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EXTENSION OF PLURISUBHARMONIC FUNCTIONS AND DYNAMICS OF POLYNOMIAL MAPPINGS

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ABSTRACT

Let X be an algebraic subvariety of $\mathbb{C}^n$ and $\overline{X}$ be its closure in $\mathbb{P}^n$. Coman-Guedj-Zeriahi proved that any plurisubharmonic function with logarithmic growth on X extends to a plurisubharmonic function with logarithmic growth on $\mathbb{C}^n$ when the germs $(\overline{X}, a)$ in $\mathbb{P}^n$ are irreducible for all $a \in \overline{X} \setminus X$. In this dissertation, we consider X for which the germ $(\overline{X}, a)$ is reducible for some $a \in \overline{X} \setminus X$ and give a necessary and sufficient condition for X so that any plurisubharmonic function with logarithmic growth on X extends to a plurisubharmonic function with logarithmic growth on $\mathbb{C}^n$.

We also study a problem in complex dynamics. Quadratic automorphisms of $\mathbb{C}^3$ are classified up to affine conjugacy into seven classes by Fornæss and Wu. Five of these classes contain maps with interesting dynamics. For these maps, Coman and Fornæss estimated the rates of escape of orbits to infinity and described the subsets of $\mathbb{C}^3$ where they occur. Using these estimates, they constructed invariant measures for the maps in three of these classes. By the work of Coman on the fourth class later, the dynamics of the maps from the first four classes is completely understood. This dissertation focuses on the dynamics of maps from the fifth class:

$$H(x, y, z) = (xy + az, x^2 + by, x),\ a \neq 0 \neq b.$$

We investigate the behaviors of H at infinity and construct a dynamically interesting closed positive current of bidimension $(1, 1)$.

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by

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Chapter 1

Introduction

This dissertation focuses on a problem of extension of plurisubharmonic functions and on the dynamics of polynomial maps in higher dimensions. Although these two problems are not directly related, some similar tools of pluripotential theory are used in working on both of them. A main tool in the study of dynamics of polynomials in the complex plane is Montel's theorem, which states that a family of holomorphic maps of the unit disk to $\mathbb{P}^1$ which omits three distinct points is normal. In higher dimension, we do not have an analogue of Montel's theorem. It was first discovered by Sibony that some similar equidistributional results for polynomials can be obtained for the maps in higher dimension using the theory of plurisubharharmonic functions and currents. Since then, the pluripotential theory has played an important role in the dynamical study of maps in higher dimensions.

1.1 Extension of Plurisubharmonic Functions

An upper semicontinuous (usc) function ϕ in $\mathbb{C}^n$ is called *plurisubharmonic* (psh) if the restriction of ϕ to any complex line in $\mathbb{C}^n$ is subharmonic. Plurisubharmonic functions relate very closely to holomorphic functions. For instance, if f is a holomorphic function on $\mathbb{C}^n$, then $\log|f|$ is a psh function. Plurisubharmonicity is a local property, hence psh functions are defined on complex manifolds naturally. Plurisubharmonic functions can also be defined on (possibly) singular sets. A set $X \subset \mathbb{C}^n$ is called an analytic variety if it is locally the zero locus of a finite number of holomorphic functions. In particular, if the holomorphic functions which define the variety can be chosen as holomorphic polynomials, then it is called an algebraic variety. Let X be an analytic variety in $\mathbb{C}^n$. A function ϕ on X is called plurisubharmonic if at any point $z \in X$, ϕ is the restriction of a psh function which is defined in a neighborhood of z in $\mathbb{C}^n$. A plurisubharmonic function ϕ on X is in the Lelong class $\mathcal{L}(X)$ if it satisfies the growth condition $\phi(z) \leq \log^+ ||z|| + C$ for all $z \in X$, where C is a constant that depends on ϕ.

Let $\mathbb{P}^n$ be the complex projective space with the Fubini-Study Kähler form ω. By the maximum principle, a plurisubharmonic function on $\mathbb{P}^n$ is constant. However there are many functions, called quasiplurisubharmonic (qpsh), in $\mathbb{P}^n$ which are locally given as the sum of psh functions and smooth functions. A quasiplurisubharmonic function ϕ is called $\omega-$plurisubharmonic ($\omega - psh$) if $dd^c\phi + \omega \geq 0$ where $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. These functions were introduced by Demailly. Guedj and Zeriahi [GZ] developed a pluripotential theory on compact Kähler manifolds using $\omega - psh$ functions. For a subvariety $X \subset \mathbb{P}^n$, we define $\omega|_X-$ psh functions on X, roughly speaking, as locally the restriction of $\omega-$psh functions (see Section

2.5 for the precise definition).

Let X be an analytic variety in $\mathbb{C}^n$. It is well know that a psh function on X extends to a globally defined psh function on $\mathbb{C}^n$ by the works of Sadullaev [SA] when X is smooth and Coltoiu [CO] when X is (possibly) singular. In [CGZ, Thereom A], Coman-Guedj-Zeriahi showed that psh functions on X extend to psh functions on $\mathbb{C}^n$ with a global growth control. In particular, their result implies that psh functions in the Lelong class $\mathcal{L}(X)$ extend to psh functions on $\mathbb{C}^n$ with an arbitrarily small additional logarithmic growth.

A function $\eta \in \mathcal{L}(\mathbb{C}^n)$ induces a $\omega - psh$ function $\tilde{\eta}$ on $\mathbb{P}^n$ (see Section 2.5 for the definition of $\tilde{\eta}$). The map $\eta \to \tilde{\eta}$ gives a one to one correspondence between $\mathcal{L}(\mathbb{C}^n)$ and the set of $\omega-$psh functions $PSH(\mathbb{P}^n, \omega)$. When $X \subset \mathbb{C}^n$ is an algebraic variety, we denote by $\overline{X}$ the closure of X in $\mathbb{P}^n$. Similarly, a function $\eta \in \mathcal{L}(X)$ induces a function $\tilde{\eta}$ on $\overline{X}$. However, $\tilde{\eta}$ is not necessarily $\omega|_{\overline{X}}$-psh on $\overline{X}$ (see Section 2.5 for an example). In [CGZ, Proposition 3.1], they show that a function $\eta \in \mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^n)$ if and only if $\tilde{\eta}$ defines an $\omega|_{\overline{X}}$-psh function on $\overline{X}$. The same proposition also implies that if the germ $(\overline{X}, a)$ is irreducible for every $a \in \overline{X} \setminus X$, then any function $\eta \in \mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^n)$.

When $(\overline{X}, a)$ is reducible at some $a \in \overline{X} \setminus X$, the extension of functions in the Lelong class of X with the same growth is not guaranteed (see [CGZ, Example 3.2]). In [Y, Theorem 1.2], we give a geometric condition saying that the extension is possible with the same growth if and only if the irreducible components of $\overline{X}$ are *linked* at infinity in an appropriate sense (see Section 3.1).

Theorem 1.1.1. *Let X be an algebraic variety in $\mathbb{C}^n$ where $n \geq 2$. Then any function in*

$\mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^n)$ if and only if for all $a \in \overline{X} \setminus X$, any two irreducible components of the germ $(\overline{X}, a)$ are linked.

For $X \subset \mathbb{C}^2$, the intersection of two irreducible components of the germ $(\overline{X}, a)$ is given by at most a finite set of points. Hence if $(\overline{X}, a)$ is reducible then the irreducible components of the germ $(\overline{X}, a)$ are not linked. Therefore Theorem 1.1.1 implies that any function in $\mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^2)$ if and only if the germs $(\overline{X}, a)$ are irreducible for all points $a \in \overline{X} \setminus X$.

We prove Theorem 1.1.1 in Section 3.2. In Section 3.3, we give interesting examples of algebraic varieties in $\mathbb{C}^n$ and check if they meet the condition of Thereom 1.1.1 for the extendibility.

1.2 Complex Dynamics

The dynamics of rational functions of one complex variable was first studied by Fatou and Julia. For a rational function $f : \mathbb{C} \to \mathbb{C}$, the Fatou set is defined as the largest open set where the sequence of iterates $\{f^n\}$ is a normal family. The complement of the Fatou set is called the Julia set. Most of the time rational functions have a chaotic behavior on Julia sets and Julia sets are usually complicated fractal subsets of $\mathbb{P}^1$.

Let us consider a polynomial p of degree d on $\mathbb{C}$ with complex coefficients. Let K be the set of points in $\mathbb{C}$ with bounded orbit $\{p^n(z)\}$. This set is called the filled-in Julia set. The Julia set is then the boundary of K. Let ϵ_ω be the Dirac measure at ω. In [Br], Brolin showed that the sequence of measures

$$\frac{(p^n)^*\epsilon_\omega}{d^n} = \frac{1}{d^n}\sum_{p^n(\omega_i)=\omega}\epsilon_{\omega_i}$$

converges weakly to μ, where μ is the harmonic measure of K with respect to the point at infinity.

In [S1], Sibony realized that this convergence can be obtained using the tools of complex potential theory, by considering the Green's function

$$G(z) = \lim_{n\to\infty}\frac{1}{d^n}\log^+|p^n(z)|.$$

Then G is a continuous subharmonic function in $\mathbb{C}$ with $G(p(z)) = dG(z)$. Also $K = \{z \in \mathbb{C} : G(z) = 0\}$, G is harmonic in $\mathbb{C}\setminus K$ and $\Delta G = \mu$ where Δ is the Laplacian operator and μ is the harmonic measure considered by Brolin.

The idea of using Green's function inspired the study of complex dynamics in higher dimension. The maps of the form

$$h(z,w) = (P(z) - aw, z),$$

where P is a polynomial in $\mathbb{C}$ of degree $d \geq 2$ are called *Hénon* maps. In [BS, FS1, H], the Green's function of Hénon maps was defined by,

$$G^\pm(z,w) = \lim_{n\to\infty}\frac{1}{2^n}\log^+||h^{\pm n}(z,w)||.$$

The Green's functions $G^{\pm}$ are continuous psh functions on $\mathbb{C}^2$ and pluriharmonic outside of the sets

$$K^{\pm} = \{(z,w) : h^{\pm n}(z,w) \text{ is bounded}\},$$

respectively. Also $K^+ = \{G^+ = 0\}$ and $K^- = \{G^- = 0\}$. The currents defined by $\mu^+ = dd^c G^+$ and $\mu^- = dd^c G^-$ are supported on ∂K^+ and ∂K^-, respectively. These currents define an $h-$invariant ($h^*\mu = \mu$) measure $\mu = \mu^+ \wedge \mu^-$ which is supported on $K^+ \cap K^-$.

In [S2], Sibony adopted the idea of Hénon maps from $\mathbb{C}^2$ to $\mathbb{C}^n$ by introducing the notion of regular automorphism. A polynomial automorphism f of $\mathbb{C}^n$ and its inverse f^{-1} define meromorphic maps of $\mathbb{P}^n$ which are well-defined away from the indeterminacy set I^+ (resp. I^-). An automorphism f of $\mathbb{C}^n$ is called *regular* if the indeterminacy sets I^+ and I^- of f and f^{-1} are disjoint. This assumption on the indeterminacy sets gives a relation between the degree of f and the degree of f^{-1}.

The more general class of *weakly regular* automorphisms of $\mathbb{C}^n$ was studied by Guedj and Sibony in [GS]. Roughly speaking, these are the maps for which I^+ and $X^+ = f(\{t=0\} \backslash I^+)$ are disjoint where $\{t = 0\}$ is the hyperplane at infinity in $\mathbb{P}^n$.

In [FW], the quadratic polynomial automorphisms of $\mathbb{C}^3$ are classified up to affine conjugacy into 7 classes, 5 of which are dynamically interesting as they are non-linear. In [CF], Coman and Fornæss studied these classes by estimating the rates of escape of orbits to infinity and by describing the subsets of $\mathbb{C}^3$ where such orbits occur. Using these estimates, they construct invariant measures for the maps in some of the classes. The maps in 2 of these classes are the most complicated. They have the form

$$H_4(x,y,z) = (P(x,y) + az, Q(y) + x, y),$$

$$H_5(x,y,z) = (P(x,y) + az, Q(x) + by, x),$$

where $\max\{deg(P), deg(Q)\} = 2$ and $a \neq 0 \neq b$. The complication is due to their strange behavior at infinity. They both map $\{t = 0\} \setminus I^+$ onto I^-. For H_5, the extended indeterminacy set I^+_∞ consists of union of two lines, $\{t = x = 0\} \cup \{t = y = 0\}$ and there are two points, $\{[1 : 0 : 0 : 0], [0 : 1 : 0 : 0]\} = I^+_\infty \cap I^-$, where the orbits that tend to infinity with a slow rate may accumulate.

The dynamics of H_4 is explained in detail by Coman in [C]. We focus on the class H_5 and work on the maps

$$H : (x,y,z) \to (xy + az, x^2 + by, x),\ ab \neq 0,$$

for simplicity. The iterates of H will be denoted by $H^n(w) = w_n = (x_n, y_n, z_n)$ where $w = (x,y,z) \in \mathbb{C}^3$. The induced map $H : \mathbb{P}^3 \to \mathbb{P}^3$ is defined by

$$H[x : y : z : t] = [xy + azt : x^2 + byt : xt : t^2].$$

Then H is not (weakly) regular since $I^+ = \{t = x = 0\}$, $X^+ = I^- = \{t = z = 0\}$ and $X^+ \cap I^+ = [0 : 1 : 0 : 0]$.

In Section 4.1, we define the invariant sets K^- and U^- for H^{-1} and prove that the orbits

$H^{-n}(w)$ of points in U^- escape to $[0:0:1:0]$ with a super-exponential rate $(const)^{3^n}$. We also show that if the coefficient b of H has modulus $|b|>1$, then the orbits $H^{-n}(w)$ are bounded for all $w \in K^-$.

In Section 4.2, we define the H-invariant sets U^+ and $K^+ = \mathbb{C}^3 \setminus U^+$. Then we show that on U^+, the orbits $H^n(w)$ escape to infinity at the super-exponential rate $(const)^{2^n}$, while on K^+ the orbits escape to infinity at a much slower rate. In the same section, we show that unbounded orbits of points in K^+ can only cluster at two points at infinity, $[1:0:0:0]$ and $[0:1:0:0]$. If the orbits $H^n(w)$ of points in K^+ are nearby the point $[1:0:0:0]$ for every other iterate, then the orbits escape to infinity at the highest possible rate, which is $(const)^{(\sqrt{3})^n}$ and the y_{2n} coordinates stay bounded near $\pm\sqrt{-b}$. More precisely, we prove the following in Section 4.4.

Let $M_n(w) = \max\{R, 2a|z_n|, |x_n|^{3/2}, \frac{1}{\epsilon}|y_n|^{3/2}\}$ and $\mathbb{B}(a,r)$ be the ball in $\mathbb{C}^n$ with center a and radius $r>0$ which is defined by the Euclidean distance.

