June 2015

Poletsky-Stessin Hardy Spaces on the Unit Disk

Khim Raj Shrestha
Syracuse University

ABSTRACT

The holomorphic functions on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ have a remarkable property: to know the values of a holomorphic function on $\mathbb{D}$ it suffices to know only its values on the unit circle $\mathbb{T}$. However not all holomorphic functions on $\mathbb{D}$ are defined on $\mathbb{T}$ and the major problem of establishing such values (called boundary values) led to the appearance of Hardy spaces $H^p(\mathbb{D})$, $p \geq 1$. If a function lies in a Hardy space then its boundary values can be defined and its values on $\mathbb{D}$ can be obtained using standard Cauchy or Poisson formulas.

The theory of Hardy spaces $H^p(\mathbb{D})$ was well developed in the last century and the spaces became the fundamental ground for complex analysis. To create analogous spaces in higher dimensions Poletsky and Stessin introduced new spaces on hyperconvex domains in $\mathbb{C}^n$ in [20]. We call these spaces the Poletsky–Stessin Hardy spaces. Poletsky and Stessin used them to study composition operators but did not look at their detailed properties.

In this thesis we fill this gap studying Poletsky–Stessin Hardy spaces on the unit disk $\mathbb{D}$. As in [20] for their definition we use subharmonic exhaustion functions u and denote these spaces by $H^p_u(\mathbb{D})$. It was mentioned in [20] that the classical Hardy spaces form a subclass of Poletsky–Stessin Hardy spaces. Our work begins with producing an example that shows that there are subharmonic exhaustion functions u on $\mathbb{D}$ for which the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{D})$ are different from classical Hardy spaces $H^p(\mathbb{D})$. Thus we have an abundance of new function spaces to be explored.

We show that the theory of boundary values for functions in Poletsky–Stessin Hardy spaces is analogous to the classical theory of Hardy spaces and the most of the classical properties stay true for these new spaces. Since by [20] the space $H^p_u(\mathbb{D})$ lies in $H^p(\mathbb{D})$ we can use the classical boundary values for functions in $H^p_u(\mathbb{D})$. This allows us to redefine Poletsky–Stessin Hardy spaces as spaces whose boundary values belong to weighted L^p spaces on $\mathbb{T}$ and we completely characterize the weights that produce Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{D})$.

Many problems in complex analysis ask for existence of a bounded function in some class. Usually it is easier to find a function in $H^p_u(\mathbb{D})$ but they are not necessarily bounded. As an application of Poletsky–Stessin Hardy spaces we provide a reduction of such problems to the existence of a function in $H^p_u(\mathbb{D})$ and use it to give shortcuts in the proofs of the famous interpolation theorem and corona problem.

At the end of the thesis we also study the boundary behavior of functions in the Hardy spaces on the polydisk and discuss the intersection of Poletsky–Stessin spaces on bidisk.

Poletsky–Stessin Hardy Spaces on the Unit Disk

By

K. R. Shrestha

B.A., Tribhuvan University, Nepal

M.A., Tribhuvan University, Nepal

Postgraduate Diploma, ICTP, Italy

Master of Science, Syracuse University, USA

DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

Syracuse University

June 2015

Acknowledgements

I would like to express my deepest gratitude to my advisor Prof. Eugene Poletsky for his excellent guidance, immense support and encouragement. Without him this work would not have been accomplished. His contribution has helped shape my career in the field of mathematics and placed me in the position I am today.

I am sincerely thankful to the faculties of Mathematics for imparting the knowledge that significantly widened the breadth of mathematics in me. I am thankful to all the staffs for generously providing me with the facilities and support to do my study, conduct the research and carry out my other responsibilities. Emeritus Prof. Daniel Waterman had recommended me this place and showed me care throughout for which I will remain thankful forever. I am thankful to the fellow graduate students for their company. Furthermore, I am extremely grateful to the members of my defense committee for helping me getting through the final stage.

Last but not least, I would like to thank my wife Spardha Karmacharya for constantly supporting me in everyway possible. I wish to thank my family and friends for their direct and indirect support.

To

Suyog & Khushi

Contents

Acknowledgements **v**

1 Introduction **1**

2 Preliminaries **5**

2.1 Definitions 5

2.2 Hardy Space of Harmonic Functions 6

2.3 Hardy Space of Holomorphic Functions 8

2.4 Hardy Spaces on Hyperconvex Domains 10

3 Poletsky–Stessin Hardy Spaces on the Unit Disk **14**

3.1 Example 15

3.2 The Hardy spaces of harmonic functions and the measure μ_u 17

3.3 Boundary values of harmonic functions with respect to the measures $\mu_{u,r}$ 21

3.4 Boundary values of holomorphic functions with respect to the measures $\mu_{u,r}$ 26

3.5 Properties of $H_u^p(\mathbb{D})$ and Dual Spaces 31

3.6 Characterization of $H_u^p(\mathbb{D})$ 37

4 Applications 41

4.1 Duality 41

4.2 From H_u^p to H^∞ 43

4.3 Interpolation Theorem 47

4.4 Corona Theorem 49

5 Hardy Spaces on the Polydisk 52

5.1 Hardy Spaces and Poisson Integral Formula 52

5.2 The F. and M. Riesz Theorem 57

5.3 Boundary Values 59

5.4 Poletsky–Stessin Hardy Spaces on the Bidisk 62

Chapter 1

Introduction

The study of Hardy spaces was initiated by G. H. Hardy in [12] in 1914. About a decade later in 1923, F. Riesz introduced these spaces in [23] and named them after him. Initially the Hardy spaces were defined on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Later Hardy space theory was studied on more general domains. For instance, on the polydisc in [25], on the unit ball of $\mathbb{C}^n$ in [26], on simply connected domains, on Jordan domains with rectifiable boundary, on Smirnov domains and multiply connected domains in $\mathbb{C}$ in [6], on pseudoconvex domains with C^2 boundaries in [31]. In 2008, Poletsky and Stessin introduced in [20] the weighted Hardy spaces $H^p_u(\Omega)$ on hyperconvex domains $\Omega \subset \mathbb{C}^n$. For their definition they used a plurisubharmonic exhaustion function u on Ω and the Monge–Ampère measures $\mu_{u,r}$ constructed by Demailly in [3]. This appears to be the most general definition of Hardy spaces as it subsumes the classical theory of Hardy spaces. Recently M. Alan and N. Gogus in [1], S. Sahin in [29], [30], K.

R. Shrestha in [27], [28] and with E. A. Poletsky in [19] have done some extensive studies of these spaces independently. They refer to these spaces as Poletsky–Stessin Hardy spaces.

Let λ be the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Let $\alpha \in L^1(\mathbb{T})$ be a non-negative function such that $\log \alpha \in L^1(\mathbb{T})$. Among many different definitions of weighted Hardy spaces the closest to our purpose is the definition in [2] and [15], which is defined as $H^p_\alpha = N^+ \cap L^p_\alpha(\mathbb{T})$, where $L^p_\alpha(\mathbb{T})$ is the space of all functions with the finite norm

$$\|\phi\|_{\alpha,p} = \left(\int_0^{2\pi} |\phi(e^{i\theta})|^p \alpha(e^{i\theta})\, d\lambda \right)^{1/p}$$

for $0 < p < \infty$ and N^+ is the Smirnov class. If $\alpha \equiv 1$ then we will use notations H^p and $\|\cdot\|_{H^p}$. Our study shows that if Ω is the unit disk $\mathbb{D}$, the Poletsky–Stessin Hardy spaces form a subclass of weighted Hardy spaces as defined in [2] and [15].

In this work we primarily focus our study on the Poletsky–Stessin Hardy spaces on the unit disk. We will take an exhaustion function u whose total Monge–Ampère mass $\int_{\mathbb{D}} \Delta u$ is finite and the Laplacian Δu is not necessarily compactly supported. It was proved in [20] that the space $H^p_u(\mathbb{D})$, $p \geq 1$, is Banach and for all exhaustion functions u the space $H^p_u(\mathbb{D})$ is contained in the classical Hardy space $H^p(\mathbb{D})$ and for $u = \log|z|$, $H^p_u(\mathbb{D}) = H^p(\mathbb{D})$. We show by an example in Section 3.1 that, in general, $H^p_u(\mathbb{D}) \neq H^p(\mathbb{D})$. Thus we have an abundance of Poletsky–Stessin spaces to explore inside the classical Hardy spaces.

Most of our work is devoted to establishing the results for the Poletsky–Stessin

Hardy spaces analogous to those for the classical space $H^p(\mathbb{D})$. One thing that we want to understand in detail is the boundary behavior of the functions in $H^p_u(\mathbb{D})$. Since $H^p_u(\mathbb{D}) \subset H^p(\mathbb{D})$ any function $f \in H^p_u(\mathbb{D})$ has radial boundary values f^*. In classical theory, if $f \in H^p(\mathbb{D})$ then $f^* \in L^p(\lambda)$, where λ is the normalized Lebesgue measure, and the H^p-norm of f coincides with L^p-norm of f^*. The analogue of this statement holds for the functions in $H^p_u(\mathbb{D})$. For $f \in H^p_u(\mathbb{D})$ the boundary value function $f^* \in L^p_u := L^p(\mu_u)$, where μ_u is the weak-star limit of the measures $\mu_{u,r}$ used in the construction of the Poletsky–Stessin spaces, and the H^p_u-norm of f is equal to the L^p_u-norm of f^* (Theorem 3.14). The space $H^p_u(\mathbb{D})$ is isometrically isomorphic to $H^p(\mathbb{D})$ (Theorem 3.17) and, therefore, the duality of Poletsky–Stessin spaces is analogous to that of classical spaces (Theorem 3.19).

We can define the equivalence class $\mathcal{E}_u$ of exhaustion functions generating the same space $H^p_u(\mathbb{D})$ with equivalent norms. Then the class $\mathcal{E}_0$ of $u = \log|z|$ generates the space $H^p(\mathbb{D})$ with equivalent norms. However, the norms generated by the exhaustion functions in a class vary so much so that the intersection of all unit balls in these norms is the unit ball in $H^\infty(\mathbb{D})$ (Theorem 3.16).

In Section 3.6 we give a complete characterization of Poletsky–Stessin Hardy spaces as a subclass of weighted Hardy spaces as defined in [2] and [15]. For every Poletsky–Stessin space $H^p_u(\mathbb{D})$ there is a weight function $\alpha_u \in L^1(\mathbb{T})$ (see Proposition 3.4 for the definition of α_u) so that $H^p_u(\mathbb{D}) = H^p_{\alpha_u}$ (Section 3.2 and Section 3.4). Conversely, for every weighted Hardy space H^p_α where the weight function α is lower semicontinuous and $\alpha \geq c > 0$ for some constant c, there is an exhaustion function u such that $H^p_\alpha = H^p_u(\mathbb{D})$ (Theorem 3.20).

Although the weighted Hardy spaces can be studied per se there is also an expectation that they can be useful for the classical theory. If a closed convex set $A \subset H^p(\mathbb{D})$ intersects unit balls in all $H^p_u(\mathbb{D})$ for some $p > 1$ then it intersects the unit ball in $H^\infty(\mathbb{D})$ (Theorem 4.2). Thus to find bounded solutions to a linear problem it suffices to show that they exist at all $H^p_u(\mathbb{D})$ and their norms are uniformly bounded. This fact has been used to demonstrate shortcuts to the proofs of the interpolation theorem (Section 4.3) and corona problem (Section 4.4).

In Chapter 5, we study the boundary behavior of the functions in Hardy spaces on the polydisk. We prove F. and M. Riesz theorem and discuss the intersection of Poletsky–Stessin Hardy spaces on the polydisk.

Chapter 2

Preliminaries

2.1 Definitions

Definition 2.1. Let $\Omega \subset \mathbb{C}$ be an open subset. A function $u : \Omega \to \mathbb{R}$ is called *harmonic* if $h \in C^2(\Omega)$ and $\Delta u = 0$ on Ω, where Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Definition 2.2. Let $\Omega \subset \mathbb{C}$ be an open set. A function $u : \Omega \to [-\infty, \infty)$ is said to be subharmonic if

(i) u is upper semicontinuous.

(ii) if $B(z_0, r) \subset\subset \Omega$, h is harmonic on a neighborhood of $\overline{B}(z_0, r)$ and $u \leq h$ on ∂B, then $u \leq h$ on $B(z_0, r)$.

Definition 2.3. Let $\Omega \subset \mathbb{C}^n$ be a domain. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is called plurisubharmonic if u is subharmonic on each complex line, that is, $u(a\zeta + b)$ is subharmonic as a function of $\zeta \in \{\zeta \in \mathbb{C} : a\zeta + b \in \Omega\}$ for each $a, b \in \mathbb{C}^n$.

We will use the shorthand notation psh for the plurisubharmonic function.

2.2 Hardy Space of Harmonic Functions

Definition 2.4. The *Hardy space* $h^p(\mathbb{D}), 0 < p < \infty$, consists of the harmonic functions $u : \mathbb{D} \to \mathbb{R}$ satisfying

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta < \infty$$

and the space $h^\infty(\mathbb{D})$ consists of the harmonic functions u such that

$$\sup_{z\in\mathbb{D}} |u(z)| < \infty.$$

Here $\mathbb{D}$ is the unit disk.

For $p \geq 1$,

$$\|u\|_{h^p} = \left(\sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

$$\|u\|_\infty = \sup_{z\in\mathbb{D}} |u(z)|$$

is a norm and $h^p(\mathbb{D})$ is Banach. Also it is clear that, if $0 < p < q < \infty$ then

$$h^\infty(\mathbb{D}) \subset h^q(\mathbb{D}) \subset h^p(\mathbb{D}).$$

Theorem 2.1. *If $u \in h^p(\mathbb{D}), 1 < p < \infty$, then there exists an $f \in L^p[0, 2\pi]$ with*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) f(t)\, dt$$

where

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, z = re^{i\theta}$$

is the Poisson kernel. Same holds if $p = \infty$. If $p = 1$ there exists a finite signed measure on $[0, 2\pi]$ such that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it})\, d\mu(t).$$

We have the following converse to Theorem 2.1.

Theorem 2.2. *Let $f \in L^p[0, 2\pi], p \geq 1$. Then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) f(t)\, dt \in h^p(\mathbb{D})$$

and if μ is a finite signed measure on $[0, 2\pi]$ then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it})\, d\mu(t) \in h^1(\mathbb{D}).$$

The following is the Fatou's theorem.

Theorem 2.3. *Let $f \in L^p[0, 2\pi]$, $p \geq 1$ and let*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(t)\, dt.$$

Then $u(re^{i\theta}) \to f(\phi)$ a.e. in ϕ as $re^{i\theta} \to e^{i\phi}$.

Hence it is clear that if $u \in h^p$, $p > 1$, then $u(re^{i\theta}) \to f(\theta)$ as $r \to 1$ almost everywhere for some $f \in L^p[0, 2\pi]$ and

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) f(t)\, dt.$$

2.3 Hardy Space of Holomorphic Functions

Definition 2.5. The Hardy space of holomorphic functions $H^p(\mathbb{D}), p > 0$, consists of holomorphic functions f that satisfy

$$\|f\|_{H^p} = \left(\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \text{ when } 0 < p < \infty$$

and

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty, \text{ when } p = \infty.$$

As before, $\|.\|_{H^p}$ is a norm for $p \ge 1$ and $H^p(\mathbb{D})$ endowed with this norm is Banach. Also it is clear that, if $0 < p < q < \infty$ then

$$H^\infty(\mathbb{D}) \subset H^q(\mathbb{D}) \subset H^p(\mathbb{D}).[11, Ch\, IX, Sec.4]$$

If $f \in H^p(\mathbb{D})$, $p > 0$ then

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(e^{i\theta})$$

exists almost everywhere and

$$f^*(e^{i\theta}) \in L^p[0, 2\pi].$$

For $p \geq 1$ we have

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(re^{i\theta}, e^{it}) f^*(e^{it})\, dt. \quad [14, Ch.II, Sec.B] \tag{2.1}$$

Theorem 2.4. *[6, Theorem 2.6]If $f \in H^p$, $p > 0$, then*

$$\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta \tag{2.2}$$

$$\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p\, d\theta = 0. \tag{2.3}$$

We have the following duality for H^p spaces [14, Ch. VII].

Theorem 2.5. *If $1 < p < \infty$ and $1/p + 1/q = 1$, then dual of H^p is $L^q/H^q(0)$ and the dual of L^q/H^q is $H^p(0)$ where $H^p(0) = zH^p$. Similarly, dual of H^1 is $L^\infty/H^\infty(0)$ and dual of L^1/H^1 is $H^\infty(0)$.*

We can consider the space $H^p(\mathbb{D})$ as a $\|\cdot\|_{L^p}$-closed subspace of $L^p(\mathbb{T})$. Let $F \in L^p(\mathbb{T})$, $1 < p < \infty$. The distance from F to $H^p(\mathbb{D})$ is given by

$$\|F - H^p\|_{L^p} = \inf\{\|F - h\|_{L^p}; h \in H^p\}.$$

The duality result in Theorem 2.5 provides some theorems about approximation by H^p functions [14, p. 143].

