Pion-Pion Scattering in Two Dimensions

Joseph Schechter
Department of Physics, Syracuse University, Syracuse, NY

D. Delphenich
Syracuse University

S. Vaidya
Syracuse University

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D. Delphenich[*], J. Schechter [†] and S. Vaidya [‡]

Department of Physics, Syracuse University,

Syracuse, N. Y. 13244-1130, U. S. A.

Abstract

Massive two-flavor QED_2 is known to have many similarities to the two-flavor QCD_4. Here we compare the $\pi - \pi$ scattering amplitudes (actually an analog process in QED_2) of the two theories. The QED_2 amplitude is computed from the bosonized version of the model while the QCD_4 amplitude is computed from an effective low energy chiral Lagrangian. A number of interesting features are noted. For example, the contribution of the two-dimensional Wess-Zumino-Witten (WZW) term in QED_2 is structurally identical to the vector meson exchange contribution in QCD_4. Also it is shown that the QED_2 amplitude computed at tree level is a reasonable approximation to the known exact strong coupling solution.

11.10.Kk, 11.30.Rd, 13.75.Lb

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[*]delpheni@suhep.syr.edu

[†]schechte@@suhep.syr.edu

[‡]sachin@suhep.phy.syr.edu

I. INTRODUCTION

Two dimensional massive QED (QED_2) [1–8] and four dimensional QCD (QCD_4) both are asymptotically free, confining theories with non-trivial vacuums labeled by an angular parameter θ. This similarity has been exploited for theoretical purposes in a great body of work, nicely summarized in [9,10] for example. The similarity is enhanced if one examines the non-Abelian bosonized [11,12] version [5,13] of massive multi-flavor QED_2. In this version the action is given by Eqs. (2.3) and (2.4) below together with the additional piece

$$-\frac{e^2}{8\pi^2} \int (\theta + i \ln \det U)^2 d^2 x, \tag{1.1}$$

which decouples as the coupling strength e gets large. Now it will not escape the attentive reader that this looks identical to the low energy *effective* action made from pion fields for ordinary QCD_4, if only allowance is made for the two extra dimensions which must appear. Specifically, the non-linear sigma model terms in the form (2.3) look like those in, for instance, [14], the term (2.4) looks like the one in [15] and the term involving θ above looks like the one in [16–21]. Since some exact results are known for QED_2 it is reasonable to expect to learn something new about the effective Lagrangian approach to QCD_4. At least, it is interesting to test the accuracy of the tree approximation in QED_2; that is not so easy to do in QCD_4 where the exact analytic results are not known.

In an earlier paper [13], among other things, the two-point functions of the two theories were compared. This led to a simple understanding, at the tree level, of the stability of the equality $m(\pi^+) = m(\pi^0)$ to addition of a minimal isotopic spin violating term in the strong coupling limit of two flavor massive QED_2. This was seen to be the analog of the fact that in ordinary QCD, the $\pi^+ - \pi^0$ mass difference is essentially unrelated to the up-down quark mass difference. In the present paper we will carry out the tree level comparison of the two theories for their four-point functions, i.e. the $\pi - \pi$ scattering amplitudes. It will be seen that a number of surprising features emerge. These features are related to the not so innocuous fact that the two dimensional analog pion field has *positive G*-parity, unlike the four dimensional case. This has the consequence that a three pion vertex is allowed in two dimensions and appears in the "topological" Wess-Zumino-Witten [11,12] term. The form of the scattering amplitude which results from pion exchange is not similar to anything in the four dimensional scattering amplitude computed from an effective chiral Lagrangian of only pseudoscalars. Rather, it is identical in structure to the *vector* meson exchange graphs in the four dimensional model based on a chiral Lagrangian of pseudoscsalars *plus* vectors. It has been known for many years that the addition of vector mesons to the pion effective Lagrangian substantially improves the tree level predictions. It is intriguing that an exact bosonized analog model leads to just this type of structure.

We will specifically focus attention on massive, isospin invariant two-flavor QED_2 [4] in the strong coupling approximation. The stable light particle spectrum consists of an analog pion (pseudoscalar, isotriplet) and an analog sigma meson (scalar isosinglet). The theory will be treated in the bosonized format. There are two possible ways to bosonize. In the Abelian method [4], the Lagrangian is constructed only from the π^0 field. The π^+ and π^- states appear as "static" solitons while the π^0 appears again as a time-dependent "breather" soliton. The σ appears as a second breather soliton. In the non-Abelian method,

the Lagrangian is constructed from the complete π triplet. The triplet appears again as a static soliton and still[1] once more as the first breather. The σ appears as the second breather.

We shall work here with the tree level non-Abelian bosonized action, which has the advantage of manifest isotopic spin invariance. On the other hand, for the consideration of the solitons of the model, the Abelian bosonized version is much easier to work with since there is less redundancy. Note that the σ state only appears non-perturbatively (as a soliton). We can, as is done in four dimensional effective theories [22], add it in an *ad hoc* way at the tree level, remembering that a more exact treatment of the model would render the tree level term unnecessary. The question of a σ-meson in the four dimensional effective QCD Lagrangian is very timely inasmuch as a number of authors [23–38] have recently provided evidence that such a state ought to exist.

