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An Approach to Permutation Symmetry for the Electroweak Theory

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An approach to permutation symmetry for the electroweak theory

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Abstract

The form of the leptonic mixing matrix emerging from experiment has, in the last few years, generated a lot of interest in the so-called tribimaximal type. This form may be naturally associated with the possibility of a discrete permutation symmetry (S_3) among the three generations. However, trying to implement this attractive symmetry has resulted in some problems and it seems to have fallen out of favor. We suggest an approach in which the S_3 holds to first approximation, somewhat in the manner of the old $SU(3)$ flavor symmetry of the three flavor quark model. It is shown that in the case of the neutrino sector, a presently large experimentally allowed region can be fairly well described in this first approximation. We briefly discuss the nature of the perturbations which are the analogs of the Gell-Mann Okubo perturbations but confine our attention for the most part to the S_3 invariant model. We postulate that the S_3 invariant mass spectrum consists of non zero masses for the (τ, b, t) and zero masses for the other charged fermions but approximately degenerate masses for the three neutrinos. The mixing matrices are assumed to be trivial for the charged fermions but of tribimaximal type for the neutrinos in the first approximation. It is shown that this can be implemented by allowing complex entries for the mass matrix and spontaneous breakdown of the S_3 invariance of the Lagrangian.

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I. INTRODUCTION

It is generally considered that a full understanding of the masses and associated mixings of the quarks and leptons is one of the chief unsolved problems in elementary particle physics. While a great deal of experimental knowledge about quark masses and mixings has been available for quite a long time, it is only in the last few years that very detailed information on lepton masses and mixings has been found from a series of remarkable experiments involving neutrino oscillations (See for some examples, refs [1, 2, 3, 4, 5, 6, 7, 8]). These results can be expected to provide valuable clues toward the solution of this mass and mixing problem. One such clue is the fact that the leptonic mixing matrix is now known to be somewhat close to what is called the "tribimaximal" form; actually a number of interesting discussions [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] of this form have been presented over a period of years. For immediate convenience this form may be read from Eq.(16) below and it can be seen that one column has three equal entries while another has two entries of equal magnitude with the third zero. Such a structure is natural in the context of permutation symmetry since it is a characteristic one which brings the basis of the defining representation of S_3 (the permutation group on three objects) to the basis in which it is decomposed into two and one dimensional irreducible pieces. Of course, S_3 is natural in the context of the standard model of elementary particles, which contains three parallel families of fermions. It has also been often discussed in the literature [20, 21, 22, 23].

However, it seems that this symmetry can not be exact; otherwise there would be two exactly degenerate neutrinos, which is not the case. It is possible that the situation may be analogous to another three flavor theory- the old $SU(3)$ model in which the three flavors turned out to represent the u, d and s quarks rather than the three families we now know about. An initial $SU(3)$ flavor symmetry for the then unknown strong interaction Hamiltonian was assumed to be broken, according to the Gell Mann Okubo hypothesis [24], as:

$$H = H_0 + H_3^3, \tag{1}$$

where H_0 is $SU(3)$ invariant and H_3^3 breaks the symmetry , leaving the iso-spin subgroup invariant. Actually, another symmetry breaking term, which breaks the iso-spin subgroup but preserves a different $SU(2)$ subgroup is also required. A possible analogy to the S_3 symmetry case might be to postulate that the standard electroweak Lagrangian density has a decomposition like

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}' + \mathcal{L}'', \tag{2}$$

where, on the right hand side, the first term is assumed to be S_3 invariant, the second term only preserves one S_2 subgroup of S_3 and the third term preserves a different S_2 subgroup of S_3. There is a presumption that the first term is the dominant one.

In the present paper we will concentrate on the fully S_3 invariant term, $\mathcal{L}_0$ of Eq.(2). We must specify what are the first approximation fermion masses and the first approximation mixing matrices for which we are aiming. In the cases of the charged leptons and the up and down quarks, the first approximation to the masses is conventionally taken to mean assigning the heaviest fermion of each of the three families (τ, b, t) a non-zero mass and setting the masses of the others to zero. Considering that the quark CKM angles are all small it seems reasonable to consider the charged fermion mixing matrices be just the unit matrices in first approximation. On the other hand we would like to take the tribimaximal form as the first S_3 invariant approximation to the neutrino mixing matrix. The question of what is a good first approximation to the spectrum of neutrino masses is perhaps not so clear. We will argue that an approximately degnerate neutrino spectrum is the most reasonable choice. Our job will be to demonstrate that the same S_3 symmetry can give rise to different patterns of masses and mixings for the electrically charged fermions and the neutrinos.

In this paper we will work completely in the framework of the electroweak theory, postponing any possible inclusion in a grand unified theory. We will assume only a minimal number of fermion fields so that the neutrinos will be initially taken to be Majorana type. On the other hand we won't fully specify the Higgs fields in advance. We will also show that the model can be modified for the case of Dirac neutrinos.

In section II we will give the needed notation for the fermion fields and shall show how the permutation group, S_3 may act on them. Section III first contains a discussion of the form of the neutrino mass matrix under the assumption of S_3 invariance. It is shown how it can be diagonalized by a matrix which is, up to a diagonal phase matrix and a possible rotation in a two dimensional subspace, the same as the tribimaximal form. Then, an argument is given that the largest part of the solution space for the three neutrino masses derived from the known mass differences corresponds to almost complete degeneracy. Also in section III, we note that this can easily be achieved by making use of the freedom to have complex constants in the mass matrix. This results in a physical Majorana phase in the mixing matrix when the neutrinos are introduced as Majorana particles.

