A Single-Field Finite-Difference Time-Domain Formulations for Electromagnetic Simulations

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ABSTRACT

In this dissertation, a set of general purpose single-field finite-difference time-domain updating equations for solving electromagnetic problems is derived. The formulation uses a single-field expression for full-wave solution. This formulation can provide numerical results similar to those obtained using the traditional formulation with less required computer resources.

Traditional finite-difference time-domain updating equations are based on Maxwell's curl equations whereas the single-field updating equations used here are based on the vector wave equation. General formulations are derived for normal and oblique incidence plane wave cases for linear, isotropic, homogeneous and non-dispersive as well as dispersive media.

To compare the single-field updating equations with the traditional ones, two-dimensional transverse magnetic, two-dimensional transverse electric and one-dimensional electromagnetic problems are solved. Fields generated by a current sheet and a filament electric current are calculated for one and two-dimensional formulations, respectively. Performance analyses of the single–field formulation in terms of CPU time, memory requirement, stability, dispersion, and accuracy are presented. Based on the simulations of several two-dimensional problems excited by a filament of electric current, it was observed that the single-field method is more efficient than the traditional one in terms of speed and memory requirements.

One scattering problem consisting of three infinitely long dielectric cylinders excited by an obliquely incident plane wave and another scattering problem consisting of a point source exciting a dispersive sphere, utilizing Lorentz-Drude model, are also formulated and analyzed. The numerical results obtained confirmed the validity and efficiency of the single–field formulations.

A SINGLE-FIELD FINITE-DIFFERENCE TIME-DOMAIN FORMULATION FOR ELECTROMAGNETIC SIMULATIONS

By

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DISSERTATION

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1. INTRODUCTION

1.1. Finite-Difference Time-Domain (FDTD) Method

The first paper on FDTD was published in 1966 by Kane Yee [1]. Though Yee was the first person to develop the algorithm, the term "finite-difference time-domain" and its acronym "FDTD" were coined by Allen Taflove in 1980 [2]. The correct numerical stability criterion for Yee's algorithm was determined by Taflove and Brodwin; they also reported the first sinusoidal steady-state FDTD solutions of two- and three-dimensional electromagnetic wave interactions with material structures [3]. In 1981, Mur published the first numerically stable, second-order accurate, absorbing boundary condition (ABC) for Yee's grid [4]. Thanks to the improvements in computational power, interest in FDTD solution of Maxwell's equations has increased almost exponentially since 1980s. FDTD is and will likely remain one of the dominant computational electrodynamics techniques due to its simple and versatile nature as well as its ability to utilize developments in computer hardware and software architecture.

Yee's insight was to choose a geometry for spatially sampling the electric and magnetic field vector components which robustly represents both the differential and integral forms of Maxwell's equations [2]. The technique divides the problem geometry into spatial grids as shown in Figure 1.1 where electric and magnetic field components are placed at certain discrete positions in space and it solves Maxwell's equations in time-domain at discrete time instances [5]. Yee used an electric field (E) grid, which was offset from the magnetic field (H) grid in time as well as space, to derive updating equations that are used to calculate present values of field by using the past values throughout the entire computational domain. The updating equations march E and H fields in a leap-frog fashion and move forward in time.

The starting point for the construction of the traditional FDTD algorithm is Maxwell's time-domain curl equations. The vector form of Maxwell's curl equations is decomposed into six scalar equations for three-dimensional space:

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma^e E_x - J_{i,x}) \tag{1.1}$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma^e E_y - J_{i,y}\right) \tag{1.2}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma^e E_z - J_{i,z}) \tag{1.3}$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \sigma^m H_x - M_{i,x}) \tag{1.4}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \sigma^m H_y - M_{i,y}) \tag{1.5}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \sigma^m H_z - M_{i,z}) \tag{1.6}$$

where E is the electric field strength in volts per meter, H is the magnetic field strength in amperes per meter, J_i is the impressed electric current density in amperes per square meter, M_i is the impressed magnetic current density in volts per square meter, ε is the permittivity ($\varepsilon_0 \approx 8.854 \times 10^{-12}$) in farads per meter and μ is the permeability ($\mu_0 = 4\pi \times 10^{-7}$) in henrys per meter.

The grid based nature of the method makes it memory-hog; therefore the FDTD problem domain must have the minimum size possible. The computational domain is bounded with proper absorbing boundary to prevent any reflection from the FDTD problem domain walls. Accuracy, speed and required memory of the simulation are directly related to the absorbing boundary used. Various ABCs have been studied to provide better reflection performance while minimizing the extra memory requirements for the absorbing boundaries [4,6,7].

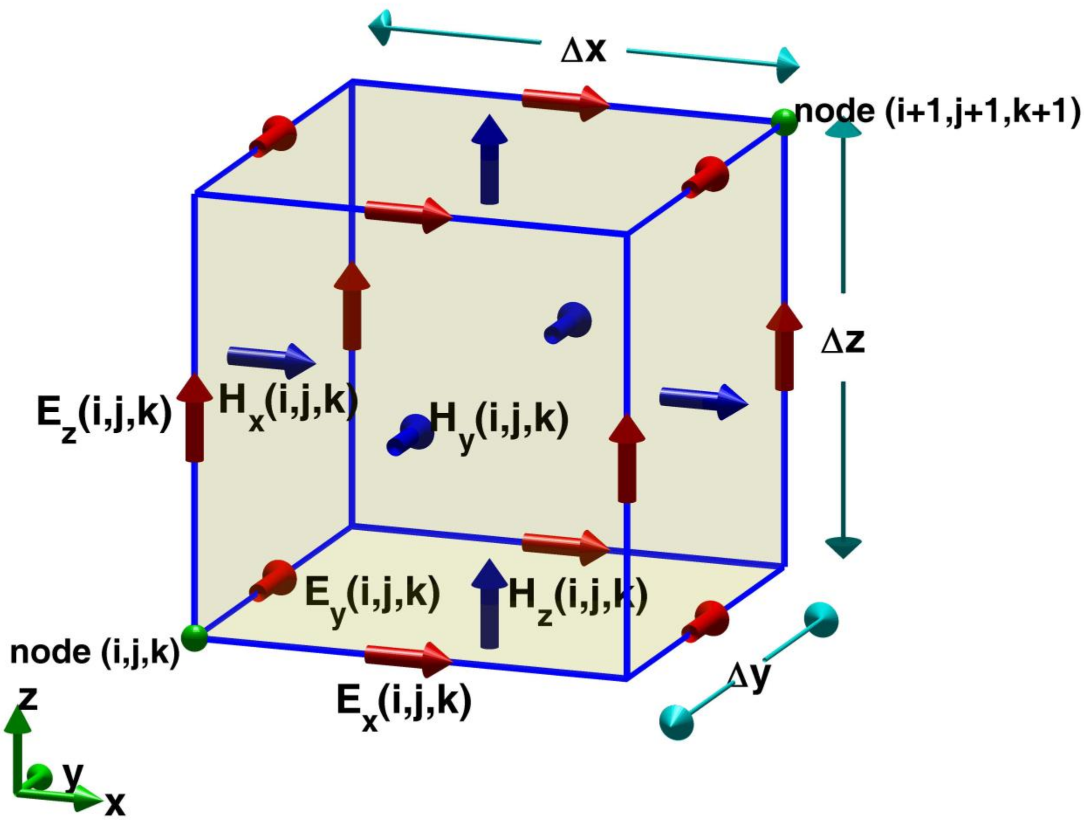

Fig. 1.1 The Yee-cell [5].

FDTD can be used for the simulation of numerous kinds of electromagnetic problems: microstrip circuits, waveguide structures, electromagnetic coupling and propagation. FDTD can also easily handle many complex structures which are either quite challenging or currently impossible to solve analytically or with other numerical methods.

FDTD gives field solutions for transient problems. Time-domain results can be easily transformed to frequency domain if necessary; therefore, with one single computation, simulation results can be obtained over a wide frequency range.

1.2. Simulation Challenges with FDTD Method

FDTD has some issues to be handled with care such as stability, numerical dispersion, long computational time and large memory requirement for big size problems.

The stability, hence the accuracy, of the solution is guaranteed by the choice of the sampling period in time and space i.e., Δt, Δx, Δy and Δz. The numerical stability of the FDTD method is determined by the Courant-Friedrichs-Lewy (CFL) condition, which simply requires that a wave cannot be allowed to travel more than one cell size in space during one time step [8]. The existence of instability exposes itself as the development of divergent spurious fields in the problem space as the FDTD iterations proceed [5].

Due to the discretized nature of the procedure, error is inevitably introduced into the solution because of the finite-difference approximation of derivatives of continuous functions. One of the consequences of this error is the difference between *c* and the velocity of propagation of the numerical solution for a wave even in homogenous free space. The difference between the phase velocity numerically obtained by the FDTD method and the exact phase velocity is known as numerical dispersion [5]. A general discussion of numerical stability and dispersion, including the derivation of two and three dimensional numerical dispersion relations, other factors affecting the numerical dispersion and strategies to reduce the associated errors, can be found in [2].

1.3. Motivation

Although the FDTD method is widely used in the field of computational electromagnetics, the long computational time and large memory requirement have always been a concern with the technique. Extensive research has been done to improve the accuracy and speed of the method and different ABCs are developed to provide more accurate results [4,6,7]. An improvement in speed of the method, however, has relied almost solely on progress in computer hardware and software architecture.

The equivalency of the vector wave equation to the Maxwell's curl equations, hence the existence of alternative formulations such as scalar and vector wave equations, has been known. Therefore, they are used as alternative formulations to be utilized with numerical techniques. Peterson et al. studied the scalar wave equation for the analysis of two-dimensional inhomogeneous dielectric bodies illuminated by normally-incident fields with the finite element method (FEM) [9, 10]; he also developed ABCs for the vector wave equation to be used with FEM [11]. Gedney and Navsariwala provided an unconditionally stable finite element time-domain solution of the vector wave equation [12]. There is not much published work investigating these formulations as a complete alternative to the traditional Yee algorithm; Aoyagi et al. investigated a possible combination of scalar and vector wave equations as well as a scalar wave equation and Maxwell's equations [13]; however both approaches lose generality since they require partitioning of the problem domain; Okoniewski discussed the application of the vector wave equation approach to an inhomogeneous wave-guide structure by using transverse field components [14]. Chu et al. studied the FDTD modeling of optical guided-wave devices based on the Yee algorithm and investigated the scalar wave equation and its semivectorial version for the simulation of optical guided-wave devices, but the vector nature of the electromagnetic waves is either completely or partially ignored [15-19].

This study investigates a general single-field approach to derive the FDTD updating equations in a way that only one field component will be calculated and updated inside the iteration loop to eliminate iteration steps required to update the other field components. The single-field FDTD is an effort at reduction of FDTD variables in a Yee grid to only the three components of a single field variable, either E or H, while maintaining the ability to analyze full vector source injection. Since one field (*E* or *H*) can be derived from the other field, whenever needed, the proposed method, hence, is able to provide simulation results that can be obtained from traditional FDTD updating equations. The paradigm presented proposes to use single-field for the simulation. However, it does not impose and is not limited to any particular field term (E or H), because each field formulation has advantage over the others for some set of problems. For example, the electric field formulation (E formulation) is the most advantageous for two-dimensional transverse magnetic (2D TM) problems whereas the magnetic field formulation (H formulation) is a better choice for two-dimensional transverse electric (2D TE) problems. From software point of view, however, this does not imply that the lines of code and memory allocations will be doubled for the proposed technique. Due to the symmetry in Maxwell's equations and dual relations between the field and the source terms, one set of equations (E or H) can be transformed into a general purpose simulation software that can handle either case (TM or TE) with advantage in speed and memory. The described technique has the ability to analyze a large class of two-dimensional structures including arbitrary material types (perfect conductors, lossy dielectric and magnetic material types). In addition to the arbitrary material types, various illumination sources (plane wave and arbitrarily positioned electric and magnetic sources) can also be handled.

The significance of the improvement in speed and memory usage of the two-dimensional formulations is that there are many interesting geometries that lend themselves to a two-dimensional analysis where bodies are assumed to be infinite in the longitudinal dimension. This includes (*i*) structures that are long in one dimension and are, thus, naturally two-dimensional problems, (*ii*) three-dimensional problems in which significant insight into the physics of the problem can be gained by a two-dimensional analysis, and (*iii*) structures that can be described by their E-plane and H-plane patterns [20].

In this dissertation, *a single-field finite-difference time-domain formulation for electromagnetic simulations* is derived and its accuracy and performance are investigated with various problems. The main focus is on solving two-dimensional electromagnetic problems with the presented formulation as the single-field formulation shows obvious advantage over the traditional one for two-dimensional problems.

In Chapter 2, single-field finite-difference time-domain updating equations are derived for linear, homogeneous, isotropic and nondispersive media. In Chapter 3, one-dimensional updating equations are derived and their performance is analyzed with an example. Chapter 4 presents the derivation and various analyses e.g., accuracy, speed, etc. for the two-dimensional single-field formulation. Chapter 5 has a detailed investigation of the single-field formulation for electromagnetic problems in the case of oblique plane wave incidence, and Chapter 6 includes the derivation and the numerical validation of single-field FDTD updating equations for dispersive media. Chapter 7 concludes the dissertation.

2. SINGLE-FIELD FDTD UPDATING EQUATIONS

Characteristic behavior of electromagnetic fields can be specified by constructing an FDTD algorithm of Maxwell's time-domain equations. For linear, isotropic and nondispersive media; starting with Maxwell's curl equations, one can obtain a vector wave equation and solve it as scalar equations for its Cartesian coordinates.

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} - (\boldsymbol{M}_i + \sigma^m \boldsymbol{H}) \tag{2.1}$$

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{J}_i + \sigma^e \boldsymbol{E}) \tag{2.2}$$

Taking the curl of (2.1), we have:

$$\nabla \times \left(\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} - (\boldsymbol{M}_i + \sigma^m \boldsymbol{H})\right) \tag{2.3}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \nabla \times \frac{\partial \boldsymbol{H}}{\partial t} - \nabla \times \boldsymbol{M}_i - \sigma^m (\nabla \times \boldsymbol{H}) \tag{2.4}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}) - \nabla \times \boldsymbol{M}_i - \sigma^m (\nabla \times \boldsymbol{H}) \tag{2.5}$$

Substituting (2.2) into (2.5)

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{J}_i + \sigma^e \boldsymbol{E})\right) - \nabla \times \boldsymbol{M}_i - \sigma^m \left(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{J}_i + \sigma^e \boldsymbol{E})\right) \tag{2.6}$$

Using the following vector identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \tag{2.7}$$

Rearranging the terms in (2.6), one can obtain a vector wave equation as

$$\begin{aligned} &\nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) \\ &\quad = \sigma^m \sigma^e \boldsymbol{E} + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial \boldsymbol{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \nabla \times \boldsymbol{M}_i + \sigma^m \boldsymbol{J}_i + \mu\frac{\partial \boldsymbol{J}_i}{\partial t} \end{aligned} \tag{2.8}$$

To solve (2.8) with finite-difference time-domain method, we have to decompose it into its Cartesian components. The following three sections will detail the FDTD single-field formulation for a three dimensional computational domain.

2.1. Derivation of the Updating Equation for the x Component

Cartesian component of (2.8) in x direction can be written as

$$\begin{aligned} &\nabla^2 E_x - (\nabla(\nabla \cdot \boldsymbol{E}))_x \\ &\quad = \sigma^m \sigma^e E_x + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_x}{\partial t} + \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} + \left(\frac{\partial M_{i,z}}{\partial y} - \frac{\partial M_{i,y}}{\partial z}\right) \\ &\quad + \sigma^m J_{i,x} + \mu\frac{\partial J_{i,x}}{\partial t} \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial x \partial y} + \frac{\partial^2 E_z}{\partial x \partial z}\right) \\ &\quad = \sigma^m \sigma^e E_x + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_x}{\partial t} + \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} + \left(\frac{\partial M_{i,z}}{\partial y} - \frac{\partial M_{i,y}}{\partial z}\right) \\ &\quad + \sigma^m J_{i,x} + \mu\frac{\partial J_{i,x}}{\partial t} \end{aligned} \tag{2.10}$$

$$\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}-\frac{\partial^2 E_y}{\partial x\partial y}-\frac{\partial^2 E_z}{\partial x\partial z}$$
$$=\sigma^m\sigma^e E_x+(\mu\sigma^e+\varepsilon\sigma^m)\frac{\partial E_x}{\partial t}+\mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}+\left(\frac{\partial M_{i,z}}{\partial y}-\frac{\partial M_{i,y}}{\partial z}\right) \quad (2.11)$$
$$+\sigma^m J_{i,x}+\mu\frac{\partial J_{i,x}}{\partial t}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (2.11) at the corresponding electric field node, i.e. E_x. The time derivatives are evaluated at the nth time step; therefore:

$$\frac{\partial E_x(i,j,k)}{\partial t}=\frac{E_x^{n+1}(i,j,k)-E_x^{n-1}(i,j,k)}{2\Delta t} \quad (2.12)$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial t^2}=\frac{E_x^{n+1}(i,j,k)-2E_x^{n}(i,j,k)+E_x^{n-1}(i,j,k)}{(\Delta t)^2} \quad (2.13)$$

$$\frac{\partial J_{i,x}(i,j,k)}{\partial t}=\frac{J_{i,x}^{n+1}(i,j,k)-J_{i,x}^{n-1}(i,j,k)}{2\Delta t} \quad (2.14)$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial y^2}=\frac{E_x^{n}(i,j+1,k)-2E_x^{n}(i,j,k)+E_x^{n}(i,j-1,k)}{(\Delta y)^2} \quad (2.15)$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial z^2}=\frac{E_x^{n}(i,j,k+1)-2E_x^{n}(i,j,k)+E_x^{n}(i,j,k-1)}{(\Delta z)^2} \quad (2.16)$$

For the other electric field components, we have to consider their positions in the Yee-cell [1] as shown in Figures 2.1 and 2.2.

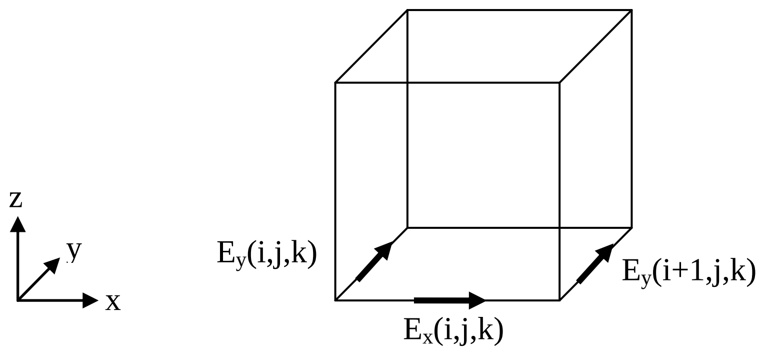

Fig. 2.1 The positions of E_y field components with respect to E_x in the Yee-cell.

$$\frac{\partial E_y(i,j,k)}{\partial y} = \frac{E_y^n(i,j,k) - E_y^n(i,j-1,k)}{\Delta y} \tag{2.17}$$

$$\frac{\partial E_y(i,j,k)}{\partial x} = \frac{E_y^n(i+1,j,k) - E_y^n(i,j,k)}{\Delta x} \tag{2.18}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial x \partial y} = \frac{E_y^n(i+1,j,k) - E_y^n(i+1,j-1,k) - E_y^n(i,j,k) + E_y^n(i,j-1,k)}{\Delta x \Delta y} \tag{2.19}$$

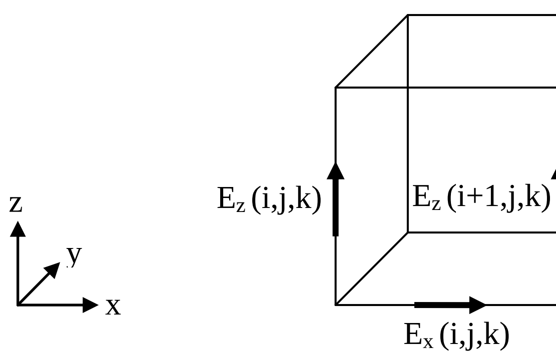

Fig. 2.2 The positions of E_z field components with respect to E_x in the Yee-cell.

$$\frac{\partial E_z(i,j,k)}{\partial x} = \frac{E_z^n(i+1,j,k) - E_z^n(i,j,k)}{\Delta x} \tag{2.20}$$

$$\frac{\partial E_z(i,j,k)}{\partial z} = \frac{E_z^n(i,j,k) - E_z^n(i,j,k-1)}{\Delta z} \tag{2.21}$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial x \partial z} = \frac{E_z^n(i+1,j,k) - E_z^n(i+1,j,k-1) - E_z^n(i,j,k) + E_z^n(i,j,k-1)}{\Delta x \Delta z} \quad (2.22)$$

The spatial derivatives of the magnetic sources are determined according to their positions in the Yee-cell as shown in Figures 2.3 and 2.4.

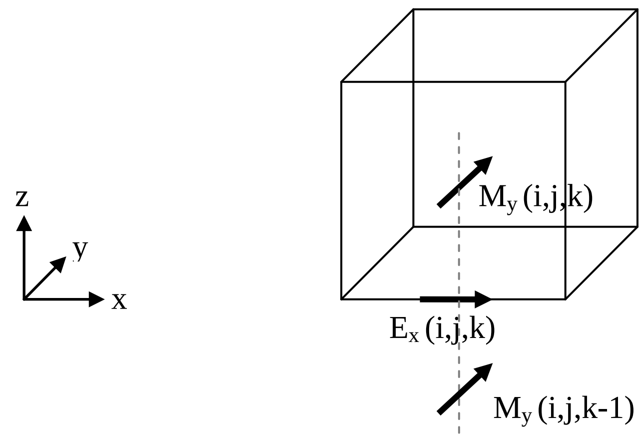

Fig. 2.3 The positions of M_y with respect to E_x in the Yee-cell.

$$\frac{\partial M_{i,y}(i,j,k)}{\partial z} = \frac{M_{i,y}^n(i,j,k) - M_{i,y}^n(i,j,k-1)}{\Delta z} \quad (2.23)$$

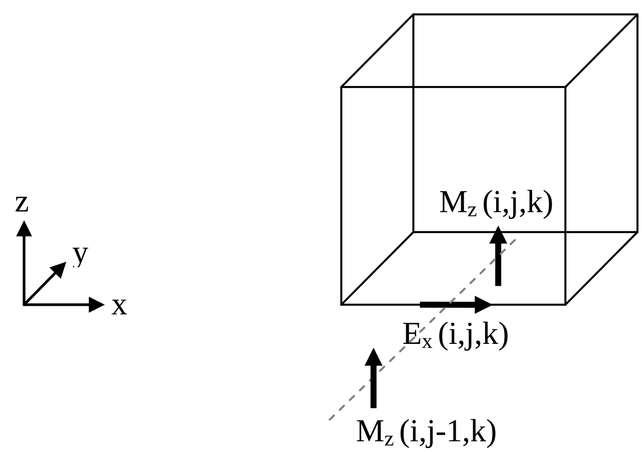

Fig. 2.4 The positions of M_z with respect to E_x in the Yee-cell.

$$\frac{\partial M_{i,z}(i,j,k)}{\partial y} = \frac{M_{i,z}^{n}(i,j,k) - M_{i,z}^{n}(i,j-1,k)}{\Delta y} \tag{2.24}$$

Inserting (2.12)-(2.24) into (2.11), we have

$$\begin{aligned}
&\frac{E_x^n(i,j+1,k) - 2E_x^n(i,j,k) + E_x^n(i,j-1,k)}{(\Delta y)^2} \\
&+ \frac{E_x^n(i,j,k+1) - 2E_x^n(i,j,k) + E_x^n(i,j,k-1)}{(\Delta z)^2} \\
&- \frac{E_y^n(i+1,j,k) - E_y^n(i+1,j-1,k) - E_y^n(i,j,k) + E_y^n(i,j-1,k)}{\Delta x \Delta y} \\
&- \frac{E_z^n(i+1,j,k) - E_z^n(i+1,j,k-1) - E_z^n(i,j,k) + E_z^n(i,j,k-1)}{\Delta x \Delta z} \\
&= \sigma^m \sigma^e E_x^n(i,j,k) + (\mu\sigma^e + \varepsilon\sigma^m)\frac{E_x^{n+1}(i,j,k) - E_x^{n-1}(i,j,k)}{2\Delta t} \\
&+ \mu\varepsilon \frac{E_x^{n+1}(i,j,k) - 2E_x^n(i,j,k) + E_x^{n-1}(i,j,k)}{(\Delta t)^2} \\
&+ \left(\frac{M_{i,z}^n(i,j,k) - M_{i,z}^n(i,j-1,k)}{\Delta y} - \frac{M_{i,y}^n(i,j,k) - M_{i,y}^n(i,j,k-1)}{\Delta z}\right) \\
&+ \sigma^m J_{i,x}^n(i,j,k) + \mu \frac{J_{i,x}^{n+1}(i,j,k) - J_{i,x}^{n-1}(i,j,k)}{2\Delta t}
\end{aligned} \tag{2.25}$$

We can simplify the updating equation for E_x as

$$\begin{aligned} E_x^{n+1}(i,j,k) = {} & C_{ex}^{ex,n}(i,j,k)[E_x^n(i,j,k)] \;\; + C_{ex}^{ex,n-1}(i,j,k)[E_x^{n-1}(i,j,k)] \\ & + C_{ex}^{ex,n,y}(i,j,k)[E_x^n(i,j+1,k) + E_x^n(i,j-1,k)] \\ & + C_{ex}^{ex,n,z}(i,j,k)[E_x^n(i,j,k+1) + E_x^n(i,j,k-1)] \\ & + C_{ex}^{ey,n,xy}(i,j,k)\big[E_y^n(i+1,j,k) - E_y^n(i+1,j-1,k) - E_y^n(i,j,k) \\ & + E_y^n(i,j-1,k)] \\ & + C_{ex}^{ez,n,xz}(i,j,k)[E_z^n(i+1,j,k) - E_z^n(i+1,j,k-1) - E_z^n(i,j,k) \\ & + E_z^n(i,j,k-1)] + C_{ex}^{mz,n,y}(i,j,k)[M_z^n(i,j,k) - M_z^n(i,j-1,k)] \\ & + C_{ex}^{my,n,z}(i,j,k)\big[M_y^n(i,j,k) - M_y^n(i,j,k-1)\big] \\ & + C_{ex}^{jx,n}(i,j,k)\big[J_{i,x}^n(i,j,k)\big] \\ & + C_{ex}^{jx,t}(i,j,k)\big[J_{i,x}^{n+1}(i,j,k) - J_{i,x}^{n-1}(i,j,k)\big] \end{aligned} \tag{2.26}$$

where

$$C_x(i,j,k) = -\frac{2\Delta t^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{2.27}$$

$$C_{ex}^{ex,n}(i,j,k) = C_x(i,j,k)\left(\frac{2}{(\Delta y)^2} + \frac{2}{(\Delta z)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{2.28}$$

$$C_{ex}^{ex,n-1}(i,j,k) = C_x(i,j,k)\left(\frac{\mu\varepsilon}{\Delta t^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{2.29}$$

$$C_{ex}^{ex,n,y}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{2.30}$$

$$C_{ex}^{ex,n,z}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{2.31}$$

$$C_{ex}^{ey,n,xy}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{2.32}$$

$$C_{ex}^{ez,n,xz}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta x\Delta z}\right) \tag{2.33}$$

$$C_{ex}^{mz,n,y}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta y}\right) \tag{2.34}$$

$$C_{ex}^{my,n,z}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta z}\right) \tag{2.35}$$

$$C_{ex}^{jx,n}(i,j,k) = C_x(i,j,k)(\sigma^m) \tag{2.36}$$

$$C_{ex}^{jx,t}(i,j,k) = C_x(i,j,k)\left(\frac{\mu}{2\Delta t}\right) \tag{2.37}$$

2.2. Derivation of the Updating Equation for the y Component

Cartesian component of (2.8) in y direction can be written as

$$\begin{aligned}\nabla^2 E_y - \left(\nabla(\nabla\cdot\boldsymbol{E})\right)_y \\ &= \sigma^m\sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} + \left(\frac{\partial M_{i,x}}{\partial z} - \frac{\partial M_{i,z}}{\partial x}\right) \\ &+ \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{2.38}$$

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} - \left(\frac{\partial^2 E_x}{\partial x\partial y} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\partial z}\right) \\ &= \sigma^m\sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} + \left(\frac{\partial M_{i,x}}{\partial z} - \frac{\partial M_{i,z}}{\partial x}\right) \\ &+ \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{2.39}$$

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{\partial^2 E_x}{\partial x\partial y} - \frac{\partial^2 E_z}{\partial y\partial z} \\ &= \sigma^m\sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} + \left(\frac{\partial M_{i,x}}{\partial z} - \frac{\partial M_{i,z}}{\partial x}\right) \\ &+ \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{2.40}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (2.40) at the corresponding electric field node, i.e. E_y. The time derivatives are evaluated at the n^{th} time step; therefore:

$$\frac{\partial E_y(i,j,k)}{\partial t} = \frac{E_y^{n+1}(i,j,k) - E_y^{n-1}(i,j,k)}{2\Delta t} \tag{2.41}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial t^2} = \frac{E_y^{n+1}(i,j,k) - 2E_y^{n}(i,j,k) + E_y^{n-1}(i,j,k)}{(\Delta t)^2} \tag{2.42}$$

$$\frac{\partial J_{i,y}(i,j,k)}{\partial t} = \frac{J_{i,y}^{n+1}(i,j,k) - J_{i,y}^{n-1}(i,j,k)}{2\Delta t} \tag{2.43}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial x^2} = \frac{E_y^{n}(i+1,j,k) - 2E_y^{n}(i,j,k) + E_y^{n}(i-1,j,k)}{(\Delta x)^2} \tag{2.44}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial z^2} = \frac{E_y^{n}(i,j,k+1) - 2E_y^{n}(i,j,k) + E_y^{n}(i,j,k-1)}{(\Delta z)^2} \tag{2.45}$$

For the other electric field components we have to consider their positions in the Yee-cell as shown in Figures 2.5 and 2.6.

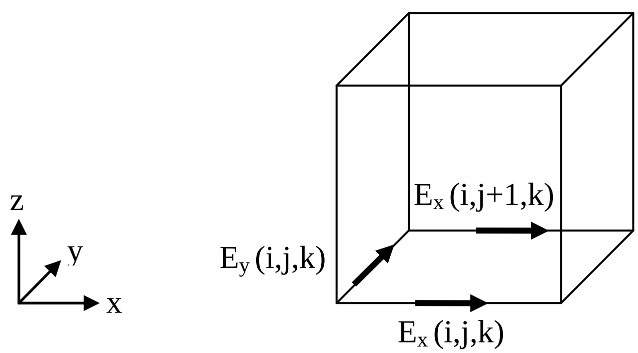

Fig. 2.5 The positions of E_x field components with respect to E_y in the Yee-cell.

$$\frac{\partial E_x(i,j,k)}{\partial x} = \frac{E_x^{n}(i,j,k) - E_x^{n}(i-1,j,k)}{\Delta x} \tag{2.46}$$

$$\frac{\partial E_x(i,j,k)}{\partial y} = \frac{E_x^{n}(i,j+1,k) - E_x^{n}(i,j,k)}{\Delta y} \tag{2.47}$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial x \partial y} = \frac{E_x^n(i,j+1,k) - E_x^n(i-1,j+1,k) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta y} \quad (2.48)$$

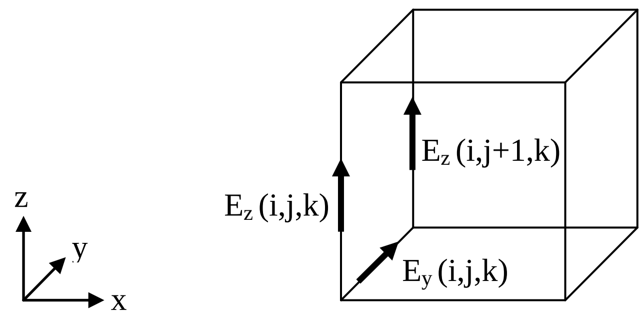

Fig. 2.6 The positions of E_z field components with respect to E_y in the Yee-cell.

$$\frac{\partial E_z(i,j,k)}{\partial y} = \frac{E_z^n(i,j+1,k) - E_z^n(i,j,k)}{\Delta y} \quad (2.49)$$

$$\frac{\partial E_z(i,j,k)}{\partial z} = \frac{E_z^n(i,j,k) - E_z^n(i,j,k-1)}{\Delta z} \quad (2.50)$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial z \partial y} = \frac{E_z^n(i,j+1,k) - E_z^n(i,j+1,k-1) - E_z^n(i,j,k) + E_z^n(i,j,k-1)}{\Delta y \Delta z} \quad (2.51)$$

The spatial derivatives of the magnetic sources are determined according to their positions in the Yee cell as shown in Figures 2.7 and 2.8.

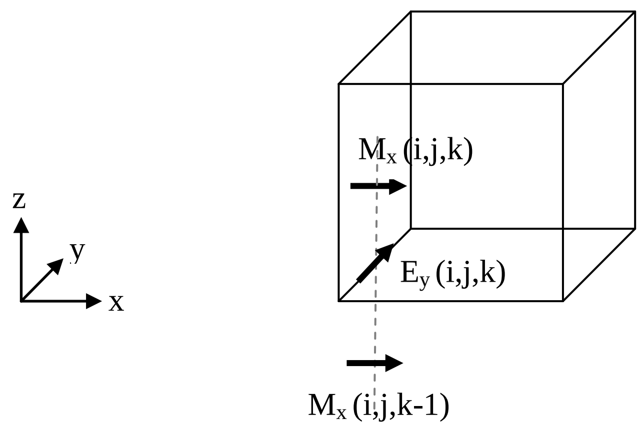

Fig. 2.7 The positions of M_x with respect to E_y in the Yee-cell.

$$\frac{\partial M_{i,x}(i,j,k)}{\partial z} = \frac{M_{i,x}^{n}(i,j,k) - M_{i,x}^{n}(i,j,k-1)}{\Delta z} \tag{2.52}$$

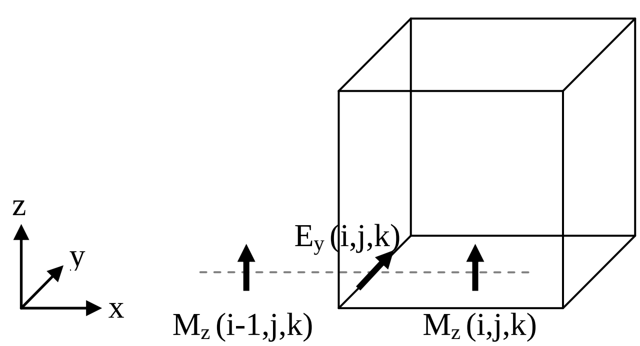

Fig. 2.8 The positions of M_z with respect to E_y in the Yee-cell.

$$\frac{\partial M_{i,z}(i,j,k)}{\partial x} = \frac{M_{i,z}^{n}(i,j,k) - M_{i,z}^{n}(i-1,j,k)}{\Delta x} \tag{2.53}$$

Inserting (2.41)-(2.53) into (2.40), we have

$$
\begin{aligned}
&\frac{E_y^n(i+1,j,k)-2E_y^n(i,j,k)+E_y^n(i-1,j,k)}{(\Delta x)^2} \\
&+\frac{E_y^n(i,j,k+1)-2E_y^n(i,j,k)+E_y^n(i,j,k-1)}{(\Delta z)^2} \\
&-\frac{E_x^n(i,j+1,k)-E_x^n(i-1,j+1,k)-E_x^n(i,j,k)+E_x^n(i-1,j,k)}{\Delta x\Delta y} \\
&-\frac{E_z^n(i,j+1,k)-E_z^n(i,j+1,k-1)-E_z^n(i,j,k)+E_z^n(i,j,k-1)}{\Delta y\Delta z} \\
&=\sigma^m\sigma^e E_y^n(i,j,k)+(\mu\sigma^e+\varepsilon\sigma^m)\frac{E_y^{n+1}(i,j,k)-E_y^{n-1}(i,j,k)}{2\Delta t} \\
&+\mu\varepsilon\frac{E_y^{n+1}(i,j,k)-2E_y^n(i,j,k)+E_y^{n-1}(i,j,k)}{(\Delta t)^2} \\
&+\left(\frac{M_{i,x}^n(i,j,k)-M_{i,x}^n(i,j,k-1)}{\Delta z}-\frac{M_{i,z}^n(i,j,k)-M_{i,z}^n(i-1,j,k)}{\Delta x}\right) \\
&+\sigma^m J_{i,y}^n(i,j,k)+\mu\frac{J_{i,y}^{n+1}(i,j,k)-J_{i,y}^{n-1}(i,j,k)}{2\Delta t}
\end{aligned}
\tag{2.54}
$$

We can simplify the updating equation for E_y as

$$
\begin{aligned}
E_y^{n+1}(i,j,k)=\;&C_{ey}^{ey,n}(i,j,k)\left[E_y^n(i,j,k)\right]+C_{ey}^{ey,n-1}(i,j,k)\left[E_y^{n-1}(i,j,k)\right] \\
&+C_{ey}^{ey,n,x}(i,j,k)\left[E_y^n(i+1,j,k)+E_y^n(i-1,j,k)\right] \\
&+C_{ey}^{ey,n,z}(i,j,k)\left[E_y^n(i,j,k+1)+E_y^n(i,j,k-1)\right] \\
&+C_{ey}^{ex,n,xy}(i,j,k)[E_x^n(i,j+1,k)-E_x^n(i-1,j+1,k)-E_x^n(i,j,k) \\
&+E_x^n(i-1,j,k)] \\
&+C_{ey}^{ez,n,yz}(i,j,k)[E_z^n(i,j+1,k)-E_z^n(i,j+1,k-1)-E_z^n(i,j,k) \\
&+E_z^n(i,j,k-1)]+C_{ey}^{mx,n,z}(i,j,k)\left[M_{i,x}^n(i,j,k)-M_{i,x}^n(i,j,k-1)\right] \\
&+C_{ey}^{mz,n,x}(i,j,k)\left[M_{i,z}^n(i,j,k)-M_{i,z}^n(i-1,j,k)\right] \\
&+C_{ey}^{jy,n}(i,j,k)\left[J_{i,y}^n(i,j,k)\right] \\
&+C_{ey}^{jy,t}(i,j,k)\left[J_{i,y}^{n+1}(i,j,k)-J_{i,y}^{n-1}(i,j,k)\right]
\end{aligned}
\tag{2.55}
$$

where

$$C_y(i,j,k) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{2.56}$$

$$C_{ey}^{ey,n}(i,j,k) = C_y(i,j,k)\left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta z)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{2.57}$$

$$C_{ey}^{ey,n-1}(i,j,k) = C_y(i,j,k)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{2.58}$$

$$C_{ey}^{ey,n,x}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{2.59}$$

$$C_{ey}^{ey,n,z}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{2.60}$$

$$C_{ey}^{ex,n,xy}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{2.61}$$

$$C_{ey}^{ez,n,yz}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{2.62}$$

$$C_{ey}^{mx,n,z}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta z}\right) \tag{2.63}$$

$$C_{ey}^{mz,n,x}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta x}\right) \tag{2.64}$$

$$C_{ey}^{jy,n}(i,j,k) = C_y(i,j,k)(\sigma^m) \tag{2.65}$$

$$C_{ey}^{jy,t}(i,j,k) = C_y(i,j,k)\left(\frac{\mu}{2\Delta t}\right) \tag{2.66}$$

2.3. Derivation of the Updating Equation for the z Component

Cartesian component of (2.8) in z direction can be written as

$$\nabla^2 E_z - \left(\nabla(\nabla\cdot\boldsymbol{E})\right)_z = \sigma^m\sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \left(\frac{\partial M_{i,y}}{\partial x} - \frac{\partial M_{i,x}}{\partial y}\right) + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \quad (2.67)$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} - \left(\frac{\partial^2 E_x}{\partial x\partial z} + \frac{\partial^2 E_y}{\partial y\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right) = \sigma^m\sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \left(\frac{\partial M_{i,y}}{\partial x} - \frac{\partial M_{i,x}}{\partial y}\right) + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \quad (2.68)$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_x}{\partial x\partial z} - \frac{\partial^2 E_y}{\partial y\partial z} = \sigma^m\sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \left(\frac{\partial M_{i,y}}{\partial x} - \frac{\partial M_{i,x}}{\partial y}\right) + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \quad (2.69)$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equations (2.69) at the corresponding electric field node, i.e. E_z. The time derivatives are evaluated at the nth time step; therefore:

$$\frac{\partial E_z(i,j,k)}{\partial t} = \frac{E_z^{n+1}(i,j,k) - E_z^{n-1}(i,j,k)}{2\Delta t} \quad (2.70)$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial t^2} = \frac{E_z^{n+1}(i,j,k) - 2E_z^{n}(i,j,k) + E_z^{n-1}(i,j,k)}{(\Delta t)^2} \quad (2.71)$$

$$\frac{\partial J_{i,z}(i,j,k)}{\partial t} = \frac{J_{i,z}^{n+1}(i,j,k) - J_{i,z}^{n-1}(i,j,k)}{2\Delta t} \quad (2.72)$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial x^2} = \frac{E_z^{n}(i+1,j,k) - 2E_z^{n}(i,j,k) + E_z^{n}(i-1,j,k)}{(\Delta x)^2} \quad (2.73)$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial y^2} = \frac{E_z^n(i,j+1,k) - 2E_z^n(i,j,k) + E_z^n(i,j-1,k)}{(\Delta y)^2} \tag{2.74}$$

For the other electric field components, we have to consider their positions in the Yee-cell as shown in Figures 2.9 and 2.10.