Section 2 contains the set up of the model and the computation of the tree level analog $\pi - \pi$ scattering amplitudes for arbitrary particle charges. These are displayed using the redundant set (more so in two dimensions than in four dimensions) of Mandelstam variables. Comparison is made with four-dimensional $\pi - \pi$ scattering and some interesting features are noted. In Section 3 we mainly focus on the specific case of the analog $\pi^+ - \pi^-$ scattering and eliminate the redundancy of the Mandelstam variable description by specializing to the transmission and reflection amplitudes. The exact solution in the strong coupling limit, based on known results [39,40] for the sine-Gordon theory, is written in Section 4. It is observed that this limit corresponds to pure reflectionless scattering. However, the theory is not trivial because it contains two bound state poles. The tree level amplitude is shown to give an accurate, approximate value for the residue at the analog pion pole. Finally, Section 5 contains a brief summary, discussion of the significance of these results and some directions for future work.

II. SCATTERING IN THE TREE LEVEL NON-ABELIAN BOSONIZED MODEL

Many authors have observed that the ordinary low energy $3 + 1$ dimensional QCD has a number of striking similarities to $1 + 1$ dimensional two-flavor QED [9,10]. Now, in the ordinary QCD case, the scattering of pions is already quite well described near threshold by the tree level treatment of the effective Lagrangian of pions (Of course, further higher order improvements can be implemented in the chiral perturbation scheme [41–44]). It is thus interesting to investigate the tree level scattering of the analog "pions" in the two-dimensional, two-flavor QED where, as we will discuss, an exact answer is available for comparison. This may be a useful step in obtaining a deeper understanding of four-dimensional QCD and its relation to the two-dimensional model. In fact, we shall see an initially unexpected correspondence between the two theories. Naturally there are important differences as well. One question of interest concerns the low mass σ-meson and its two-dimensional analog. There has been a great deal of recent work [23–38] on the possibility of an experimental σ-meson in QCD. At the same time the two-dimensional model is known [4] to contain an analog

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[1]This is the interpretation of [13]. A slightly different interpretation is given in [5].

σ-meson with mass $m(\sigma) \approx \sqrt{3}m(\pi)$; this is however a bound state rather than a scattering resonance.

In this section we shall calculate the tree-level analog pion-pion scattering S-matrix starting from the bosonized two-flavor QED Lagrangian. Since this Lagrangian only contains "pion" fields we shall also consider adding a suitable extra piece to get the σ-meson pole at the tree level.

The relevant non-Abelian bosonization method was developed by Witten [11] and first applied to two-flavor QED_2 by Gepner [5]. Here we shall follow the notation of [13]; the bosonized action is given in (2.10) of this reference. In the strong coupling limit, an η-type particle (pseudoscalar isosinglet) decouples and the effective action is given in terms of a unitary unimodular matrix field $U(x)$ which transforms under left and right chiral transformations U_L, U_R, as $U(x) \to U_L U(x) U_R^\dagger$ and which has the decomposition

$$U(x) = e^{i\sqrt{4\pi}\phi(x)}, \quad \phi(x) = \frac{1}{\sqrt{2}}\vec{\tau}.\vec{\pi}(x). \tag{2.1}$$

Here the τ_i are the Pauli matrices and the $\pi_i(x)$ are the analog pion fields. Actually all the π_i carry zero electric charge in two flavor QED_2 but for convenience we will assign the names

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2), \quad \pi^0 = \pi_3. \tag{2.2}$$

The low energy bosonized action of multi-flavor QED_2 reads

$$\Gamma = \int d^2x[-\frac{1}{8\pi}Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{m^2}{2}Tr(U + U^\dagger - 2)] + \Gamma_{WZW} \tag{2.3}$$

where the third (Wess-Zumino-Witten) term [11,12] may be compactly written using the matrix one-form $\alpha = dUU^\dagger$ as

$$\Gamma_{\text{WZW}} = \frac{1}{12\pi}\int_{M^3} \text{Tr}\,(\alpha^3). \tag{2.4}$$

Here, M^3 is a three-dimensional manifold whose boundary ∂M^3 is the two-dimensional Minkowski space. Note that as written, (2.3) and (2.4) can be used for an arbitrary number N_f of flavors; we are however specializing to the case $N_f = 2$ by restricting the matrix U to be a 2×2 matrix in form. Furthermore we have, unlike [13], restricted U to satisfy $detU = 1$ as is appropriate for low energies where the pseudoscalar isosinglet may be considered infinitely heavy.