Section IV contains a discussion of the mass matrix of the charged leptons when it is assumed that these particles acquire mass using just the standard Higgs field. There is no problem with the S_3 invariance allowing for a mass spectrum in which only the τ lepton is massive, as desired. However, there is a problem with getting a trivial charged lepton mixing matrix, as needed to realize the tribimaximal form for the overall lepton mixing matrix. To solve this problem one might imagine backing up and allowing various row and column permutations for both the neutrino and charged lepton mixing matrices (there are 36 x 36 possibilities) and possible two dimensional rotations in the two dimensional degenerate subspaces of each in the hope that the product comes out to be of tribimaximal form. This hope is not realized. We give a simple proof that the tribimaximal form can not be obtained in this manner.

In section V, we point out a possible alternative treatment of the charged lepton masses and mixings which seems able to give in the first approximation, the desired trivial mixing matrix and non-zero mass only for the tau lepton. Instead of requiring the right handed charged leptons to transform under S_3 we consider them to be singlets under S_3 so that the relevant discrete group becomes just S_{3L}. To construct an S_3 invariant Yukawa term with the Higgs fields now requires that we introduce three different Higgs doublets, one associated with each fermion family. Furthermore, the Higgs potential (which may involve other possible Higgs fields too) should allow for spontaneous breakdown of the S_3 symmetry in order to obtain the desired mass spectrum and trivial mixing. As an application of the first approximation model with three approximately degenerate neutrinos we calculate, in this section, the leptonic factor for neutrinoless beta decay. It depends on the value of the degenerate neutrino masses and the single Majorana phase in the model.

The up and down quark masses and mixings are briefly discussed in section VI. Since we want to have masses only for the b and t quarks and trivial mixing in first approximation we use exactly the same model for the Yukawa terms as in the case of the charged leptons, just mentioned. Note that only the left handed quark fields, not the right handed ones, are assigned to transform under S_3. In section VII, we briefly discuss the form of the characteristic matrices which reflect the invariance under any S_2 subgroup of S_3. These may be considered as plausible perturbations to give higher approximations to this model. We count the number of parameters corresponding to the sum of the S_3 matrix and either one perturbation matrix or two perturbation matrices invariant under different S_2 subgroups. Finally. section VIII contains a brief summary and discussion.

II. PERMUTATION SYMMETRY

For a general orientation it may be useful to start from the part of the electroweak theory which contains just the minimal set of fundamental fermions and their interactions with the $SU(2)_L$ x $U(1)$ gauge fields:

$$\mathcal{L} = -\bar{L}_a\gamma_\mu\mathcal{D}_\mu L_a - \bar{e}_{Ra}\gamma_\mu\mathcal{D}_\mu e_{Ra} - \bar{q}_a\gamma_\mu\mathcal{D}_\mu q_a - \bar{u}_{Ra}\gamma_\mu\mathcal{D}_\mu u_{Ra} - \bar{d}_{Ra}\gamma_\mu\mathcal{D}_\mu d_{Ra} + \cdots, \tag{3}$$

where a=(1,2,3) are the generation indices, $\mathcal{D}_\mu$ are the appropriate covariant derivatives for each term and

$$L_a = \begin{bmatrix} \rho_a \\ e_{La} \end{bmatrix}, \qquad q_a = \begin{bmatrix} u_{La} \\ d_{La} \end{bmatrix}. \tag{4}$$

The notation is conventional except that we are denoting ρ_a as the two component neutrino field belonging to generation a. We will need the charged current leptonic weak interaction terms contained in this equation:

$$\mathcal{L}_{cc} = \frac{ig}{\sqrt{2}} W_\mu^- \bar{\hat{e}}_L\gamma_\mu K\hat{\rho} + \quad h.c. + \cdots, \tag{5}$$

where g is the weak coupling constant and W_μ^- is the charged intermediate vector boson field. The hatted fields correspond to mass eigenstates and are related to the original ones as:

$$\rho = U\hat{\rho}, \qquad e_L = W\hat{e}_L. \tag{6}$$

Here, U and W are 3x3 unitary matrices. Finally the leptonic mixing matrix is defined as

$$K = W^\dagger U. \tag{7}$$

It is clear that the part of the Lagrangian given in Eq.(3) (which seems to be the most securely established) possesses a global $U(3)^5$ symmetry- one $U(3)$ symmetry for each term. Subgroups of this very large group are candidates for extra symmetries which might be imposed when adding the Higgs interaction terms and potential. Discrete symmetries have the feature that they do not necessarily imply the existence of new Goldstone bosons when spontaneous breakdown to $U(1)_{EM}$ occurs. The discrete symmetry which is staring us in the face is the generational permutation symmetry S_3 and has been discussed by many people. Of course, there are a number of possibilities for implementing this symmetry. The simplest would be to have all the 5 S_3's act in the same way on all the fermions. In the following we will discuss other possibilities too. The specific symmetry transformations take the form:

$$L_a \rightarrow S_{ab} L_b, \qquad e_{Ra} \rightarrow S_{ab} e_{Rb}, \cdots \tag{8}$$

where the permutation matrices S are the 6 matrices:

$$\begin{aligned} S^{(1)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ S^{(23)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S^{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad S^{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{9}$$

For our purpose here it will be sufficient to use this (defining) reducible representation of S_3.