Theorem 2.6. *Let $F \in L^p(\mathbb{T})$, $1 < p < \infty$, and $1/p + 1/q = 1$. Then*

$$\|F - H^p\|_{L^p} = \sup\left\{\left|\int_0^{2\pi} F(e^{i\theta})g(e^{i\theta})\, d\theta\right|; g \in H^q(0) \text{ and } \|g\|_{L^q} = 1\right\}.$$

This supremum is attained, that is, there is a $g_0 \in H^q(0)$ with $\|g_0\|_{L^q} = 1$ such that

$$\|F - H^p\|_{L^p} = \int_0^{2\pi} F(e^{i\theta})g_0(e^{i\theta})\, d\theta.$$

2.4 Hardy Spaces on Hyperconvex Domains

Let $\Omega \subset \mathbb{C}^n$ be a domain. If there is a continuous negative plurisubharmonic function u on Ω such that $u(z) \to 0$ as z goes to $\partial\Omega$ then Ω is called hyperconvex. Such a function u is called the exhaustion function.

For $r < 0$ define,

$$B_{u,r} = \{z \in \Omega : u(z) < r\}$$

$$S_{u,r} = \{z \in \Omega : u(z) = r\}.$$

The operators d and d^c are given by $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$. For $\varphi \in C^2(\Omega)$ we have

$$dd^c\varphi = 2i \sum \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \, dz_j \wedge d\overline{z}_k.$$

We set $u_r = \max\{u, r\}$. Demailly in [4] introduced the positive measures

$$\mu_{u,r} = (dd^c u_r)^n - \chi_{\Omega \setminus B_{u,r}} (dd^c u)^n, \qquad r \in (-\infty, 0)$$

supported on $S_{u,r}$. The measures $\mu_{u,r}$ are called the family of Monge–Ampère measures associated with the exhaustion function u. In [3, Theorem 1.7] Demailly has proved the following formula which is fundamental to our study.

Theorem 2.7 (Lelong–Jensen formula)**.** *Let φ be any plurisubharmonic function on Ω. Then for every $r < 0$, φ is $\mu_{u,r}$-integrable and*

$$\int_{S_{u,r}} \varphi \, d\mu_{u,r} = \int_{B_{u,r}} \varphi (dd^c u)^n + \int_{B_{u,r}} (r - u) \, dd^c\varphi \wedge (dd^c u)^{n-1}. \qquad (2.4)$$

The following corollary is immediate consequence of Theorem 2.7 [3, Corollary 1.9].

Corollary 2.8. *If φ is a non-negative plurisubharmonic function then $r \mapsto \mu_{u,r}(\varphi)$ is an increasing function of r on $(-\infty, 0)$.*

The total mass of $\mu_{u,r}$ is given by

$$\|\mu_{u,r}\| = \mu_{u,r}(1) = \int_{B_{u,r}} (dd^c u)^n.$$

The following theorem [3, Theorem 3.1] shows that the Monge–Ampère measures $\mu_{u,r}$ extend naturally to the boundary $\partial\Omega$.

Theorem 2.9. *Let $u : \Omega \to [-\infty, 0)$ be a psh continuous exhaustion function. Suppose that the total Monge–Ampère mass of u is finite, that is,*

$$\int_{\Omega} (dd^c u)^n < \infty.$$

Then the measures $\mu_{u,r}$ converge weak-$*$ in $C^*(\overline{\Omega})$ to a positive measure μ_u of total mass $\int_{\Omega}(dd^c u)^n$ supported by $\partial\Omega$ as $r \to 0^-$.

The measure μ_u is called the boundary Monge–Ampère measure associated with u.

Using the measures $\mu_{u,r}$, E. A. Poletsky and M. I. Stessin introduced in [20] the weighted Hardy spaces associated with an exhaustion u which we call the Poletsky–Stessin Hardy spaces and denote by $H^p_u(\Omega)$ or simply by H^p_u whenever there is no confusion about the domain.

Definition 2.6. The space $H_u^p(\Omega)$, $p > 0$, consists of all holomorphic functions f in Ω satisfying the growth condition

$$\|f\|_{H_u^p}^p = \limsup_{r \to 0^-} \int_{S_{u,r}} |f|^p \, d\mu_{u,r} < \infty. \tag{2.5}$$

By Corollary 2.8 the integral on the right is an increasing function of r. So we can replace the lim sup in (2.5) with lim. By Theorem 2.7 and the monotone convergence theorem it follows that,

$$\|f\|_{H_u^p}^p = \int_\Omega |f|^p \, (dd^c u)^n - \int_\Omega u \, dd^c |f|^p \wedge (dd^c u)^{n-1}. \tag{2.6}$$

For $p \geq 1$, $\|\cdot\|_{H_u^p}$ defines a norm on H_u^p and with this norm the spaces H_u^p are Banach ([20, Theorem 4.1]).

Every exhaustion function u on Ω generates a Poletsky–Stessin Hardy space and thus there is an abundance of such spaces. The following theorem ([20, Corollary 3.2]) helps determine the inclusion between these spaces.

Theorem 2.10. *Let u and v be continuous psh exhaustion functions on Ω and $bv(z) \leq u(z)$ near $\partial\Omega$ for some constant $b > 0$. Then $H_v^p \subset H_u^p$ and $\|f\|_{H_u^p} \leq b^n \|f\|_{H_v^p}$.*

It is clear from this theorem that if for some exhaustions u, v there is a constant $b > 0$ such that

$$bv \leq u \leq b^{-1} v \tag{2.7}$$

near $\partial\Omega$ then the spaces they generate are same with equivalent norms, that is, $H_u^p = H_v^p$ ([20, Corollary 3.3]. For the class of exhaustion functions u on Ω with compactly supported $(dd^c u)^n$ the inequality (2.7) holds automatically ([20, Lemma 3.4]). Thus the exhaustion functions in this class generate the same space. The following theorem ([20, Proposition 3.5]) shows that these are the largest Poletsky–Stessin Hardy spaces.

Theorem 2.11. *Let u be a psh exhaustion function on Ω such that $(dd^c u)^n$ has compact support and let v be a continuous psh exhaustion function on Ω then there is a constant C such that $\|f\|_{H^p_u} \leq C\|f\|_{H^p_v}$ and $H^p_v \subset H^p_u$.*

Chapter 3

Poletsky–Stessin Hardy Spaces on the Unit Disk

In our study we will take $\Omega = \mathbb{D}$, where $\mathbb{D}$ is the unit disk. Let $u : \mathbb{D} \to [-\infty, 0)$ be a continuous subharmonic exhaustion function on $\mathbb{D}$ such that $u(z) \to 0$ as $|z| \to 1$. Let us denote by $\mathcal{E}$ the set of all continuous negative subharmonic exhaustion functions u on $\mathbb{D}$ with finite total Monge–Ampère mass, that is,

$$\int_{\mathbb{D}} \Delta u < \infty.$$

The equation (2.6) takes the form

$$\|f\|^p_{H^p_u} = \int_{\mathbb{D}} |f|^p \, \Delta u - \int_{\mathbb{D}} u \Delta |f|^p, \tag{3.1}$$

Denote by $\mathcal{E}_0$ the class of continuous negative subharmonic functions on $\mathbb{D}$ such that the measure Δu is compactly supported. Since the relation (2.7) holds for the exhaustions $u \in \mathcal{E}_0$, they generate the same Poletsky–Stessin Hardy space $H_u^p(\mathbb{D})$ with the equivalent norms and these are the largest Poletsky–Stessin Hardy spaces.

The classical Hardy spaces correspond to $u(z) = \log|z| \in \mathcal{E}_0$ (see Section 4 in [20]) and will be denoted by H^p. From this the following two things are apparent:

1. By Hopf's Lemma the Poletsky–Stessin Hardy spaces stay inside the classical Hardy spaces, that is, $H_u^p \subset H^p$.

2. Classical Hardy spaces are particular type of Poletsky–Stessin Hardy spaces. Hence the classical theory is subsumed in this new theory.

3.1 Example

Since the spaces $H_u^p(\mathbb{D}) \subset H^p(\mathbb{D})$ for all $u \in \mathcal{E}$ and $H_u^p(\mathbb{D}) = H^p(\mathbb{D})$ for $u \in \mathcal{E}_0$, a question arises naturally whether there are exhaustions $u \in \mathcal{E}$ for which $H_u^p(\mathbb{D}) \neq H^p(\mathbb{D})$. We construct a subharmonic function $u \in \mathcal{E}$ on $\mathbb{D}$ for which $H_u^p(\mathbb{D}) \neq H^p(\mathbb{D})$.

Lemma 3.1. *If* $0 < \beta < 1$ *the integral*

$$\int_0^1 \log\left|\frac{s-t}{1-ts}\right| \frac{ds}{(1-s)^\beta}, \quad 0 < t < 1,$$

tends to 0 *as* $t \to 1$.

Proof. Write

$$\int_0^1 \log\left|\frac{s-t}{1-ts}\right| \frac{ds}{(1-s)^\beta} = \int_0^t \log\left(\frac{t-s}{1-ts}\right) \frac{ds}{(1-s)^\beta} + \int_t^1 \log\left(\frac{s-t}{1-ts}\right) \frac{ds}{(1-s)^\beta}$$

$$= \text{I} + \text{II}\,.$$

Make a substitution of $s = \dfrac{x+t}{1+tx}$ in II to get

$$\begin{aligned}
\text{II } &= (1+t)(1-t)^{1-\beta} \int_0^1 \frac{\log x}{(1-x)^\beta (1+tx)^{2-\beta}}\, dx \\
&\geq (1+t)(1-t)^{1-\beta} \int_0^1 \frac{\log x}{(1-x)^\beta}\, dx \\
&\to 0 \quad \text{as } t \to 1 \text{ when } 0 < \beta < 1.
\end{aligned}$$

Again, make substitution of $s = \dfrac{t-x}{1-tx}$ in I to get

$$\begin{aligned}
\text{I } &= (1+t)(1-t)^{1-\beta} \int_0^t \frac{\log x}{(1+x)^\beta (1-tx)^{2-\beta}}\, dx \\
&\geq (1+t)(1-t)^{1-\beta} \int_0^t \frac{\log x}{(1-tx)^{2-\beta}}\, dx \\
&= t(1+t)(1-t)^{1-\beta} \int_0^1 \frac{\log(tx)}{(1-t^2x)^{2-\beta}}\, dx \\
&\geq t(1+t)(1-t)^{1-\beta} \left(\int_0^1 \frac{\log t}{(1-t^2x)^{2-\beta}}\, dx + \int_0^1 \frac{\log x}{(1-x)^{2-\beta}}\, dx \right) \\
&\to 0 \text{ as } t \to 1 \text{ when } 0 < \beta < 1.
\end{aligned}$$

Thus $u(t) \to 0$ as $t \to 1$ when $0 < \beta < 1$. □

Now define a function $u : \mathbb{D} \to [-\infty, 0)$ by

$$u(z) = \int_0^1 \log \left| \frac{z-s}{1-sz} \right| \frac{ds}{(1-s)^\beta},$$

where β is a number between 0 and 1. The function u is subharmonic. If $z, w \in \mathbb{D}$, then by the inequality (see [21, Lemma 4.5.7])

$$\left| \frac{|z| - |w|}{1 - |w||z|} \right| \leq \left| \frac{z-w}{1-\bar{w}z} \right|$$

and Lemma 3.1 it follows that $u(z) \to 0$ as $|z| \to 1$. Also

$$\int_{\mathbb{D}} \Delta u = \int_0^1 \frac{dx}{(1-x)^\beta} < \infty.$$

Thus $u \in \mathcal{E}$.

Theorem 3.2. *For* $\frac{1-\beta}{p} \leq \alpha < \frac{1}{p}$ *the function*

$$f(z) = \frac{1}{(1-z)^\alpha}$$

is in $H^p(\mathbb{D})$ *but not in* $H^p_u(\mathbb{D})$.

Proof. The function $f(z) = \frac{1}{(1-z)^\alpha}$ belongs to $H^p(\mathbb{D})$ for every $\alpha < \frac{1}{p}$ ([22, Ch. I, Prop. 1.3]). On the other hand, by (2.6)

$$\|f\|^p_{H^p_u} \geq \int_{\mathbb{D}} |f|^p \, \Delta u = \int_0^1 \frac{1}{(1-x)^{p\alpha+\beta}} \, dx = \infty$$

when $p\alpha + \beta \geq 1$. Hence $f(z) \notin H^p_u(\mathbb{D})$ for $\alpha \geq \frac{1-\beta}{p}$.

□

3.2 The Hardy spaces of harmonic functions and the measure μ_u

Let us denote by $h^p_u(\mathbb{D})$, $p > 1$, $u \in \mathcal{E}$, the space of harmonic functions h on $\mathbb{D}$ such that

$$\|h\|^p_{u,p} = \lim_{r \to 0^-} \int_{S_{u,r}} |h|^p \, d\mu_{u,r} < \infty.$$

By Theorem 2.10, $h^p_u(\mathbb{D}) \subset h^p(\mathbb{D})$. Thus if $h \in h^p_u(\mathbb{D})$, then h has radial boundary values h^* on $\partial\mathbb{D} = \mathbb{T}$.

Henceforth throughout this document λ is the normalized Lebesgue measure on $\mathbb{T}$, i.e. $\int_{\mathbb{T}} d\lambda = 1$. We have the following theorem.

Theorem 3.3. *Let $h \in h^p_u(\mathbb{D})$, $p > 1$. Then $h^* \in L^p_u(\mathbb{T}) := L^p(\mathbb{T}, \mu_u)$ and*

$$\|h\|_{u,p} = \|h^*\|_{L^p_u}.$$

Proof. The least harmonic majorant on $\mathbb{D}$ of the subharmonic function $|h|^p$ is the Poisson integral of $|h^*|^p$. By the Riesz Decomposition Theorem

$$|h(w)|^p = \int_{\mathbb{T}} |h^*(e^{i\theta})|^p P(w, e^{i\theta})\, d\lambda(\theta) + \int_{\mathbb{D}} G(w,z)\Delta|h|^p(z),$$

where P is the Poisson kernel and G is the Green kernel.

By Lelong–Jensen formula and the monotone convergence theorem we have

$$\|h\|^p_{u,p} = \int_{\mathbb{D}} |h|^p \Delta u - \int_{\mathbb{D}} u\Delta|h|^p.$$

Again by the Riesz formula,

$$u(z) = \int_{\mathbb{D}} G(z,w)\Delta u(w). \tag{3.2}$$

Hence, by Fubini–Tonnelli's Theorem and the symmetry of the Green kernel

$$\int_{\mathbb{D}} u(z)\Delta|h|^p(z) = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} G(w,z)\Delta|h|^p(z) \right) \Delta u(w)$$

and

$$\begin{aligned}\|h\|_{u,p}^p &= \int_{\mathbb{D}} \left(|h(w)|^p - \int_{\mathbb{D}} G(w,z)\Delta|h|^p(z) \right) \Delta u(w) \\ &= \int_{\mathbb{D}} \left(\int_{\mathbb{T}} |h^*(e^{i\theta})|^p P(w,e^{i\theta})\, d\lambda(\theta) \right) \Delta u(w) \\ &= \int_{\mathbb{T}} \left(\int_{\mathbb{D}} P(w,e^{i\theta})\Delta u(w) \right) |h^*(e^{i\theta})|^p \, d\lambda(\theta).\end{aligned}$$

Let

$$\alpha_u(e^{i\theta}) = \int_{\mathbb{D}} P(w,e^{i\theta})\Delta u(w). \tag{3.3}$$

Then

$$\|h\|_{u,p}^p = \int_{\mathbb{T}} |h^*(e^{i\theta})|^p \alpha_u(e^{i\theta})\, d\lambda(\theta).$$

Let ϕ be a continuous function on $\mathbb{T}$ and let h be its harmonic extension to $\mathbb{D}$. Then $h^* = \phi$ and by Theorem 2.9

$$\|h\|_{u,p}^p = \int_{\mathbb{T}} |\phi(e^{i\theta})|^p \, d\mu_u(\theta).$$

Hence $\mu_u = \alpha_u \lambda$ and $\alpha_u \in L^1(\lambda)$. Consequently, for any $h \in h_u^p(\mathbb{D})$

$$\|h\|_{u,p}^p = \int_{\mathbb{T}} |h^*(e^{i\theta})|^p \, d\mu_u(\theta).$$

□

We normalize the exhaustion function u assuming that $\int_{\mathbb{D}} \Delta u = 1$. The class of such exhaustion functions will be denoted by $\mathcal{E}_1$.

From the proof of Theorem 3.3 it follows that the measure μ_u is absolutely continuous with respect to the Lebesgue measure λ and the weight function α_u has the following properties.

Proposition 3.4. *Let $u \in \mathcal{E}_1$. Then the measure $\mu_u = \alpha_u \lambda$, where the function $\alpha_u(e^{i\theta})$ has the following properties:*

(i) $\alpha_u(e^{i\theta}) \in L^1(\lambda)$ *and* $\|\alpha_u\|_{L^1(\lambda)} = 1$.

(ii) $\alpha_u(e^{i\theta}) = \int_{\mathbb{D}} P(z, e^{i\theta})\, \Delta u(z)$.

(iii) $\alpha_u(e^{i\theta})$ *is lower semicontinuous.*

(iv) $\alpha_u(e^{i\theta}) \geq c$ *on* $\mathbb{T}$ *for some* $c > 0$.

(v) $\alpha_u(e^{i\theta})$ *need not to be necessarily bounded.*

Proof. Everything except (iii), (iv) and (v) follow from the proof of the theorem above. Let $e^{i\theta_j} \to e^{i\theta_0}$ in $\mathbb{T}$. By Fatou's lemma

$$\liminf_{j\to\infty} \alpha_u(e^{i\theta_j}) = \liminf_{j\to\infty} \int_{\mathbb{D}} P\left(z, e^{i\theta_j}\right) \Delta u(z) \geq \int_{\mathbb{D}} P\left(z, e^{i\theta_0}\right) \Delta u(z) = \alpha_u(e^{i\theta_0}).$$

This proves (iii).