Theorem 1.2.1. *On K^+ if $|x_{2n}|^2 > M_{2n}(w)$ for all n then for any $r>0$, $y_{2n} \in \mathbb{B}(\pm\sqrt{-b}, r)$ for all $n > N$, for some $N = N(r)$. Moreover $H^n(w) \approx C^{(\sqrt{3})^n}$ for some $C = C(w)$.*

In [GS, Theorem 3.1], given an automorphism f of $\mathbb{C}^k$ such that f^{-1} is weakly regular and I^- is f-attracting, they construct an f-invariant positive closed current σ_s of bidimension (s,s) supported on $\overline{K}^+$, where $\dim X^- = s-1$. Using this current, a dynamically interesting invariant measure is constructed by $\mu = \sigma_s \wedge T_-^s$ where $T_- = \lim(1/d_-^n)(f^{-n})^*\omega$. Their results apply to our map H only when $|b|>1$, since I^- is H-attracting if and only if $|b|>1$ (see [GS, Lemma 5.4]). We also know that I^+ is not H^{-1}-attracting regardless of $|b|$. However $I^+ \ni X^- = [0:0:1:0]$ is H^{-1}-attracting and this allows us to construct an H^{-1} invariant current σ of bidimension $(1,1)$ on $\mathbb{P}^3$ which is supported on $\partial\overline{K^-}$. Namely we prove the following in Section 4.3 :

Theorem 1.2.2. *There exists a closed positive current σ of bidimension $(1,1)$ on $\mathbb{P}^3$ such that $(H^{-1})^*\sigma = 2\sigma$ and supp $\sigma \subset \partial\overline{K^-}$.*

We note here that an H^{-1}-invariant measure can not be constructed using only the powers of T_- since $T_- \wedge T_- = 0$ in $\mathbb{C}^3$ (see Section 4.3). However the H^{-1}-invariant current σ which is constructed in Theorem 1.2.2 may lead to an H^{-1}-invariant measure by wedging with T_+.

Chapter 2

Preliminaries

In this chapter, we provide some basic tools of pluripotential theory, analytic spaces and complex dynamics which are used in the thesis. For more details of these concepts, we refer the reader to the book of Klimek [K] and Demailly [D2], [D3].

2.1 Plurisubharmonic Functions

Let $z = (z_1, ..., z_n)$ be the standard coordinates in $\mathbb{C}^n$. The *Euclidean norm* in $\mathbb{C}^n$ is defined by

$$||z|| = (z_1\bar{z}_1 + ... + z_n\bar{z}_n)^{1/2}.$$

Let Ω be an open subset of $\mathbb{C}^n$ and let $u : \Omega \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. The function u is said to be *plurisubharmonic* if for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $\lambda \longmapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in \Omega\}$. We denote by $\mathcal{PSH}(\Omega)$, the set of all plurisubharmonic functions in Ω.

The following theorem gives an equivalent definition of plurisubharmonic functions.

Theorem 2.1.1. *Let* $u : \Omega \longrightarrow [-\infty, \infty)$ *be upper semicontinuous and not identically* $-\infty$ *on any connected component of* $\Omega \subset \mathbb{C}^n$*. Then* $u \in \mathcal{PSH}(\Omega)$ *if and only if for each* $a \in \Omega$ *and* $b \in \mathbb{C}^n$ *such that*

$$\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega,$$

we have

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{it} b) dt.$$

It should be noted that plurisubharmonicity is a local property. Hence one can define plurisubharmonic functions on complex manifolds in the same way. Let $\Omega \subset \mathbb{C}^n$ be open. If $\Omega \neq \mathbb{C}^n$, define

$$\Omega_\epsilon := \{z \in \Omega : \quad dist(z, \partial\Omega) > \epsilon\}$$

for $\epsilon > 0$. If $\Omega = \mathbb{C}^n$, we set $\Omega_\epsilon = \mathbb{C}^n$. The following theorem shows that plurisubharmonic functions can be approximated by smooth plurisubharmonic functions u_ϵ defined in smaller domains Ω_ϵ contained in Ω.

Theorem 2.1.2. *Let* Ω *be an open subset of* $\mathbb{C}^n$*, and let* $u \in \mathcal{PSH}(\Omega)$*. Then there exists a sequence* $\{u_\epsilon\}_{\epsilon>0} \subset \mathcal{C}^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$ *such that* u_ϵ *decreases with decreasing* ϵ*, and* $\lim_{\epsilon\to 0} u_\epsilon(z) = u(z)$ *for each* $z \in \Omega$*.*

Note that by an example of Bedford [B], in general it is not possible to approximate a plurisubharmonic function by smooth plurisubharmonic functions which are defined in the original domain Ω.

The following theorem describes some properties of sequences of plurisubharmonic functions.

Theorem 2.1.3. *Let Ω be an open subset of $\mathbb{C}^n$.*

(i) If $u, v \in \mathcal{PSH}(\Omega)$ then $\max(u, v) \in \mathcal{PSH}(\Omega)$.

(ii) The family $\mathcal{PSH}(\Omega)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathcal{PSH}(\Omega)$, then $\alpha u + \beta v \in \mathcal{PSH}(\Omega)$.

(iii) If Ω is connected and $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ is a decreasing sequence then $u = \lim_{j\to\infty} u_j \in \mathcal{PSH}(\Omega)$ or $u \equiv -\infty$.

(iv) Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is plurisubharmonic in Ω, where

$$u^*(y) = \limsup_{\substack{z\to y \\ z\in\Omega}} u(z), \quad y \in \bar{\Omega}$$

Two psh functions can be glued together to produce another psh function as follows.

Proposition 2.1.4. *Let Ω be a domain in $\mathbb{C}^n$ and $V \subset \Omega$ be an open subset. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(V)$, and*

$$\limsup_{\substack{z\to y \\ z\in V}} v(z) \le u(y),\ y \in \partial V \cap \Omega,$$

then

$$w = \begin{cases} \max\{u, v\} & \text{in } V \\ u & \text{in } \Omega \setminus V \end{cases}$$

is plurisubharmonic in Ω.

2.2 Currents

Let Ω be an open set in $\mathbb{C}^n$. By $\mathcal{D}^{p,q}(\Omega)$ we denote the space of compactly supported smooth forms of bidegree (p,q). Any such form can be written as

$$\phi = \sum_{|I|=p,|J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J,$$

where $dz_I = dz_{i_1} \wedge ... \wedge dz_{i_p}$ with $1 \leq i_1 < ... < i_p \leq n$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}$ with $1 \leq j_1 < ... < j_q \leq n$. A continuous linear functional on the space $\mathcal{D}^{p,q}(\Omega)$ is called a current of bidimension (p,q) (or bidegree $(n-p, n-q)$). The space of all currents is denoted by $\mathcal{D}'_{p,q}(\Omega)$. A current $S \in \mathcal{D}'_{p,q}(\Omega)$ can be represented as a differential form of bidegree $(n-p, n-q)$ with distributional coefficients:

$$S = \sum_{|I'|=n-p,|J'|=n-q} S_{I',J'} dz_{I'} \wedge d\bar{z}_{J'}.$$

The differential operators d and d^c are defined by

$$d = \partial + \overline{\partial},\ d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$$

where

$$\partial u = \sum_j \frac{\partial u}{\partial z_j} dz_j \text{ and } \overline{\partial} u = \sum_j \frac{\partial u}{\partial \overline{z_j}} d\bar{z}_j, \quad \text{for} \quad u \in \mathcal{C}^1(\Omega).$$

Note that

$$dd^c = \frac{i}{\pi}\partial\bar{\partial}$$

and if $u \in \mathcal{C}^2(\Omega)$, then

$$dd^c u = \frac{i}{\pi}\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

If S is a current of bidimension (p,p), then the currents dS, d^cS are defined by

$$\langle dS, \phi\rangle = -\langle S, d\phi\rangle \quad \text{and} \quad \langle d^cS, \phi\rangle = -\langle S, d^c\phi\rangle .$$

So $\langle dd^cS, \phi\rangle = \langle S, dd^c\phi\rangle$.

A current S of bidimension (p,p) is called positive if for each test form

$$\phi = i\alpha_1 \wedge \bar{\alpha}_1 \wedge ... \wedge i\alpha_p \wedge \bar{\alpha}_p$$

we have $< S, \phi > \geq 0$, where $\alpha_j \in \mathcal{D}^{1,0}(\Omega)$.

For $u \in \mathcal{PSH}(\Omega)$ and $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ we have that dd^cu is a current of bidegree $(1,1)$ and

$$dd^c u(\varphi) = \int_\Omega u dd^c(\varphi).$$

Theorem 2.2.1. *If $u \in \mathcal{PSH}(\Omega)$, then dd^cu is a positive current of bidegree $(1,1)$. If S is closed positive current of bidegree $(1,1)$, then for every $z_0 \in \Omega$ there exists a neighborhood U of z_0 and a plurisubharmonic function u in U such that $dd^cu = S$ in U.*

Let M and N be complex manifolds of dimension m and n respectively. Let $f : M \to N$ be a smooth map and S be a current of bidimension (p,p) on M. Suppose that f is a proper map on the support of S, that is, for any compact subset X of N, $f^{-1}(X) \cap supp\ S$ is compact. The *pushforward* f_*S of S is defined by

$$< f_*S, \phi >=< S, f^*\phi >,\ \phi \in \mathcal{D}^{p,p}(N).$$

Let $f : M \to N$ be a smooth submersion. The map $f_* : \mathcal{D}^{m-p,m-p} \to \mathcal{D}^{n-p,n-p}$ is defined by

$$< f_*\psi, \phi >=< \psi, f^*\phi >,\ \psi \in \mathcal{D}^{m-p,m-p}(M),\ \phi \in \mathcal{D}^{p,p}(N).$$

Then the *pullback* f^*S of a current $S \in \mathcal{D}'_{n-p,n-p}(N)$ is defined by the formula

$$< f^*S, \psi >=< S, f_*\psi >,\ \psi \in \mathcal{D}^{m-p,m-p}(M).$$

Let M be a Kähler manifold with Kähler form ω. For a positive current S of bidimension (p,p) on M, the *trace measure* of S is defined by

$$\sigma_S = S \wedge \frac{\omega^p}{p!}.$$

Then the *mass* of a current S over a compact set $K \subset M$ is

$$|S|_K := \int_K S \wedge \frac{\omega^p}{p!}.$$

When M is compact we denote the *total mass* of the current S by

$$||S|| = \int S \wedge \frac{\omega^p}{p!}.$$

Let $\mathbb{P}^k$ be complex projective space and $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ be the canonical projection. Let S be a closed positive current on $\mathbb{P}^k$ of bidegree $(1,1)$. Since π is a submersion, we can consider the pull back π^*S and extend it to $\mathbb{C}^{k+1}$. By a theorem of Skoda [Sk], π^*S is a closed positive $(1,1)$ current in $\mathbb{C}^{k+1}$. Then there exists a unique plurisubharmonic function u such that $dd^c u = \pi^*S$, $u(\lambda z) = c \log |\lambda| + u(z)$ and $||S|| = c$ for some $c > 0$ and for all $\lambda \in \mathbb{C}$ (see [FS2] for details). This u is called the *potential* of the current S.

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominant rational map and S be a closed positive $(1,1)$ current on $\mathbb{P}^k$ with the potential u. Then there exists $F = (F_0, ..., F_k)$, where $F_j's$ are homogeneous polynomials of degree d on $\mathbb{C}^{k+1}$, which represents f, that is, $\pi \circ F = f \circ \pi$. We define f^*S by

$$\pi^*(f^*S) = dd^c(u \circ F).$$

Then f^*S is a closed positive $(1,1)$ current on $\mathbb{P}^k$ with potential $u \circ F$ and

$$||f^*S|| = d||S||.$$

Let S be a closed positive current of bidimension (p,p) in an open set $\Omega \subset \mathbb{C}^m$. Let $\beta = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$ denote the standard Kähler form on $\mathbb{C}^m$. The trace measure $|S|$ of S is defined by

$$|S| = \frac{1}{p!} S \wedge \beta^p.$$

If u is a locally integrable psh function with respect to the measure $|S|$ then the product uS is a current on Ω. In this case the exterior product of currents is defined by

$$dd^c u \wedge S := dd^c(uS).$$

This is a closed positive current of bidimension $(p-1, p-1)$ in Ω. In particular when u is locally bounded, the exterior product $dd^c u \wedge S$ is defined for all S. $dd^c u \wedge S$ can also be defined in more general settings when u is not locally bounded below everywhere (see [D2] for more details).

2.3 Complex Analytic Sets

An analytic set is a set which can be defined locally by the zero locus of finitely many holomorphic functions. Complex submanifolds can be considered as examples of analytic sets. However, not all analytic sets are complex manifolds as they may have singularities. Here we will give some basic properties of analytic sets. For a more detailed discussion, we refer to the book of Gunning [G] and Demailly [D2]

Definition 2.3.1. A *holomorphic subvariety* X of an open set $U \subset \mathbb{C}^n$ is a subset $X \subset U$ with the property that for each point $x \in U$ there exists an open neighborhood U_x of x in U and finitely many holomorphic functions $f_1, ..., f_m$ in U_x such that

$$X \cap U_x = \{z \in U_x : f_1(z) = ... = f_m(z) = 0\}.$$

The functions $f_1, ..., f_m$ are called *local defining* functions of X. X is called an *algebraic* *subvariety* if polynomials can be chosen as defining functions.

Let us state some basic properties of analytic subvarieties.

Theorem 2.3.2. *Let U be an open subset of $\mathbb{C}^n$.*

(i) Union and intersection of two analytic subvarieties of U are also analytic subvarieties of U.

(ii) A proper analytic subvariety of an open set $U \subset \mathbb{C}$ is discrete.

(iii) A proper analytic subvariety X of U is nowhere dense in U. $U \setminus X$ is connected if U is connected.

We will focus on local properties of analytic subvarieties. Let X_1 and X_2 be two analytic subvarieties of open neighborhoods U_1 and U_2 of a point $x \in \mathbb{C}^n$. $X_1 \sim X_2$ if there is a neighborhood V of x such that $V \subset U_1 \cap U_2$ and $X_1 \cap V = X_2 \cap V$. This is an equivalence relation and an equivalence class is called a *germ of an analytic subvariety* at the point x. A germ X at x will be denoted by (X, x).

Definition 2.3.3. A germ (X, x) is said to be *reducible* if it has a decomposition $(X, x) = (X_1, x) \cup (X_2, x)$ where the germs $(X_j, x) \neq (X, x)$, $j = 1, 2$. A germ is *irreducible* if it is not reducible.

Let (X, x) be a germ of an analytic subvariety of $U \subset \mathbb{C}^n$. We denote by $\mathcal{J}_{X,x}$ the ideal of the ring $\mathcal{O}_{U,x}$ (germs of holomorphic functions on U at x) which vanish on the germ (X, x).

Proposition 2.3.4. *A germ (X, x) is irreducible if and only if $\mathcal{J}_{X,x}$ is a prime ideal of the ring $\mathcal{O}_{U,x}$.*

The ring $\mathcal{O}_{U,x}$ is Noetherian, i.e. every ideal $\mathcal{J}$ of $\mathcal{O}_{U,x}$ is finitely generated. The following theorem is a consequence of this fact.

Theorem 2.3.5. *Any germ (X, x) of an analytic subvariety has a unique finite decomposition*

$$(X, x) = \bigcup_{1 \leq k \leq N} (X_k, x)$$

where the germs (X_k, x) are irreducible and $(X_j, x) \not\subset (X_k, x)$ for $j \neq k$.

