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Multiflavor Massive Schwinger Model With Non-Abelian Bosonization

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Abstract

We revisit the treatment of the multiflavor massive Schwinger model by non-Abelian Bosonization. We compare three different approximations to the low-lying spectrum: i) reading it off from the bosonized Lagrangian (neglecting interactions), ii) semi-classical quantization of the static soliton, iii) approximate semi-classical quantization of the “breather” solitons. A number of new points are made in this process. We also suggest a different “effective low-energy Lagrangian” for the theory which permits easy calculation of the low-energy scattering amplitudes. It correlates an exact mass formula of the system with the requirement of the Mermin-Wagner theorem.

I. INTRODUCTION

The study of two-dimensional field theories has been extremely useful [1] for understanding many aspects of the realistic four-dimensional cases. In a very interesting paper [2], Coleman analyzed the multi-flavor generalization of two-dimensional electrodynamics [3]. The well-known Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \bar{\psi}_f[\gamma_\mu(\partial_\mu - ie\mathcal{A}_\mu) + m']\psi_f, \quad (1.1)$$

where $F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ and summation is to be understood over the flavor index f . (Here $\bar{\psi}_f = \psi_f^\dagger\gamma_2$ and one may choose $\gamma_1 = \sigma_1, \gamma_2 = \sigma_2$.) Since the electric charge has the dimension of mass in this theory it is meaningful to define the strong coupling regime as

$$e \gg m', \quad (1.2)$$

where m' is the common fermion mass. (It is also interesting to allow different masses, m_f for each fermion.)

The natural presentation [2] of the theory in the strong coupling regime is its bosonized form [4]. Then the large quantity e^2 ends up just multiplying a quadratic (mass) term and does not complicate the interactions. The resulting Lagrangian has a lot of similarity to the low energy effective meson Lagrangian used for describing QCD. Since some exact results are known for the two-dimensional case we may hope to learn more about various aspects of the QCD effective Lagrangian. That is, in fact, our motivation for looking at this model and sets the framework of our analysis.

Coleman [2] used an Abelian bosonization technique and showed that the lowest state in the 2-flavor model is a "meson" with quantum numbers $I^{PG} = 1^{-+}$ ¹. He pointed out that the first excited state has the quantum numbers $I^{PG} = 0^{++}$ and obeys the exact mass relation

$$m(0^{++}) = \sqrt{3}m(1^{-+}). \quad (1.3)$$

In addition, there are an infinite number of unstable mesons in the model. At a much larger mass scale there appears the $I^{PG} = 0^{--}$ meson, which would lie rather low in the weak coupling limit [2].

¹ $G = e^{i\pi I_y}C$, where C is the charge parity, is the usual G parity. Note that $\bar{\psi}\gamma_5\psi$ goes to $-\bar{\psi}\gamma_5\psi$ under charge conjugation ($\gamma_5 = -i\gamma_1\gamma_2$ here), unlike the four-dimensional case.

A complicating feature in the treatment of [2] is that the lowest-lying physical states emerge in a very asymmetrical manner. The members of this $I^{PG} = 1^{-+}$ triplet, in fact, variously emerge as a soliton, an anti-soliton and a soliton-anti-soliton bound state (or "breather"). It is possible to give a symmetrical treatment by using the more recently discovered non-Abelian bosonization technique [5]. Gepner [6] carried out this analysis, showing that the 1^{-+} triplet could be treated symmetrically as the collective excitation of the classical soliton solution in the non-Abelian model. This method of treating the meson states is similar to that employed in the treatment [7] of three-flavor baryons in the four-dimensional Skyrme model [8].

In the present note we shall investigate some aspects of the non-Abelian bosonization of the model in more detail. As a preliminary, we point out that some interesting things can be said about the low-lying 1^{-+} triplet at the level of the non-Abelian Lagrangian itself, without going to the soliton sectors. In this way, for example, we may easily relate two of Coleman's "three things I don't understand" [2] to the situation in the QCD meson spectrum. We address a problem concerning the true lowest-lying state which appeared in [6]. There it was found that, at the semi-classical level, $m(0^{++}) < m(1^{-+})$, which would make the $I^{PG} = 0^{++}$ meson lowest-lying. This was interpreted as a deficiency of the approximation in treating the breather modes. We investigate further the breather modes here and develop a quantitative approximation method for treating their excitations. We point out that a natural alternative interpretation of the model yields $m(0^{++}) > m(1^{-+})$, in agreement with Coleman. This is welcome since the semi-classical treatment of soliton collective modes has usually given a nice understanding of at least the overall features of the baryon spectrum. A procedural difference from [6] here, which yields the same result, involves starting from the free bosonized theory and then gauging it, rather than bosonizing the interacting theory as a whole. We also give a slightly different treatment of the soliton collective quantization.

Finally, we investigate the possibility of an approximate low-energy effective Lagrangian description of multiflavor QED_2 rather than the exact bosonized description. A low-energy effective Lagrangian has an advantage over the exact bosonized theory in that it can con-

tain all the low-lying particles. Hence the tree-level scattering amplitudes computed from this Lagrangian should be good approximations at low energy. Furthermore we show that taking the linear sigma model as the effective Lagrangian leads to a correlation between the special mass formula (1.3) and the Mermin-Wagner theorem [9,10] on the impossibility of the spontaneous breakdown of a continuous symmetry in two dimensions.

In section 2 we show how the non-Abelian bosonized multiflavor QED_2 Lagrangian can be derived by a suitable “gauging” of Witten’s bosonized Lagrangian [5] representing a multiplet of free fermi fields. Section 3 contains a discussion showing how certain puzzling features of the multiflavor theory can be understood at the tree level of the resulting theory. The analogy to low-energy particle physics phenomena is pointed out. We go beyond the tree approximation by exploiting the semi-classical quantization of the classical solitons of the model. The well-known time-independent and time-dependent (breather) solitons are discussed in section 4. Section 5 contains a treatment of the semi-classical quantization of the static solitons. In section 6 the same method is applied to the time-dependent solitons by making a kind of Born-Oppenheimer approximation which requires computing the time-averaged “moment of inertia” of the soliton. Details of this calculation are given in Appendix A. Section 7 contains a comparison of the alternative approaches to the spectrum given in sections 3, 5, and 6. The need for an effective Lagrangian is explained and it is argued that the linear sigma model is a suitable candidate. It is shown to lead to an understanding of the mass relation (1.3) and is used to find the low-energy scattering amplitude.