Let $v(z) = \log|z|$. By Hopf's lemma there is a constant $c > 0$ such that $cu(z) < v(z)$ near $\mathbb{T}$. It follows from [3, Theorem 3.8] that $\mu_v \leq c\mu_u$. Since $\mu_v = \lambda$, (iv) follows.

For the exhaustion function constructed in Section 3.1,

$$\int_{\mathbb{D}} P(z,1)\Delta u = \int_0^1 \frac{1+x}{1-x} \cdot \frac{1}{(1-x)^{\beta}}\, dx = \infty$$

when $\beta > 0$. This proves (v). □

In the proof of the Theorem 3.3 we have deduced the norm of the functions $h \in h^p_u(\mathbb{D})$, $p > 1$ to

$$\|h\|^p_{u,p} = \int_{\partial\mathbb{D}} \left(\int_{\mathbb{D}} P(w, e^{i\theta})\, \Delta u(w) \right) |h^*(e^{i\theta})|^p \, d\lambda.$$

Since $\frac{\partial}{\partial n} G(z,w)|_{z=e^{i\theta}} = P(e^{i\theta}, w)$, from the Riesz formula (3.2) we get

$$\frac{\partial u}{\partial n}(e^{i\theta}) = \int_{\mathbb{D}} P(w, e^{i\theta})\, \Delta u(w)$$

and therefore the norm can be written as

$$\|h\|^p_{u,p} = \int_{\partial\mathbb{D}} \frac{\partial u}{\partial n}(e^{i\theta}) |h^*(e^{i\theta})|^p \, d\lambda.$$

From this deduction it is clear that if $u \in \mathcal{E}$ is such that $\frac{\partial u}{\partial n}(e^{i\theta})$ is bounded then $h^p_u(\mathbb{D}) = h^p(\mathbb{D})$, $p > 1$.

3.3 Boundary values of harmonic functions with respect to the measures $\mu_{u,r}$

While functions in $h^p_u(\mathbb{D})$, $p > 1$, have radial limits μ_u-a.e., we are interested in the analogs of more subtle classical properties of boundary values. For example, if $h \in h^p(\mathbb{D})$ then it is known that the measures $h(re^{i\theta})\lambda(\theta)$ converge weak-$*$ in $C^*(\mathbb{T})$ to $h^*(e^{i\theta})\lambda(\theta)$ as $r \to 1^-$.

In this section we will establish the analogs of these statements.

Theorem 3.5. *Let $h \in h^p_u(\mathbb{D})$, $p > 1$. Then the measures $\{h\mu_{u,r}\}$ converge weak-$*$ to $h^*\mu_u$ in $C^*(\overline{\mathbb{D}})$ when $r \to 0^-$.*

Proof. Since the space $C(\overline{\mathbb{D}})$ is separable the weak-$*$ topology on the balls in $C^*(\overline{\mathbb{D}})$ is metrizable. Thus it suffices to show that for any sequence $r_j \nearrow 0$ and any $\phi \in C(\overline{\mathbb{D}})$ we have

$$\lim_{j\to\infty} \int_{S_{u,r_j}} \phi h \, d\mu_{u,r_j} = \int_{\partial\mathbb{D}} \phi h^* \, d\mu_u.$$

We introduce functions

$$p_r(e^{i\theta}) = \int_{S_{u,r}} P(z, e^{i\theta}) \, d\mu_{u,r}(z) = \int_{B_{u,r}} P(z, e^{i\theta}) \, \Delta u(z),$$

where the last equality follows from Theorem 2.7. Hence $p_r(e^{i\theta}) \nearrow \alpha_u(e^{i\theta})$.

Let $\varepsilon > 0$ be given. The uniform continuity of ϕ implies that there is $\delta > 0$ such that $|\phi(z) - \phi(e^{i\theta})| < \varepsilon$ when $|z - e^{i\theta}| \le \delta$. On the other hand, there exists $0 < s < 1$ such that for $|z| > s$, $|P(z, e^{i\theta})| < \varepsilon$ when $|z - e^{i\theta}| > \delta$. Hence, when r is sufficiently close to 0,

$$\begin{aligned}
&\left| \int_{S_{u,r}} \phi(z) P(z, e^{i\theta}) \, d\mu_{u,r}(z) - \int_{S_{u,r}} \phi(e^{i\theta}) P(z, e^{i\theta}) \, d\mu_{u,r}(z) \right| \\
\le & \int_{S_{u,r} \setminus \overline{\mathbb{D}}(e^{i\theta},\delta)} |\phi(z) - \phi(e^{i\theta})| P(z, e^{i\theta}) \, d\mu_{u,r}(z) \\
& + \int_{S_{u,r} \cap \overline{\mathbb{D}}(e^{i\theta},\delta)} |\phi(z) - \phi(e^{i\theta})| P(z, e^{i\theta}) \, d\mu_{u,r}(z) \\
\le & 2M\varepsilon + \varepsilon p_r(e^{i\theta}),
\end{aligned}$$

where $\mathbb{D}(e^{i\theta}, \delta)$ is the disk of radius δ and center at $e^{i\theta}$ M is the uniform norm of ϕ on $\overline{\mathbb{D}}$.

Now,

$$\begin{aligned}\int_{S_{u,r}} \phi(z)h(z)\,d\mu_{u,r}(z) &= \int_{S_{u,r}} \phi(z)\left(\int_{\mathbb{T}} h^*(e^{i\theta})P(z,e^{i\theta})\,d\lambda(\theta)\right)\,d\mu_{u,r}(z)\\ &= \int_{\mathbb{T}} h^*(e^{i\theta})\left(\int_{S_{u,r}} \phi(z)P(z,e^{i\theta})\,d\mu_{u,r}(z)\right)\,d\lambda(\theta).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\int_{S_{u,r}} \phi(z)h(z)\,d\mu_{u,r}(z) - \int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})\,d\mu_u(\theta)\right|\\ \le&\left|\int_{S_{u,r}} \phi(z)h(z)\,d\mu_{u,r}(z) - \int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})p_r(e^{i\theta})\,d\lambda(\theta)\right|\\ &+\left|\int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})p_r(e^{i\theta})\,d\lambda(\theta) - \int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})\,d\mu_u(\theta)\right|\\ =&\left|\int_{\mathbb{T}} h^*(e^{i\theta})\left(\int_{S_{u,r}} (\phi(z)-\phi(e^{i\theta}))P(z,e^{i\theta})\,d\mu_{u,r}(z)\right)\,d\lambda(\theta)\right|\\ &+\left|\int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})\left(p_r(e^{i\theta})-\alpha_u(e^{i\theta})\right)\,d\lambda(\theta)\right|\\ \le&\varepsilon\int_{\mathbb{T}} \left|h^*(e^{i\theta})\right|(2M+p_r(e^{i\theta}))\,d\lambda(\theta) + M\int_{\mathbb{T}} \left|h^*(e^{i\theta})\right|\left|p_r(e^{i\theta})-\alpha_u(e^{i\theta})\right|\,d\lambda(\theta).\end{aligned}$$

Now,

$$\begin{aligned}\int_{\mathbb{T}} \left|h^*(e^{i\theta})\right|(2M+p_r(e^{i\theta}))\,d\lambda(\theta) \le& \int_{\mathbb{T}} \left|h^*(e^{i\theta})\right|(2M+\alpha_u(e^{i\theta}))\,d\lambda(\theta)\\ \le& 2M\|h^*\|_{L^p} + \|h\|_{u,p}.\end{aligned}$$

Since $\left|p_r(e^{i\theta})-\alpha_u(e^{i\theta})\right| \searrow 0$ and $\left|p_r(e^{i\theta})-\alpha_u(e^{i\theta})\right| < \alpha_u(e^{i\theta})$ with $\left|h^*(e^{i\theta})\right|\alpha_u(e^{i\theta}) \in L^1(\lambda)$, by the monotone convergence theorem,

$$\int_{\mathbb{T}} \left|h^*(e^{i\theta})\right|\left|p_r(e^{i\theta})-\alpha_u(e^{i\theta})\right|\,d\lambda(\theta) \to 0$$

Thus, since ε is arbitraty,

$$\left|\int_{S_{u,r}} \phi(z)h(z)\,d\mu_{u,r}(z) - \int_{\mathbb{T}} \phi(e^{i\theta})h^*(e^{i\theta})\,d\mu_u(\theta)\right| \to 0.$$

The proof is complete. □

It was proved by Demailly in [3] that the measures $\mu_{u,r}$ converge weak-$*$ in $C^*(\overline{\mathbb{D}})$ as $r \to 0^-$. The corollary below shows that they converge weak-$*$ also in the dual of $h^p_u(\mathbb{D})$.

Corollary 3.6. *If $p > 1$, then the measures $\mu_{u,r}$ converge weak-$*$ to μ_u in the dual of $h^p_u(\mathbb{D})$ when $r \to 0^-$.*

Proof. For $\phi \in C(\overline{\mathbb{D}})$, from the theorem above we have

$$\lim_{r\to 0^-} \int_{S_{u,r}} \phi h \, d\mu_{u,r} = \int_{\mathbb{T}} \phi h^* \, d\mu_u$$

for every $h \in h^p_u(\mathbb{D})$. In particular, if we take $\phi \equiv 1$ we get

$$\lim_{r\to 0^-} \int_{S_{u,r}} h \, d\mu_{u,r} = \int_{\mathbb{T}} h^* \, d\mu_u$$

for every $h \in h^p_u(\mathbb{D})$. The corollary follows. □

In [17] Poletsky introduced the weak and strong limit values for a sequence $\{\phi_j\}$ of Borel functions defined on compact subsets K_j of a compact metric space K with respect to a sequence of regular Borel measures μ_j supported by K_j and converging weak-$*$ in $C^*(K)$ to a finite measure μ. If the measures $\{\phi_j\mu_j\}$ converge weak-$*$ in $C^*(K)$ to a measure $\phi_*\mu$, then the function ϕ_* is called the *weak limit values* of $\{\phi_j\}$.

We say that the sequence $\{\phi_j\}$ has a *strong limit values* on $\operatorname{supp}\mu = K_0$ with respect to $\{\mu_j\}$ if there is a μ-measurable function ϕ^* on K_0 such that for any $b > a$

and any $\epsilon, \delta > 0$ there is j_0 and an open set $O \subset K$ containing $G(a, b) = \{x \in K_0 : a \le \phi^*(x) < b\}$ such that

$$\mu_j(\{\phi_j < a - \epsilon\} \cap O) + \mu_j(\{\phi_j > b + \epsilon\} \cap O) < \delta$$

when $j \ge j_0$. The function ϕ^* is called the *strong limit values* of $\{\phi_j\}$.

Following the definition in [17], we say that a function $h \in h^p_u(\mathbb{D})$ has *boundary values* with respect to the measures $\mu_{u,r}$ if it has strong limit values with respect to $\{\mu_{u,r_j}\}$ for any sequence $r_j \nearrow 0$ and these strong limit values do not depend on the choice of a sequence.

The following three theorems are the results in [17] which are useful to study the boundary values of functions in our spaces.

Theorem 3.7. *Suppose that $\{\phi_j\}$ has the strong limit values on K_0 equal to ϕ^*. Then any two choices of ϕ^* coincide μ-a.e. The sequences $\{c\phi_j\}$ and $\{|\phi_j|^p\}$ have strong limit values and $(c\phi)^* = c\phi^*$ and $(|\phi|^p)^* = |\phi^*|^p$.*

Theorem 3.8. *Suppose that a sequence $\{\phi_j\}$ has the strong limit values ϕ^*. If* $\limsup_{j\to\infty} \|\phi_j\|_{L^p(K_j,\mu_j)} = A < \infty$*, $p > 1$, then $\|\phi^*\|_{L^p(K,\mu)} \le A$.*

Theorem 3.9. *Let $\{\phi_j\}$ has weak limit values and* $\limsup_{j\to\infty} \|\phi_j\|_{L^p(K_j,\mu_j)} < \infty$ *for some $p > 1$. Let the measures $\{|\phi_j|^p \mu_j\}$ converge weak-$*$ to ν. If*

$$\nu(K) = \int_K |\phi_*|^p \, d\mu$$

then the sequence $\{\phi_j\}$ has the strong limit values equal to ϕ_*.

The functions in $h^p_u(\mathbb{D})$, $p > 1$, have boundary values in the sense of Poletsky.

Theorem 3.10. *Let $h \in h^p_u(\mathbb{D})$, $p > 1$. Then h has the boundary values equal to h^* with respect to $\{\mu_{u,r}\}$.*

Proof. Let r_j be any increasing sequence of numbers converging to 0. By Theorem 3.5 the measures $h\mu_{u,r}$ converge weak-$*$ in $C^*(\overline{\mathbb{D}})$ to the measure $h^*\mu_u$. By Theorem 3.3

$$\lim_{j\to\infty} \int_{S_{u,r_j}} |h|^p \, d\mu_{u,r_j} = \int_{\mathbb{T}} |h^*|^p \, d\mu_u.$$

By Theorem 3.9 the sequence of the function $h|_{S_{u,r_j}}$ has the strong boundary values equal to h^*. $\square$

3.4 Boundary values of holomorphic functions with respect to the measures $\mu_{u,r}$

In this section we prove results analogous to those in two previous sections but for $p > 0$. To consider the Hardy spaces for $0 < p \leq 1$ we need a factorization theorem.

From the classical theory we know that every function $f \in H^p(\mathbb{D})$, $p > 0$, $f \not\equiv 0$ can be factored into $f(z) = \beta(z)g(z)$ where $\beta(z)$ is a Blaschke product with same zeros as f and g is a non-vanishing function in $H^p(\mathbb{D})$ with $\|g\|_{H^p} = \|f\|_{H^p}$. Let us show that similar results hold for the functions in $H^p_u(\mathbb{D})$.

Theorem 3.11. *Let $f(z) \in H^p_u(\mathbb{D})$, $p > 0$ and $f(z) \not\equiv 0$. Then there exists a function*

$g(z) \in H^p_u(\mathbb{D})$, $g(z) \neq 0$ in $\mathbb{D}$, such that

$$f(z) = \beta(z)g(z) \quad \textit{and} \quad \|g\|_{H^p_u} = \|f\|_{H^p_u},$$

where $\beta(z)$ is a Blaschke product having the same zeros as f.

Proof. We mimic the proof of the classical version [10, Theorem 2.3]. Let $\{a_j\}$ be the zeros of $f(z)$ in $\mathbb{D}$ not necessarily all distinct. We may assume that $a_j \neq 0$ for all j since otherwise if 0 is the zero of order m then we write $f(z) = z^m \tilde{f}(z)$ and work with $\tilde{f}(z)$. Then

$$\beta(z) = \prod_{j=1}^{\infty} \frac{-\overline{a}_j}{|a_j|} \frac{z - a_j}{1 - \overline{a}_j z}.$$

From classical theory we have $g(z) = \frac{f(z)}{\beta(z)} \in H^p(\mathbb{D})$. We show that $g(z) \in H^p_u(\mathbb{D})$.

Write

$$g_N(z) = \frac{f(z)}{\beta_N(z)}, \quad \text{where } \beta_N(z) = \prod_{j=1}^{N} \frac{-\overline{a}_j}{|a_j|} \frac{z - a_j}{1 - \overline{a}_j z}.$$

For fixed N, $|\beta_N(z)| \to 1$ uniformly as $|z| \to 1$. So for given $\varepsilon > 0$ there exists $\rho_0 > 0$ such that $|\beta_N(z)| > 1 - \varepsilon$ when $|z| > \rho_0$. Thus near $\mathbb{T}$ we have

$$|g_N(z)| < \frac{|f(z)|}{1 - \varepsilon}.$$

Since ε is arbitrary and $\mu_{u,r}(|f|^p)$ is an increasing function of r, it follows that

$$\int_{S_{u,r}} |g_N(z)|^p \, d\mu_{u,r} \leq \|f\|^p_{H^p_u}.$$

Since $|g_N(z)| \nearrow |g(z)|$, by the monotone convergence theorem,

$$\int_{S_{u,r}} |g(z)|^p \, d\mu_{u,r} = \lim_{N \to \infty} \int_{S_{u,r}} |g_N(z)|^p \, d\mu_{u,r} \leq \|f\|^p_{H^p_u}.$$

Hence $\|g\|_{H^p_u} \le \|f\|_{H^p_u}$. The reverse inequality is trivial because $|f(z)| \le |g(z)|$ in $\mathbb{D}$. Thus $\|g\|_{H^p_u} = \|f\|_{H^p_u}$. This completes the proof. $\square$

Since $H^p_u(\mathbb{D}) \subset H^p(\mathbb{D})$, any $f \in H^p_u(\mathbb{D})$ has radial limits $f^*(e^{i\theta})$ λ-a.e. But it is not clear that $\|f\|_{H^p_u} \ge \|f^*\|_{L^p_u}$. The theory of weak and strong limit values in [17] provides sufficient conditions for this estimate. To implement these conditions we have to show the existence of strong limit values for $f \in H^p_u(\mathbb{D})$.

