(X)$,

$$h(W(B), W(C)) \leq s \bullet h(B, C).$$

Its unique fixed point, $A \in \mathcal{H}(X)$, obeys

$$A = W(A) = \bigcup_{n=1}^N w_n(A)$$

and is given by $A = \lim_{n \rightarrow \infty} W^{on}(B)$ for any $B \in \mathcal{H}(X)$.

An IFS also has one fixed point that does not move after the contractions are applied. This point is the limit of the system.

Definition VIII.2 *The fixed point $A \in \mathcal{H}(X)$ described in the theorem is called the attractor of the IFS.*

IX. Linear and Affine Transformations

The contractions we will use are all *linear transformation*, that is, maps in which the big picture is contracted into a smaller image of the picture such that the *zero*, or point of the large picture on the origin, goes to the *zero* in the small picture such that the map preserves addition and scalar multiplication on points. This smaller picture is a replica of the large picture just a fraction of the size. An *affine transformation* is a linear transformation plus a shift so that the small image is no longer at the origin. Both linear and affine transformations are used to produce fractal pictures.

To make a linear contraction map, we multiply by a matrix denoted A . Since we are concerned with contraction maps, the values in A will be greater than 0 and less than 1.

Theorem IX.1 *Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a linear transformation. Then there exists a unique matrix A such that for all $x \in \mathbf{R}^2$*

$$T(\bar{x}_n) = A\bar{x}.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix in \mathbf{R}^2 ; that is,

$$A = [T(e_1) \dots T(e_n)].$$

Example: If a linear transformation has matrix $A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{2}{9} \end{bmatrix}$, then the

transformation is $T_1(\bar{x}) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{2}{9} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$. If we set $T(\bar{x}) = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, then $x_n = \frac{1}{4}x$ and

$$y_n = \frac{2}{9}y.$$

If an affine transformation has matrix $A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{2}{9} \end{bmatrix}$ and shift $b = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{5} \end{bmatrix}$, then the

transformation is $T_2(\bar{x}) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{2}{9} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3}{7} \\ \frac{4}{5} \end{bmatrix}$. If we set $T(\bar{x}) = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, then

$$x_n = \frac{1}{4}x + \frac{3}{7} \text{ and } y_n = \frac{2}{9}y + \frac{4}{5}.$$

Example: Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the transformation that rotates each point in \mathbf{R}^2 about the origin through an angle θ , with a counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation.

Solution: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$. By Theorem

$$\text{IX.1, } A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

X. Algorithms

There are two different algorithms that can be used to compute fractals from iterated function systems. The first method is known as the *Deterministic Algorithm* and the second method is known as the *Random Iteration Algorithm*.

In the *Deterministic Algorithm*, you start with any point x in the metric space, $\mathcal{H}(X)$. You then apply all N contraction mappings in the iterated function system to x and get N new points. You then apply all N contraction mappings to each of the N new points found and get N^N new points. This process is continued infinite times. The sequence of all the points found approaches the fixed point or limit and creates the fractal picture.

The picture that results is a combination of smaller versions of the original picture. The fixed limit point ensures the property of *self-similarity*. This means that all the small pictures are similar to the original large picture only contracted and shifted in some cases.

Algorithm X.1 The Deterministic Algorithm. Let $\{X; w_1, w_2, \dots, w_N\}$ be an IFS.

Choose a compact set $A_0 \subset \mathbf{R}^2$. Then compute successively $A_n = W^{on}(A)$

according to

$$A_{n+1} = \bigcup_{j=1}^n w_j(A_n) \text{ for } n = 1, 2, \dots$$

Thus construct a sequence $\{A_n : n = 1, 2, 3, \dots\} \subset \mathcal{H}(X)$. Then by Theorem 7.1 the sequence $\{A_n\}$ converges to the attractor of the IFS in the Hausdorff metric.

The *Random Iteration Algorithm* is what was used to generate my fractal.

This algorithm is similar to the Deterministic Algorithm except instead of applying all N contraction mappings to each point found, one of the N contraction mappings is chosen at random and applied to each point found. The limit of this sequence of points also approaches the fixed point only the points jump around more making the picture not as clearly defined.