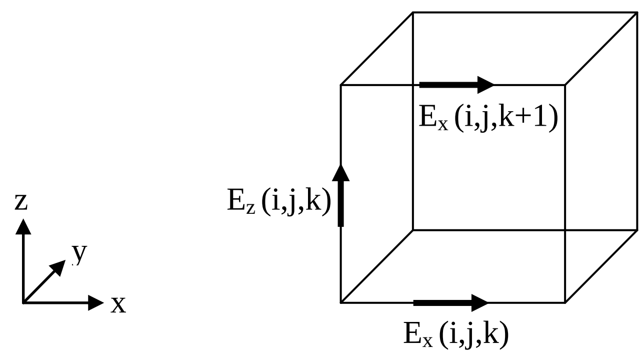

Fig. 2.9 The positions of E_x field components with respect to E_z in the Yee-cell.

$$\frac{\partial E_x(i,j,k)}{\partial z} = \frac{E_x^n(i,j,k+1) - E_x^n(i,j,k)}{\Delta z} \tag{2.75}$$

$$\frac{\partial E_x(i,j,k)}{\partial x} = \frac{E_x^n(i,j,k) - E_x^n(i-1,j,k)}{\Delta x} \tag{2.76}$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial x \partial z} = \frac{E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta z} \tag{2.77}$$

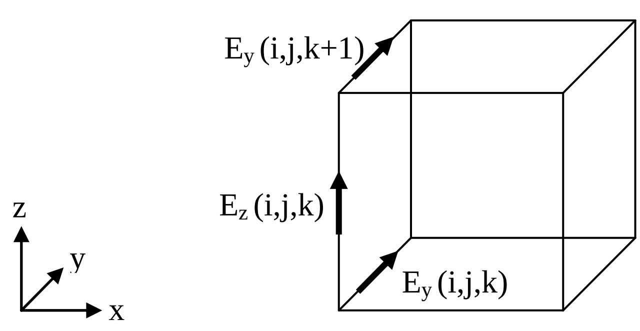

Fig. 2.10 The positions of E_y field components with respect to E_z in the Yee-cell.

$$\frac{\partial E_y(i,j,k)}{\partial y} = \frac{E_y^n(i,j,k) - E_y^n(i,j-1,k)}{\Delta y} \tag{2.78}$$

$$\frac{\partial E_y(i,j,k)}{\partial z} = \frac{E_y^n(i,j,k+1) - E_y^n(i,j,k)}{\Delta z} \tag{2.79}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial y \partial z} = \frac{E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1) + E_y^n(i,j-1,k)}{\Delta y \Delta z} \tag{2.80}$$

The spatial derivatives of the magnetic sources are determined according to their positions in the Yee cell as shown in Figures 2.11 and 2.12.

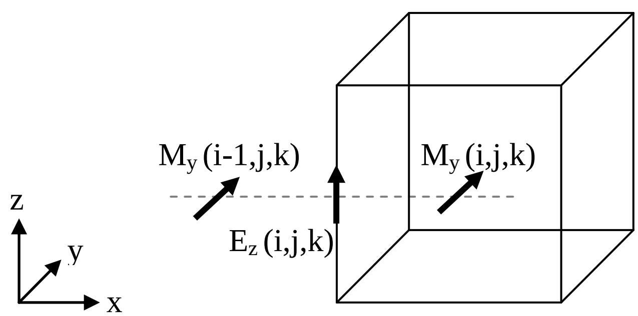

Fig. 2.11 The positions of M_y with respect to E_z in the Yee-cell.

$$\frac{\partial M_{i,y}(i,j,k)}{\partial x} = \frac{M_{i,y}^n(i,j,k) - M_{i,y}^n(i-1,j,k)}{\Delta x} \tag{2.81}$$

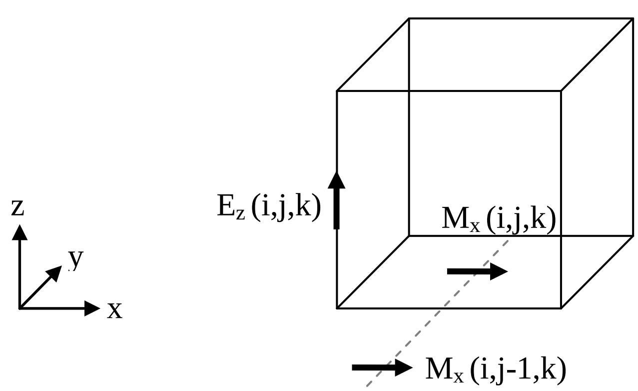

Fig. 2.12 The positions of M_x with respect to E_z in the Yee-cell.

$$\frac{\partial M_{i,x}(i,j,k)}{\partial y} = \frac{M_{i,x}^{n}(i,j,k) - M_{i,x}^{n}(i,j-1,k)}{\Delta y} \tag{2.82}$$

Inserting (2.70)-(2.82) into (2.69), we have

$$\begin{aligned}
&\frac{E_z^n(i+1,j,k) - 2E_z^n(i,j,k) + E_z^n(i-1,j,k)}{(\Delta x)^2} \\
&+ \frac{E_z^n(i,j+1,k) - 2E_z^n(i,j,k) + E_z^n(i,j-1,k)}{(\Delta y)^2} \\
&- \frac{E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta z} \\
&- \frac{E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1) + E_y^n(i,j-1,k)}{\Delta y \Delta z} \\
&= \sigma^m \sigma^e E_z^n(i,j,k) + (\mu\sigma^e + \varepsilon\sigma^m) \frac{E_z^{n+1}(i,j,k) - E_z^{n-1}(i,j,k)}{2\Delta t} \\
&+ \mu\varepsilon \frac{E_z^{n+1}(i,j,k) - 2E_z^n(i,j,k) + E_z^{n-1}(i,j,k)}{(\Delta t)^2} + \frac{M_{i,y}^n(i,j,k) - M_{i,y}^n(i-1,j,k)}{\Delta x} \\
&- \frac{M_{i,x}^n(i,j,k) - M_{i,x}^n(i,j-1,k)}{\Delta y} + \sigma^m J_{i,z}^n(i,j,k) + \mu \frac{J_{i,z}^{n+1}(i,j,k) - J_{i,z}^{n-1}(i,j,k)}{2\Delta t}
\end{aligned} \tag{2.83}$$

We can simplify the updating equation for E_z as

$$\begin{aligned}
E_z^{n+1}(i,j,k) = {} & C_{ez}^{ez,n}(i,j,k)[E_z^n(i,j,k)] + C_{ez}^{ez,n-1}(i,j,k)[E_z^{n-1}(i,j,k)] \\
&+ C_{ez}^{ez,n,x}(i,j,k)[E_z^n(i+1,j,k) + E_z^n(i-1,j,k)] \\
&+ C_{ez}^{ez,n,y}(i,j,k)[E_z^n(i,j+1,k) + E_z^n(i,j-1,k)] \\
&+ C_{ez}^{ex,n,xz}(i,j,k)[E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) \\
&+ E_x^n(i-1,j,k)] \\
&+ C_{ez}^{ey,n,yz}(i,j,k)\big[E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1) \\
&+ E_y^n(i,j-1,k)\big] + C_{ez}^{my,n,x}(i,j,k)\big[M_{i,y}^n(i,j,k) - M_{i,y}^n(i-1,j,k)\big] \\
&+ C_{ez}^{mx,n,y}(i,j,k)\big[M_{i,x}^n(i,j,k) - M_{i,x}^n(i,j-1,k)\big] \\
&+ C_{ez}^{jz,n}(i,j,k)\big[J_{i,z}^n(i,j,k)\big] \\
&+ C_{ez}^{jz,t}(i,j,k)\big[J_{i,z}^{n+1}(i,j,k) - J_{i,z}^{n-1}(i,j,k)\big]
\end{aligned} \tag{2.84}$$

where

$$C_z(i,j,k) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{2.85}$$

$$C_{ez}^{ez,n}(i,j,k) = C_z(i,j,k)\left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{2.86}$$

$$C_{ez}^{ez,n-1}(i,j,k) = C_z(i,j,k)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{2.87}$$

$$C_{ez}^{ez,n,x}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{2.88}$$

$$C_{ez}^{ez,n,y}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{2.89}$$

$$C_{ez}^{ex,n,xz}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta x\Delta z}\right) \tag{2.90}$$

$$C_{ez}^{ey,n,yz}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{2.91}$$

$$C_{ez}^{my,n,x}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta x}\right) \tag{2.92}$$

$$C_{ez}^{mx,n,y}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta y}\right) \tag{2.93}$$

$$C_{ez}^{jz,n}(i,j,k) = C_z(i,j,k)(\sigma^m) \tag{2.94}$$

$$C_{ez}^{jz,t}(i,j,k) = C_z(i,j,k)\left(\frac{\mu}{2\Delta t}\right) \tag{2.95}$$

3. ONE-DIMENSIONAL SINGLE-FIELD FDTD UPDATING EQUATIONS

For the one-dimensional case, we assume that there is no field variation in z and y directions, i.e., $\frac{\partial}{\partial z} = \frac{\partial}{\partial y} = 0$ and the wave is propagating in the x direction. Based on this assumption, we can derive one-dimensional single-field FDTD updating equations by starting with (2.8) and decomposing it into its Cartesian components. Since only plane waves propagate in this one-dimensional domain and the assumed propagation direction is x, E_x component is therefore equal to zero.

3.1 Derivation of 1D Updating Equations

3.1.1. 1D Updating Equation for the y Component

Cartesian component of (2.8) in y direction can be written as

$$\nabla^2 E_y - (\nabla(\nabla \cdot \boldsymbol{E}))_y = \sigma^m \sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial M_{i,z}}{\partial x} + \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t} \tag{3.1}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \sigma^m \sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial M_{i,z}}{\partial x} + \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t} \tag{3.2}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equations (3.2) at the corresponding electric field node, i.e. E_y. The time derivatives are evaluated at the n^{th} time step; therefore:

$$\frac{\partial E_y(i)}{\partial t} = \frac{E_y^{n+1}(i) - E_y^{n-1}(i)}{2\Delta t} \tag{3.3}$$

$$\frac{\partial^2 E_y(i)}{\partial t^2} = \frac{E_y^{n+1}(i) - 2E_y^{n}(i) + E_y^{n-1}(i)}{(\Delta t)^2} \tag{3.4}$$

$$\frac{\partial J_{i,y}(i)}{\partial t} = \frac{J_{i,y}^{n+1}(i) - J_{i,y}^{n-1}(i)}{2\Delta t} \tag{3.5}$$

$$\frac{\partial^2 E_y(i)}{\partial x^2} = \frac{E_y^{n}(i+1) - 2E_y^{n}(i) + E_y^{n}(i-1)}{(\Delta x)^2} \tag{3.6}$$

The magnetic sources are associated with the magnetic field, so their positions in the Yee-cell are determined accordingly as shown in Figure 3.1.

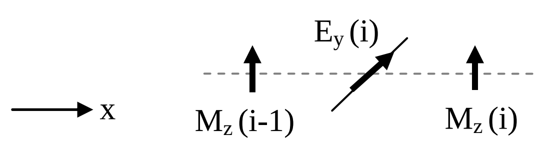

Fig. 3.1 The positions of M_z with respect to E_y in the Yee-cell.

$$\frac{\partial M_{i,z}(i)}{\partial x} = \frac{M_{i,z}^{n}(i) - M_{i,z}^{n}(i-1)}{\Delta x} \tag{3.7}$$

Inserting (3.3)-(3.7) into (3.2), we have

$$\frac{E_y^n(i+1)-2E_y^n(i)+E_y^n(i-1)}{(\Delta x)^2} \\ = \sigma^m\sigma^e E_y^n(i) + (\mu\sigma^e+\varepsilon\sigma^m)\frac{E_y^{n+1}(i)-E_y^{n-1}(i)}{2\Delta t} \\ + \mu\varepsilon\frac{E_y^{n+1}(i)-2E_y^n(i)+E_y^{n-1}(i)}{(\Delta t)^2} - \frac{M_{i,z}^n(i)-M_{i,z}^n(i-1)}{\Delta x} \\ + \sigma^m J_{i,y}^n(i) + \mu\frac{J_{iy}^{n+1}(i)-J_{iy}^{n-1}(i)}{2\Delta t} \tag{3.8}$$

We can simplify the updating equation for E_y as

$$E_y^{n+1}(i) = C_{ey}^{ey,n}(i)\left[E_y^n(i)\right] + C_{ey}^{ey,n-1}(i)\left[E_y^{n-1}(i)\right] \\ + C_{ey}^{ey,n,x}(i)\left[E_y^n(i+1)+E_y^n(i-1)\right] \\ + C_{ey}^{mz,n,x}(i)\left[M_{i,z}^n(i)-M_{i,z}^n(i-1)\right] + C_{ey}^{jy,n}(i)\left[J_{i,y}^n(i)\right] \\ + C_{ey}^{jy,t}(i)\left[J_{i,y}^{n+1}(i)-J_{i,y}^{n-1}(i)\right] \tag{3.9}$$

where

$$C_y(i) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e+\varepsilon\sigma^m)+2\mu\varepsilon} \tag{3.10}$$

$$C_{ey}^{ey,n}(i) = C_y(i)\left(\frac{2}{(\Delta x)^2}-\frac{2\mu\varepsilon}{(\Delta t)^2}+\sigma^m\sigma^e\right) \tag{3.11}$$

$$C_{ey}^{ey,n-1}(i) = C_y(i)\left(\frac{\mu\varepsilon}{(\Delta t)^2}-\frac{(\mu\sigma^e+\varepsilon\sigma^m)}{2\Delta t}\right) \tag{3.12}$$

$$C_{ey}^{ey,n,x}(i) = -C_y(i)\left(\frac{1}{(\Delta x)^2}\right) \tag{3.13}$$

$$C_{ey}^{mz,n,x}(i) = -C_y(i)\left(\frac{1}{\Delta x}\right) \tag{3.14}$$

$$C_{ey}^{jy,n}(i) = C_y(i)(\sigma^m) \tag{3.15}$$

$$C_{ey}^{jy,t}(i) = C_y(i)\left(\frac{\mu}{2\Delta t}\right) \tag{3.16}$$

3.1.2. 1D Updating Equation for the z Component

Cartesian component of (2.8) in z direction can be written as

$$\nabla^2 E_z - (\nabla(\nabla \cdot \boldsymbol{E}))_z = \sigma^m \sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \frac{\partial M_{i,y}}{\partial x} + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \tag{3.17}$$

$$\frac{\partial^2 E_z}{\partial x^2} = \sigma^m \sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \frac{\partial M_{i,y}}{\partial x} + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \tag{3.18}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equations (3.18) at the corresponding electric field node, i.e. E_z. The time derivatives are evaluated at the n^{th} time step; therefore:

$$\frac{\partial E_z(i)}{\partial t} = \frac{E_z^{n+1}(i) - E_z^{n-1}(i)}{2\Delta t} \tag{3.19}$$

$$\frac{\partial^2 E_z(i)}{\partial t^2} = \frac{E_z^{n+1}(i) - 2E_z^{n}(i) + E_z^{n-1}(i)}{(\Delta t)^2} \tag{3.20}$$

$$\frac{\partial J_{i,z}(i)}{\partial t} = \frac{J_{i,z}^{n+1}(i) - J_{i,z}^{n-1}(i)}{2\Delta t} \tag{3.21}$$

$$\frac{\partial^2 E_z(i)}{\partial x^2} = \frac{E_z^{n}(i+1) - 2E_z^{n}(i) + E_z^{n}(i-1)}{(\Delta x)^2} \tag{3.22}$$

The magnetic sources are associated with the magnetic field, so their positions in the Yee-cell are determined accordingly as shown in Figure 3.2.

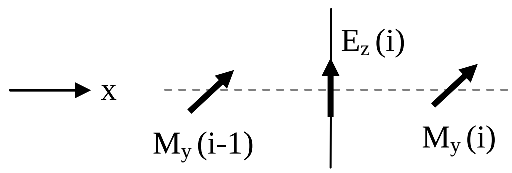

Fig. 3.2 The positions of M_y with respect to E_z in the Yee-cell.

$$\frac{\partial M_{i,y}(i)}{\partial x} = \frac{M_{i,y}^{n}(i) - M_{i,y}^{n}(i-1)}{\Delta x} \tag{3.23}$$

Inserting (3.19)-(3.23) into (3.18), we have

$$\begin{aligned} &\frac{E_z^n(i+1) - 2E_z^n(i) + E_z^n(i-1)}{(\Delta x)^2} \\ &\qquad = \sigma^m \sigma^e E_z^n(i) + (\mu\sigma^e + \varepsilon\sigma^m)\frac{E_z^{n+1}(i) - E_z^{n-1}(i)}{2\Delta t} \\ &\qquad + \mu\varepsilon \frac{E_z^{n+1}(i) - 2E_z^n(i) + E_z^{n-1}(i)}{(\Delta t)^2} + \frac{M_{i,y}^n(i) - M_{i,y}^n(i-1)}{\Delta x} \\ &\qquad + \sigma^m J_{i,z}^n(i) + \mu \frac{J_{i,z}^{n+1}(i) - J_{i,z}^{n-1}(i)}{2\Delta t} \end{aligned} \tag{3.24}$$

We can simplify the updating equation for E_z as

$$\begin{aligned} E_z^{n+1}(i) = {} & C_{ez}^{ez,n}(i)[E_z^n(i)] + C_{ez}^{ez,n-1}(i)[E_z^{n-1}(i)] \\ & + C_{ez}^{ez,n,x}(i)[E_z^n(i+1) + E_z^n(i-1)] \\ & + C_{ez}^{my,n,x}(i)\left[M_{i,y}^n(i) - M_{i,y}^n(i-1)\right] + C_{ez}^{jz,n}(i)\left[J_{i,z}^n(i)\right] \\ & + C_{ez}^{jz,t}(i)\left[J_{i,z}^{n+1}(i) - J_{i,z}^{n-1}(i)\right] \end{aligned} \tag{3.25}$$

where

$$C_z(i) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{3.26}$$

$$C_{ez}^{ez,n}(i) = C_z(i)\left(\frac{2}{(\Delta x)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{3.27}$$

$$C_{ez}^{ez,n-1}(i) = C_z(i)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{3.28}$$

$$C_{ez}^{ez,n,x}(i) = -C_z(i)\left(\frac{1}{(\Delta x)^2}\right) \tag{3.29}$$

$$C_{ez}^{my,n,x}(i) = C_z(i)\left(\frac{1}{\Delta x}\right) \tag{3.30}$$

$$C_{ez}^{jz,n}(i) = C_z(i)(\sigma^m) \tag{3.31}$$

$$C_{ez}^{jz,t}(i) = C_z(i)\left(\frac{\mu}{2\Delta t}\right) \tag{3.32}$$

3.2 Performance Analysis

Next, we examine a one-dimensional electromagnetic problem, as given in [5]. The electric field components, due to a *z*-directed electric current sheet placed at the center of a problem space filled with air between two parallel perfectly electric conducting (PEC) plates extending to infinity in *y* and *z* directions, are computed.

Figure 3.3 shows the problem geometry along with field distributions at t = 0.3 ns. The current sheet placed at the center, namely x = 0.5 m, generates two waves in both sides in opposite directions: solid line represents the electric field whereas corresponding magnetic field multiplied by the free space characteristic impedance is depicted as the dashed line. PECs are located at x = 0 and x = 1m. Electric field values are calculated based on single-field FDTD formulation along x axis. In addition, magnetic field component, namely H_y, is also calculated directly from electric field values for visualization purpose.

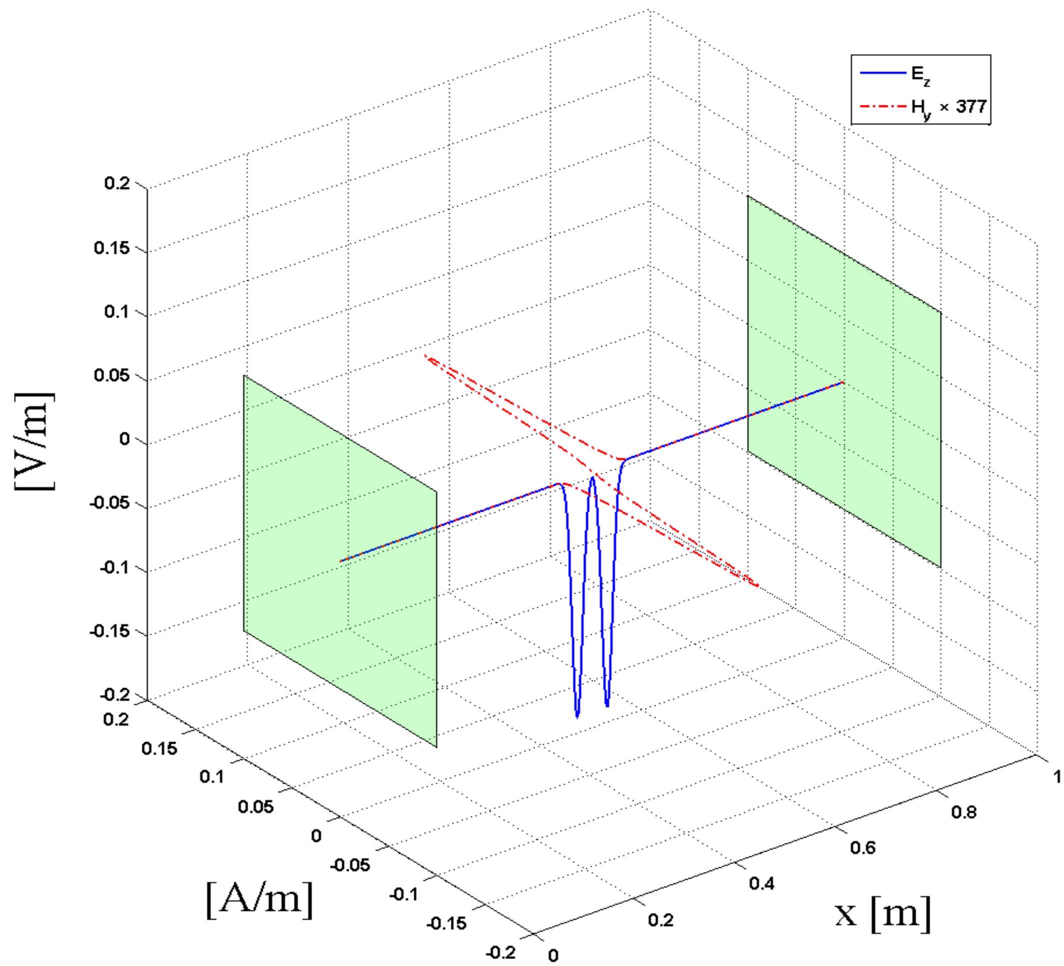

Fig. 3.3 One-dimensional problem configuration [5].

Figure 3.4 shows the comparison of the CPU times required by the single-field and the traditional formulations for different sizes of one-dimensional computational domain. There is approximately 20% improvement in the simulation speed. This decrease in simulation time is because the single-field formulation has three floating-point multiplication operations per node (FLMOPn) in the FDTD loop as opposed to the traditional one having four as shown in Table 3.1.

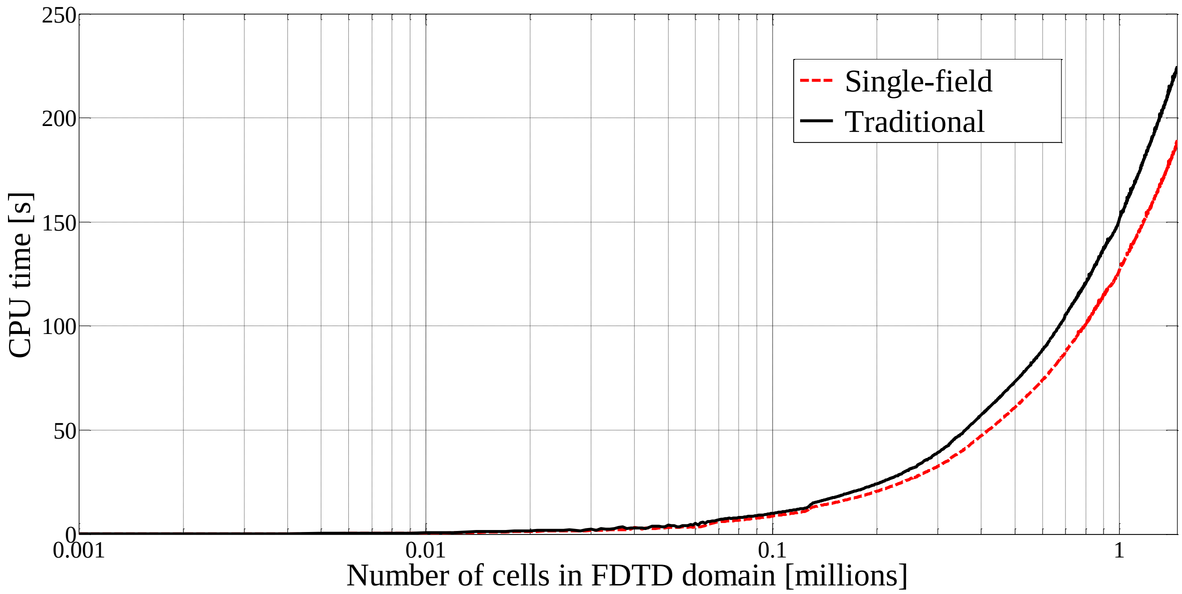

Fig. 3.4 Comparison of CPU time performances in 1D.

Required FLMOPn and the number of memory allocations for field terms per node (MAFTn) are tabulated in Table 3.1. The traditional formulation requires 20% more memory than the single-field formulation does. With these results, we can conclude that the single-field formulation is slightly advantageous over the traditional one for one-dimensional computational domains.

Table 3.1: The required FLMOPn and MAFTn for 1D case.

Formulations	# FLMOPn	# MAFTn
Single-Field	3	3 coefficients + 2 fields = 5
Traditional	4	3 coefficients + 3 fields = 6
Improvement	% 25	% 17

4. TWO-DIMENSIONAL SINGLE-FIELD FDTD UPDATING EQUATIONS

For the two-dimensional case, we assume that there is no field variation in the z direction, i.e., $\frac{\partial}{\partial z} = 0$. Based on this assumption, we can derive two-dimensional single-field FDTD updating equations by starting with (2.8) and decomposing it to its Cartesian components.

4.1. Derivation of 2D Updating Equations

4.1.1. 2D Updating Equation for the x Component

Cartesian component of (2.8) in x direction can be written as

$$\nabla^2 E_x - (\nabla(\nabla \cdot \boldsymbol{E}))_x = \sigma^m \sigma^e E_x + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_x}{\partial t} + \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} + \frac{\partial M_{i,z}}{\partial y} + \sigma^m J_{i,x} + \mu\frac{\partial J_{i,x}}{\partial t} \tag{4.1}$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} - \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial x \partial y}\right) = \sigma^m \sigma^e E_x + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_x}{\partial t} + \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} + \frac{\partial M_{i,z}}{\partial y} + \sigma^m J_{i,x} + \mu\frac{\partial J_{i,x}}{\partial t} \tag{4.2}$$

$$\frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_y}{\partial x \partial y} = \sigma^m \sigma^e E_x + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_x}{\partial t} + \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} + \frac{\partial M_{i,z}}{\partial y} + \sigma^m J_{i,x} + \mu\frac{\partial J_{i,x}}{\partial t} \tag{4.3}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (4.3) at the corresponding electric field node, i.e. E_x. The time derivatives are evaluated at the n^{th} time step; therefore:

$$\frac{\partial E_x(i,j)}{\partial t} = \frac{E_x^{n+1}(i,j) - E_x^{n-1}(i,j)}{2\Delta t} \tag{4.4}$$

$$\frac{\partial^2 E_x(i,j)}{\partial t^2} = \frac{E_x^{n+1}(i,j) - 2E_x^{n}(i,j) + E_x^{n-1}(i,j)}{(\Delta t)^2} \tag{4.5}$$

$$\frac{\partial J_{i,x}(i,j)}{\partial t} = \frac{J_{i,x}^{n+1}(i,j) - J_{i,x}^{n-1}(i,j)}{2\Delta t} \tag{4.6}$$

$$\frac{\partial^2 E_x(i,j)}{\partial y^2} = \frac{E_x^{n}(i,j+1) - 2E_x^{n}(i,j) + E_x^{n}(i,j-1)}{(\Delta y)^2} \tag{4.7}$$

For the y-directed electric field components, we have to consider their positions in the Yee-cell as shown in Figure 4.1.

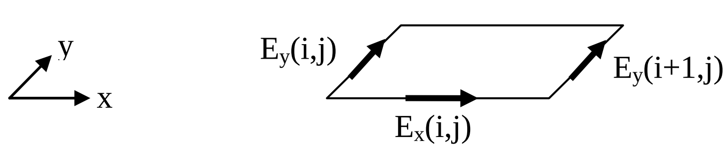

Fig. 4.1 The positions of E_y field components with respect to E_x in the Yee-cell.

$$\frac{\partial E_y(i,j)}{\partial y} = \frac{E_y^n(i,j) - E_y^n(i,j-1)}{\Delta y} \tag{4.8}$$

$$\frac{\partial E_y(i,j)}{\partial x} = \frac{E_y^n(i+1,j) - E_y^n(i,j)}{\Delta x} \tag{4.9}$$

$$\frac{\partial^2 E_y(i,j)}{\partial x \partial y} = \frac{E_y^n(i+1,j) - E_y^n(i+1,j-1) - E_y^n(i,j) + E_y^n(i,j-1)}{\Delta x \Delta y} \tag{4.10}$$

The magnetic sources are associated with the magnetic field, so their positions in the Yee-cell are determined accordingly, as shown in Figure 4.2.

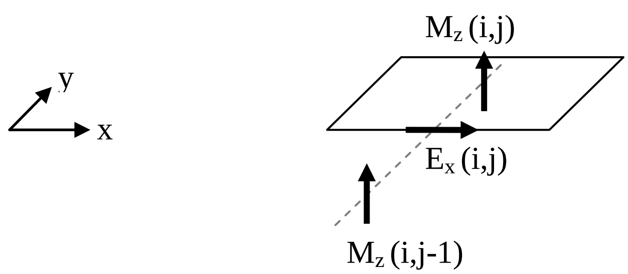

Fig. 4.2 The positions of M_z with respect to E_x in the Yee-cell.

$$\frac{\partial M_{i,z}(i,j)}{\partial y} = \frac{M_{i,z}^n(i,j) - M_{i,z}^n(i,j-1)}{\Delta y} \tag{4.11}$$

Inserting (4.4) - (4.11) into (4.3), we have

$$
\begin{aligned}
&\frac{E_x^n(i,j+1) - 2E_x^n(i,j) + E_x^n(i,j-1)}{(\Delta y)^2} \\
&\qquad - \frac{E_y^n(i+1,j) - E_y^n(i+1,j-1) - E_y^n(i,j) + E_y^n(i,j-1)}{\Delta x \Delta y} \\
&\qquad = \sigma^m \sigma^e E_x^n(i,j) + (\mu\sigma^e + \varepsilon\sigma^m)\frac{E_x^{n+1}(i,j) - E_x^{n-1}(i,j)}{2\Delta t} \\
&\qquad + \mu\varepsilon \frac{E_x^{n+1}(i,j) - 2E_x^n(i,j) + E_x^{n-1}(i,j)}{(\Delta t)^2} \\
&\qquad + \left(\frac{M_{i,z}^n(i,j) - M_{i,z}^n(i,j-1)}{\Delta y}\right) + \sigma^m J_{i,x}^n(i,j) \\
&\qquad + \mu \frac{J_{i,x}^{n+1}(i,j) - J_{i,x}^{n-1}(i,j)}{2\Delta t}
\end{aligned}
\tag{4.12}
$$

We can simplify the updating equation for E_x as

$$
\begin{aligned}
E_x^{n+1}(i,j) = {} & C_{ex}^{ex,n}(i,j)[E_x^n(i,j)] + C_{ex}^{ex,n-1}(i,j)[E_x^{n-1}(i,j)] \\
& + C_{ex}^{ex,n,y}(i,j,k)[E_x^n(i,j+1) + E_x^n(i,j-1)] \\
& + C_{ex}^{ey,n,xy}(i,j)\big[E_y^n(i+1,j) - E_y^n(i+1,j-1) - E_y^n(i,j) \\
& + E_y^n(i,j-1)\big] + C_{ex}^{mz,n,y}(i,j)\big[M_{i,z}^n(i,j) - M_{i,z}^n(i,j-1)\big] \\
& + C_{ex}^{jx,n}(i,j)\big[J_{i,x}^n(i,j)\big] + C_{ex}^{jx,t}(i,j)\big[J_{i,x}^{n+1}(i,j) - J_{i,x}^{n-1}(i,j)\big]
\end{aligned}
\tag{4.13}
$$

where

$$C_x(i,j) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{4.14}$$

$$C_{ex}^{ex,n}(i,j) = C_x(i,j)\left(\frac{2}{(\Delta y)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{4.15}$$

$$C_{ex}^{ex,n-1}(i,j) = C_x(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{4.16}$$

$$C_{ex}^{ex,n,y}(i,j) = -C_x(i,j)\left(\frac{1}{(\Delta y)^2}\right) \tag{4.17}$$

$$C_{ex}^{ey,n,xy}(i,j) = C_x(i,j)\left(\frac{1}{\Delta x \Delta y}\right) \tag{4.18}$$

$$C_{ex}^{mz,n,y}(i,j) = C_x(i,j)\left(\frac{1}{\Delta y}\right) \tag{4.19}$$

$$C_{ex}^{jx,n}(i,j) = C_x(i,j)(\sigma^m) \tag{4.20}$$

$$C_{ex}^{jx,t}(i,j) = C_x(i,j)\left(\frac{\mu}{2\Delta t}\right) \tag{4.21}$$

4.1.2. 2D Updating Equation for the y Component

Cartesian component of (2.8) in y direction can be written as

$$\begin{aligned}\nabla^2 E_y - (\nabla(\nabla \cdot \boldsymbol{E}))_y \\ = \sigma^m \sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} + \left(\frac{\partial M_{i,x}}{\partial z} - \frac{\partial M_{i,z}}{\partial x}\right) \\ + \sigma^m J_{i,y} + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{4.22}$$

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} - \left(\frac{\partial^2 E_x}{\partial x \partial y} + \frac{\partial^2 E_y}{\partial y^2}\right) \\ = \sigma^m \sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial M_{i,z}}{\partial x} + \sigma^m J_{i,y} \\ + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{4.23}$$

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_x}{\partial x \partial y} = \sigma^m \sigma^e E_y + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_y}{\partial t} + \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial M_{i,z}}{\partial x} + \sigma^m J_{i,y} \\ + \mu\frac{\partial J_{i,y}}{\partial t}\end{aligned} \tag{4.24}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equations (4.24) at the corresponding electric field node, i.e. E_y. The time derivatives are evaluated at the n[th] time step; therefore:

$$\frac{\partial E_y(i,j)}{\partial t} = \frac{E_y^{n+1}(i,j) - E_y^{n-1}(i,j)}{2\Delta t} \tag{4.25}$$

$$\frac{\partial^2 E_y(i,j)}{\partial t^2} = \frac{E_y^{n+1}(i,j) - 2E_y^{n}(i,j) + E_y^{n-1}(i,j)}{(\Delta t)^2} \tag{4.26}$$

$$\frac{\partial J_{i,y}(i,j)}{\partial t} = \frac{J_{i,y}^{n+1}(i,j) - J_{i,y}^{n-1}(i,j)}{2\Delta t} \tag{4.27}$$

$$\frac{\partial^2 E_y(i,j)}{\partial x^2} = \frac{E_y^{n}(i+1,j) - 2E_y^{n}(i,j) + E_y^{n}(i-1,j)}{(\Delta x)^2} \tag{4.28}$$

For the x-directed electric field components, we have to consider their positions in the Yee-cell, as shown in Figure 4.3.