Now for a perturbative treatment of $\pi - \pi$ scattering we should expand (2.3) up to fourth order in the number of pion fields. The quadratic terms arise from the first two terms of (2.3) and give the Lagrangian

$$\mathcal{L}^{(2)} = -\frac{1}{2}\partial_\mu\vec{\pi}.\partial_\mu\vec{\pi} - \frac{m_\pi^2}{2}\vec{\pi}.\vec{\pi}, \tag{2.5}$$

wherein we have identified

$$m_\pi = 2\sqrt{\pi}m. \tag{2.6}$$

We have used Eq. (2.1) in obtaining this result. The first two terms of (2.3) also yield quartic terms which may be simplified to

$$\mathcal{L}^{(4)} = \frac{\pi}{3}[(\partial_\mu\vec{\pi}.\partial_\mu\vec{\pi})(\vec{\pi}.\vec{\pi}) - (\vec{\pi}.\partial_\mu\vec{\pi})^2] + \frac{\pi}{12}m_\pi^2(\vec{\pi}.\vec{\pi})^2. \tag{2.7}$$

Finally the Wess-Zumino-Witten term yields a cubic interaction of the pion fields;

$$\mathcal{L}^{(3)} = \frac{i\sqrt{2\pi}}{3}\epsilon_{\mu\nu}\epsilon_{jkl}\pi_j\partial_\mu\pi_k\partial_\nu\pi_l \tag{2.8}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. In obtaining (2.8), we used Stokes' theorem as

$$\int_{M^3} \mathrm{Tr}(d\phi\, d\phi\, d\phi) = \int_{M^3} d\mathrm{Tr}(\phi\, d\phi\, d\phi) = \int_{\partial M^3} \mathrm{Tr}(\phi\, d\phi\, d\phi). \tag{2.9}$$

Judging from usual experience with four dimensional physics, Eq. (2.8) may appear startling. While it may seem to be ruled out because the two-dimensional pions are pseudoscalar, the $\epsilon_{\mu\nu}$ factor rescues parity (This will not work in four dimensions.). In four dimensions, a three-pion vertex is also ruled out because the pion carries a negative G-parity. In two dimensions however a pseudoscalar bilinear $\bar\psi\gamma_5\psi$ picks up a minus sign under charge conjugation so, taking the isotopic spin factor in $G = e^{i\pi I_2}C$ into account, the two-dimensional analog pion carries *positive* G-parity.

The momentum space trilinear pion interaction (essentially $i\mathcal{L}^{(3)}$ from (2.8)) or "Feynman rule" is:

$$2\sqrt{2\pi}\epsilon_{ijk}\epsilon_{\mu\nu}p_\mu^{(j)}p_\nu^{(k)}. \tag{2.10}$$

This corresponds to the diagram in Fig. 1.

We are interested in the analog pion-pion scattering reaction:

$$\pi_i(p_1) + \pi_j(p_2) \rightarrow \pi_k(p_1') + \pi_l(p_2'), \tag{2.11}$$

where i, j, k, l are the isospin indices and the $(p_i)_\mu$ are the two-momenta. The usual Mandelstam variables are

$$s = -(p_1 + p_2)^2, \ t = -(p_1 - p_1')^2, \ u = -(p_1 - p_2')^2, \tag{2.12}$$
$$s + t + u = 4m_\pi^2. \tag{2.13}$$

In the present 1+1 dimensional case, these are somewhat redundant as the only two physical possibilities are (a) forward scattering in the center-of-mass frame:

$$t = 0, \quad u = 4m_\pi^2 - s, \tag{2.14}$$

and (b) backward scattering in the center-of-mass frame:

$$u = 0, \quad t = 4m_\pi^2 - s, \tag{2.15}$$

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With the one-particle state normalization

$$< \pi_j(p')|\pi_i(p) >= \delta_{ij}\delta(p - p'), \tag{2.16}$$

the standard crossing-symmetric parameterization of the scattering matrix element for the reaction (2.11) is

$$
\begin{aligned}
< \pi_k(p_1')\pi_l(p_2')|S|\pi_i(p_1)\pi_j(p_2) >= \\
\delta_{ik}\delta_{jl}\delta(p_1 - p_1')\delta(p_2 - p_2') + \delta_{il}\delta_{jk}\delta(p_1 - p_2')\delta(p_2 - p_1') + \\
\frac{i}{4}\frac{1}{\sqrt{E_1 E_2 E_1' E_2'}}\delta(p_1 + p_2 - p_1' - p_2')\delta(E_1 + E_2 - E_1' - E_2') \times \\
\times \left[\delta_{ij}\delta_{kl}A(s,t,u) + \delta_{ik}\delta_{jl}A(t,s,u) + \delta_{il}\delta_{jk}A(u,t,s)\right]
\end{aligned} \tag{2.17}
$$

This is a convenient form for perturbation theory and for comparison with the four-dimensional case. All the dynamics is contained in the function $A(s,t,u)$.