Definition 2.3.6. Let X be an analytic subvariety of an open set $U \subset \mathbb{C}^n$ and $x \in X$. x is called a *regular point* if $X \cap V$ is a complex submanifold of V for some neighborhood V of x. x is *singular* if it is not regular. The set of all regular (singular) points will be denoted by X_{reg} (X_{sing}), respectively.

It follows from the definition that X_{reg} is an open subset of X. Hence X_{sing} a closed subset of X. Connected components of X_{reg} are complex submanifolds of $U \setminus X_{sing}$.

Proposition 2.3.7. *If (X, x) is irreducible, there exist arbitrarily small neighborhoods V of x such that $X_{reg} \cap V$ is dense and connected in $X \cap V$.*

This proposition enables us to define the dimension of a germ of an analytic variety.

Definition 2.3.8. The dimension of an irreducible germ of an analytic variety (X, x) is defined by

$$\dim(X, x) = \dim(X_{reg}, x).$$

If (X, x) has several irreducible components (X_j, x) then we define

$$\dim(X, x) = \max\{\dim(X_j, x)\}, \ \operatorname{codim}(X, x) = n - \dim(X, x).$$

An analytic variety of codimension 1 is called *hypersurface*. Let (X, x) be an irreducible hypersurface. Then (X, x) is defined by the prime ideal $J_{X,x} = (f)$ for some irreducible $f \in \mathcal{O}_{\mathbb{C}^n,x}$. In other words, an irreducible hypersurface is given locally by the zero set of a single irreducible holomorphic function. More generally, any hypersurface is given locally by the zero set of a holomorphic function.

Next we state the Weierstrass preparation theorem (WPT) which is a very useful tool in the study of several complex variables. A *Weierstrass polynomial* is a polynomial in z_n of the form

$$z_n^d + c_1(z')z_n^{d-1} + ... + c_d(z')$$

where the coefficients $c_i(z')$ are holomorphic functions of $z' = (z_1, ..., z_{n-1})$ on some small polydisc in $\mathbb{C}^{n-1}$ vanishing at the origin. We will use WPT to write down a hypersurface locally as the zero set of a Weierstrass polynomial.

Theorem 2.3.9 (Weierstrass preparation theorem)**.** *Let f be a holomorphic function on a neighborhood of 0 in $\mathbb{C}^n$. Assume that $f(0) = 0$ and $f \not\equiv 0$. By change of coordinates, we may assume that $f(0, ..., 0, z_n) \not\equiv 0$. Then there exists a unique Weierstrass polynomial $g_{z'}(z_n) = g(z', z_n)$ and a holomorphic function h on some polydisc contained in the neighborhood of 0 such that $f = gh$ and $h(0) \neq 0$.*

It follows from WPT that any Weierstrass polynomial can be written as a product of irreducible Weierstrass polynomials. It also follows that if g is irreducible as a Weierstrass polynomial, then it is irreducible as an element of $\mathcal{O}_{\mathbb{C}^n,0}$.

Let (X, x) be an irreducible hypersurface defined by $f \in \mathcal{O}_{\mathbb{C}^n,x}$. By the Weierstrass preparation theorem, $f = gh$ for some Weierstrass polynomial g and a holomorphic function

h defined locally near x with $h(x) \neq 0$. Thus the germ (X, x) can also be defined by the Weierstrass polynomial g. This helps us to study the irreducible components of the germ (X, x) as these components are defined by irreducible factors of g which are also Weierstrass polynomials.

2.4 Plurisubharmonic Functions on Analytic Spaces

Definition 2.4.1. Let X be an analytic subvariety of $\mathbb{C}^n$. A function $\phi : X \to [-\infty, +\infty)$ is called *plurisubharmonic* (psh) if $\phi \not\equiv -\infty$ on any open subset of X and every point $z \in X$ has a neighborhood U in $\mathbb{C}^n$ so that $\phi = u|_X$ for some psh function u on U.

In [FN, Theorem 5.3.1] they prove that psh functions on an analytic variety can also be defined as follows.

Theorem 2.4.2. *Let X be an analytic subvariety of $\mathbb{C}^n$. An usc function $\phi : X \to [-\infty, +\infty)$ is psh if and only if for any holomorphic map f of the unit disc $\Delta \subset \mathbb{C}$ to X, $\phi \circ f$ is subharmonic or $\phi \circ f = -\infty$.*

Using this description of psh functions, Demailly [D3, Theorem 1.7] proved the following extension result.

Theorem 2.4.3. *Let X be a locally irreducible analytic variety and Y be an analytic subset of empty interior in X. Let ϕ be a psh function on $X \setminus Y$ which is locally bounded above near Y. Then there exists a unique psh function ϕ^* on X extending ϕ, given by*

$$\phi^*(y) = \limsup_{X \in X \setminus Y, x \to y} \phi(x), \quad y \in Y.$$

Let ϕ be a locally integrable function on X with respect to the area measure given by the restriction of $\beta^p/p!$ to the irreducible components of X of dimension p. Then ϕ is called *weakly psh* if ϕ is locally bounded above and $dd^c\phi \geq 0$ on X_{reg}. A weakly psh function on X is not necessarily equal almost everywhere to a psh function on X. For example, we take $X : z_1z_2 = 0$ in $\mathbb{C}^2$ and define ϕ on X by $\phi(z_1, 0) = 1$ and $\phi(0, z_2) = 0$ if $z_2 \neq 0$. It is clear that ϕ is weakly psh on X, however by Theorem 2.4.2 and the maximum principle, there is no psh function on X which is almost everywhere equal to ϕ.

We have the following characterization of weakly psh functions due to Demailly [D3].

Theorem 2.4.4. *For $\phi : X \to [-\infty, \infty)$, TFAE:*

(i) ϕ is weakly psh on X.

(ii) ϕ coincides almost everywhere with a psh function ϕ_r on X_{reg} and ϕ_r is locally bounded above near each point on X_{sing}.

2.5 ω-plurisubharmonic Functions on $\mathbb{P}^n$

We denote by $\mathbb{P}^n$ the complex projective space and consider the standard embedding

$$z \in \mathbb{C}^n \hookrightarrow [1 : z] \in \mathbb{P}^n,$$

where $[t : z]$ denotes the homogeneous coordinates on $\mathbb{P}^n$. Let ω be the Fubini-Study Kähler form on $\mathbb{P}^n$ with the potential function $\rho(t, z) = \log \sqrt{|t|^2 + ||z||^2}$, that is, $\pi^*\omega = dd^c\rho$ where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical map. We call ϕ a *quasiplurisubharmonic* (*qpsh*) function in $\mathbb{P}^n$ if ϕ is locally the sum of a psh function and a smooth function. Let

$L^1(\mathbb{P}^n, [-\infty, +\infty))$ be denote the integrable functions on $\mathbb{P}^n$ with respect to the measure ω^n. Then the class of ω- plurisubharmonic ($\omega - psh$) functions on $\mathbb{P}^n$ is defined by

$$PSH(\mathbb{P}^n, \omega) = \{\phi \in L^1(\mathbb{P}^n, [-\infty, +\infty)) : \ \phi \ \text{ qpsh}, \ dd^c\phi + \omega \geq 0\}.$$

The following theorem gives some basic properties of $\omega - psh$ functions. These are the consequences of the properties of psh functions. For details, we refer [GZ].

Theorem 2.5.1.

(i) If $u, v \in PSH(\mathbb{P}^n, \omega)$ *then* $\max(u, v), \ \frac{u+v}{2}, \ \log[e^u + e^v] \in PSH(\mathbb{P}^n, \omega)$.

(ii) If $\{u_j\}_{j\in\mathbb{N}} \subset PSH(\mathbb{P}^n, \omega)$ *is a decreasing sequence then* $u = \lim_{j\to\infty} u_j \in PSH(\mathbb{P}^n, \omega)$ *or* $u \equiv -\infty$.

(iii) If $\{u_j\}_{j\in\mathbb{N}} \subset PSH(\mathbb{P}^n, \omega)$ *is an increasing and uniformly bounded above sequence then the upper semicontinuous regularization* u^* *of* $u := \lim u_j$ *is* $\omega -$ *plurisubharmonic in* $\mathbb{P}^n$.

By $\mathcal{L}(\mathbb{C}^n)$ we denote the *Lelong class* of psh functions ϕ on $\mathbb{C}^n$ which verify $\phi(z) \leq \log^+ ||z|| + C$ for all $z \in \mathbb{C}^n$, where C is a constant that depends on ϕ.

The following theorem shows that there is a one-to-one correspondence between the Lelong class $\mathcal{L}(\mathbb{C}^n)$ and $PSH(\mathbb{P}^n, \omega)$. Although this is a folklore result, we provide a proof of it here for the convenience of the reader.

Theorem 2.5.2. *The mapping*

$$F : PSH(\mathbb{P}^n, \omega) \to \mathcal{L}(\mathbb{C}^n), \ F(\phi)(z) = \rho(1, z) + \phi([1 : z]), \tag{2.1}$$

is well defined. Its inverse $F^{-1} : \mathcal{L}(\mathbb{C}^n) \to PSH(\mathbb{P}^n, \omega)$ *is given by* $F^{-1}(\eta) = \tilde{\eta}$*, where*

$$\tilde{\eta}([t:z]) = \begin{cases} \eta(z) - \rho(1,z) & \textit{if } t = 1, \\ \limsup_{\mathbb{C}^n \ni [1:\zeta] \to [0:z]} (\eta(\zeta) - \rho(1,\zeta)) & \textit{if } t = 0. \end{cases}$$

Proof. Let $\phi \in PSH(\mathbb{P}^n, \omega)$. Since ϕ is usc and $\mathbb{P}^n$ is compact, ϕ is globally bounded above. Thus $F\phi$ has logarithmic growth. $F\phi$ is usc since ϕ is usc. We also have

$$dd^c(F\phi(z)) = dd^c(\phi([1:z]) + \rho(1,z)) = dd^c\phi + \omega \geq 0.$$

Hence $F(\phi) \in \mathcal{L}(\mathbb{C}^n)$.

Conversely, let $\eta \in \mathcal{L}(\mathbb{C}^n)$. We will show that $\tilde{\eta} \in PSH(\mathbb{P}^n, \omega)$. Let $a \in \{t \neq 0\} \subset \mathbb{P}^n$. Near a, $\tilde{\eta}$ is given by the qpsh function $\eta(z) - \rho(1,z)$ and

$$dd^c\tilde{\eta} + \omega = dd^c(\eta(z) - \rho(1,z)) + \omega = dd^c\eta(z) \geq 0.$$

Thus $\tilde{\eta}$ is ω-psh in $\{t \neq 0\} \subset \mathbb{P}^n$.

Let a be point in the hyperplane at infinity. Without loss of generality, we may assume that $a = [0:1:0:...:0]$. Let $U_a = \{[t:1:z_2:...:z_n] \in \mathbb{P}^n : \ |t| < 1\}$ be a neighborhood of a contained in the chart $\{z_1 \neq 0\} \subset \mathbb{P}^n$. In the neighborhood U_a, $\tilde{\eta}$ is given by

$$\tilde{\eta}([t:z]) = \begin{cases} \eta(t,1,z_2,...,z_n) - \rho(t,1,z_2,...,z_n) & \text{if } t \neq 0, \\ \limsup_{\{t\neq 0\} \ni [t:\zeta] \to [0:z]} (\eta(t,\zeta) - \rho(t,\zeta)) & \text{if } t = 0. \end{cases}$$

We construct

$$\eta_k([t:z]) = \begin{cases} \eta(t,1,z_2,...,z_n) - \rho(t,1,z_2,...,z_n) + \frac{1}{k}\log|t| & \text{if } [t:z] \in U_a \setminus \{t=0\}, \\ -\infty & \text{if } \ [t:z] \in U_a \cap \{t=0\}. \end{cases}$$

Then η_k is an increasing sequence of ω-psh functions in U_a which is uniformly bounded above and

$$\lim_{k\to\infty} \eta_k([t:z]) = \eta(t,z) - \rho(t,z) \text{ in } U_a \setminus \{t=0\}.$$

Thus $(\lim_{k\to\infty} \eta_k([t:z]))^* = \tilde{\eta}$ in U_a. By Theorem 2.5.1 (iii), $\tilde{\eta}$ is ω-psh in U_a.

□

Let X be an algebraic subvariety of $\mathbb{P}^n$. An usc function $\phi : X \to [-\infty, +\infty)$ is called $\omega|_X-$ *psh* if $\phi \not\equiv -\infty$ on any open subset of X and if there exist an open cover $\{U_i\}_{i\in I}$ of X in $\mathbb{P}^n$, psh functions ϕ_i and ρ_i defined on U_i where ρ_i is smooth and $dd^c\rho_i = \omega$, so that $\rho_i + \phi = \phi_i$ holds on $X \cap U_i$ for all $i \in I$. The class of $\omega|_X-$ psh functions on X is denoted by $PSH(X, \omega|_X)$.

We will say a function $\phi : X \to [-\infty, \infty)$ is *weakly* $\omega|_X$*-psh* if it is bounded above on X and it is $\omega|_{X_{reg}} - psh$ on the set X_{reg} of regular points of X. This definition is analogous to weakly psh functions which are defined in the previous section.

Let X be an algebraic subvariety of $\mathbb{C}^n$. We define the Lelong class $\mathcal{L}(X)$ in a similar way as we define $\mathcal{L}(\mathbb{C}^n)$. By $\overline{X}$ we denote the closure of X in $\mathbb{P}^n$ so $\overline{X}$ is an algebraic subvariety of $\mathbb{P}^n$. It is natural to ask if there is a correspondence between $\mathcal{L}(X)$ and $PSH(\overline{X}, \omega|_{\overline{X}})$ like the one between $\mathcal{L}(\mathbb{C}^n)$ and $PSH(\mathbb{P}^n, \omega)$. However the map $F : PSH(\overline{X}, \omega|_{\overline{X}}) \to \mathcal{L}(X)$ which is defined similarly as in (2.1) is not necessarily surjective. In fact any $\eta \in \mathcal{L}(X)$ induces an usc function

$$\tilde{\eta}([t:z]) := \begin{cases} \eta(z) - \rho(1,z) & \text{if } t = 1, z \in X, \\ \limsup_{X \ni [1:\zeta] \to [0:z]} (\eta(\zeta) - \rho(1,\zeta)) & \text{if } t = 0, \end{cases} \tag{2.2}$$

on $\overline{X}$. However $\tilde{\eta}$ is not necessarily $\omega|_{\overline{X}} - psh$ on $\overline{X}$. It is in general only weakly $\omega|_{\overline{X}}$-psh. For example, let $X = \{y = 0\} \cup \{y = 1\} \subset \mathbb{C}^2$ and $\eta(x, y)$ be defined by

$$\eta(z) := \begin{cases} \rho(1,z) & \text{if } z = (x,0) \\ \rho(1,z) + 1 & \text{if } z = (x,1), \end{cases}$$

where $z = (x, y)$. Then η is psh on X and it has a logarithmic growth on X. Hence $\eta \in \mathcal{L}(X)$. Note that $\overline{X} = X \cup a \subset \mathbb{P}^2$, $a = [0 : 1 : 0]$. For the induced function $\tilde{\eta}$, we have that $\tilde{\eta}(a) = 1$ and $\tilde{\eta}([t : x : 0]) = 0$ when $t \neq 0$. Then $(\tilde{\eta} + \rho)(a) = 1$, $(\tilde{\eta} + \rho)([t : z]) \approx 0$ on the line $\{y = 0\} \subset \mathbb{P}^2$ near a and hence the subaverage inequality fails for $\tilde{\eta} + \rho$ near a on $\{y = 0\}$.