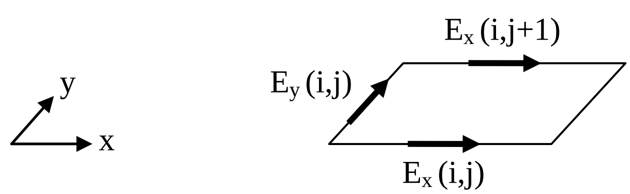

Fig. 4.3 The positions of E_x field components with respect to E_y in the Yee-cell.

$$\frac{\partial E_x(i,j)}{\partial x} = \frac{E_x^{n}(i,j) - E_x^{n}(i-1,j)}{\Delta x} \tag{4.29}$$

$$\frac{\partial E_x(i,j)}{\partial y} = \frac{E_x^{n}(i,j+1) - E_x^{n}(i,j)}{\Delta y} \tag{4.30}$$

$$\frac{\partial^2 E_x(i,j)}{\partial x \partial y} = \frac{E_x^n(i,j+1) - E_x^n(i-1,j+1) - E_x^n(i,j) + E_x^n(i-1,j)}{\Delta x \Delta y} \tag{4.31}$$

The magnetic sources are associated with the magnetic field, so their positions in the Yee-cell are determined accordingly, as shown in Figure 4.4.

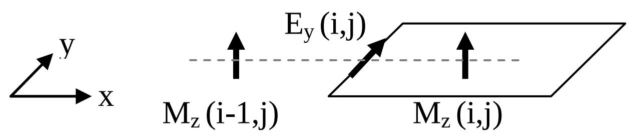

Fig. 4.4 The positions of M_z with respect to E_y in the Yee-cell.

$$\frac{\partial M_{i,z}(i,j)}{\partial x} = \frac{M_{i,z}^n(i,j) - M_{i,z}^n(i-1,j)}{\Delta x} \tag{4.32}$$

Inserting (4.25) - (4.32) into (4.24), we have

$$\begin{aligned} &\frac{E_y^n(i+1,j) - 2E_y^n(i,j) + E_y^n(i-1,j)}{(\Delta x)^2} \\ &\quad - \frac{E_x^n(i,j+1) - E_x^n(i-1,j+1) - E_x^n(i,j) + E_x^n(i-1,j)}{\Delta x \Delta y} \\ &\quad = \sigma^m \sigma^e E_y(i,j) + (\mu\sigma^e + \varepsilon\sigma^m)\frac{E_y^{n+1}(i,j) - E_y^{n-1}(i,j)}{2\Delta t} \\ &\quad + \mu\varepsilon \frac{E_y^{n+1}(i,j) - 2E_y^n(i,j) + E_y^{n-1}(i,j)}{(\Delta t)^2} - \frac{M_{i,z}^n(i,j) - M_{i,z}^n(i-1,j)}{\Delta x} \\ &\quad + \sigma^m J_{i,y}^n(i,j) + \mu \frac{J_{i,y}^{n+1}(i,j) - J_{i,y}^{n-1}(i,j)}{2\Delta t} \end{aligned} \tag{4.33}$$

We can simplify the updating equation for E_y as

$$
\begin{aligned}
E_y^{n+1}(i,j) = {} & C_{ey}^{ey,n}(i,j)\left[E_y^n(i,j)\right] + C_{ey}^{ey,n-1}(i,j)\left[E_y^{n-1}(i,j)\right] \\
& + C_{ey}^{ey,n,x}(i,j)\left[E_y^n(i+1,j) + E_y^n(i-1,j)\right] \\
& + C_{ey}^{ex,n,xy}(i,j)[E_x^n(i,j+1) - E_x^n(i-1,j+1) - E_x^n(i,j) \\
& + E_x^n(i-1,j)] + C_{ey}^{mz,n,y}(i,j)\left[M_{i,z}^n(i,j) - M_{i,z}^n(i-1,j)\right] \\
& + C_{ey}^{jy,n}(i,j)\left[J_{i,y}^n(i,j)\right] + C_{ey}^{jy,t}(i,j)\left[J_{i,y}^{n+1}(i,j) - J_{i,y}^{n-1}(i,j)\right]
\end{aligned}
\tag{4.34}
$$

where

$$C_y(i,j) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e + \varepsilon\sigma^m) + 2\mu\varepsilon} \tag{4.35}$$

$$C_{ey}^{ey,n}(i,j) = C_y(i,j)\left(\frac{2}{(\Delta x)^2} - \frac{2\mu\varepsilon}{(\Delta t)^2} + \sigma^m\sigma^e\right) \tag{4.36}$$

$$C_{ey}^{ey,n-1}(i,j) = C_y(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{4.37}$$

$$C_{ey}^{ey,n,x}(i,j) = -C_y(i,j)\left(\frac{1}{(\Delta x)^2}\right) \tag{4.38}$$

$$C_{ey}^{ex,n,xy}(i,j) = C_y(i,j)\left(\frac{1}{\Delta x\Delta y}\right) \tag{4.39}$$

$$C_{ey}^{mz,n,y}(i,j) = -C_y(i,j)\left(\frac{1}{\Delta x}\right) \tag{4.40}$$

$$C_{ey}^{jy,n}(i,j) = C_y(i,j)(\sigma^m) \tag{4.41}$$

$$C_{ey}^{jy,t}(i,j) = C_y(i,j)\left(\frac{\mu}{2\Delta t}\right) \tag{4.42}$$

4.1.3. 2D Updating Equation for the z Component

Cartesian component of (2.8) in z direction can be written as

$$\nabla^2 E_z - \left(\nabla(\nabla \cdot \boldsymbol{E})\right)_z = \sigma^m \sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \left(\frac{\partial M_{i,y}}{\partial x} - \frac{\partial M_{i,x}}{\partial y}\right) + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \tag{4.43}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \sigma^m \sigma^e E_z + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_z}{\partial t} + \mu\varepsilon\frac{\partial^2 E_z}{\partial t^2} + \left(\frac{\partial M_{i,y}}{\partial x} - \frac{\partial M_{i,x}}{\partial y}\right) + \sigma^m J_{i,z} + \mu\frac{\partial J_{i,z}}{\partial t} \tag{4.44}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (4.44) at the corresponding electric field node, i.e. E_z. The time derivatives are evaluated at the n^{th} time step; therefore:

$$\frac{\partial E_z(i,j)}{\partial t} = \frac{E_z^{n+1}(i,j) - E_z^{n-1}(i,j)}{2\Delta t} \tag{4.45}$$

$$\frac{\partial^2 E_z(i,j)}{\partial t^2} = \frac{E_z^{n+1}(i,j) - 2E_z^{n}(i,j) + E_z^{n-1}(i,j)}{(\Delta t)^2} \tag{4.46}$$

$$\frac{\partial J_{i,z}(i,j)}{\partial t} = \frac{J_{i,z}^{n+1}(i,j) - J_{i,z}^{n-1}(i,j)}{2\Delta t} \tag{4.47}$$

$$\frac{\partial^2 E_z(i,j)}{\partial x^2} = \frac{E_z^{n}(i+1,j) - 2E_z^{n}(i,j) + E_z^{n}(i-1,j)}{(\Delta x)^2} \tag{4.48}$$

$$\frac{\partial^2 E_z(i,j)}{\partial y^2} = \frac{E_z^{n}(i,j+1) - 2E_z^{n}(i,j) + E_z^{n}(i,j-1)}{(\Delta y)^2} \tag{4.49}$$

The magnetic sources are associated with the magnetic field, so their positions in the Yee-cell are determined accordingly, as shown in Figures 4.5 and 4.6.

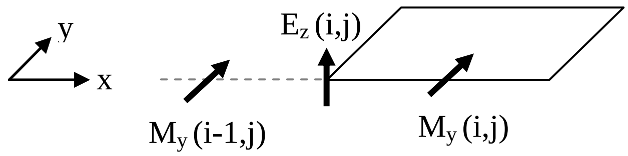

Fig. 4.5 The positions of M_y with respect to E_z in the Yee-cell.

$$\frac{\partial M_{i,y}(i,j)}{\partial x} = \frac{M_{i,y}^{n}(i,j) - M_{i,y}^{n}(i-1,j)}{\Delta x} \tag{4.50}$$

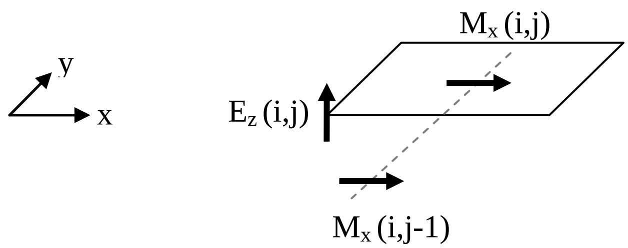

Fig. 4.6 The positions of M_x with respect to E_z in the Yee-cell.

$$\frac{\partial M_{i,x}(i,j)}{\partial y} = \frac{M_{i,x}^{n}(i,j) - M_{i,x}^{n}(i,j-1)}{\Delta y} \tag{4.51}$$

Inserting (4.45)-(4.51) into (4.44), we have

$$\frac{E_z^n(i+1,j)-2E_z^n(i,j)+E_z^n(i-1,j)}{(\Delta x)^2}+\frac{E_z^n(i,j+1)-2E_z^n(i,j)+E_z^n(i,j-1)}{(\Delta y)^2}$$
$$\begin{aligned}
&= \sigma^m\sigma^e E_z^n(i,j) + (\mu\sigma^e+\varepsilon\sigma^m)\frac{E_z^{n+1}(i,j)-E_z^{n-1}(i,j)}{2\Delta t}\\
&+ \mu\varepsilon\frac{E_z^{n+1}(i,j)-2E_z^n(i,j)+E_z^{n-1}(i,j)}{(\Delta t)^2}\\
&+ \frac{M_{i,y}^n(i,j)-M_{i,y}^n(i-1,j)}{\Delta x}-\frac{M_{i,x}^n(i,j)-M_{i,x}^n(i,j-1)}{\Delta y}\\
&+ \sigma^m J_{i,z}^n(i,j)+\mu\frac{J_{i,z}^{n+1}(i,j)-J_{i,z}^{n-1}(i,j)}{2\Delta t}
\end{aligned} \tag{4.52}$$

We can simplify the updating equation for E_z as

$$\begin{aligned}
E_z^{n+1}(i,j) = {} & C_{ez}^{ez,n}(i,j)[E_z^n(i,j)] + C_{ez}^{ez,n-1}(i,j)[E_z^{n-1}(i,j)]\\
&+ C_{ez}^{ez,n,x}(i,j)[E_z^n(i+1,j)+E_z^n(i-1,j)]\\
&+ C_{ez}^{ez,n,y}(i,j)[E_z^n(i,j+1)+E_z^n(i,j-1)]\\
&+ C_{ez}^{my,n,x}(i,j)\left[M_{i,y}^n(i,j)-M_{i,y}^n(i-1,j)\right]\\
&+ C_{ez}^{mx,n,y}(i,j)\left[M_{i,x}^n(i,j)-M_{i,x}^n(i,j-1)\right] + C_{ez}^{jz,n}(i,j)\left[J_{i,z}^n(i,j)\right]\\
&+ C_{ez}^{jz,t}(i,j)\left[J_{i,z}^{n+1}(i,j)-J_{i,z}^{n-1}(i,j)\right]
\end{aligned} \tag{4.53}$$

where

$$C_z(i,j) = -\frac{2(\Delta t)^2}{\Delta t(\mu\sigma^e+\varepsilon\sigma^m)+2\mu\varepsilon} \tag{4.54}$$

$$C_{ez}^{ez,n}(i,j) = C_z(i,j)\left(\frac{2}{(\Delta x)^2}+\frac{2}{(\Delta y)^2}-\frac{2\mu\varepsilon}{(\Delta t)^2}+\sigma^m\sigma^e\right) \tag{4.55}$$

$$C_{ez}^{ez,n-1}(i,j) = C_z(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2}-\frac{(\mu\sigma^e+\varepsilon\sigma^m)}{2\Delta t}\right) \tag{4.56}$$

$$C_{ez}^{ez,n,x}(i,j) = -C_z(i,j)\left(\frac{1}{(\Delta x)^2}\right) \tag{4.57}$$

$$C_{ez}^{ez,n,y}(i,j) = -C_z(i,j)\left(\frac{1}{(\Delta y)^2}\right) \tag{4.58}$$

$$C_{ez}^{my,n,x}(i,j) = C_z(i,j)\left(\frac{1}{\Delta x}\right) \tag{4.59}$$

$$C_{ez}^{mx,n,y}(i,j) = -C_z(i,j)\left(\frac{1}{\Delta y}\right) \tag{4.60}$$

$$C_{ez}^{jz,n}(i,j) = C_z(i,j)(\sigma^m) \tag{4.61}$$

$$C_{ez}^{jz,t}(i,j) = C_z(i,j)\left(\frac{\mu}{2\Delta t}\right) \tag{4.62}$$

4.2. A 2D TM Problem with a Filament Electric Current

A two-dimensional problem is constructed as free space with a *z*-directed impressed electric current located at the origin, as depicted in Figure 4.7.

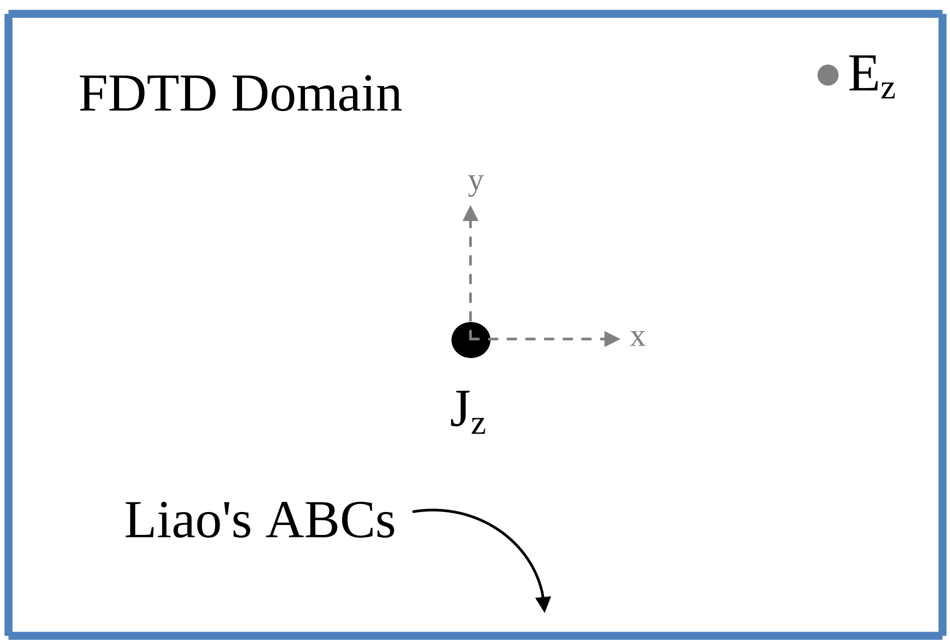

Fig. 4.7 2D TM problem configuration.

The current density has a Gaussian waveform with magnitude of 1 [Amp/m]. Electric fields generated by the traditional and the single-field formulations are compared in time and frequency domains; the stability and dispersion analyses are also performed for both. Since the real benefit of the single-field formulation is the time required to run the simulation and the required memory size, the two formulations are run for different domain sizes and the CPU times required to complete the simulation are recorded. CPU time verses domain size is plotted for both formulations. To get a better insight for the simulation time and memory usage, required FLMOPn and MAFTn will be tabulated.

4.2.1. Stability Comparison

Stability analysis was conducted by changing the value of discrete time, i.e., Δt, and observing the change in the field values generated by the single-field and the traditional 2D updating equations.

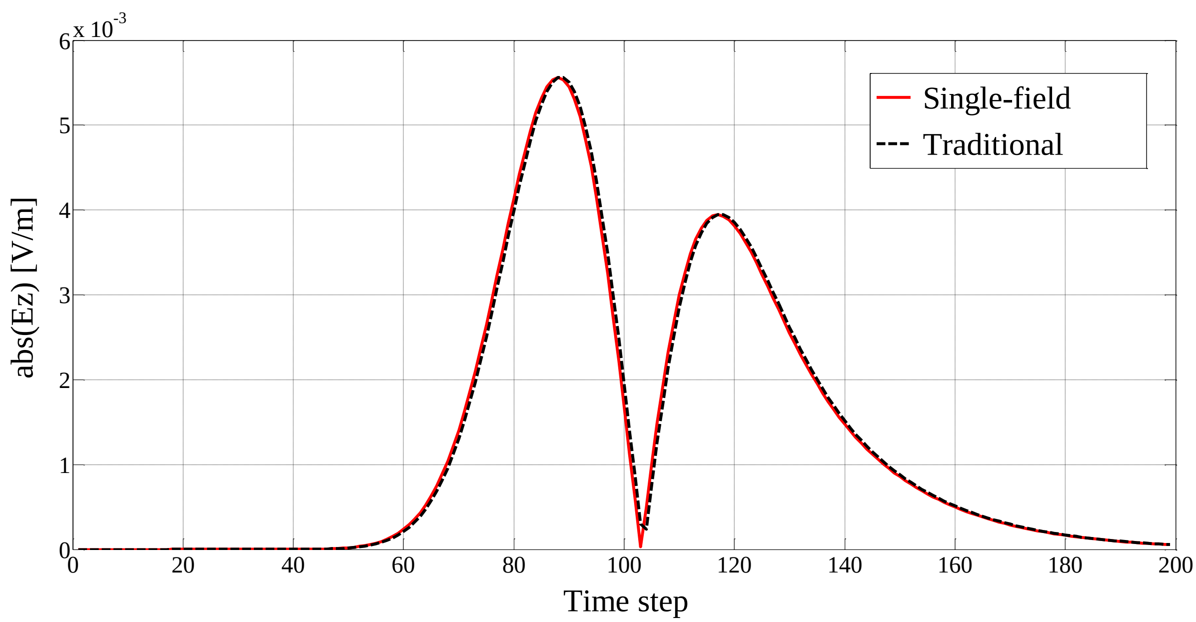

Fig. 4.8 Field comparison for Δt = 2.35 ps.

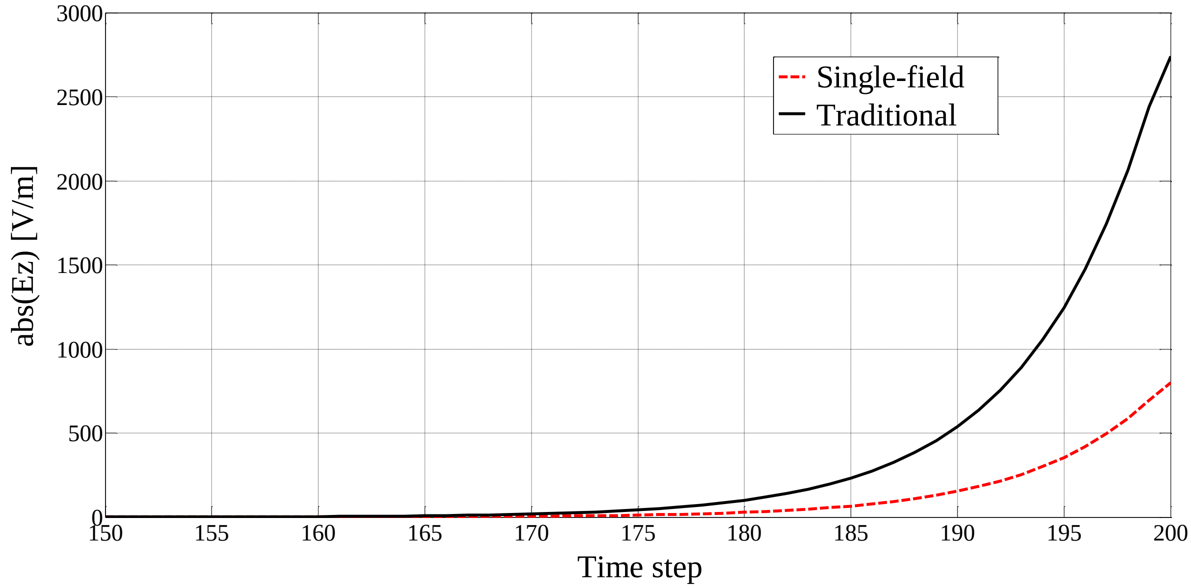

Fig. 4.9 Field comparison for Δt = 2.37 ps.

The Courant-Friedrichs-Lewy (CFL) condition [8] requires that the time increment Δt be 2.35 ps for a stable result if the space increments in both directions, Δx and Δy, are 1 mm. Figure 4.8 shows the field comparison of such stable simulation results calculated at point (8, 8) mm in a 20 mm x 20 mm problem domain as shown in Figure 4.7. If we set it to 2.37 ps, the single-field and the traditional formulations show divergence from optimum field values. Figure 4.9 shows the divergence in terms of magnitude of the field versus time step. The single-field formulation provides comparatively less divergent results than the traditional formulation does, since it requires less numerical computation.

4.2.2. Dispersion Analysis

Dispersion is defined as the variation of a propagating wave's velocity with frequency. The analysis is done for E_z component of the electric field under the assumption of lossless medium and monochromatic traveling wave solution

$$E_z^n(i,j,k) = E_{0,z}e^{j(\omega n\Delta t - k_x i_x \Delta_x - k_y i_y \Delta_y)} \tag{4.63}$$

where k_x and k_y are the x and y components of the numerical wavevector; i_x, and i_y are space indices. By substituting this field expression into the single-field updating equation for E_z, and using $PPW = \frac{\lambda_n}{h}$, $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ and $\frac{c\Delta t}{h} = 0.5$, one can obtain

$$\frac{c_n}{c_0} = \frac{\lambda_n}{\lambda} = \frac{2PPW}{\pi}\sin^{-1}\left\{\frac{1}{4} - \left[\frac{1}{4}cos\left(\frac{\pi}{PPW}(cos\,\alpha + sin\,\alpha)\right)cos\left(\frac{\pi}{PPW}(cos\,\alpha - sin\,\alpha)\right)\right]\right\}^{1/2} \tag{4.64}$$

where PPW is the number of points in wavelength discretization, c_n is the numerical velocity, λ_n is the numerical wavelength, and α is the angle between the direction of the propagating wave and the positive x-axis. (4.64) gives the ratio of the velocities or wavelengths as a function of PPW and α. A detailed derivation of dispersion analysis procedure for single-field formulation is given in Appendix A.

Figure 4.10 shows the variation of the normalized numerical phase velocity (c_n / c_0) versus points per wavelength discretization (PPW) in two-dimensional FDTD grid.

Dispersion performance of the single-field formulation shows a characteristic similar to the traditional formulation as given in [21].

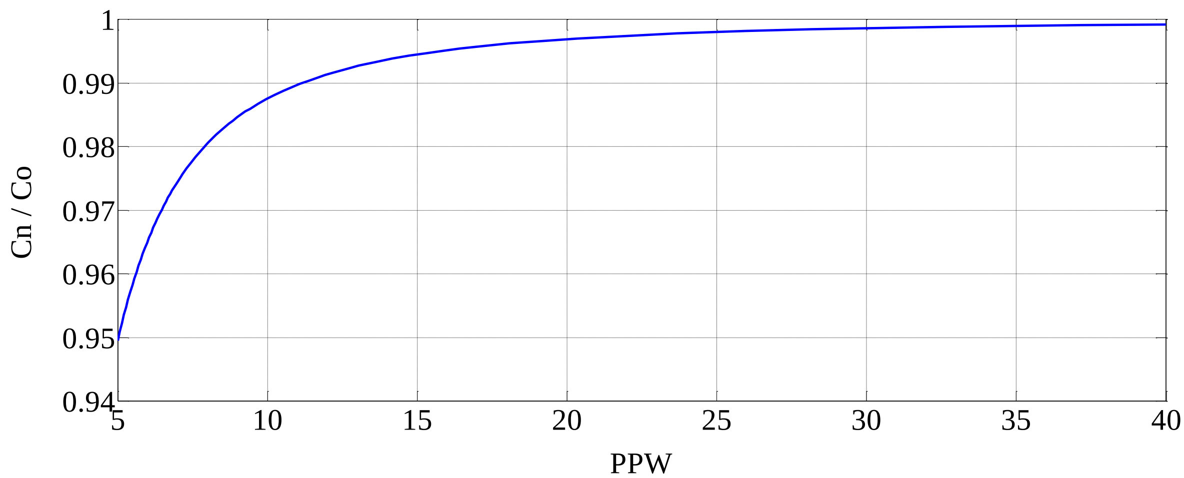

Fig. 4.10 Dispersion performance of the single-field formulation.

4.2.3. CPU Time Analysis

Figure 4.11 shows the CPU time the formulations require to complete a simulation of corresponding size for 1800 time steps with 0.0694% difference in calculated field values. X axis represents the number of grids used to characterize the problem.

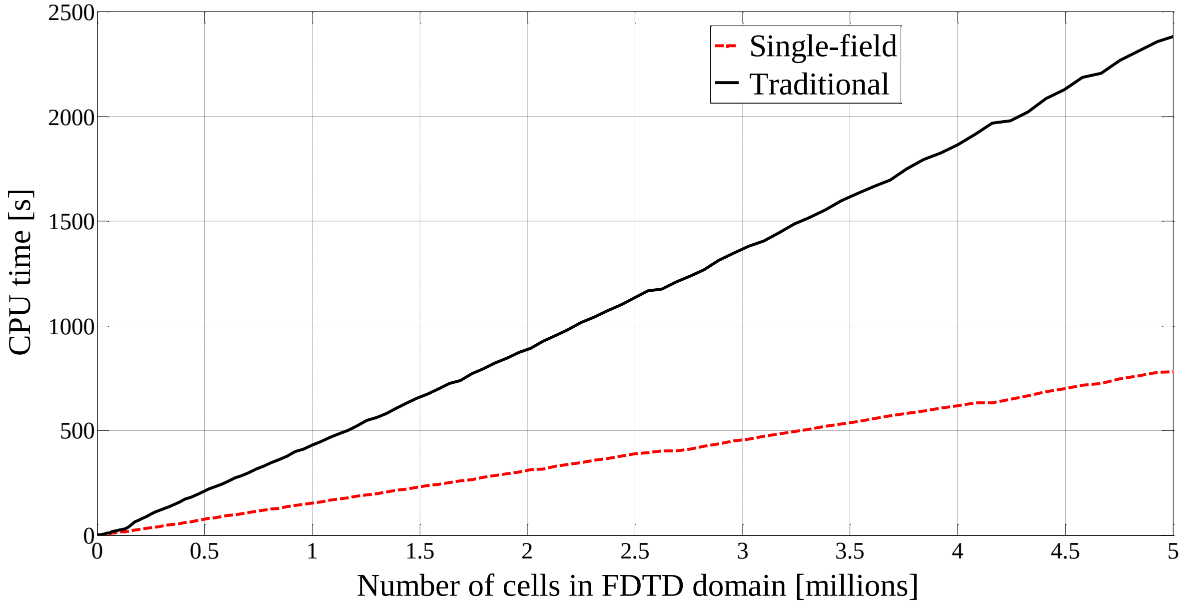

Fig. 4.11 Comparison of CPU time performances in 2D.

A speed up factor is calculated according to the formula given in (4.65) for different problem sizes and plotted in Fig. 4.11 for different number of time steps.

$$Speed\ up\ Factor = \frac{CPU\ Time\ (Traditional)}{CPU\ Time\ (Single - field)} \tag{4.65}$$

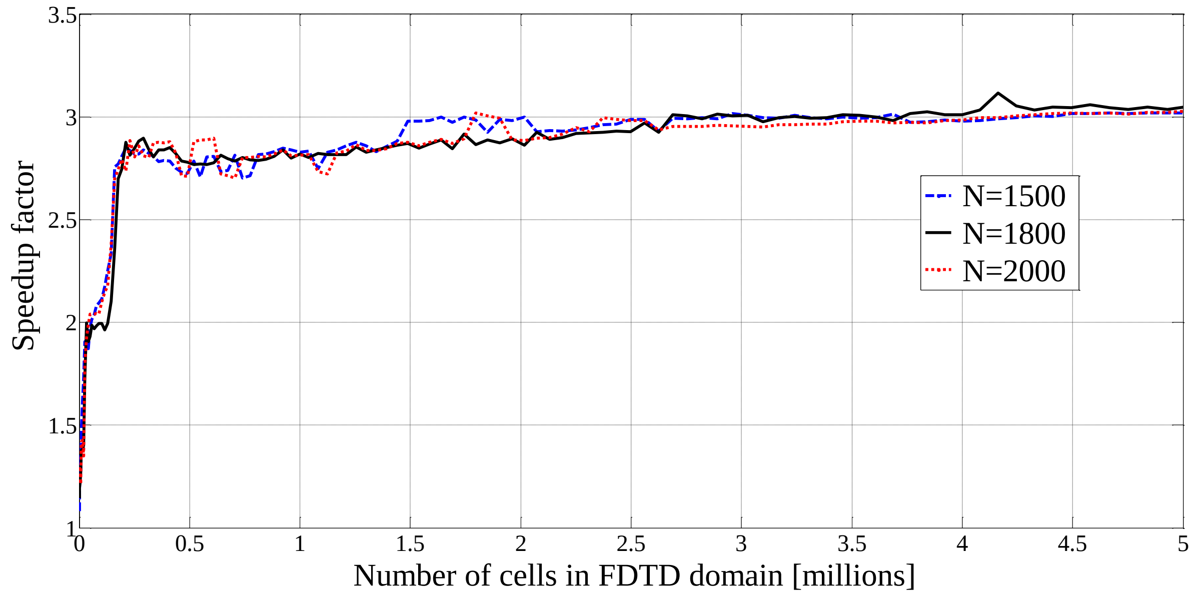

Fig. 4.12 Speedup factor in 2D.

The single-field formulation appears to be three times faster than the traditional one for domain sizes close to three million cells. Higher speed factors are expected for larger domains as evident from the trend in Figure 4.12. This speed up is because the single-field formulation has four FLMOPn inside the FDTD time marching loop as opposed to the traditional one having seven. The specifications of the computing system used for the simulations are given in Appendix C.

4.2.4. Memory Usage Analysis

Table 4.1 shows the number of FLMOPn, floating-point addition operation per node (FLAOPn) and MAFTn. The single-field formulation requires 40% less memory to simulate the same size problem than the traditional formulation.

Table 4.1. The required FLMOPn, FLAOPn and MAFTn for 2D formulations.

Formulations	# FLAOPn	# FLMOPn	# MAFTn
Single-field	5	4	4 coefficients + 2 fields = 6
Traditional	8	7	7 coefficients + 3 fields = 10
Improvement	% 37.5	% 43	% 40

4.3. A 2D TE Problem with a Filament Magnetic Current

A two-dimensional problem is constructed as free space with a *z*-directed impressed magnetic current located at the origin. The current density has a Gaussian waveform with magnitude of 1 [V/m]. Magnetic fields generated by the traditional and the single-field formulations are compared in time and frequency domains; stability and dispersion analyses are also performed for both. For the TE problem, H field-based single-field formulation is used as given in Appendix B. Due to the symmetry in the formulation and duality in the problem, merits for CPU time, memory requirements, stability and dispersion are the same as for the TM problem given in section 4.2. Therefore, Figure 4.11 and Figure 4.12, and Table 4.1 show the performance of the single-field formulation for 2D TE problems as well.

4.4. A 2D TM Scattered Field Problem

An infinite line of a constant electric current is placed parallel and in the vicinity of a circular conducting cylinder of infinite length.

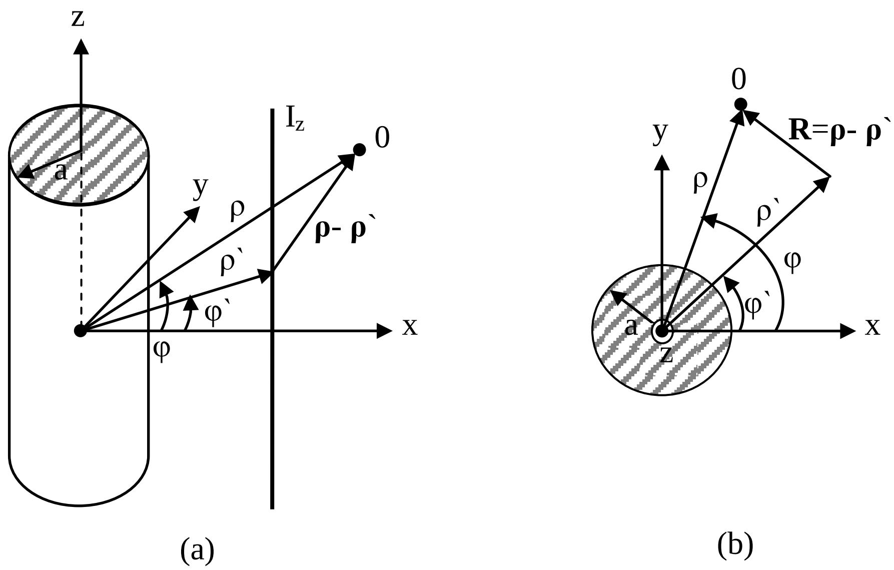

Fig. 4.13 A line source near a circular cylinder. (a) Side view. (b) Top view [22].

We will examine here the scattering of the cylindrical waves by the cylinder for $\rho \geq \rho'$. The analytical solution for the total electric field is given in [22] as

$$E_\rho^t = E_\varphi^t = 0 \tag{4.66}$$

$$E_z^t = -\frac{\beta^2 I_e}{4\omega\varepsilon} \sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho) \left[J_n(\beta\rho) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta\rho')} H_n^{(2)}(\beta\rho') \right] e^{jn(\varphi-\varphi')} \tag{4.67}$$

where ρ is the distance from the center of the cylinder to the field point, its range is 0.1-1.1 m, ρ' is the distance from the center of the cylinder to the source point, its value is 0.1 m, φ is the azimuth angle of the field point and φ' is the azimuth angle of the source point, its value is 0, *a* is the radius of the conducting cylinder and its value is 0.01 m. For the numerical simulation, the spatial and temporal steps used are Δx = 1 mm, Δy = 1 mm and Δt = 2.2407 ps, respectively. The cylinder is modeled in FDTD domain by stair-casing. For the analytical solution two hundred terms are used for the Hankel function summation and the frequency is set to 1 GHz. Electric field is computed with the single-field and the traditional formulation at one thousand different spatial points on the x axis in time-domain and converted to frequency domain to compare with the analytical solution results. Error for single-field and traditional formulations with respect to the analytical solution is calculated according to (4.68) and their performances are shown in Figures 4.14 and 4.15. The single-field and the traditional formulations show similar performance in terms of accuracy.

$$Error = 20 \times \log_{10}\left(\frac{abs(Numerical\ value - Analytical\ value)}{\max\left(abs(Analytical\ value)\right)}\right) \tag{4.68}$$

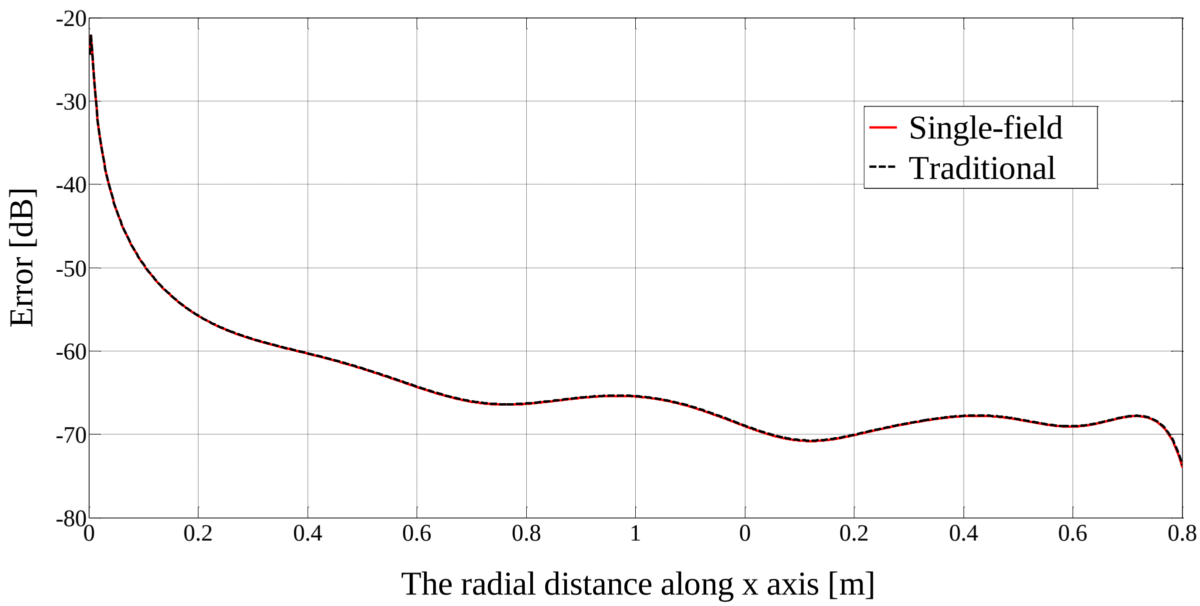

Fig. 4.14 Comparison of the numerical solutions with the analytical solution; magnitude.

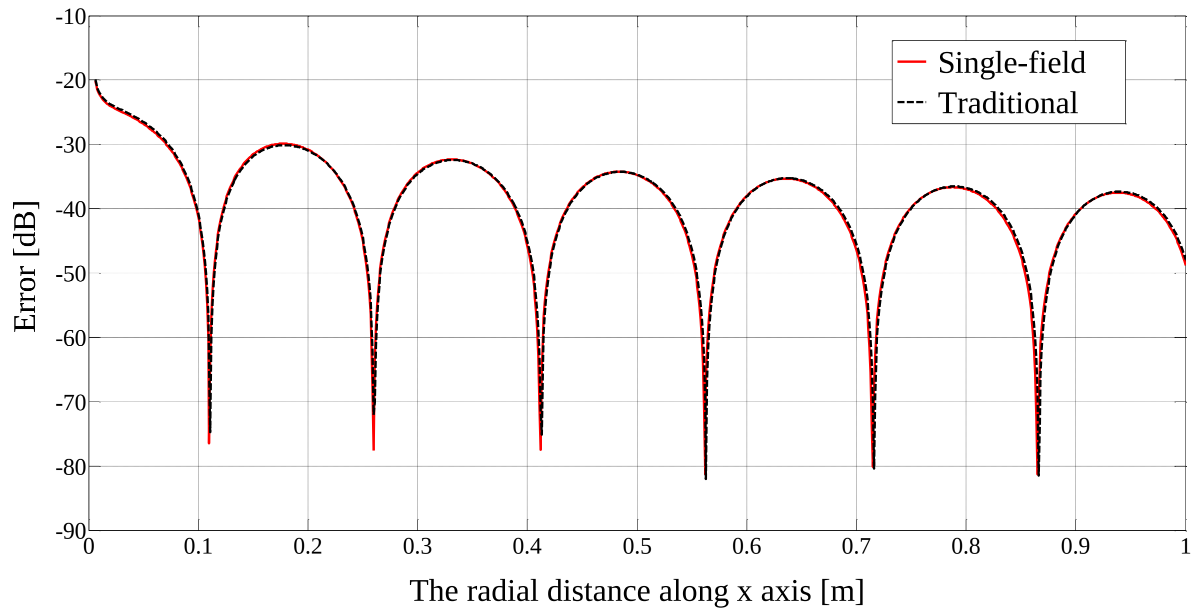

Fig. 4.15 Comparison of the numerical solutions with the analytical solution; phase.