Algorithm X.2 The Random Iteration Algorithm. Let $\{X; w_1, w_2, \dots, w_N\}$ be an IFS, where probability $p_i > 0$ has been assigned to w_i for $i = 1, 2, \dots, N$, where

$\sum_{i=1}^N p_i = 1$. Choose $x_0 \in X$ and then choose recursively, independently,

$$x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), \dots, w_N(x_{n-1})\} \text{ for } n = 1, 2, 3, \dots,$$

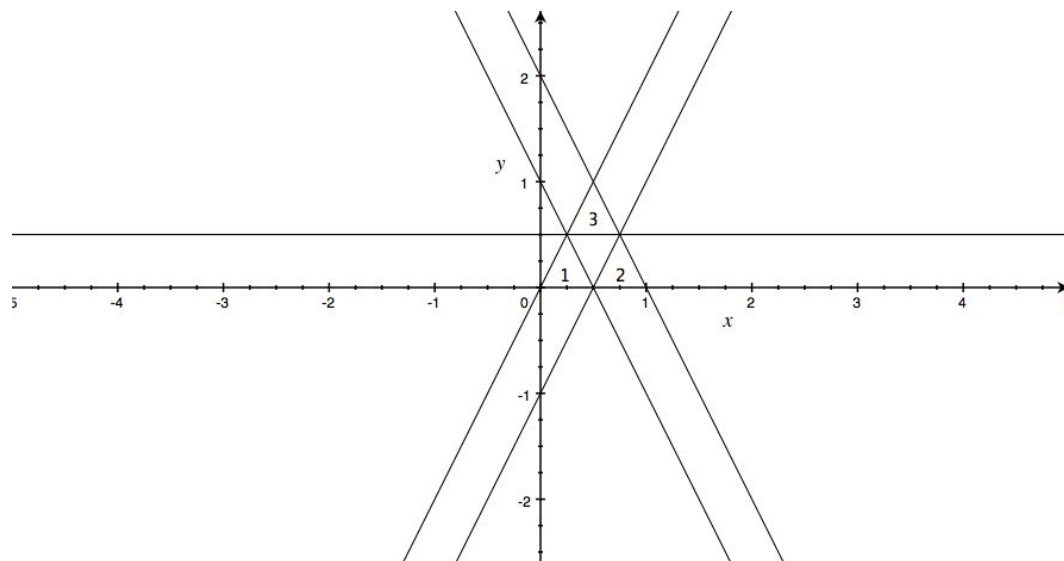
where the probability of the event $x_n = w_i(x_{n-1})$ is p_i . Thus, construct a sequence

$$\{x_n : n = 0, 1, 2, 3, \dots\} \subset X.$$

XI. Construction of a Classic Fractal

The triangle at the top of the first page with infinite triangles inside of it is an example of a fractal. It is known as the Sierpinski triangle. The Sierpinski triangle utilizes contraction mappings to create smaller images that form the large image.

This image is one large triangle made up of 3 smaller, self-similar triangles, which are each made up of many smaller, self-similar triangles of itself. This is a fractal. To produce this fractal, we must use both linear and affine transformations. We show how to get each of those smaller 3 triangles using 1 linear transformation and 2 affine transformations.



We are focusing on the large triangle, which is broken up into three small triangles. We label the bottom left triangle 1, the bottom right triangle 2, and the top triangle 3. There are two necessary steps to follow when drawing a

fractal. One, the picture must line up with the origin, and two, the image must fit in a 1×1 square. Triangle 1 is lined up with the origin and if we rescale the x - y axis, the picture fits within a 1×1 square. Recall that the point at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is labeled e_1 and the point at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is labeled e_2 . We want images of e_1 and e_2 under the transformation since Theorem IX.1 then gives the formula.

To send the fractal to any of the three triangles, we must first do a linear transformation and line the small triangle up with the origin and determine where e_1 and e_2 are. Then if the triangle is shifted away from the origin, we must determine the shift and add that shift to the linear transformation making an affine transformation.

Lining triangle 1 up with the origin, e_1 is at $\left(\frac{1}{2}, 0\right)$ and e_2 is at $\left(0, \frac{1}{2}\right)$. In the picture, triangle 1 remains against the origin. So the transformation of

$$\text{triangle 1 is } T_1(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}.$$

Lining triangle 2 up with the origin, e_1 is at $\left(\frac{1}{2}, 0\right)$ and e_2 is at $\left(0, \frac{1}{2}\right)$. This

triangle is shifted away from the origin $\left(\frac{1}{2}, 0\right)$. So the transformation of

$$\text{triangle 2 is } T_2(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

When triangle 3 is lined up with the origin, once again e_1 is at $\left(\frac{1}{2}, 0\right)$ and e_2 is

at $\left(0, \frac{1}{2}\right)$. This triangle is then shifted away from the origin $\left(\frac{1}{4}, \frac{1}{2}\right)$. So the

$$\text{transformation of triangle 3 is } T_3(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

When either the deterministic or random iteration algorithm is applied to the

$$\text{IFS with } T_1(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}, T_2(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \text{ and}$$

$$T_3(\bar{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}, \text{ the Sierpinski triangle is produced.}$$