II. BOSONIZED ACTION

First, we shall write down the bosonized version of the free fermion terms in the Lagrangian (2.1). It is built from the $N_f \times N_f$ unitary matrix field $U(x)$ which transforms as

$$U(x) \rightarrow U_L U(x) U_R^{-1} \tag{2.1}$$

under the global chiral $U_L(N_f) \times U_R(N_f)$ transformation $\psi_{L,R} \rightarrow U_{L,R}\psi_{L,R}$. Here $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi$. There are three pieces:

$$\Gamma_{free} = \Gamma_\sigma + \Gamma_m + \Gamma_{WZW}. \quad (2.2)$$

Γ_σ and Γ_m are essentially the usual kinetic and mass terms of the non-linear sigma model:

$$\Gamma_\sigma + \Gamma_m = \int d^2x \left[-\frac{1}{8\pi} \text{Tr}(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{2} m^2 \text{Tr}(U + U^\dagger - 2) \right], \quad (2.3)$$

where [1] m is essentially proportional to m' in (1.1). Clearly the first term in (2.3) is chiral invariant while the second has the same chiral transformation property as the mass term in (1.1). The characteristic Wess-Zumino-Witten term necessary for non-Abelian bosonization [5] may be compactly written, using the matrix one-form $\alpha = dUU^\dagger$, as

$$\Gamma_{WZW} = \frac{1}{12\pi} \int_{M^3} \text{Tr}(\alpha^3), \quad (2.4)$$

where M^3 is a three-dimensional manifold whose boundary is the two-dimensional Minkowski space.

Now let us "gauge" the set of N_f bosonized massive Dirac fields represented by (2.2).

We can always include a gauge-invariant piece Γ_γ containing just the electromagnetic fields:

$$\Gamma_\gamma = \int d^2x \left\{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{ie\theta}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu} \right\}. \quad (2.5)$$

The second term, labeled by the angular parameter θ , describes the effect of a background electric field [2]. It violates parity invariance and is the analog of the θ parameter in 4-dimensional QCD [11]. We shall, for the most part, consider only the $\theta = 0$ case in the present paper. Finally, and most importantly, we must include the matter-gauge field interaction. At the fermion level it is, of course, obtained by replacing $\partial_\mu \psi_f$ by $(\partial_\mu - ie\mathcal{A}_\mu)\psi_f$ so that the change in $\partial_\mu \psi_f$ under a local U(1) gauge transformation $\psi_f \rightarrow e^{i\Lambda(x)}\psi_f$ is canceled by the transformation of the gauge field $\mathcal{A}_\mu \rightarrow \mathcal{A}_\mu + \frac{1}{e}\partial_\mu \Lambda$. At the bosonic level there is a problem with this approach since the basic field $U(x)$ represents only electrically neutral objects (mesons) and should thus remain invariant under a U(1) gauge transformation. It

would, at first, seem that the free bosonized action (2.2) is gauge-invariant as it stands so there is no need to couple it to the $U(1)$ gauge field \mathcal{A}_μ . We seem to have reached a dead end!

However, the situation for the three-dimensional term (2.4) is not really clear and, in any event, we have the obligation to demand a consistent gauging of the bosonized massless free field terms $\Gamma_\sigma + \Gamma_{WZW}$ with respect to non-Abelian flavor transformations (under which U does transform). We shall thus add terms to make Γ_{WZW} invariant under local $U(N_f)$ vector-type transformations and afterwards specialize to the electromagnetic $U(1)_{EM}$ subgroup. Under a local infinitesimal $U(N_f)$ vector-type transformation one has

$$\delta U = i[E, U], \quad \delta A = \frac{1}{e}dE + i[E, A], \quad (2.6)$$

where $E = E^\dagger$ and A is the matrix one-form of $U(N_f)$ gauge fields. Eq. (2.4) can now be gauged iteratively [12]. Its variation under (2.6) is seen, with the help of Stokes's theorem to be partially canceled by the variation of the additional term $\frac{ie}{4\pi} \int_{M^2} Tr[A(\alpha + \beta)]$, where $\beta = U^\dagger dU$. The remaining variation of the new term is canceled by the variation of the term $\frac{e^2}{4\pi} \int_{M^2} Tr[UAU^\dagger A]$ and the procedure terminates. Now if we specialize A_μ to the desired $U(1)$ component by setting $A_\mu = \mathcal{A}_\mu \mathbf{1}$ we see that the second new term vanishes while the first becomes the interaction term

$$\Gamma_{int} = -\frac{e}{2\pi} \int d^2x \epsilon_{\alpha\beta} \mathcal{A}_\alpha Tr(\partial_\beta UU^\dagger). \quad (2.7)$$

The total bosonized action for multiflavor QED is then the sum of (2.3), (2.4), (2.5) and (2.7). As a check on this procedure we may calculate the electromagnetic current

$$J_\mu^{EM} = \frac{\delta\Gamma}{\delta\mathcal{A}_\mu} |_{\mathcal{A}_\mu=0} = -\frac{e}{2\pi} \epsilon_{\mu\nu} Tr(\partial_\nu UU^\dagger). \quad (2.8)$$

The action may be further simplified by making use of the fact that there is no propagating photon degree of freedom in the two-dimensional theory; then the photon field may be "integrated out." This is conveniently accomplished by the substitution [13] $F_{\mu\nu} = \epsilon_{\mu\nu} F$. The field F obeys the equation of motion:

$$F = \frac{ie}{2\pi}(\theta + i \ln \det U), \quad (2.9)$$

wherein $Tr(\partial_\nu U U^\dagger) = \partial_\nu \ln \det U$ was used. Substituting (2.9) back into Γ gives

$$\Gamma = \int d^2x \left[-\frac{1}{8\pi} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{m^2}{2} Tr(U + U^\dagger - 2) - \frac{e^2}{8\pi^2} (\theta + i \ln \det U)^2 \right] + \Gamma_{WZW}. \quad (2.10)$$