4.5. A 2D TM Problem with Dielectric and PEC Scatterers

A two-dimensional problem is constructed as free space with a *z*-directed impressed electric current located at the origin. The current density has a Gaussian waveform with magnitude of 1 [V/m]. A dielectric square of size 1 mm with dielectric constant 2.2 is located at (0.5, 1.5) mm and a square PEC of size 1 mm is located at (-0.5, -1.5) mm. In addition, one dielectric circular media of radius 1 mm with dielectric constant 2.2 and another circular media with dielectric constant 3.2, relative permeability 1.4, electric conductivity 0.5, and magnetic conductivity 0.3 are located at (-2, 1) mm and (2, -1) mm, respectively [5].

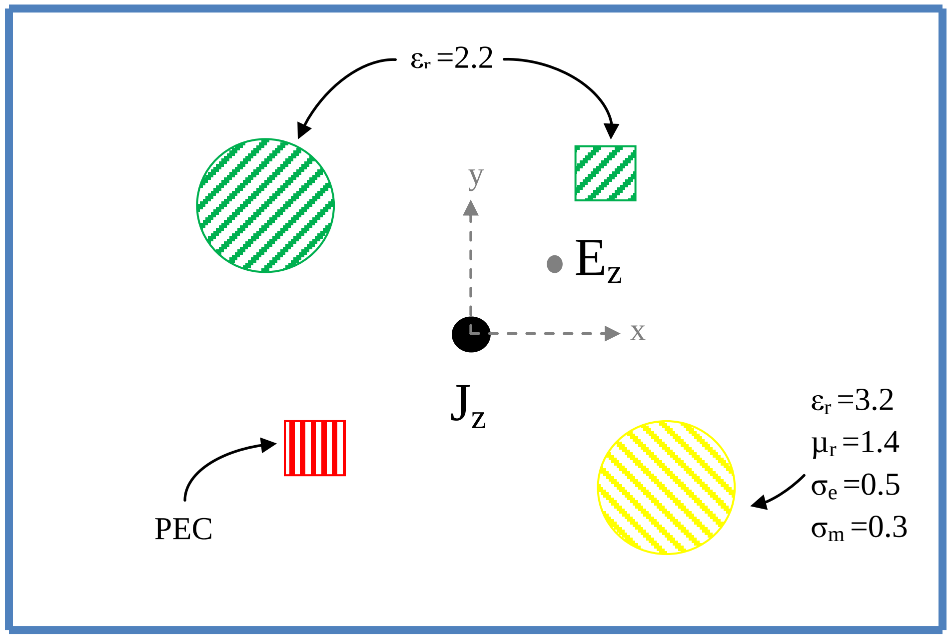

Fig. 4.16 2DTM problem with an electric line current in the presence of objects of different materials and shapes.

Electric field values sampled at (0.8, 0.8) mm by the traditional and the single-field formulations are compared in time and frequency domain. Figure 4.17 shows a comparison between field values calculated by both formulations in time-domain. Figures 4.18 and 4.19 show a comparison of magnitude and phase of the field values in frequency domain, respectively.

CPU time comparison is also performed for this configuration and the resulting speed up is the same as the one shown in Fig. 4.11, as expected. This example is of great significance for the validity of the single-field formulation as it includes non-zero electric and magnetic conductivity in addition to dielectric and magnetic property in scatterers.

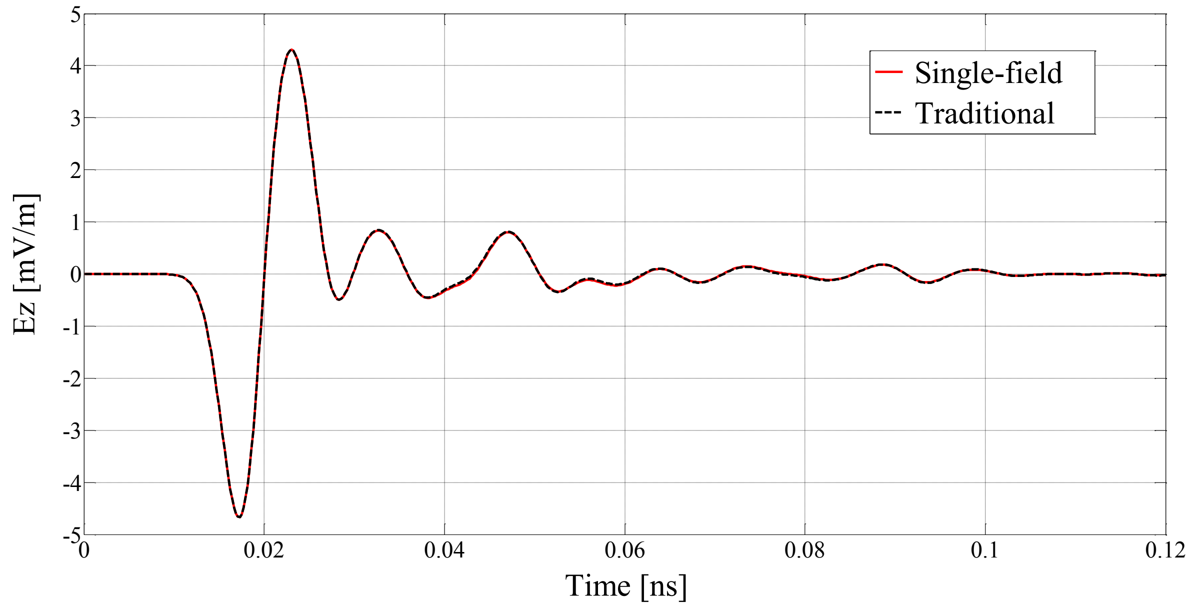

Fig. 4.17 Time-domain comparison.

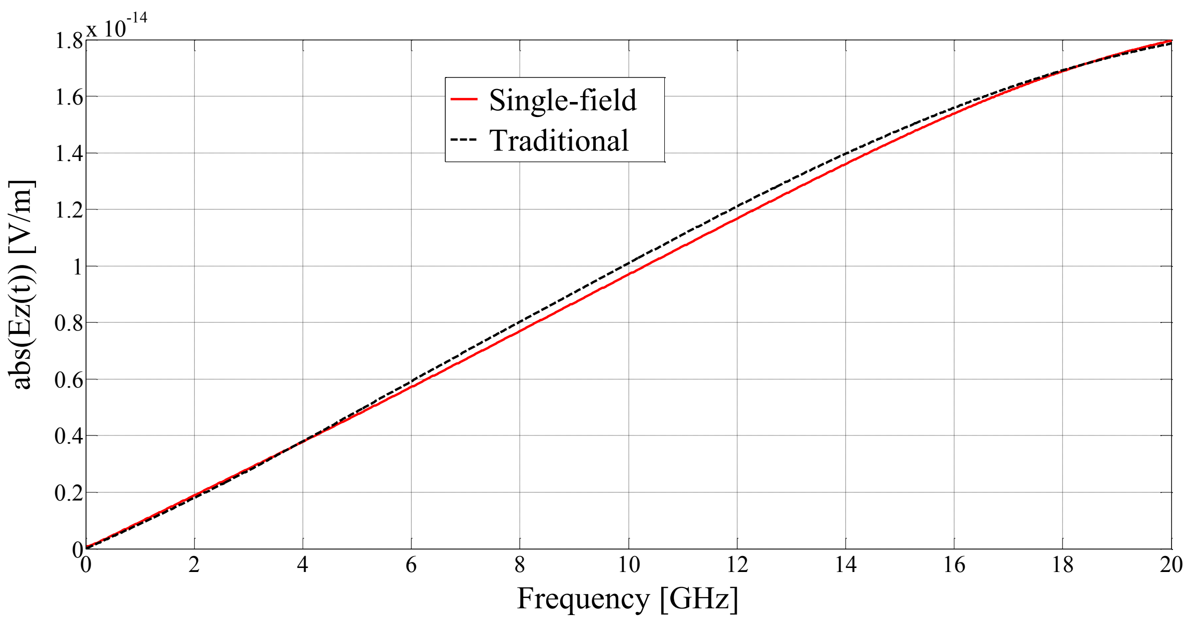

Fig. 4.18 Frequency domain comparison: magnitude.

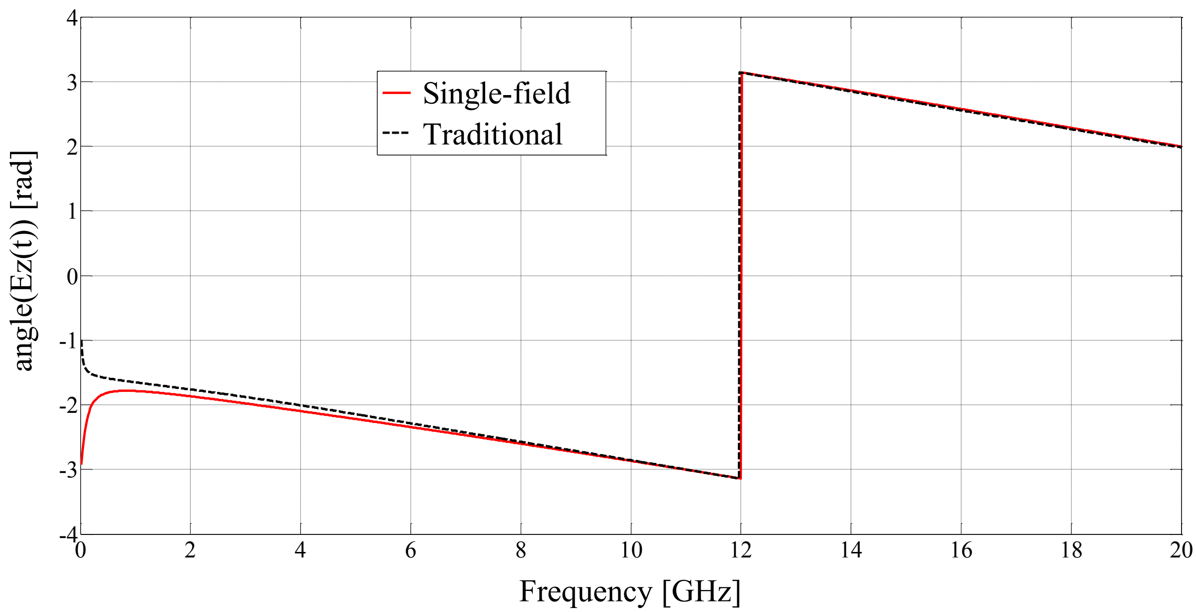

Fig. 4.19 Frequency domain comparison: phase.

4.6. 2D Analysis of a Horn Antenna

FDTD solution is produced for a sectoral (2D) PEC horn antenna excited by a sinusoidal voltage in a TE_z computational domain. The computational domain is truncated by a Liao absorbing boundary condition (ABC). The ABC is introduced to eliminate reflections from the grid truncation and to simulate outgoing traveling wave propagation in an unbounded medium. The horn is modeled by setting the necessary FDTD update equation coefficients to represent the PEC material walls.

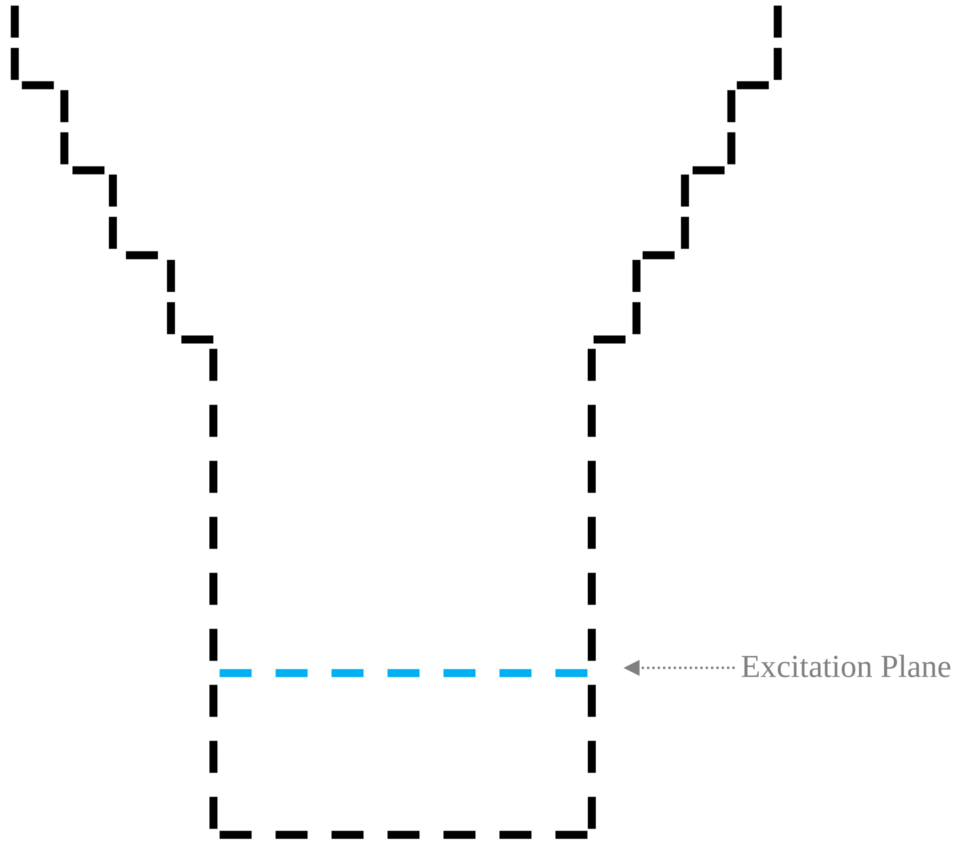

Fig. 4.20 TE_z 2D horn antenna configuration in the FDTD computational domain.

Figure 4.20 shows how the horn antenna is modeled for the FDTD method, E_y field excites the antenna on the excitation plane. The flare section of the horn is staircased to conform with the Cartesian coordinates used. The simulation is run with the following data: time step 4.23 ps, the frequency of excitation 9.84252 GHz, spatial discretization in x and y 2.5 mm and the wavelegth 30.5 m [23]. This application is a good example of structures that can be characterized by their principal plane patterns. This 2D analysis of the geometry gives substantial engineering insight to the behavior of the antenna with minimum memory requirement and computational time.

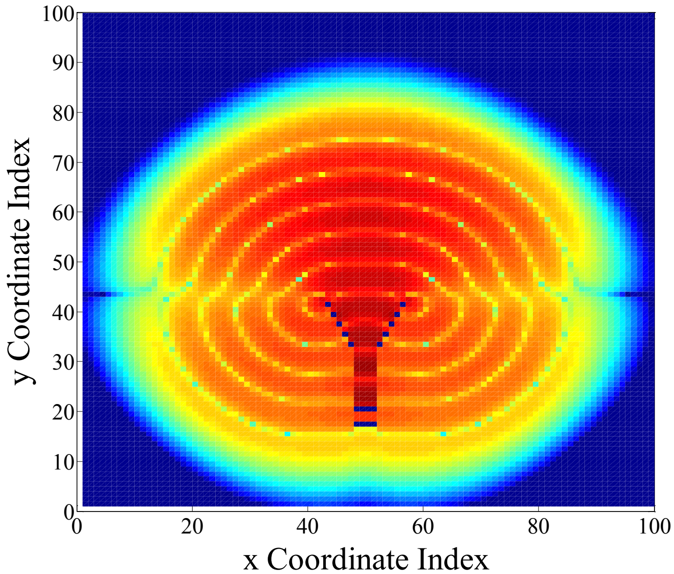

Fig. 4.21 E_y field at 0.5 ns: The single-field formulation.

The y component of the radiated electric field is given for visual comparison between the single-field and the traditional formulations. Figures 4.21 and 4.22 show that both formulations' simulation results are in good match.

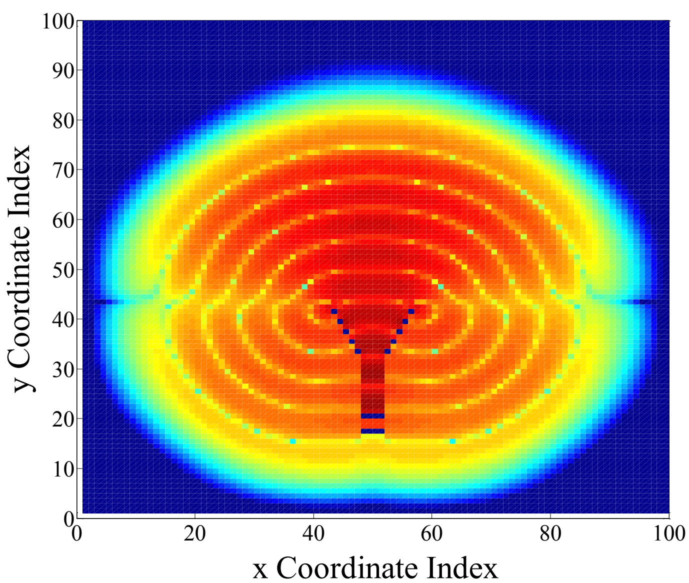

Fig. 4.22 E_y field at 0.5 ns: The traditional formulation.

5. SINGLE-FIELD FDTD UPDATING EQUATIONS for OBLIQUE INCIDENCE

Starting with Maxwell's equations for the incident and the total field, one can obtain the vector wave equation and solve it for each component of the Cartesian coordinate system as scalar equations.

One can write Maxwell's equations for incident field in free space as

$$\nabla \times \boldsymbol{E}_{inc}(t) = -\mu_0 \frac{\partial \boldsymbol{H}_{inc}(t)}{\partial t} \tag{5.1}$$

$$\nabla \times \boldsymbol{H}_{inc}(t) = \varepsilon_0 \frac{\partial \boldsymbol{E}_{inc}(t)}{\partial t} \tag{5.2}$$

and for the total field as

$$\nabla \times \boldsymbol{E}_{tot}(t) = -\mu \frac{\partial \boldsymbol{H}_{tot}(t)}{\partial t} - \sigma^m \boldsymbol{H}_{tot}(t) \tag{5.3}$$

$$\nabla \times \boldsymbol{H}_{tot}(t) = \varepsilon \frac{\partial \boldsymbol{E}_{tot}(t)}{\partial t} + \sigma^e \boldsymbol{E}_{tot}(t) \tag{5.4}$$

The total field is comprised of incident and scattered field components

$$\boldsymbol{E}_{tot} = \boldsymbol{E}_{inc} + \boldsymbol{E}_{scat} \tag{5.5}$$

$$\boldsymbol{H}_{tot} = \boldsymbol{H}_{inc} + \boldsymbol{H}_{scat} \tag{5.6}$$

Taking the curl of (5.1) and (5.3), we have:

$$\nabla \times \left(\nabla \times \boldsymbol{E}_{inc} = -\mu_0 \frac{\partial \boldsymbol{H}_{inc}}{\partial t} \right) \tag{5.7}$$

$$\nabla \times \nabla \times \boldsymbol{E}_{inc} = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}_{inc}) \tag{5.8}$$

$$\nabla \times \nabla \times \boldsymbol{E}_{inc} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}_{inc}}{\partial t^2} \tag{5.9}$$

$$\nabla \times \nabla \times \boldsymbol{E}_{tot} = -\mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}_{tot}) - \sigma^m (\nabla \times \boldsymbol{H}_{tot}) \tag{5.10}$$

Using (5.4), (5.5) and (5.9)

$$\begin{aligned} &\left(-\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}_{inc}}{\partial t^2} \right) + \nabla \times \nabla \times \boldsymbol{E}_{scat} \\ &\qquad = -\mu \frac{\partial}{\partial t} \left(\varepsilon \frac{\partial \boldsymbol{E}_{tot}}{\partial t} + \sigma^e \boldsymbol{E}_{tot} \right) - \sigma^m \left(\varepsilon \frac{\partial \boldsymbol{E}_{tot}}{\partial t} + \sigma^e \boldsymbol{E}_{tot} \right) \end{aligned} \tag{5.11}$$

Using the following vector identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla (\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \tag{5.12}$$

$$\begin{aligned} &\nabla^2 \boldsymbol{E}_{scat} - \nabla (\nabla \cdot \boldsymbol{E}_{scat}) \\ &\qquad = (\mu\varepsilon - \mu_0 \varepsilon_0) \frac{\partial^2 \boldsymbol{E}_{inc}}{\partial t^2} + \mu\varepsilon \frac{\partial^2 \boldsymbol{E}_{scat}}{\partial t^2} + (\mu \sigma^e + \varepsilon \sigma^m) \frac{\partial \boldsymbol{E}_{tot}}{\partial t} \\ &\qquad + \sigma^m \sigma^e \boldsymbol{E}_{tot} \end{aligned} \tag{5.13}$$

To implement (5.13) with finite-difference time-domain method, we have to decompose it to its Cartesian components. Moreover, we assume no variation for field magnitude in z direction, but the variation in phase of the field can be obtained from the phase expression of the field.

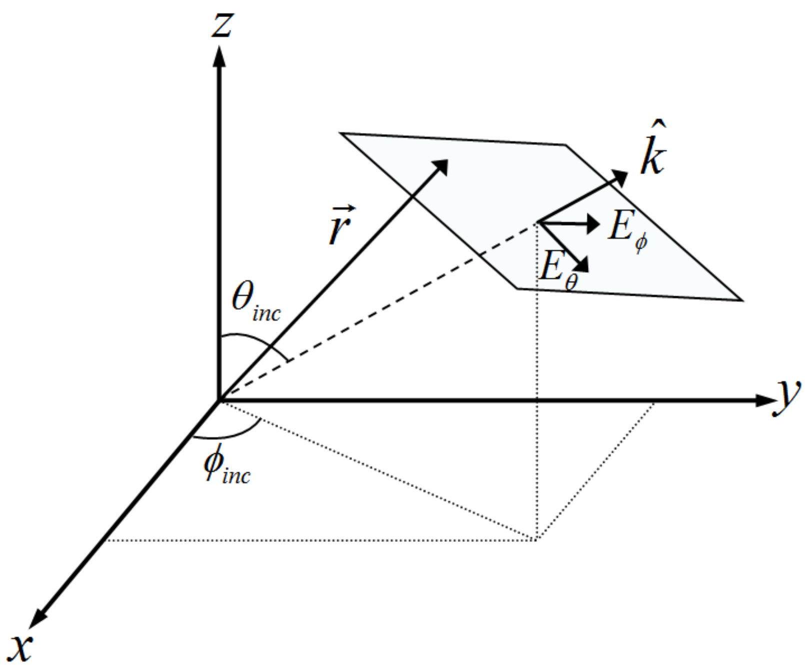

Fig. 5.1 Obliquely incident electric field.

The phase expression of a time-harmonic incident plane field, as shown in Figure 5.1, can be written as

$$e^{j(\boldsymbol{k}\cdot\boldsymbol{r})} \tag{5.14}$$

where the wave vector **k** and the position vector **r** are expressed in Cartesian coordinates as

$$\boldsymbol{k} = k_0(\hat{x}\, sin\, \theta_{inc}\, cos\, \varphi_{inc} + \hat{y}\, sin\, \theta_{inc}\, sin\, \varphi_{inc} + \hat{z}\, cos\, \theta_{inc}) \tag{5.15}$$

$$\boldsymbol{r} = \hat{x}x + \hat{y}y + \hat{z}z \tag{5.16}$$

where

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} \tag{5.17}$$

A general expression for the incident field with a time delay t_0 and spatial shift l_0, as depicted in Figure 5.1, can be written as

$$\boldsymbol{E}_{inc} = (E_\theta \hat{\theta} + E_\varphi \hat{\varphi}) \times f\left((t - t_0) - \frac{1}{c}\left(\hat{k} \cdot \boldsymbol{r} - l_0\right)\right) \tag{5.18}$$

Expressions for the incident plane wave for oblique incidence case are given in Table 5.1.

Table 5.1. The obliquely incident plane wave field expressions.

$TE_z\ \left(E_\theta = 0\ \&\ E_\varphi = 1\right)$	$TM_z\ \left(E_\theta = 1\ \&\ E_\varphi = 0\right)$
$E_{inc,x} = -\sin\varphi_{inc} \times f(t,x,y)$	$E_{inc,x} = \cos\theta_{inc}\cos\varphi_{inc} \times f(t,x,y)$
$E_{inc,y} = \cos\varphi_{inc} \times f(t,x,y)$	$E_{inc,y} = \cos\theta_{inc}\sin\varphi_{inc} \times f(t,x,y)$
$E_{inc,z} = 0$	$E_{inc,z} = -\sin\theta_{inc} \times f(t,x,y)$
$H_{inc,x} = \frac{1}{\eta_0}\cos\theta_{inc}\cos\varphi_{inc} \times f(t,x,y)$	$H_{inc,x} = \frac{1}{\eta_0}\sin\varphi_{inc} \times f(t,x,y)$
$H_{inc,y} = \frac{1}{\eta_0}\cos\theta_{inc}\sin\varphi_{inc} \times f(t,x,y)$	$H_{inc,y} = -\frac{1}{\eta_0}\cos\varphi_{inc} \times f(t,x,y)$
$H_{inc,z} = -\frac{1}{\eta_0}\sin\theta_{inc} \times f(t,x,y)$	$H_{inc,z} = 0$

Using (5.14) and (5.15), one can derive the following identities for the variation in z direction

$$\frac{\partial}{\partial z} = jk_0\, cos\, \theta_{inc} \tag{5.19}$$

$$\frac{\partial}{\partial z} = \sqrt{\mu_0 \varepsilon_0}\, cos\, \theta_{inc} \frac{\partial}{\partial t} \tag{5.20}$$

$$\frac{\partial^2}{\partial z^2} = (jk_0 \cos \theta_{inc})^2 \tag{5.21}$$

$$\frac{\partial^2}{\partial z^2} = \mu_0 \varepsilon_0 (\cos \theta_{inc})^2 \frac{\partial^2}{\partial t^2} \tag{5.22}$$

At this point, we have two options to continue with; either to replace the spatial derivatives with a constant, namely (5.19) and (5.21) or to harness the assumption that the fields are time-harmonic or can be decomposed into harmonic components, hence replace the spatial derivatives with their time derivative equivalents, namely (5.20) and (5.22). Since there is no published work that uses the latter approach to compare with, the former (constant-k) approach will be evaluated in the following section to compare with the traditional formulation [24, 25].

5.1. Derivation of the Updating Equations for Oblique Case

5.1.1. Updating Equation for the x Component

Cartesian component of (5.13) incorporated with (5.19) and (5.21) in x direction can be written as

$$\begin{aligned} \frac{\partial^2 E_{scat,x}}{\partial y^2} + (jk_0 \cos \theta_{inc})^2 E_{scat,x} - \frac{\partial^2 E_{scat,y}}{\partial x \partial y} - (jk_0\, cos\, \theta_{inc}) \frac{\partial E_{scat,z}}{\partial x} \\ = (\mu\varepsilon - \mu_0 \varepsilon_0) \frac{\partial^2 E_{inc,x}}{\partial t^2} + \mu\varepsilon \frac{\partial^2 E_{scat,x}}{\partial t^2} + (\mu\sigma^e + \varepsilon\sigma^m) \frac{\partial E_{tot,x}}{\partial t} \\ + \sigma^m \sigma^e E_{tot,x} \end{aligned} \tag{5.23}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (5.23) at the corresponding electric field node, i.e. E_x. The time derivatives are evaluated at the nth time step; therefore:

$$\frac{\partial E_x(i,j)}{\partial t} = \frac{E_x^{n+1}(i,j) - E_x^{n-1}(i,j)}{2\Delta t} \tag{5.24}$$

$$\frac{\partial^2 E_x(i,j)}{\partial t^2} = \frac{E_x^{n+1}(i,j) - 2E_x^{n}(i,j) + E_x^{n-1}(i,j)}{(\Delta t)^2} \tag{5.25}$$

$$\frac{\partial^2 E_x(i,j)}{\partial y^2} = \frac{E_x^{n}(i,j+1) - 2E_x^{n}(i,j) + E_x^{n}(i,j-1)}{(\Delta y)^2} \tag{5.26}$$

$$\frac{\partial^2 E_y(i,j)}{\partial x \partial y} = \frac{E_y^{n}(i+1,j) - E_y^{n}(i+1,j-1) - E_y^{n}(i,j) + E_y^{n}(i,j-1)}{\Delta x \Delta y} \tag{5.27}$$

$$\frac{\partial E_z}{\partial x} = \frac{E_z^{n}(i+1,j) - E_z^{n}(i,j)}{\Delta x} \tag{5.28}$$

Inserting (5.24) - (5.28) into (5.23), we have

$$
\begin{aligned}
&\left(\frac{E^{n}_{scat,x}(i,j+1) - 2E^{n}_{scat,x}(i,j) + E^{n}_{scat,x}(i,j-1)}{(\Delta y)^2}\right) \\
&+ (jk_0 \cos\theta_{inc})^2 \left(E^{n}_{scat,x}(i,j)\right) \\
&- \left(\frac{E^{n}_{scat,y}(i+1,j) - E^{n}_{scat,y}(i+1,j-1) - E^{n}_{scat,y}(i,j) + E^{n}_{scat,y}(i,j-1)}{\Delta x \Delta y}\right) \\
&- (jk_0\, cos\,\theta_{inc}) \left(\frac{E^{n}_{scat,z}(i+1,j) - E^{n}_{scat,z}(i,j)}{\Delta x}\right) \\
&= (\mu\varepsilon - \mu_0\varepsilon_0) \left(\frac{E^{n+1}_{inc,x}(i,j) - 2E^{n}_{inc,x}(i,j) + E^{n-1}_{inc,x}(i,j)}{(\Delta t)^2}\right) \\
&+ \mu\varepsilon \left(\frac{E^{n+1}_{scat,x}(i,j) - 2E^{n}_{scat,x}(i,j) + E^{n-1}_{scat,x}(i,j)}{(\Delta t)^2}\right) \\
&+ (\mu\sigma^e + \varepsilon\sigma^m) \left(\frac{E^{n+1}_{scat,x}(i,j) - E^{n-1}_{scat,x}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e \left(E^{n}_{scat,x}(i,j)\right) \\
&+ (\mu\sigma^e + \varepsilon\sigma^m) \left(\frac{E^{n+1}_{inc,x}(i,j) - E^{n-1}_{inc,x}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e \left(E^{n}_{inc,x}(i,j)\right)
\end{aligned} \tag{5.29}
$$

We can simplify the updating equation for E_x as

$$
\begin{aligned}
E^{n+1}_{scat,x}(i,j) = {} & C^{ex}_{ex}(i,j)\left[E^{n}_{scat,x}(i,j)\right] + C^{ex,n-1}_{ex}(i,j)\left[E^{n-1}_{scat,x}(i,j)\right] \\
& + C^{ex,n,y}_{ex}(i,j)\left[E^{n}_{scat,x}(i,j+1) + E^{n}_{scat,x}(i,j-1)\right] \\
& + C^{ey,n,xy}_{ex}(i,j)\left[E^{n}_{scat,y}(i+1,j) - E^{n}_{scat,y}(i+1,j-1)\right. \\
& \left. - E^{n}_{scat,y}(i,j) + E^{n}_{scat,y}(i,j-1)\right] \\
& + C^{ez,n,\theta,x}_{ex}(i,j)\left[E^{n}_{scat,z}(i+1,j) - E^{n}_{scat,z}(i,j)\right] \\
& + C^{exinc,n+1}_{ex}(i,j)\left[E^{n+1}_{inc,x}(i,j)\right] + C^{exinc,n}_{ex}(i,j)\left[E^{n}_{inc,x}(i,j)\right] \\
& + C^{exinc,n-1}_{ex}(i,j)\left[E^{n-1}_{inc,x}(i,j)\right]
\end{aligned} \tag{5.30}
$$

where

$$
C_x(i,j) = -\left(\frac{\mu\varepsilon}{(\Delta t)^2} + \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right)^{-1} \tag{5.31}
$$

$$
C^{ex}_{ex}(i,j) = C_x(i,j)\left(\sigma^m\sigma^e - (jk_0\cos\theta_{inc})^2 - \frac{2\mu\varepsilon}{(\Delta t)^2} + \frac{2}{(\Delta y)^2}\right) \tag{5.32}
$$

$$C_{ex}^{ex,n-1}(i,j) = C_x(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \quad (5.33)$$

$$C_{ex}^{ex,n,y}(i,j) = -C_x(i,j)\left(\frac{1}{(\Delta y)^2}\right) \quad (5.34)$$

$$C_{ex}^{ey,n,xy}(i,j) = C_x(i,j)\left(\frac{1}{\Delta x \Delta y}\right) \quad (5.35)$$

$$C_{ex}^{ez,n,\theta,x}(i,j) = C_x(i,j)\left(\frac{jk_0\, cos\, \theta_{inc}}{\Delta x}\right) \quad (5.36)$$

$$C_{ex}^{exinc,n+1}(i,j) = C_x(i,j)\left(\frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t} - \frac{(\mu_0\varepsilon_0 - \mu\varepsilon)}{(\Delta t)^2}\right) \quad (5.37)$$

$$C_{ex}^{exinc,n}(i,j) = C_x(i,j)\left(\frac{2(\mu_0\varepsilon_0 - \mu\varepsilon)}{(\Delta t)^2} + \sigma^m\sigma^e\right) \quad (5.38)$$

$$C_{ex}^{exinc,n-1}(i,j) = -C_x(i,j)\left(\frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t} + \frac{(\mu_0\varepsilon_0 - \mu\varepsilon)}{(\Delta t)^2}\right) \quad (5.39)$$

5.1.2. Updating Equation for the y Component

Cartesian component of (5.13) with (5.19) and (5.21) incorporated in y direction can be written as

$$\begin{aligned}\frac{\partial^2 E_{scat,y}}{\partial x^2} + (jk_0 \cos\theta_{inc})^2 E_{scat,y} - \frac{\partial^2 E_{scat,x}}{\partial x \partial y} - (jk_0\, cos\, \theta_{inc})\frac{\partial E_{scat,z}}{\partial y} \\ = (\mu\varepsilon - \mu_0\varepsilon_0)\frac{\partial^2 E_{inc,y}}{\partial t^2} + \mu\varepsilon\frac{\partial^2 E_{scat,y}}{\partial t^2} + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_{tot,y}}{\partial t} \\ + \sigma^m\sigma^e E_{tot,y}\end{aligned} \quad (5.40)$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (5.40) at the corresponding electric field node, i.e. E_y. The time derivatives are evaluated at the n[th] time step; therefore:

$$\frac{\partial E_y(i,j)}{\partial t} = \frac{E_y^{n+1}(i,j) - E_y^{n-1}(i,j)}{2\Delta t} \tag{5.41}$$

$$\frac{\partial^2 E_y(i,j)}{\partial t^2} = \frac{E_y^{n+1}(i,j) - 2E_y^{n}(i,j) + E_y^{n-1}(i,j)}{(\Delta t)^2} \tag{5.42}$$

$$\frac{\partial^2 E_y(i,j)}{\partial x^2} = \frac{E_y^{n}(i+1,j) - 2E_y^{n}(i,j) + E_y^{n}(i-1,j)}{(\Delta x)^2} \tag{5.43}$$

$$\frac{\partial^2 E_x(i,j)}{\partial x \partial y} = \frac{E_x^{n}(i+1,j) - E_x^{n}(i+1,j-1) - E_x^{n}(i,j) + E_x^{n}(i,j-1)}{\Delta x \Delta y} \tag{5.44}$$

$$\frac{\partial E_z(i,j)}{\partial y} = \frac{E_z^{n}(i,j+1) - E_z^{n}(i,j)}{\Delta y} \tag{5.45}$$

Inserting (5.41) - (5.45) into (5.40), we have

$$
\begin{aligned}
&\left(\frac{E^n_{scat,y}(i+1,j) - 2E^n_{scat,y}(i,j) + E^n_{scat,y}(i-1,j)}{(\Delta x)^2}\right)\\
&+ (jk_0\cos\theta_{inc})^2\left(E^n_{scat,y}(i,j)\right)\\
&- \left(\frac{E^n_{scat,x}(i+1,j) - E^n_{scat,x}(i+1,j-1) - E^n_{scat,x}(i,j) + E^n_{scat,x}(i,j-1)}{\Delta x\Delta y}\right)\\
&- (jk_0\,cos\,\theta_{inc})\left(\frac{E^n_{scat,z}(i,j+1) - E^n_{scat,z}(i,j)}{\Delta y}\right)\\
&= (\mu\varepsilon - \mu_0\varepsilon_0)\left(\frac{E^{n+1}_{inc,y}(i,j) - 2E^n_{inc,y}(i,j) + E^{n-1}_{inc,y}(i,j)}{(\Delta t)^2}\right)\\
&+ \mu\varepsilon\left(\frac{E^{n+1}_{scat,y}(i,j) - 2E^n_{scat,y}(i,j) + E^{n-1}_{scat,y}(i,j)}{(\Delta t)^2}\right)\\
&+ (\mu\sigma^e + \varepsilon\sigma^m)\left(\frac{E^{n+1}_{scat,y}(i,j) - E^{n-1}_{scat,y}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e\left(E^n_{scat,y}(i,j)\right)\\
&+ (\mu\sigma^e + \varepsilon\sigma^m)\left(\frac{E^{n+1}_{inc,y}(i,j) - E^{n-1}_{inc,y}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e\left(E^n_{inc,y}(i,j)\right)
\end{aligned}
\tag{5.46}
$$