Theorem 3.12. *Any function $f \in H^p_u(\mathbb{D})$, $p > 1$, has weak limit values equal to f^* with respect to the measures $\{\mu_{u,r}\}$.*

Proof. Follows directly from Theorem 3.5. $\square$

Theorem 3.13. *Let $f \in H^p_u(\mathbb{D})$, $p > 1$. Then $|f|$ has the boundary values equal to $|f^*|$ with respect to $\{\mu_{u,r}\}$.*

Proof. For $f \in H^p_u(\mathbb{D})$, $\mathbf{Re}\, f$ and $\mathbf{Im}\, f \in h^p_u(\mathbb{D})$. Hence the corollary follows from Theorem 3.7 and 3.10 by writing $|f|^2 = (\mathbf{Re}\, f)^2 + (\mathbf{Im}\, f)^2$. $\square$

Now we prove the main result of the section:

Theorem 3.14. *Let $f \in H^p(\mathbb{D})$, $p > 0$. Then $f \in H^p_u(\mathbb{D})$ if and only if $f^*(e^{i\theta}) \in L^p_u$. Moreover, $\|f\|_{H^p_u} = \|f^*\|_{L^p_u}$.*

Proof. First, we prove the theorem for $p > 1$. Let $f^* \in L^p_u$. There exists $f^*_j \in C(\mathbb{T})$ such that

$$\|f^*_j - f^*\|_{L^p_u} \to 0 \quad \text{as } j \to \infty.$$

By Proposition 3.4,

$$\|f_j^* - f^*\|_{L^p(\lambda)} \to 0 \quad \text{as } j \to \infty.$$

We know that $f(z)$ is the Poisson integral of its boundary value $f^*(e^{i\theta})$ [6, Theorem 3.1], that is,

$$f(z) = \int_0^{2\pi} P(z, e^{i\theta}) f^*(e^{i\theta})\, d\lambda(\theta).$$

If we take

$$f_j(z) = \int_0^{2\pi} P(z, e^{i\theta}) f_j^*(e^{i\theta})\, d\lambda(\theta)$$

by Hölder's inequality,

$$\begin{aligned} |f_j(z) - f(z)| &= \left| \int_0^{2\pi} \left(f_j^*(e^{i\theta}) - f^*(e^{i\theta}) \right) P(z, e^{i\theta})\, d\lambda(\theta) \right| \\ &\le \left(\int_0^{2\pi} \left| f_j^*(e^{i\theta}) - f^*(e^{i\theta}) \right|^p \, d\lambda(\theta) \right)^{\frac{1}{p}} \left(\int_0^{2\pi} P^q(z, e^{i\theta})\, d\lambda(\theta) \right)^{\frac{1}{q}}. \end{aligned}$$

The last integral is, evidently, bounded on compact sets in $\mathbb{D}$ and hence $f_j \to f$ uniformly on compacta. Therefore

$$\lim_{j\to\infty} \int_{S_{u,r}} |f_j|^p \, d\mu_{u,r} = \int_{S_{u,r}} |f|^p \, d\mu_{u,r}.$$

The weak-$*$ convergence of $\mu_{u,r}$ gives

$$\lim_{r\to 0^-} \int_{S_{u,r}} |f_j|^p \, d\mu_{u,r} = \int_{\mathbb{T}} |f_j|^p \, d\mu_u.$$

Since $f_j(z)$ is harmonic, $|f_j|^p$ is subharmonic and by Corollary 2.8, $\mu_{u,r}(|f_j|^p)$ is an increasing function of r. It follows, for each j, that

$$\int_{S_{u,r}} |f_j|^p \, d\mu_{u,r} \le \int_{\mathbb{T}} |f_j|^p \, d\mu_u = \int_{\mathbb{T}} |f_j^*|^p \, d\mu_u.$$

Hence

$$\int_{S_{u,r}} |f|^p \, d\mu_{u,r} = \lim_{j\to\infty} \int_{S_{u,r}} |f_j|^p \, d\mu_{u,r} \le \lim_{j\to\infty} \int_{\mathbb{T}} |f_j^*|^p \, d\mu_u = \int_{\mathbb{T}} |f^*|^p \, d\mu_u.$$

Therefore $\|f\|_{H_u^p} \le \|f^*\|_{L_u^p}$ and $f \in H_u^p(\mathbb{D})$.

Let $f \in H_u^p(\mathbb{D})$. Then by Corollary 3.13, $|f|$ has the boundary values $|f^*|$ with respect to $\{\mu_{u,r}\}$. By Theorem 3.8, it follows that

$$\|f^*\|_{L_u^p} \le \|f\|_{H_u^p}.$$

Hence $f^* \in L_u^p$ and $\|f\|_{H_u^p} = \|f^*\|_{L_u^p}$.

Now we prove the theorem for $0 < p \le 1$. Let $f \in H^p(\mathbb{D})$. Then we have the factorization $f(z) = \beta(z)g(z)$ where $\beta(z)$ is a Blaschke product and $g(z)$ is a non-vanishing function in $H^p(\mathbb{D})$. Suppose $f^* \in L_u^p$. Since $|f^*| = |g^*|$ λ-a.e. (and hence μ_u-a.e.), $g^* \in L_u^p$. It follows from the proof for $p > 1$ and the fact that $g^{\frac{p}{2}} \in H^2(\mathbb{D})$ and $(g^*)^{\frac{p}{2}} \in L_u^2$ that

$$\|g^{\frac{p}{2}}\|_{H_u^2} \le \|(g^*)^{\frac{p}{2}}\|_{L_u^2}.$$

This implies

$$\|g\|_{H_u^p} \le \|g^*\|_{L_u^p}.$$

Since $|f(z)| \le |g(z)|$ in $\mathbb{D}$ we get

$$\|f\|_{H_u^p} \le \|f^*\|_{L_u^p}$$

and hence $f \in H_u^p(\mathbb{D})$.

On the other hand if $f \in H^p_u(\mathbb{D})$ then by Theorem 3.11, $f(z) = \beta(z)g(z)$ where $g(z)$ is a non-vanishing function in $H^p_u(\mathbb{D})$. Since $g^{\frac{p}{2}} \in H^2_u(\mathbb{D})$, $|g^{\frac{p}{2}}|$ has the boundary values $|(g^{\frac{p}{2}})^*|$ with respect to $\{\mu_{u,r}\}$. Then by Theorem 3.8,

$$\|(g^{\frac{p}{2}})^*\|_{L^2_u} \leq \|g^{\frac{p}{2}}\|_{H^2_u}.$$

This implies

$$\|g^*\|_{L^p_u} \leq \|g\|_{H^p_u}$$

and hence

$$\|f^*\|_{L^p_u} \leq \|f\|_{H^p_u}.$$

Thus $f^* \in L^p_u$ and $\|f\|_{H^p_u} = \|f^*\|_{L^p_u}$. □

3.5 Properties of $H^p_u(\mathbb{D})$ and Dual Spaces

Note that $H^p_u(\mathbb{D})$ is not a closed subspace of $H^p(\mathbb{D})$ because both spaces contain $H^\infty(\mathbb{D})$. However, the closed balls in $H^p_u(\mathbb{D})$ are closed in $H^p(\mathbb{D})$.

Theorem 3.15. *The closed unit ball*

$$B_{u,p}(1) = \{f \in H^p_u(\mathbb{D}) : \|f\|_{H^p_u} \leq 1\}$$

in $H^p_u(\mathbb{D})$, $p > 0$, is closed in $H^p(\mathbb{D})$.

Proof. The case $p = \infty$ is obvious. Let $\{f_j\} \subset B_{u,p}(1)$ be such that $f_j \to f$ in $H^p(\mathbb{D})$, i.e.

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \left|f_j(re^{i\theta}) - f(re^{i\theta})\right|^p \, d\lambda(\theta) \to 0 \quad \text{as } j \to \infty.$$

By formula (3.2) in [20] if $|z| < r$ then

$$|f(z) - f_j(z)|^p \leq \int_{|w|=r} |f(re^{i\theta}) - f_j(re^{i\theta})|^p \, d\lambda(\theta) \leq \|f_j - f\|_{H^p}.$$

Hence the functions $f_j \to f$ uniformly on compacta.

Now

$$\int_{S_{u,r}} |f_j(z)|^p \, d\mu_{u,r} \to \int_{S_{u,r}} |f(z)|^p \, d\mu_{u,r}$$

for all $r < 0$. Therefore

$$\lim_{r \to 0^-} \int_{S_{u,r}} |f(z)|^p \, d\mu_{u,r} \leq 1,$$

showing that $f \in B_{u,p}(1)$. □

For $u \in \mathcal{E}$, define $\mathcal{E}_u = \{v \in \mathcal{E} : bv \leq u \leq b^{-1}v$ for some constant $b > 0$ near $\mathbb{T}\}$. It has been discussed in [20] that all the exhaustions in $\mathcal{E}_u$ generate the same Poletsky–Stessin Hardy space $H^p_u(\mathbb{D})$ with equivalent norms. Let us take the class $\mathcal{E}_0$ which corresponds to the exhaustion function $u(z) = \log|z|$. Then all the exhaustions in $\mathcal{E}_0$ generate the classical Hardy space $H^p(\mathbb{D})$, with equivalent norms and this is the largest space in our class.

However as we show below the norms generated by exhaustions in $\mathcal{E}_0 \cap \mathcal{E}_1$ differ so much that the intersection of all unit balls in these norms is the unit ball in $H^\infty(\mathbb{D})$.

For $u \in \mathcal{E}$ define the ball of radius R in $H^p_u(\mathbb{D})$ by

$$B_{u,p}(R) = \{f \in H^p_u(\mathbb{D}) : \|f\|_{H^p_u} \leq R\}.$$

Let $B_\infty(R) = \{f \in H^\infty(\mathbb{D}) : |f| \leq R\}$.

Theorem 3.16. *For $p > 0$,*

$$\bigcap_{u \in \mathcal{E}_0 \cap \mathcal{E}_1} B_{u,p}(1) = B_\infty(1).$$

Proof. The inclusion $B_\infty(1) \subset \bigcap_{u\in\mathcal{E}_0\cap\mathcal{E}_1} B_{u,p}(1)$ is clear. For the other way around, let $f \in H^\infty(\mathbb{D}) \setminus B_\infty(1)$. Since $|f^*|^p \in L^1(\mathbb{T})$, by the Fatou's theorem

$$\int_{\mathbb{T}} P(re^{i\varphi}, e^{i\theta})|f^*(e^{i\theta})|^p \, d\lambda \to |f^*(e^{i\varphi})|^p$$

λ-a.e. on $\mathbb{T}$. Hence there exists $A \subset \mathbb{T}$ with $\lambda(A) > 0$ such that

(i) $|f^*(e^{i\varphi})| > 1$ and

(ii) $\int_{\mathbb{T}} P(re^{i\varphi}, e^{i\theta})|f^*(e^{i\theta})|^p \, d\lambda \to |f^*(e^{i\varphi})|^p$

for every $e^{i\varphi} \in A$. We may suppose that $1 \in A$.

Since $u(z) = \int_{\mathbb{D}} G(z, w)\, \Delta u(w)$, where $G(z, w)$ is the Green's function for the unit disk, and $\frac{\partial}{\partial n} G(z, w)|_{z=e^{i\theta}} = P(w, e^{i\theta})$,

$$\frac{\partial u}{\partial n}(e^{i\theta}) = \int_{\mathbb{D}} P(w, e^{i\theta})\, \Delta u(w) = \alpha_u(e^{i\theta}).$$

Also we have for $f \in H^p_u(\mathbb{D})$,

$$\|f\|^p_{H^p_u} = \int_{\mathbb{T}} \frac{\partial u}{\partial n}(e^{i\theta})|f^*(e^{i\theta})|^p \, d\lambda.$$

Let $t_k \nearrow 1$ and $u_k(z) = G(z, t_k)$. Then

$$\begin{aligned} \|f\|^p_{H^p_{u_k}} &= \int_{\mathbb{T}} P(t_k, e^{i\theta})|f^*(e^{i\theta})|^p \, d\lambda \\ &\longrightarrow |f^*(1)|^p \end{aligned}$$

as $k \to \infty$ because $1 \in A$. Hence $f \notin \bigcap_{u\in\mathcal{E}_0} B_{u,p}(1)$. The theorem follows. □

Recall from Proposition 3.4 that we have $\mu_u = \alpha_u \lambda$ where $\alpha_u \in L^1(\lambda)$ and $\alpha_u \geq c > 0$ for some constant c. Moreover, α_u is lower semicontinuous. Hence, there exists an increasing sequence of positive smooth functions α_n converging to α_u pointwise. Define

$$\tilde{\alpha}(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \alpha_u(e^{i\theta})\, d\lambda(\theta)$$
$$\tilde{\alpha}_n(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \alpha_n(e^{i\theta})\, d\lambda(\theta).$$

Clearly $\tilde{\alpha}$, $\tilde{\alpha}_j \in \mathcal{O}(\mathbb{D})$, so the functions $A(z) = e^{\tilde{\alpha}(z)}$ and $A_n(z) = e^{\tilde{\alpha}_n(z)} \in \mathcal{O}(\mathbb{D})$. Moreover, The functions $\tilde{\alpha}_n$ and A_n extend smoothly to the boundary, $|A^*(e^{i\theta})| = \alpha_u(e^{i\theta})$ and $|A_n^*(e^{\theta})| = \alpha_n(e^{i\theta})$.

Theorem 3.17. *The space $H^p_u(\mathbb{D})$ is isometrically isomorphic to $H^p(\mathbb{D})$.*

Proof. First, we show that if $f \in H^p_u(\mathbb{D})$ then $A^{1/p} f \in H^p(\mathbb{D})$. Clearly $A_n^{1/p} f \in H^p(\mathbb{D})$. Then by formula (9) in [11, IX.4],

$$\int_0^{2\pi} |A_n(re^{i\theta})||f(re^{i\theta})|^p\, d\lambda(\theta) \leq \int_0^{2\pi} |A_n^*(e^{i\theta})||f^*(e^{i\theta})|^p\, d\lambda(\theta).$$

Since $A_n^{1/p} f$ converges to $A^{1/p} f$ uniformly on compact subsets of $\mathbb{D}$, for $0 < r < 1$,

$$\begin{aligned} \int_0^{2\pi} |A(re^{i\theta})||f(re^{i\theta})|^p\, d\lambda &= \lim_{n\to\infty} \int_0^{2\pi} |A_n(re^{i\theta})||f(re^{i\theta})|^p\, d\lambda(\theta) \\ &\leq \lim_{n\to\infty} \int_0^{2\pi} |A_n^*(e^{i\theta})||f^*(e^{i\theta})|^p\, d\lambda(\theta) \\ &= \|f\|^p_{H^p_u}. \end{aligned}$$

The last equality above follows from the monotone convergence theorem. Thus $A^{1/p} f \in H^p(\mathbb{D})$.

Now, define an operator

$$\Phi : H^p_u(\mathbb{D}) \to H^p(\mathbb{D})$$

$$f \mapsto A^{1/p} f.$$

Clearly Φ is linear. Since

$$\int_0^{2\pi} |A^*(e^{i\theta})||f^*(e^{i\theta})|^p \, d\lambda = \int_0^{2\pi} |f^*(e^{i\theta})|^p \alpha_u(e^{i\theta}) \, d\lambda = \int_{\mathbb{T}} |f^*|^p \, d\mu_u,$$

we have $\|A^{1/p} f\|_{H^p} = \|f\|_{H^p_u}$. So Φ is an isometry.

Let $f \in H^p(\mathbb{D})$. Since $|A(z)| \geq c > 0$, $A^{-1/p} f \in H^p(\mathbb{D})$. It follows from the identity

$$\int_{\mathbb{T}} |A^*|^{-1} |f^*|^p \, d\mu_u = \int_{\mathbb{T}} |f^*|^p \, d\lambda$$

together with Theorem 3.14 that $A^{-1/p} f \in H^p_u(\mathbb{D})$. Thus Φ is a surjective linear isometry. We are done. □

In the theorem above we have established that $H^p_u = B^{1/p} H^p$, where $B(z) = 1/A(z)$. Also it is clear that $L^p_u = (B^*)^{1/p} L^p(\lambda)$. In order to describe the duality of the space H^p_u, $p \geq 1$, we need to identify the annihilator of H^p_u in $(L^p_u)^* = L^q_u$, where $1/p + 1/q = 1$. The annihilator turns out to be what we expect based on the knowledge of classical theory.

Theorem 3.18. *The annihilator of H^p_u, $p \geq 1$, in $(L^p_u)^*$,*

$$(H^p_u)^\perp = \left\{ g^* \in L^q_u : \int_{\partial\mathbb{D}} g^* f^* \, d\mu_u = 0 \text{ for all } f \in H^p_u \right\}$$

is isometrically isomorphic to

$$H^q_u(0) = \{g \in H^q_u(\mathbb{D}) : g(0) = 0\}\,.$$

Proof. Let $g^* \in (H^p_u)^\perp$, $p > 1$. Define

$$\tilde{g}^* := g^*(B^*)^{1/p}\alpha_u.$$

Observe that

$$\begin{aligned}\int_{\partial\mathbb{D}} |\tilde{g}^*|^q \, d\lambda &= \int_{\partial\mathbb{D}} |g^*|^q |B^*|^{q/p} \alpha_u^q \, d\lambda \\ &= \int_{\partial\mathbb{D}} |g^*|^q \, d\mu_u\end{aligned}$$

and since $B^{1/p} z^n \in H^p_u$,

$$\begin{aligned}\int_{\partial\mathbb{D}} \tilde{g}^* e^{in\theta} \, d\lambda &= \int_{\partial\mathbb{D}} g^* \left((B^*)^{1/p} e^{in\theta}\right) \, d\lambda \\ &= 0.\end{aligned}$$

Hence there is $\tilde{g} \in H^q(0)$ such that $\tilde{g}^*$ is the boundary value of $\tilde{g}$ and the association $g^* \mapsto g := B^{1/q}\tilde{g}$ gives isometric isomorphism between $(H^p_u)^\perp$ and $H^p_u(0)$. It just remains to show that this is surjective.