2.6 Dynamics of Polynomial Automorphisms of $\mathbb{C}^k$

2.6.1 Regular Polynomial Automorphisms of $\mathbb{C}^k$

Let $f = (f_1, f_2, ..., f_k)$ be a polynomial automorphism of $\mathbb{C}^k$ of degree $d = \max_i(\deg f_i)$. By homogenizing the polynomials f_i, f defines a map $\bar{f} = [F_1 : ... : F_k : t^d] : \mathbb{P}^k \to \mathbb{P}^k$ where $[z_1 : ... : z_k : t]$ are the homogeneous coordinates of $\mathbb{P}^k$ and $F_j(z,t) = t^d f_j(z/t)$. For simplicity, we will use f for $\bar{f}$. The extended map f of $\mathbb{P}^k$ is not defined on the *indeterminacy set* $I = \pi(F^{-1}\{0\})$ where $F = (F_1, ..., F_k, t^d)$ is the map from $\mathbb{C}^{k+1}$ to $\mathbb{C}^{k+1}$ and $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$

is the canonical map.

A rational map $f : \mathbb{P}^k \to \mathbb{P}^k$ is called *algebraically stable* if there does not exist any integer n and a hypersurface $V \subset \mathbb{P}^k$ such that all the components of F^n are zero on $\pi^{-1}(V)$. For an algebraically stable map f of degree d, f^n has degree d^n (see [S1, Proposition 1.4.3]).

An automorphism of $\mathbb{C}^k$ is called *regular* if $I^+ \cap I^- = \emptyset$ where I^+ and I^- denote the indeterminacy sets of f and f^{-1}. Regular maps are algebraically stable. Indeed, $f(\{t = 0\} \setminus I^+) \subset I^-$ for any polynomial automorphism f of $\mathbb{C}^k$. If f is regular, then no points in $\{t = 0\} \setminus I^+$ can be sent to I^+ as $I^+ \cap I^- = \emptyset$.

As examples of regular automorphisms of $\mathbb{C}^2$, we consider the Hénon maps:

$$f(x, y) = (p(x) + ay, x),$$

where p is a polynomial of degree $d \geq 2$ and $a \neq 0$. The extension of f to $\mathbb{P}^2$ is given by

$$f([x : y : t]) = \left[t^d p\left(\frac{x}{t}\right) + ayt^{d-1} : xt^{d-1} : t^d\right]$$

and the inverse is given by

$$f^{-1}([x : y : t]) = \left[yt^{d-1} : \frac{1}{a}\left(xt^{d-1} - t^d p\left(\frac{y}{t}\right)\right) : t^d\right].$$

Then the indeterminacy sets of f and f^{-1} are two points, $I^+ = [0 : 1 : 0]$ and $I^- = [1 : 0 : 0]$. Hence f is a regular automorphism of $\mathbb{P}^2$. Any automorphism of $\mathbb{C}^2$ is either conjugate to a finite composition of Hénon maps or *elementary* maps of the form

$$e(x, y) = (ax + p(y), by + c),$$

where p is a polynomial of degree $d \geq 2$ (see [FM]). Elementary maps have simple dynamics, thus Hénon maps play an important role in the study of the dynamics of polynomial automorphisms of $\mathbb{C}^2$. Hénon maps were studied extensively in the papers [BS],[FS1],[H] and [HR].

Regular automorphisms were first introduced by Sibony as a generalization of Hénon maps in the higher dimensions $\mathbb{C}^k$, $k > 2$, and studied in [S2]. A subset X of $\mathbb{C}^k$ (or $\mathbb{P}^k$) is said to be *attracting* for f if there exists a neighborhood U of X such that $f(U) \Subset U$ and $X = \cap_{n\geq 0} f^n(U)$. We define the set of points with bounded orbits by

$$K^+ = \{z \in \mathbb{C}^k : f^n(z) \text{ is bounded}\},$$

and denote the complement by $U^+ = \mathbb{C}^k \setminus K^+$. K^- and U^- are defined similarly for the inverse map f^{-1}. The Green's functions for f and the inverse f^{-1} are defined by

$$G^+(z) = \lim \frac{1}{d^n} \log^+ |f^n(z)|,$$

and

$$G^-(z) = \lim \frac{1}{d_-^n} \log^+ |f^{-n}(z)|,$$

where d and d_- are the degrees of f and f^{-1}. In [S2, Proposition 2.2.6], Sibony showed that for a regular automorphism of $\mathbb{C}^k$, $G^{\pm}$ are continuous in $\mathbb{C}^k$ and $K^+ = \{G^+ = 0\}$, $K^- = \{G^- = 0\}$. The closures $\overline{K^+}$ and $\overline{K^-}$ of K^+ and K^- in $\mathbb{P}^k$ satisfy

$$\overline{K^+} \subset K^+ \cup I^+ \text{ and } \overline{K^-} \subset K^- \cup I^-.$$

Moreover, I^+ is attracting for f^{-1} and I^- is attracting for f.

Let f be an algebraically stable polynomial automorphism. A decreasing sequence $\{X_j\}$ of analytic sets is defined by

$$X_1 = \overline{f(\{t = 0\} \setminus I^+)}, ..., X_j = \overline{f(X_{j-1} \setminus I^+)}.$$

This sequence is stationary as it is decreasing. We denote the limit set of the sequence $\{X_j\}$ by $X^+ = X_s$ for some s. Note that $X^+ \neq \emptyset$ since f is algebraically stable. X^+ is contained in I^-.

Proposition 2.6.1. *[S1] Let f be a regular automorphism of $\mathbb{C}^k$. Then X^+ is f-attracting. If X^+ is a point, then G^+ is pluriharmonic in U^+ and the Green's current $T_+ := dd^cG^+$ is supported on ∂K^+.*

Let us consider the induced meromorphic map $\bar{f}$ by f in $\mathbb{P}^k$. We will still denote $\bar{f}$ by f. f is called *dominant* if the Jacobian $J(f)$ is not identically zero. The Green's current $\overline{T}_+$ of f in $\mathbb{P}^k$ is defined by the following theorem.

Theorem 2.6.2. *[S1] Let f be a dominant and algebraically stable meromorphic map of degree d on $\mathbb{P}^k$. The sequence of currents $\left\{\frac{1}{d^n}(f^n)^*(\omega)\right\}$ converges to $\overline{T}_+$ and $f^*(\overline{T}_+) = d \cdot \overline{T}_+$.*

For a polynomial automorphism f of $\mathbb{C}^k$, $d \leq (d_-)^{k-1}$ where d and d_- are the degrees of f and f^{-1}. (see [S1, Proposition 2.3.1]). When f is regular, we have a more precise relation between the degrees of f and f^{-1}.

Proposition 2.6.3. *[S1] Let f be a regular automorphism of $\mathbb{C}^k$. Then*

$$d^l = (d_-)^{k-l}$$

where $\dim I^+ = k - l - 1$. *Moreover,* $\dim I^+ + \dim I^- = k - 2$.

Let f be a regular automorphism of $\mathbb{C}^k$ with $\dim I^+ = k - l - 1$. Since the potential of the current $\overline{T}_+$ is locally bounded outside I^+, the currents $\overline{T}_+^j$ are well-defined for $j \leq l + 1$ by [S1, Theorem A.6.4]. Similarly, the currents $\overline{T}_-^j$ are well-defined for $j \leq k - l + 1$ as the codimension of I^- is $k - l + 1$. Then an f-invariant measure μ in $\mathbb{C}^k$ is defined by

$$\mu = T_+^l \wedge T_-^{k-l}.$$

Indeed, $f^*\mu = f^*T_+^l \wedge f^*T_-^{k-l} = d^l T_+^l \wedge \frac{1}{(d_-)^{k-l}} T_-^{k-l} = \mu$.

2.6.2 Weakly Regular Automorphisms of $\mathbb{C}^k$

A polynomial map f of $\mathbb{C}^k$ is called *weakly regular* if $X^+ \cap I^+ = \emptyset$. This definition was first introduced by Guedj and Sibony in [GS]. Note that $X^+ \subset I^-$, hence regular maps are weakly regular by definition. The Green's function G^+ and the Green's current T_+ are defined similarly as in the case of regular maps. Let $U^+ = \{z \in \mathbb{C}^k : f^n(z) \to X^+\}$ be the basin of attraction of X^+, $\mathcal{K}^+ := \mathbb{C}^k \setminus U^+$, $K^+ = \{z \in \mathbb{C}^k : f^n(z) \text{ is bounded}\} \subset \mathcal{K}^+$.

Theorem 2.6.4. *[GS] Let f be an automorphism of $\mathbb{C}^k$ such that f^{-1} is weakly regular and I^- is an attracting set for f. Then $K^- = \mathcal{K}^- = \{z : G^-(z) = 0\}$ and $K^+ = \mathbb{C}^k \backslash \mathcal{B}(I^-)$ where $\mathcal{B}(I^-)$ is the basin of attraction of I^-. Moreover, $\overline{K}^+ \cap \{t = 0\} = X^- = \overline{\partial K^+} \cap \{t = 0\}$.*

In [GS], with the same assumptions as in Theorem 2.6.4, they construct an f-invariant current of bidimension (s, s) supported on $\overline{K}^+$, where $\dim X^- = s - 1$.

Theorem 2.6.5. *[GS] Let f be an automorphism of $\mathbb{C}^k$ such that f^{-1} is weakly regular and I^- is an attracting set for f. Then there is a positive closed current σ_s of bidimension (s, s) supported on $\overline{K}^+$ which satisfies $f^*\sigma_s = d_-^s \sigma_s$ and $||\sigma_s||_{\mathbb{P}^k} = ||\sigma_s||_{\mathbb{C}^k} = 1$. Moreover, if $d_-^s > d_+^{k-s-1}$, then*

$$\frac{1}{d_-^{ns}} (f^n)^* (\omega^{k-s}) \to \sigma_s$$

in the weak sense of currents, where ω is the Fubini-Study form on $\mathbb{P}^k$.

Using the current σ_s, an f-invariant measure supported in the compact set $K = \{z : (f^n(z))_{n=-\infty}^{\infty}$ is bounded$\}$ can be constructed by $\mu := \sigma_s \wedge T_-^s$ (see [GS, Theorem 4.1]).

2.7 Quadratic Polynomial Automorphisms of $\mathbb{C}^3$

Quadratic polynomial automorphisms of $\mathbb{C}^3$ have been classified up to affine conjugacy into seven classes by Fornæss and Wu [FW]. Two of these classes consist of affine automorphisms and elementary polynomial automorphisms

$$E(x, y, z) = (P(y, z) + ax, Q(z) + by, cz + d),$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$. These maps have simple dynamics as they reduce to two dimensional maps (see [FW]). The other five classes are given by

$$H_1(x, y, z) = (P(x, z) + ay, Q(z) + x, cz + d)$$

$$H_2(x, y, z) = (P(y, z) + ax, Q(y) + bz, y)$$

$$H_3(x, y, z) = (P(x, z) + ay, Q(x) + z, x)$$

$$H_4(x, y, z) = (P(x, y) + az, Q(y) + x, y)$$

$$H_5(x, y, z) = (P(x, y) + az, Q(x) + by, x)$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$.

Note that some of these automorphisms are not regular. In [CF], Coman and Fornæss studied the dynamics of maps from these five classes. For all these classes, they construct the set U^+ of points whose orbit escapes to infinity at the highest super-exponential rate $(const)^{2^n}$. In contrast to regular maps, the set $K^+ = \mathbb{C}^3 \setminus U^+$ does not consist only of points with bounded orbits. The Green's functions $G^\pm$ are pluriharmonic in $U^\pm$ and $K^\pm = \{G^\pm = 0\}$. Hence the Green's currents $\mu^\pm = dd^c G^\pm$ are supported on $\partial K^\pm$. For these maps, it is impossible to construct invariant measures using only the currents $\mu^\pm$ as in the case of regular maps, since $\mu^+ \wedge \mu^+ = \mu^- \wedge \mu^- = 0$. For some of these maps $\deg H = 2$, $\deg H^{-1} = 3$ and hence $H^*\mu^+ = 2\mu^+$ and $H^*\mu^- = \frac{1}{3}\mu^-$. In [CF] for the first two classes H_1 and H_2, they construct an invariant measure by wedging $\mu^+ \wedge \mu^-$ with a current of integration along a hypersurface whose intersection with $K^+ \cap K^-$ is H-invariant.

To understand the dynamics of these maps, one looks at their behavior at infinity $\{t = 0\}$. For the maps in H_1, H_2 and H_3, the second iterate H^2 maps $\{t = 0\} \setminus I^+$ to a single point. The maps in H_4 and H_5 map $\{t = 0\} \setminus I^+$ onto I^-. Hence their dynamics are more complicated than that of the maps in the first three classes.

In [C], Coman described the dynamics of the automorphisms of the form H_4 in detail. For the maps in class H_4, there is a subset $B^+ \subset K^+$ of points with orbits converging to $[1 : 0 : 0 : 0]$ at infinity with a rate of $(const)^{(3/2)^n}$. B^+ is an open set in K^+ however it has empty interior in $\mathbb{C}^3$. Points in the complement $K_b = K^+ \setminus B^+$ have bounded orbits. In the other words, orbits of points in K^+ are either bounded or escape to $[1 : 0 : 0 : 0]$ with the rate $(const)^{(3/2)^n}$. The partial Green's function

$$g(w) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \log^+ ||H^n(w)||$$

is well-defined for $w \in K^+$ and $K_b^+ = \{g = 0\}$. The function g is continuous on K^+ hence the probability measure

$$\mu = dd^c g \wedge \mu^+ \wedge \mu^- := dd^c(g\mu^+ \wedge \mu^-)$$

is well-defined. Moreover, μ is supported in $K_b^+ \cap K^-$ and μ is H-invariant. Indeed,

$$H^*\mu = H^*(dd^c g) \wedge H^*(\mu^+) \wedge H^*(\mu^-) = \frac{3}{2} dd^c g \wedge 2\mu^+ \wedge \frac{1}{3}\mu^- = \mu,$$

since $g(H(w)) = \frac{3}{2} g(w)$.

In this thesis, we will focus on the maps of the form H_5. For simplicity, it suffices to consider

$$H(x, y, z) = (xy + az, x^2 + by, x)$$

where $a \neq 0$ and $b \neq 0$. The induced map $H : \mathbb{P}^3 \to \mathbb{P}^3$ is defined by

$$H[x : y : z : t] = [xy + azt : x^2 + byt : xt : t^2].$$

Then H is not (weakly) regular since $I^+ = \{t = x = 0\}$, $X^+ = I^- = \{t = z = 0\}$ and $X^+ \cap I^+ = [0 : 1 : 0 : 0]$. $X^- = [0 : 0 : 1 : 0]$ hence H^{-1} is weakly regular and I^- is H-attracting if and only if $|b| > 1$ by [GS, Lemma 5.4]. Therefore Theorem 2.6.5 applies to H only when $|b| > 1$ and we can construct an invariant measure for H in this case. In [CF], they construct the sets K^+ and U^+ for H_5. We have the growth estimate $||H^n(\omega)|| \leq C(\omega)^{(\sqrt{3})^n}$ for orbits of the points in K^+. Exact growth on K^+ is not known. There are two points at infinity, $[0 : 1 : 0 : 0]$ and $[1 : 0 : 0 : 0]$ at which orbits of points in K^+ may accumulate. Hence it is difficult to make a filtration of K^+ as in case of H_4.