We can simplify the updating equation for E_y as

$$
\begin{aligned}
E^{n+1}_{scat,y}(i,j) = {} & C^{ey,n}_{ey}(i,j)\left[E^n_{scat,y}(i,j)\right] + C^{ey,n-1}_{ey}(i,j)\left[E^{n-1}_{scat,y}(i,j)\right]\\
&+ C^{ey,n,x}_{ey}(i,j)\left[E^n_{scat,y}(i+1,j) + E^n_{scat,y}(i-1,j)\right]\\
&+ C^{ex,n,xy}_{ey}(i,j)\left[E^n_{scat,x}(i+1,j) - E^n_{scat,x}(i+1,j-1)\right.\\
&\left.- E^n_{scat,x}(i,j) + E^n_{scat,x}(i,j-1)\right]\\
&+ C^{ez,n,\theta,xt}_{ey}(i,j)\left[E^n_{scat,z}(i,j+1) - E^n_{scat,z}(i,j)\right]\\
&+ C^{eincy,n+1}_{ey}(i,j)\left[E^{n+1}_{inc,y}(i,j)\right] + C^{eincy,n}_{ey}(i,j)\left[E^n_{inc,y}(i,j)\right]\\
&+ C^{eincy,n-1}_{ey}(i,j)\left[E^{n-1}_{inc,y}(i,j)\right]
\end{aligned}
\tag{5.47}
$$

where

$$
C_y(i,j) = \left(-\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right)^{-1} \tag{5.48}
$$

$$C_{ey}^{ey,n}(i,j) = C_y(i,j)\left(\sigma^m\sigma^e - (jk_0 \cos\theta_{inc})^2 - \frac{2\mu\varepsilon}{(\Delta t)^2} + \frac{2}{(\Delta x)^2}\right) \tag{5.49}$$

$$C_{ey}^{ey,n-1}(i,j) = C_y(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{5.50}$$

$$C_{ey}^{ey,n,x}(i,j) = -C_y(i,j)\left(\frac{1}{(\Delta x)^2}\right) \tag{5.51}$$

$$C_{ey}^{ex,n,xy}(i,j) = C_y(i,j)\left(\frac{1}{\Delta x\Delta y}\right) \tag{5.52}$$

$$C_{ey}^{ez,n,\theta,xt}(i,j) = C_y(i,j)\left(\frac{jk_0 \cos\theta_{inc}}{\Delta y}\right) \tag{5.53}$$

$$C_{ey}^{eincy,n+1}(i,j) = C_y(i,j)\left(\frac{(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2} + \frac{\mu\sigma^e + \varepsilon\sigma^m}{2\Delta t}\right) \tag{5.54}$$

$$C_{ey}^{eincy,n}(i,j) = C_y(i,j)\left(\sigma^m\sigma^e - \frac{2(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2}\right) \tag{5.55}$$

$$C_{ey}^{eincy,n-1}(i,j) = C_y(i,j)\left(\frac{(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2} - \frac{\mu\sigma^e + \varepsilon\sigma^m}{2\Delta t}\right) \tag{5.56}$$

5.1.3. Updating Equation for the z Component

Cartesian component of (5.13) incorporated with (5.19) and (5.21) in z direction can be written as

$$\begin{aligned}\frac{\partial^2 E_{scat,z}}{\partial x^2} + \frac{\partial^2 E_{scat,z}}{\partial y^2} - (jk_0 \cos\theta_{inc})\frac{\partial E_{scat,x}}{\partial x} - (jk_0 \cos\theta_{inc})\frac{\partial E_{scat,y}}{\partial y} \\ = (\mu\varepsilon - \mu_0\varepsilon_0)\frac{\partial^2 E_{inc,z}}{\partial t^2} + \mu\varepsilon\frac{\partial^2 E_{scat,z}}{\partial t^2} + (\mu\sigma^e + \varepsilon\sigma^m)\frac{\partial E_{tot,z}}{\partial t} \\ + \sigma^m\sigma^e E_{tot,z}\end{aligned} \tag{5.57}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (5.57) at the corresponding electric field node, i.e. E_z. The time derivatives are evaluated at the n[th] time step; therefore:

$$\frac{\partial E_z(i,j)}{\partial t} = \frac{E_z^{n+1}(i,j) - E_z^{n-1}(i,j)}{2\Delta t} \tag{5.58}$$

$$\frac{\partial^2 E_z(i,j)}{\partial t^2} = \frac{E_z^{n+1}(i,j) - 2E_z^{n}(i,j) + E_z^{n-1}(i,j)}{(\Delta t)^2} \tag{5.59}$$

$$\frac{\partial^2 E_z(i,j)}{\partial x^2} = \frac{E_z^{n}(i+1,j) - 2E_z^{n}(i,j) + E_z^{n}(i-1,j)}{(\Delta x)^2} \tag{5.60}$$

$$\frac{\partial^2 E_z(i,j)}{\partial y^2} = \frac{E_z^{n}(i,j+1) - 2E_z^{n}(i,j) + E_z^{n}(i,j-1)}{(\Delta y)^2} \tag{5.61}$$

$$\frac{\partial E_x(i,j)}{\partial x} = \frac{E_x^{n}(i,j) - E_x^{n}(i-1,j)}{\Delta x} \tag{5.62}$$

$$\frac{\partial E_y(i,j)}{\partial y} = \frac{E_y^{n}(i,j) - E_y^{n}(i,j-1)}{\Delta y} \tag{5.63}$$

Inserting (5.58) - (5.63) into (5.57), we have

$$
\begin{aligned}
&\left(\frac{E^n_{scat,z}(i+1,j) - 2E^n_{scat,z}(i,j) + E^n_{scat,z}(i-1,j)}{(\Delta x)^2}\right) \\
&\quad + \left(\frac{E^n_{scat,z}(i,j+1) - 2E^n_{scat,z}(i,j) + E^n_{scat,z}(i,j-1)}{(\Delta y)^2}\right) \\
&\quad - jk_0 \cos\theta_{inc} \left(\frac{E^n_{scat,x}(i,j) - E^n_{scat,x}(i-1,j)}{\Delta x}\right) \\
&\quad - jk_0 \cos\theta_{inc} \left(\frac{E^n_{scat,y}(i,j) - E^n_{scat,y}(i,j-1)}{\Delta y}\right) \\
&\quad = (\mu\varepsilon - \mu_0\varepsilon_0)\left(\frac{E^{n+1}_{inc,z}(i,j) - 2E^n_{inc,z}(i,j) + E^{n-1}_{inc,z}(i,j)}{(\Delta t)^2}\right) \\
&\quad + \mu\varepsilon\left(\frac{E^{n+1}_{scat,z}(i,j) - 2E^n_{scat,z}(i,j) + E^{n-1}_{scat,z}(i,j)}{(\Delta t)^2}\right) \\
&\quad + (\mu\sigma^e + \varepsilon\sigma^m)\left(\frac{E^{n+1}_{scat,z}(i,j) - E^{n-1}_{scat,z}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e\left(E^n_{scat,z}(i,j)\right) \\
&\quad + (\mu\sigma^e + \varepsilon\sigma^m)\left(\frac{E^{n+1}_{inc,z}(i,j) - E^{n-1}_{inc,z}(i,j)}{2\Delta t}\right) + \sigma^m\sigma^e\left(E^n_{inc,z}(i,j)\right)
\end{aligned} \tag{5.64}
$$

We can simplify the updating equation for E_z as

$$
\begin{aligned}
E^{n+1}_{scat,z}(i,j) = {} & C^{ez,n}_{ez}(i,j)\left[E^n_{scat,z}(i,j)\right] + C^{ez,n-1}_{ez}(i,j)\left[E^{n-1}_{scat,z}(i,j)\right] \\
& + C^{ez,n,x}_{ez}(i,j)\left[E^n_{scat,z}(i+1,j) + E^n_{scat,z}(i-1,j)\right] \\
& + C^{ez,n,y}_{ez}(i,j)\left[E^n_{scat,z}(i,j+1) + E^n_{scat,z}(i,j-1)\right] \\
& + C^{ex,n,\theta,xt}_{ez}(i,j)\left[E^n_{scat,x}(i,j) - E^n_{scat,x}(i-1,j)\right] \\
& + C^{ey,n,\theta,yt}_{ez}(i,j)\left[E^n_{scat,y}(i,j) - E^n_{scat,y}(i,j-1)\right] \\
& + C^{eincz,n+1}_{ez}(i,j)\left[E^{n+1}_{inc,z}(i,j)\right] + C^{eincz,n}_{ez}\left[E^n_{inc,z}(i,j)\right] \\
& + C^{eincz,n-1}_{ez}(i,j)\left[E^{n-1}_{inc,z}(i,j)\right]
\end{aligned} \tag{5.65}
$$

where

$$
C_z(i,j) = -\left(\frac{\mu\varepsilon}{(\Delta t)^2} + \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right)^{-1} \tag{5.66}
$$

$$C_{ez}^{ez,n}(i,j) = C_z(i,j)\left(\sigma^m\sigma^e - \frac{2\mu\varepsilon}{(\Delta t)^2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right) \tag{5.67}$$

$$C_{ez}^{ez,n-1}(i,j) = C_z(i,j)\left(\frac{\mu\varepsilon}{(\Delta t)^2} - \frac{(\mu\sigma^e + \varepsilon\sigma^m)}{2\Delta t}\right) \tag{5.68}$$

$$C_{ez}^{ez,n,x}(i,j) = -C_z(i,j)\left(\frac{1}{(\Delta x)^2}\right) \tag{5.69}$$

$$C_{ez}^{ez,n,y}(i,j) = -C_z(i,j)\left(\frac{1}{(\Delta y)^2}\right) \tag{5.70}$$

$$C_{ez}^{ex,n,\theta,xt}(i,j) = C_z(i,j)\left(\frac{jk_0 \cos\theta_{inc}}{\Delta x}\right) \tag{5.71}$$

$$C_{ez}^{ey,n,\theta,yt}(i,j) = C_z(i,j)\left(\frac{jk_0 \cos\theta_{inc}}{\Delta y}\right) \tag{5.72}$$

$$C_{ez}^{eincz,n+1}(i,j) = C_z(i,j)\left(\frac{(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2} + \frac{\mu\sigma^e + \varepsilon\sigma^m}{2\Delta t}\right) \tag{5.73}$$

$$C_{ez}^{eincz,n}(i,j) = C_z(i,j)\left(\sigma^m\sigma^e - \frac{2(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2}\right) \tag{5.74}$$

$$C_{ez}^{eincz,n-1}(i,j) = C_z(i,j)\left(\frac{(\mu\varepsilon - \mu_0\varepsilon_0)}{(\Delta t)^2} - \frac{\mu\sigma^e + \varepsilon\sigma^m}{2\Delta t}\right) \tag{5.75}$$

5.2. Accuracy Analysis

Problem geometry is constructed as shown in Figures 5.2 and 5.3. Three dielectric cylinders are located on y axis, as a plane wave is obliquely incident towards -x direction. Scattered fields are sampled at 500 points on the x-axis (0-0.5 m, 0). The dielectric cylinders have radius of 1 cm and dielectric constant 4. Center-to-center distance is 3 cm. The incident wave has a Gaussian shape with maximum frequency 15 GHz, the azimuth angle (φ_{inc}) is 0 with respect to the x axis and its angle of incidence (θ_{inc}) is 30 degrees with respect to the z axis. Figures 5.4 to 5.9 show the field comparison of the single-field and the traditional formulation. In both formulations, constant-k approach is utilized, and the k_0 value is set according to the following formula

$$k = \omega\sqrt{\mu\varepsilon}\cos\theta \tag{5.76}$$

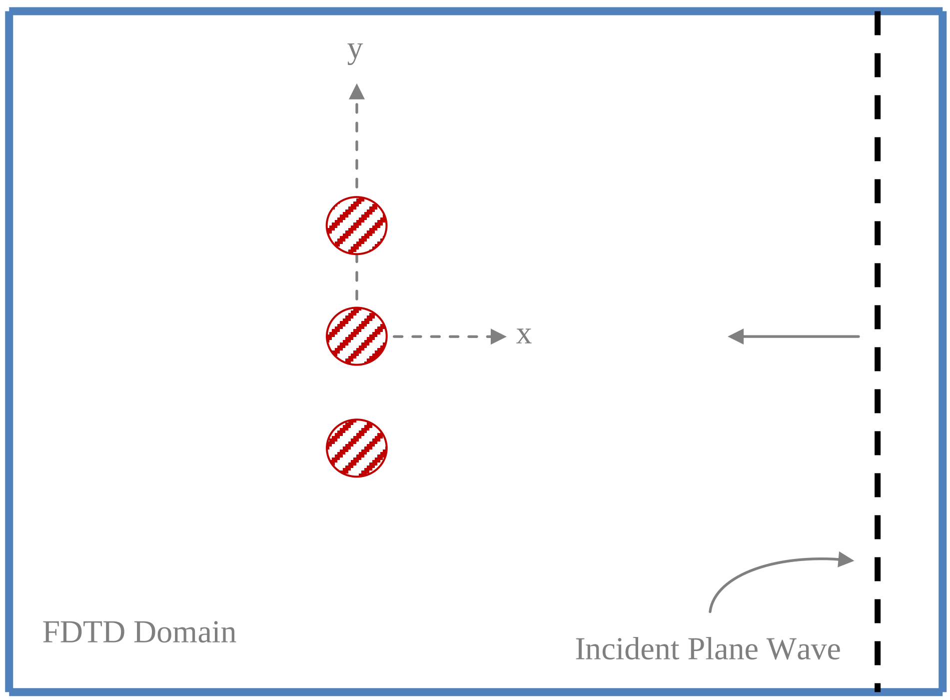

Fig. 5.2 Three dielectric cylinders subject to an obliquely incident plane wave.

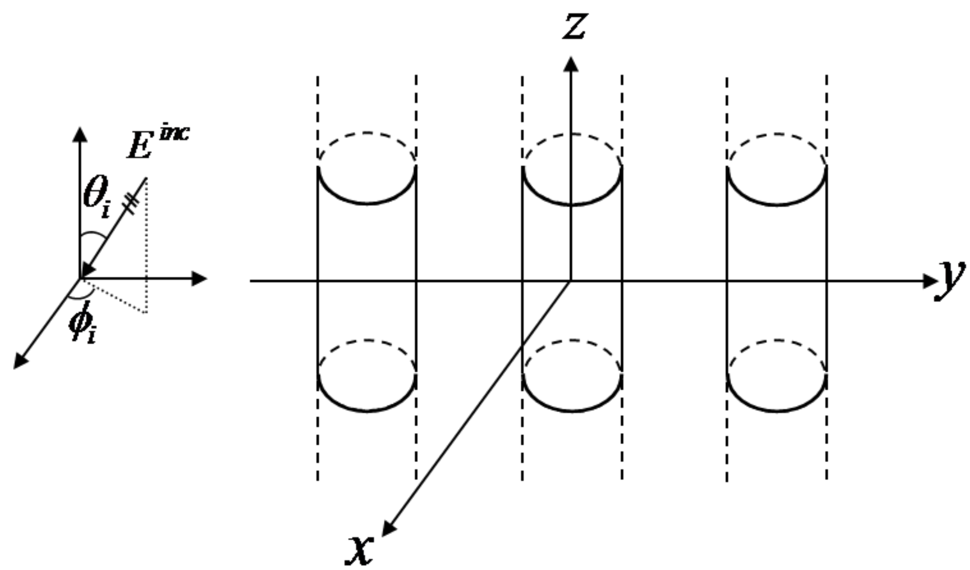

Fig. 5.3 Arrangement of the cylinders.

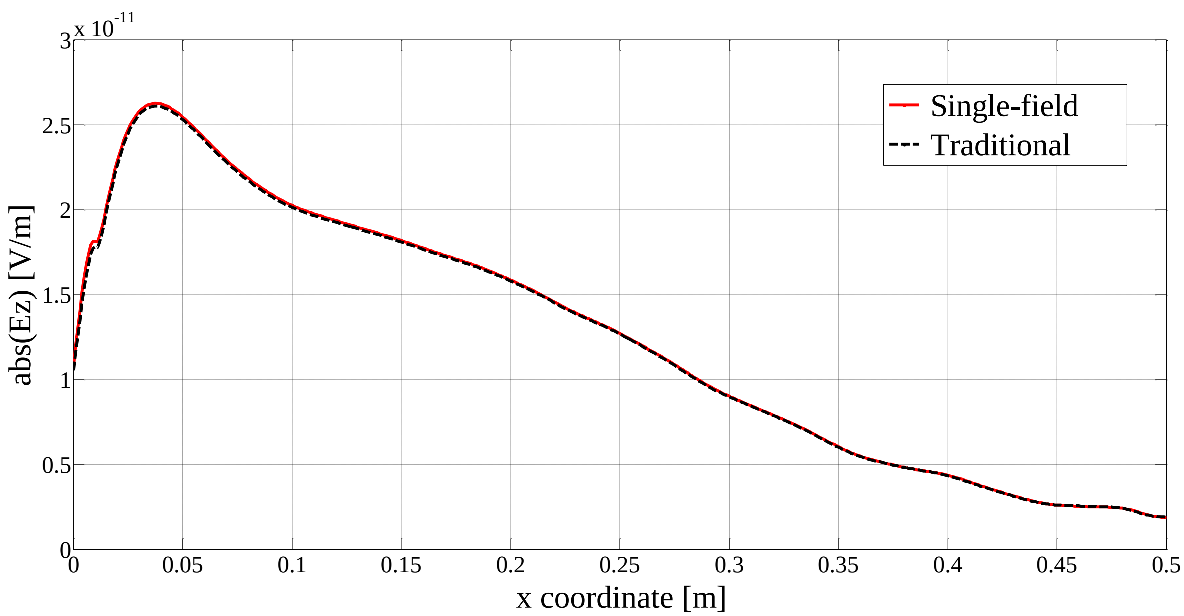

Fig. 5.4 Electric field comparison for f = 5 GHz: magnitude.

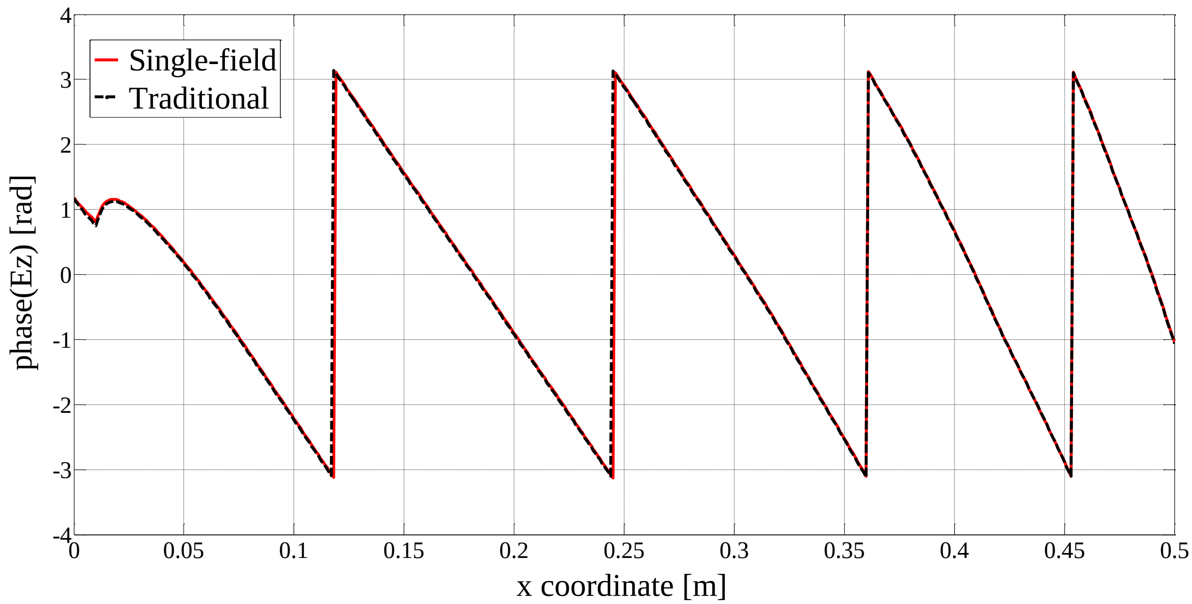

Fig. 5.5 Electric field comparison for f = 5 GHz: phase.

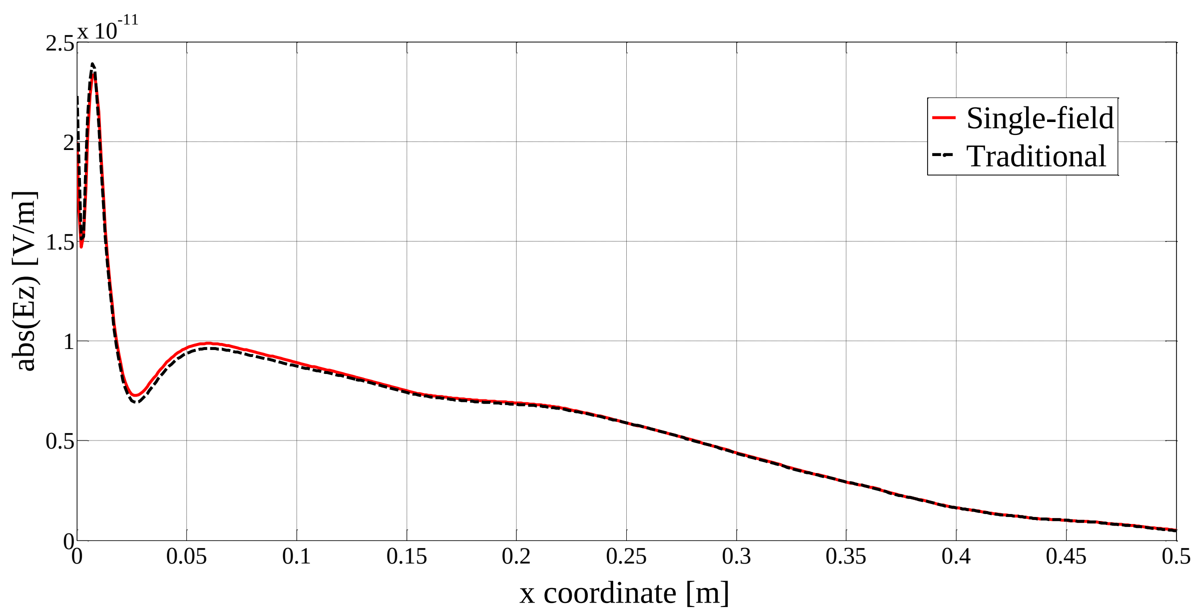

Fig. 5.6 Electric field comparison for f = 10 GHz: magnitude.

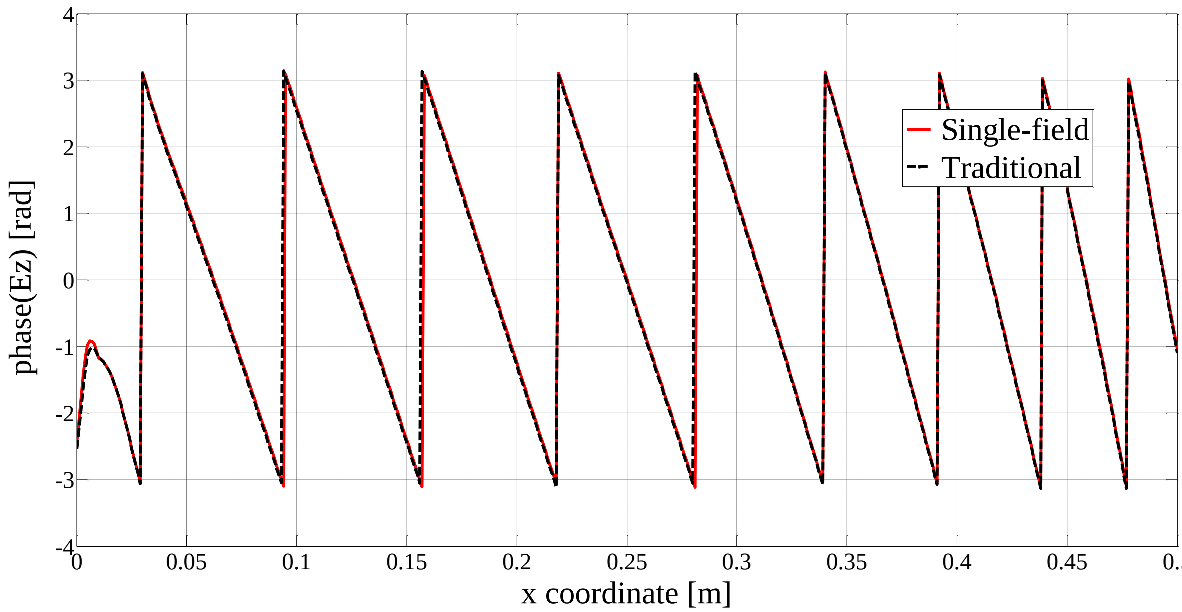

Fig. 5.7 Electric field comparison for f = 10 GHz: phase.

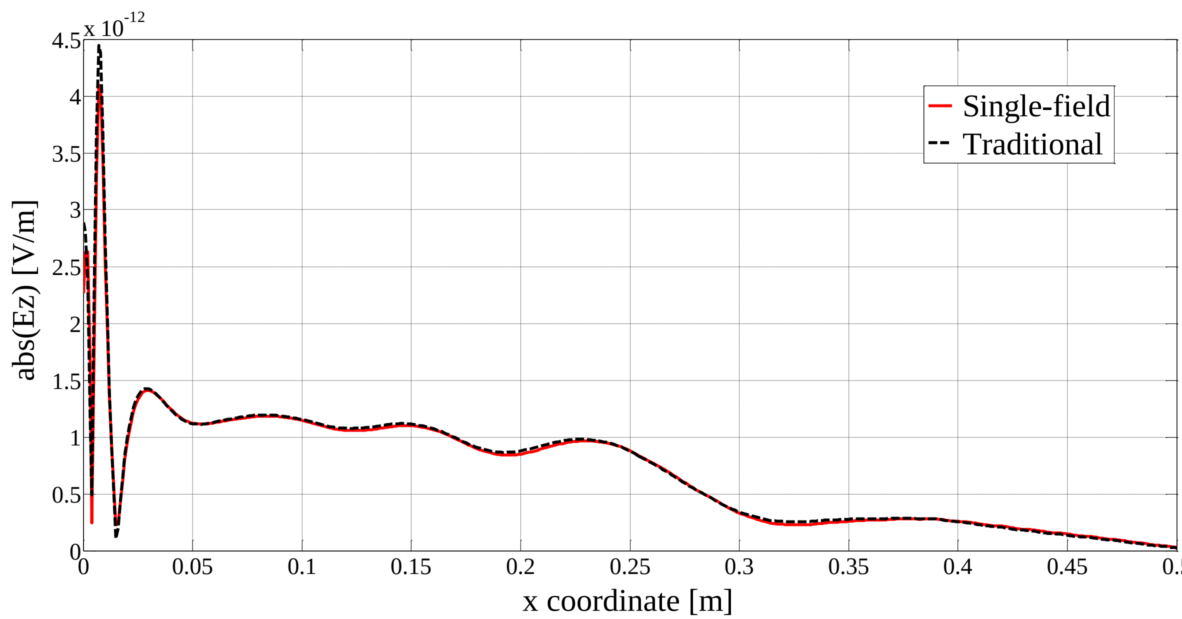

Fig. 5.8 Electric field comparison for f = 15 GHz: magnitude.

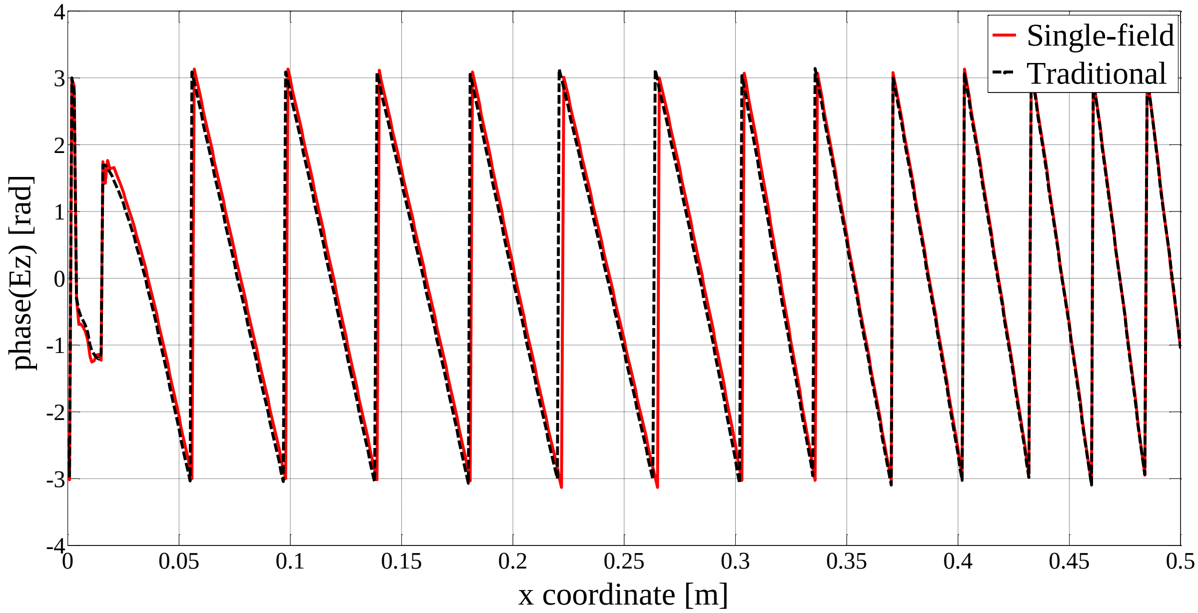

Fig. 5.9 Electric field comparison for f = 15 GHz: phase.

Figures 5.4 to 5.9 show a good agreement between the single-field and the traditional formulation. The k value is calculated as 91, 182 and 273 for incidence angle of 30 degrees and for frequency of 5, 10 and 15 GHz, respectively, according to the Equation (5.76).

5.3. CPU Time Analysis

Figure 5.10 shows the CPU time the formulations require to complete a simulation of corresponding size. X axis represents the number of cells used to characterize the problem. Though in oblique incidence case the single-field formulation has to use three updating equations as opposed to the normal incident case where only one updating equation is used to solve any 2D TE or TM problem, the single-field formulation is still faster than the traditional one as shown in Figure 5.10. This result reinforces the argument that the single-field formulation is faster than the traditional one for any 2D problem in the case of both normal and oblique incidence.

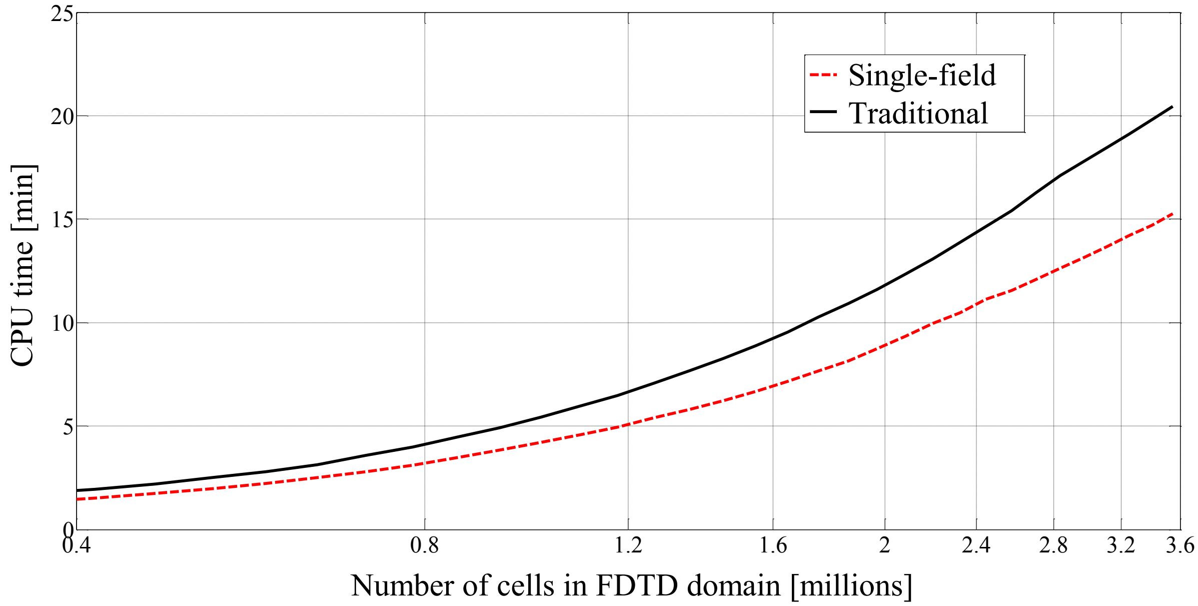

Fig. 5.10 CPU time comparison for 2D oblique case.

5.4. Memory Usage Analysis

Memory requirements for both formulations can be compared by counting the number of coefficients used in the updating equations in addition to the scattered and incident field terms needed. The traditional formulation has six updating equations for constant-k approach, such as

$$\begin{aligned} E_{scat,x}^{n+1}(i,j) = C_1(i,j)\left[E_{scat,x}^{n}(i,j)\right] + C_2(i,j)\left[H_{scat,z}^{n+0.5}(i,j) - H_{scat,z}^{n+0.5}(i,j-1)\right] \\ + C_3(i,j)\left[H_{scat,y}^{n+0.5}(i,j)\right] + C_4(i,j)\left[E_{inc,x}^{n+1}(i,j)\right] + C_5(i,j)\left[E_{inc,x}^{n}(i,j)\right] \end{aligned} \tag{5.77}$$

Table 5.2 shows the number of FLMOPn and MAFTn required for the single-field and the traditional formulations. Given memory allocations are for updating equations only, problem domain and material related memory allocations are not mentioned here since they apply in both formulations. The single-field formulation seems to require less memory; therefore it can handle bigger problems than the traditional one with the same amount of memory.

Table 5.2. Required FLMOPn and MAFTn for oblique incidence case.

Formulations	# FLMOPn	# MAFTn
Single-field	25	25 coefficients + 15 fields (6 scat. + 9 inc.) = 40
Traditional	30	30 coefficients + 18 fields (6 scat. + 12 inc.) = 48
Improvement	% 17	% 17

6. SINGLE-FIELD FDTD UPDATING EQUATIONS FOR DISPERSIVE MEDIA

Characteristic behavior of electromagnetic fields inside dispersive media can be analyzed by Lorentz-Drude (LD) model. The traditional formulation incorporated with the LD model has been used extensively to simulate various materials and geometries for scientific and practical applications [26-28]; the updating equations are given in Appendix D. One can incorporate LD model in Maxwell's curl equations in frequency domain, and transform the resulting equations to time-domain.

Starting with Maxwell's curl equations in frequency domain and harnessing the auxiliary differential equation (ADE) approach [29], one can obtain a vector wave equation and solve it for each component of Cartesian coordinate system as scalar equations.

6.1. Lorentz-Drude Model for Permittivity

The LD model for permittivity is given by [2] as

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D\omega} + \frac{\Delta\varepsilon_L\omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2} \tag{6.1}$$

where ε_∞ is the relative permittivity at infinite frequency, ω_{pD} is the Drude pole frequency, Γ_D is the inverse of the pole relaxation time, $\Delta\varepsilon_L$ is the change in relative permittivity due to the Lorentz pole pair, ω_{pL} is the frequency of the pole pair (the undamped resonant frequency of the medium), and Γ_L is the damping coefficient.

Maxwell's equations in frequency domain for time-harmonic fields with $e^{j\omega t}$ dependence are given by

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} \tag{6.2}$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} \tag{6.3}$$

And the constitutive relations for linear, isotropic and homogeneous media are defined as

$$\boldsymbol{D} = \varepsilon\, \boldsymbol{E} \tag{6.4}$$

$$\boldsymbol{B} = \mu\, \boldsymbol{H} \tag{6.5}$$

Assuming a constant permeability, taking the curl of (6.2) and using (6.3), we have:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -j\omega\mu(\nabla \times \boldsymbol{H}) \tag{6.6}$$

$$\nabla \times (\nabla \times \boldsymbol{E}) = -j\omega\mu(\mathrm{j}\omega\varepsilon\mathbf{E}) \tag{6.7}$$

$$\nabla \times (\nabla \times \boldsymbol{E}) = \omega^2\mu\varepsilon_0(\varepsilon_\mathrm{r})\mathbf{E} \tag{6.8}$$

Substituting (6.1) into (6.8)

$$\nabla \times (\nabla \times \boldsymbol{E}) = \omega^2\mu\varepsilon_0\left(\varepsilon_\infty + \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D\omega} + \frac{\Delta\varepsilon_L\omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2}\right)\mathbf{E} \tag{6.9}$$

Rearranging (6.9) and introducing two terms $\boldsymbol{J}_D$ and $\boldsymbol{P}_L$

$$\nabla \times (\nabla \times \boldsymbol{E}) = \omega^2 \mu \varepsilon_0 \varepsilon_\infty \mathbf{E} + \boldsymbol{J}_D + \boldsymbol{P}_L \tag{6.10}$$

where $\boldsymbol{J}_D$ and $\boldsymbol{P}_L$ represents the Drude part and the Lorentz part, respectively.

$$\boldsymbol{J}_D = \omega^2 \mu \varepsilon_0 \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D \omega} \boldsymbol{E} \tag{6.11}$$

$$\boldsymbol{P}_L = \omega^2 \mu \varepsilon_0 \frac{\Delta\varepsilon_L \omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2} \boldsymbol{E} \tag{6.12}$$

Replacing $j\omega$ terms with $\frac{\partial}{\partial \mathrm{t}}$ and arranging the terms, (6.11) and (6.12) can be written as

$$\frac{\partial^2 \boldsymbol{J}_D}{\partial t^2} = -\mu\varepsilon_0 \omega_{pD}^2 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \Gamma_D \frac{\partial \boldsymbol{J}_D}{\partial t} \tag{6.13}$$

$$\frac{\partial^2 \boldsymbol{P}_L}{\partial t^2} = -\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \Gamma_L \frac{\partial \boldsymbol{P}_L}{\partial t} - \omega_{pL}^2 \boldsymbol{P}_L \tag{6.14}$$

Revisiting (6.10), and replacing ω^2 term with $-\frac{\partial^2}{\partial \mathrm{t}^2}$, one can obtain the vector wave equation

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu\varepsilon_0 \varepsilon_\infty \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \boldsymbol{J}_D + \boldsymbol{P}_L \tag{6.15}$$

Now, we can decompose (6.15) into its Cartesian components.