Let $g \in H^q_u(0)$. Then $g = B^{1/q}\tilde{g}$ for some $\tilde{g} \in H^q(0)$. Define

$$g^* := \frac{\tilde{g}^*}{(B^*)^{1/p}\alpha_u}.$$

Observe that

$$\int_{\partial\mathbb{D}} |g^*|^q \, d\mu_u = \int_{\partial\mathbb{D}} |\tilde{g}^*|^q \, d\lambda$$

and since every $f \in H^p_u$ is given by $f = B^{1/p}\tilde{f}$ for some $\tilde{f} \in H^p$,

$$\int_{\partial\mathbb{D}} g^* f^* \, d\mu_u = \int_{\partial\mathbb{D}} \tilde{g}^* \tilde{f}^* \, d\lambda = 0.$$

Hence $g^* \in (H^p_u)^\perp$ and $g^* \mapsto g$.

The case $p = 1$ is handled similarly. $\square$

Now the following duality results follow from [6, Theorem 7.1 and 7.2].

Theorem 3.19. *If $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

1. *$L^q_u / H^q_u(0)$ is isometrically isomorphic to $(H^p_u)^*$.*
2. *$(L^p_u / H^p_u)^*$ is isometrically isomorphic to $H^q_u(0)$.*

3.6 Characterization of $H^p_u(\mathbb{D})$

Among many different definitions of weighted Hardy spaces the closest to our purpose is the definition in [2] and [15]. Let $\alpha \in L^1(\mathbb{T})$ be a non-negative function such that $\log \alpha \in L^1(\mathbb{T})$. Then $L^p_\alpha(\mathbb{T})$ is the space of all functions with the finite norm

$$\|\phi\|_{L^p_\alpha} = \left(\int_0^{2\pi} |\phi(e^{i\theta})|^p \alpha(e^{i\theta}) \, d\lambda \right)^{1/p}$$

for $0 < p < \infty$ and $H^p_\alpha = N^+ \cap L^p_\alpha(\mathbb{T})$, where N^+ is the Smirnov class. If $\alpha \equiv 1$ then we will use notations H^p and $\|\cdot\|_p$. The Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{D})$ thus correspond to the weight function α_u. We have established in Proposition 3.4 that the weight α_u has the following properties:

1.
$$\alpha_u(e^{i\theta}) = \int_{\mathbb{D}} P(z, e^{i\theta})\, \Delta u(z),$$
where $P(z, e^{i\theta})$ is the Poisson kernel;

2. $\|\alpha_u\|_{L^1} = 1$ if and only if $u \in \mathcal{E}_1$;

3. $\alpha_u(e^{i\theta})$ is lower semicontinuous and $\alpha_u(e^{i\theta}) \geq c$ on $\mathbb{T}$ for some $c > 0$.

Because of these restrictions on the weight function the class of Poletsky–Stessin Hardy spaces is more narrow than weighted spaces discussed above. As the following result shows these are the only restrictions on weights.

Theorem 3.20. *Let α be a measurable function on $\mathbb{T}$. Then $\alpha\, d\lambda = \mu_u$ for some $u \in \mathcal{E}_1$ if and only if α is lower semicontinuous, $\alpha(e^{i\theta}) \geq c > 0$ for some c on $\mathbb{T}$ and*
$$\int_{\mathbb{T}} \alpha\, d\lambda = 1. \tag{3.4}$$

Proof. Let $\alpha \in C(\mathbb{T})$ be a function such that $\alpha \geq c > 0$ on $\mathbb{T}$. For $0 < r < 1$ define
$$\alpha_r(e^{i\theta}) = \int_{\mathbb{T}} P(re^{i\theta}, e^{i\varphi})\alpha(e^{i\varphi})\, d\lambda(\varphi).$$
Then $\alpha_r \to \alpha$ uniformly on $\mathbb{T}$ as $r \to 1$. Clearly $\alpha_r \in C^\infty(\mathbb{T})$.

Define
$$u_r(z) = \int_{\mathbb{T}} \log \left| \frac{z - re^{i\varphi}}{1 - re^{-i\varphi} z} \right| \alpha(e^{i\varphi})\, d\lambda(\varphi).$$
Then u_r is a subharmonic exhaustion function on $\mathbb{D}$ and by the Riesz Decomposition Theorem its Laplacian Δu_r is supported by $\mathbb{T}(r) = \{z = re^{i\phi}\}$ and is equal to

$\alpha(e^{i\varphi})\, d\lambda(\varphi)$. Hence

$$\int_{\mathbb{D}} \Delta u_r(z) = \int_{\mathbb{T}} \alpha(e^{i\varphi})\, d\lambda(\varphi).$$

The weight of u_r is equal to

$$\int_{\mathbb{T}} P(re^{i\varphi}, e^{i\theta})\alpha(e^{i\varphi})\, d\lambda(\varphi) = \alpha_r(e^{i\theta}).$$

Hence any $\alpha \in C(\mathbb{T})$ can be uniformly approximated by a function β_u such that $\beta_u\, d\lambda = \mu_u$ and $u \in \mathcal{E}$.

If α is any lower semicontinuous function satisfying (3.4) and such that $\alpha \geq c > 0$ on $\mathbb{T}$, then α is the pointwise limit of an increasing sequence of continuous functions α_j such that $\alpha_j \geq c/2 > 0$ on $\mathbb{T}$. Replacing α_j with the functions $\alpha_j - 2^{-j}$ we may assume that the function $\beta_j = \alpha_j - \alpha_{j-1} \geq 2^{-j-1}$ on $\mathbb{T}$. (Here we set $\alpha_0 = 0$.) By the argument above we can approximate the functions β_j by continuous functions γ_j such that $\gamma_j \geq 2^{-j-2}$ on $\mathbb{T}$, $\gamma_j\, d\lambda = \mu_{u_j}$ for some $u_j \in \mathcal{E}$ and

$$\sum_{j=1}^{\infty} \gamma_j = \alpha.$$

Let $v_j = \max\{u_j, -2^{-j}\}$. Since for a fixed j the weak-$*$ limits of $\mu_{u_j,r}$ and $\mu_{v_j,r}$ as $r \to 0^-$ coincide we see that $\alpha_{v_j} = \alpha_{u_j} = \gamma_j$. If $v = \sum v_j$ then v is a continuous exhaustion of $\mathbb{D}$ such that $\lim_{|z|\to 1} v(z) = 0$. Moreover,

$$\int_{\mathbb{D}} \Delta v = \sum_{j=1}^{\infty} \int_{\mathbb{D}} \Delta v_j = \sum_{j=1}^{\infty} \int_{\mathbb{T}} \gamma_j = \int_{\mathbb{T}} \alpha = 1.$$

Hence $v \in \mathcal{E}_1$.

Now

$$\int_{\mathbb{D}} P(z, e^{i\theta})\Delta v(z) = \sum_{j=1}^{\infty} \int_{\mathbb{D}} P(z, e^{i\theta})\Delta v_j(z) = \sum_{j=1}^{\infty} \gamma_j(e^{i\theta}) = \alpha(e^{i\theta}).$$

Thus $\mu_v = \alpha\, d\lambda$.

The converse statements has been established in Section 3.2. $\square$

Theorem 3.20 gives complete characterization of the Poletsky–Stessin Hardy spaces as weighted spaces.

Chapter 4

Applications

4.1 Duality

Let α be a non-negative measurable function on $\mathbb{T}$ such that $\log \alpha \in L^1(\mathbb{T})$. Let $a(z)$ be a holomorphic function such that $|a(e^{i\theta})| = \alpha(e^{i\theta})$ a.e. on $[0, 2\pi]$ and a never takes the zero value. Such a function does exist and belongs to H^1 because $\log \alpha$ is integrable on $\mathbb{T}$ so we can take a harmonic function

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \log \alpha(e^{i\theta}) P(z, e^{i\theta})\, d\theta,$$

add a conjugate function g and write $a(z) = e^{h(z)+ig(z)}$.

In [2] for $f \in L^p_\alpha(\mathbb{T})$ the operator $A_p f = a^{1/p} f$ was introduced. Then

$$\|A_p f\|_p^p = \int_0^{2\pi} |f(e^{i\theta})|^p \alpha(e^{i\theta})\, d\lambda = \|f\|_{\alpha,p}^p.$$

Thus A_p is an isometrical imbedding of $L^p_\alpha(\mathbb{T})$ into $L^p(\mathbb{T})$.

We add another requirement on the weight α asking that $\alpha \geq c > 0$ on $\mathbb{T}$ for some c. Clearly, $A_p^{-1} f = \alpha^{-1/p} f$ is also an isometry and the inverse of A_p. Hence A_p is an isometric isomorphism of $L^p_\alpha(\mathbb{T})$ onto $L^p(\mathbb{T})$. Moreover, A_p maps H^p_α isometrically onto H^p.

If $\phi \in L^p_\alpha(\mathbb{T})$ then $\operatorname{dist}(\phi, H^p_\alpha) = \operatorname{dist}(A_p\phi, H^p)$. By the classical result (see [14]) for $p > 1$

$$\operatorname{dist}(A_p\phi, H^p) = \left| \sup_{g\in H^q, \|g\|_{H^q}=1} \int_0^{2\pi} a^{1/p}(e^{i\theta})\phi(e^{i\theta})e^{i\theta} g(e^{i\theta})\, d\lambda \right|. \tag{4.1}$$

Since the H^q_α-norm of $\alpha^{-1/q} g(z)$ coincides with the H^q-norm of g we can get the following duality result:

Theorem 4.1. *If $\phi \in L^p_\alpha(\mathbb{T})$ then*

$$\operatorname{dist}(\phi, H^p_\alpha) = \left| \sup_{g\in H^q_\alpha, \|g\|_{H^q_\alpha}=1} \int_{\mathbb{T}} \phi(e^{i\theta})a(e^{i\theta})e^{i\theta} g(e^{i\theta})\, d\lambda \right|.$$

Among the advantages of these spaces compared to spaces studied in [2] we can list the following. First of all, one does not need the existence of boundary values or the notion of Smirnov class to introduce these spaces. This is especially attractive for the theory of functions in several variables on non-smooth domains.

Another advantage is the existence of Carleson measures. Given a weight α a measure ν on the unit disk $\mathbb{D}$ is called α-Carleson with the constant $C(\alpha)$ if

$$\int_{\mathbb{D}} |f|^p\, d\nu \leq C(\alpha) \int_{\mathbb{T}} |f|^p \alpha\, d\lambda$$

for all $p > 1$ and all $f \in H^p_\alpha$. If $\alpha \equiv 1$ then such measure are called Carleson measures. In [15] one can find the characterisation of α-Carleson measures for α satisfying Muckenhoupt's conditions similar to the classical characterisation of Carleson measures by L. Carleson. In the case of Poletsky–Stessin Hardy spaces it follows immediately from

$$\int_{\mathbb{T}} |f|^p \, d\mu_u = \int_{\mathbb{D}} |f|^p \, \Delta u - \int_{\mathbb{D}} u \, \Delta |f|^p \tag{4.2}$$

that the measure Δu is α_u-Carleson with the constant $C(\alpha_u) = 1$. By Theorem 3.20 we see that α-Carleson measures with constant 1 exist for all lower semicontinuous weights.

Thirdly, the formula (4.2) helps to obtain additional information. For example, one can get integrability of derivative. Since $\Delta|f|^p = \frac{p^2}{4}|f|^{p-2}|f'|^2$ for all $f \in H_u^p$, $p \geq 1$, we have the inequality

$$\int_{\mathbb{T}} |f|^p \, d\mu_u \geq \frac{p^2}{4} \int_{\mathbb{D}} |u||f|^{p-2}|f'|^2 \, dx \, dy.$$

4.2 From H_u^p to H^∞

Let $u_1, \dots, u_k$ be exhaustion functions from $\mathcal{E}_1$ and let $u = (u_1, \dots, u_k)$. We say that $u \in \mathcal{E}_1^k$. Let H_u^p to be the direct product $H_{u_1}^p \times \dots \times H_{u_k}^p$ with the norm

$$\|(f_1, \dots, f_k)\|_{H_u^p} = \sum_{j=1}^{k} \|f_j\|_{H_{u_j}^p}.$$

We will use the notation $(H^p)^k$ and $\|f\|_p$ when $\alpha_{u_1} = \dots = \alpha_{u_k} = 1$. As in Section 3.5 we denote by $B_{u,p}(r)$ the closed ball of radius r centered at the origin of H_u^p.

The norm on $(H^\infty)^k$ will be defined as

$$\|f\|_\infty = \sum_{j=1}^{k} \|f_j\|_\infty$$

and $B_\infty(r)$ is the closed ball of radius r centered at the origin of $(H^\infty)^k$. Then $B_\infty(r) \subset B_{u,p}(r)$.

Theorem 4.2. *Let $A \subset (H^p)^k$, $p > 1$, be a closed convex set. Then $A \cap B_\infty(1) \neq \emptyset$ if and only if $A \cap B_{u,p}(1) \neq \emptyset$ for all exhaustion vector-functions $u = (u_1, \dots, u_k) \in \mathcal{E}_1^k$.*

Proof. Let us take $0 < \varepsilon < 1$ and suppose that $A \cap B_\infty(r_0) = \emptyset$ for $r_0 = (1-\varepsilon)^{-1}$. By the Hahn–Banach theorem there exists $g = (g_1, \dots, g_k) \in (L^q(\mathbb{T}))^k$ such that

$$\sum_{j=1}^k \mathbf{Re} \int_{\mathbb{T}} f_j g_j \, d\lambda \geq 1$$

for all $f \in A$ and

$$\sum_{j=1}^k \mathbf{Re} \int_{\mathbb{T}} f_j g_j \, d\lambda \leq 1$$

for all $f \in B_\infty(r_0)$. Multiplying f_j by appropriate constants a_j with $|a_j| = 1$ we see that

$$\sum_{j=1}^k \left| \int_{\mathbb{T}} f_j g_j \, d\lambda \right| \leq r_0^{-1} = 1 - \varepsilon$$

for all $f \in B_\infty(1)$.

Let $\tilde{g}_j(z) = g_j(z)/z$. Then $\tilde{g}_j \in L^q(\mathbb{T}) \subset L^1(\mathbb{T})$ for all j. By a duality result (see [14, VII.2]) there exist $h_j \in H^1$ and $p_j \in H^\infty$ such that $\|p_j\|_\infty = 1$, $p_j(0) = 0$ and

$$(\tilde{g}_j - h_j) p_j = |\tilde{g}_j - h_j|$$

almost everywhere.

We take $f = (f_1, \dots, f_k) \in (H^\infty)^k$ such that $f_i \equiv 0$ when $i \neq j$ and $f_j(z) = p_j(z)/z$. Clearly, $f \in B_\infty(1)$. Therefore,

$$1 - \varepsilon \geq \left| \int_{\mathbb{T}} f_j g_j \, d\lambda \right| = \left| \int_{\mathbb{T}} (\tilde{g}_j - h_j) p_j \, d\lambda \right| = \int_{\mathbb{T}} |\tilde{g}_j - h_j| \, d\lambda.$$

There is $\tilde{h}_j \in H^q$ so that $\|h_j - \tilde{h}_j\|_1 \le \varepsilon/2$. Let $\phi_j = |\tilde{g}_j - \tilde{h}_j|$. Then

$$\int_{\mathbb{T}} \phi_j \, d\lambda \le \int_{\mathbb{T}} \left(|\tilde{g}_j - h_j| + |h_j - \tilde{h}_j| \right) \, d\lambda \le 1 - \varepsilon/2.$$

And for $f \in A$,

$$\sum_{j=1}^{k} \int_{\mathbb{T}} \phi_j |f_j| \, d\lambda = \sum_{j=1}^{k} \int_{\mathbb{T}} |(g_j - z\tilde{h}_j) f_j| \, d\lambda \ge \sum_{j=1}^{k} \left| \int_{\mathbb{T}} (g_j - z\tilde{h}_j) f_j \, d\lambda \right| \ge 1.$$

Let $\tilde{\phi}_j = \max\{\phi_j, \varepsilon/4\}$. Then $\|\tilde{\phi}_j\|_1 \le \|\phi_j + \varepsilon/4\|_1 \le 1 - \varepsilon/4$. Now for $f \in A$,

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \tilde{\phi}_j \, d\lambda \ge \sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \phi_j \, d\lambda \ge 1.$$

For any $\delta > 0$ and $1 \le j \le k$ there exists $\psi_j \in C(\mathbb{T})$ such that $\psi_j \ge \varepsilon/8$, $\|\psi_j\|_1 = \|\tilde{\phi}_j\|_1$ and $\|\psi_j - \tilde{\phi}_j\|_q < \delta$.

For $f \in A$,

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \psi_j \, d\lambda \ge \sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \tilde{\phi}_j \, d\lambda - \sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| |\psi_j - \tilde{\phi}_j| \, d\lambda \ge 1 - \delta \|f\|_p.$$

By Theorem 3.20 there are exhaustion functions u_j, $1 \le j \le k$, such that $\mu_{u_j} = a_j \psi_j$, where a_j is chosen so that $\|a_j \psi_j\|_1 = 1$. Let $u = (u_1, \ldots, u_k)$. Note that $a_j \ge (1 - \varepsilon/4)^{-1}$.