III. ANALOGY TO PARTICLE PHYSICS

The form (2.10) can nowadays be recognized as essentially identical to that of the four-dimensional Lagrangian describing the pseudoscalar mesons. Coleman [2] suspected the analogy and pointed out features which were puzzling (stated as "questions I don't understand") on the fermion picture. However, the connection was slightly obscured by the use of the Abelian bosonization. Hence it may be interesting to briefly discuss this here. Let us simplify to the two-flavor case and set $\theta = 0$. Introduce the decomposition:

$$U = \exp(i\sqrt{4\pi}\phi), \quad \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad (3.1)$$

with the picturesque names

$$\pi^+ = \phi_{12}, \quad \pi^- = \phi_{21}, \quad \pi^0 = \frac{1}{\sqrt{2}}(\phi_{11} - \phi_{22}), \quad \eta = \frac{1}{\sqrt{2}}(\phi_{11} + \phi_{22}). \quad (3.2)$$

(In the present model, of course, all these fields describe electrically neutral particles.) Then the expansion of (2.10) in powers of ϕ yields the quadratic terms

$$\Gamma = - \int d^2x \left\{ \partial_\mu \pi^+ \partial_\mu \pi^- + \frac{1}{2} (\partial_\mu \pi^0)^2 + \frac{1}{2} (\partial_\mu \eta)^2 + m_\pi^2 (\pi^+ \pi^- + \frac{1}{2} \pi^0 \pi^0) + \frac{1}{2} m_\eta^2 \eta^2 \right\} + \dots, \quad (3.3)$$

where

$$m_\pi = 2\sqrt{\pi}m \approx 3.54m, \quad m_\eta = \sqrt{4\pi m^2 + \frac{2e^2}{\pi}}. \quad (3.4)$$

In the strong coupling limit, $e \gg m$, the mass of the $I^{PG} = 0^{--}$ particle η is clearly many orders of magnitude higher than the mass of the $I^{PG} = 1^{-+}$ triplet. Since e^2 only makes its appearance in the η mass term, all of the other low-lying states will be of the same order of magnitude as the π triplet (Coleman's second question). The η essentially decouples. In QC D, the fact that the η' meson is much heavier than the π triplet is usually attributed to instanton effects rather than quark annihilation graphs [14].

It is also interesting to consider what happens when we allow different masses for the fundamental fermions. In the bosonized picture of the two-flavor model this corresponds to the additional term

$$\Gamma_{\Delta} = \Delta Tr[\tau_3(U + U^{\dagger})], \quad (3.5)$$

where Δ is an isospin violation parameter with dimension $(mass)^2$ and τ_3 is the Pauli matrix. Now Coleman's first question is: why does the π triplet remain degenerate even if, for example, $\sqrt{|\frac{\Delta}{m^2}|}$ has order of magnitude 10? In the present framework it is easy to see that this is just a variant of the second question discussed above.

We consider the strong coupling situation where $e \gg \{m, \sqrt{|\Delta|}\}$. With the decomposition (3.1), Γ_{Δ} expands out as

$$\Gamma_{\Delta} = - \int d^2x [2\sqrt{2}\Delta\eta\pi^0 + \dots]. \quad (3.6)$$

This mixing between the π^0 and η fields requires us to diagonalize the matrix

$$\begin{pmatrix} m_{\pi}^2 & 2\sqrt{2}\Delta \\ 2\sqrt{2}\Delta & m_{\eta}^2 \end{pmatrix}, \quad (3.7)$$

(where m_{π} and m_{η} are given by (3.4)) in order to obtain the physical π^0 and η states and masses. The eigenvalues of (3.7) give the physical masses

$$m_{phys}^2(\pi^0) \approx m_{\pi}^2 - \frac{\Delta^2}{m_{\eta}^2}, \quad m_{phys}^2(\eta) \approx m_{\eta}^2 + \frac{\Delta^2}{m_{\eta}^2}, \quad (3.8)$$

which leads to

$$m_{phys}^2(\pi^\pm) - m_{phys}^2(\pi^0) \approx \frac{\Delta^2}{m_\eta^2}. \quad (3.9)$$

Remembering that we are working in the strong coupling approximation where $\Delta \ll m_\eta^2 \approx \frac{2e^2}{\pi}$, we see that the $\pi^\pm - \pi^0$ mass splitting vanishes as $e \rightarrow \infty$! This is essentially the same as the effect in four-dimensional QCD that the $\pi^\pm - \pi^0$ mass splitting is due to photon exchange diagrams rather than to the difference between the down and up quark masses, $m_d - m_u$. More precisely, the piece due to $(m_d - m_u)$ is proportional to $(m_d - m_u)^2$ and hence negligible (as in (3.9) above) rather than being proportional to $(m_d - m_u)$. This also follows from the isospin transformation properties of the quark mass operator. By Bose statistics, the $\pi^\pm - \pi^0$ mass difference can only be mediated by an operator satisfying $\Delta I=2$. However the quark mass terms transform as a linear combination of $\Delta I=0$ and $\Delta I=1$ pieces.

IV. CLASSICAL SOLUTIONS

We are interested in studying the strong coupling spectrum of the model by quantizing the excitations around exact classical solutions [15].. We adopt the ansatz [13] for classical solutions:

$$U_c(x, t) = \text{diag}[\exp(i2\sqrt{\pi}\chi_1(x, t)), \dots, \exp(i2\sqrt{\pi}\chi_{N_f}(x, t))]. \quad (4.1)$$

It is being assumed that U_c depends only on x_1 and $x_2 = it$, not on x_3 , the coordinate appearing in the three-dimensional term Γ_{WZW} , eq. (2.4). The structure of Γ_{WZW} then shows that it will give zero when (4.1) is substituted into it. Hence, substituting (4.1) into the total action (2.10) yields, after the usual Legendre transform, the classical Hamiltonian density,

$$\mathcal{H}_{class} = \frac{1}{2} \sum_{i=1}^{N_f} (\dot{\chi}_i^2 + (\chi_i')^2) + \frac{e^2}{2\pi} \left(\sum_i \chi_i - \frac{\theta}{2\sqrt{\pi}} \right)^2 + \sum_i m_i^2 [1 - \cos(2\sqrt{\pi}\chi_i)], \quad (4.2)$$

where we have allowed for N_f different masses, m_i ². (Here $\dot{\chi}_i = \frac{\partial \chi_i}{\partial t}$ and $\chi'_i = \frac{\partial \chi_i}{\partial x}$.) Notice that the classical Hamiltonian coincides with the Hamiltonian obtained [2] via Abelian bosonization. The boundary values of the χ_i at spatial infinity are well-known to be related to the electric charge Q (or equivalently, the "fermion number" B) of the model. This may be seen by substituting (4.1) into (2.8) to give

$$B = \frac{1}{e}Q = -\frac{i}{e} \int_{-\infty}^{+\infty} dx J_2^{EM}(x, 0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx \sum_i \chi'_i = \frac{1}{\sqrt{\pi}} \sum_i [\chi_i(\infty, 0) - \chi_i(-\infty, 0)]. \quad (4.3)$$