Our tree-level perturbation calculation yields

$$A(s,t,u) = 2\pi(s - m_\pi^2) + 2\pi\left[\frac{t(s-u)}{m_\pi^2 - t} + \frac{u(s-t)}{m_\pi^2 - u}\right] + \gamma^2\frac{(s - 2m_\pi^2)^2}{m_\sigma^2 - s}. \tag{2.18}$$

The first term arises from the contact interaction (2.7) while the second term is associated with pion exchange diagrams using the vertices from (2.8) or, perhaps more conveniently, from (2.10). The third term does not follow from the tree level treatment of the action (2.3) but was added on somewhat ad hoc grounds for comparison with the four-dimensional case. We expect, as discussed in Section 1, that a σ-particle (and hence a σ pole in the scattering amplitude) should arise as a "breather soliton" from (2.3). We may formally treat the σ as a "matter" particle according to the method of [22]; then the third term in (2.18) comes from the following addition to the interaction Lagrangian:

$$\mathcal{L}_{\sigma\pi\pi} = \frac{\gamma}{4\pi}\sigma \, \text{Tr} \, (\partial_\mu U \partial_\mu U^\dagger), \tag{2.19}$$

where γ is a real coupling constant.

It is interesting to compare the two-dimensional $\pi - \pi$ scattering amplitude (2.18) with a recently considered model [23] which gives a reasonable phenomenological description of ordinary pion scattering (presumably four-dimensional QCD) up to 1.2 GeV. That model, prompted by the $1/N_c$ expansion [45,46], starts out by writing the amplitude as the tree expansion of a chiral Lagrangian including scalar mesons. The model is formally crossing symmetric but, for arbitrary choice of parameters, may very badly violate unitarity bounds. A kind of "regularization" in the vicinity of the physical divergences at the direct channel poles is performed which formally maintains crossing symmetry. Then the arbitrary parameters are adjusted to provide cancellations which preserve the unitarity bounds (and fit the data). In this way an approximate amplitude obeying both crossing symmetry and unitarity is obtained. The unregularized amplitude (see Eqs. (C1), (C2), and (C3) of [23]) for this model is

$$A^{QCD}(s,t,u) = \frac{2(s - m_\pi^2)}{F_\pi^2} + \frac{g_{\rho\pi\pi}^2}{2m_\rho^2}\left[\frac{t(s-u)}{m_\rho^2 - t} + \frac{u(s-t)}{m_\rho^2 - u}\right]$$
$$+ \frac{\gamma_0^2}{2}\frac{(s - 2m_\pi^2)^2}{m_\sigma^2 - s} + \dots \tag{2.20}$$

where $F_\pi \approx 0.131$ GeV is the pion decay constant, m_ρ is the mass of the ρ-meson and $g_{\rho\pi\pi}$ is the $\rho\pi\pi$ coupling constant. The 3 dots stand for another scalar meson pole term which however is not expected to exist in the two-flavor version of the model. The first term of (2.20) is the "current algebra" term which is well-known to be a good approximation very close to the $\pi\pi$ threshold. It is not very surprising that it is identical, up to a numerical factor, to the first term of (2.18). Similarly it is not surprising that the third, σ-meson exchange term in (2.20), is identical up to a factor with the third term in (2.18). What is much more surprising is that the second term in (2.20), which represents the effects of the chiral-symmetric ρ-meson exchange, has the same structure (up to an overall numerical factor and with the replacement $m_\rho \to m_\pi$) as the second (WZW) term in (2.18). From a technical standpoint it may be reasonable in the sense that both the rho and the pion are isovector particles. However it is amusing to see that the analog of the two-dimensional WZW model of "pions" is not the four dimensional WZW model of pions but must also include the terms associated with the introduction of the ρ-meson in a chiral-symmetric manner.

Let us now try to exploit the above correspondence. We notice that the *first terms* of (2.18) and (2.20) satisfy

$$A(s,t,u) = \pi F_\pi^2 A^{QCD}(s,t,u). \tag{2.21}$$

Suppose we assume that the corresponding second terms also obey (2.21) in the $m_\rho = m_\pi$ limit. This then demands that

$$\frac{F_\pi^2 g_{\rho\pi\pi}^2}{2m_\rho^2} = 2, \tag{2.22}$$

which is of the form of the famous KSRF [47,48] relation (which, however, has 1 rather than 2 on the right hand side). From the present perspective, this KSRF-type relation is the analog of the special relationship which exists between the kinetic (first term of (2.3)) and the "topological" (2.4) term of the two-dimensional WZW model. In the two dimensional model, this special relationship between the kinetic and the topological terms is required [11] to obtain the correct equations of motion and currents.