If $f \in B_{u,p}(1)$ then

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| a_j \psi_j \, d\lambda \le \sum_{j=1}^{k} \|f_j\|_{H^p_{u_j}} \|a_j \psi_j\|_1^{1/q} \le 1$$

and

$$\|f\|_p = \sum_{j=1}^{k} \left(\int_{\mathbb{T}} |f_j|^p \, d\lambda \right)^{1/p} \le \left(\frac{\varepsilon}{8} \right)^{-1/p} \sum_{j=1}^{k} \|f_j\|_{H^p_{u_j}} \le \left(\frac{\varepsilon}{8} \right)^{-1/p} = c.$$

Thus if $f \in A$ and $\|f\|_p > c$ then $f \notin B_{u,p}(1)$. On the other hand if $f \in A$ and $\|f\|_p \le c$, then

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| a_j \psi_j \, d\lambda \ge (1-\varepsilon/4)^{-1}(1-c\delta).$$

Taking $\delta > 0$ so small that $(1-\varepsilon/4)^{-1}(1-c\delta) > 1$ we see that $A \cap B_{u,p}(1) = \emptyset$. Hence $A \cap B_\infty(r_0) \ne \emptyset$ for all $r_0 > 1$.

Let $\{f_n\}$ be a sequence of functions such that $f_n \in A \cap B_\infty(1+1/n)$. We may assume that $\{f_n\}$ converges uniformly on compacta to a function $f \in B_\infty(1)$. This implies that $\{f_n\}$ converges to f weakly. Since any convex closed set is weakly closed we see that $f \in A$.

The second part is trivial. □

As the following corollary shows it is possible to use the theorem above when all functions u_j are equal although constants will change.

Corollary 4.3. *Let $A \subset (H^p)^k$, $p > 1$, be a closed convex set. Suppose $A \cap B_{\mathbf{u},p}(1) \ne \emptyset$ for all exhaustion vector-functions $\mathbf{u} = (u, \ldots, u) \in \mathcal{E}_1^k$. Then $A \cap B_\infty(k) \ne \emptyset$. Conversely, if $A \cap B_\infty(1) \ne \emptyset$ then $A \cap B_{\mathbf{u},p}(1) \ne \emptyset$.*

Proof. Let $v = (v_1, \ldots, v_k) \in \mathcal{E}_1^k$. Let

$$u = \frac{1}{k} \sum_{j=1}^{k} v_j.$$

Then $u \in \mathcal{E}_1$ and by the assumption of the corollary there is $f = (f_1, \ldots, f_k) \in A \cap B_{\mathbf{u},p}(1)$, where $\mathbf{u} = (u, \ldots, u)$. Note that $v_j \ge ku$. By Corollary 3.2 in [20]

$\|f_j\|_{v_j,p} \leq k\|f_j\|_{u,p}$, $1 \leq j \leq k$. Hence $f \in B_{v,p}(k)$ and $A \cap B_{v,p}(k) \neq \emptyset$. By Theorem 4.2, $A \cap B_\infty(k) \neq \emptyset$. □

4.3 Interpolation Theorem

A sequence $\{z_j\}_1^\infty \subset \mathbb{D}$ is δ-sparse for $\delta > 0$ if

$$\inf_k \prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \geq \delta$$

for all k.

A sequence $\{z_j\} \subset \mathbb{D}$ is called interpolating if for any sequence $s = \{s_j\} \in l^\infty$ there is a function $f \in H^\infty$ such that $f(z_j) = s_j$ for all j and $\|f\|_{H^\infty} \leq C\|s\|_\infty$ and the constant C does not depend on $\|s\|_\infty$.

The famous theorem of Carleson states

Theorem 4.4. *A sequence $\{z_j\} \subset \mathbb{D}$ is interpolating if and only if it is δ-sparse for some $\delta > 0$.*

Now we can present a shorter proof of Theorem 4.4 by using the result of Section 4.2. Theorem 3.2 in [13], which is a quick consequence of the general characterization of Carleson measures, states that if a sequence $\{z_j\} \subset \mathbb{D}$ is δ-sparse then the measure

$$\nu = \sum_{j=1}^\infty (1 - |z_j|^2)\delta_{z_j}$$

is Carleson with a constant C depending only on δ.

We take an integer $N > 1$ and denote by X_N the set of all functions $f \in H^2$ such that $f(z_j) = s_j$, $1 \le j \le N$. Clearly X_N is closed and convex.

Let

$$B(z) = \prod_{j=1}^{N} \frac{z - z_j}{1 - \bar{z}_j z} \text{ and } B_k(z) = \prod_{j=1, j\neq k}^{N} \frac{z - z_j}{1 - \bar{z}_j z}, \quad k = 1, \dots, N.$$

Then any function f in X_N has the form

$$\sum_{j=1}^{N} \frac{s_j}{B_j(z_j)} B_j(z) + B(z)h(z) = \left(\sum_{j=1}^{N} \frac{s_j}{B_j(z_j)} \frac{1 - \bar{z}_j z}{z - z_j} + h(z) \right) B(z),$$

where $h \in H^2$.

We set $C_j = s_j B_j^{-1}(z_j)$ and let

$$\phi(z) = \sum_{j=1}^{N} C_j \frac{1 - \bar{z}_j z}{z - z_j}.$$

Let $u \in \mathcal{E}_1$ and let $a = a_u$ be the function introduced in Section 4.1. Then for $g \in H^2$ with $\|g\|_{H^2} = 1$ we have

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathbb{T}} \phi(z) a^{1/2}(z) g(z))\, dz = \sum_{j=1}^{N} C_j (1 - |z_j|^2) g(z_j) a^{1/2}(z_j) \\
&= \frac{\|s\|_\infty}{\delta} \int_{\mathbb{D}} |g a^{1/2}|\, d\nu \le \frac{\|s\|_\infty}{\delta} \left(\int_{\mathbb{D}} |g|^2\, d\nu \right)^{1/2} \left(\int_{\mathbb{D}} |a|\, d\nu \right)^{1/2} \\
&\le \frac{C^2 \|s\|_\infty}{\delta} \|g\|_{H^2} \|a^{1/2}\|_{H^2} = \frac{C^2 \|s\|_\infty}{\delta} = C' \|s\|_\infty.
\end{aligned}$$

Hence by (4.1) $\operatorname{dist}(\phi, H_u^2) \le C'\|s\|_\infty$ and this means that $X_N \cap B_{u,2}(C'\|s\|_\infty) \neq \emptyset$. Thus by Theorem 4.2 there is $f_N \in X_N \cap B_\infty(C'\|s\|_\infty)$. Since C' does not depend on N there is $f \in B_\infty(C'\|s\|_\infty)$ interpolating s.

The proof of necessity is quite elementary and can be found in [10].

4.4 Corona Theorem

The space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on the unit disk is a Banach algebra over $\mathbb{C}$. For every $z \in \mathbb{D}$, the point evaluation

$$f \mapsto f(z)$$

is a multiplicative homomorphism of $H^\infty(\mathbb{D})$ onto $\mathbb{C}$. Since $\mathbb{C}$ is a field the kernel of any homomorphism is a maximal ideal in $H^\infty(\mathbb{D})$. So the kernel of the point evaluation at z, that is

$$\{f \in H^\infty(\mathbb{D}) : f(z) = 0\}$$

is maximal ideal in $H^\infty(\mathbb{D})$. The set of multiplicative homomorphisms of $H^\infty(\mathbb{D})$ onto $\mathbb{C}$ is in one-to-one correspondence with the maximal ideal space $\mathfrak{M}$ of $H^\infty(\mathbb{D})$. Hence by identifying z with

$$\{f \in H^\infty(\mathbb{D}) : f(z) = 0\}$$

it follows that $\mathbb{D} \subset \mathfrak{M}$. The corona theorem which states that $\mathbb{D}$ is weak-$*$ dense in $\mathfrak{M}$ was conjectured by Kakutani in 1941 and first proved by Carleson in 1962. In 1979 T. Wolff gave two equivalent formulations of the corona theorem:

1. If $m \in \mathfrak{M}$ there is a net $\{z_\alpha\}$ in $\mathbb{D}$ with $z_\alpha \to m$ in $\mathfrak{M}$.

2. If $f_1, \cdots, f_n \in H^\infty(\mathbb{D})$ and

$$\sup_k |f_k(z)| \geq \delta > 0$$

for all $z \in \mathbb{D}$, there exist functions $g_1, \cdots, g_n \in H^\infty(\mathbb{D})$ such that

$$f_1 g_1 + \cdots + f_n g_n \equiv 1$$

on $\mathbb{D}$.

We use the result of Section 4.2 to demonstrate a shortcut to the proof of the corona problem, as formulated in statement (2) above.

Theorem 4.5. *If the functions $f_1, \ldots, f_n$ are in the unit ball of H^∞ and*

$$\sum_{j=1}^{n} |f_j|^2 \geq \delta > 0,$$

then there are functions $g_1, \ldots, g_n$ in H^∞ such that

$$\sum_{j=1}^{n} f_j g_j = 1 \tag{4.3}$$

and $\|g_j\| \leq C$, where C depends only on δ.

We will discuss only the case when $n = 2$. It suffices to prove this theorem for functions f_j that can be continuously extended to $\overline{\mathbb{D}}$ and have finitely many zeros in $\overline{\mathbb{D}}$. In this case one can easily find functions ϕ_1 and ϕ_2 smooth up to the boundary such that

$$f_1\phi_1 + f_2\phi_2 = 1.$$

To make them holomorphic we look for a function v such that

$$\bar{\partial}(\phi_1 + f_2 v) = \bar{\partial}\phi_1 + f_2\bar{\partial}v = 0$$

and

$$\bar{\partial}(\phi_2 - f_1 v) = \bar{\partial}\phi_2 - f_1\bar{\partial}v = 0.$$

Since $f_1\bar{\partial}\phi_1 + f_2\bar{\partial}\phi_2 = 0$ we see that

$$\bar{\partial}v = f_1^{-1}\bar{\partial}\phi_2 = -f_2^{-1}\bar{\partial}\phi_1 =: \psi.$$

The following lemma can be found in [10].

Lemma 4.6. *There are solutions ϕ_1 and ϕ_2 to (4.3) continuous up to the boundary such that the measure $\nu = |\psi|\, dzd\bar{z}$ is Carleson with constant C depending only on δ and $|\phi_1| + |\phi_2| \le K(\delta)$.*

Let

$$\Psi(z) = \int_{\mathbb{D}} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta d\bar{\zeta}.$$

Then $\bar{\partial}\Psi = \psi$ and for any $u \in \mathcal{E}_1$

$$\left|\int_{\mathbb{T}} \Psi(z) a_u(z) g(z)\, dz\right|^2 = \left|\int_{\mathbb{D}} \psi(\zeta) a_u(\zeta) g(\zeta)\, d\zeta d\bar{\zeta}\right|^2$$
$$\le \int_{\mathbb{D}} |a_u(\zeta)|\psi(\zeta)\, d\zeta d\bar{\zeta} \int_{\mathbb{D}} |\psi(\zeta)||a_u(\zeta) g^2(\zeta)|\, d\zeta d\bar{\zeta} \le C^2 \|g\|_{u,2}^2.$$

Thus by Theorem 4.1 $\operatorname{dist}(\Psi, H_u^2) \le C$. Hence there is $v = \Psi + h$ such that $h \in H_u^2$ and $\|v\|_{H_u^2} \le C$. Therefore the function $h_1 = \phi_1 + f_2 v$ is holomorphic, lies in H_u^2 and $\|h_1\|_{u,2} \le K(\delta) + C = R$. The same estimate holds for the function $h_2 = \phi_2 - f_1 v$.

Thus if $A \subset (H^2)^2$ is the set of all solutions (g_1, g_2) to (4.3), then $A \cap B_{u,2}(R) \ne \emptyset$ for all pairs (u, u), where $u \in \mathcal{E}_1$. Since the set A is convex and closed, by Corollary 4.3 $A \cap B_\infty(2R) \ne \emptyset$. This ends the proof.

Chapter 5

Hardy Spaces on the Polydisk

5.1 Hardy Spaces and Poisson Integral Formula

An n-harmonic function u on $\mathbb{D}^n$ is a function which is harmonic in each variable separately. Similarly, an n-subharmonic function u on $\mathbb{D}^n$ is a function which is subharmonic in each variable separately.

We will use the following notations:

$$z = (z_1, \cdots, z_n)$$

$$\zeta = (\zeta_1, \cdots, \zeta_n)$$

$$P(z, \zeta) = P(z_1, \zeta_1) \cdots P(z_n, \zeta_n)$$

where $P(z, \zeta)$ is the Poisson kernel and

$$P(z_j, \zeta_j) = \mathbf{Re}\left(\frac{\zeta_j + z_j}{\zeta_j - z_j}\right) = \frac{1 - |z_j|^2}{|\zeta_j - z_j|^2}, \quad j = 1, \cdots, n.$$

Denote by $h^p(\mathbb{D}^n)$ the space of all n-harmonic functions satisfying

$$\sup_{0\le r<1}\int_{\mathbb{T}^n} |u_r(\zeta)|^p\, dm(\zeta) < \infty \tag{5.1}$$

where dm is the normalized Lebesgue measure on $\mathbb{T}^n$ and $u_r(\zeta) = u(r\zeta_1, \cdots, r\zeta_n)$. The p -th root of (5.1) defines a norm on $h^p(\mathbb{D}^n)$ when $p \ge 1$. With this norm $h^p(\mathbb{D}^n)$ is Banach.

The following theorem [25, Theorem 2.1.2] shows that any n-harmonic functions in $\mathbb{D}^n$ continuous up to the boundary can be restored by the Poisson integral of its boundary values on the distinguished boundary.

Theorem 5.1. *If u is continuous on $\overline{\mathbb{D}^n}$ and n-harmonic in $\mathbb{D}^n$ then*

$$u(z) = \int_{\mathbb{T}^n} P(z,\zeta)u(\zeta)\, dm(\zeta)$$

for $z \in \mathbb{D}^n$.

Theorem 5.2. *Let $u \in h^p(\mathbb{D}^n)$, $p > 1$. Then there exists a function $f \in L^p(\mathbb{T}^n)$ such that*

$$u(z) = \int_{\mathbb{T}^n} P(z,\zeta)f(\zeta)\, dm(\zeta).$$

Proof. The equation (5.1) implies that there is a weakly convergent sequence u_{r_j}. Hence for $g \in L^q(\mathbb{T}^n)$

$$g \mapsto \lim_{j\to\infty}\int_{\mathbb{T}^n} g(\zeta)u_{r_j}(\zeta)\, dm(\zeta)$$

is a linear functional on $L^q(\mathbb{T}^n)$. By Riesz theorem there exists an $f \in L^p(\mathbb{T}^n)$ such that

$$\lim_{j\to\infty}\int_{\mathbb{T}^n} g(\zeta)u_{r_j}(\zeta)\, dm(\zeta) = \int_{\mathbb{T}^n} g(\zeta)f(\zeta)\, dm(\zeta).$$

Now take $g(\zeta) = P(z,\zeta)$. Then

$$u(z) = \lim_{j\to\infty} u_{r_j}(z) = \lim_{j\to\infty} \int_{\mathbb{T}^n} P(z,\zeta) u_{r_j}(\zeta)\, dm = \int_{\mathbb{T}^n} P(z,\zeta) f(\zeta)\, dm(\zeta).$$

The second equality above follows from Theorem 5.1. □

What makes the above proof work is the duality of L^p spaces. Since L^∞ is the dual of L^1, the same result holds with the same proof for $p = \infty$. Of course we have to change the statement accordingly. But since L^1 is not dual of anything, we don't have the same result for $p = 1$. Instead, since the space of finite signed measures on $\mathbb{T}^n$ is dual of the space of continuous functions $C(\mathbb{T}^n)$ we have the following result from [25, Theorem 2.1.3, (e)]

Theorem 5.3. *Let $u \in h^p(\mathbb{D}^n)$, $p = 1$. Then there exists a finite signed measure μ on $\mathbb{T}^n$ with*

$$u(z) = \int_{\mathbb{T}^n} P(z,\zeta)\, d\mu(\zeta).$$

By Theorem 5.2, for $p > 1$, the function $u \in h^p(\mathbb{D}^n)$ is the Poisson integral of a function $f \in L^p(\mathbb{T}^n)$. Is there any other connection between u and f? We know, when $n = 1$, f is the boundary value function of u and when $n > 1$ the following theorem ([25, Theorem 2.3.1]) answers this question.

Theorem 5.4. *If $f \in L^1(\mathbb{T}^n)$, if σ is a measure on $\mathbb{T}^n$ which is singular with respect to dm, and if $u = P[f + d\sigma]$, then $u^*(\zeta) = f(\zeta)$ for almost every $\zeta \in \mathbb{T}^n$, where $u^*(\zeta) = \lim_{r\to 1} u(r\zeta)$.*

Theorems 5.2 and 5.4 together imply that every function $u \in h^p(\mathbb{D}^n)$, $p > 1$, has radial limit $u^* \in L^p(\mathbb{T}^n)$ and the function u can be restored by the Poisson integral of u^*. However, for $p = 1$ we just saw in Theorem 5.3 that $u(z) = P[d\mu](z)$. By the Lebesgue decomposition theorem

$$d\mu = f\,dm + d\sigma$$

where σ is singular with respect to m and $f \in L^1(\mathbb{T}^n)$. Again by Theorem 5.4, $u^*(\zeta) = f(\zeta)$ but u can not be restored by the Poisson integral of its boundary value function unless, of course, $P[d\sigma] = 0$.