The equation of motion to be satisfied by the classical ansatz is

$$\ddot{\chi}_i - \chi_i'' + \frac{e^2}{\pi} \left(\sum_k \chi_k - \frac{\theta}{2\sqrt{\pi}} \right) + 2\sqrt{\pi} m_i^2 \sin(2\sqrt{\pi} \chi_i) = 0. \quad (4.4)$$

For definiteness in what follows we shall specialize to the parity-conserving theory by setting $\theta = 0$ and also to the case of two flavors with equal masses. If we set $\chi_1 = -\chi_2 \equiv \chi$ the two equations collapse to the sine-Gordon equation :

$$\ddot{\chi} - \chi'' + 2\sqrt{\pi} m^2 \sin(2\sqrt{\pi} \chi) = 0. \quad (4.5)$$

Both time-independent and time-dependent classical solutions are important.

i. Time-independent solution. We set $\dot{\chi}_i = 0$ and choose the boundary conditions

$$\chi_1(-\infty) = -\chi_2(-\infty) = 0, \quad \chi_1(\infty) = -\chi_2(\infty) = \sqrt{\pi}. \quad (4.6)$$

Equation (4.3) shows that setting $\chi_1 = -\chi_2$ gives zero electric charge for the solutions. Then, integrating (4.5) yields the well-known static sine-Gordon soliton

$$\chi_1(x) = -\chi_2(x) = \frac{2}{\sqrt{\pi}} \tan^{-1} \exp[2\sqrt{\pi} m x + c'], \quad (4.7)$$

where c' is an arbitrary constant specifying the soliton location. The classical energy is obtained by substituting (4.7) into (4.2):

²An interesting discussion of the unequal mass case has been given in [13], whereas [2] and [6] confine their attention to the equal-mass case.

$$E_{class} = \int_{-\infty}^{+\infty} dx \mathcal{H}_{class} = \frac{8m}{\sqrt{\pi}} \approx 4.51m. \quad (4.8)$$

It is amusing that the numerical value of E_{class} is of the same order as the lowest-lying $I^{PG} = 1^{-+}$ meson mass, m_π found in (3.4).

ii. Time-dependent solutions. There is a well-known [15,16] family of time-dependent bound solutions of the sine-Gordon equation, referred to as "breathers." They are physically interpreted as a bound soliton-anti-soliton pair ³. In our problem, these solutions read:

$$\chi_1(x, t) = -\chi_2(x, t) = \frac{2}{\sqrt{\pi}} \tan^{-1} \left[\frac{a(t)}{\cosh(bx)} \right], \quad a(t) = \eta \sin \omega t, \quad b = \eta \omega, \quad \eta = \sqrt{\frac{4\pi m^2}{\omega^2} - 1}, \quad (4.9)$$

and are characterized by an angular frequency $\omega < 2\sqrt{\pi}m$. The parameter m is the mass which appears in the bosonized Lagrangian. We obtain the classical breather energy by substituting (4.9) into (4.2):

$$E_{breather}(\omega) = \int_{-\infty}^{+\infty} dx \mathcal{H}[\chi_i(x, 0)] = \frac{8\omega}{\pi} \sqrt{\frac{4\pi m^2}{\omega^2} - 1}. \quad (4.10)$$

Both the time-independent classical solution (4.7) as well as the time-dependent classical solution (4.9) obey $\det U_c = 1$. Physically, this corresponds to the specialization to the states of the system whose masses remain finite as the electric charge e goes to infinity. As discussed in section 3, this means the neglect of the η type field which can be formally isolated by the decomposition $U = \tilde{U} \exp(i\sqrt{\frac{4\pi}{N_f}}\eta)$. \tilde{U} satisfies $\det \tilde{U} = 1$ and describes the light degrees of freedom.

V. SEMI-CLASSICAL QUANTIZATION.

In this section we review the semi-classical quantization of the static soliton solution in a slightly different way from [6] but with essentially equivalent results. We make the ansatz for the matrix field U [17],

³The anti-soliton is obtained by giving the right-hand side of (4.7) a negative sign.

$$U(x, t) = A(t)U_c(x, t)A^\dagger, \quad (5.1)$$

where $A(t)$ is, in general, an $N_f \times N_f$ special unitary matrix and the classical solution U_c is allowed, for later purposes, to also depend on time. We now substitute this into the bosonized action (2.10). The first integral yields

$$\dots + \frac{1}{4\pi} \int d^2x \text{Tr}(U_c A^\dagger \dot{A} U_c^\dagger A^\dagger \dot{A} - (A^\dagger \dot{A})^2 + [U_c^\dagger, \dot{U}_c] A^\dagger \dot{A}), \quad (5.2)$$

where the three dots stand for the A -independent piece. Notice that when an Abelian ansatz like (4.1) is taken, the last term in (5.2) vanishes so no dependence on \dot{U}_c remains in the non-classical piece of the Lagrangian. For the three-dimensional integral in (2.10) we get

$$\Gamma_{WZW}[U] = \Gamma_{WZW}[A(t)U_c(x, t)A^\dagger(t)] = \Gamma_{WZW}[U_c] + \frac{1}{4\pi} \int d^2x \text{Tr}(A^\dagger \dot{A}(U_c^\dagger U_c' + U_c' U_c^\dagger)). \quad (5.3)$$

The collective variable to be quantized which appears in (5.2) and (5.3) is clearly $A^\dagger(t)\dot{A}(t)$. This is an angular-velocity type quantity which, in the two flavor case of present interest may be decomposed as

$$A^\dagger \dot{A} = \frac{i}{2} \boldsymbol{\Omega}(t) \cdot \boldsymbol{\tau}, \quad (5.4)$$

where the $\boldsymbol{\tau}$ are the Pauli matrices. The ansatz in this case reads

$$U(x, t) = A(t) \begin{pmatrix} e^{i2\sqrt{\pi}\chi_1} & 0 \\ 0 & e^{-i2\sqrt{\pi}\chi_1} \end{pmatrix} A^\dagger(t), \quad (5.5)$$

with χ_1 given in (4.7). Using (5.2), (5.3) and (5.4) then gives the collective Lagrangian

$$L_{coll} = -\frac{8m}{\sqrt{\pi}} + \frac{1}{2} \mathcal{I}(\Omega_1^2 + \Omega_2^2) - \frac{1}{\sqrt{\pi}} [\chi_1(\infty) - \chi_1(-\infty)] \Omega_3, \quad (5.6)$$

wherein,

$$\mathcal{I} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \sin^2[2\sqrt{\pi}\chi_1(x)] = \frac{2}{3\pi^{3/2}m}. \quad (5.7)$$

The first term in (5.5) represents the classical soliton mass in (4.8). The second term comes from (5.2). The third term comes from (5.3); using the boundary condition (4.6) we can see

that the coefficient of Ω_3 in (5.6) is simply -1. Finally, the quantity in (5.7) will be seen to represent a “moment of inertia” for rotations in isospin space. It determines the excitation spectrum and its explicit evaluation is discussed in Appendix A.