III. NEUTRINO MASS MATRIX

The simplest way to give masses to the three 2-component Majorana neutrino fields is to add to the theory, in addition to the usual complex Higgs doublet, a complex Higgs triplet:

$$h = \begin{bmatrix} h^+/\sqrt{2} & h^{++} \\ h^0 & -h^+/\sqrt{2} \end{bmatrix}. \tag{10}$$

A vacuum expectation value for h^0 will then generate the neutrino mass terms:

$$\mathcal{L}_{mass} = -\frac{1}{2}\rho^T \sigma_2 M_\nu \rho + \quad h.c. + \qquad \cdots \tag{11}$$

To begin with, M_ν is an arbitrary symmetric (but not necessarily real) 3x3 matrix. Requiring S_3 invariance of the Lagrangian demands invariance of this term under the transformation $\rho_a \rightarrow S_{ab}\rho_b$, where S stands for any of the permutation matrices in Eq. (9). This requires:

$$[S, M_\nu] = 0, \tag{12}$$

for all six matrices S. By explicitly evaluating the commutators we obtain the solution:

$$M_\nu = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \equiv \alpha \mathbf{1} + \beta d. \tag{13}$$

α and β are, in general complex numbers while d is usually called the "democratic" matrix. We remark that the result of Eq.(13) would be the same even for the case of Dirac neutrinos, where the initial matrix is not required to be symmetric.

The form of Eq. (11) indicates that the correct "diagonalization" of M_ν should be:

$$U^T M_\nu U = \hat{M}_\nu, \tag{14}$$

where $\hat{M}_\nu$ is a real, diagonal matrix and U is a unitary matrix. Let us perform this diagonalization while making contact with the historical background of the subject. It is easy to verify that M_ν may be brought to diagonal (but not necessarily real) form by a real orthogonal matrix, R as:

$$R^T(\alpha \mathbf{1} + \beta d)R = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha + 3\beta & 0 \\ 0 & 0 & \alpha \end{bmatrix}. \tag{15}$$

The real matrix, R is typically chosen as:

$$R = \begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \equiv \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix}, \tag{16}$$

wherein we have explicitly shown how R is constructed out of the three eigenvectors (columns in R) of M_ν. In order to bring M_ν to real diagonal form we first define the (Majorana-type) phases, σ and τ as:

$$\alpha + 3\beta = |\alpha + 3\beta| e^{2i\sigma}, \qquad \alpha = |\alpha| e^{2i\tau} \tag{17}$$

and the diagonal matrix

$$P = \begin{bmatrix} e^{-i\tau} & 0 & 0 \\ 0 & e^{-i\sigma} & 0 \\ 0 & 0 & e^{-i\tau} \end{bmatrix}. \tag{18}$$

Now we note that the choice

$$U = RP \tag{19}$$

yields, as desired, real positive neutrino masses:

$$U^T(\alpha\mathbf{1} + \beta d)U = \begin{bmatrix} |\alpha| & 0 & 0 \\ 0 & |\alpha + 3\beta| & 0 \\ 0 & 0 & |\alpha| \end{bmatrix}. \tag{20}$$

Most previous workers on models of this type have considered M_ν to be a real matrix, which results in a little simplification. Let us discuss this case first. It is clear from a purely mathematical standpoint, that the choice of matrix R has a large amount of arbitrariness. First, there is nothing a priori wrong with permuting the three eigenvectors, $\vec{r}_i$ in Eq. (16) in any way. Second, there is nothing wrong with permuting the three rows of R in any way. In fact, there is a larger freedom of making an arbitrary rotation in the two dimensional subspace of the degenerate eigenvectors $\vec{r}_1$ and $\vec{r}_3$. This arbitrariness leaves $\vec{r}_2$ invariant up to its column placement. This structure is a reflection of the possibility of decomposing the three dimensional representation of S_3 into its irreducible two and one dimensional pieces. To illustrate the rotational freedom, we may use, instead of R,

$$R' \equiv \begin{bmatrix} \vec{s}_1 & \vec{r}_2 & \vec{s}_3 \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix} \begin{bmatrix} cos\xi & 0 & -sin\xi \\ 0 & 1 & 0 \\ sin\xi & 0 & cos\xi \end{bmatrix}. \tag{21}$$

Notice that $\vec{r}_2$ is left invariant under this rotation. Furthermore, the matrix element, R'_{13} is no longer zero for general ξ.

The particular choice of R written in Eq. (16) is called the "tribimaximal" matrix. It seems to agree with present experimental indications when it is identified with the lepton mixing matrix, K in Eq. (7). (Of course, the identification of R with K requires the assumption that W is at least approximately the unit matrix.) Especially, the R_{13} element is known to be small and could be zero. However this model can not be exactly correct because it leads to two degenerate neutrino masses, which is ruled out since both $m_2^2 - m_1^2$ (from solar neutrino experiments) and $m_3^2 - m_2^2$ (from atmospheric neutrino experiments) are non-zero and not equal to each other. Thus the S_3 symmetry can only be a kind of first approximation to Nature. Nevertheless it may be a good first approximation and hence interesting to study in detail. That is the point of view we will take here.

In [14], Harrison and Scott have argued that, even if one accepts the predicted two-fold degeneracy, the S_3 model is still not a good first approximation because the two degenerate masses belong to columns 1 and 3 rather than columns 1 and 2. Masses 1 and 2 are expected to be closer to each other because the solar neutrino mass difference is a lot smaller than the atmospheric neutrino mass difference. Here we would like to point out that this conclusion can be altered if we allow for complex values of α and β. In that case, we can easily achieve a situation in which all three neutrinos are degenerate. This is a valid first approximation to the neutrino spectrum for a relatively large range of neutrino masses. One may notice this fact by adopting a typical choice [25] of best fit neutrino mass differences:

$$\begin{aligned} A \equiv m_2^2 - m_1^2 &= 6.9 \times 10^{-5} eV^2, \\ B \equiv |m_3^2 - m_2^2| &= 2.6 \times 10^{-3} eV^2. \end{aligned} \tag{22}$$

The uncertainty in these numbers is roughly 25 per cent. From these two numbers we may-with a two fold ambiguity due to the unknown sign of $(m_3^2 - m_2^2)$- determine masses m_1 and m_2 for a given choice of m_3. We denote the type I solution to be the case when m_3 is greater than m_1 and m_2 and the type II solution to be the case when m_3 is less than m_1 and m_2. Note that there is an upper limit on m_3 corresponding to the cosmological bound from structure formation [26]:

$$m_1 + m_2 + m_3 < 0.7 \ eV. \tag{23}$$

The results are shown as Table I, parts of which have already appeared in refs. [27]. It should be remarked that the number of digits given for the neutrino masses are only for comparing them with each other. They clearly do not reflect the uncertainties of the experimental determinations in Eq.(22).