Chapter 3

Extension of Plurisubharmonic Functions in the Lelong Class

3.1 Introduction

Let X be an algebraic subvariety of $\mathbb{C}^n$ for $n \geq 2$. By $\overline{X}$ we denote the closure of X in $\mathbb{P}^n$ so $\overline{X}$ is an algebraic subvariety of $\mathbb{P}^n$. A psh function η with logarithmic growth on X ($\eta \in \mathcal{L}(X)$) induces an usc function $\tilde{\eta}$ on $\overline{X}$ by

$$\tilde{\eta}([t:z]) := \begin{cases} \eta(z) - \rho(1,z) & \text{if } t = 1, z \in X, \\ \limsup_{X \ni [1:\zeta] \to [0:z]} (\eta(\zeta) - \rho(1,\zeta)) & \text{if } t = 0, \end{cases} \tag{3.1}$$

As we discussed in Section 2.5, $\tilde{\eta}$ is not necessarily $\omega|_{\overline{X}} - psh$ on $\overline{X}$. It is in general only *weakly* $\omega - psh$, that is, it is bounded above on $\overline{X}$ and $\omega|_{\overline{X_r}} - psh$ on $\overline{X_r}$, where $\overline{X_r}$ is the regular part of $\overline{X}$.

We denote by $\mathcal{L}_\gamma(X)$, where γ is a positive constant, the Lelong class of psh functions on X which verify $\phi(z) \leq \gamma \log^+ ||z|| + C$ for all $z \in X$, where C is a constant that depends on ϕ. For an analytic subvariety $X \subset \mathbb{C}^n$ [CGZ, Theorem A] implies that any function $\phi \in \mathcal{L}(X)$ has an extension in $\mathcal{L}_\gamma(\mathbb{C}^n)$ for every $\gamma > 1$. In [CGZ, Section 3] the question whether this additional arbitrarily small growth is necessary on an algebraic variety to have an extension is addressed. More precisely, is every psh function with logarithmic growth on an algebraic variety $X \subset \mathbb{C}^n$ the restriction of a function in $\mathcal{L}(\mathbb{C}^n)$? The following is proved.

Proposition 3.1.1. *[CGZ] Let $X \subset \mathbb{C}^n$ be an algebraic variety and $\eta \in \mathcal{L}(X)$. The following are equivalent:*

(i) There exists $\psi \in \mathcal{L}(\mathbb{C}^n)$ so that $\psi = \eta$ on X.

(ii) $\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})$.

(iii) For every point $a \in \overline{X} \setminus X$, the following holds: if (X_j, a) are irreducible components of the germ $(\overline{X}, a)$, then the value

$$\limsup_{X_j \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1, \zeta))$$

is independent of j.

In particular, if the germs $(\overline{X}, a)$ are irreducible for all points $a \in \overline{X} \setminus X$ then $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$.

Here we consider the converse of the last statement : If X is such that the germ $(\overline{X}, a)$ is reducible for some $a \in \overline{X} \setminus X$, then is there always a function in $\mathcal{L}(X)$ which does not extend to a function in $\mathcal{L}(\mathbb{C}^n)$? That is, is the inclusion $\mathcal{L}(\mathbb{C}^n)|_X \subseteq \mathcal{L}(X)$ strict?

We need to give a definition before we state our main result answering the above question.

Let X_k be the irreducible components of the germ $(\overline{X}, a)$. We will say that X_i and X_j are *linked* if there exist some irreducible components $X_{i_k}'s$ such that all the intersections $X_i \cap X_{i_1} \cap \mathbb{C}^n$, $X_{i_1} \cap X_{i_2} \cap \mathbb{C}^n$,..., $X_{i_m} \cap X_j \cap \mathbb{C}^n$ have positive dimension. Now we can state our main result.

Theorem 3.1.2. *Let X be an algebraic variety in $\mathbb{C}^n$ where $n \geq 2$. Then any function in $\mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^n)$ if and only if for all $a \in \overline{X} \setminus X$, any two irreducible components of the germ $(\overline{X}, a)$ are linked.*

The proof of this theorem will be given in Section 3.2. In Section 3.3, we will consider some well known examples of algebraic varieties X and check whether our condition in Theorem 3.1.2 holds for X, hence whether $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$. In cases where $\mathcal{L}(X) \neq \mathcal{L}(\mathbb{C}^n)|_X$ we will construct a function $\eta \in \mathcal{L}(X)$ which has no extension in $\mathcal{L}(\mathbb{C}^n)$. By [CGZ, Theorem A] we know that there is an extension of η in $\mathcal{L}_\gamma(\mathbb{C}^n)$ for all $\gamma > 1$. Here we give such an extension of η explicitly. In Example 3.3.1 and Example 3.3.2, we have smooth varieties of $\mathbb{P}^n$. Therefore for all $a \in \overline{X} \setminus X$, the germ $(\overline{X}, a)$ is irreducible and $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$ by Proposition 3.1.1. In Example 3.3.3 and Example 3.3.4, the germ $(\overline{X}, a)$ is irreducible for any singular point $a \in \overline{X} \setminus X$. Thus by Proposition 3.1.1, $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$. In Example 3.3.5, there is only one singular point $a \in \overline{X} \setminus X$ and at this point the irreducible components of the germ $(\overline{X}, a)$ intersect along a line which is not contained in the hyperplane at infinity. Thus irreducible components of the germ $(\overline{X}, a)$ are linked and $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$ by Theorem 3.1.2. In Example 3.3.6, there are three singular points in $\overline{X} \setminus X$. At two of these points the germ $(\overline{X}, a)$ is irreducible. At the other point the germ $(\overline{X}, a)$ has two irreducible components which are linked. Thus by Theorem 3.1.2 $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$. In Example 3.3.7, Example 3.3.8 and Example 3.3.9 for some singular point $a \in \overline{X} \setminus X$, the germ $(\overline{X}, a)$ has two irreducible components whose intersection lies in the hyperplane at infinity. Therefore these irreducible components are not linked and by Theorem 3.1.2 $\mathcal{L}(X) \neq \mathcal{L}(\mathbb{C}^n)|_X$. In these cases we construct a function $\eta \in \mathcal{L}(X)$ which has no extension in $\mathcal{L}(\mathbb{C}^n)$. Then we give an explicit extension of η in $\mathcal{L}_\gamma(\mathbb{C}^n)$ for any $\gamma > 1$.

3.2 Proof of the Theorem 3.1.2

We need some lemmas to prove Theorem 3.1.2.

Lemma 3.2.1. *Let X be as in Theorem 3.1.2 and let $a \in \overline{X} \setminus X$. If two irreducible components X_i and X_j of the germ $(\overline{X}, a)$ are not linked then $(\overline{X}, a) = \tilde{X}_i \cup \tilde{X}_j$ where $\tilde{X}_i$ and $\tilde{X}_j$ are germs of subvarieties of $\overline{X}$ at a such that $\tilde{X}_i \cap \tilde{X}_j \cap \mathbb{C}^n = \emptyset$.*

Proof. Let (X_k, a), $k \in I$, be irreducible components of the germ $(\overline{X}, a)$. We take $\tilde{X}_i = X_i \cup \{\cup_{k\in K} X_k\}$ and $\tilde{X}_j = X_j \cup \{\cup_{k\notin K} X_k\}$ where $K = \{k \in I : X_k \text{ linked to } X_i\}$. We claim that $\tilde{X}_i \cap \tilde{X}_j \cap \mathbb{C}^n$ has dimension 0. Otherwise one of the irreducible component of $\tilde{X}_i$ is linked to an irreducible component of $\tilde{X}_j$. Consequently this irreducible component of $\tilde{X}_j$ is linked to X_i and this contradicts the definition of the set K.

Let Y be an irreducible component of the germ $\tilde{X}_i \cap \tilde{X}_j$ at a. Then the previous claim implies that $Y \subset \{t = 0\}$, hence $\tilde{X}_i \cap \tilde{X}_j \subset \{t = 0\}$. Thus $\tilde{X}_i \cap \tilde{X}_j \cap \mathbb{C}^n = \emptyset$. □

The following lemma will show that for a qpsh function v on a germ of an irreducible analytic variety (X, p), $\limsup_{z\to p} v(z)$ is attained along the complement $X \setminus Y$ for any proper germ of subvariety (Y, p) of (X, p). Although it is well known, we will include its proof for the convenience of the reader.

Lemma 3.2.2. *Let $(Y,p) \subset (X,p)$ be germs of analytic varieties in $\mathbb{C}^n$ such that* $\dim(X,p) = k > 0$, *(X,p) is irreducible and $(Y,p) \neq (X,p)$. Then*

$$\limsup_{X \setminus Y \ni z \to p} v(z) = v(p),$$

for any qpsh function v on (X,p).

Proof. We construct a non-constant holomorphic function $f : \Delta_\epsilon \to X$ such that $f(0) = p$ and $f(\Delta_\epsilon) \cap Y = \{p\}$ where Δ_ϵ is a disc of radius ϵ in $\mathbb{C}$. Let $\pi : \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^k$ be the projection map onto the first k coordinates. By Local Parametrization Theorem (see [D2, Theorem 4.19 on page 95]) there is a choice of coordinates in $\mathbb{C}^n$ such that the restriction of the projection map $\pi : X \cap U \to U'$ is a finite, proper, holomorphic map where U and U' are some neighborhoods of $p \in \mathbb{C}^n$ and $0 \in \mathbb{C}^k$ with $\pi(p) = 0$. By Remmert's Proper Mapping Theorem (see [D2, Theorem 8.8 on page 118]) $\pi(Y \cap U) \subset U'$ is an analytic subvariety. Since π is a finite map, $\dim \pi(Y \cap U) = \dim Y < k$ by [D2, Lemma 8.1 on page 118]. Let $B_r \subset U'$ be a polydisc in $\mathbb{C}^k$ centered at 0 with radius $r > 0$. Let $a \in B_{\frac{r}{2}} \setminus \pi(Y \cap U)$. We define a holomorphic map ϕ from unit disc $\Delta \subset \mathbb{C}$ to $B_r \subset U'$ by $\phi(\zeta) = 2a\zeta$. Then $\phi(0) = 0$ and $\phi(\frac{1}{2}) = a \notin \pi(Y \cap U)$. Thus $\phi^{-1}(\pi(Y \cap U))$ is a proper subvariety of $\Delta \subset \mathbb{C}$. This implies that 0 is an isolated point in $\phi^{-1}(\pi(Y \cap U))$. We take a smaller disc Δ_ρ such that $\phi(\Delta_\rho) \cap \pi(Y \cap U) = \{0\}$. $\pi^{-1}(\phi(\Delta_\rho))$ is an analytic subvariety of $X \cap U$ and its dimension is 1 since π is finite. By parametrization of curves (see [D2, Example 4.27 on page 98]) there is a non-constant holomorphic map

$$f : \Delta_\epsilon \subset \mathbb{C} \to \pi^{-1}(\phi(\Delta_\rho)) \subset X,$$

with $f(0) = p$ for some disc Δ_ϵ of radius ϵ. It follows that $f(\Delta_\epsilon) \cap Y = \{p\}$.

Let v be a *qpsh* function on the germ (X, p). Since v is locally the sum of a psh function and smooth function, it is enough to prove the lemma when v is psh. Since $v \circ f$ is subharmonic in Δ_ϵ

$$v(p) = v(f(0)) = \limsup_{0 \neq t \to 0} v \circ f(t) \leq \limsup_{X \setminus Y \ni z \to p} v(z) \leq \limsup_{X \ni z \to p} v(z) = v(p).$$

□

Proof of Theorem 3.1.2. First we assume that the germ $(\overline{X}, a)$ has two irreducible components X_i and X_j which are not linked. Then we will show that there exists a $\eta \in \mathcal{L}(X)$ which has no extension in $\mathcal{L}(\mathbb{C}^n)$. For simplicity, we can assume that $a = [0 : 0 : ... : 1] \in \{t = 0\} \subset \mathbb{P}^n$. We take a neighborhood $\overline{V}_M$ of a where

$$V_M = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : |z_n| > M \max(1, |z_1|, |z_2|, ..., |z_{n-1}|)\}.$$

For M big enough $X \cap V_M = Y_1 \cup Y_2$, and $\overline{Y_1} \cap \overline{Y_2} \subset \{t = 0\}$ where we can choose $\overline{Y_1} = \tilde{X}_i$ and $\overline{Y_2} = \tilde{X}_j$ as in Lemma 3.2.1. Let

$$u(z_1, ..., z_n) := \max\{\log |z_1|, \log |z_2|, ..., \frac{1}{2} \log |z_n|\}.$$

Note that $u \in \mathcal{L}(\mathbb{C}^n)$. We will show that

$$u(z) + 2\log M \geq \rho(1, z) \text{ on } \partial V_M, \tag{3.2}$$

if M is sufficiently large. In order to prove (3.2) we consider four cases for $z \in \partial V_M$.

Case 1 : $|z_n| = M$ for $z = (z_1, ..., z_n) \in \partial V_M$. Then

$$\begin{aligned}\rho(1, z) &= \frac{1}{2}\log(1 + |z_1|^2 + ... + |z_n|^2) \\ &\leq \frac{1}{2}\log(n + M^2) \leq \frac{1}{2}\log(2M^2) \leq 2\log M,\end{aligned}$$

when M is big enough. On the other hand $u(z_1, ..., z_n) = \frac{1}{2}\log M$ and inequality (3.2) is satisfied.

Case 2 : $|z_n| = M|z_1|$ and $|z_n| < |z_1|^2$. Then $u(z_1, ..., z_n) = \log|z_1|$ and

$$\begin{aligned}\rho(1, z_1, ..., z_n) &= \frac{1}{2}\log(1 + |z_1|^2 + ... + |z_n|^2) \\ &\leq \frac{1}{2}\log((n + 1)|z_n|^2) = \frac{1}{2}\log(n + 1) + \log|z_n| \\ &\leq 2\log M + \log|z_1|.\end{aligned}$$

Thus inequality (3.2) is satisfied in this case.

Case 3 : $|z_n| = M|z_1|$ and $|z_n| \geq |z_1|^2$. Then

$$u(z_1, ..., z_n) = \frac{1}{2}\log|z_n| = \frac{1}{2}(\log|z_1| + \log M)$$

and

$$\begin{aligned}\rho(1, z_1, ..., z_n) &\leq \frac{1}{2}\log(n+1) + \log|z_n| = \frac{1}{2}\log(n+1) + \log|z_1| + \log M \\ &\leq \frac{1}{2}\log|z_1| + \frac{3}{2}\log M + \frac{1}{2}\log(n+1),\end{aligned}$$

since $|z_1| \leq M$. Therefore (3.2) is satisfied in this case too.