III. FORMULAS FOR PARTICULAR REACTIONS

In the previous section we discussed the formal analogy between the two-flavor QED_2 and QCD_4 scattering amplitudes at tree level. Now let us concentrate on the two-dimensional scattering itself in more detail. We will consider various "charged" meson reactions with appropriate two-dimensional kinematics. Eq.(2.17) contains information about the scattering of pions with all "charges". It is standard (see for example p 178 of [49]) to consider

linear combinations $T^{(I)}(s,t,u)$ corresponding to scattering states of definite isotopic spin $I = 0, 1, 2$:

$$T^{(0)}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s), \qquad (3.1)$$

$$T^{(1)}(s,t,u) = A(t,s,u) - A(u,t,s), \qquad (3.2)$$

$$T^{(2)}(s,t,u) = A(t,s,u) + A(u,t,s). \qquad (3.3)$$

Amplitudes for scattering "pions" with definite "charges" are related to these; for example

$$T^{(+-)} = \frac{1}{3}T^{(0)} + \frac{1}{2}T^{(1)} + \frac{1}{6}T^{(2)} = A(s,t,u) + A(t,s,u), \qquad (3.4)$$

$$T^{(+0)} = \frac{1}{2}T^{(1)} + \frac{1}{2}T^{(2)} = A(t,s,u), \qquad (3.5)$$

$$T^{(++)} = T^{(2)} = A(t,s,u) + A(u,t,s), \qquad (3.6)$$

etc.

It is also desirable to eliminate the large redundancy in the kinematical description of two dimensional scattering when the Mandelstam variables (2.12) are used. We should specialize to the two cases: forward and backward scattering in the center of mass frame as specified in (2.14) and (2.15). This may be conveniently implemented by re-expressing the overall energy-momentum conservation delta function in (2.17) as

$$\delta(p_1 + p_2 - p_1' - p_2')\delta(E_1 + E_2 - E_1' - E_2') =$$
$$\left| \frac{p_1}{E_1} - \frac{p_2}{E_2} \right|^{-1} \left[\delta(p_1 - p_1')\delta(p_2 - p_2') + \delta(p_1 - p_2')\delta(p_2 - p_1') \right]. \qquad (3.7)$$

This can be verified by multiplying both sides by an arbitrary "test function" and integrating. The first term on the right hand side of (3.7) enforces a forward scattering evaluation while the second term yields the backward scattering evaluation. A needed factor in (2.17) is evaluated as

$$\frac{\left| \frac{p_1}{E_1} - \frac{p_2}{E_2} \right|^{-1}}{\sqrt{E_1 E_2 E_1' E_2'}} = \frac{2}{\sqrt{s(s - 4m_\pi^2)}}. \qquad (3.8)$$

We will mainly be interested in the $\pi^+\pi^- \to \pi^+\pi^-$ reaction. According to (3.4) it corresponds to the linear combination $A(s,t,u) + A(t,s,u)$. The S-matrix, after using (3.7), can be written as the sum of a "transmission" piece (proportional to $\delta(p_1 - p_1')\delta(p_2 - p_2')$) and a "reflection" piece proportional to $\delta(p_1 - p_2')\delta(p_2 - p_1')$:

$$S^{(+-)} = \delta(p_1 - p_1')\delta(p_2 - p_2')S_T^{(+-)} + \delta(p_1 - p_2')\delta(p_2 - p_1')S_R^{(+-)}. \qquad (3.9)$$

With the help of (2.14), (2.15) and (3.8) we find from (2.17) and (2.18):

$$S_T^{(+-)} = 1 + \frac{i}{2\sqrt{s(s - 4m_\pi^2)}} \left[2\pi(s - 2m_\pi^2) + 2\pi \frac{s(s - 4m_\pi^2)}{m_\pi^2 - s} \right.$$

$$+ \gamma^2 \left(\frac{(s - 2m_\pi^2)^2}{m_\sigma^2 - s} + \frac{4m_\pi^4}{m_\sigma^2} \right) \Bigg], \tag{3.10}$$

$$S_R^{(+-)} = \frac{i}{2\sqrt{s(s - 4m_\pi^2)}} \left[4\pi m_\pi^2 + 2\pi s(4m_\pi^2 - s) \left(\frac{1}{m_\pi^2 - s} - \frac{1}{3m_\pi^2 - s} \right) \right.$$
$$\left. + \gamma^2 (s - 2m_\pi^2)^2 \left(\frac{1}{m_\sigma^2 - s} - \frac{1}{(4m_\pi^2 - m_\sigma^2) - s} \right) \right]. \tag{3.11}$$

The 1 on the left hand side of (3.10) but not (3.11) is due to resolving the unit operator in (2.17) analogously to the resolution of the amplitude in (3.4). As in ordinary one-dimensional quantum mechanics, $S_T^{(+-)} \to 1$ and $S_R^{(+-)} \to 0$ when the interaction vanishes. Another check is provided by considering the S-matrix for $\pi^+ - \pi^+$ scattering. In this case Bose statistics prohibits any distinction between forward and backward scattering. The appropriate linear combination (see (3.6)) for $\pi^+ \pi^+ \to \pi^+ \pi^+$ is $A(t, s, u) + A(u, t, s)$ and it is easy to see that both (2.14) and (2.15) give the same results for this quantity. Then (2.17) becomes

$$S^{(++)} = [\delta(p_1 - p_1')\delta(p_2 - p_2') + \delta(p_1 - p_2')\delta(p_2 - p_1')] \times$$
$$\times \left[1 + \frac{i}{2\sqrt{s(s - 4m_\pi^2)}} \left(2\pi(2m_\pi^2 - s) + 2\pi\frac{s(s - 4m_\pi^2)}{s - 3m_\pi^2} \right) \right.$$
$$\left. + \gamma^2 \left(\frac{(s - 2m_\pi^2)^2}{m_\sigma^2 - 4m_\pi^2 + s} + \frac{4m_\pi^4}{m_\sigma^2} \right) \right], \tag{3.12}$$

which does not distinguish between forward and backward scattering.