In the large range 0.3 eV $> m_3 >$ 0.15eV, degeneracy of the three neutrino masses is a good first approximation to the spectrum. It is clearly less good in the lower range 0.1 eV $> m_3 >$ 0.001 eV. Incidentally, it is seen that only in the small range just above m_3 =0.0517 eV does one have anything like a usual hierarchy. An amusing feature is that the viability of S_3 symmetry as a good first order approximation selects a range of favored neutrino masses.

Actually, at first glance it may not be apparent how Eq. (20) allows for a degenerate mass spectrum. To achieve an approximately degenerate mass spectrum we impose:

$$|\alpha| = |\alpha + 3\beta| + \epsilon, \tag{24}$$

type	m_1(eV)	m_2(eV)	m_3(eV)
I	0.2955	0.2956	0.3
II	0.3042	0.3043	0.3
I	0.2446	0.2447	0.25
II	0.2550	0.2551	0.25
I	0.1932	0.1934	0.2
II	0.2062	0.2064	0.2
I	0.1408	0.1411	0.15
II	0.1582	0.1584	0.15
I	0.0856	0.0860	0.1
II	0.1119	0.1123	0.1
I	0.0305	0.0316	0.06
II	0.0783	0.0787	0.06
I	0.0000	0.0083	0.0517
II	0.0643	0.0648	0.04
II	0.0541	0.0548	0.02
II	0.0506	0.0512	0.005
II	0.0503	0.0510	0.001

TABLE I: Typical solutions for (m_1, m_2) as m_3 is lowered from about the highest value which is experimentally reasonable. In the type I solutions m_3 is the largest mass while in the type II solutions m_3 is the smallest mass.

where ϵ is a real number which may be as small as one likes. A picture of this equation with $\epsilon = 0$, where α and $\alpha + 3\beta$ are displayed as 2 dimensional vectors making up the equal sides of an isosceles triangle is given in Fig.1. The angle, $\psi = 2(\sigma - \tau)$ between them must obey (to first order in ϵ):

$$sin(\frac{\psi}{2}) = \frac{3|\beta|}{2|\alpha|}(1 + \frac{\epsilon}{2|\alpha|}). \tag{25}$$

It is interesting to contrast this solution with one in which the coefficient β in Eq.(13) vanishes [22, 23]. Then the three neutrino masses are obviously degenerate but R in Eq.(15) can be any orthogonal matrix. However, in the present case, for any finite ϵ no matter how

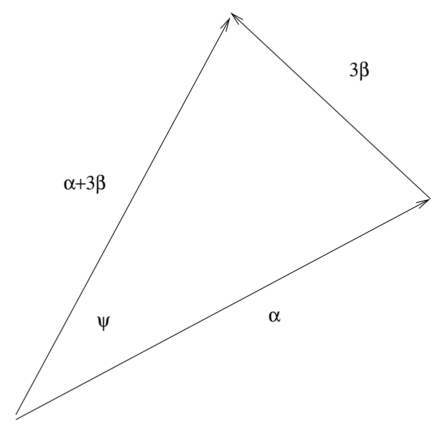

FIG. 1: Isosceles triangle representing Eq.(24) in the $\epsilon = 0$ limit.

small, R can have at most a two dimensional degenerate subspace.We will see that there is a physical effect which persists for small ϵ.

Since only experiment can tell us eventually whether neutrinos are of Majorana or Dirac type, it seems worthwhile to show that the same results may be obtained in case the neutrinos are of Dirac type. In that case we have the two fields $\nu_L \equiv \rho$ and ν_R so the mass term becomes $-\bar{\nu}_R M_\nu \nu_L + h.c.$. The diagonalization equation which replaces Eq.(14) is:

$$U_R^\dagger N_\nu U_L = \hat{M}_\nu. \tag{26}$$

If, for example, both ν_L and ν_R transform as triplets under the same S_3, we will again have the same form of M_ν as given in Eq. (13). The bi-diagonalizing matrices then become

$$U_L = RP, \qquad U_R = RP^*, \tag{27}$$

where R is given in Eq.(16) and P is given in Eq.(18). This replaces Eq.(19) so the neutrino contribution to the overall lepton mixing matrix is just the same. In this case, however, the extra phases, P in U_L may be transformed away. There is no Majorana phase and, of course, no neutrinoless double beta decay. Nevertheless, the complex phase plays a role in the mass eigenvalues, allowing all three neutrinos to be degenerate with non zero β.