Case 4 : $|z_n| = M|z_i|$ for some $i : 2, ..., n-1$. Then the same argument as above works. Thus we obtain the inequality (3.2). We consider

$$\eta(z_1, ..., z_n) = \begin{cases} \max(u(z_1, ..., z_n) + 2\log M, \rho(1, z_1, ..., z_n)) & \text{on } Y_1, \\ u(z_1, ..., z_n) + 2\log M & \text{on } X \setminus Y_1. \end{cases}$$

Let $p \in Y_1 \cap \partial V_M$ and let $U_p \subset \mathbb{C}^n$ be a neighborhood of p disjoint from Y_2. It follows that $\eta = \psi|_X$, where

$$\psi(z_1, ..., z_n) = \begin{cases} \max(u(z_1, ..., z_n) + 2\log M, \rho(1, z_1, ..., z_n)) & \text{on } U_p \cap V_M, \\ u(z_1, ..., z_n) + 2\log M & \text{on } U_p \setminus V_M. \end{cases}$$

The inequality (3.2) implies that ψ is psh on U_p. Therefore η is psh on $U_p \cap X$, hence on X. Since $u \in \mathcal{L}(X)$, $\eta \in \mathcal{L}(X)$. Let

$$V_{M'} = \{(z_1 : z_2 : ... : z_n) \in \mathbb{C}^n : |z_n| > M' \max(1, |z_1|, |z_2|, ..., |z_{n-1}|)\},$$

where $M' > e^{2k}M^4$ and k is any positive number. In $V_{M'}$, $\rho(1,z) > \log|z_n|$ and

$$\begin{aligned} u(z) + 2\log M &\leq \max(\log|z_n| - \log M', \frac{1}{2}\log|z_n|) + 2\log M \\ &\leq \log|z_n| - \frac{1}{2}\log M' + 2\log M < \log|z_n| - k \leq \rho(1,z) - k. \end{aligned}$$

Thus $\eta(z) = \rho(1,z)$ on Y_1 near the point a and $\eta(z) - \rho(1,z) < -k$ on Y_2 near the point a. These imply that

$$\limsup_{Y_1 \ni [1:z_1:...:z_n] \to a} (\eta(z_1,...,z_n) - \rho(1,z_1,...,z_n)) = 0$$

and

$$\limsup_{Y_2 \ni [1:z_1:...:z_n] \to a} (\eta(z_1,...,z_n) - \rho(1,z_1,...,z_n)) = -\infty.$$

Hence by Proposition 3.1.1, $\eta \in \mathcal{L}(X)$ does not extend in $\mathcal{L}(\mathbb{C}^n)$.

Now we assume that any two irreducible components of the germ $(\overline{X}, a)$ are linked for any $a \in \overline{X} \setminus X$. Let $\eta \in \mathcal{L}(X)$ and X_i and X_j be arbitrary irreducible components of $(\overline{X}, a)$. By the assumption there exist some irreducible components $X_{i_k}{}'s$ such that all the intersections $X_i \cap X_{i_1} \cap \mathbb{C}^n$, $X_{i_1} \cap X_{i_2} \cap \mathbb{C}^n$,..., $X_{i_m} \cap X_j \cap \mathbb{C}^n$ have positive dimension. Let C be a positive dimensional irreducible analytic subvariety of $X_i \cap X_{i_1}$ which is not contained in $\{t = 0\} \subset \mathbb{P}^3$. $\eta|_{X_i \cap \{t=1\}}$ induces a function $\tilde{\eta}$ on $X_i \subset \mathbb{P}^n$ defined by

$$\tilde{\eta}([t:z]) = \begin{cases} \eta(z) - \rho(1,z) & \text{if } t = 1,\ z \in X_i \cap \mathbb{C}^n \\ \limsup_{X_i \ni [1:\zeta] \to [0:z]} (\eta(\zeta) - \rho(1,\zeta)) & \text{if } t = 0,\ [0,z] \in X_i. \end{cases}$$

Since X_i is locally irreducible near a, [D3, Theorem 1.7] implies that $\tilde{\eta}$ is $\omega|_{X_i}$-psh on X_i.

Then $\tilde{\eta}|_C$ is $\omega|_C$ - psh on C. It follows that

$$\begin{aligned}\limsup_{X_i \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) &= \tilde{\eta}(a) = \tilde{\eta}|_C(a) = \limsup_{\mathbb{C}^n \cap C \ni [1:z] \to a} \tilde{\eta}|_C([1:z]) \\ &= \limsup_{C \ni [1:z] \to a} (\eta(z) - \rho(1,z)).\end{aligned}$$

Note that the third equality above follows from the Lemma 3.2.2. By changing X_i with X_{i_1} above, we obtain that

$$\limsup_{X_{i_1} \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = \limsup_{C \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = \limsup_{X_i \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)).$$

By applying the same argument as above to the other irreducible components X_{i_k} of the germ $(\overline{X}, a)$ we conclude that

$$\limsup_{X_i \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = \limsup_{X_j \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)).$$

It follows from Proposition 3.1.1 that $\eta \in \mathcal{L}(X)$ extends in $\mathcal{L}(\mathbb{C}^n)$. □

Remark 3.2.3. For $X \subset \mathbb{C}^2$ the intersection of two irreducible components of the germ $(\overline{X}, a)$ is given by at most a finite set of points. Thus any two irreducible components of the germ the $(\overline{X}, a)$ are not linked when $(\overline{X}, a)$ is reducible. Therefore Theorem 3.1.2 has the following immediate corollary in dimension two:

Corollary 3.2.4. *Let X be an algebraic variety in $\mathbb{C}^2$. Then any function in $\mathcal{L}(X)$ extends to a function in $\mathcal{L}(\mathbb{C}^2)$ if and only if the germs $(\overline{X}, a)$ are irreducible for all points $a \in \overline{X} \setminus X$.*

3.3 Examples

In this section, we study some well known examples of algebraic varieties X and check that whether all functions in $\mathcal{L}(X)$ extend to a function in $\mathcal{L}(\mathbb{C}^n)$. In cases where $\mathcal{L}(X) \neq \mathcal{L}(\mathbb{C}^n)|_X$ we construct a function $\eta \in \mathcal{L}(X)$ which has no extension in $\mathcal{L}(\mathbb{C}^n)$. Then we find explicitly an extension of η in $\mathcal{L}_\gamma(\mathbb{C}^n)$ where $\gamma > 1$.

Example 3.3.1. Let X be the surface in $\mathbb{C}^3$ given by equation $x^4 + y^4 + z^4 + 1 = 0$. The closure of X in $\mathbb{P}^3$ is

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : t^4 + x^4 + y^4 + z^4 = 0\}.$$

This surface is known as the Fermat quartic surface in $\mathbb{P}^3$. One checks that $\overline{X}$ is a smooth surface. Thus the germ $(\overline{X}, a)$ is irreducible for all $a \in \overline{X} \setminus X$ and $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$ by Proposition 3.1.1.

Example 3.3.2. We consider the Segre map

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

given by $\sigma(([s_0 : s_1], [t_0 : t_1])) = [s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1]$. The image of σ is the surface in $\mathbb{P}^3$ with the equation $tz - xy = 0$. One checks that it is a smooth surface. Therefore $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$ by Proposition 3.1.1.

Example 3.3.3. Let X be the surface in $\mathbb{C}^3$ given by equation $x^2 - y^2z = 0$. This surface is called Whitney's umbrella. The closure of X in $\mathbb{P}^3$ is given by

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : x^2 t - y^2 z = 0\}.$$

Let $H = \{t = 0\}$ be the hyperplane at infinity in $\mathbb{P}^3$. Then $\overline{X} \cap H = \{t = y = 0\} \cup \{t = z = 0\}$. First we will find the singular points of $\overline{X}$ in the hyperplane at infinity. In $\{x \neq 0\}$, $\overline{X}$ is defined by $f(t, y, z) = t - y^2 z = 0$. The gradient $\nabla f = (1, -2yz, -y^2)$ never vanishes. Thus $\overline{X}$ does not have any singular point in the open set $\{x \neq 0\} \subset \mathbb{P}^3$. In $\{y \neq 0\}$, $\overline{X}$ is defined by $f(t, x, z) = x^2 t - z = 0$. Again $\nabla f = (x^2, 2tx, -1)$ never vanishes. Thus $\overline{X}$ does not have any singular point in the open set $\{y \neq 0\} \subset \mathbb{P}^3$. In $\{z \neq 0\}$, $\overline{X}$ is defined by $f(t, x, z) = x^2 t - y^2 = 0$. Then $\nabla f = (x^2, 2tx, -2y)$ vanishes only when $x = y = 0$. Thus $a = [0 : 0 : 0 : 1]$ is the only singular point of $\overline{X}$ in the hyperplane at infinity.

Near a, $\overline{X}$ is given by

$$\{(t, x, y) \in \mathbb{C}^3 : f(t, x, y) = y^2 - x^2 t = 0\},$$

near the origin in $\mathbb{C}^3$. We will show that the germ $(\overline{X}, a)$ is irreducible. This is equivalent to show that f is irreducible in $\mathcal{O}_{\mathbb{C}^3,0}$. Since f is a Weierstrass polynomial, it is enough to show that f is irreducible in $\mathcal{O}_{\mathbb{C}^2,0}[y]$. Suppose that f is reducible in $\mathcal{O}_{\mathbb{C}^2,0}[y]$. Then $f(t, x, y) = (y - g(t, x))(y - h(t, x)) = 0$ where $g \in \mathcal{O}_{\mathbb{C}^2,0}$ and $h \in \mathcal{O}_{\mathbb{C}^2,0}$ with $g(0, 0) = h(0, 0) = 0$. It follows that $gh = -x^2 t$ and $g + h = 0$. Since t divides gh, we may assume without loss of generality that $t|g$. Since $h = -g$, $t|h$. Then $t^2|gh$ which is a contradiction. Thus the germ $(\overline{X}, a)$ is irreducible and $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$ by Theorem 3.1.2.

Example 3.3.4. Let X be the surface in $\mathbb{C}^3$ given by equation $xyz + xy + yz + zx = 0$. Then

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : F(t, x, y, z) = txy + xyz + yzt + ztx = 0\}.$$

$\overline{X}$ is known as Cayley's cubic nodal surface in $\mathbb{P}^3$. Let $H = \{t = 0\} \subset \mathbb{P}^3$ be the hyperplane at infinity. One checks that

$$\overline{X} \cap H = \{x = t = 0\} \cup \{y = t = 0\} \cup \{z = t = 0\}$$

and the only singular points of $\overline{X}$ in H are $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. Note that $\overline{X}$ is symetric with respect to x, y, z. Without loss of generality we can take $a = [0 : 1 : 0 : 0] \in \overline{X} \setminus X$. Near a

$$\overline{X} = \{(t, y, z) \in \mathbb{C}^3 : ty + yz + zt + yzt = 0\},$$

near the origin in $\mathbb{C}^3$. We change the coordinates by $y = y' + z'$, $z = y' - z'$, $t = t'$. Then near $(0, 0, 0)$

$$\overline{X} = \{(t', y', z') \in \mathbb{C}^3 : f(t', y', z') = y'^2 + \frac{2t'}{1 + t'}y' - z'^2 = 0\}.$$

We will show that the germ $(\overline{X}, a)$ is irreducible. Suppose that the Weierstrass polynomial f is reducible in $\mathcal{O}_{\mathbb{C}^2,0}[y']$. Then $f(t', y', z') = (y' - g(t', z'))(y' - h(t', z'))$ where $g \in \mathcal{O}_{\mathbb{C}^2,0}$

and $h \in \mathcal{O}_{\mathbb{C}^2,0}$ with $g(0,0) = h(0,0) = 0$. We have

$$g(t', z')h(t', z') = -z'^2 \tag{3.3}$$

and

$$g(t', z') + h(t', z') = \frac{-2t'}{1+t'}. \tag{3.4}$$

Equation (3.3) above implies $z'|g$ or $z'|h$. If z' divides both, then $z'|g+h$ but this contradicts to equation (3.4) above. Thus we may assume that $z' \not| g$. It follows that $h(t', z') = z'^2\tilde{h}(t', z')$ for some $\tilde{h} \in \mathcal{O}_{\mathbb{C}^2,0}$. This with equation (3.3) and equation (3.4) above imply that

$$g(t', z')h(t', z') = \frac{-2t'}{1+t'}z'^2\tilde{h}(t', z') - \tilde{h}^2(t', z')z'^4 = -z'^2.$$

When $t' = 0$ above equation becomes $z'^2\tilde{h}^2(0, z') = 1$ which is a contradiction since $\tilde{h}(0, z')$ is holomorphic near 0 in $\mathbb{C}$. Therefore the germ $(\overline{X}, a)$ is irreducible and by Proposition 3.1.1, $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$.

Example 3.3.5. Let X be the surface in $\mathbb{C}^3$ given by equation $zx^2 + zy^2 - 2xy = 0$. This surface is called Plücker's conoid. Then

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : F(t, x, y, z) = zx^2 + zy^2 - 2xyt = 0\}.$$

$H \cap \overline{X} = \{t = z(x^2 + y^2) = 0\}$. One verifies that $a = [0 : 0 : 0 : 1] \in \overline{X} \setminus X$ is the only singular point of $\overline{X}$ in the hyperplane at infinity. Near a, $\overline{X}$ is given by

$$\{(t,x,y) \in \mathbb{C}^3 : f(t,x,y) = x^2 - 2xty + y^2 = 0\},$$

near the origin in $\mathbb{C}^3$. In a neighborhood of the origin, $f = f_1 f_2$ where

$$f_1(t,x,y) = x - y(t + \sqrt{t^2-1}) \text{ and } f_2(t,x,y) = x - y(t - \sqrt{t^2-1}).$$

Here we take a branch of the root function with $\sqrt{-1} = i$. One verifies that $\{f_1 = 0\} \cap \{f_2 = 0\} = \{x = y = 0\}$ near the origin in $\mathbb{C}^3$. Thus the germ $(\overline{X}, a)$ has two irreducible components whose intersection lies along a line not contained in the hyperplane at infinity. So these irreducible components are linked and $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$ by Theorem 3.1.2.

Example 3.3.6. Let X be the surface in $\mathbb{C}^3$ given by equation $xy^2 + y^2z^2 + z^2x^2 - xyz = 0$. Then

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : xty^2 + y^2z^2 + z^2x^2 - xyzt = 0\}.$$

This surface is called the Roman (Steiner) surface. $H \cap \overline{X} = \{t = z = 0\} \cup \{t = x^2 + y^2 = 0\}$. One checks that $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$ are the only singular points of $\overline{X}$ in the hyperplane at infinity.

Near $a = [0 : 1 : 0 : 0]$, $\overline{X}$ is given by

$$\{(t,y,z) \in \mathbb{C}^3 : f(t,y,z) = z^2 - \frac{yt}{1+y^2} z + \frac{y^2 t}{1+y^2} = 0\},$$

near $(0,0,0)$. We claim that the Weierstrass polynomial $f \in \mathcal{O}_{\mathbb{C}^2,0}[z]$ is irreducible. Otherwise

$$f(t,y,z) = (z - g(t,y))(z - h(t,y)),$$

where $g \in \mathcal{O}_{\mathbb{C}^2,0}$ and $h \in \mathcal{O}_{\mathbb{C}^2,0}$ with $g(0,0) = h(0,0) = 0$. We have

$$gh = \frac{y^2 t}{1+y^2}, \tag{3.5}$$

and

$$g + h = \frac{yt}{1+y^2}. \tag{3.6}$$

Since t divides gh, we may assume without loss of generality that $t|g$. By equation (3.6) $t|h$. This implies that $t^2|gh$ which contradicts equation (3.5). Thus the germ $(\overline{X}, a)$ is irreducible.