XII. Final Fractal

For my final fractal, I decided to do a picture of my initials KMM. To do this, I first drew out my initials on a piece of graph paper. I made them fit into a 1×1 square by rescaling each of the unit squares to be $\frac{1}{12}$ of a square. Next I

labeled e_1 at the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and e_2 at the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

I then broke each letter up into 4 different sections so that there were 12 total sections. I split the “K” by cutting the long piece on the left in half and designating each leg as a section, and I split both “M”s by designating each leg as a section and cutting the top part that dips inwards in half, making two parallelograms. The lower portion of the long left part of the “K” was segment 1. The upper portion of the long left part of the “K” was segment 2. The upper branch of the “K” was segment 3. The lower branch of the “K” was segment 4. The left leg of the first “M” was segment 5. The downward slanted top piece on the first “M” was segment 6. The upward slanted top piece on the first “M” was segment 7. The right leg of the first “M” was segment 8. The left leg of the second “M” was segment 9. The downward slanted top piece on the second “M” was segment 10. The upward slanted top piece on the second “M” was segment 11. And the right leg of the second “M” was segment 12. The image KMM repeating infinite times was sent to each of these sections using linear and affine transformations. To find each of these transformations, I used the same method used to find the Sierpinski triangle.

To find transformation 1, which was the lower portion of the long left part of

the “K,” the piece was lined up with the origin. I found e_1 at $\begin{bmatrix} 1 \\ 12 \\ 0 \end{bmatrix}$ and e_2 at $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

This image was no shifted away from the origin so transformation 1 is

$$T_1(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}.$$

To find transformation 2, which was the upper portion of the long left part of

the “K,” the section was lined up with the origin. I found e_1 at $\begin{bmatrix} 1 \\ 12 \\ 0 \end{bmatrix}$ and e_2 at

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. This piece in my drawing was then shifted $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. So transformation 2 is

$$T_2(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

To find transformation 3, which was the upper branch of the “K,” I lined the

segment up with the origin and found e_1 at $\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$ and e_2 at $\begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$. This segment

was then shifted $\begin{bmatrix} 1 \\ 12 \\ 1 \\ 3 \end{bmatrix}$. Transformation 3 is $T_3(\bar{x}) = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{12} \\ \frac{1}{3} \end{bmatrix}$.

I found transformations 4- 12 using the same method described for segments

1- 3.

$$T_4(\bar{x}) = \begin{bmatrix} \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{12} \\ \frac{1}{3} \end{bmatrix}$$

$$T_5(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

$$T_6(\bar{x}) = \begin{bmatrix} \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$T_7(\bar{x}) = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$T_8(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{7}{12} \\ 0 \end{bmatrix}$$

$$T_9(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix}$$

$$T_{10}(\bar{x}) = \begin{bmatrix} \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{5}{6} \end{bmatrix}$$

$$T_{11}(\bar{x}) = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{5}{6} \\ \frac{1}{2} \end{bmatrix}$$

$$T_{12}(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{11}{12} \\ 0 \end{bmatrix}$$

Once all of the 12 transformations were calculated, I used the computer program Maple to make my fractal. To use Maple for this purpose, I first had to change each of my transformation matrices into the x_n, y_n form. So for

example $T_2(\bar{x}) = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ where $T(\bar{x}) = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ was rewritten as

$$\bar{x}_n = \frac{1}{12}x \text{ and } \bar{y}_n = \frac{1}{2}y + \frac{1}{2}.$$

Maple uses the Random Iteration Algorithm. With my fractal, it randomly chose 1 of the 12 equations of my IFS to apply to each point it found. The sequence of all the points found approached the fixed point and made the fractal picture. I used the program that Professor Banerjee created to make fractals. I programmed it to plot 20,000 points and discard the first 100