Let us focus on the transmission amplitude $S_T^{(+-)}$ for definiteness. Note again that the term proportional to γ^2 does not follow from the perturbative treatment of the original action (2.3). It has been introduced to mimic the σ meson exchange (in analogy to the treatment of 2 flavor QCD_4) since it is known that a σ meson with mass satisfying $m_\sigma^2 = 3m_\pi^2$ should exist in the strong coupling limit. Without the γ^2 term, the model just differs from the WZW action [11] by the mass term. If the mass term is also dropped we would have just the WZW action which describes the free theory. As another check, we observe that $S_T^{(+-)}$ does in fact go to 1 when γ and m_π^2 are set to zero. When $\gamma = 0$, the large s behaviors are

$$S_T^{(+-)} \to 1 + \frac{i\pi m_\pi^2}{s}, \quad S_R^{(+-)} \to \frac{i6\pi m_\pi^6}{s^3}. \tag{3.13}$$

$S_T^{(+-)}$ has poles at $s = m_\pi^2$ and $s = m_\sigma^2 = 3m_\pi^2$. Both are below the threshold at $s = 4m_\pi^2$ and therefore to be interpreted as bound states. For comparison with the work in the next section we give the residues:

$$\mathrm{Res}[S_T^{(+-)}, s = m_\pi^2] = \mp\sqrt{3}\pi m_\pi^2, \tag{3.14}$$

$$\mathrm{Res}[S_T^{(+-)}, s = 3m_\pi^2] = \mp\frac{\gamma^2 m_\pi^2}{2\sqrt{3}}, \tag{3.15}$$

where the $\mp$ corresponds to the different sign choices for the square root in (3.10).

IV. CONNECTION WITH EXACT RESULTS

Coleman [4] has argued that two flavor massive QED_2 with isotopic spin invariance reduces simply to the sine-Gordon theory in the strong coupling limit, when attention is focussed on the light particles of the theory. This follows from the treatment of the model by the Abelian bosonization technique. That approach requires two pseudoscalar fields χ_1 and χ_2 which enter into the Lagrangian as

$$\mathcal{L}_{\text{Abelian}} = -\frac{1}{2}\sum_{i=1,2}\partial_\mu\chi_i\partial_\mu\chi_i - \frac{e^2}{2\pi}(\sum_i\chi_i - \frac{\theta}{2\pi})^2 - m^2\sum_i[1-\cos(2\sqrt{\pi}\chi_i)] \qquad (4.1)$$

where θ, (which will be set to zero henceforth), represents the background electric field of the underlying theory. The Lagrangian (4.1) has the same form as the one for the classical soliton ansatz of the non-Abelian bosonized model (see Eq.(4.2) of [13]). It is convenient to define π^0 and η as

$$\pi^0 = \frac{\chi_1 - \chi_2}{\sqrt{2}}, \quad \eta = \frac{\chi_1 + \chi_2}{\sqrt{2}}. \qquad (4.2)$$

Notice that both the electric charge e and the mass parameter m have the same units. When $e \gg m$ (i.e. strong coupling) the η field becomes very heavy and decouples. We are then left with a special case of the sine-Gordon model:

$$\mathcal{L}_{\text{Abelian}} \rightarrow -\frac{1}{2}(\partial_\mu\pi^0)^2 + 2m^2\cos(\sqrt{2\pi}\pi^0). \qquad (4.3)$$

This enables us to read off a tree level π^0 mass of $2\sqrt{\pi}m \approx 3.54m$. The π^+ and π^- particles are hidden from sight in (4.3) but appear [4] as solitons and anti-solitons with mass

$$M = \frac{8m}{\sqrt{\pi}} \approx 4.51m. \qquad (4.4)$$

At first glance it appears that the $\pi^\pm$ masses differ from the π^0 mass. However the theory also contains two "breather" solitons with masses given by [50]:

$$M_n = 2M\sin\left(\frac{n\pi}{6}\right), \quad n = 1, 2. \qquad (4.5)$$

This formula is argued to be exact when M includes radiative corrections to the classical soliton mass. The $n = 1$ breather has mass $M_1 = M$ and is identified by Coleman as the π^0, thereby restoring the isotopic spin invariance. The tree level pion mass is considered to be just a rough (to about 20% accuracy) approximation. Finally the $n = 2$ breather is identified as the isosinglet sigma with mass $m_\sigma = \sqrt{3}m_\pi$.