6.1.1. Updating Equation for the x Component

Cartesian component of (6.15) in x direction can be written as

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_y}{\partial x \partial y}+\frac{\partial^2 E_z}{\partial x \partial z}-\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right)=-\mu\varepsilon_0\varepsilon_\infty\frac{\partial^2 E_x}{\partial t^2}+J_{D,x}+P_{L,x} \qquad (6.16)$$

$$\frac{\partial^2 E_y}{\partial x \partial y}+\frac{\partial^2 E_z}{\partial x \partial z}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}=-\mu\varepsilon_0\varepsilon_\infty\frac{\partial^2 E_x}{\partial t^2}+J_{D,x}+P_{L,x} \qquad (6.17)$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (6.17) at the corresponding electric field node, i.e. E_x. The time derivatives are evaluated at the n[th] time step, therefore:

$$\frac{\partial^2 E_x(i,j,k)}{\partial t^2}=\frac{E_x^{n+1}(i,j,k)-2E_x^n(i,j,k)+E_x^{n-1}(i,j,k)}{(\Delta t)^2} \qquad (6.18)$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial y^2}=\frac{E_x^n(i,j+1,k)-2E_x^n(i,j,k)+E_x^n(i,j-1,k)}{(\Delta y)^2} \qquad (6.19)$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial z^2}=\frac{E_x^n(i,j,k+1)-2E_x^n(i,j,k)+E_x^n(i,j,k-1)}{(\Delta z)^2} \qquad (6.20)$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial x \partial y}=\frac{E_y^n(i+1,j,k)-E_y^n(i+1,j-1,k)-E_y^n(i,j,k)+E_y^n(i,j-1,k)}{\Delta x \Delta y} \qquad (6.21)$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial x \partial z}=\frac{E_z^n(i+1,j,k)-E_z^n(i+1,j,k-1)-E_z^n(i,j,k)+E_z^n(i,j,k-1)}{\Delta x \Delta z} \qquad (6.22)$$

Inserting (6.18) - (6.22) into (6.17), we have

$$\frac{E_y^n(i+1,j,k)-E_y^n(i+1,j-1,k)-E_y^n(i,j,k)+E_y^n(i,j-1,k)}{\Delta x\Delta y} + \frac{E_z^n(i+1,j,k)-E_z^n(i+1,j,k-1)-E_z^n(i,j,k)+E_z^n(i,j,k-1)}{\Delta x\Delta z} - \frac{E_x^n(i,j+1,k)-2E_x^n(i,j,k)+E_x^n(i,j-1,k)}{(\Delta y)^2} - \frac{E_x^n(i,j,k+1)-2E_x^n(i,j,k)+E_x^n(i,j,k-1)}{(\Delta z)^2} = -\mu\varepsilon_0\varepsilon_\infty\frac{E_x^{n+1}(i,j,k)-2E_x^n(i,j,k)+E_x^{n-1}(i,j,k)}{(\Delta t)^2} + \mathrm{J}_{D,x}^n(i,j,k) + \mathrm{P}_{L,x}^n(i,j,k) \tag{6.23}$$

We can simplify the updating equation for E_x as

$$\begin{aligned} E_x^{n+1}(i,j,k) = {} & C_{ex}^{ex,n}(i,j,k)[E_x^n(i,j,k)] + C_{ex}^{ex,n-1}(i,j,k)[E_x^{n-1}(i,j,k)] \\ & + C_{ex}^{ex,n,y}(i,j,k)[E_x^n(i,j+1,k)+E_x^n(i,j-1,k)] \\ & + C_{ex}^{ex,n,z}(i,j,k)[E_x^n(i,j,k+1)+E_x^n(i,j,k-1)] \\ & + C_{ex}^{ey,n,xy}(i,j,k)\big[E_y^n(i+1,j,k)-E_y^n(i+1,j-1,k)-E_y^n(i,j,k) \\ & + E_y^n(i,j-1,k)] \\ & + C_{ex}^{ez,n,xz}(i,j,k)[E_z^n(i+1,j,k)-E_z^n(i+1,j,k-1)-E_z^n(i,j,k) \\ & + E_z^n(i,j,k-1)] + C_x(i,j,k)\big[P_{L,x}^n(i,j,k)+J_{D,x}^n(i,j,k)\big] \end{aligned} \tag{6.24}$$

where

$$C_x(i,j,k) = \frac{(\Delta t)^2}{\mu\varepsilon_0\varepsilon_\infty} \tag{6.25}$$

$$C_{ex}^{ex,n}(i,j,k) = -C_x(i,j,k)\left(-\frac{2\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2} + \frac{2}{(\Delta y)^2} + \frac{2}{(\Delta z)^2}\right) \tag{6.26}$$

$$C_{ex}^{ex,n-1}(i,j,k) = -C_x(i,j,k)\left(\frac{\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2}\right) = -1 \tag{6.27}$$

$$C_{ex}^{ex,n,y}(i,j,k) = C_x(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{6.28}$$

$$C_{ex}^{ex,n,z}(i,j,k) = C_x(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{6.29}$$

$$C_{ex}^{ey,n,xy}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta x \Delta y}\right) \tag{6.30}$$

$$C_{ex}^{ez,n,xz}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta x \Delta z}\right) \tag{6.31}$$

The x component of the Drude part can be written as

$$\frac{\partial^2 J_{D,x}}{\partial t^2} = -\mu\varepsilon_0\omega_{pD}^2\frac{\partial^2 E_x}{\partial t^2} - \Gamma_D\frac{\partial J_{D,x}}{\partial t} \tag{6.32}$$

$$\frac{\partial^2 J_{D,x}(i,j,k)}{\partial t^2} = \frac{J_{D,x}^{n+1}(i,j,k) - 2J_{D,x}^{n}(i,j,k) + J_{D,x}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.33}$$

$$\frac{\partial J_{D,x}(i,j,k)}{\partial t} = \frac{J_{D,x}^{n+1}(i,j,k) - J_{D,x}^{n-1}(i,j,k)}{2\Delta t} \tag{6.34}$$

$$\begin{aligned} &\frac{J_{D,x}^{n+1}(i,j,k) - 2J_{D,x}^{n}(i,j,k) + J_{D,x}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad = -\mu\varepsilon_0\omega_{pD}^2\frac{E_x^{n+1}(i,j,k) - 2E_x^{n}(i,j,k) + E_x^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad - \Gamma_D\frac{J_{D,x}^{n+1}(i,j,k) - J_{D,x}^{n-1}(i,j,k)}{2\Delta t} \end{aligned} \tag{6.35}$$

We can simplify the updating equation for $J_{D,x}$ as

$$\begin{aligned} J_{D,x}^{n+1}(i,j,k) = {} & C_{JDx}^{jdx,n}(i,j,k)J_{D,x}^{n}(i,j,k) + C_{JDx}^{jdx,n-1}(i,j,k)J_{D,x}^{n-1}(i,j,k) \\ & + C_{JDx}^{ex,n}(i,j,k)[E_x^{n+1}(i,j,k) - 2E_x^{n}(i,j,k) + E_x^{n-1}(i,j,k)] \end{aligned} \tag{6.36}$$

where

$$C_{JDx}^{jdx,n}(i,j,k) = \frac{4}{2+\Gamma_D \Delta t} \tag{6.37}$$

$$C_{JDx}^{jdx,n-1}(i,j,k) = \frac{\Delta t \Gamma_D - 2}{2+\Gamma_D \Delta t} \tag{6.38}$$

$$C_{JDx}^{ex,n}(i,j,k) = -\frac{2\mu\varepsilon_0\omega_{pD}^2}{2+\Gamma_D \Delta t} \tag{6.39}$$

The x component of the Lorentz part can be written as

$$\frac{\partial^2 P_{L,x}}{\partial t^2} = -\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2 \frac{\partial^2 E_x}{\partial t^2} - \Gamma_L \frac{\partial P_{L,x}}{\partial t} - \omega_{pL}^2 P_{L,x} \tag{6.40}$$

$$\frac{\partial^2 P_{L,x}(i,j,k)}{\partial t^2} = \frac{P_{L,x}^{n+1}(i,j,k) - 2P_{L,x}^{n}(i,j,k) + P_{L,x}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.41}$$

$$\frac{\partial P_{L,x}(i,j,k)}{\partial t} = \frac{P_{L,x}^{n+1}(i,j,k) - P_{L,x}^{n-1}(i,j,k)}{2\Delta t} \tag{6.42}$$

$$\begin{aligned} &\frac{P_{L,x}^{n+1}(i,j,k) - 2P_{L,x}^{n}(i,j,k) + P_{L,x}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\quad = -\frac{\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2}{(\Delta t)^2}\left[E_x^{n+1}(i,j,k) - 2E_x^{n}(i,j,k) + E_x^{n-1}(i,j,k)\right] \\ &\quad - \Gamma_L \frac{P_{L,x}^{n+1}(i,j,k) - P_{L,x}^{n-1}(i,j,k)}{2\Delta t} - \omega_{pL}^2 P_{L,x}^{n}(i,j,k) \end{aligned} \tag{6.43}$$

We can simplify the updating equation for $P_{L,x}$ as

$$\begin{aligned} P_{L,x}^{n+1}(i,j,k) &= C_{PLx}^{plx,n}(i,j,k) P_{L,x}^{n}(i,j,k) + C_{PLx}^{plx,n-1}(i,j,k) P_{L,x}^{n-1}(i,j,k) \\ &+ C_{PLx}^{ex,n}(i,j,k)\left[E_x^{n+1}(i,j,k) - 2E_x^{n}(i,j,k) + E_x^{n-1}(i,j,k)\right] \end{aligned} \tag{6.44}$$

where

$$C_{PLx}^{plx,n}(i,j,k)=\frac{4-2(\Delta t)^2\omega_{pL}^2}{2+\Gamma_L\Delta t} \tag{6.45}$$

$$C_{PLx}^{plx,n-1}(i,j,k)=\frac{\Gamma_L\Delta t-2}{2+\Gamma_L\Delta t} \tag{6.46}$$

$$C_{PLx}^{ex,n}(i,j,k)=-\frac{2\mu\varepsilon_0\Delta\varepsilon_L\omega_{pL}^2}{2+\Gamma_L\Delta t} \tag{6.47}$$

Equations (6.24), (6.36) and (6.44) constitute the updating equations in x direction.

6.1.2. Updating Equation for the y Component

Cartesian component of (6.15) in y direction can be written as

$$\left(\frac{\partial^2 E_x}{\partial x\partial y}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_z}{\partial y\partial z}\right)-\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}\right)=-\mu\varepsilon_0\varepsilon_\infty\frac{\partial^2 E_y}{\partial t^2}+J_{D,y}+P_{L,y} \tag{6.48}$$

$$-\frac{\partial^2 E_y}{\partial x^2}-\frac{\partial^2 E_y}{\partial z^2}+\frac{\partial^2 E_x}{\partial x\partial y}+\frac{\partial^2 E_z}{\partial y\partial z}=-\mu\varepsilon_0\varepsilon_\infty\frac{\partial^2 E_y}{\partial t^2}+J_{D,y}+P_{L,y} \tag{6.49}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (6.49) at the corresponding electric field node, i.e. E_y. The time derivatives are evaluated at the n^{th} time step, therefore:

$$\frac{\partial^2 E_y(i,j,k)}{\partial t^2}=\frac{E_y^{n+1}(i,j,k)-2E_y^n(i,j,k)+E_y^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.50}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial x^2}=\frac{E_y^n(i+1,j,k)-2E_y^n(i,j,k)+E_y^n(i-1,j,k)}{(\Delta x)^2} \tag{6.51}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial z^2} = \frac{E_y^n(i,j,k+1) - 2E_y^n(i,j,k) + E_y^n(i,j,k-1)}{(\Delta z)^2} \tag{6.52}$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial x \partial y} = \frac{E_x^n(i,j+1,k) - E_x^n(i-1,j+1,k) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta y} \tag{6.53}$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial y \partial z} = \frac{E_z^n(i,j+1,k) - E_z^n(i,j+1,k-1) - E_z^n(i,j,k) + E_z^n(i,j,k-1)}{\Delta y \Delta z} \tag{6.54}$$

Inserting (6.50) - (6.54) into (6.49), we have

$$\begin{aligned}
&- \frac{E_y^n(i+1,j,k) - 2E_y^n(i,j,k) + E_y^n(i-1,j,k)}{(\Delta x)^2} \\
&- \frac{E_y^n(i,j,k+1) - 2E_y^n(i,j,k) + E_y^n(i,j,k-1)}{(\Delta z)^2} \\
&+ \frac{E_x^n(i,j+1,k) - E_x^n(i-1,j+1,k) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta y} \\
&+ \frac{E_z^n(i,j+1,k) - E_z^n(i,j+1,k-1) - E_z^n(i,j,k) + E_z^n(i,j,k-1)}{\Delta y \Delta z} \\
&= -\mu \varepsilon_0 \varepsilon_\infty \frac{E_y^{n+1}(i,j,k) - 2E_y^n(i,j,k) + E_y^{n-1}(i,j,k)}{(\Delta t)^2} + J_{D,y} + P_{L,y}
\end{aligned} \tag{6.55}$$

We can simplify the updating equation for E_y as

$$\begin{aligned}
E_y^{n+1}(i,j,k) = {} & C_{ey}^{ey,n}(i,j,k)\left[E_y^n(i,j,k)\right] + C_{ey}^{ey,n-1}(i,j,k)\left[E_y^{n-1}(i,j,k)\right] \\
& + C_{ey}^{ey,n,x}(i,j,k)\left[E_y^n(i+1,j,k) + E_y^n(i-1,j,k)\right] \\
& + C_{ey}^{ey,n,z}(i,j,k)\left[E_y^n(i,j,k+1) + E_y^n(i,j,k-1)\right] \\
& + C_{ey}^{ex,n,xy}(i,j,k)[E_x^n(i,j+1,k) - E_x^n(i-1,j+1,k) - E_x^n(i,j,k) \\
& + E_x^n(i-1,j,k)] \\
& + C_{ey}^{ez,n,yz}(i,j,k)[E_z^n(i,j+1,k) - E_z^n(i,j+1,k-1) - E_z^n(i,j,k) \\
& + E_z^n(i,j,k-1)] + C_y(i,j,k)\left[\mathrm{J}_{D,y}^n(i,j,k) + \mathrm{P}_{L,y}^n(i,j,k)\right]
\end{aligned} \tag{6.56}$$

where

$$C_y(i,j,k) = \frac{(\Delta t)^2}{\mu\varepsilon_0\varepsilon_\infty} \tag{6.57}$$

$$C_{ey}^{ey,n}(i,j,k) = -C_y(i,j,k)\left(-\frac{2\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta z)^2}\right) \tag{6.58}$$

$$C_{ey}^{ey,n-1}(i,j,k) = -C_y(i,j,k)\left(\frac{\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2}\right) = -1 \tag{6.59}$$

$$C_{ey}^{ey,n,x}(i,j,k) = C_y(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{6.60}$$

$$C_{ey}^{ey,n,z}(i,j,k) = C_y(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{6.61}$$

$$C_{ey}^{ex,n,xy}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{6.62}$$

$$C_{ey}^{ez,n,yz}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{6.63}$$

The y component of the Drude part can be written as

$$\frac{\partial^2 J_{D,y}}{\partial t^2} = -\mu\varepsilon_0\omega_{pD}^2\frac{\partial^2 E_y}{\partial t^2} - \Gamma_D\frac{\partial J_{D,y}}{\partial t} \tag{6.64}$$

$$\frac{\partial^2 J_{D,y}(i,j,k)}{\partial t^2} = \frac{J_{D,y}^{n+1}(i,j,k) - 2J_{D,y}^{n}(i,j,k) + J_{D,y}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.65}$$

$$\frac{\partial J_{D,y}(i,j,k)}{\partial t} = \frac{J_{D,y}^{n+1}(i,j,k) - J_{D,y}^{n-1}(i,j,k)}{2\Delta t} \tag{6.66}$$

$$\begin{aligned}\frac{J_{D,y}^{n+1}(i,j,k) - 2J_{D,y}^{n}(i,j,k) + J_{D,y}^{n-1}(i,j,k)}{(\Delta t)^2}&\\ = -\mu\varepsilon_0\omega_{pD}^2\frac{E_y^{n+1}(i,j,k) - 2E_y^{n}(i,j,k) + E_y^{n-1}(i,j,k)}{(\Delta t)^2}&\\ - \Gamma_D\frac{J_{D,y}^{n+1}(i,j,k) - J_{D,y}^{n-1}(i,j,k)}{2\Delta t}&\end{aligned} \tag{6.67}$$

We can simplify the updating equation for $J_{D,y}$ as

$$J_{D,y}^{n+1}(i,j,k) = C_{JDy}^{jdy,n}(i,j,k)J_{D,y}^{n}(i,j,k) + C_{JDy}^{jdy,n-1}(i,j,k)J_{D,y}^{n-1}(i,j,k) + C_{JDy}^{ey,t}(i,j,k)\left[E_y^{n+1}(i,j,k) - 2E_y^{n}(i,j,k) + E_y^{n-1}(i,j,k)\right] \tag{6.68}$$

where

$$C_{JDy}^{jdy,n}(i,j,k) = \frac{4}{2+\Gamma_D \Delta t} \tag{6.69}$$

$$C_{JDy}^{jdy,n-1}(i,j,k) = \frac{\Delta t \Gamma_D - 2}{2+\Gamma_D \Delta t} \tag{6.70}$$

$$C_{JDy}^{ey,n}(i,j,k) = -\frac{2\mu \varepsilon_0 \omega_{pD}^2}{2+\Gamma_D \Delta t} \tag{6.71}$$

The y component of the Lorentz part can be written as

$$\frac{\partial^2 P_{L,y}}{\partial t^2} = -\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2 \frac{\partial^2 E_y}{\partial t^2} - \Gamma_L \frac{\partial P_{L,y}}{\partial t} - \omega_{pL}^2 P_{L,y} \tag{6.72}$$

$$\frac{\partial^2 P_{L,y}(i,j,k)}{\partial t^2} = \frac{P_{L,y}^{n+1}(i,j,k) - 2P_{L,y}^{n}(i,j,k) + P_{L,y}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.73}$$

$$\frac{\partial P_{L,y}(i,j,k)}{\partial t} = \frac{P_{L,y}^{n+1}(i,j,k) - P_{L,y}^{n-1}(i,j,k)}{2\Delta t} \tag{6.74}$$

$$\begin{aligned} \frac{P_{L,y}^{n+1}(i,j,k) - 2P_{L,y}^{n}(i,j,k) + P_{L,y}^{n-1}(i,j,k)}{(\Delta t)^2} &= -\frac{\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2}{(\Delta t)^2}\left[E_y^{n+1}(i,j,k) - 2E_y^{n}(i,j,k) + E_y^{n-1}(i,j,k)\right] \\ &- \Gamma_L \frac{P_{L,y}^{n+1}(i,j,k) - P_{L,y}^{n-1}(i,j,k)}{2\Delta t} - \omega_{pL}^2 P_{L,y}^{n}(i,j,k) \end{aligned} \tag{6.75}$$

We can simplify the updating equation for $P_{L,y}$ as

$$P_{L,y}^{n+1}(i,j,k) = C_{PLy}^{ply,n}(i,j,k)P_{L,y}^{n}(i,j,k) + C_{PLy}^{ply,n-1}(i,j,k)P_{L,y}^{n-1}(i,j,k) + C_{PLy}^{ey,n}(i,j,k)\left[E_y^{n+1}(i,j,k) - 2E_y^{n}(i,j,k) + E_y^{n-1}(i,j,k)\right] \tag{6.76}$$

where

$$C_{PLy}^{ply,n}(i,j,k) = \frac{4 - 2(\Delta t)^2\omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.77}$$

$$C_{PLy}^{ply,n-1}(i,j,k) = \frac{\Gamma_L \Delta t - 2}{2 + \Gamma_L \Delta t} \tag{6.78}$$

$$C_{PLy}^{ey,n}(i,j,k) = -\frac{2\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.79}$$

Equations (6.56), (6.68) and (6.76) constitute the updating equations in y direction.

6.1.3. Updating Equation for the z Component

Cartesian component of (6.15) in z direction can be written as

$$\left(\frac{\partial^2 E_x}{\partial x \partial z} + \frac{\partial^2 E_y}{\partial y \partial z} + \frac{\partial^2 E_z}{\partial z^2}\right) - \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}\right) = -\mu\varepsilon_0\varepsilon_\infty \frac{\partial^2 E_z}{\partial t^2} + J_{D,z} + P_{L,z} \tag{6.80}$$

$$-\frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_x}{\partial x \partial z} + \frac{\partial^2 E_y}{\partial y \partial z} = -\mu\varepsilon_0\varepsilon_\infty \frac{\partial^2 E_z}{\partial t^2} + J_{D,z} + P_{L,z} \tag{6.81}$$

To derive the FDTD updating equations for the electric fields, we have to evaluate all the spatial derivatives in equation (6.81) at the corresponding electric field node, i.e. E_z. The time derivatives are evaluated at the n^{th} time step, therefore:

$$\frac{\partial^2 E_z(i,j,k)}{\partial t^2} = \frac{E_z^{n+1}(i,j,k) - 2E_z^n(i,j,k) + E_z^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.82}$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial x^2} = \frac{E_z^n(i+1,j,k) - 2E_z^n(i,j,k) + E_z^n(i-1,j,k)}{(\Delta x)^2} \tag{6.83}$$

$$\frac{\partial^2 E_z(i,j,k)}{\partial y^2} = \frac{E_z^n(i,j+1,k) - 2E_z^n(i,j,k) + E_z^n(i,j-1,k)}{(\Delta y)^2} \tag{6.84}$$

$$\frac{\partial^2 E_x(i,j,k)}{\partial x \partial z} = \frac{E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta z} \tag{6.85}$$

$$\frac{\partial^2 E_y(i,j,k)}{\partial y \partial z} = \frac{E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1) + E_y^n(i,j-1,k)}{\Delta y \Delta z} \tag{6.86}$$

Inserting (6.82)-(6.86) into (6.81), we have

$$\begin{aligned}
&-\frac{E_z^n(i+1,j,k) - 2E_z^n(i,j,k) + E_z^n(i-1,j,k)}{(\Delta x)^2} \\
&-\frac{E_z^n(i,j+1,k) - 2E_z^n(i,j,k) + E_z^n(i,j-1,k)}{(\Delta y)^2} \\
&+\frac{E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) + E_x^n(i-1,j,k)}{\Delta x \Delta z} \\
&+\frac{E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1) + E_y^n(i,j-1,k)}{\Delta y \Delta z} \\
&= -\mu \varepsilon_0 \varepsilon_\infty \frac{E_z^{n+1}(i,j,k) - 2E_z^n(i,j,k) + E_z^{n-1}(i,j,k)}{(\Delta t)^2} + J_{D,z} + P_{L,z}
\end{aligned} \tag{6.87}$$

We can simplify the updating equation for E_z as

$$\begin{aligned}E_z^{n+1}(i,j,k) = {} & C_{ez}^{ez,n}(i,j,k)[E_z^n(i,j,k)] + C_{ez}^{ez,n-1}(i,j,k)[E_z^{n-1}(i,j,k)] \\ & + C_{ez}^{ez,n,x}(i,j,k)[E_z^n(i+1,j,k) + E_z^n(i-1,j,k)] \\ & + C_{ez}^{ez,n,y}(i,j,k)[E_z^n(i,j+1,k) + E_z^n(i,j-1,k)] \\ & + C_{ez}^{ey,n,xz}(i,j,k)[E_x^n(i,j,k+1) - E_x^n(i-1,j,k+1) - E_x^n(i,j,k) \\ & + E_x^n(i-1,j,k)] \\ & + C_{ez}^{ez,n,yz}(i,j,k)\left[E_y^n(i,j,k+1) - E_y^n(i,j,k) - E_y^n(i,j-1,k+1)\right. \\ & \left. + E_y^n(i,j-1,k)\right] + C_z(i,j,k)\left[P_{L,z}^n(i,j,k) + J_{D,z}^n(i,j,k)\right]\end{aligned} \tag{6.88}$$

where

$$C_z(i,j,k) = \frac{(\Delta t)^2}{\mu\varepsilon_0\varepsilon_\infty} \tag{6.89}$$

$$C_{ez}^{ez,n}(i,j,k) = -C_z(i,j,k)\left(-\frac{2\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right) \tag{6.90}$$

$$C_{ez}^{ez,n-1}(i,j,k) = -C_z(i,j,k)\left(\frac{\mu\varepsilon_0\varepsilon_\infty}{(\Delta t)^2}\right) = -1 \tag{6.91}$$

$$C_{ez}^{ez,n,x}(i,j,k) = C_z(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{6.92}$$

$$C_{ez}^{ez,n,y}(i,j,k) = C_z(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{6.93}$$

$$C_{ez}^{ey,n,xz}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta x\Delta z}\right) \tag{6.94}$$

$$C_{ez}^{ez,n,yz}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{6.95}$$

The z component of the Drude part can be written as

$$\frac{\partial^2 J_{D,z}}{\partial t^2} = -\mu\varepsilon_0\omega_{pD}^2\frac{\partial^2 E_z}{\partial t^2} - \Gamma_D\frac{\partial J_{D,z}}{\partial t} \tag{6.96}$$

$$\frac{\partial^2 J_{D,z}(i,j,k)}{\partial t^2} = \frac{J_{D,z}^{n+1}(i,j,k) - 2J_{D,z}^{n}(i,j,k) + J_{D,z}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.97}$$

$$\frac{\partial J_{D,z}(i,j,k)}{\partial t} = \frac{J_{D,z}^{n+1}(i,j,k) - J_{D,z}^{n-1}(i,j,k)}{2\Delta t} \tag{6.98}$$

$$\begin{aligned} &\frac{J_{D,z}^{n+1}(i,j,k) - 2J_{D,z}^{n}(i,j,k) + J_{D,z}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad = -\mu\varepsilon_0\omega_{pD}^2 \frac{E_z^{n+1}(i,j,k) - 2E_z^{n}(i,j,k) + E_z^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad - \Gamma_D \frac{J_{D,z}^{n+1}(i,j,k) - J_{D,z}^{n-1}(i,j,k)}{2\Delta t} \end{aligned} \tag{6.99}$$

We can simplify the updating equation for $J_{D,z}$ as

$$\begin{aligned} J_{D,z}^{n+1}(i,j,k) = {} & C_{JDz}^{jdz,n}(i,j,k) J_{D,z}^{n}(i,j,k) + C_{JDz}^{jdz,n-1}(i,j,k) J_{D,z}^{n-1}(i,j,k) \\ & + C_{JDz}^{ez,n}(i,j,k)[E_z^{n+1}(i,j,k) - 2E_z^{n}(i,j,k) + E_z^{n-1}(i,j,k)] \end{aligned} \tag{6.100}$$

where

$$C_{JDz}^{jdz,n}(i,j,k) = \frac{4}{2 + \Gamma_D \Delta t} \tag{6.101}$$

$$C_{JDz}^{jdz,n-1}(i,j,k) = \frac{\Delta t \Gamma_D - 2}{2 + \Gamma_D \Delta t} \tag{6.102}$$

$$C_{JDz}^{ez,n}(i,j,k) = -\frac{2\mu\varepsilon_0\omega_{pD}^2}{2 + \Gamma_D \Delta t} \tag{6.103}$$

The z component of the Lorentz part can be written as

$$\frac{\partial^2 P_{L,z}}{\partial t^2} = -\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2 \frac{\partial^2 E_z}{\partial t^2} - \Gamma_L \frac{\partial P_{L,z}}{\partial t} - \omega_{pL}^2 P_{L,z} \tag{6.104}$$

$$\frac{\partial^2 P_{L,z}(i,j,k)}{\partial t^2} = \frac{P_{L,z}^{n+1}(i,j,k) - 2P_{L,z}^{n}(i,j,k) + P_{L,z}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.105}$$

$$\frac{\partial P_{L,z}(i,j,k)}{\partial t} = \frac{P_{L,z}^{n+1}(i,j,k) - P_{L,z}^{n-1}(i,j,k)}{2\Delta t} \tag{6.106}$$

$$\begin{aligned} &\frac{P_{L,z}^{n+1}(i,j,k) - 2P_{L,z}^{n}(i,j,k) + P_{L,z}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\quad = -\frac{\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2}{(\Delta t)^2}\left[E_z^{n+1}(i,j,k) - 2E_z^{n}(i,j,k) + E_z^{n-1}(i,j,k)\right] \\ &\quad - \Gamma_L \frac{P_{L,z}^{n+1}(i,j,k) - P_{L,z}^{n-1}(i,j,k)}{2\Delta t} - \omega_{pL}^2 P_{L,z}^{n}(i,j,k) \end{aligned} \tag{6.107}$$

We can simplify the updating equation for $P_{L,z}$ as

$$\begin{aligned} P_{L,z}^{n+1}(i,j,k) &= C_{PLz}^{plz,n}(i,j,k) P_{L,z}^{n}(i,j,k) + C_{PLz}^{plz,n-1}(i,j,k) P_{L,z}^{n-1}(i,j,k) \\ &+ C_{PLz}^{ez,n}(i,j,k)\left[E_z^{n+1}(i,j,k) - 2E_z^{n}(i,j,k) + E_z^{n-1}(i,j,k)\right] \end{aligned} \tag{6.108}$$

where

$$C_{PLz}^{plz,n}(i,j,k) = \frac{4 - 2(\Delta t)^2 \omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.109}$$

$$C_{PLz}^{plz,n-1}(i,j,k) = \frac{\Gamma_L \Delta t - 2}{2 + \Gamma_L \Delta t} \tag{6.110}$$

$$C_{PLz}^{ez,n}(i,j,k) = -\frac{2\mu\varepsilon_0 \Delta\varepsilon_L \omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.111}$$

Equations (6.88), (6.100) and (6.108) constitute the updating equations in z direction.

6.2. Lorentz-Drude Model for Permeability

The LD model for permeability can be written as

$$\mu_r(\omega) = \mu_\infty + \frac{\omega_{pD}^2}{j^2\omega^2 + j\mathit{\Gamma}_D\omega} + \frac{\mathit{\Delta}\mu_L\omega_{pL}^2}{j^2\omega^2 + j\omega\mathit{\Gamma}_L + \omega_{pL}^2} \tag{6.112}$$

where μ_∞ is the relative permeability at infinite frequency, ω_{pD} is the Drude pole frequency, Γ_D is the inverse of the pole relaxation time, $\Delta\mu_L$ is the change in relative permeability due to the Lorentz pole pair, ω_{pL} is the frequency of the pole pair (the undamped resonant frequency of the medium), Γ_L is the damping coefficient.

Assuming constant permittivity, taking the curl of (6.3), using (6.2), (6.4) and (6.5), we have:

$$\nabla \times (\nabla \times \boldsymbol{H}) = j\omega(\nabla \times \boldsymbol{D}) \tag{6.113}$$

$$\nabla \times (\nabla \times \boldsymbol{H}) = j\omega\varepsilon(-j\omega\boldsymbol{B}) \tag{6.114}$$

$$\nabla \times (\nabla \times \boldsymbol{H}) = \omega^2\varepsilon\mu_0\mu_r\mathbf{H} \tag{6.115}$$

Substituting (6.112) into (6.115)

$$\nabla \times (\nabla \times \boldsymbol{H}) = \omega^2\varepsilon\mu_0\left(\mu_\infty + \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D\omega} + \frac{\mathit{\Delta}\mu_L\omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2}\right)\mathbf{H} \tag{6.116}$$

Rearranging (6.116) and introducing two terms $\boldsymbol{K}_D$ and $\boldsymbol{M}_L$

$$\nabla \times (\nabla \times \boldsymbol{H}) = \omega^2\varepsilon\mu_0\mu_\infty\mathbf{H} + \boldsymbol{K}_D + \boldsymbol{M}_L \tag{6.117}$$

where $\boldsymbol{K}_D$ and $\boldsymbol{M}_L$ represent the Drude part and the Lorentz part, respectively.

$$\boldsymbol{K}_D = \omega^2 \varepsilon \mu_0 \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D \omega} \boldsymbol{H} \tag{6.118}$$

$$\boldsymbol{M}_L = \omega^2 \varepsilon \mu_0 \frac{\Delta\mu_L \omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2} \boldsymbol{H} \tag{6.119}$$

Replacing $j\omega$ terms with $\frac{\partial}{\partial \mathrm{t}}$ and arranging terms, (6.118) and (6.119) can be written as

$$\frac{\partial^2 \boldsymbol{K_D}}{\partial t^2} = -\varepsilon\mu_0 \omega_{pD}^2 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \Gamma_D \frac{\partial \boldsymbol{K}_D}{\partial t} \tag{6.120}$$

$$\frac{\partial^2 \boldsymbol{M_L}}{\partial t^2} = -\varepsilon\mu_0 \Delta\mu_L \omega_{pL}^2 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \Gamma_L \frac{\partial \boldsymbol{M}_L}{\partial t} - \omega_{pL}^2 \boldsymbol{M}_L \tag{6.121}$$

Revisiting (6.117), and replacing ω^2 term with $-\frac{\partial^2}{\partial \mathrm{t}^2}$, one can obtain the vector wave equation

$$\nabla \times (\nabla \times \boldsymbol{H}) = -\varepsilon\mu_0\mu_\infty \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \boldsymbol{K}_D + \boldsymbol{M}_L \tag{6.122}$$

Now, we can decompose (6.122) into its Cartesian components

6.2.1. Updating Equation for the x Component

Cartesian component of (6.122) in x direction can be written as

$$\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_y}{\partial x\partial y}+\frac{\partial^2 H_z}{\partial x\partial z}-\left(\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_x}{\partial z^2}\right)=-\varepsilon\mu_0\mu_\infty\frac{\partial^2 H_x}{\partial t^2}+K_{D,x}+M_{L,x} \quad (6.123)$$

$$\frac{\partial^2 H_y}{\partial x\partial y}+\frac{\partial^2 H_z}{\partial x\partial z}-\frac{\partial^2 H_x}{\partial y^2}-\frac{\partial^2 H_x}{\partial z^2}=-\varepsilon\mu_0\mu_\infty\frac{\partial^2 H_x}{\partial t^2}+K_{D,x}+M_{L,x} \quad (6.124)$$

To derive the FDTD updating equations for the magnetic fields, we have to evaluate all the spatial derivatives in equation (6.124) at the corresponding magnetic field node, i.e. H_x. The time derivatives are evaluated at the n^{th} time step, therefore:

$$\frac{\partial^2 H_x(i,j,k)}{\partial t^2}=\frac{H_x^{n+1}(i,j,k)-2H_x^n(i,j,k)+H_x^{n-1}(i,j,k)}{(\Delta t)^2} \quad (6.125)$$

$$\frac{\partial^2 H_x(i,j,k)}{\partial y^2}=\frac{H_x^n(i,j+1,k)-2H_x^n(i,j,k)+H_x^n(i,j-1,k)}{(\Delta y)^2} \quad (6.126)$$

$$\frac{\partial^2 H_x(i,j,k)}{\partial z^2}=\frac{H_x^n(i,j,k+1)-2H_x^n(i,j,k)+H_x^n(i,j,k-1)}{(\Delta z)^2} \quad (6.127)$$

$$\frac{\partial^2 H_y(i,j,k)}{\partial x\partial y}=\frac{H_y^n(i+1,j,k)-H_y^n(i+1,j-1,k)-H_y^n(i,j,k)+H_y^n(i,j-1,k)}{\Delta x\Delta y} \quad (6.128)$$

$$\frac{\partial^2 H_z(i,j,k)}{\partial x\partial z}=\frac{H_z^n(i+1,j,k)-H_z^n(i+1,j,k-1)-H_z^n(i,j,k)+H_z^n(i,j,k-1)}{\Delta x\Delta z} \quad (6.129)$$

Inserting (6.125) - (6.129) into (6.124), we have

$$
\begin{aligned}
&\frac{H_y^n(i+1,j,k) - H_y^n(i+1,j-1,k) - H_y^n(i,j,k) + H_y^n(i,j-1,k)}{\Delta x \Delta y} \\
&+ \frac{H_z^n(i+1,j,k) - H_z^n(i+1,j,k-1) - H_z^n(i,j,k) + H_z^n(i,j,k-1)}{\Delta x \Delta z} \\
&- \frac{H_x^n(i,j+1,k) - 2H_x^n(i,j,k) + H_x^n(i,j-1,k)}{(\Delta y)^2} \\
&- \frac{H_x^n(i,j,k+1) - 2H_x^n(i,j,k) + H_x^n(i,j,k-1)}{(\Delta z)^2} \\
&= -\varepsilon \mu_0 \mu_\infty \frac{H_x^{n+1}(i,j,k) - 2H_x^n(i,j,k) + H_x^{n-1}(i,j,k)}{(\Delta t)^2} + K_{D,x}^n(i,j,k) + M_{L,x}^n(i,j,k)
\end{aligned}
\tag{6.130}
$$

We can simplify the updating equation for H_x as

$$
\begin{aligned}
H_x^{n+1}(i,j,k) = {} & C_{hx}^{hx,n}(i,j,k)[H_x^n(i,j,k)] + C_{hx}^{hx,n-1}(i,j,k)[H_x^{n-1}(i,j,k)] \\
& + C_{hx}^{hx,n,y}(i,j,k)[H_x^n(i,j+1,k) + H_x^n(i,j-1,k)] \\
& + C_{hx}^{hx,n,z}(i,j,k)[H_x^n(i,j,k+1) + H_x^n(i,j,k-1)] \\
& + C_{hx}^{hy,n,xy}(i,j,k)\big[H_y^n(i+1,j,k) - H_y^n(i+1,j-1,k) - H_y^n(i,j,k) \\
& + H(i,j-1,k)] \\
& + C_{hx}^{hz,n,xz}(i,j,k)[H_z^n(i+1,j,k) - H_z^n(i+1,j,k-1) - H_z^n(i,j,k) \\
& + H_z^n(i,j,k-1)] + C_x(i,j,k)\big[K_{D,x}^n(i,j,k) + M_{L,x}^n(i,j,k)\big]
\end{aligned}
\tag{6.131}
$$

where

$$
C_x(i,j,k) = \frac{(\Delta t)^2}{\varepsilon \mu_0 \mu_\infty} \tag{6.132}
$$

$$
C_{hx}^{hx,n}(i,j,k) = -C_x(i,j,k)\left(-\frac{2\varepsilon\mu_0\mu_\infty}{(\Delta t)^2} + \frac{2}{(\Delta y)^2} + \frac{2}{(\Delta z)^2}\right) \tag{6.133}
$$

$$
C_{hx}^{hx,n-1}(i,j,k) = -C_x(i,j,k)\left(\frac{\varepsilon\mu_0\mu_\infty}{(\Delta t)^2}\right) = -1 \tag{6.134}
$$

$$
C_{hx}^{hx,n,y}(i,j,k) = C_x(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{6.135}
$$

$$
C_{hx}^{hx,n,z}(i,j,k) = C_x(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{6.136}
$$

$$C_{hx}^{hy,n,xy}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta x \Delta y}\right) \tag{6.137}$$

$$C_{hx}^{hz,n,xz}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta x \Delta z}\right) \tag{6.138}$$

The x component of the Drude part can be written as

$$\frac{\partial^2 K_{D,x}}{\partial t^2} = -\varepsilon\mu_0\omega_{pD}^2\frac{\partial^2 H_x}{\partial t^2} - \Gamma_D\frac{\partial K_{D,x}}{\partial t} \tag{6.139}$$

$$\frac{\partial^2 K_{D,x}(i,j,k)}{\partial t^2} = \frac{K_{D,x}^{n+1}(i,j,k) - 2K_{D,x}^{n}(i,j,k) + K_{D,x}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.140}$$

$$\frac{\partial K_{D,x}(i,j,k)}{\partial t} = \frac{K_{D,x}^{n+1}(i,j,k) - K_{D,x}^{n-1}(i,j,k)}{2\Delta t} \tag{6.141}$$

$$\begin{aligned}\frac{K_{D,x}^{n+1}(i,j,k) - 2K_{D,x}^{n}(i,j,k) + K_{D,x}^{n-1}(i,j,k)}{(\Delta t)^2} \\ = -\varepsilon\mu_0\omega_{pD}^2\frac{H_x^{n+1}(i,j,k) - 2H_x^{n}(i,j,k) + H_x^{n-1}(i,j,k)}{(\Delta t)^2} \\ - \Gamma_D\frac{K_{D,x}^{n+1}(i,j,k) - K_{D,x}^{n-1}(i,j,k)}{2\Delta t}\end{aligned} \tag{6.142}$$

We can simplify the updating equation for $K_{D,x}$ as

$$\begin{aligned}K_{D,x}^{n+1}(i,j,k) = C_{KDx}^{kdx,n}(i,j,k)K_{D,x}^{n}(i,j,k) + C_{KDx}^{kdx,n-1}(i,j,k)K_{D,x}^{n-1}(i,j,k) \\ + C_{KDx}^{hx,n}(i,j,k)[H_x^{n+1}(i,j,k) - 2H_x^{n}(i,j,k) + H_x^{n-1}(i,j,k)]\end{aligned} \tag{6.143}$$

where

$$C_{KDx}^{kdx,n}(i,j,k) = \frac{4}{2 + \Gamma_D\Delta t} \tag{6.144}$$

$$C_{KDx}^{kdx,n-1}(i,j,k) = \frac{\Delta t \Gamma_D - 2}{2 + \Gamma_D \Delta t} \tag{6.145}$$

$$C_{KDx}^{hx,n}(i,j,k) = -\frac{2\varepsilon\mu_0\omega_{pD}^2}{2 + \Gamma_D \Delta t} \tag{6.146}$$

The x component of the Lorentz part can be written as

$$\frac{\partial^2 M_{L,x}}{\partial t^2} = -\varepsilon\mu_0\Delta\mu_L\omega_{pL}^2\frac{\partial^2 H_x}{\partial t^2} - \Gamma_L\frac{\partial M_{L,x}}{\partial t} - \omega_{pL}^2 M_{L,x} \tag{6.147}$$

$$\frac{\partial^2 M_{L,x}(i,j,k)}{\partial t^2} = \frac{M_{L,x}^{n+1}(i,j,k) - 2M_{L,x}^{n}(i,j,k) + M_{L,x}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.148}$$

$$\frac{\partial M_{L,x}(i,j,k)}{\partial t} = \frac{M_{L,x}^{n+1}(i,j,k) - M_{L,x}^{n-1}(i,j,k)}{2\Delta t} \tag{6.149}$$

$$\begin{aligned}\frac{M_{L,x}^{n+1}(i,j,k) - 2M_{L,x}^{n}(i,j,k) + M_{L,x}^{n-1}(i,j,k)}{(\Delta t)^2}& \\ = -\frac{\varepsilon\mu_0\Delta\mu_L\omega_{mL}^2}{(\Delta t)^2}&\left[H_x^{n+1}(i,j,k) - 2H_x^{n}(i,j,k) + H_x^{n-1}(i,j,k)\right] \\ - \Gamma_L\frac{M_{L,x}^{n+1}(i,j,k) - M_{L,x}^{n-1}(i,j,k)}{2\Delta t}& - \omega_{\mathrm{pL}}^2 M_{L,x}^{n}(i,j,k)\end{aligned} \tag{6.150}$$

We can simplify the updating equation for $M_{L,x}$ as

$$\begin{aligned}M_{L,x}^{n+1}(i,j,k) = {}& C_{MLx}^{mlx,n}(i,j,k)M_{L,x}^{n}(i,j,k) + C_{MLx}^{mlx,n-1}(i,j,k)M_{L,x}^{n-1}(i,j,k) \\ &+ C_{MLx}^{ex,n}(i,j,k)\left[H_x^{n+1}(i,j,k) - 2H_x^{n}(i,j,k) + H_x^{n-1}(i,j,k)\right]\end{aligned} \tag{6.151}$$

where

$$C_{MLx}^{mlx,n}(i,j,k) = \frac{4 - 2(\Delta t)^2\omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.152}$$

$$C_{MLx}^{mlx,n-1}(i,j,k) = \frac{\Gamma_L \Delta t - 2}{2 + \Gamma_L \Delta t} \tag{6.153}$$

$$C_{MLx}^{ex,n}(i,j,k) = -\frac{2\varepsilon\mu_0 \Delta\mu_L \omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.154}$$

Equations (6.131), (6.143) and (6.151) constitute the updating equations in x direction.