because the first 100 points plotted were not as close to the fixed point and therefore more sporadic as opposed to being close to other points creating the fractal picture I was aiming to make.

```

restart:
with(plots):
x:=10;y:=0:
n:=20000:
n1:=100:
for i from 1 to n do
j:=rand(1..12):
c:=j();
if (c=1) then
  xn:=(1/12)*x:
  yn:=0.5*y+0.5:
elif (c=2) then
  xn:=(1/12)*x:
  yn:=0.5*y:
elif (c=3) then
  xn:=0.25*x+(1/12):
  yn:=0.5*x+(1/6)*y+(1/3):
elif (c=4) then
  xn:=(1/6)*x+(1/12):
  yn:=- (1/3)*x+(1/6)*y+(1/3):
elif (c=5) then
  xn:=(1/12)*x+(1/3):
  yn:=- (1/6)*x+(2/3)*y+(1/6):
elif (c=6) then
  xn:=(1/6)*x+(1/3):
  yn:=- (1/3)*x+(1/6)*y+0.75:
elif (c=7) then
  xn:=(1/6)*x+0.5:
  yn:=(1/3)*x+(1/6)*y+0.5:
elif (c=8) then
  xn:=(1/12)*x+(7/12):
  yn:=(1/6)*x+(2/3)*y:
elif (c=9) then
  xn:=(1/12)*x+(2/3):
  yn:=- (1/6)*x+(2/3)*y+(1/6):
elif (c=10) then
  xn:=(1/6)*x+(2/3):
  yn:=- (1/3)*x+(1/6)*y+(5/6):
elif (c=11) then
  xn:=(1/6)*x+(5/6):

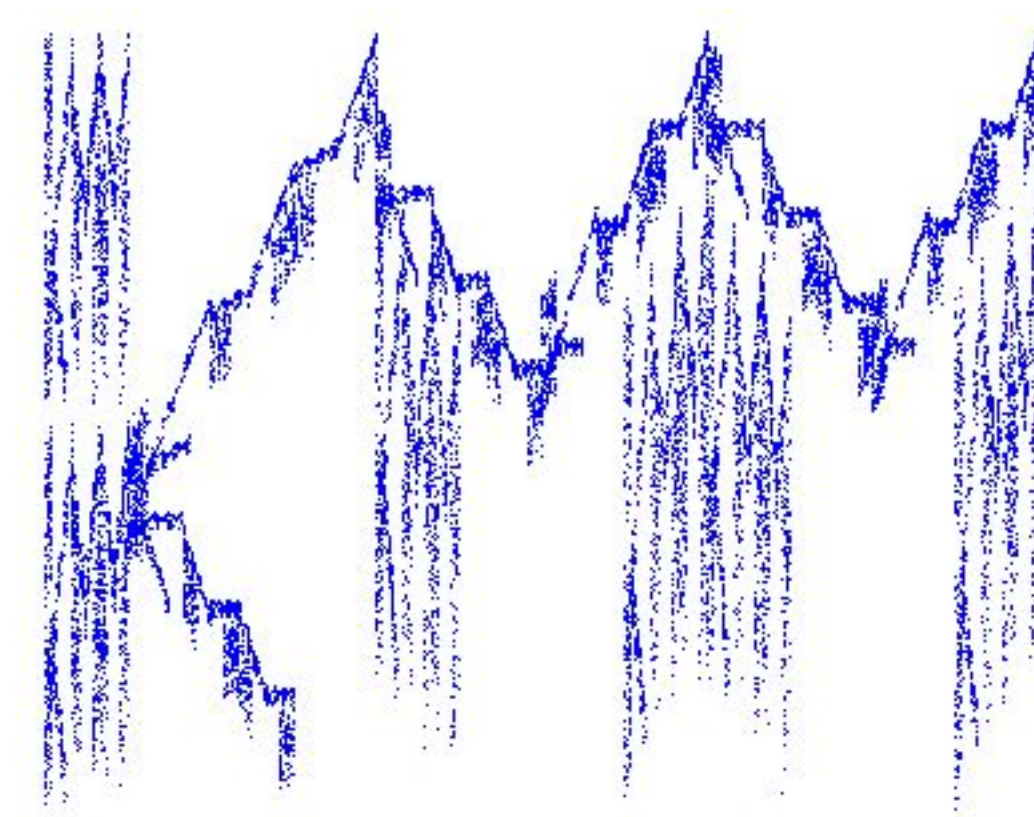
```

```

    yn:=(1/3)*x+(1/6)*y+0.5:
else
    xn:=(1/12)*x+(11/12):
    yn:=(1/6)*x+(2/3)*y:
end if;
xx[i]:=xn: yy[i]:=yn:
x:=xn: y:=yn:
end do:
##
a:=[seq(xx[i],i=n1..n)]:
b:=[seq(yy[i],i=n1..n)]:
pair:=(x,y)->[x,y]:
P:=zip(pair,a,b):
plot(P,symbolsize=1,color=blue,style=point,axes=none);

```

After plugging this information into Maple, the following fractal appeared.



If you look closely at this picture you will see “KMM” repeated many times in the 12 different sections I explained earlier.

While I calculated a fractal in a purely mathematical way, fractals also occur naturally. They can be found throughout nature in things such as pinecones, leaves, and shorelines. Zooming in on a small part of a shoreline, it continues to look like the original large shoreline even as the area of the shoreline decreases in size. This advancement in mathematics has allowed us to mimic the work of nature and understand just how complicated it is. With continued advancement, I wonder what sort of thing could be discovered next?