IV. CHARGED LEPTON MASS MATRIX

The charged leptons may simply obtain their masses using the conventional complex doublet Higgs field,

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}, \tag{28}$$

in the Lagrangian term,

$$\mathcal{L} = -\frac{\sqrt{2}}{v}\bar{e}_{Ra}(M_e)_{ab}\Phi^\dagger L_b + h.c. + \cdots, \tag{29}$$

where $v = \sqrt{2} < \phi^0 >$ and $\phi^0 = (v + \tilde{\phi}^0)/\sqrt{2}$. Here, M_e is the pre-diagonal charged lepton mass matrix. This matrix may, in general, be brought to a real diagonal form $\hat{M}_e$ by the bi-unitary transformation,

$$M_e = U_e \hat{M}_e W^\dagger, \tag{30}$$

where,

$$M_e^\dagger M_e = W \hat{M}_e^2 W^\dagger, \qquad M_e M_e^\dagger = U_e \hat{M}_e^2 U_e^\dagger. \tag{31}$$

This leads to the mass diagonal, hatted, fields:

$$e_L = W\hat{e}_L, \qquad e_R = U_e \hat{e}_R. \tag{32}$$

So far we haven't discussed the imposition of S_3 symmetry on M_e. Because the fields ρ_a and e_{La} both belong to the same $SU(2)_L$ doublet we should, to maintain the $SU(2)_L$ symmetry, require e_{La} to transform in the same manner as ρ_a. However e_{Ra} is not required to do so. Nevertheless that is the obvious first guess so let us initially require invariance of the Lagrangian under both transformations in Eq. (8). Then we must also impose

$$[S, M_e] = 0, \tag{33}$$

for all six permutation matrices. As before, this has the solution,

$$M_e = \gamma \mathbf{1} + \delta d, \tag{34}$$

where d is the previously defined democratic matrix while γ and δ are two complex numbers. $\hat{M}_e$ will then have have the doubly degenerate real positive eigenvalue $|\gamma|$ as well as the non degenerate eigenvalue $|\gamma + 3\delta|$. We define phases σ' and τ' by,

$$\gamma + 3\delta = |\gamma + 3\delta| e^{2i\sigma'} \qquad \gamma = |\gamma| e^{2i\tau'}. \tag{35}$$

If these phases are used to define a diagonal matrix, P' of the form,

$$P' = \begin{bmatrix} e^{-i\tau'} & 0 & \\ 0 & e^{-i\sigma'} & 0 \\ 0 & 0 & e^{-i\tau'} \end{bmatrix} \tag{36}$$

we can satisfy Eq.(30) with the identifications,

$$W = R'P' \qquad U_e = R'P'^*, \tag{37}$$

where R' is a real orthogonal matrix of the generalized (as discussed around Eq. (21)) tribimaximal type. We remark that it is not necessary for the ordering of the eigenvalues in $\hat{M}_e$ (and correspondingly in Eq.(36) to be the same as in $\hat{M}_\nu$. As discussed above, we can also permute the rows of R' and make an arbitrary rotation in the degenerate subspace. A reasonable first approximation to the charged lepton mass spectrum would seem to consist of a single massive τ^- and two massless (or small mass) others (e^- and μ^-). This can easily be achieved by setting γ to zero and identifying the third eigenvalue to be the non-degenerate one.

However, it is not so easy to get the correct lepton mixing matrix which, using Eq.(7), takes the form,

$$K = P'^* R'^T R P. \tag{38}$$

Note that the phase matrix P'^* can be eliminated by a rephasing of the field $\hat{\hat{e}}_L$, which sits to the left of it in the physical interaction term, Eq.(5). The question is whether the product $R'^T R$ can have anything like the tribimaximal form. Ideally, one might like to keep R as the tribimaximal matrix and arrange for R' to approximate the unit matrix. This could be achieved if the complex number δ in Eq. (34) could be made zero. However we have seen that it is necessary for γ to be zero in order to give a realistic first approximation to the charged lepton mass spectrum. Clearly, having $\gamma = \delta = 0$ is not viable. Then, R' can not be the unit matrix but must be of generalized tribimaximal type. It is not expected that R' would be the same as R since we have seen that a different ordering of mass eigenvalues is appropriate. In general there are 36 x 36 discrete possibilities for the matrix $R'^T R$ corresponding to 6 possible row exchanges and six possible column exchanges for each and also the possibilities of two independent rotations in the two degenerate subspaces. At first, it seems to be a daunting task to study them. Nevertheless we will demonstrate a simple way to see that there are no acceptable solutions that would give a resulting K anything like the tribimaximal form desired.

For this task it is convenient to represent each of the relevant matrices as a single row, having elements which are column vectors as in Eqs. (16) and (21). To anchor the notation we introduce unit vectors:

$$\vec{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{39}$$

Now the matrices $R = [\vec{r}_1, \vec{r}_2, \vec{r}_3]$ and $R' = [\vec{t}_1, \vec{t}_2, \vec{t}_3]$ may be written as row vectors with indexed column vector elements according to:

$$\vec{r}_a = \vec{n}_b R_{ba}, \qquad \vec{t}_a = \vec{n}_b R'_{ba}. \tag{40}$$

From the first of these equations we may write $\vec{n}_c = \vec{r}_a R^{-1}_{ac}$. Substituting this into the second of Eqs. (40) we find $\vec{t}_f = \vec{r}_a R^{-1}_{ac} R'_{cf}$, which may be more compactly written as:

$$\vec{t}_f = \vec{r}_a \tilde{K}_{af}. \tag{41}$$

Here $\tilde{K} = KP^*$ is the non-phase part of the lepton mixing matrix. In other words, the lepton mixing matrix is simply displayed as the transformation between the columns of R and R'. The key point for our present purpose is that one of the $\vec{t}_f$ and one of the $\vec{r}_a$ must be:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{42}$$

This is seen to be the case for any generalized tribimaximal matrix which diagonalizes permutation invariant mass matrices in Eq. (21). It is also clear that an arbitrary permutation of rows in the diagonalizing matrix does not change this vector. Without any loss of generality let the invariant vector in Eq.(42) occur as $\vec{t}_1$ and $\vec{r}_2$. Then Eq.(41) becomes,

$$\vec{t}_1 = \vec{r}_1 \tilde{K}_{11} + \vec{t}_1 \tilde{K}_{21} + \vec{r}_3 \tilde{K}_{31}. \tag{43}$$