6.2.2. Updating Equation for the y Component

Cartesian component of (6.122) in y direction can be written as

$$\begin{aligned}\left(\frac{\partial^2 H_x}{\partial x \partial y} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_z}{\partial y \partial z}\right) - \left(\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2}\right) \\ = -\varepsilon\mu_0\mu_\infty \frac{\partial^2 H_y}{\partial t^2} + K_{D,y} + M_{L,y}\end{aligned} \tag{6.155}$$

$$-\frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_y}{\partial z^2} + \frac{\partial^2 H_x}{\partial x \partial y} + \frac{\partial^2 H_z}{\partial y \partial z} = -\varepsilon\mu_0\mu_\infty \frac{\partial^2 H_y}{\partial t^2} + K_{D,y} + M_{L,y} \tag{6.156}$$

To derive the FDTD updating equations for the magnetic fields, we have to evaluate all the spatial derivatives in equations (6.156) at the corresponding magnetic field node, i.e. H_y. The time derivatives are evaluated at the n^{th} time step, therefore:

$$\frac{\partial^2 H_y(i,j,k)}{\partial t^2} = \frac{H_y^{n+1}(i,j,k) - 2H_y^n(i,j,k) + H_y^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.157}$$

$$\frac{\partial^2 H_y(i,j,k)}{\partial x^2} = \frac{H_y^n(i+1,j,k) - 2H_y^n(i,j,k) + H_y^n(i-1,j,k)}{(\Delta x)^2} \tag{6.158}$$

$$\frac{\partial^2 H_y(i,j,k)}{\partial z^2} = \frac{H_y^n(i,j,k+1) - 2H_y^n(i,j,k) + H_y^n(i,j,k-1)}{(\Delta z)^2} \tag{6.159}$$

$$\frac{\partial^2 H_x(i,j,k)}{\partial x \partial y} = \frac{H_x^n(i,j+1,k) - H_x^n(i-1,j+1,k) - H_x^n(i,j,k) + H_x^n(i-1,j,k)}{\Delta x \Delta y} \tag{6.160}$$

$$\frac{\partial^2 H_z(i,j,k)}{\partial y \partial z} = \frac{H_z^n(i,j+1,k) - H_z^n(i,j+1,k-1) - H_z^n(i,j,k) + H_z^n(i,j,k-1)}{\Delta y \Delta z} \tag{6.161}$$

Inserting (6.157) - (6.161) into (6.156), we have

$$\begin{aligned}
&\frac{H_x^n(i,j+1,k) - H_x^n(i-1,j+1,k) - H_x^n(i,j,k) + H_x^n(i-1,j,k)}{\Delta x \Delta y} \\
&+ \frac{H_z^n(i,j+1,k) - H_z^n(i,j+1,k-1) - H_z^n(i,j,k) + H_z^n(i,j,k-1)}{\Delta y \Delta z} \\
&- \frac{H_y^n(i+1,j,k) - 2H_y^n(i,j,k) + H_y^n(i-1,j,k)}{(\Delta x)^2} \\
&- \frac{H_y^n(i,j,k+1) - 2H_y^n(i,j,k) + H_y^n(i,j,k-1)}{(\Delta z)^2} \\
&= -\varepsilon \mu_0 \mu_\infty \frac{H_y^{n+1}(i,j,k) - 2H_y^n(i,j,k) + H_y^{n-1}(i,j,k)}{(\Delta t)^2} + K_{D,y}^n(i,j,k) \\
&+ M_{L,y}^n(i,j,k)
\end{aligned} \tag{6.162}$$

We can simplify the updating equation for H_y as

$$\begin{aligned}
H_y^{n+1}(i,j,k) = {} & C_{hy}^{hy,n}(i,j,k)\left[H_y^n(i,j,k)\right] + C_{hy}^{hy,n-1}(i,j,k)\left[H_y^{n-1}(i,j,k)\right] \\
& + C_{hy}^{hy,n,x}(i,j,k)\left[H_y^n(i+1,j,k) + H_y^n(i-1,j,k)\right] \\
& + C_{hy}^{hy,n,z}(i,j,k)\left[H_y^n(i,j,k+1) + H_y^n(i,j,k-1)\right] \\
& + C_{hy}^{hx,n,xy}(i,j,k)[H_x^n(i,j+1,k) - H_x^n(i-1,j+1,k) - H_x^n(i,j,k) \\
& + H_x^n(i-1,j,k)] \\
& + C_{hy}^{hz,n,yz}(i,j,k)[H_z^n(i,j+1,k) - H_z^n(i,j+1,k-1) - H_z^n(i,j,k) \\
& + H_z^n(i,j,k-1)] + C_y(i,j,k)\left[K_{D,y}^n(i,j,k) + M_{L,y}^n(i,j,k)\right]
\end{aligned} \tag{6.163}$$

where

$$C_y(i,j,k) = \frac{(\Delta t)^2}{\varepsilon\mu_0\mu_\infty} \tag{6.164}$$

$$C_{hy}^{hy,n}(i,j,k) = -C_y(i,j,k)\left(-\frac{2\varepsilon\mu_0\mu_\infty}{(\Delta t)^2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta z)^2}\right) \tag{6.165}$$

$$C_{hy}^{hy,n-1}(i,j,k) = -C_y(i,j,k)\left(\frac{\varepsilon\mu_0\mu_\infty}{(\Delta t)^2}\right) = -1 \tag{6.166}$$

$$C_{hy}^{hy,n,x}(i,j,k) = C_y(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{6.167}$$

$$C_{hy}^{hy,n,z}(i,j,k) = C_y(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{6.168}$$

$$C_{hy}^{hx,n,xy}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{6.169}$$

$$C_{hy}^{hz,n,yz}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{6.170}$$

The y component of the Drude part can be written as

$$\frac{\partial^2 M_{D,y}}{\partial t^2} = -\varepsilon\mu_0\omega_{pD}^2\frac{\partial^2 H_y}{\partial t^2} - \Gamma_D\frac{\partial M_{D,y}}{\partial t} \tag{6.171}$$

$$\frac{\partial^2 M_{D,y}(i,j,k)}{\partial t^2} = \frac{M_{D,y}^{n+1}(i,j,k) - 2M_{D,y}^{n}(i,j,k) + M_{D,y}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.172}$$

$$\frac{\partial M_{D,y}(i,j,k)}{\partial t} = \frac{M_{D,y}^{n+1}(i,j,k) - M_{D,y}^{n-1}(i,j,k)}{2\Delta t} \tag{6.173}$$

$$\frac{M_{D,y}^{n+1}(i,j,k) - 2M_{D,y}^{n}(i,j,k) + M_{D,y}^{n-1}(i,j,k)}{(\Delta t)^2} = -\varepsilon\mu_0\omega_{pD}^2 \frac{H_y^{n+1}(i,j,k) - 2H_y^{n}(i,j,k) + H_y^{n-1}(i,j,k)}{(\Delta t)^2} - \Gamma_D \frac{M_{D,y}^{n+1}(i,j,k) - M_{D,y}^{n-1}(i,j,k)}{2\Delta t} \tag{6.174}$$

We can simplify the updating equation for $M_{D,y}$ as

$$M_{D,y}^{n+1}(i,j,k) = C_{MDy}^{mdy,n}(i,j,k)M_{D,y}^{n}(i,j,k) + C_{MDy}^{mdy,n-1}(i,j,k)M_{D,y}^{n-1}(i,j,k) + C_{MDy}^{ey,n}(i,j,k)\left[H_y^{n+1}(i,j,k) - 2H_y^{n}(i,j,k) + H_y^{n-1}(i,j,k)\right] \tag{6.175}$$

where

$$C_{MDy}^{mdy,n}(i,j,k) = \frac{4}{2 + \Gamma_D\Delta t} \tag{6.176}$$

$$C_{MDy}^{mdy,n-1}(i,j,k) = \frac{\Delta t\Gamma_D - 2}{2 + \Gamma_D\Delta t} \tag{6.177}$$

$$C_{MDy}^{ey,n}(i,j,k) = -\frac{2\varepsilon\mu_0\omega_{pD}^2}{2 + \Gamma_D\Delta t} \tag{6.178}$$

The y component of the Lorentz part can be written as

$$\frac{\partial^2 M_{L,y}}{\partial t^2} = -\varepsilon\mu_0\Delta\mu_L\omega_{pL}^2 \frac{\partial^2 H_y}{\partial t^2} - \Gamma_L \frac{\partial M_{L,y}}{\partial t} - \omega_{pL}^2 M_{L,y} \tag{6.179}$$

$$\frac{\partial^2 M_{L,y}(i,j,k)}{\partial t^2} = \frac{M_{L,y}^{n+1}(i,j,k) - 2M_{L,y}^{n}(i,j,k) + M_{L,y}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.180}$$

$$\frac{\partial M_{L,y}(i,j,k)}{\partial t} = \frac{M_{L,y}^{n+1}(i,j,k) - M_{L,y}^{n-1}(i,j,k)}{2\Delta t} \tag{6.181}$$

(6.182)

$$\frac{M_{L,y}^{n+1}(i,j,k) - 2M_{L,y}^{n}(i,j,k) + M_{L,y}^{n-1}(i,j,k)}{(\Delta t)^2}$$
$$= -\frac{\varepsilon\mu_0\Delta\mu_L\omega_{pL}^2}{(\Delta t)^2}\left[H_y^{n+1}(i,j,k) - 2H_y^{n}(i,j,k) + H_y^{n-1}(i,j,k)\right]$$
$$- \Gamma_L\frac{M_{L,y}^{n+1}(i,j,k) - M_{L,y}^{n-1}(i,j,k)}{2\Delta t} - \omega_{pL}^2 M_{L,y}^{n}(i,j,k)$$

We can simplify the updating equation for $M_{L,y}$ as

$$M_{L,y}^{n+1}(i,j,k) = C_{MLy}^{mly,n}(i,j,k)M_{L,y}^{n}(i,j,k) + C_{MLy}^{mly,n-1}(i,j,k)M_{L,y}^{n-1}(i,j,k) + C_{MLy}^{ey,n}(i,j,k)\left[H_y^{n+1}(i,j,k) - 2H_y^{n}(i,j,k) + H_y^{n-1}(i,j,k)\right] \tag{6.183}$$

where

$$C_{MLy}^{mly,n}(i,j,k) = \frac{4 - 2(\Delta t)^2\omega_{pL}^2}{2 + \Gamma_L\Delta t} \tag{6.184}$$

$$C_{MLy}^{mly,n-1}(i,j,k) = \frac{\Gamma_L\Delta t - 2}{2 + \Gamma_L\Delta t} \tag{6.185}$$

$$C_{MLy}^{ey,n}(i,j,k) = -\frac{2\varepsilon\mu_0\Delta\mu_L\omega_{pL}^2}{2 + \Gamma_L\Delta t} \tag{6.186}$$

Equations (6.163), (6.175) and (6.183) constitute the updating equations in y direction.

6.2.3. Updating Equation for the z Component

Cartesian component of (6.122) in z direction can be written as

$$\left(\frac{\partial^2 H_x}{\partial x \partial z} + \frac{\partial^2 H_y}{\partial y \partial z} + \frac{\partial^2 H_z}{\partial z^2}\right) - \left(\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2}\right) = -\varepsilon\mu_0\mu_\infty \frac{\partial^2 H_z}{\partial y^2} + K_{D,z} + M_{L,z} \tag{6.187}$$

$$\frac{\partial^2 H_x}{\partial x \partial z} + \frac{\partial^2 H_y}{\partial y \partial z} - \frac{\partial^2 H_z}{\partial x^2} - \frac{\partial^2 H_z}{\partial y^2} = -\varepsilon\mu_0\mu_\infty \frac{\partial^2 H_z}{\partial y^2} + K_{D,z} + M_{L,z} \tag{6.188}$$

To derive the FDTD updating equations for the magnetic fields, we have to evaluate all the spatial derivatives in equations (6.188) at the corresponding magnetic field node, i.e. H_z. The time derivatives are evaluated at the n^{th} time step, therefore:

$$\frac{\partial^2 H_z(i,j,k)}{\partial t^2} = \frac{H_z^{n+1}(i,j,k) - 2H_z^n(i,j,k) + H_z^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.189}$$

$$\frac{\partial^2 H_z(i,j,k)}{\partial x^2} = \frac{H_z^n(i+1,j,k) - 2H_z^n(i,j,k) + H_z^n(i-1,j,k)}{(\Delta x)^2} \tag{6.190}$$

$$\frac{\partial^2 H_z(i,j,k)}{\partial y^2} = \frac{H_z^n(i,j+1,k) - 2H_z^n(i,j,k) + H_z^n(i,j-1,k)}{(\Delta y)^2} \tag{6.191}$$

$$\frac{\partial^2 H_x(i,j,k)}{\partial x \partial z} = \frac{H_x^n(i,j,k+1) - H_x^n(i-1,j,k+1) - H_x^n(i,j,k) + H_x^n(i-1,j,k)}{\Delta x \Delta z} \tag{6.192}$$

$$\frac{\partial^2 H_y(i,j,k)}{\partial y \partial z} = \frac{H_y^n(i,j,k+1) - H_y^n(i,j,k) - H_y^n(i,j-1,k+1) + H_y^n(i,j-1,k)}{\Delta y \Delta z} \tag{6.193}$$

Inserting (6.189) - (6.193) into (6.188), we have

$$\frac{H_x^n(i,j,k+1) - H_x^n(i-1,j,k+1) - H_x^n(i,j,k) + H_x^n(i-1,j,k)}{\Delta x \Delta z}$$
$$+ \frac{H_y^n(i,j,k+1) - H_y^n(i,j,k) - H_y^n(i,j-1,k+1) + H_y^n(i,j-1,k)}{\Delta y \Delta z}$$
$$- \frac{H_z^n(i+1,j,k) - 2H_z^n(i,j,k) + H_z^n(i-1,j,k)}{(\Delta x)^2} \tag{6.194}$$
$$- \frac{H_z^n(i,j+1,k) - 2H_z^n(i,j,k) + H_z^n(i,j-1,k)}{(\Delta y)^2}$$
$$= -\varepsilon\mu_0\mu_\infty \frac{H_z^{n+1}(i,j,k) - 2H_z^n(i,j,k) + H_z^{n-1}(i,j,k)}{(\Delta t)^2} + K_{D,z}^n(i,j,k) + M_{L,z}^n(i,j,k)$$

We can simplify the updating equation for H_z as

$$\begin{aligned} H_z^{n+1}(i,j,k) = {} & C_{hz}^{hz,n}(i,j,k)[H_z^n(i,j,k)] + C_{hz}^{hz,n-1}(i,j,k)[H_z^{n-1}(i,j,k)] \\ & + C_{hz}^{hz,n,x}(i,j,k)[H_z^n(i+1,j,k) + H_z^n(i-1,j,k)] \\ & + C_{hz}^{hz,n,y}(i,j,k)[H_z^n(i,j+1,k) + H_z^n(i,j-1,k)] \\ & + C_{hz}^{hy,n,xz}(i,j,k)[H_x^n(i,j,k+1) - H_x^n(i-1,j,k+1) - H_x^n(i,j,k) \\ & + H_x^n(i-1,j,k)] \\ & + C_{hz}^{hz,n,yz}(i,j,k)\big[H_y^n(i,j,k+1) - H_y^n(i,j,k) - H_y^n(i,j-1,k+1) \\ & + H_y^n(i,j-1,k)] + C_z(i,j,k)\big[K_{D,z}^n(i,j,k) + M_{L,z}^n(i,j,k)\big] \end{aligned} \tag{6.195}$$

where

$$C_z(i,j,k) = \frac{(\Delta t)^2}{\varepsilon\mu_0\mu_\infty} \tag{6.196}$$

$$C_{hz}^{hz,n}(i,j,k) = -C_z(i,j,k)\left(-\frac{2\varepsilon\mu_0\mu_\infty}{(\Delta t)^2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right) \tag{6.197}$$

$$C_{hz}^{hz,n-1}(i,j,k) = -C_z(i,j,k)\left(\frac{\varepsilon\mu_0\mu_\infty}{(\Delta t)^2}\right) = -1 \tag{6.198}$$

$$C_{hz}^{hz,n,x}(i,j,k) = C_z(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{6.199}$$

$$C_{hz}^{hz,n,y}(i,j,k) = C_z(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{6.200}$$

$$C_{hz}^{hy,n,xz}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta x \Delta z}\right) \tag{6.201}$$

$$C_{hz}^{hz,n,yz}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta y \Delta z}\right) \tag{6.202}$$

The z component of the Drude part can be written as

$$\frac{\partial^2 K_{D,z}}{\partial t^2} = -\varepsilon\mu_0\omega_{pD}^2 \frac{\partial^2 H_z}{\partial t^2} - \Gamma_D \frac{\partial K_{D,z}}{\partial t} \tag{6.203}$$

$$\frac{\partial^2 K_{D,z}(i,j,k)}{\partial t^2} = \frac{K_{D,z}^{n+1}(i,j,k) - 2K_{D,z}^{n}(i,j,k) + K_{D,z}^{n-1}(i,j,k)}{(\Delta t)^2} \tag{6.204}$$

$$\frac{\partial K_{D,z}(i,j,k)}{\partial t} = \frac{K_{D,z}^{n+1}(i,j,k) - K_{D,z}^{n-1}(i,j,k)}{2\Delta t} \tag{6.205}$$

$$\begin{aligned} &\frac{K_{D,z}^{n+1}(i,j,k) - 2K_{D,z}^{n}(i,j,k) + K_{D,z}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad = -\varepsilon\mu_0\omega_{pD}^2 \frac{H_z^{n+1}(i,j,k) - 2H_z^{n}(i,j,k) + H_z^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad - \Gamma_D \frac{K_{D,z}^{n+1}(i,j,k) - K_{D,z}^{n-1}(i,j,k)}{2\Delta t} \end{aligned} \tag{6.206}$$

We can simplify the updating equation for $K_{D,z}$ as

$$\begin{aligned} K_{D,z}^{n+1}(i,j,k) = {} & C_{KDz}^{kdz,n}(i,j,k)K_{D,z}^{n}(i,j,k) + C_{KDz}^{kdz,n-1}(i,j,k)K_{D,z}^{n-1}(i,j,k) \\ & + C_{KDz}^{ez,n}(i,j,k)[H_z^{n+1}(i,j,k) - 2H_z^{n}(i,j,k) + H_z^{n-1}(i,j,k)] \end{aligned} \tag{6.207}$$

where

$$C_{KDz}^{kdz,n}(i,j,k) = \frac{4}{2 + \Gamma_D \Delta t} \tag{6.208}$$

$$C_{KDz}^{kdz,n-1}(i,j,k) = \frac{\Delta t\Gamma_D - 2}{2 + \Gamma_D \Delta t} \quad (6.209)$$

$$C_{KDz}^{ez,n}(i,j,k) = -\frac{2\varepsilon\mu_0\omega_{pD}^2}{2 + \Gamma_D \Delta t} \quad (6.210)$$

The z component of the Lorentz part can be written as

$$\frac{\partial^2 M_{L,z}}{\partial t^2} = -\varepsilon\mu_0 \Delta\mu_L \omega_{pL}^2 \frac{\partial^2 H_z}{\partial t^2} - \Gamma_L \frac{\partial M_{L,z}}{\partial t} - \omega_{pL}^2 M_{L,z} \quad (6.211)$$

$$\frac{\partial^2 M_{L,z}(i,j,k)}{\partial t^2} = \frac{M_{L,z}^{n+1}(i,j,k) - 2M_{L,z}^{n}(i,j,k) + M_{L,z}^{n-1}(i,j,k)}{(\Delta t)^2} \quad (6.212)$$

$$\frac{\partial M_{L,z}(i,j,k)}{\partial t} = \frac{M_{L,z}^{n+1}(i,j,k) - M_{L,z}^{n-1}(i,j,k)}{2\Delta t} \quad (6.213)$$

$$\begin{aligned} &\frac{M_{L,z}^{n+1}(i,j,k) - 2M_{L,z}^{n}(i,j,k) + M_{L,z}^{n-1}(i,j,k)}{(\Delta t)^2} \\ &\qquad = -\frac{\varepsilon\mu_0 \Delta\mu_L \omega_{pL}^2}{(\Delta t)^2}\left[H_z^{n+1}(i,j,k) - 2H_z^{n}(i,j,k) + H_z^{n-1}(i,j,k)\right] \\ &\qquad - \Gamma_L \frac{M_{L,z}^{n+1}(i,j,k) - M_{L,z}^{n-1}(i,j,k)}{2\Delta t} - \omega_{pL}^2 M_{L,z}^{n}(i,j,k) \end{aligned} \quad (6.214)$$

We can simplify the updating equation for $M_{L,z}$ as

$$\begin{aligned} M_{L,z}^{n+1}(i,j,k) &= C_{MLz}^{mlz,n}(i,j,k) M_{L,z}^{n}(i,j,k) + C_{MLz}^{mlz,n-1}(i,j,k) M_{L,z}^{n-1}(i,j,k) \\ &+ C_{MLz}^{ez,n}(i,j,k)\left[H_z^{n+1}(i,j,k) - 2H_z^{n}(i,j,k) + H_z^{n-1}(i,j,k)\right] \end{aligned} \quad (6.215)$$

where

$$C_{MLz}^{mlz,n}(i,j,k) = \frac{4 - 2(\Delta t)^2 \omega_{pL}^2}{2 + \Gamma_L \Delta t} \quad (6.216)$$

$$C_{MLz}^{mlz,n-1}(i,j,k) = \frac{\Gamma_L \Delta t - 2}{2 + \Gamma_L \Delta t} \tag{6.217}$$

$$C_{MLz}^{ez,n}(i,j,k) = -\frac{2\varepsilon\mu_0 \Delta\mu_L \omega_{pL}^2}{2 + \Gamma_L \Delta t} \tag{6.218}$$

Equations (6.195), (6.207) and (6.215) constitute the updating equations in z direction.

6.3. A Note on Performance

Single-field FDTD updating equations are derived for three-dimensional dispersive media by harnessing Lorentz-Drude model. One can reduce the formulations presented here to two-dimensional dispersive problems and take advantage of the improvement in speed and memory requirement as presented in Chapter 4, because the dispersive media updating equations have structure similar to the ones derived for general non-dispersive media. The only addition is the Lorentz and Drude parts that also exist in the traditional method. This chapter shows that the single-field approach is applicable for dispersive as well as nondispersive media.

6.4. Numerical Validation

To verify the numerical validity of the derived formulations, a three-dimensional problem is set up. To perform the simulation of the structure, the single-field formulations incorporated with LD modeled permittivity as given in Section 6.1 are used along with the traditional formulations given in Appendix D.

The problem space includes a sphere of radius 18 nm located at the origin with the following properties: $\varepsilon_\infty = 5.9673$, $\omega_{pD} = 2\pi(2.1136 \times 10^{15})$, $\Gamma_D = 2\pi(15.92 \times 10^{12})$,

$\omega_{pL} = 2\pi(650.07 \times 10^{12})$, $\Gamma_L = 2\pi(104.84 \times 10^{12})$ and $\Delta\varepsilon_L = 1.09$. There is a point field-source that updates x component of the electric field at (60, 60, 20) nm with a Gaussian pulse; the total field is sampled in three different locations: (60, 60, 40) nm, (60, 60, 70) nm and (60, 60, 100) nm. Figures 6.1, 6.2 and 6.3 show the comparison of field values calculated by the single-field and the traditional formulations. The entire FDTD domain is 120 nm x 120 nm x 120 nm, and the sphere is located at (60, 60, 60) nm.

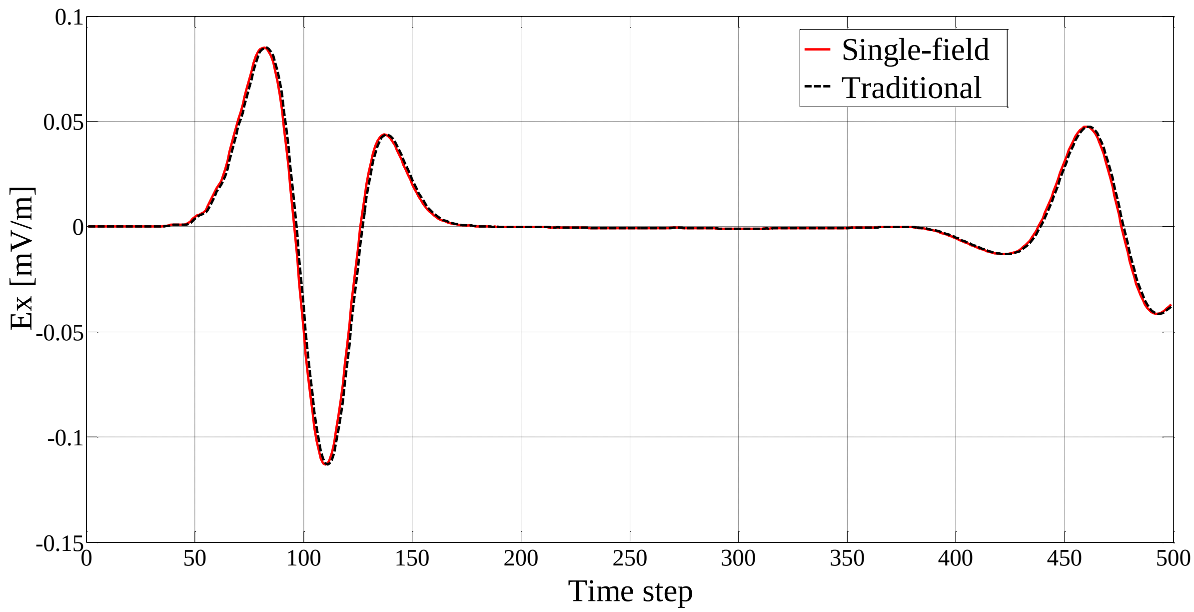

Fig. 6.1 E_x field sampled at (60, 60, 40) nm.

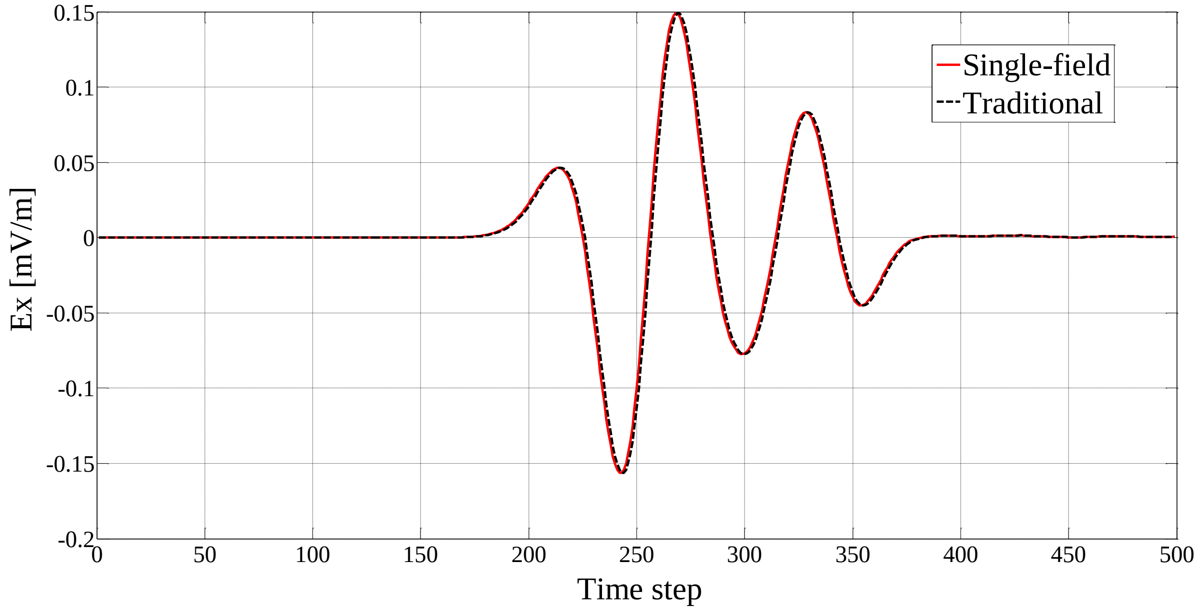

Fig. 6.2 E_x field sampled at (60, 80, 70) nm.

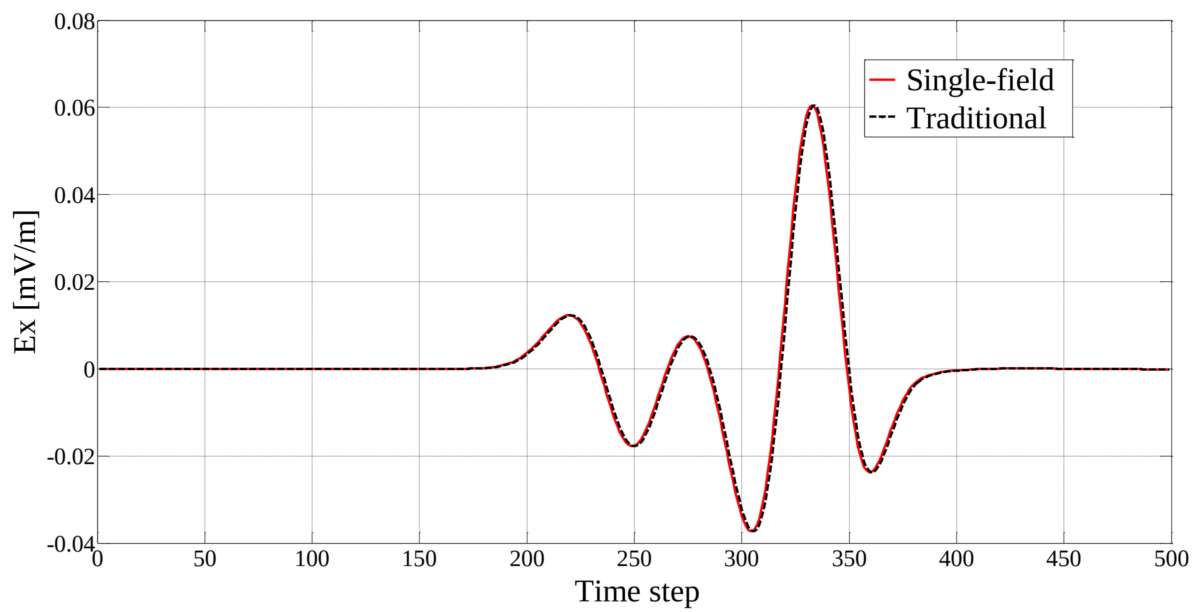

Fig. 6.3 E_x field sampled at (60, 80, 100) nm.

7. CONCLUSION

Single-field finite-difference time-domain updating equations based on single-field have been derived for three-, two- and one-dimensional electromagnetic problems. Although the single-field approach can be applied to either field, that is, we can develop single-field FDTD updating equations based on E or H field, electric field based updating equations are used for the derivation and verification purposes. Liao's absorbing boundary condition is used whenever needed.

One-dimensional case of the single-field formulation is evaluated with an example geometry, and it is observed that the single-field formulation is 20% faster than the traditional one, and provides around 17% memory reduction for solving the same size problem.

The single-field formulation has a great advantage in two-dimensional case. A two-dimensional TM problem is constructed with an electric current source, and the field away from the source is calculated by the single-field and the traditional formulations. First, the stability and dispersion analyses are performed. Then, the speed and memory analyses follow; the single-field formulation happens to be almost three times faster and requires about 40% less memory than its traditional counterpart. A two-dimensional TE problem evaluation is also discussed to show that the single-field formulation is advantageous for two-dimensional TE as well as TM problems. A scattering problem of an infinite line-current in the vicinity of a circular conducting cylinder is simulated with both formulations and the results are compared with respect to the analytical solution; the two FDTD formulations show similar accuracy characteristics. Another TM problem is considered to test the ability of the single-field formulation in handling simulations that include PECs and dielectric and magnetic scatterers with non-zero electric and magnetic conductivity. A sectoral (2D) horn antenna that is of great significance in practice is also simulated with both formulations; generated fields are plotted for visual comparison and they are in good agreement.

In addition to the normal incidence case, oblique incidence case is also considered; oblique incidence FDTD updating equations are derived and compared with the traditional formulation in terms of accuracy, speed and memory requirements. The single-field formulation is as advantageous in terms of speed and memory requirements in oblique incidence case as it is in the case of normal incidence.

Finally, general FDTD updating equations based on single-field are derived for dispersive media. Two cases are studied: (i) E-based single-field FDTD updating equations with constant permeability and Lorentz-Drude (LD) modeled permittivity and (ii) H-based single-field FDTD updating equations with constant permittivity and LD modeled permeability. It is shown that single-field formulation can be obtained for dispersive media, too. Numerical validation is performed with a three-dimensional problem that includes a dispersive sphere. Results generated by the single-field and the traditional formulations are in good agreement.

The single-field FDTD formulation, in overall, is faster and requires less memory for any two-dimensional TE and TM problems with normal as well as oblique incident waves. This is the main contribution of this dissertation. Another contribution is the derivation and validation of single-field FDTD formulation for dispersive media analysis.

Future studies would be to investigate the compatibility of the single-field approach with the software and hardware acceleration techniques such as parallel programming and

Compute Unified Device Architecture (CUDA) [30, 31]. Moreover, this single-field approach can be extended to finite-difference frequency-domain formulation.