The next step is to quantize (5.6). The canonical momenta (for an implicit parameterization of the matrix A) may be taken as

$$J_k = \frac{\partial L_{coll}}{\partial \Omega_k} = \begin{cases} \mathcal{I}\Omega_k & k = 1, 2 \\ -1 & k = 3. \end{cases} \quad (5.8)$$

These yield true dynamical momenta only for $k = 1, 2$, but amount to a constraint for $k = 3$. This is analogous to the quantization of the SU(3) Skyrme model [7]. For quantization we may introduce an operator J_3 which, together with J_1 and J_2 , satisfies the SU(2) algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$. However, we must restrict the allowed states to those obeying

$$J_3|allowed\rangle = -|allowed\rangle. \quad (5.9)$$

The collective Hamiltonian is

$$H_{coll} = -L_{coll} + \frac{\partial L_{coll}}{\partial \Omega_i}\Omega_i = \frac{8m}{\sqrt{\pi}} + \frac{1}{2\mathcal{I}}\mathbf{J}^2 - \frac{1}{2\mathcal{I}}(J_3)^2. \quad (5.10)$$

After introducing the SU(2) adjoint representation matrix

$$D_{ij}(A) = \frac{1}{2}Tr(\tau_i A \tau_j A^\dagger), \quad (5.11)$$

we define $I_i = -D_{ij}(A)J_j$, which can be shown to satisfy $\mathbf{I}^2 = \mathbf{J}^2$, as well as $[I_i, I_j] = i\epsilon_{ijk}I_k$.

Then, finally, acting on allowed states, the collective Hamiltonian may be put in the form

$$H_{coll} = \frac{8m}{\sqrt{\pi}} + \frac{1}{2\mathcal{I}}\mathbf{I}^2 - \frac{1}{2\mathcal{I}}, \quad (5.12)$$

which describes a rigid rotator in isospin space. A basis for the space of states on which the operators in this model act consists of the SU(2) representation matrices $D_{m,-m'}^{(I)*}(A)$ for isotopic spin I . The isotopic spin operator \mathbf{I} acts on the left index of this matrix while the operator \mathbf{J} acts on the right index. The constraint (5.9) requires that allowed wavefunctions be of the form $D_{m,1}^{(I)*}(A)$ for a meson with $I_3 = m$. Since a state with $I_3 = 1$ is evidently

required for an allowed representation we learn that the possible excited states associated with the classical solution (4.7) are mesons with isospin $I = 1, 2, 3, \dots$. In the fermionic picture these correspond to multi-fermion-anti-fermion states for $I > 1$. In the bosonic picture, the $I > 1$ states are bound states of the fundamental meson in (3.1).

Using the result for the moment of inertia (5.7) in (5.12) finally yields for the mass of the isospin= I meson,

$$\frac{m(I)}{m} \approx 4.514 + 4.176[I(I + 1) - 1], \quad I \geq 1. \quad (5.13)$$

The meson with $I = 1$ has the mass, $m(1) = 8.690m$. We interpret this meson as corresponding to the fundamental one in the bosonized Lagrangian. The mass obtained by directly reading the coefficient of the quadratic term in (3.3) is $3.545m$; the different value obtained is interpreted as arising from the different method of approximation being employed. Since the fundamental meson has parity= -1 , we expect the parity of meson I to be $(-1)^I$ in the picture where the meson I is a bound state of I fundamental ones. Noticing that $m(2) > 2m(1)$, we see that the decay $2 \rightarrow 1 + 1$ is energetically allowed. It is also easy to see that the decay $I \rightarrow (I - 1) + 1$ is energetically allowed. Hence, in the present approximation, only the $I = 1$ meson is expected to be stable.

VI. QUANTIZED BREATHER MODES AND THEIR EXCITATIONS.

The quantization of the classical breather solutions in (4.9) is more involved than the quantization of the static soliton in (4.7). Whereas the latter has the fixed mass (4.8), the classical breathers exist for a continuous family of energies as seen in (4.10). It is necessary to find the discrete quantum “orbits” by a semi-classical technique like the old Bohr-Sommerfeld method. Afterwards, one can get excited isotopic spin states by quantization of the variable $A(t)$ in (5.1). The general picture is very similar to the “bound state” approach to the strange baryons [18] in the Skyrme model.

A quick way to find the Bohr-Sommerfeld energies was discussed in [16]. Since the energy difference between two neighboring semi-classical (large quantum number n) levels is

the classical angular frequency of periodic motion ω , the number of levels in energy interval dE will be $dn = \frac{dE}{\omega(E)}$. Using $\omega(E)$ from (4.10) and integrating to find n yields

$$E_n = 2M \sin\left(\frac{n\pi}{8}\right), \quad (6.1)$$

where n is an integer and $M = \frac{8m}{\sqrt{\pi}}$ is the soliton mass given in (4.8). The corresponding angular frequencies are given by

$$\omega_n = (2\sqrt{\pi})m \cos\left(\frac{n\pi}{8}\right). \quad (6.2)$$

Inspection of (6.1) shows that the discrete energies are $E_1 \approx 0.765M$, $E_2 \approx 1.414M$ and $E_3 \approx 1.848M$. The value $E_4 = 2M$ corresponds to zero angular frequency. Remember that the breathers are classical solutions of the sine-Gordon equation (4.5). In that context, a simple physical interpretation was given in [19]. Expanding the argument of the sine in (6.1) yields

$$E_n \approx n(2m\sqrt{\pi}) + \dots \quad (6.3)$$

Now $(2m\sqrt{\pi})$ is recognized from (3.4) as m_π , the mass of the fundamental meson degree of freedom of the bosonized theory in the approximation where meson-meson interactions are neglected. Thus it is natural to interpret the E_n solution as a bound state of n fundamental mesons. Then the breather solution E_1 would be, in fact, a third alternative description of the fundamental meson. The breather solution E_2 corresponds to a $\pi\pi$ bound state, etc. This is the interpretation adopted by Coleman [2] in the Abelian quantization case, and is the one we shall adopt. Note that the Hamiltonian for our classical ansatz (4.2) agrees with the Hamiltonian for Coleman's Abelian bosonization so the classical solution should be the same. On the other hand, in ref. [6] the breather solution E_1 was identified with the $\pi\pi$ bound state.