Finding the exact solution for the scattering matrix in the sine-Gordon model is a by now classic problem which has been solved and elucidated by several authors [39]. It is carried out for a more general sine-Gordon model than (4.3); there is an extra parameter γ which shows up by (4.5) now reading $M_n = 2M\sin(n\gamma/16)$. Clearly we are dealing with the special case

$$\gamma = \frac{8\pi}{3}. \tag{4.6}$$

The construction [39] is based on four principles:
(i) Crossing symmetry.
(ii) Unitarity of the S-matrix.
(iii) Trilinear relation derived from the extra (infinite number of) conservation laws associated with the sine-Gordon theory.
(iv) Absence of the "CDD pole" ambiguity.

It is convenient to employ the rapidity variable θ_i for each pion so that the momentum and energy become

$$p_i = m_\pi \sinh \theta_i, \quad E_i = m_\pi \cosh \theta_i. \tag{4.7}$$

The relevant variable is

$$\theta \equiv \theta_1 - \theta_2 \tag{4.8}$$

in terms of which the Mandelstam variable reads

$$s = 4m_\pi^2 \cosh^2(\theta/2). \tag{4.9}$$

We see from (3.10) that, for example, $S_T^{(+-)}$ depends only on s, or equivalently on θ.

The exact solution[2] for the soliton-anti-soliton transmission amplitude in the sine-Gordon model is:

$$S_T^*(\theta) = \prod_{l=0}^{\infty} \frac{\Gamma(\frac{l\gamma}{16\pi} - \frac{i\theta}{2\pi})\Gamma(\frac{(l-1)\gamma}{16\pi} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{l\gamma}{16\pi} - \frac{i\theta}{2\pi})\Gamma(-\frac{1}{2} + \frac{(l-1)\gamma}{16\pi} - \frac{i\theta}{2\pi})} \times$$
$$\times \frac{\Gamma(\frac{3}{2} + \frac{l\gamma}{16\pi} + \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} + \frac{(l-1)\gamma}{16\pi} + \frac{i\theta}{2\pi})}{\Gamma(1 + \frac{l\gamma}{16\pi} + \frac{i\theta}{2\pi})\Gamma(1 + \frac{(l-1)\gamma}{16\pi} + \frac{i\theta}{2\pi})}. \tag{4.10}$$

This rather complicated formula simplifies for the special case, as in (4.6) when $\gamma = 8\pi/n$, where n is an integer. Using $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we then get

$$S_T(\theta) = e^{in\pi} \prod_{k=1}^{n-1} \frac{e^{\theta - i(\pi k/n)} + 1}{e^\theta + e^{-i(\pi k/n)}} \tag{4.11}$$

In the physical scattering region $\theta > 0$ this is just a pure phase factor so there is no attenuation of the incoming wave. Furthermore, it is easy to see that we also directly have $S_R(\theta) = 0$, verifying that there is no reflected wave. This special case was first discussed in [51].

Our model requires us to set $n = 3$ in (4.11). It is amusing to note that the strong coupling limit of massive two-flavor QED_2 is not merely an integrable model but one which

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[2]We have complex conjugated (4.11) of [39] in order that it reduce to their (4.12).

corresponds to reflectionless scattering. Even though (4.11) is a pure phase in the physical region, its general analytic structure is of interest. The $n = 3$ case has two poles in the unphysical region where θ is pure imaginary. These are at:

$$\theta = \frac{\pi i}{3}, (s = 3m_\pi^2) \text{where} S_T(\theta) = \frac{2\sqrt{3}i}{\theta - i\pi/3} + \dots,$$

$$\theta = \frac{2\pi i}{3}, (s = m_\pi^2) \text{where} S_T(\theta) = \frac{-2\sqrt{3}i}{\theta - i2\pi/3} + \dots \tag{4.12}$$

and correspond respectively to the σ and π bound states. In fact the prediction of the pole position of the exact scattering result is used [4] to argue for the exactness of the DHN formula (4.5). To transform (4.12) to the s-plane it is sufficient to note that, near the poles, we may replace

$$\theta - \theta_0 = [s(\theta) - s(\theta_0)]/\left[\frac{ds}{d\theta}\right]_{\theta = \theta_0} \tag{4.13}$$

with $ds/d\theta = 2m_\pi^2\sinh\theta$. Then the residues at the bound state poles in the s-plane are

$$\text{Res}[S_T, s = m_\pi^2] = 6m_\pi^2, \tag{4.14}$$
$$\text{Res}[S_T, s = 3m_\pi^2] = -6m_\pi^2, \tag{4.15}$$

Now let us compare these exact results with the tree level results we obtained in (3.14, 3.15). The residue at the pion pole $\mp\sqrt{3}\pi m_\pi^2 \approx \mp 5.44 m_\pi^2$ agrees to within ten percent if we adopt the lower sign. This is encouraging since it again indicates that the tree level results may be close to the exact ones. Of course, we cannot compare the magnitude[3] of the residue at the sigma pole since it was introduced in an *ad hoc* way and involves the undetermined factor γ^2.