Since the three vectors on the right hand side are linearly independent, we conclude that $\tilde{K}_{21} = 1$ and in addition $\tilde{K}_{11} = \tilde{K}_{31} = 0$. Similarly, the inverse relation $\vec{r}_b = \vec{t}_c \tilde{K}^{-1}_{cb}$ yields again $\tilde{K}_{21} = 1$ and also $\tilde{K}_{22} = \tilde{K}_{23} = 0$. Thus the matrix $\tilde{K}$ must have the form,

$$\tilde{K} = \begin{bmatrix} 0 & x & x \\ 1 & 0 & 0 \\ 0 & x & x \end{bmatrix}, \tag{44}$$

where the x's stand for elements yet to be determined. Since $\tilde{K}$ is a real orthogonal matrix we can finally write:

$$\tilde{K} = \begin{bmatrix} 0 & \pm c & \pm s \\ 1 & 0 & 0 \\ 0 & -s & c \end{bmatrix}, \tag{45}$$

where s and c stand for the sine and cosine of some angle. The key point is that there are four zero elements of the matrix and one independent parameter. Therefore it represents, up to some possible reflections and relabeling of axes, a rotation in a two dimensional subspace. This structure clearly does not depend on the choices of $\vec{r}_a$ and $\vec{t}_b$ we chose to identify with the invariant vector of Eq.(42). The tribimaximal matrix of Eq.(16) and the generalized tribimaximal mixing matrix of Eq.(21) can not be obtained from this too simple form.

To sum up the work so far, we have seen that S_3 symmetry based on the transformations properties for L_a and e_{Ra} as given in Eq.(8) has some very encouraging features for a first approximation to lepton properties. First, the neutrino mass spectrum is consistent with three approximately degenerate masses. In addition, the predicted neutrino mixing matrix is consistent with the tribimaximal matrix, as experiment suggests. Also, the charged lepton mass pattern can be made consistent with a heavy τ^- and zero mass μ^- and e^-. Things would be fine if the charged lepton mixing matrix could be made consistent with the unit matrix so that, as in Eq.(7), the lepton mixing matrix, K could be identified with the tribimaximal form (up to the possibility of Majorana phases). Unfortunately, we have just shown that this is not possible. We will now investigate a modification of the field transformation properties which looks able to solve this problem.

For clarification we remark that the problem above would not have arisen if we had not assumed the restriction on the S_3 invariant form of the charged lepton spectrum that only the τ particle has mass. If we could tolerate a first approximation in which (e, μ, τ) are all degenerate, then the choice of $\delta = 0$ in Eq.(34) would work. However that does not seem physically reasonable.

V. MODIFIED APPROACH FOR CHARGED LEPTONS

As mentioned around Eq.(8), there is some freedom to modify the type of the S_3 transformations we assume. In the last section we assumed that L_a and e_{Ra} transform in the same way so we were identifying the S_{3L} and S_{3R} transformations. Now let us assume that e_{Ra} does not transform under S_{3L}; i.e., it transforms as a singlet under S_{3L}. Then, in order to construct an invariant charged lepton mass term we need to assume that there are three Higgs doublets which belong to the (reducible) defining representations of S_{3L}:

$$\Phi_a = \begin{bmatrix} \phi_a^+ \\ \phi_a^0 \end{bmatrix}, \tag{46}$$

where $\phi_a^0 = (v_a + \tilde{\phi}_a^0)/\sqrt{2}$. The charged lepton masses may then arise from a term,

$$\mathcal{L} = -\sqrt{2} \sum_{c,b} \bar{e}_{Rc} \sum_a \Phi_a^\dagger G_{ab}^{(c)} L_b + \quad h.c., \tag{47}$$

where $G_{ab}^{(c)}$ is a matrix of coupling constants whose form will be restricted by the S_{3L} symmetry (The S_{3R} symmetry is not being implemented now). The charged lepton mass matrix is:

$$(M_e)_{cb} = \sum_a v_a G_{ab}^{(c)}, \tag{48}$$

so that the charged lepton mass term in the Lagrangian looks conventional:

$$\mathcal{L} = -\sum_{c,b} \bar{e}_{Rc} (M_e)_{cb} e_{Lb} + \quad h.c. \tag{49}$$

As before, the S_{3L} invariance demands that the three matrices, $G^{(c)}$ have the forms:

$$G^{(c)} = \gamma^{(c)} \mathbf{1} + \delta^{(c)} d, \tag{50}$$

wherein the six quantities $\gamma^{(c)}$ and $\delta^{(c)}$ are all undetermined complex numbers. Then the predicted form for M_e is:

$$M_e = \begin{bmatrix} v_1\gamma^{(1)} + \lambda\delta^{(1)} & v_2\gamma^{(1)} + \lambda\delta^{(1)} & v_3\gamma^{(1)} + \lambda\delta^{(1)} \\ v_1\gamma^{(2)} + \lambda\delta^{(2)} & v_2\gamma^{(2)} + \lambda\delta^{(2)} & v_3\gamma^{(2)} + \lambda\delta^{(2)} \\ v_1\gamma^{(3)} + \lambda\delta^{(3)} & v_2\gamma^{(3)} + \lambda\delta^{(3)} & v_3\gamma^{(3)} + \lambda\delta^{(3)} \end{bmatrix}, \tag{51}$$

where we have denoted the sum of the three vacuum values as $\lambda = v_1 + v_2 + v_3$.

Our goal is to see if Eq.(51) can yield a mass spectrum with a single massive charged lepton and the unit matrix for W, the charged lepton mixing matrix defined in Eq.(6). This can be simply achieved if first, the three arbitrary constants $\delta^{(a)}$ are all set to zero together with the two arbitrary constants $\gamma^{(1)}$ and $\gamma^{(2)}$. Second, since just the bottom row remains, we must have a spontaneous breakdown structure in the Higgs potential which results in the vanishing of the expectation values v_1 and v_2 but not v_3. The τ^- mass then becomes, in the first approximation limit, $v_3\gamma^{(3)}$.