It should be remarked that a more accurate (argued to be exact) quantization of the breathers was introduced by Dashen, Hasslacher and Neveu (DHN) [20] and used in [2]. It requires the simple modification of (6.1) to

$$E_n(DHN) = 2M \sin\left(\frac{n\pi}{6}\right), \quad (6.4)$$

where M is now the soliton mass with the inclusion of quantum corrections. In this case the only discrete levels are $E_1 = M$ and $E_2 \approx 1.732M$. Thus the level E_3 found in the Bohr-Sommerfeld approximation is apparently spurious.

Now let us consider the semi-classical treatment of the isospin excitations around (separately) the Bohr-Sommerfeld bound state levels E_1 and E_2 . We again substitute (5.1), but this time with U_c given by the breather solution [(4.9) plus (5.5)], into the action. The work of section 5, in which U_c was also allowed to depend on time, shows that the analog of the collective Lagrangian (5.6) becomes

$$L_{coll}^{(n)} = +L_{class}^{(n)} + \frac{1}{2}\mathcal{I}_n(t)(\Omega_1^2 + \Omega_2^2). \quad (6.5)$$

Here, $L_{class}^{(n)}$ ($n = 1, 2$ for the present case) is the classical Lagrangian whose Legendre transform yields the levels E_n in (6.1). Note that the analog of the last term in (5.6) doesn't appear since (4.9) shows that $\chi_1(x, t = \pm\infty) = 0$. The remaining new feature is that the moment of inertia depends on time in a complicated way:

$$\mathcal{I}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \sin^2 \left\{ 4 \tan^{-1} \left[\frac{\eta \sin \omega t}{\cosh(\eta \omega x)} \right] \right\}, \quad (6.6)$$

where η is given in (4.9). In Appendix A we show that this may be integrated analytically to yield

$$\mathcal{I}(t) = \frac{4}{3\pi b} \frac{1}{(a^2 + 1)^{7/2}} \left[-(6a^5 - 6a^3 + 3a) \ln(\sqrt{a^2 + 1} - a) + (2a^6 - 4a^4 + 9a^2) \sqrt{a^2 + 1} \right], \quad (6.7)$$

where $a(t)$ and b are given in (4.9). (Note that the right-hand side actually is an even function of a .) Plots of $\mathcal{I}(t)$ for two particular choices of parameters are shown in Figs. 1 and 2. As expected, the plot looks roughly like a rectified sine curve for the E_1 soliton and flattens out into a double square wave as the energy increases.

We will treat (6.5) in an approximate way based on two assumptions. First, since the underlying classical motion is periodic, it seems natural to replace $\mathcal{I}_n(t)$ by its average over a period $2\pi/\omega_n$:

$$\overline{\mathcal{I}}_n = \frac{\omega_n}{2\pi} \int_0^{2\pi/\omega_n} dt \mathcal{I}(\sin \omega_n t, b). \quad (6.8)$$

This integral is calculated numerically for the appropriate values of ω_n . Secondly, in order to get the correct value isotopic spin=1 for the fundamental meson it is necessary to consider the collective quantization component of the isotopic spin \mathbf{I}^{coll} as an addition to the isotopic spin of the bound state solution itself \mathbf{I}^{bs} ($I = 1, I_3=0$ for the fundamental meson according to [2]):

$$\mathbf{I} = \mathbf{I}^{bs} + \mathbf{I}^{coll}. \quad (6.9)$$

Following section 5 we then obtain the collective Hamiltonian from (6.5) as

$$H_{coll}^{(n)} = E_n + \frac{1}{2\overline{\mathcal{I}}_n} (\mathbf{I}^{coll})^2. \quad (6.10)$$

In this case, unlike (5.12), there is no additional restriction on the allowed values of $(\mathbf{I}^{coll})^2$; the eigenvalue $(\mathbf{I}^{coll})^2 = 0$ is now acceptable.

From (6.9), using (4.9) and (6.2), it may be seen that $\overline{\mathcal{I}}_n$ scales as $1/\omega_n$. Then (6.10) yields a tower of energy levels for each BS quantized frequency, ω_n :

$$H_{coll}^{(n)} = m \left[\frac{16}{\sqrt{\pi}} \sin\left(\frac{n\pi}{8}\right) + \frac{\sqrt{\pi}}{\omega_n \overline{\mathcal{I}}_n} I_{coll} (I_{coll} + 1) \cos\left(\frac{n\pi}{8}\right) \right], \quad (6.11)$$

where, from the numerical integration of (6.8),

$$\omega_n \overline{\mathcal{I}}_n = \begin{cases} 0.742 & n = 1 \\ 0.658 & n = 2 \\ 0.328 & n = 3. \end{cases} \quad (6.12)$$

(For comparison, using the DHN frequencies would have given $\omega_1 \overline{\mathcal{I}}_1 = 0.796$ and $\omega_2 \overline{\mathcal{I}}_2 = 0.426$.) Let us now examine this spectrum. First consider the $n = 1$ tower. The first state has $I^{coll} = 0$ and mass $3.457m$. Using the assumption (6.9) and Coleman's identification $I^{bs}(1) = 1$ we get $I = 1$ for this state which is therefore presumed to be yet a third approximation to the lowest-lying meson state of the model ($I^{PG} = 1^{-+}$). The second level on the $n = 1$ tower has $I^{coll} = 1$ and mass $7.841m$. A state on this level is clearly massive

enough to decay to two fundamental mesons and would then be unstable. Similarly it is easy to check that all higher states are heavier than the sum of the preceding level mass and the fundamental meson mass. Now consider the $n = 2$ tower. The first state has $I^{coll} = 0$ and mass = $6.382m$. Again using (6.9) and Coleman's identification $I^{bs}(2) = 0$ we identify this as the $I^{PG} = 0^{++}$ meson. Its mass, according to this BS quantization is 1.846 times that of the fundamental meson. (If we had used the DHN quantization it would be $\sqrt{3}$ times as massive). The next level on the $n = 2$ tower has a mass = $10.190m$. It is clearly heavy enough to decay into $1^{-+} + 0^{++}$. Similarly all higher states of the $n = 2$ tower are massive enough to decay into the preceding one $+1^{-+}$. The $n = 3$ tower will be considered spurious.