Finally, we expect that, when one goes to higher orders in perturbation theory, S_T given in (3.10) will exponentiate and S_R given in (3.11) will get cancelled. A possible hint of this may be perceived in the large s behavior shown in (3.13)- S_R is seen to fall off very much faster with increasing s than does S_T.

V. DISCUSSION

We calculated the tree level analog $\pi - \pi$ scattering amplitude in the strong coupling limit of massive two flavor QED_2. A characteristic new feature, compared to the four dimensional

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[3]However, with the same choice for the signs in (3.14, 3.15), it appears that the signs of the two residues in (4.14, 4.15) should be the same. Usually a "wrong sign" residue is associated with a "ghost" particle. However it is also possible, at least in the scattering regime, for rescattering effects to change the effective sign. An example is provided in the case of the $f_0(980)$ particle in $\pi\pi$ scattering, in Section IV A of [23].

case, is the presence of a three point pion vertex. This comes from the WZW term and is allowed because the two dimensional pion has positive G-parity.

The resulting pion exchange contribution has the identical dependence on kinematical variables (appropriately restricted) as the vector meson exchange contribution in the theory based on a four dimensional effective low energy Lagrangian for QCD. Since the analog QED_2 theory represents an *exact* bosonization it seems that there is a sense in which the "minimal" QCD effective Lagrangian should include both pseudoscalars and vectors. Of course, this does not exactly agree with the organization of the chiral perturbation theory expansion [41–44]. However in that approach, many of the leading order "counterterms" are dominated [52,53] by vector meson exchange. For a tree level treatment, as suggested by the $1/N_c$ expansion, the vectors are very important phenomenologically [23]. In addition, when calculating the properties of nucleons-as-solitons derived from the low energy Lagrangian, the presence of vector mesons is crucial for a satisfactory understanding of the "short-distance" effects like neutron-proton mass splitting [54], "proton spin current" [55], and heavy baryon hyperfine splitting [56–58]. In any event, it seems worthwhile to further contemplate the connection between the QCD effective Lagrangian and its dimensionally reduced version.

In Section 4 we compared the tree level $\pi - \pi$ scattering in QED_2 with the known exact result in the sine-Gordon theory. It was pointed out (though it is an elementary observation from existing results) that $\pi^+\pi^-$ scattering in strong coupling QED_2 is not merely given by a known analytic formula but is actually reflectionless. This, of course, can only be approximated at tree level. However the model has two bound states so its analytic structure is of great interest. The locations of the pion and sigma poles satisfying $m_\sigma^2 = 3m_\pi^2$ have been well documented in the literature. Here we pointed out that the residue at the pion pole is quite well described (to about 10% accuracy) by the tree level calculation. Certainly it would be desirable in the future to extend the perturbative tree level calculation to higher orders.

The triviality of the scattering in the two dimensional theory is clearly different from the four dimensional QCD case. Another difference concerns the question of spontaneous breakdown of chiral symmetry, which is well-known to be a characteristic feature of the QCD_4 effective low energy Lagrangian (when the quark mass terms are neglected). On the other hand, the spontaneous breakdown of chiral symmetry is ruled out in the two dimensional case, according to the Mermin-Wagner-Coleman theorem [59,60]. One may wonder how this feature gets displayed at the effective Lagrangian level, since it is not manifestly evident from the bosonized action (2.3). A heuristic way of understanding this was discussed in [13,61] using an old-fashioned linear sigma model [62] containing both π and σ fields. This model is not an exact bosonization and does not faithfully mirror all the desired properties of the two dimensional theory. Nevertheless, it contains a manifest potential function which lets one conclude that the predicted ratio $R = m_\sigma^2/m_\pi^2 = 3$ corresponds to a theory which will *not* be spontaneously broken when the parameter m in (2.3) is set to zero. A study of the topography of the potential using the methods of "catastrophe theory" suggests [61] that a generalized spontaneous breakdown regime is related to the range $R > 9$ which is expected to hold in QCD. This type of analysis also seems like a promising direction for future work.
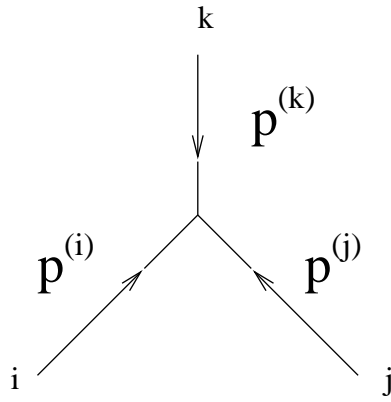
FIG. 1. Feynman diagram for the cubic pion interaction. For example $p^{(i)}$ is the 2-momentum of the pion with isotopic spin index i. Also $p^{(i)} + p^{(j)} + p^{(k)} = 0$.

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