Let $a = [0:0:1:0]$. Near a, the germ $\overline{X}$ is given by

$$\{(t,x,z) \in \mathbb{C}^3 : f(t,x,z) = z^2 - \frac{xt}{1+x^2} z + \frac{xt}{1+x^2} = 0\}.$$

We claim that the Weierstrass polynomial $f \in \mathcal{O}_{\mathbb{C}^2,0}[z]$ is irreducible. Otherwise $f(t,x,z) = (z - g(t,x))(z - h(t,x))$ where $g \in \mathcal{O}_{\mathbb{C}^2,0}$ and $h \in \mathcal{O}_{\mathbb{C}^2,0}$ with $g(0,0) = h(0,0) = 0$. It follows that

$$gh = \frac{xt}{1+x^2}, \tag{3.7}$$

and

$$g + h = \frac{xt}{1+x^2}. \tag{3.8}$$

Since t divides gh, we may assume without loss of generality that $t|g$. By equation (3.8) $t|h$ and this implies that $t^2|gh$ which contradicts equation (3.7). Thus the germ $(\overline{X}, a)$ is irreducible.

Let $a = [0:0:0:1]$. Near a, $\overline{X}$ is given by

$$\{(t,x,y) \in \mathbb{C}^3 : f(t,x,y) = y^2 - \frac{xt}{1+xt}y + \frac{x^2}{1+xt} = 0\},$$

near the origin in $\mathbb{C}^3$. The function f can be written as $f = f_1 f_2$ where

$$f_1(t,x,y) = y - \frac{x}{2(1+xt)}(t + \sqrt{t^2 - 4xt - 4}),$$

$$f_2(t,x,y) = y - \frac{x}{2(1+xt)}(t - \sqrt{t^2 - 4xt - 4}).$$

Here we take a branch of root function with $\sqrt{-4} = 2i$. One checks that $\{f_1 = 0\} \cap \{f_2 = 0\} = \{x = y = 0\}$ near $(0,0,0)$. Thus the germ $(\overline{X}, a)$ has two irreducible components whose intersection lies along a line not contained in the hyperplane at infinity. Therefore these irreducible components are linked and by Theorem 3.1.2, $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^3)|_X$.

The following example is a generalization of [CGZ, Example 3.2].

Example 3.3.7. Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $X = \{z_{m+1} = ... = z_n = 0\} \cup \{z_{m+1} = ... = z_n = 1\} \subset \mathbb{C}^n$ be an m dimensional subvariety of $\mathbb{C}^n$. Let $\rho(t,z) = \log \sqrt{|t|^2 + ||z||^2}$. The function

$$\eta(z) = \begin{cases} \rho(1,z) & \text{if } z \in X_1 = \{z_{m+1} = ... = z_n = 0\}, \\ \rho(1,z) + 1 & \text{if } z \in X_2 = \{z_{m+1} = ... = z_n = 1\}, \end{cases}$$

is in $\mathcal{L}(X)$ and

$$\limsup_{X_1 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = 0, \ \limsup_{X_2 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = 1,$$

where $a = [0 : 1 : 0 : .. : 0] \in \overline{X} \setminus X$. Proposition 3.1.1 implies that η does not extend in $\mathcal{L}(\mathbb{C}^n)$. However by [CGZ, Theorem A], we can find an extension with arbitrarily small additional growth. Explicitly we take

$$\tilde{\eta}(z) = \rho(1,z) + \epsilon \log |1 + z_n(e^{\frac{1}{\epsilon}} - 1)|.$$

Then $\tilde{\eta} \in \mathcal{L}_{1+\epsilon}(\mathbb{C}^n)$ and $\tilde{\eta}|_X = \eta$.

The following example is a generalization of [CGZ, Example 3.3].

Example 3.3.8. Let X be given by the equation $z_1 z_n = z_1^3 + 1$. It is clear that X is irreducible in $\mathbb{C}^n$. The closure $\overline{X}$ of X in $\mathbb{P}^n$ is given by

$$\overline{X} = \{[t : z_1 : ... : z_n] \in \mathbb{P}^n : z_1 z_n t = z_1^3 + t^3\} = X \cup \{t = z_1 = 0\}.$$

We take $a = [0 : ... : 0 : 1] \in \overline{X} \setminus X$. First we will show that the germ $(\overline{X}, a)$ has two irreducible components X_1 and X_2 whose intersection lies in the hyperplane at infinity. Let $(s_0, ..., s_{n-1})$ be affine coordinates near $a \in \{z_n \neq 0\}$ where $s_0 = \frac{t}{z_n}$, $s_i = \frac{z_i}{z_n}$. In these coordinates the germ $(\overline{X}, a)$ is defined by $s_0 s_1 = s_0^3 + s_1^3$. We change the coordinate s_0 by $u = s_0 + s_1$. In the new coordinates the germ $(\overline{X}, a)$ is defined by the Weierstrass polynomial $f(u, s_1, ..., s_{n-1}) = s_1^2 - s_1 u + \frac{u^3}{3u+1} = 0$ and $f = f_1 f_2$ where f_1 and f_2 are germs of holomorphic functions in $\mathcal{O}_{\mathbb{C}^n,0}$ defined by

$$f_1(u, s_1, ..., s_{n-1}) = s_1 - \frac{1}{2}u\left(1 + \sqrt{1 - \frac{4u}{3u+1}}\right),$$

$$f_2(u, s_1, ..., s_{n-1}) = s_1 - \frac{1}{2}u\left(1 - \sqrt{1 - \frac{4u}{3u+1}}\right).$$

Then $\{f_1 = 0\} \cap \{f_2 = 0\} = \{s_1 = u = 0\}$ near a. In the original coordinates, $\{f_1 = 0\} \cap \{f_2 = 0\} = \{s_0 = s_1 = 0\}$. Thus the germ $(\overline{X}, a)$ has two irreducible components X_1 and X_2 whose intersection is contained in the hyperplane at infinity. Therefore X_1 and X_2 are not linked and by Theorem 3.1.2, $\mathcal{L}(X) \neq \mathcal{L}(\mathbb{C}^n)|_X$.

Now we will give an explicit example of a function in $\mathcal{L}(X)$ which has no extension in $\mathcal{L}(\mathbb{C}^n)$. Let

$$\psi(z) = \max(\log|z_n - z_1^2|, 2\log|z_1| + 1) \text{ and } \eta := \psi|_X.$$

Using that $z_n = z_1^2 + 1/z_1$ on X,

$$\eta(z) = \max\left(\log\left|\frac{1}{z_1}\right|, \log\left|z_n - \frac{1}{z_1}\right| + 1\right).$$

Thus $\eta \in \mathcal{L}(X)$. On X, in the coordinates $(s_0, ..., s_{n-1})$ near a, we have $s_1 \neq 0$ as $z_1 \neq 0$ and $s_0 \neq 0$ since $X \subset \mathbb{C}^n = \{t \neq 0\}$. In these coordinates near a, the functions η and ρ are given by

$$\eta(s_0, ..., s_{n-1}) = \max\left(\log\left|\frac{s_0}{s_1}\right|, 2\log\left|\frac{s_1}{s_0}\right| + 1\right),$$

and

$$\rho(s_0, ..., s_{n-1}) = \log\left(1 + \left|\frac{s_1}{s_0}\right|^2 + ... + \left|\frac{s_{n-1}}{s_0}\right|^2 + \frac{1}{|s_0|^2}\right)^{\frac{1}{2}} = \log\frac{1}{|s_0|} + o(1),$$

as $(s_0, ..., s_{n-1}) \to (0, ..., 0)$.

On the germ (X_1, a) in the coordinates $(s_0, ..., s_{n-1})$,

$$f_1(s_0, ..., s_{n-1}) = -s_0 - \frac{1}{2}(s_0 + s_1)O(|s_0 + s_1|) = 0.$$

This implies that

$$\left|\frac{s_1}{s_0}\right| = \frac{2 + O(|s_0 + s_1|)}{O(|s_0 + s_1|)} \to \infty,$$

as $(s_0, s_1) \to (0, 0)$.

On the germ (X_2, a) in the coordinates $(s_0, ..., s_{n-1})$,

$$f_2(s_0, ..., s_{n-1}) = s_1 + \frac{1}{2}(s_0 + s_1)O(|s_0 + s_1|) = 0.$$

This implies that

$$\left|\frac{s_0}{s_1}\right| = \frac{2 + O(|s_0 + s_1|)}{O(|s_0 + s_1|)} \to \infty,$$

as $(s_0, s_1) \to (0,0)$. Since $s_0 s_1 = s_0^3 + s_1^3$ on $\overline{X}$ near a,

$$\begin{aligned}
\limsup_{X_1 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) &= \limsup_{(s_0,s_1)\to(0,0)} \left(2\log\left|\frac{s_1}{s_0}\right| + 1 - \log\frac{1}{|s_0|}\right) \\
&= \limsup_{(s_0,s_1)\to(0,0)} \log\left|\frac{s_1^2}{s_0}\right| + 1 \\
&= \limsup_{(s_0,s_1)\to(0,0)} \log\left|1 - \frac{s_0^2}{s_1}\right| + 1 = 1.
\end{aligned}$$

Similarly we obtain that

$$\begin{aligned}
\limsup_{X_2 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) &= \limsup_{(s_0,s_1)\to(0,0)} \left(\log\left|\frac{s_0}{s_1}\right| - \log\frac{1}{|s_0|}\right) \\
&= \limsup_{(s_0,s_1)\to(0,0)} \log\left|\frac{s_0^2}{s_1}\right| \\
&= \limsup_{(s_0,s_1)\to(0,0)} \log\left|1 - \frac{s_1^2}{s_0}\right| = 0.
\end{aligned}$$

Thus by Proposition 3.1.1, η has no extension in $\mathcal{L}(\mathbb{C}^n)$. However we know from [CGZ, Theorem A] that with an arbitrarily small additional growth η has an extension in $\mathcal{L}_\gamma(\mathbb{C}^n)$ where $\gamma > 1$. We will give an explicit extension using a similar idea given in the proof of [BL, Proposition 3.3].

Let $\Theta = \{z \in \mathbb{C}^n : |z_1 z_n - z_1^3 - 1| < e^{-3}\}$. Clearly $X \subset \Theta$. In $\bar{\Theta} \subset \mathbb{C}^n$ when $|z_1| < 2$, ψ

has logarithmic growth. In $\bar{\Theta}$ when $|z_1| \geq 2$, $\psi(z) = \log|z_1|^2 + 1$ and

$$|z_1|^2 - \frac{\delta+1}{2} < |z_n| < |z_1|^2 + \frac{\delta+1}{2}, \tag{3.9}$$

where $\delta = e^{-3}$. Thus ψ has logarithmic growth in $\bar{\Theta}$. One can easily check that $\psi(z) \leq \log^+||z|| + 3$ in $\bar{\Theta}$. Indeed, if $z \in \Theta$ and $|z_1| < 2$ then

$$\psi(z) \leq \max(\log(|z_n| + 4), \log 4 + 1) \leq \max(\log(5|z_n|), 3) \leq \log^+||z|| + 3.$$

If $z \in \Theta$ and $|z_1| \geq 2$ then inequality (3.9) implies that

$$\begin{aligned} \psi(z) &= \log|z_1|^2 + 1 \leq \log(|z_n| + 1) + 1 \\ &\leq \max(\log 2, \log(2|z_n|)) + 1 \leq \log^+||z|| + 3. \end{aligned}$$

Let $\phi(z) = \epsilon(\frac{1}{3}\log|z_1 z_n - z_1^3 - 1| + 1)$ where $\epsilon > 0$. Then $\phi \in \mathcal{L}_\epsilon(\mathbb{C}^n)$, $\phi = -\infty$ on X and $\phi \geq 0$ on $\mathbb{C}^n \setminus \Theta$. We now define

$$\tilde{\eta}(z) = \begin{cases} \max(\psi(z), \log^+||z|| + \phi(z) + 3) & \text{if } z \in \Theta, \\ \log^+||z|| + \phi(z) + 3 & \text{if } z \in \mathbb{C}^n \setminus \Theta. \end{cases}$$

We have $\psi(z) \leq \log^+||z|| + \phi(z) + 3$ on $\partial\Theta$. So $\tilde{\eta} \in PSH(\mathbb{C}^n)$. Since $\psi \in \mathcal{L}(\Theta)$ and $\phi \in \mathcal{L}_\epsilon(\mathbb{C}^n)$, $\tilde{\eta} \in \mathcal{L}_{1+\epsilon}(\mathbb{C}^n)$ and $\tilde{\eta}|_X = \eta$.

Example 3.3.9. Let X be the surface in $\mathbb{C}^3$ with the equation $3z - 3xy + x^3 = 0$. Then

$$\overline{X} = \{[t : x : y : z] \in \mathbb{P}^3 : 3zt^2 - 3xyt + x^3 = 0\}.$$

This surface is called Cayley's ruled cubic surface. One verifies that $\overline{X} \cap H = \{t = x = 0\}$ and all the points on the line $\{t = x = 0\}$ are singular.

Let (s_0, s_1, s_2) be affine coordinates near $a = [0 : 0 : 1 : 1] \in \{z \neq 0\}$ where $s_0 = \frac{t}{z}$, $s_1 = \frac{x}{z}$ and $s_2 = \frac{y}{z}$. In these coordinates the germ $(\overline{X}, a)$ is defined by

$$\{(s_0, s_1, s_2) \in \mathbb{C}^3 : f(s_0, s_1, s_2) = s_0^2 - s_1 s_2 s_0 + \frac{s_1^3}{3} = 0\},$$

near $(0, 0, 1)$. The function f can be written as

$$f(s_0, s_1, s_2) = f_1(s_0, s_1, s_2) f_2(s_0, s_1, s_2)$$

where

$$f_1(s_0, s_1, s_2) = s_0 - \frac{s_1 s_2}{2}\left(1 - \sqrt{1 - \frac{4s_1}{3s_2^2}}\right)$$

and

$$f_2(s_0, s_1, s_2) = s_0 - \frac{s_1 s_2}{2}\left(1 + \sqrt{1 - \frac{4s_1}{3s_2^2}}\right)$$

are holomorphic near $(0, 0, 1)$. In a small neighborhood of $(0, 0, 1)$,

$$\{f_1 = 0\} \cap \{f_2 = 0\} = \{s_0 = s_1 = 0\}.$$

Thus the germ $(\overline{X}, a)$ has two irreducible components whose intersection lies along a line contained in the hyperplane at infinity. Therefore these irreducible components are not linked and by Theorem 3.1.2, $\mathcal{L}(X) \neq \mathcal{L}(\mathbb{C}^3)|_X$.