Thus, it seems the present interpretation and approximation in the treatment of the non-Abelian bosonization can lead to the same stable particle spectrum as the presumed exact spectrum obtained by Coleman [2] in the Abelian bosonization approach using the results of the DHN analysis [20]. In particular, the numerical values of the averaged moments of inertia obtained are consistent with the instability of the higher levels on the towers in the non-Abelian approach. It would be interesting, however, to introduce additional "microscopic" coordinates associated with the soliton and anti-soliton components of the breather in order to verify the assumption (6.9) and to determine all the quantum numbers of the allowed states on the higher levels.

VII. ADDITIONAL DISCUSSION AND AN EFFECTIVE LAGRANGIAN

We have given a different treatment of the non-Abelian bosonized version of multiflavor QED_2 from that presented in [6]. The new features included are: i) starting by "gauging" the bosonized free theory (section 2), ii) using the manifestly symmetric form of the bosonized Lagrangian to emphasize the analogy to particle physics (section 3), iii) treating the collective quantization around the static soliton in closer analogy to the Skyrme model discussions in four-dimensional theories (section 5), and iv) a more detailed discussion of the collective quantization around the breather solutions (section 6).

It seems worthwhile to summarize the masses of the stable mesons obtained in the different approximations to the bosonized theory. Reading the mass term from the perturbative Lagrangian (3.3) (that is, neglecting interactions) yields $m(1^{-+}) = 3.545m$. Calculating the mass as the first level of the collective Hamiltonian (5.12) built around the static soliton solution yields $m(1^{-+}) = 8.690m$. Finally, the approximate treatment of the $n = 1$ breather Bohr-Sommerfeld level in (6.11) yields $m(1^{-+}) = 3.457m$ while the $n = 2$ level yields $m(0^{++}) \approx \sqrt{3}m(1^{-+})$. It is claimed [2] that the ratio $m(0^{++}) = \sqrt{3}m(1^{-+})$ is exact, while [16] there is no special reason for different approximation methods to yield especially close results.

Compared to the treatment of the model by Abelian bosonization, the non-Abelian bosonization is advantageous in getting a general understanding of the model as illustrated in the treatment of section 3. On the other hand, it seems fair to say that treatment of the solitons is more complicated in the non-Abelian approach. This is because the extra symmetry of the Lagrangian introduces extra zero modes, which require collective quantization in the non-Abelian case. The virtue of treating solitons in the non-Abelian approach is that the work may be used to illuminate some aspects of four-dimensional Skyrme model calculations. One example of interest is the study of unequal mass corrections for soliton bound states; this would be relevant in the bound state picture of strange and heavy baryons [18]. Another example concerns the possible relevance to the Skyrmion treatment of nucleon-anti-nucleon annihilation [21].

Both the abelian and non-abelian bosonizations yield exact representations of the fermionic theory. The non-abelian version has the advantage that the “charged pions” (π^\pm in the notation of section 3) are present in the Lagrangian to give manifest isospin invariance (in the two-flavor case). Now, both versions have the undesirable feature that the other stable particle in the theory - the $I^{PG} = 0^{++}$ particle which we now denote as σ - does *not* appear in the Lagrangian. In fact, it arises in a rather arcane manner. This raises the question of whether it is possible to find a different Lagrangian which also includes the σ . This should not necessarily be an exact representation of the theory but it should be a

good approximation in the “low-energy” region. We shall now see that such a Lagrangian can be found and furthermore gives a physical motivation for the basic mass relation

$$m(\sigma) = \sqrt{3}m(\pi). \quad (7.1)$$

We search for an effective Lagrangian which contains the low-lying particles and respects the underlying symmetries of the exact theory. It is desired to model the strong coupling regime of (1.1) taking, for simplicity here, the two-flavor case. The underlying symmetry of the massless theory is manifestly $U(2)_L \times U(2)_R$ which is, however, intrinsically broken to $SU(2)_L \times SU(2)_R \times U(1)_V$ by quantum corrections (the usual $U(1)_A$ anomaly). A common mass term for both flavors will further break the symmetry to $SU(2)_V$. Since we are in the strong coupling regime the η particle ($I^{PG} = 0^{--}$) is essentially decoupled from the low-energy theory, as discussed in section 3. Taking those facts into account it is clear that the *linear* $SU(2)$ sigma model [22] is a good candidate to describe low-energy two-flavor QED_2 . In this model the field multiplet contains just the π and σ fields as desired. We write the Lagrangian density as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - \frac{1}{2}(\partial_\mu \sigma)^2 - V, \quad (7.2)$$

$$V = A(\sigma^2 + \boldsymbol{\pi}^2 - \lambda)^2 - B\sigma, \quad (7.3)$$

where $A > 0$, B and λ are three real constants. The $B\sigma$ term manifestly breaks the $SU(2)_L \times SU(2)_R$ symmetry down to $SU(2)_V$ and represents the effect of the fermion mass terms. We shall assume that this model is valid for $B = 0$ as well as for small $B \neq 0$. We will work at tree level here.