The Higgs potential appropriate for this model may be rather complicated since many additional Higgs fields could very well be present. One characteristic term illustrating how a basic permutation invariant quadratic form appears is:

$$V = k_1 \left[\Phi_a^\dagger(\epsilon\delta_{ab} + \zeta d_{ab})\Phi_b - k_2\right] \left[\Phi_e^\dagger(\epsilon\delta_{ef} + \zeta d_{ef})\Phi_f - k_2\right]^* + \quad \cdots, \tag{52}$$

where k_1,k_2,ϵ and ζ are constants while d_{ab} stands for elements of the democratic matrix.

So it seems that by using, instead of a single Higgs doublet, three Higgs doublets linked to the three leptonic families by S_{3L} invariance, we can obtain a suitable first approximation to the lepton mass spectrum and to the leptonic mixing matrix K. Specifically, the charged leptons have a trivial mixing matrix and only the heaviest one is massive. On the other hand, in this same approximation, the three neutrino masses are approximately degenerate and have a mixing matrix which can be taken to be of the tribimaximal form, Eq.(19) or its generalization to include a rotation in the two dimensional degenerate subspace.

Associated with this model in the case where the neutrinos are formulated as Majorana particles, is the existence of the Majorana phase ψ defined in Eq.(25). To illustrate the importance of this CP violating Majorana phase we calculate the characteristic leptonic factor $|m_{ee}|$ for neutrinoless double beta decay in the present model with three approximately degenerate neutrinos having mass, $m = |\alpha|$. The general formula is:

$$|m_{ee}| = |m_1(K_{11})^2 + m_2(K_{12})^2 + m_3(K_{13})^2|, \tag{53}$$

where the lepton mixing matrix elements are obtained by identifying K with U given in Eqs.(19), (18) and (16). This yields with ϵ negligible,

$$|m_{ee}| = \frac{m}{3}\sqrt{5 + 4cos\psi}. \tag{54}$$

It is easy to check that this result would not change if we computed the matrix elements K_{ab} using the rotated form of R given in Eq.(21) rather than the one in Eq.(16). Clearly

$|m_{ee}|$ obeys the inequality,

$$m \geq |m_{ee}| \geq m/3. \tag{55}$$

The present experimental bound on this quantity is [28],

$$|m_{ee}| < (0.35 - 1.30)eV, \tag{56}$$

which is not much greater than the range of m for which the approximate degeneracy of all three neutrinos holds. Thus this kind of model may be tested in the next few years.

VI. QUARKS

The up and down quark mass matrices clearly have a reasonable first approximation in which the heaviest of each is massive while the other two have no mass. In the same approximation it is reasonable to have a unit mixing matrix. Then the situation for each is the same as for the charged leptons, as just discussed. It thus seems natural to use the same setup as for the charged leptons, with three Higgs fields transforming under S_{3L}. The quark mass terms in the Lagrangian would take the form,

$$\mathcal{L} = -\sqrt{2}\sum_{c,b} \bar{d}_{Rc} \sum_{a} \Phi_a^\dagger B_{ab}^{(c)} q_b + i\sqrt{2}\sum_{c,b} \bar{u}_{Rc} \sum_{a} \Phi_a^T A_{ab}^{(c)} \tau_2 q_b + \quad h.c., \tag{57}$$

which yields the up and down quark mass matrices as:

$$(M_d)_{cb} = \sum_a v_a B_{ab}^{(c)}, \quad (M_u)_{cb} = \sum_a v_a A_{ab}^{(c)}. \tag{58}$$

The predictions of S_3 invariance are

$$A^{(c)} = \eta^{(c)}\mathbf{1} + \theta^{(c)} d, \quad B^{(c)} = \iota^{(c)}\mathbf{1} + \kappa^{(c)} d. \tag{59}$$

As in the charged lepton case the first approximation to the quark mass spectrum yields,

$$m_t = v_3 \eta^{(3)}, \quad m_b = v_3 \iota^{(3)}, \tag{60}$$

while the other quark masses remain zero. For completeness we mention that the up and down mass matrices in the general case are brought to diagonal forms by the bi-unitary transformations:

$$M_u = U_u \hat{M}_u W_u^\dagger, \quad M_d = U_d \hat{M}_d W_d^\dagger. \tag{61}$$

This leads to the diagonal (hatted) states,

$$u_L = W_u \hat{u}_L, \quad u_R = U_u \hat{u}_R, \quad d_L = W_d \hat{d}_L, \quad d_R = U_d \hat{d}_R, \tag{62}$$

and the hadronic part of the charged current weak interaction,

$$\mathcal{L}_{cc} = \frac{ig}{\sqrt{2}} W_\mu^+ \hat{\bar{u}}_L \gamma_\mu C \hat{d}_L + \quad h.c. + \cdots . \tag{63}$$

Here $C = W_u^\dagger W_d$ is the quark mixing (or CKM) matrix. In the first approximation under consideration $C = \mathbf{1}$.

VII. HIGHER ORDER APPROXIMATIONS

Next let us briefly consider the perturbations which might be employed to improve the first approximation fermion spectra and mixing matrices. For this purpose, we first summarize the general form of the S_3 invariant matrices which appear in Eqs. (13), (34), (50) and (59) as:

$$\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} . \tag{64}$$

This matrix contains two complex parameters. As we have seen the non-reality plays an important role in obtaining the proposed first approximation to the neutrino mass spectrum. Of course, the parameters will be different for fermions of each electric charge. The assumption of a grand unified theory would give additional relations among them. Note also that the S_3 invariant matrix and possible perturbations of it directly correspond to the the mass matrix for the assumed Majorana neutrinos but not for the charged leptons and quarks. In the latter cases, the effects of the three Higgs mesons being assumed must be folded in as in Eqs.(48) and (58). An additional difference is that the matrix for the Majorana neutrino masses must be a symmetric one.