Also in [25] it has been proved that if $f \in L^p(\mathbb{T}^n), 1 \le p < \infty$, and $u = P[f]$ then u_r converges to f in the L^p-norm as $r \to 1$, i.e. $\lim_{r\to 1} \|u_r - f\|_{L^p} = 0$. But when $p = 1$ we have the weak-$*$ convergence.

Theorem 5.5. *Let $f(z) = P[d\mu](z)$ with μ a finite signed measure on $\mathbb{T}^n$. Then $f_r\,dm \to d\mu$ weak-$*$ as $r \to 1$.*

Proof. Let $\varphi \in C(\mathbb{T}^n)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{T}^n} \varphi(\zeta) f_r(\zeta)\,dm(\zeta) - \int_{\mathbb{T}^n} \varphi(\zeta)\,d\mu(\zeta)\right| \\
=&\left|\int_{\mathbb{T}^n} \varphi(\zeta)\left(\int_{\mathbb{T}^n} P(r\zeta,\eta)\,d\mu(\eta)\right) dm(\zeta) - \int_{\mathbb{T}^n} \varphi(\eta)\,d\mu(\eta)\right| \\
=&\left|\int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\eta,\zeta)\varphi(\zeta)\,dm(\zeta)\right) d\mu(\eta) - \int_{\mathbb{T}^n} \varphi(\eta)\,d\mu(\eta)\right| \\
=&\left|\int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\eta,\zeta)\varphi(\zeta)\,dm(\zeta) - \varphi(\eta)\right) d\mu(\eta)\right| \\
\to& 0
\end{aligned}$$

because the inner integral goes to zero uniformly on η. Hence $f_r\, dm \to d\mu$ weak-$*$ as $r \to 1$. □

We define $H^p(\mathbb{D}^n)$, $0 < p < \infty$, to be the class of all holomorphic functions $f \in \mathbb{D}^n$ for which

$$\sup_{0 \le r < 1} \int_{\mathbb{T}^n} |f_r(\zeta)|^p \, dm < \infty$$

and $H^\infty(\mathbb{D}^n)$ is the space of all bounded holomorphic functions in $\mathbb{D}^n$.

Since $|f|^p$ is n-subharmonic, sup in the definition can be replaced by lim as $r \to 1$.

It is known that if $f \in H^p(\mathbb{D}^n)$, $0 < p < \infty$, then f has a non-tangential limit at almost all points of $\mathbb{T}^n$ [32, Ch. XVII, Theorem 4.8]. We denote this limit by f^* as in [25] and call it a boundary value function for f. Moreover, we have the following results from Rudin (see [25, Theorem 3.4.2 and 3.4.3]).

Theorem 5.6. *If $f \in H^p(\mathbb{D}^n), 0 < p < \infty$, then $f^* \in L^p(\mathbb{T}^n)$ and*

1. $\lim_{r\to 1} \int_{\mathbb{T}^n} |f_r|^p \, dm = \int_{\mathbb{T}^n} |f^*|^p \, dm$

2. $\lim_{r\to 1} \int_{\mathbb{T}^n} |f_r - f^*|^p \, dm = 0.$

When $p \ge 1$ the function in $H^p(\mathbb{D}^n)$ can be represented by the Poisson integral of its boundary value function.

Theorem 5.7. *If $f \in H^1(\mathbb{D}^n)$, then*

$$f(z) = \int_{\mathbb{T}^n} P(z, \zeta) f^*(\zeta) \, dm.$$

(The case $n = 1$ can be found in [24, Theorem 17.11].)

Proof. Since for every $z \in \mathbb{D}^n$, $P(z,\zeta)$ is bounded on $\mathbb{T}^n$, by Theorem 5.6 (2)

$$\begin{aligned}
&\left| \int_{\mathbb{T}^n} P(z,\zeta) f_r(\zeta)\, dm(\zeta) - \int_{\mathbb{T}^n} P(z,\zeta) f^*(\zeta)\, dm(\zeta) \right| \\
\leq &\int_{\mathbb{T}^n} P(z,\zeta) |f_r(\zeta) - f^*(\zeta)|\, dm(\zeta) \\
\rightarrow &0.
\end{aligned}$$

Now by [25, Theorem 2.1.2]

$$\begin{aligned}
f(z) &= \lim_{r\to 1} f_r(z) \\
&= \lim_{r\to 1} \int_{\mathbb{T}^n} P(z,\zeta) f_r(\zeta)\, dm(\zeta) \\
&= \int_{\mathbb{T}^n} P(z,\zeta) f^*(\zeta)\, dm(\zeta).
\end{aligned}$$

□

5.2 The F. and M. Riesz Theorem

Now we generalize the F. and M. Riesz theorem.

Theorem 5.8. *Let μ be a complex Borel measure on $\mathbb{T}^n$. If*

$$\int_{\mathbb{T}^n} e^{i(k\theta)}\, d\mu(\theta) = 0$$

for $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ with at least one $k_j, j = 1, 2, \cdots, n$ positive, where $(k\theta) = k_1\theta_1 + \cdots + k_n\theta_n$, then μ is absolutely continuous with respect to dm.

(When $n = 1$ see [24, Theorem 17.13].)

Proof. Define $f(z) = P[d\mu](z)$. Then, with the notations

$$\begin{aligned} z &= (z_1, \cdots, z_n) \text{ with } z_j = r_j e^{i\theta_j}, j = 1, \cdots, n \\ r^{|k|} &= r_1^{|k_1|} \cdots r_n^{|k_n|} \\ (k \cdot \theta) &= k_1\theta_1 + \cdots + k_n\theta_n \\ (k \cdot t) &= k_1 t_1 + \cdots + k_n t_n \end{aligned}$$

and using the series representation for the Poisson kernel, we get

$$\begin{aligned} f(z) &= \int_{\mathbb{T}^n} P(z, e^{it})\, d\mu(t) \\ &= \int_{\mathbb{T}^n} \left(\sum_{k \in \mathbb{Z}^n} r^{|k|} e^{i(k\cdot\theta)} e^{-i(k\cdot t)} \right) d\mu(t) \\ &= \sum_{k \in \mathbb{Z}^n} \left(\int_{\mathbb{T}^n} e^{-i(k\cdot t)} d\mu(t) \right) r^{|k|} e^{i(k\cdot\theta)} \\ &= \sum_{k \in \mathbb{Z}^n_+} c_k z^k \end{aligned}$$

where $c_k = \int_{\mathbb{T}^n} e^{-i(k\cdot t)}\, d\mu(t)$ and $z_k = r^{|k|} e^{i(k\cdot\theta)}$. Notice that all other integrals in the above sum vanish by the hypothesis. Thus $f(z)$ is holomorphic.

For $0 \le r < 1$,

$$\begin{aligned} \int_{\mathbb{T}^n} |f_r(\zeta)|\, dm(\zeta) &= \int_{\mathbb{T}^n} \left| \int_{\mathbb{T}^n} P(r\zeta, \eta)\, d\mu(\eta) \right| dm(\zeta) \\ &\le \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\zeta, \eta)\, d|\mu|(\eta) \right) dm(\zeta) \\ &= \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\zeta, \eta)\, dm(\zeta) \right) d|\mu|(\eta) \\ &= \|\mu\|. \end{aligned}$$

Thus $f \in H^1(\mathbb{D}^n)$ and by Theorem 5.7, $f(z) = P[f^*](z)$, where f^* is the boundary value function for f. Now the uniqueness of the Poisson integral representation shows that

$$d\mu = f^* dm$$

and the proof is completed. □

5.3 Boundary Values

Do the boundary values of functions in $H^p(\mathbb{D}^n)$ exist on the non-distinguished boundary? Now we look into this question.

Let $\{j_1, \cdots, j_k\}$ and $\{i_1, \cdots, i_l\}$ be disjoint sets of indices such that their union is $\{1, \cdots, n\}$ where $j_1 < j_2 < \cdots < j_k$ and $i_1 < i_2 < \cdots < i_l$. Define the sections of $\mathbb{D}^n$ as follows

$$\mathbb{D}^n_{z_{j_1},\cdots,z_{j_k}} = \{(z_1, \cdots, z_n) \in \mathbb{D}^n : z_{j_1}, \cdots, z_{j_k} \text{ are fixed}\}$$

and define $f_{z_{j_1},\cdots,z_{j_k}} = f|_{\mathbb{D}^n_{z_{j_1},\cdots,z_{j_k}}}$. We will write $f_{z_{j_1},\cdots,z_{j_k}}(z_{i_1}, \cdots, z_{i_l})$ instead of $f_{z_{j_1},\cdots,z_{j_k}}(z_1, \cdots, z_n)$.

We will see below that for $f \in H^p(\mathbb{D}^n), 1 \leq p < \infty$, the non-tangential limit of $f_{z_{j_1},\cdots,z_{j_k}}$ exists at almost all points of the distinguished boundary of the section $\mathbb{D}^n_{z_{j_1},\cdots,z_{j_k}}$ which is $\mathbb{T}^l$ and the function $f_{z_{j_1},\cdots,z_{j_k}}$ can be restored by the Poisson integral of this limit.

Theorem 5.9. *Let $f \in H^p(\mathbb{D}^n)$, $1 \leq p < \infty$. Then $f_{z_{j_1},\cdots,z_{j_k}} \in H^p(\mathbb{D}^l)$.*

Proof. Without loss of generality we suppose that $\{j_1, \cdots, j_k\} = \{1, \cdots, k\}$. Let's use the following notations for the Poisson kernels

$$P_j(\zeta_j) = \begin{cases} P(z_j, \zeta_j) & j = 1, \cdots, k \\ \\ P(r\xi_j, \zeta_j) & j = k+1, \cdots, n \end{cases}$$

where $|\xi_j| = 1$. Then, for $0 < r < 1$, by Theorem 5.7

$$f_{z_1,\cdots,z_k}(r\xi_{k+1}, \cdots, r\xi_n) = \int_{\mathbb{T}^n} P_1(\zeta_1) \cdots P_n(\zeta_n) f^*(\zeta_1, \cdots, \zeta_n)\, dm_n.$$

For $p > 1$, by Hölder and Fubini

$$\begin{aligned}
&\int_{\mathbb{T}^l} |f_{z_1,\cdots,z_k}(r\xi_{k+1}, \cdots, r\xi_n)|^p dm_l \\
=&\int_{\mathbb{T}^l} \left| \int_{\mathbb{T}^n} P_1(\zeta_1) \cdots P_n(\zeta_n) f^*(\zeta_1, \cdots, \zeta_n)\, dm_n \right|^p dm_l \\
\leq&\int_{\mathbb{T}^l} \left(\int_{\mathbb{T}^n} P_1(\zeta_1) \cdots P_n(\zeta_n) |f^*(\zeta_1, \cdots, \zeta_n)|^p\, dm_n \right) dm_l \\
=&\int_{\mathbb{T}^n} P_1(\zeta_1) \cdots P_k(\zeta_k) |f^*(\zeta_1, \cdots, \zeta_n)|^p \left(\int_{\mathbb{T}^l} P_{k+1}(\zeta_{k+1}) \cdots P_n(\zeta_n)\, dm_l \right) dm_n \\
\leq&\frac{2^k}{(1-|z_1|)\cdots(1-|z_k|)} \int_{\mathbb{T}^n} |f^*(\zeta_1, \cdots, \zeta_n)|^p dm_n.
\end{aligned}$$

The same estimate holds for $p = 1$ also. The last quantity above is independent of r and is finite by Theorem 5.6. Thus the theorem is proved. □

The following corollary is immediate.

Corollary 5.10. *If $f \in H^p(\mathbb{D}^n), 1 \leq p < \infty$, then the non-tangential limit $f^*_{z_{j_1},\cdots,z_{j_k}}$ of the function $f_{z_{j_1},\cdots,z_{j_k}}$ exists almost everywhere on $\mathbb{T}^l$ and belongs to $L^p(\mathbb{T}^l)$.*

The following theorems are the direct consequences of Theorems 5.6 and 5.7.

Theorem 5.11. *If $1 \le p < \infty$ and $f \in H^p(\mathbb{D}^n)$, then*

1. $\lim_{r\to 1} \int_{\mathbb{T}^l} |(f_{z_{j_1},\cdots,z_{j_k}})_r|^p \, dm_l = \int_{\mathbb{T}^l} |f^*_{z_{j_1},\cdots,z_{j_k}}|^p \, dm_l$

2. $\lim_{r\to 1} \int_{\mathbb{T}^l} |(f_{z_{j_1},\cdots,z_{j_k}})_r - f^*_{z_{j_1},\cdots,z_{j_k}}|^p \, dm_l = 0$

where $(f_{z_{j_1},\cdots,z_{j_k}})_r(\zeta_{i_1},\cdots,\zeta_{i_l}) = f_{z_{j_1},\cdots,z_{j_k}}(r\zeta_{i_1},\cdots,r\zeta_{i_l})$.

Theorem 5.12. *If $f \in H^1(\mathbb{D}^n)$, then*

$$f_{z_{j_1},\cdots,z_{j_k}}(z_{i_1},\cdots,z_{i_l}) = \int_{\mathbb{T}^l} P(z_{i_1},\zeta_{i_1})\cdots P(z_{i_l},\zeta_{i_l}) f^*_{z_{j_1},\cdots,z_{j_k}}(\zeta_{i_1},\cdots,\zeta_{i_l}) \, dm_l.$$

Theorem 5.13. *Let f be a holomorphic function in $\mathbb{D}^n$. If $1 \le p < \infty$ and*

$$\sup_{\substack{(z_{j_1},\cdots,z_{j_k}) \\ |z_{j_1}|=\cdots=|z_{j_k}|}} \|f_{z_{j_1},\cdots,z_{j_k}}\|_{H^p(\mathbb{D}^{n-k})} = M < \infty,$$

for some integer k, $1 \le k \le n$, then $f \in H^p(\mathbb{D}^n)$.

Proof. For simplicity we take $\{j_1,\cdots,j_k\} = \{1,\cdots,k\}$. Now for $0 \le r < 1$,

$$\begin{aligned}
&\int_{\mathbb{T}^n} |f(r\zeta_1,\cdots,r\zeta_n)|^p \, dm_n \\
=&\int_{\mathbb{T}^k} \left(\int_{\mathbb{T}^{n-k}} |f(r\zeta_1,\cdots,r\zeta_n)|^p \, dm_{n-k} \right) dm_k \\
\le&\int_{\mathbb{T}^k} \left(\sup_{0\le t<1} \int_{\mathbb{T}^{n-k}} |f(r\zeta_1,\cdots,r\zeta_k,t\zeta_{k+1},\cdots,t\zeta_n)|^p \, dm_{n-k} \right) dm_k \\
=&\int_{\mathbb{T}^k} \|f_{r\zeta_1,\cdots,r\zeta_k}\|^p_{H^p(\mathbb{D}^{n-k})} dm_k \\
\le& M^p.
\end{aligned}$$

Thus $f \in H^p(\mathbb{D}^n)$. □

5.4 Poletsky–Stessin Hardy Spaces on the Bidisk

Let u be a negative continuous plurisubharmonic function on the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$$

such that $u(z_1, z_2) \to 0$ as $(z_1, z_2) \to (\zeta_1, \zeta_2) \in \partial\mathbb{D}^2$. Following Demailly [3], for $r < 0$ we define

$$S_u(r) = \left\{(z_1, z_2) \in \mathbb{D}^2 : u(z_1, z_2) = r\right\} \text{ and } B_u(r) = \{(z_1, z_2) \in \mathbb{D}^2 : u(z_1, z_2) < r\}.$$

For convenience we will write $z = (z_1, z_2)$. Associated with this u, Demailly in [3] has defined the positive measure $\mu_{u,r}$, which we call the Monge-Ampère measure, by

$$\mu_{u,r} = (dd^c u_r)^2 - \chi_{\mathbb{D}^2 \setminus B_u(r)} (dd^c u)^2$$

where $u_r = \max\{u, r\}$. These measures are supported by the level sets $S_u(r)$. Demailly has proved the following [3, Theorem 1.7].

Theorem 5.14 (Lelong–Jensen Formula)**.** *For all $r < 0$ every plurisubharmonic function φ on $\mathbb{D}^2$ is $\mu_{u,r}$-integrable and*

$$\mu_{u,r}(\varphi) = \int_{B_u(r)} \varphi (dd^c u)^2 + \int_{B_u(r)} (r - u)(dd^c \varphi) \wedge (dd^c u).$$

Denote by $\mathcal{E}(\mathbb{D}^2)$ the set of all continuous negative plurisubharmonic functions u on $\mathbb{D}^2$ and equal to zero on $\partial\mathbb{D}^2$ whose Monge–Ampère mass is finite, i.e.

$$\int_{\mathbb{D}^2} (dd^c u)^2 < \infty$$

and denote by $\mathcal{E}_1(\mathbb{D}^2)$ the set of those $u \in \mathcal{E}(\mathbb{D}^2)$ for which $\int_{\mathbb{D}^2}(dd^c u)^2 = 1$.