To treat this model it is necessary to impose the extremum condition

$$\left\langle \frac{\partial V}{\partial \sigma} \right\rangle = 4A\langle \sigma \rangle (\langle \sigma \rangle^2 - \lambda) - B = 0, \quad (7.4)$$

where $\langle \pi_i \rangle = 0$ was taken to agree with parity or isospin invariance. We must also demand stability:

$$m_\sigma^2 \equiv \left\langle \frac{\partial^2 V}{\partial \sigma^2} \right\rangle = 4A(3\langle \sigma \rangle^2 - \lambda) \geq 0, \quad m_\pi^2 \equiv \left\langle \frac{\partial^2 V}{\partial \pi_3 \partial \pi_3} \right\rangle = 4A(\langle \sigma \rangle^2 - \lambda) \geq 0. \quad (7.5)$$

Now we can see what is special about $m(\sigma) = \sqrt{3}m(\pi)$; substituting this relation into (7.5) yields the result $\lambda = 0$. Consider the zero mass case $B = 0$. Eq. (7.4) shows there are two possible solutions ⁴ for the “condensate” $\langle \sigma \rangle$: $\langle \sigma \rangle = 0$ and $\langle \sigma \rangle = \lambda$. If λ vanishes we guarantee that $\langle \sigma \rangle = 0$. This is, in fact, what is required by the Mermin-Wagner theorem [9,10] which forbids, for spacetime dimension ≤ 2 , a non-zero condensate which spontaneously breaks a symmetry. Such an object would signify here the spontaneous breakdown of chiral $SU(2)$ to $SU(2)_V$ which is not allowed in two dimensions. (In the one-flavor case the $U(1)_A$ is already explicitly broken, so these considerations are not relevant.) The situation is very different from the usual four-dimensional σ model in which a condensate exists for $B = 0$ and is maintained as a small non-zero B is turned on. The particular mass relation (7.1) is seen to unambiguously force the unusual two-dimensional behavior.

When B is turned on, the condensate is determined from (7.4) as $\langle \sigma \rangle = (\frac{B}{4A})^{1/3}$. Furthermore $m_\pi^2 = m_\sigma^2/3 = 4A\langle \sigma \rangle^2$. The present formulation has the nice feature that it enables the simple calculation of meson scattering amplitudes which are expected to be accurate in the very low energy region. For this purpose we introduce the shifted field

$$\tilde{\sigma} = \sigma - \langle \sigma \rangle, \quad (7.6)$$

and rewrite the Lagrangian (7.2) as

$$\mathcal{L} = -\frac{1}{2}[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \tilde{\sigma})^2 + m_\pi^2(\boldsymbol{\pi}^2 + 3\tilde{\sigma}^2)] - g_3(\tilde{\sigma} \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \tilde{\sigma}^3) - g_4[(\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 + \tilde{\sigma}^4 + 2\boldsymbol{\pi}^2 \tilde{\sigma}^2], \quad (7.7)$$

where $g_3 = 4A\langle \sigma \rangle$ and $g_4 = A$. Using this Lagrangian we may compute the tree-level scattering amplitude for $\pi_i(p_1) + \pi_j(p_2) \rightarrow \pi_k(p'_1) + \pi_l(p'_2)$ as

⁴Note that the usual spontaneous breakdown situation in the four-dimensional sigma model corresponds to $\lambda > 0$ and $\langle \sigma \rangle = \lambda$. The $\langle \sigma \rangle = 0$ solution is seen from (7.5) to be unstable.

$$A(s, t, u)\delta_{ij}\delta_{kl} + A(t, s, u)\delta_{ik}\delta_{jl} + A(u, t, s)\delta_{il}\delta_{jk}, \quad (7.8)$$

where $s = -(p_1 + p_2)^2$, $t = -(p_1 - p'_1)^2$, and $u = -(p_1 - p'_2)^2$, with

$$A(s, t, u) = -\frac{m_\pi^2}{\langle\sigma\rangle^2} \left(\frac{2s - 5m_\pi^2}{s - 3m_\pi^2} \right). \quad (7.9)$$

The characteristic feature of this amplitude is the sigma pole below the threshold at $s_{th} = 4m_\pi^2$. This is in marked contrast to the four-dimensional case where the sigma mass is not restricted and in fact, the fairly accurate ‘‘current algebra theorem’’ [23] is obtained by taking $m_\sigma \rightarrow \infty$. Because the sigma pole lies so close to the threshold in the present case, we may reasonably expect it to dominate the low-energy amplitude. Loop corrections should become necessary as one goes away from the threshold region. The accuracy of the model itself away from the threshold region requires more investigation. Further work beyond these encouraging initial results, on the low-energy effective Lagrangian approach to multiflavor QED_2 will be reported elsewhere.

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APPENDIX A

Here we evaluate the moment of inertia integrals. For the static soliton case in (5.6),

$$\mathcal{I} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \sin^2 4\theta(x) = \frac{8}{\pi} \int_{-\infty}^{\infty} dx (\sin^2\theta - 5\sin^4\theta + 8\sin^6\theta - 4\sin^8\theta), \quad (A.1)$$

with $\sin^2\theta = 1/(1 + e^{-4\sqrt{\pi}mx})$, we make the substitution $y = e^{4\sqrt{\pi}mx}$ to obtain,

$$\mathcal{I} = \frac{2}{\pi^{3/2}m} \int_0^\infty dy \frac{(1-y)^2}{(1+y)^4} = \frac{2}{3\pi^{3/2}m}. \quad (A.2)$$

Next consider the time-dependent case in (6.6). A similar substitution to the one made above yields

$$\mathcal{I} = \frac{16}{\pi b} [a^2 I_1 - 5a^4 I_2 + 8a^6 I_3 - 4a^8 I_4], \quad (A.3)$$

where,

$$I_k(a) = \int_1^\infty \frac{dx}{(x^2 + a)^k \sqrt{x^2 - 1}}. \quad (\text{A.4})$$

From the indefinite integral

$$\int \frac{dx}{(x^2 + a^2)\sqrt{x^2 - 1}} = \frac{-1}{a\sqrt{a^2 + 1}} \ln \left[\frac{x\sqrt{a^2 + 1} - a\sqrt{x^2 - 1}}{\sqrt{x^2 + a^2}} \right], \quad (\text{A.5})$$

we find

$$I_1(a) = \frac{-\ln(\sqrt{a^2 + 1} - a)}{a\sqrt{a^2 + 1}}. \quad (\text{A.6})$$

The other I_k 's may be obtained by differentiating $I_1(a)$ as

$$I_2(a) = -\frac{I_1'}{2a}, \quad I_3(a) = -\frac{1}{8a^3}I_1' + \frac{1}{8a^2}I_1'', \quad I_4(a) = -\frac{1}{16a^5}I_1' + \frac{1}{16a^4}I_1'' - \frac{1}{48a^3}I_1''', \quad (\text{A.7})$$

where a prime indicates differentiation with respect to a . Putting everything together gives (6.7).

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Figure Captions

Fig. 1. Plot of $\mathcal{I}_1(t)$ for $m = 1$.

Fig. 2. Plot of $\mathcal{I}_2(t)$ for $m = 1$.