If we assume that the perturbations are invariant under only an S_2 subgroup of S_3, there are three choices. These correspond to two element subgroups containing, besides the identity element, $S^{(12)}$, $S^{(13)}$ or $S^{(23)}$. Requiring, a general matrix, M to commute with these separately gives, respectively the S_2 invariant forms:

$$\begin{bmatrix} a' & b' & c' \\ b' & a' & c' \\ d' & d' & e' \end{bmatrix}, \quad \begin{bmatrix} a'' & c'' & b'' \\ d'' & e'' & d'' \\ b'' & c'' & a'' \end{bmatrix}, \quad \begin{bmatrix} e''' & d''' & d''' \\ c''' & a''' & b''' \\ c''' & b''' & a''' \end{bmatrix}. \tag{65}$$

Each of these possible perturbation matrices contains five complex parameters. In case of application to the Majorana neutrino masses, there will be only four complex parameters since we have the corresponding symmetric structures:

$$\begin{bmatrix} a' & b' & c' \\ b' & a' & c' \\ c' & c' & e' \end{bmatrix}, \quad \begin{bmatrix} a'' & c'' & b'' \\ c'' & e'' & c'' \\ b'' & c'' & a'' \end{bmatrix}, \quad \begin{bmatrix} e''' & c''' & c''' \\ c''' & a''' & b''' \\ c''' & b''' & a''' \end{bmatrix}. \tag{66}$$

The simplest perturbation scheme would consist of the original form, Eq.(64) plus one of the three matrices in Eq.(65) [or Eq.(66) for the Majorana neutrino case]. Each of the three possible sums can be seen to have five [four] complex parameters, as one can absorb the two S_3 invariant matrix parameters in the parameters of the perturbing matrix. Presumably, one of these three possibilities would be the best.

The perturbation scheme which might be the closest analog of the generalized Gell-Mann Okubo type mentioned in the Introduction would contain the S_3 invariant part together with two of the matrices in Eq.(65)[Eq.(66)] . It can be seen that one would get the same number, eight [six], of complex parameters regardless of which set of two is selected. In fact, regardless of which set is chosen, the resulting sum of three matrices would be the same. In the case of the Majorana neutrino matrix, the six complex parameters already comprise the most general matrix. For the charged fermion matrices, nine parameters are needed for the most general possibility; however, all of the perturbation matrices satisfy the relation:

$$M_{12} + M_{23} + M_{31} = M_{32} + M_{21} + M_{13}, \tag{67}$$

thereby eliminating one complex parameter. Clearly, there is no point in considering a perturbation made as the sum of all three matrices in Eq.(65).

The most practical second approximation would seem to involve choosing just one of the matrices of Eq.(65) [Eq.(66)]. A promising choice of perturbation, where the S_2 subgroup includes the element $S^{(23)}$ has been investigated by a number of authors [29, 30, 31].

VIII. SUMMARY AND DISCUSSION

For a long time the permutation invariance, S_3 of the three known generations has been considered a very natural assumption for understanding the non-trivial spectrum of fundamental fermion masses and their associated mixings. Recently, experimental data on neutrino oscillations have pointed to a leptonic mixing matrix which is of the so called tribimaximal form; this is in essence the transformation from the natural three dimensional basis of S_3 to the irreducible basis and therefore can be regarded as another motivation for this group. We have attempted to spell out in detail a possible way in which this attractive S_3 permutation invariance can be consistently implemented in the standard electroweak theory.

We started by reviewing the fact that exact S_3 invariance is not correct and suggesting that the situation might be analogous to an older one where a symmetry group was very plausible but the exact dynamics were unknown–the SU(3) symmetry of the three quark model. A similar proposal is that the S_3 invariance holds for a dominant piece but that there are other (presumably relatively small) pieces which preserve different S_2 subgroups of S_3. In this paper we concentrated on looking at the first approximation of exact S_3 invariance. We showed, by comparison with results of analysis of neutrino oscillation experiments (See Table I) that there is a large range of allowed neutrino mass sets where near degeneracy of all three neutrino masses is a very good approximation. So we regarded a near degenerate neutrino mass spectrum together with a tribimaximal type neutrino mixing matrix as the S_3 invariant form to be obtained. At the same time we considered that the S_3 invariant structures for the charged leptons and quarks should contains masses for the (τ, t, b) states and zero masses for the others. The S_3 invariant mixing matrices for the charged fermions are all considered to be just the unit matrix (no mixing to leading order).

It is not completely trivial to obtain a model which achieves this desired spectrum. To get it, we included the features of i) Majorana type CP violation, ii) S_3 invariance for only the left fermions, iii) Inclusion of three Higgs doublets linked to the three generations and transforming under S_3 and iv) spontaneous breakdown of S_3 due to only one of those three doublets developing a vacuum value. The motivation is given in section IV in which it is shown that, very generally, one can not get a tribimaximal form for the overall lepton mixing matrix together with the given mass spectrum of the charged leptons in the simplest way one would imagine.

A physical prediction of the model when the neutrinos are assumed to be of Majorana type is obtained for the neutrinoless double beta decay factor m_{ee} already in the S_3 invariant limit (See Eq. (54)). This gives a bound which should be testable in the relatively near future if the three neutrinos belong to the energy range where they are approximately degenerate.

One natural extension of this approach would be to look at its embedding in a grand unified theory. The next step would be to investigate in detail the case when a perturbation breaking the symmetry down to an S_2 subgroup is included. As discussed in Section VII, this would introduce two additional complex parameters for the neutrino mass matrix.

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