In [20] Poletsky and Stessin introduced the spaces $H^p_u(\mathbb{D}^2)$, which are defined to be the space of all holomorphic functions on $\mathbb{D}^2$ for which

$$\limsup_{r\to 0^-} \mu_{u,r}(|f|^p) < \infty.$$

We call these spaces the Poletsky–Stessin Hardy spaces. These spaces are contained in the classical spaces, that is, $H^p_u(\mathbb{D}^2) \subset H^p(\mathbb{D}^2)$. Since $\mu_{u,r}(|f|^p)$ is an increasing function of r the $\limsup$ in the definition can be replaced by lim. For $p \geq 1$, the p^{th} root of

$$\|f\|^p_{H^p_u} = \lim_{r\to 0^-} \mu_{u,r}(|f|^p)$$

defines a norm and with this norm $H^p_u(\mathbb{D}^2)$ is Banach [20, Theorem 4.1]. The Poletsky–Stessin Hardy spaces on the unit disk have been studied in detail in [1], [29], [27], [19] and [28].

In [18], Poletsky has shown that the intersection of all Poletsky–Stessin Hardy spaces $H^p_u(D), p \geq 1$, where D is a strongly pseudoconvex domain D with C^2 boundary, is $H^\infty(D)$, the space of bounded holomorphic functions. Hence it immediately follows that the intersection of all $H^p_u(\mathbb{D})$ is $H^\infty(\mathbb{D})$. We will prove this result for the polydisk. It is enough to consider the bidisk.

Let $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $\alpha = (\alpha_1, \alpha_2), 0 < \alpha_1, \alpha_2 < \pi/2$. Following [32] we define the approach region $T_\alpha(\zeta)$ as

$$T_\alpha(\zeta) = T_{\alpha_1}(\zeta_1) \times T_{\alpha_2}(\zeta_2)$$

where $T_{\alpha_j}(\zeta_j)$ is the Stolz angle at $\zeta_j \in \mathbb{T}$ with vertex angle $2\alpha_j$. Here we will consider only the congruent symmetric approach regions meaning that the Stolz angles are symmetric with respect to the radius to ζ_j and the vertex angles are equal, i.e. $\alpha_1 = \alpha_2$. Following [18] we define the Green ball of radius $0 < r < 1$ and center at w to be the set

$$G(w, r) = \{z \in \mathbb{D}^2 : g(z, w) < \log r\}$$

where $g(z, w)$ is the Green function for $\mathbb{D}^2$ with pole at w. The Green function for $\mathbb{D}^2$ is explicitly given by

$$g(z, w) = \log \max \left\{ \left| \frac{z_1 - w_1}{1 - \overline{w_1} z_1} \right|, \left| \frac{z_2 - w_2}{1 - \overline{w_2} z_2} \right| \right\}.$$

Hence it follows that

$$G(w, r) = \left\{ z_1 \in \mathbb{D} : \left| \frac{z_1 - w_1}{1 - \overline{w_1} z_1} \right| < r \right\} \times \left\{ z_2 \in \mathbb{D} : \left| \frac{z_2 - w_2}{2 - \overline{w_2} z_2} \right| < r \right\}.$$

Lemma 5.15. *Let* $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ *and* $0 < r < 1$. *For any* $0 < t < 1$ *there exists* $0 < \alpha < \pi/2$ *such that* $G(t\zeta, r) \subset T_\alpha(\zeta)$ *where* $t\zeta = (t\zeta_1, t\zeta_2)$ *and* $T_\alpha(\zeta) = T_\alpha(\zeta_1) \times T_\alpha(\zeta_2)$.

Proof. Observe that

$$\left\{ z_j \in \mathbb{D} : \left| \frac{z_j - t\zeta_j}{1 - t\overline{\zeta_j} z_j} \right| < r \right\}$$

is the image of the disk $\{|w_j| < r\} \subset \mathbb{C}$ under the conformal map

$$w_j \mapsto \frac{w_j + t\zeta_j}{1 + t\overline{\zeta_j} w_j}$$

which is a disk contained in $\mathbb{D}$ with center at

$$\frac{t(1-r^2)}{1-r^2t^2}\zeta_j$$

and radius equal to

$$\frac{r(1-t^2)}{1-r^2t^2}.$$

The tangents to this disk that pass through ζ_j make an angle of

$$\alpha = \arcsin\left(\frac{r(1+t)}{1+tr^2}\right)$$

with the radius to ζ_j. Hence

$$\left\{z_j \in \mathbb{D} : \left|\frac{z_j - t\zeta_j}{1-t\overline{\zeta_j}z_j}\right| < r\right\} \subset T_\alpha(\zeta_j)$$

for $j = 1, 2$ and $G(t\zeta, r) \subset T_\alpha(\zeta)$. Since for fixed $0 < r < 1$

$$t \mapsto \frac{r(1+t)}{1+tr^2}$$

is an increasing function of $t \in [0, 1]$ we have

$$0 < \frac{r(1+t)}{1+tr^2} \leq \frac{2r}{1+r^2} < 1.$$

From this it follows that

$$0 < \alpha \leq \arcsin\left(\frac{2r}{1+r^2}\right) < \frac{\pi}{2}.$$

$\square$

Remark 5.1. For fixed $0 < r < 1$

$$t \mapsto \frac{r(1-t^2)}{1-r^2t^2}$$

is a decreasing function of $t \in [0, 1]$ that decreases to zero as $t \to 1$. Therefore we can make the size of the Green ball $G(t\zeta, r)$ as small as we want simply by choosing t close enough to 1.

The plurisubharmonic envelope $E\phi$ of a continuous function ϕ on a domain $\Omega \subset \mathbb{C}^n$ is the maximal plurisubharmonic function on Ω less than or equal to ϕ. For a sequence of functions $\{u_j\} \subset \mathcal{E}(\mathbb{D}^2)$, we denote by $E\{u_j\}$ the envelope of $\inf\{u_j\}$. The following lemma [18, Theorem 3.3] gives the estimate on the Monge–Ampère mass of the envelope.

Lemma 5.16. *If Ω is a strongly hyperconvex domain and continuous plurisubharmonic functions $\{u_j\} \subset \mathcal{E}(\Omega)$, then*

$$\int_\Omega (dd^c E\{u_j\})^n \leq \sum \int_\Omega (dd^c u_j)^n.$$

Theorem 5.17. *Let f be a holomorphic function on $\mathbb{D}^2$. Suppose that f has non-tangential limits at points $\{\zeta_j\} \subset \mathbb{T}^2$ and $\lim_{j\to\infty} |f^*(\zeta_j)| = \infty$. Then for any $p \geq 1$ there exists $u \in \mathcal{E}_1(\mathbb{D}^2)$ such that $f \notin H^p_u(\mathbb{D}^2)$.*

We will mimic the proof of this theorem from Poletsky's manuscript [18].

Proof. Let us take a sequence $\{a_j\}$ of positive numbers such that

$$\sum_{j=1}^{\infty} a_j < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j^2 |f^*(\zeta_j)|^p = \infty.$$

For $0 < t_j < 1$ we write $G_j = G(t_j\zeta_j, e^{-1})$. By Lemma 5.15 there exists $0 < \alpha_j < \pi/2$ such that $G_j \subset T_{\alpha_j}(\zeta_j)$. Now we inductively construct a sequence $\{t_k\}, 0 < t_k < 1$, satisfying certain conditions. Choose any $0 < t_1 < 1$. Suppose that $t_1, \cdots, t_{k-1}$ have already been chosen. Now chose $0 < t_k < 1$ so that the following conditions are satisfied:

(i) $|f| > |f^*(\zeta_k)|/2$ on G_k

(ii) $G_k \cap G_j = \phi$

(iii) $g(z, t_k\zeta_k) > -a_j/2^{k+1}$ on G_j

(iv) $a_j g(z, t_j\zeta_j) > -a_k/2^{j+1}$ on G_k

for $1 \leq j \leq k-1$. The conditions (i) and (ii) can be achieved simply by taking t_k close enough to 1. Since $G_j, j < k$, and G_k are disjoint, $g(z, t_k\zeta_k) \to 0$ uniformly on G_j as $t_k \to 1$. Hence (iii) can be achieved for t_k close enough to 1. Since $g(z, t_j\zeta_j) = 0$ when $z \in \partial\mathbb{D}^2$, we can choose t_k so close to 1 that

$$G_k \subset \bigcap_{j=1}^{k-1} \left\{ z \in \mathbb{D}^2 : a_j g(z, t_j\zeta_j) > -a_k/2^{j+1} \right\}.$$

Thus (iv) can be achieved.

Define

$$u_j(z) = a_j \max\{g(z, t_j\zeta_j), -2\}.$$

Note that if F is an open set in $\mathbb{D}^2$ containing $G(t_j\zeta_j, e^{-2})$ then

$$\int_F (dd^c u_j)^2 = a_j^2.$$

Let $u = E\{u_j\}$. Since the series $v = \sum_{j=1}^{\infty} u_j$ converges uniformly on $\overline{\mathbb{D}^2}$, $v \in \mathcal{E}(\mathbb{D}^2)$. So $u \geq v$ is a continuous plurisubharmonic function on $\mathbb{D}^2$ equal to 0 on $\partial\mathbb{D}^2$. By

Lemma 5.16,

$$\int_{\mathbb{D}^2} (dd^c u)^2 \leq \sum_{j=1}^{\infty} \int_{\mathbb{D}^2} (dd^c u_j)^2 = \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Hence $u \in \mathcal{E}(\mathbb{D}^2)$.

Now we evaluate $\int_{G_k} (dd^c u)^2$. Observe that $u_k \geq u \geq v$ on $\mathbb{D}^2$. By the conditions on the choices of t_j, on ∂G_k we get

$$-a_k \geq u \geq -\sum_{j=1}^{k-1} \frac{a_k}{2^{j+1}} - a_k - \sum_{j=k+1}^{\infty} \frac{a_k}{2^{j+1}} \geq -\frac{3}{2} a_k.$$

Hence $u + 3a_k/2 \geq 0$ on ∂G_k and the set $F_k = \{6(u + \frac{3}{2}a_k) < u_k\}$ compactly belongs to G_k. Moreover, if $z \in \partial G(t_k \zeta_k, e^{-2})$ then

$$6\left(u(z) + \frac{3}{2}a_k\right) \leq 6\left(u_k(z) + \frac{3}{2}a_k\right) = -3a_k < -2a_k = u_k(z).$$

Thus $G(t_k\zeta_k, e^{-2}) \subset F_k$. By the comparison principle

$$36 \int_{G_k} (dd^c u)^2 = \int_{G_k} (dd^c 6(u(z) + \frac{3}{2}a_k))^2 \geq \int_{F_k} (dd^c u_k)^2 = a_k^2.$$

Hence by Lelong–Jensen formula

$$\|f\|_{H_u^p}^p \geq \int_{\mathbb{D}^2} |f|^p (dd^c u)^2 \geq \sum_{k=1}^{\infty} \int_{G_k} |f|^p (dd^c u)^2 \geq \frac{1}{36 \cdot 2^p} \sum_{k=0}^{\infty} |f^*(\zeta_k)|^p a_k^2 = \infty.$$

Hence $f \notin H^p(\mathbb{D}^2)$. □

The following corollary shows the existence of nontrivial Poletsky–Stessin Hardy spaces on the bidisk.

Corollary 5.18. *For every $p \geq 1$ there exists a function $u \in \mathcal{E}_1(\mathbb{D}^2)$ such that $H_u^p(\mathbb{D}^2) \subsetneq H^p(\mathbb{D}^2)$.*

Proof. Take $f \in H^p(\mathbb{D}^2)$ that is unbounded. Then the non-tangential limit f^* on $\mathbb{T}^2$ must be unbounded because otherwise

$$f(z) = \int_{\mathbb{T}^2} P(z,\zeta) f^*(\zeta)\, dm$$

would imply that $f(z)$ is bounded. So there exists a set of points $\{\zeta_j\} \in \mathbb{T}^2$ such that $\lim_{j\to\infty} |f^*(\zeta_j)| = \infty$. Hence the corollary follows from Theorem 5.17. $\square$

Now we prove the most important theorem of this section.

Theorem 5.19. *Let $p \geq 1$. Then*

$$\bigcap_{u\in\mathcal{E}_1(\mathbb{D}^2)} H^p_u(\mathbb{D}^2) = H^\infty(\mathbb{D}^2).$$

Proof. Let $f \in \bigcap_{u\in\mathcal{E}_1(\mathbb{D}^2)} H^p_u(\mathbb{D}^2)$. Then the non-tangential limit f^* on $\mathbb{T}^2$ is bounded because otherwise by Theorem 5.17 there would exist a $u \in \mathcal{E}_1(\mathbb{D}^2)$ such that $f \notin H^p_u(\mathbb{D}^2)$. Thus, since f^* is bounded,

$$f(z) = \int_{\mathbb{T}^2} P(z,\zeta) f^*(\zeta)\, dm$$

implies that $f \in H^\infty(\mathbb{D}^2)$. $\square$

Bibliography

[1] M. A. Alan, N. G. Goğuş, *Poletsky-Stessin-Hardy spaces in the plane*, Complex Analysis and Operator Theory, DOI:10.1007/s11785-013-03342.

[2] Bonilla, A.; Perez-Gonzalez, F.; Stray, A.; Trujillo-Gonzalez, R. *Approximation in weighted Hardy spaces*, J. Anal. Math. **73** (1997), 65-89.

[3] J. P. Demailly, *Mesures de Monge-Ampr̀e et mesures pluriharmoniques*, Math. Z., **194** (1987), 519–564.

[4] J. P. Demailly, *Mesures de Monge–Ampre et caractrisation gomtrique des varits algbriques affines,* Mm. Soc. Math. France (N. S.) **19** (1985) 1-124.

[5] J. P. Demailly, *Complex Analytic and Differential Geometry,* (unpublished manuscript).

[6] P. Duren, *Theory of H^p Spaces*, Academic Press, Inc, 1970.

[7] P. Duren, A. Schuster *Bergman Spaces* American Mathematical Society, 2004.

[8] J. E. Fornaess, B. Stensones, *Lectures on Counterexamples in Several Complex Variables,* Princeton University Press, 1987.

[9] Theodore W. Gamelin, *Complex Analysis,* Springer Science + Business Media, Inc., 2001.

[10] J. B. Garnett, *Bounded Analytic Functions*, Springer Science + Business Media, LLC, 2007.

[11] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, 1961.

[12] G. H. Hardy, *The mean value of the modulus of an analytic function*, Proc. London Math. Soc., (1915), 269–277.

[13] L. Hörmander, L^p *estimates fot (pluri)-subharmonic functions,* Math. Scand., **20** (1967), 65–78

[14] Paul Koosis, *Introduction to* H_p *Spaces*, Cambridge University Press, 1998.

[15] McPhail, J. Darrell, *A weighted interpolation problem for analytic functions,* Studia Math., **96** (1990), 105--116.

[16] J. Mashreghi, *Representation Theorems in Hardy Spaces*, Cambridge University Press, 2009.

[17] E. A. Poletsky, *Weak and Strong Limit Values*, J. Geom. Anal. (to appear), arXiv:1105.1365.

[18] E. A. Poletsky, *Projective limits of Poletsky–Stessin Hardy Spaces*, arXiv:1503.00575.

[19] E. A. Poletsky, K. R. Shrestha, *On Weighted Hardy Spaces on the Unit Disk*, arXiv:1503.00535.

[20] E. A. Poletsky, M. I. Stessin, *Hardy and Bergman Spaces on Hyperconvex Domains and Their Composition Operatiors*, Indiana Univ. Math. J. **57** (2008), 2153-2201.

[21] Th. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[22] Fulvio Ricci, *Hardy Spaces in One Complex Variable*, http://homepage.sns.it/fricci/papers/hardy.pdf.

[23] F. Riesz, *Über die Randwerte eiuer analytischen Funktion*, Mathematische Zeitschrift 1923, Vol. 18, Issue 1, pp 87–95.

[24] Walter Rudin, *Real and Complex Analysis, 3rd ed.*, McGraw Hill, 1987.

[25] Walter Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc. New York 1969.

[26] Walter Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, 1980.

[27] K. R. Shrestha, *Boundary Values Properties of Functions in Weighted Hardy Spaces,* arXiv:1309.6561

[28] K. R. Shrestha, *Weighted Hardy spaces on the unit disk*, Complex Analysis and Operator Theory, DOI 10.1007/s11785-014-0427-6

[29] S. Şahin, *Poletsky-Stessin Hardy spaces on domains bounded by an analytic Jordan curve in* $\mathbb{C}$, Complex Variables and Elliptic Equations: An International Journal, DOI: 10.1080/17476933.2014.1001112.

[30] S. Şahin, *Poletsky–Stessin Hardy Spaces on complex ellipsoids in* $\mathbb{C}^n$, Complex Anal. and Oper. Theory, DOI 10.1007/s11785-014-0440-9

[31] E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables,* Princeton University Press, Princeton, 1972.

[32] A. Zygmund, *Trigonometric Series, Third Edition*, Cambridge University Press, 2002.

BIOGRAPHICAL DATA

NAME OF AUTHOR: Khim Raj Shrestha

PLACE OF BIRTH: Gulmi, Nepal

DATE OF BIRTH: September 20, 1979

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

Tribhuvan University, Kirtipur, Nepal

International Centre for Theoretical Physics (ICTP), Trieste, Italy

Syracuse University, Syracuse, New York

DEGREES AWARDED:

Bachelor of Arts in Mathematics, 2000, Tribhuvan University

Master of Arts in Mathematics, 2003, Tribhuvan University

Postgraduate Diploma in Mathematics, 2007, ICTP

Marster of Science in Mathematics, 2015, Syracuse University