APPENDIX A

Dispersion Analysis in Two-dimensional Problem Space

Start with the updating equation for E_z component

$$\begin{aligned} E_z^{n+1}(i,j) = C_{ez}^{ez,n-1}(i,j)[E_z^{n-1}(i,j)] + C_{ez}^{ez,n}(i,j)[E_z^{n}(i,j)] \\ + C_{ez}^{ez,x}(i,j)[E_z^{n}(i+1,j) + E_z^{n}(i-1,j)] \\ + C_{ez}^{ez,y}(i,j)[E_z^{n}(i,j+1) + E_z^{n}(i,j-1)] \end{aligned} \tag{A.1}$$

Consider a lossless medium and assume the following monochromatic traveling-wave trial solutions, then

$$E_z^{n}(i,j) = E_{0,z}e^{j(\omega n\Delta t - k_x i_x \Delta_x - k_y i_y \Delta_y)} \tag{A.2}$$

$$E_z^{n-1}(i,j) = E_{0,z}e^{j(\omega(n-1)\Delta t - k_x i_x \Delta_x - k_y i_y \Delta_y)} \tag{A.3}$$

$$E_z^{n}(i+1,j) = E_{0,z}e^{j(\omega n\Delta t - k_x (i_x+1) \Delta_x - k_y i_y \Delta_y)} \tag{A.4}$$

$$E_z^{n}(i-1,j) = E_{0,z}e^{j(\omega n\Delta t - k_x (i_x-1) \Delta_x - k_y i_y \Delta_y)} \tag{A.5}$$

$$E_z^{n}(i,j+1) = E_{0,z}e^{j(\omega n\Delta t - k_x i_x \Delta_x - k_y (i_y+1) \Delta_y)} \tag{A.6}$$

$$E_z^{n}(i,j-1) = E_{0,z}e^{j(\omega n\Delta t - k_x i_x \Delta_x - k_y (i_y-1) \Delta_y)} \tag{A.7}$$

where k_x and k_y are the x and y components of the numerical wavevector, and i_x and i_y are space indices. By substituting those field expressions into the updating equation, one may obtain

$$
\begin{aligned}
E_{0,z}e^{j\left(\omega(n+1)\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)} \\
&= C_{ez}^{ez,n-1}(i,j)\left[E_{0,z}e^{j\left(\omega(n-1)\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,n}(i,j)\left[E_{0,z}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,x}(i,j)\left[E_{0,z}e^{j\left(\omega n\Delta t-k_x(i_x+1)\Delta_x-k_y i_y\Delta_y\right)}\right. \\
&\left.+ E_{0,z}e^{j\left(\omega n\Delta t-k_x(i_x-1)\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,y}(i,j)\left[E_{0,z}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y\left(i_y+1\right)\Delta_y\right)}\right. \\
&\left.+ E_{0,z}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y\left(i_y-1\right)\Delta_y\right)}\right]
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
e^{j(\omega\Delta t)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)} \\
&= C_{ez}^{ez,n-1}(i,j)\left[e^{j(-\omega\Delta t)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,n}(i,j)\left[e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,x}(i,j)\left[e^{j(-k_x\Delta_x)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right. \\
&\left.+ e^{j(k_x\Delta_x)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right] \\
&+ C_{ez}^{ez,y}(i,j)\left[e^{j(-k_y\Delta_y)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right. \\
&\left.+ e^{j(k_y\Delta_y)}e^{j\left(\omega n\Delta t-k_x i_x\Delta_x-k_y i_y\Delta_y\right)}\right]
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
e^{j(\omega\Delta t)} = C_{ez}^{ez,n-1}(i,j)\left[e^{j(-\omega\Delta t)}\right] + C_{ez}^{ez,n}(i,j) + C_{ez}^{ez,x}(i,j)\left[e^{j(-k_x\Delta_x)} + e^{j(k_x\Delta_x)}\right] \\
+ C_{ez}^{ez,y}(i,j)\left[e^{j(-k_y\Delta_y)} + e^{j(k_y\Delta_y)}\right]
\end{aligned} \tag{A.10}
$$

Coefficient values for a lossless medium are

$$C_{ez}^{ez,n-1}(i,j) = -1 \tag{A.11}$$

$$C_{ez}^{ez,n}(i,j) = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\left(\frac{2}{(\Delta t)^2}\mu_z\varepsilon_z - \frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}\right) \tag{A.12}$$

$$C_{ez}^{ez,x}(i,j) = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{(\Delta x)^2} \tag{A.13}$$

$$C_{ez}^{ez,y}(i,j) = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{(\Delta y)^2} \tag{A.14}$$

Assuming that $\Delta y = \Delta x = h$

$$e^{j(\omega\Delta t)} + e^{j(-\omega\Delta t)} = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\left(\frac{2}{(\Delta t)^2}\mu_z\varepsilon_z - \frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}\right) + \frac{\Delta t^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left[\left(e^{j(-k_x h)} + e^{j(k_x h)}\right) + \left(e^{j(-k_y h)} + e^{j(k_y h)}\right)\right] \tag{A.15}$$

$$2\cos(\omega\Delta t) = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\left(\frac{2}{(\Delta t)^2}\mu_z\varepsilon_z - \frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}\right) + \frac{\Delta t^2}{\mu_z\varepsilon_z}\frac{2}{h^2}\left[\cos(k_x h) + \cos(k_y h)\right] \tag{A.16}$$

$$\cos(\omega\Delta t) = \left(1 - \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{2}{h^2}\right) + \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left[\cos(k_x h) + \cos(k_y h)\right] \tag{A.17}$$

$$\cos(\omega\Delta t) - 1 = -\frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{2}{h^2} + \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left[\cos(k_x h) + \cos(k_y h)\right] \tag{A.18}$$

$$-2\left(\sin\left(\frac{\omega\Delta t}{2}\right)\right)^2 = -\frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{2}{h^2} + \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left[\cos(k_x h) + \cos(k_y h)\right] \tag{A.19}$$

$$-2\left(\sin\left(\frac{\omega\Delta t}{2}\right)\right)^2 = -\frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{2}{h^2} + \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left[2\cos\left(\frac{k_x h + k_y h}{2}\right)\cos\left(\frac{k_x h - k_y h}{2}\right)\right] \tag{A.20}$$

$$\left(\sin\left(\frac{\omega\Delta t}{2}\right)\right)^2 = \frac{(\Delta t)^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left\{1 - \left[\cos\left(h\frac{k_x + k_y}{2}\right)\cos\left(h\frac{k_x - k_y}{2}\right)\right]\right\} \tag{A.21}$$

$$\left(\sin\left(\frac{\omega\Delta t}{2}\right)\right)^2 = \frac{\Delta t^2}{\mu_z\varepsilon_z}\frac{1}{h^2}\left\{1 - \left[\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha - \sin\alpha)\right)\right]\right\} \tag{A.22}$$

$$k_x = \frac{2\pi}{\lambda_n}\cos\alpha \tag{A.23}$$

$$k_y = \frac{2\pi}{\lambda_n}\sin\alpha \tag{A.24}$$

$$\omega = \frac{2\pi c}{\lambda} \tag{A.25}$$

$$PPW = \frac{\lambda_n}{h} \tag{A.26}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{A.27}$$

$$\frac{c\Delta t}{h} = 0.5 \tag{A.28}$$

$$\left(\sin\left(\frac{\omega \Delta t}{2}\right)\right)^2 = \frac{1}{4}\left\{1 - \left[\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha - \sin\alpha)\right)\right]\right\} \tag{A.29}$$

$$\sin\left(\frac{\omega \Delta t}{2}\right) = \left\{\frac{1}{4} - \left[\frac{1}{4}\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha - \sin\alpha)\right)\right]\right\}^{1/2} \tag{A.30}$$

$$\frac{\omega \Delta t}{2} = \sin^{-1}\left\{\frac{1}{4} - \left[\frac{1}{4}\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha - \sin\alpha)\right)\right]\right\}^{1/2} \tag{A.31}$$

$$\frac{\Delta t \pi c}{\lambda} = \sin^{-1}\left\{\frac{1}{4} - \left[\frac{1}{4}\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{h\pi}{\lambda_n}(\cos\alpha - \sin\alpha)\right)\right]\right\}^{1/2} \tag{A.32}$$

$$\begin{aligned}\frac{\lambda_n}{\lambda} = \frac{2PPW}{\pi}\sin^{-1}\Bigg\{\frac{1}{4} \\ - \left[\frac{1}{4}\cos\left(\frac{\pi}{PPW}(\cos\alpha + \sin\alpha)\right)\cos\left(\frac{\pi}{PPW}(\cos\alpha - \sin\alpha)\right)\right]\Bigg\}^{1/2}\end{aligned} \tag{A.33}$$

APPENDIX B

Single-Field FDTD Updating Equations Based on H-field

Starting with Maxwell's curl equations:

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} - (\boldsymbol{M}_i + \sigma^m \boldsymbol{H}) \quad \text{(B.1)}$$

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{J}_i + \sigma^e \boldsymbol{E}) \quad \text{(B.2)}$$

Taking the curl of (B.2) and following a procedure similar to that presented in Chapter 2, one can obtain the H-field vector wave equation as

$$\begin{aligned} &\nabla^2 \boldsymbol{H} - \nabla(\nabla \cdot \boldsymbol{H}) \\ &= \sigma^e \sigma^m \boldsymbol{H} + (\sigma^m \varepsilon + \sigma^e \mu) \frac{\partial \boldsymbol{H}}{\partial t} + \varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \nabla \times \boldsymbol{J}_i + \sigma^e \boldsymbol{M}_i + \varepsilon \frac{\partial \boldsymbol{M}_i}{\partial t} \end{aligned} \quad \text{(B.3)}$$

To find the H-based single-field updating equations, (B.3) is decomposed into its Cartesian components and necessary difference equations are substituted according to their positions in the Yee-cell. Consequently, the following updating equations can be derived.

$$\begin{aligned}H_x^{n+1}(i,j,k) = {} & C_{hx}^{hx,n}(i,j,k)H_x^n(i,j,k) + C_{hx}^{hx,n-1}(i,j,k)[H_x^{n-1}(i,j,k)] \\ & + C_{hx}^{hx,n,y}(i,j,k)[H_x^n(i,j+1,k) + H_x^n(i,j-1,k)] \\ & + C_{hx}^{hx,n,z}(i,j,k)[H_x^n(i,j,k+1) + H_x^n(i,j,k-1)] \\ & + C_{hx}^{hy,n,xy}(i,j,k)\big[H_y^n(i,j,k+1) - H_y^n(i,j,k) - H_y^n(i-1,j,k+1) \\ & + H_y^n(i-1,j,k)] \\ & + C_{hx}^{hz,n,xz}(i,j,k)[H_z^n(i+1,j,k) - H_z^n(i+1,j,k-1) - H_z^n(i,j,k) \\ & + H_z^n(i,j,k-1)] + C_{hx}^{jz,n,y}(i,j,k)\big[J_{i,z}^n(i,j+1,k) - J_{i,z}^n(i,j,k)\big] \\ & + C_{hx}^{jy,n,z}(i,j,k)\big[J_{i,y}^n(i,j,k+1) - J_{i,y}^n(i,j,k)\big] \\ & + C_{hx}^{mx,n}(i,j,k)M_{i,x}(i,j,k) \\ & + C_{hx}^{mx,t}(i,j,k)\big[M_{i,x}^{n+1}(i,j,k) - M_{i,x}^{n-1}(i,j,k)\big]\end{aligned} \tag{B.4}$$

where

$$C_x(i,j,k) = -\frac{2\Delta t^2}{\Delta t(\mu_x\sigma_x^e + \varepsilon_x\sigma_x^m) + 2\mu_x\varepsilon_x} \tag{B.5}$$

$$C_{hx}^{hx,n}(i,j,k) = C_x(i,j,k)\left(\frac{2}{(\Delta y)^2} + \frac{2}{(\Delta z)^2} - \frac{2\mu_x\varepsilon_x}{\Delta t^2} + \sigma_x^m\sigma_x^e\right) \tag{B.6}$$

$$C_{hx}^{hx,n-1}(i,j,k) = C_x(i,j,k)\left(\frac{\mu_x\varepsilon_x}{\Delta t^2} - \frac{(\mu_x\sigma_x^e + \varepsilon_x\sigma_x^m)}{2\Delta t}\right) \tag{B.7}$$

$$C_{hx}^{hx,n,y}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{B.8}$$

$$C_{hx}^{hx,n,z}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{B.9}$$

$$C_{hx}^{hy,n,xy}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{B.10}$$

$$C_{hx}^{hz,n,xz}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta x\Delta z}\right) \tag{B.11}$$

$$C_{hx}^{jz,n,y}(i,j,k) = -C_x(i,j,k)\left(\frac{1}{\Delta y}\right) \tag{B.12}$$

$$C_{hx}^{jy,n,z}(i,j,k) = C_x(i,j,k)\left(\frac{1}{\Delta z}\right) \tag{B.13}$$

$$C_{hx}^{mx,n}(i,j,k) = C_x(i,j,k)(\sigma_x^e) \tag{B.14}$$

$$C_{hx}^{mx,t}(i,j,k) = C_x(i,j,k)\left(\frac{\varepsilon_x}{2\Delta t}\right) \tag{B.15}$$

$$\begin{aligned} H_y^{n+1}(i,j,k) = {} & C_{hy}^{hy,n}(i,j,k)H_y^n(i,j,k) + C_{hy}^{hy,n-1}(i,j,k)\left[H_y^{n-1}(i,j,k)\right] \\ & + C_{hy}^{hy,n,y}(i,j,k)\left[H_y^n(i+1,j,k) + H_y^n(i-1,j,k)\right] \\ & + C_{hy}^{hy,n,z}(i,j,k)\left[H_y^n(i,j,k+1) + H_y^n(i,j,k-1)\right] \\ & + C_{hy}^{hx,n,xy}(i,j,k)[H_x^n(i+1,j,k) - H_x^n(i+1,j-1,k) - H_x^n(i,j,k) \\ & + H_x^n(i,j-1,k)] \\ & + C_{hy}^{hz,n,yz}(i,j,k)[H_z^n(i,j,k+1) - H_z^n(i,j-1,k+1) - H_z^n(i,j,k) \\ & + H_z^n(i,j-1,k)] + C_{hy}^{jz,n,x}(i,j,k)\left[J_{i,z}^n(i+1,j,k) - J_{i,z}^n(i,j,k)\right] \\ & + C_{hy}^{jy,n,z}(i,j,k)\left[J_{i,x}^n(i,j,k+1) - J_{i,x}^n(i,j,k)\right] \\ & + C_{hy}^{my,n}(i,j,k)M_{i,y}(i,j,k) \\ & + C_{hy}^{my,t}(i,j,k)\left[M_{i,y}^{n+1}(i,j,k) - M_{i,y}^{n-1}(i,j,k)\right] \end{aligned} \tag{B.16}$$

where

$$C_y(i,j,k) = -\frac{2\Delta t^2}{\Delta t\left(\mu_y\sigma_y^e + \varepsilon_y\sigma_y^m\right) + 2\mu_y\varepsilon_y} \tag{B.17}$$

$$C_{hy}^{hy,n}(i,j,k) = C_y(i,j,k)\left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta z)^2} - \frac{2\mu_y\varepsilon_y}{\Delta t^2} + \sigma_y^m\sigma_y^e\right) \tag{B.18}$$

$$C_{hy}^{hy,n-1}(i,j,k) = C_y(i,j,k)\left(\frac{\mu_y\varepsilon_y}{\Delta t^2} - \frac{\left(\mu_y\sigma_y^e + \varepsilon_y\sigma_y^m\right)}{2\Delta t}\right) \tag{B.19}$$

$$C_{hy}^{hy,n,y}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{B.20}$$

$$C_{hy}^{hy,n,z}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{(\Delta z)^2}\right) \tag{B.21}$$

$$C_{hy}^{hy,n,xy}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta x\Delta y}\right) \tag{B.22}$$

$$C_{hy}^{hz,n,yz}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta y\Delta z}\right) \tag{B.23}$$

$$C_{hy}^{jz,n,x}(i,j,k) = C_y(i,j,k)\left(\frac{1}{\Delta x}\right) \tag{B.24}$$

$$C_{hy}^{jy,n,z}(i,j,k) = -C_y(i,j,k)\left(\frac{1}{\Delta z}\right) \tag{B.25}$$

$$C_{hy}^{my,n}(i,j,k) = C_y(i,j,k)\left(\sigma_y^e\right) \tag{B.26}$$

$$C_{hy}^{my,t}(i,j,k) = C_y(i,j,k)\left(\frac{\varepsilon_y}{2\Delta t}\right) \tag{B.27}$$

$$\begin{aligned}
H_z^{n+1}(i,j,k) = {} & C_{hz}^{hz,n}(i,j,k)H_z^n(i,j,k) + C_{hz}^{hz,n-1}(i,j,k)[H_z^{n-1}(i,j,k)] \\
& + C_{hz}^{hz,n,x}(i,j,k)[H_z^n(i+1,j,k) + H_z^n(i-1,j,k)] \\
& + C_{hz}^{hz,n,y}(i,j,k)[H_z^n(i,j+1,k) + H_z^n(i,j-1,k)] \\
& + C_{hz}^{hx,n,xz}(i,j,k)[H_x^n(i+1,j,k) - H_x^n(i,j,k) - H_x^n(i+1,j,k-1) \\
& + H_x^n(i,j,k-1)] \\
& + C_{hz}^{hy,n,yz}(i,j,k)\big[H_y^n(i,j+1,k) - H_y^n(i,j,k) - H_y^n(i,j+1,k-1) \\
& + H_y^n(i,j,k-1)] + C_{hz}^{jy,n,x}(i,j,k)\big[J_{i,y}^n(i+1,j,k) - J_{i,y}^n(i,j,k)\big] \\
& + C_{hz}^{jx,n,y}(i,j,k)\big[J_{i,x}^n(i,j+1,k) - J_{i,x}^n(i,j-1,k)\big] \\
& + C_{hz}^{mz,n}(i,j,k)M_{i,z}^n(i,j,k) \\
& + C_{hz}^{mz,t}(i,j,k)\big[M_{i,z}^{n+1}(i,j,k) - M_{i,z}^{n-1}(i,j,k)\big]
\end{aligned} \tag{B.28}$$

where

$$C_z(i,j,k) = -\frac{2\Delta t^2}{\Delta t(\mu_z\sigma_z^e + \varepsilon_z\sigma_z^m) + 2\mu_z\varepsilon_z} \tag{B.29}$$

$$C_{hz}^{hz,n}(i,j,k) = C_z(i,j,k)\left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} - \frac{2\mu_z\varepsilon_z}{\Delta t^2} + \sigma_z^m\sigma_z^e\right) \tag{B.30}$$

$$C_{hz}^{hz,n-1}(i,j,k) = C_z(i,j,k)\left(\frac{\mu_z\varepsilon_z}{\Delta t^2} - \frac{(\mu_z\sigma_z^e + \varepsilon_z\sigma_z^m)}{2\Delta t}\right) \tag{B.31}$$

$$C_{hz}^{hz,n,x}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{(\Delta x)^2}\right) \tag{B.32}$$

$$C_{hz}^{hz,n,y}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{(\Delta y)^2}\right) \tag{B.33}$$

$$C_{hz}^{hx,n,xz}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta x \Delta z}\right) \tag{B.34}$$

$$C_{hz}^{hy,n,yz}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta y \Delta z}\right) \tag{B.35}$$

$$C_{hz}^{jy,n,x}(i,j,k) = -C_z(i,j,k)\left(\frac{1}{\Delta x}\right) \tag{B.36}$$

$$C_{hz}^{jx,n,y}(i,j,k) = C_z(i,j,k)\left(\frac{1}{\Delta y}\right) \tag{B.37}$$

$$C_{hz}^{mz,n}(i,j,k) = C_z(i,j,k)(\sigma_z^e) \tag{B.38}$$

$$C_{hz}^{mz,t}(i,j,k) = C_z(i,j,k)\left(\frac{\varepsilon_z}{2\Delta t}\right) \tag{B.39}$$

APPENDIX C

The Computing System Information

All of the simulations presented in this dissertation are done with a system whose specifications are given in Table C.1 below.

Table C.1 The computing system specifications.

Processor	Intel(R) Core(TM) i7 CPU 920 @ 2.67 GHz
Memory	6.00 GB
System Type	64-bit Operating System
Operating System	Windows 7 Professional
Programming Language/Compiler	Matlab v.7.8.0.347 (R2009a) 32-bit (win32)

APPENDIX D

The Traditional Updating Equations with Lorentz-Drude Model for Permittivity

Following a procedure similar to the one presented in Section 6.1, we can derive the traditional FDTD updating equations for dispersive media based on Lorentz-Drude model. The formulations are developed with constant permeability and LD-modeled permittivity as given in (6.1). Moreover, this procedure can provide the dual formulations for media with a constant permittivity and LD-modeled permeability.

Incorporating (6.1) in (6.3), and making use of (6.4), one can obtain

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon_0\varepsilon_r(\omega)\boldsymbol{E} \tag{D.1}$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon_0\left(\varepsilon_\infty + \frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D\omega} + \frac{\Delta\varepsilon_L\omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2}\right)\boldsymbol{E} \tag{D.2}$$

Introducing the Drude and the Lorentz terms,

$$\boldsymbol{J}_D = j\omega\varepsilon_0\frac{\omega_{pD}^2}{j^2\omega^2 + j\Gamma_D\omega}\boldsymbol{E} \tag{D.3}$$

$$\boldsymbol{P}_L = \frac{\Delta\varepsilon_L\omega_{pL}^2}{j^2\omega^2 + j\omega\Gamma_L + \omega_{pL}^2}\boldsymbol{E} \tag{D.4}$$

Revisiting (D.2), and using (D.3) and (D.4)

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon_0\varepsilon_\infty\boldsymbol{E} + \boldsymbol{J}_D + j\omega\varepsilon_0\boldsymbol{P}_L \tag{D.5}$$

Replacing $j\omega$ terms with $\frac{\partial}{\partial \mathrm{t}}$ and arranging the equations, (D.3) and (D.4) can be written in time-domain as

$$\varepsilon_0\omega_{pD}^2\boldsymbol{E} = \frac{\partial \boldsymbol{J}_D}{\partial t} + \Gamma_D\boldsymbol{J}_D \tag{D.6}$$

$$\Delta\varepsilon_L\omega_{pL}^2\boldsymbol{E} = \frac{\partial^2 \boldsymbol{P}_L}{\partial t^2} + \Gamma_L\frac{\partial \boldsymbol{P}_L}{\partial t} + \omega_{pL}^2\boldsymbol{P}_L \tag{D.7}$$

We can write (D.5) and (6.2) in time domain as

$$\nabla \times \boldsymbol{H} = \varepsilon_0\varepsilon_\infty\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J}_D + \varepsilon_0\frac{\partial \boldsymbol{P}_L}{\partial t} \tag{D.8}$$

$$\nabla \times \boldsymbol{E} = -\mu_0\frac{\partial \boldsymbol{H}}{\partial t} \tag{D.9}$$

The traditional FDTD updating equations will be based on the time-domain equations (D.6) to (D.9). Those vector equations are decomposed into their Cartesian components and differentiation terms are discretized accordingly to obtain the following updating equations for electric field, magnetic field, Drude and Lorentz terms. C terms represent constant coefficients in terms of medium characteristics.

The x component of the electric field, magnetic field, the Lorentz part and the Drude part updating equations are

$$\begin{aligned} E_x^{n+1}(i,j,k) = {} & C_{ex}^1(i,j,k)[E_x^n(i,j,k)] \\ & + C_{ex}^2(i,j,k)\left[H_z^{n+\frac{1}{2}}(i,j,k) - H_z^{n+\frac{1}{2}}(i,j-1,k)\right] \\ & + C_{ex}^3(i,j,k)\left[H_y^{n+\frac{1}{2}}(i,j,k) - H_y^{n+\frac{1}{2}}(i,j,k-1)\right] \\ & + C_{ex}^4(i,j,k)[J_x^n(i,j,k)] \\ & + C_{ex}^5(i,j,k)[P_x^n(i,j,k)] \\ & + C_{ex}^6(i,j,k)[P_x^{n-1}(i,j,k)] \end{aligned} \tag{D.10}$$

$$\begin{aligned} H_x^{n+\frac{1}{2}}(i,j,k) = {} & H_x^{n-\frac{1}{2}}(i,j,k) + C_{hx}^1(i,j,k)[E_z^n(i,j+1,k) - E_z^n(i,j,k)] \\ & + C_{hx}^2(i,j,k)\left[E_y^n(i,j,k+1) - E_y^n(i,j,k)\right] \end{aligned} \tag{D.11}$$

$$\begin{aligned} J_x^{n+1}(i,j,k) = {} & C_{jx}^1(i,j,k)[J_x^n(i,j,k)] \\ & + C_{jx}^2(i,j,k)[E_x^{n+1}(i,j,k) + E_x^n(i,j,k)] \end{aligned} \tag{D.12}$$

$$\begin{aligned} P_x^{n+1}(i,j,k) = {} & C_{px}^1(i,j,k)[P_x^n(i,j,k)] \\ & + C_{px}^2(i,j,k)[P_x^{n-1}(i,j,k)] \\ & + C_{px}^3(i,j,k)[E_x^{n+1}(i,j,k) + E_x^n(i,j,k)] \end{aligned} \tag{D.13}$$

The y component of the electric field, magnetic field, the Lorentz part and the Drude part updating equations are

$$
\begin{aligned}
E_y^{n+1}(i,j,k) = {} & C_{ey}^1(i,j,k)\left[E_y^n(i,j,k)\right] \\
& + C_{ey}^2(i,j,k)\left[H_z^{n+\frac{1}{2}}(i,j,k) - H_z^{n+\frac{1}{2}}(i-1,j,k)\right] \\
& + C_{ey}^3(i,j,k)\left[H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n+\frac{1}{2}}(i,j,k-1)\right] \\
& + C_{ey}^4(i,j,k)\left[J_y^n(i,j,k)\right] \\
& + C_{ey}^5(i,j,k)\left[P_y^n(i,j,k)\right] \\
& + C_{ey}^6(i,j,k)\left[P_y^{n-1}(i,j,k)\right]
\end{aligned}
\tag{D.14}
$$

$$
\begin{aligned}
H_y^{n+\frac{1}{2}}(i,j,k) = {} & H_y^{n-\frac{1}{2}}(i,j,k) + C_{hy}^1(i,j,k)[E_z^n(i+1,j,k) - E_z^n(i,j,k)] \\
& + C_{hy}^2(i,j,k)[E_x^n(i,j,k+1) - E_x^n(i,j,k)]
\end{aligned}
\tag{D.15}
$$

$$
\begin{aligned}
J_y^{n+1}(i,j,k) = {} & C_{jy}^1(i,j,k)\left[J_y^n(i,j,k)\right] \\
& + C_{jy}^2(i,j,k)[E_x^{n+1}(i,j,k) + E_x^n(i,j,k)]
\end{aligned}
\tag{D.16}
$$

$$
\begin{aligned}
P_y^{n+1}(i,j,k) = {} & C_{py}^1(i,j,k)\left[P_y^n(i,j,k)\right] \\
& + C_{py}^2(i,j,k)\left[P_y^{n-1}(i,j,k)\right] \\
& + C_{py}^3(i,j,k)\left[E_y^{n+1}(i,j,k) + E_y^n(i,j,k)\right]
\end{aligned}
\tag{D.17}
$$

The z component of the electric field, magnetic field, the Lorentz part and the Drude part updating equations are

$$\begin{aligned}E_z^{n+1}(i,j,k) = {} & C_{ez}^1(i,j,k)[E_z^n(i,j,k)] \\ & + C_{ez}^2(i,j,k)\left[H_y^{n+\frac{1}{2}}(i,j,k) - H_y^{n+\frac{1}{2}}(i-1,j,k)\right] \\ & + C_{ez}^3(i,j,k)\left[H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n+\frac{1}{2}}(i,j-1,k)\right] \\ & + C_{ez}^4(i,j,k)[J_z^n(i,j,k)] \\ & + C_{ez}^5(i,j,k)[P_z^n(i,j,k)] \\ & + C_{ez}^6(i,j,k)[P_z^{n-1}(i,j,k)]\end{aligned} \tag{D.18}$$

$$\begin{aligned}H_z^{n+\frac{1}{2}}(i,j,k) = {} & H_z^{n-\frac{1}{2}}(i,j,k) + C_{hz}^1(i,j,k)\left[E_y^n(i+1,j,k) - E_y^n(i,j,k)\right] \\ & + C_{hz}^2(i,j,k)[E_x^n(i,j+1,k) - E_x^n(i,j,k)]\end{aligned} \tag{D.19}$$

$$\begin{aligned}J_z^{n+1}(i,j,k) = {} & C_{jz}^1(i,j,k)[J_z^n(i,j,k)] \\ & + C_{jz}^2(i,j,k)[E_z^{n+1}(i,j,k) + E_z^n(i,j,k)]\end{aligned} \tag{D.20}$$

$$\begin{aligned}P_z^{n+1}(i,j,k) = {} & C_{pz}^1(i,j,k)[P_z^n(i,j,k)] \\ & + C_{pz}^2(i,j,k)[P_z^{n-1}(i,j,k)] \\ & + C_{pz}^3(i,j,k)[E_z^{n+1}(i,j,k) + E_z^n(i,j,k)]\end{aligned} \tag{D.21}$$

The constant coefficients are given as follows

$$\Omega_x(i,j,k) = \frac{\varepsilon_0 C_{px}^1(i,j,k)}{\varepsilon_0\varepsilon_\infty} + 1 + \frac{\Delta t C_{jx}^2(i,j,k)}{2\varepsilon_0\varepsilon_\infty} \tag{D.22}$$

$$C_{ex}^1(i,j,k) = \frac{1}{\Omega_x} - \frac{\Delta t C_{jx}^1(i,j,k)}{2\Omega_x\varepsilon_0\varepsilon_\infty} - \frac{\varepsilon_0 C_{px}^1(i,j,k)}{\Omega_x\varepsilon_0\varepsilon_\infty} \tag{D.23}$$

$$C_{ex}^2(i,j,k) = \frac{\Delta t}{\Omega_x\varepsilon_0\varepsilon_\infty\Delta y} \tag{D.24}$$

$$C_{ex}^3(i,j,k) = -\frac{\Delta t}{\Omega_x\varepsilon_0\varepsilon_\infty\Delta z} \tag{D.25}$$

$$C_{ex}^4(i,j,k) = -\frac{\Delta t(1 + C_{jx}^1(i,j,k))}{2\Omega_x\varepsilon_0\varepsilon_\infty} \tag{D.26}$$

$$C_{ex}^5(i,j,k) = -\frac{\varepsilon_0(C_{px}^2(i,j,k) - 1)}{\Omega_x\varepsilon_0\varepsilon_\infty} \tag{D.27}$$

$$C_{ex}^6(i,j,k) = \frac{\varepsilon_0 C_{px}^3(i,j,k)}{\Omega_x\varepsilon_0\varepsilon_\infty} \tag{D.28}$$

$$C_{hx}^1(i,j,k) = -\frac{\Delta t}{\mu\Delta y} \tag{D.29}$$

$$C_{hx}^2(i,j,k) = \frac{\Delta t}{\mu\Delta z} \tag{D.30}$$

$$C_{jx}^1(i,j,k) = \frac{\left(1 - \frac{\Delta t\Gamma_D}{2}\right)}{\left(1 + \frac{\Delta t\Gamma_D}{2}\right)} \tag{D.31}$$

$$C_{jx}^2(i,j,k) = \frac{\frac{\Delta t\omega_{pD}^2\varepsilon_0}{2}}{\left(1 + \frac{\Delta t\Gamma_D}{2}\right)} \tag{D.32}$$

$$C_{px}^1(i,j,k) = \frac{\frac{\Delta t^2\Delta\varepsilon_L\omega_{pL}^2}{2}}{\left(1 + \Delta t\Gamma_L + \frac{\Delta t^2\omega_{pL}^2}{2}\right)} \tag{D.33}$$

$$C_{px}^{2}(i,j,k)=\frac{\left(2+\Delta t\Gamma_L-\frac{\Delta t^2\omega_{pL}^2}{2}\right)}{\left(1+\Delta t\Gamma_L+\frac{\Delta t^2\omega_{pL}^2}{2}\right)} \tag{D.34}$$

$$C_{px}^{3}(i,j,k)=-\frac{1}{\left(1+\Delta t\Gamma_L+\frac{\Delta t^2\omega_{pL}^2}{2}\right)} \tag{D.35}$$

$$\Omega_y(i,j,k)=\frac{\varepsilon_0 C_{py}^{1}(i,j,k)}{\varepsilon_0\varepsilon_\infty}+1+\frac{\Delta t C_{jy}^{2}(i,j,k)}{2\varepsilon_0\varepsilon_\infty} \tag{D.36}$$

$$C_{ey}^{1}(i,j,k)=\frac{1}{\Omega_y}-\frac{\Delta t C_{jy}^{1}(i,j,k)}{2\Omega_y\varepsilon_0\varepsilon_\infty}-\frac{\varepsilon_0 C_{py}^{1}(i,j,k)}{\Omega_y\varepsilon_0\varepsilon_\infty} \tag{D.37}$$

$$C_{ey}^{2}(i,j,k)=-\frac{\Delta t}{\Omega_y\varepsilon_0\varepsilon_\infty\Delta x} \tag{D.38}$$

$$C_{ey}^{3}(i,j,k)=\frac{\Delta t}{\Omega_y\varepsilon_0\varepsilon_\infty\Delta z} \tag{D.39}$$

$$C_{ey}^{4}(i,j,k)=-\frac{\Delta t(1+C_{jy}^{1}(i,j,k))}{2\Omega_y\varepsilon_0\varepsilon_\infty} \tag{D.40}$$

$$C_{ey}^{5}(i,j,k)=-\frac{\varepsilon_0(C_{py}^{2}(i,j,k)-1)}{\Omega_y\varepsilon_0\varepsilon_\infty} \tag{D.41}$$

$$C_{ey}^{6}(i,j,k)=\frac{\varepsilon_0 C_{py}^{3}(i,j,k)}{\Omega_y\varepsilon_0\varepsilon_\infty} \tag{D.42}$$

$$C_{hy}^{1}(i,j,k)=\frac{\Delta t}{\mu\Delta x} \tag{D.43}$$

$$C_{hy}^{2}(i,j,k)=-\frac{\Delta t}{\mu\Delta z} \tag{D.44}$$

$$C_{jy}^{1}(i,j,k)=\frac{\left(1-\frac{\Delta t\Gamma_D}{2}\right)}{\left(1+\frac{\Delta t\Gamma_D}{2}\right)} \tag{D.45}$$

$$C_{jy}^{2}(i,j,k)=\frac{\frac{\Delta t\omega_{pD}^{2}\varepsilon_0}{2}}{\left(1+\frac{\Delta t\Gamma_D}{2}\right)} \tag{D.46}$$

$$C_{py}^{1}(i,j,k)=\frac{\frac{\Delta t^2\Delta\varepsilon_L\omega_{pL}^{2}}{2}}{\left(1+\Delta t\Gamma_L+\frac{\Delta t^2\omega_{pL}^{2}}{2}\right)} \tag{D.47}$$

$$C_{py}^{2}(i,j,k)=\frac{\left(2+\Delta t\Gamma_L-\frac{\Delta t^2\omega_{pL}^{2}}{2}\right)}{\left(1+\Delta t\Gamma_L+\frac{\Delta t^2\omega_{pL}^{2}}{2}\right)} \tag{D.48}$$

$$C_{py}^{3}(i,j,k)=-\frac{1}{\left(1+\Delta t\Gamma_L+\frac{\Delta t^2\omega_{pL}^{2}}{2}\right)} \tag{D.49}$$

$$\Omega_z(i,j,k)=\frac{\varepsilon_0 C_{pz}^{1}(i,j,k)}{\varepsilon_0\varepsilon_\infty}+1+\frac{\Delta t C_{jz}^{2}(i,j,k)}{2\varepsilon_0\varepsilon_\infty} \tag{D.50}$$

$$C_{ez}^{1}(i,j,k)=\frac{1}{\Omega_z}-\frac{\Delta t C_{jz}^{1}(i,j,k)}{2\Omega_z\varepsilon_0\varepsilon_\infty}-\frac{\varepsilon_0 C_{pz}^{1}(i,j,k)}{\Omega_z\varepsilon_0\varepsilon_\infty} \tag{D.51}$$

$$C_{ez}^{2}(i,j,k)=\frac{\Delta t}{\Omega_z\varepsilon_0\varepsilon_\infty\Delta x} \tag{D.52}$$

$$C_{ez}^{3}(i,j,k)=-\frac{\Delta t}{\Omega_y\varepsilon_0\varepsilon_\infty\Delta y} \tag{D.53}$$

$$C_{ez}^{4}(i,j,k)=-\frac{\Delta t(1+C_{jz}^{1}(i,j,k))}{2\Omega_z\varepsilon_0\varepsilon_\infty} \tag{D.54}$$

$$C_{ez}^{5}(i,j,k)=-\frac{\varepsilon_0(C_{pz}^{2}(i,j,k)-1)}{\Omega_z\varepsilon_0\varepsilon_\infty} \tag{D.55}$$

$$C_{ez}^{6}(i,j,k)=\frac{\varepsilon_0 C_{pz}^{3}(i,j,k)}{\Omega_z\varepsilon_0\varepsilon_\infty} \tag{D.56}$$

$$C_{hz}^{1}(i,j,k)=-\frac{\Delta t}{\mu\Delta x} \tag{D.57}$$

$$C_{hz}^{2}(i,j,k) = \frac{\Delta t}{\mu \Delta y} \tag{D.58}$$

$$C_{jz}^{1}(i,j,k) = \frac{\left(1 - \frac{\Delta t \Gamma_D}{2}\right)}{\left(1 + \frac{\Delta t \Gamma_D}{2}\right)} \tag{D.59}$$

$$C_{jz}^{2}(i,j,k) = \frac{\frac{\Delta t \omega_{pD}^{2} \varepsilon_0}{2}}{\left(1 + \frac{\Delta t \Gamma_D}{2}\right)} \tag{D.60}$$

$$C_{pz}^{1}(i,j,k) = \frac{\frac{\Delta t^{2} \Delta \varepsilon_L \omega_{pL}^{2}}{2}}{\left(1 + \Delta t \Gamma_L + \frac{\Delta t^{2} \omega_{pL}^{2}}{2}\right)} \tag{D.61}$$

$$C_{pz}^{2}(i,j,k) = \frac{\left(2 + \Delta t \Gamma_L - \frac{\Delta t^{2} \omega_{pL}^{2}}{2}\right)}{\left(1 + \Delta t \Gamma_L + \frac{\Delta t^{2} \omega_{pL}^{2}}{2}\right)} \tag{D.62}$$

$$C_{pz}^{3}(i,j,k) = -\frac{1}{\left(1 + \Delta t \Gamma_L + \frac{\Delta t^{2} \omega_{pL}^{2}}{2}\right)} \tag{D.63}$$

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