Now we will give an explicit example of a function in $\mathcal{L}(X)$ which does not extend to $\mathbb{C}^3$ with logarithmic growth. Let X_1, X_2 be two irreducible components of the germ $(\overline{X}, a)$ defined by f_1 and f_2 respectively. Let C_1 be the curve $\{(x, \frac{x^2+x}{3}, \frac{x^2}{3}) : x \in \mathbb{C}\} \subset X$. In the coordinates (s_0, s_1, s_2) near a, C_1 is given by the equations $s_0 = \frac{s_1^2}{3}$ and $s_2 = \frac{s_1}{3} + 1$. On C_1 near a

$$\begin{aligned} f_1(s_0, s_1, s_2) &= f_1\left(\frac{s_1^2}{3}, s_1, 1 + \frac{s_1}{3}\right) \\ &= \frac{s_1^2}{3} - \left(\frac{s_1}{2} + \frac{s_1^2}{6}\right)\left(1 - \sqrt{1 - \frac{4s_1}{3(\frac{s_1}{3}+1)^2}}\right) = 0. \end{aligned}$$

Therefore C_1 contained in X_1 near a.

Let C_2 be the curve $\{(1, y, y - \frac{1}{3}) : y \in \mathbb{C}\} \subset X$. In the coordinates (s_0, s_1, s_2) near a, C_1 is given by the equations $s_2 = \frac{s_0}{3} + 1$, $s_1 = s_0$. On C_2 near a

$$\begin{aligned} f_2(s_0, s_1, s_2) &= f_2\left(s_0, s_0, \frac{s_0}{3} + 1\right) \\ &= s_0 - \frac{1}{2}\left(\frac{s_0^3}{3} + s_0\right)\left(1 + \sqrt{1 - \frac{4s_0}{3(\frac{s_0}{3}+1)^2}}\right) = 0. \end{aligned}$$

Thus C_2 contained in X_2 near a.

Let $\psi(x, y, z) = \max(\log|z|, 2\log|x|) \in PSH(\mathbb{C}^3)$ and $\eta := \psi|_X$. First we show that $\eta \in \mathcal{L}(X)$. When $|z| > |x|^2$, $\eta(x, y, z) = \log|z|$. Hence we may assume that $|z| \leq |x|^2$. It follows that $\eta(x, y, z) = \log|x|^2$ and $|y| > \frac{|x|^2}{3} - \frac{|z|}{|x|} > \frac{|x|^2}{3} - |x| > \frac{|x|^2}{6}$ when $|x| > 6$ on X.

Thus $\eta \in \mathcal{L}(X)$.

Note that $\overline{C_1} = C_1 \cup \{a\}$ and $X_1 \supset C_1 \ni (x, \frac{x^2+x}{3}, \frac{x^2}{3}) \to a$ as $x \to \infty$. $\eta|_{X_i \cap \{t=1\}}$ induces a function $\tilde{\eta}_i$ on $X_i \subset \mathbb{P}^3$ defined near a by

$$\tilde{\eta}_i([t:\tau]) = \begin{cases} \eta(\tau) - \rho(1,\tau) & t = 1, \\ \limsup_{X_i \cap \mathbb{C}^3 \ni [1:\zeta] \to [0:\tau]} (\eta(\zeta) - \rho(1,\zeta)) & t = 0, \end{cases}$$

where $\tau = (x, y, z)$. Since X_i is locally irreducible near a, [D3, Theorem 1.7] implies that $\tilde{\eta}_i$ is $\omega|_{X_i} - psh$ on X_i and $\tilde{\eta}_i|_{\bar{C}_i}$ is $\omega|_{\bar{C}_i} - psh$ on $\bar{C}_i$. Then

$$\begin{aligned} \limsup_{X_1 \cap \mathbb{C}^3 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) &= \tilde{\eta}_1(a) = \tilde{\eta}_1|_{\bar{C}_1}(a) = \limsup_{C_1 \ni \tau \to a} \tilde{\eta}_1|_{\bar{C}_1}(\tau) \\ &= \limsup_{x \to \infty} \left(\eta\left(x, \frac{x^2+x}{3}, \frac{x^2}{3}\right) - \rho\left(1, x, \frac{x^2+x}{3}, \frac{x^2}{3}\right)\right) = \log\left(\frac{3}{\sqrt{2}}\right). \end{aligned}$$

The third equality holds by Lemma 3.2.2. Similarly

$$\begin{aligned} \limsup_{X_2 \cap \mathbb{C}^3 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) &= \tilde{\eta}_2(a) = \tilde{\eta}_2|_{\bar{C}_2}(a) = \limsup_{C_2 \ni \tau \to a} \tilde{\eta}_2|_{\bar{C}_2}(\tau) \\ &= \limsup_{y \to \infty} \left(\eta\left(1, y, y - \frac{1}{3}\right) - \rho\left(1, 1, y, y - \frac{1}{3}\right)\right) = \log\left(\frac{1}{\sqrt{2}}\right). \end{aligned}$$

By Proposition 3.1.1, η does not extend in $\mathcal{L}(\mathbb{C}^3)$.

We know from [CGZ, Theorem A] that with an arbitrarily small additional growth η extends in $\mathcal{L}_\gamma(\mathbb{C}^3)$ where $\gamma > 1$. We will give an explicit extension using a similar idea given in the proof of [BL, Proposition 3.3].

Let $\Theta = \{(x, y, z) \in \mathbb{C}^3 : |3z - 3xy + x^3| < e^{-3}\}$. Clearly X is contained in Θ. First we show that $\psi \in \mathcal{L}(\Theta)$. When $|x| < 6$, $\psi(x, y, z) \le \log|z| + 4$. When $|z| > |x|^2$, $\psi(x, y, z) =$

$\log|z|$. Hence we can assume that $|z| \leq |x|^2$ and $|x| \geq 6$. It follows that $|y - \frac{z}{x} - \frac{x^2}{3}| < \frac{e^{-3}}{18}$. Therefore $|y| > \frac{|x|^2}{12}$ and

$$\psi(x,y,z) = \log|x|^2 \leq \log|y| + \log 12.$$

Thus in Θ, $\psi(x,y,z) \leq \log||(x,y,z)|| + 4$. That is, $\psi \in \mathcal{L}(\Theta)$. Let

$$\phi(x,y,z) = \epsilon\left(\frac{1}{3}\log|3z - 3xy + x^3| + 1\right),$$

where $\epsilon > 0$. Then $\phi \in \mathcal{L}_\epsilon(\mathbb{C}^3)$, $\phi = -\infty$ on X and $\phi \geq 0$ on $\mathbb{C}^3 \setminus \Theta$. We now define

$$\tilde{\eta}(\tau) = \begin{cases} \max(\psi(\tau), \log||\tau|| + \phi(\tau) + 4) & \text{if } \tau = (x,y,z) \in \Theta, \\ \log||\tau|| + \phi(\tau) + 4 & \text{if } \tau \in \mathbb{C}^3 \setminus \Theta. \end{cases}$$

Since we have $\psi(\tau) \leq \log||\tau|| + \phi(\tau) + 4$ on $\partial\Theta$, $\tilde{\eta} \in PSH(\mathbb{C}^3)$. As $\psi \in \mathcal{L}(\Theta)$ and $\phi \in \mathcal{L}_\epsilon(\mathbb{C}^3)$, $\tilde{\eta} \in \mathcal{L}_{1+\epsilon}(\mathbb{C}^3)$ and $\tilde{\eta}|_X = \eta$.

Chapter 4

Dynamics of the Automorphisms in the Class H_5

Automorphisms in H_5 are of the form

$$(x, y, z) \to (P(x, y) + az, Q(x) + by, x)$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$. Some of these maps are not regular and for simplicity, we will consider the irregular automorphisms

$$H(x, y, z) = (xy + az, x^2 + by, x)$$

where $a \neq 0 \neq b$. We note that most of the results that we prove here for this simplified maps hold in fact for an arbitrary irregular map in H_5.

Let $\mathbb{P}^3$ be the projective space with coordinates $[x : y : z : t]$. We identify $\mathbb{C}^3$ with $\{t = 1\}$ in $\mathbb{P}^3$. H defines a birational map of $\mathbb{P}^3$ by

$$H([x : y : z : t]) = [xy + azt : x^2 + byt : xt : t^2].$$

The indeterminacy sets of H and H^{-1} are $I^+ = \{t = x = 0\}$ and $I^- = \{t = z = 0\}$. The extended indeterminacy set $I^+_\infty = \cup_{j=1}^{\infty} I_{H^j} = I^+ \cup \{t = y = 0\}$, $X^+ = H(\{t = 0\} \setminus I^+) = I^-$ and $X^- = H(\{t = 0\} \setminus I^-)$=[0:0:1:0]. We will start with the dynamics of H^{-1} which is less complicated than the dynamics of the map itself.

4.1 Dynamics of H^{-1}

H^{-1} is given by

$$H^{-1}(x, y, z) = \left(z, \frac{y - z^2}{b}, \frac{1}{a} \left(x - z\frac{y - z^2}{b} \right) \right).$$

We will denote the iterates of H^{-1} by $H^{-n}(w) = (x_n, y_n, z_n)$, where $w = (x, y, z) \in \mathbb{C}^3$. Let us define the sets

$$\begin{aligned} V^+ &= \{w \in \mathbb{C}^3 : |z| > \max\{R, |x|, (1 + \delta)|y|^{1/2}\}\}, \\ U^- &= \cup_{n=0}^{\infty} H^n(V^+), \\ K^- &= \mathbb{C}^3 \setminus U^-, \end{aligned}$$

where $\delta > 0$ and R is sufficiently big. In [CF, Lemma 6.1, Lemma 6.2], it was shown that the H^{-1}-orbits of points in U^- converge locally uniformly on U^- to $[0:0:1:0]$ with super-exponential rate $(const)^{3^n}$, and for $w \in K^-$, $||H^{-n}(w)|| \leq C^n M(w)$ where $C > 1$ and $M(w) > 0$ depends on w continuously. We will show that when $|b| > 1$, the H^{-1}-orbits of points in K^- are actually bounded. Let's assume that $|b| = 1 + \delta$, $\delta > 0$.

Proposition 4.1.1. *(i) There exists $R_0 > 1$ such that for all $R > R_0$, we have $H^{-1}(V^+) \subset V^+$ and if $w = (x, y, z) \in V^+$ then*

$$C_1|z|^3 < |z_1| < C_2|z|^3$$

where C_1 and C_2 depend on δ.

(ii) $||H^{-n}(w)||$ is bounded when $w \in K^-$.

Proof. Let $\alpha > 0$ be a constant satisfying $\alpha + \frac{1}{(1+\delta)^2} < 1$ and $R_0 = \left(\frac{|b|}{\alpha}\right)^{\frac{1}{2}}$. First we note that on V^+,

$$|x| < |z|,\ |y| < \frac{|z|^2}{(1+\delta)^2} \text{ and } |z| > R > R_0 = \left(\frac{|b|}{\alpha}\right)^{\frac{1}{2}}. \tag{4.1}$$

It follows from (4.1) that on V^+ we have

$$\begin{aligned} \left|z_1 - \frac{z^3}{ab}\right| &= \left|\frac{x}{a} - \frac{zy}{ab}\right| < \frac{|z|}{|a|} + \frac{|z|^3}{(1+\delta)^2|ab|} \\ &< \left(\frac{|b|}{|z|^2} + \frac{1}{(1+\delta)^2}\right)\frac{|z|^3}{|ab|} \leq \left(\alpha + \frac{1}{(1+\delta)^2}\right)\frac{|z|^3}{|ab|}, \end{aligned}$$

which implies the estimate in part (i).

On V^+, $R < |z|$, $|x_1| = |z|$ and

$$\begin{aligned}(1+\delta)|y_1|^{1/2} &\leq \frac{(1+\delta)}{|b|^{1/2}}|y-z^2|^{1/2} \leq \frac{(1+\delta)}{|b|^{1/2}}\left(\frac{|z|^2}{(1+\delta)^2}+|z|^2\right)^{1/2} \\ &= \frac{(1+\delta)}{|b|^{1/2}}\left(1+\frac{1}{(1+\delta)^2}\right)^{1/2}|z|.\end{aligned}$$

Since $|z_1| > C_1|z|^3$, these estimates imply that $H^{-1}(V^+) \subset V^+$ when R is big enough.

We will show part (ii) now. By definition, on K^- we have that

$$|z_n| \leq \max\{R, |x_n|, (1+\delta)|y_n|^{\frac{1}{2}}\} =: M_n$$

for all $n \geq 0$ and $|x_{n+1}| = |z_n| \leq M_n$.

Let us consider the case when $|z_n| \leq 1$. Then

$$\begin{aligned}(1+\delta)|y_{n+1}|^{\frac{1}{2}} \leq (1+\delta)\left(\frac{|y_n|}{|b|}+\frac{1}{|b|}\right)^{\frac{1}{2}} &< (1+\delta)\left(\frac{M_n^2}{(1+\delta)^2|b|}+\frac{1}{|b|}\right)^{\frac{1}{2}} \\ &= \left(\frac{M_n^2}{|b|}+\frac{(1+\delta)^2}{|b|}\right)^{\frac{1}{2}} \\ &\leq M_n\left(\frac{1}{|b|}+\frac{(1+\delta)^2}{|b|R^2}\right)^{\frac{1}{2}} < M_n.\end{aligned}$$

When $|z_n| > 1$

$$(1+\delta)|y_{n+1}|^{\frac{1}{2}} = (1+\delta)\left|\frac{x_n}{z_n}-\frac{az_{n+1}}{z_n}\right|^{\frac{1}{2}} < (1+\delta)\left(M_n+|a|M_{n+1}\right)^{\frac{1}{2}}.$$

We also have that $|x_{n+1}| = |z_n| \leq M_n$. Hence

$$M_{n+1} \leq \max\{M_n, (1+\delta)(M_n + |a|M_{n+1})^{1/2}\}.$$

If the right hand side of the above inequality is equal to M_n then $M_{n+1} \leq M_n$ and we are done. Hence we can assume that $M_{n+1}^2 \leq (1+\delta)^2(M_n + |a|M_{n+1})$. Choosing R big enough we obtain that

$$M_{n+1} < \frac{M_{n+1}^2 - (1+\delta)^2|a|M_{n+1}}{(1+\delta)^2} < M_n.$$

Therefore $||H^n(w)||$ is bounded for $w \in K^-$. □

We now construct the Green's function of H^{-1}. Note that $\deg H^{-n} = 3^n$. We let

$$G_n(w) = \frac{1}{3^n}\log^+ ||H^{-n}(w)|| \text{ and } \tilde{G}_n(w) = \frac{1}{3^n}\log^+ |z_n|.$$

The estimates in [CF, Lemma 6.1, Lemma 6.2] imply that G_n and $\tilde{G}_n$ converge locally uniformly to the same Green's function G^-. Hence the Green's function G^- is pluriharmonic on U^-, $K^- = \{G^- = 0\}$ and $G^- \circ H^{-1} = 3G^-$. We define the Green's current by $\mu^- = dd^cG^-$. Then $H^*\mu^- = \frac{1}{3}\mu^-$ and supp $\mu^- = \partial K^-$.

4.2 Dynamics of H

We will denote the nth iteration of H by $H^n(w) = w_n = (x_n, y_n, z_n)$. For $\epsilon > 0$ and R big enough, we define the sets

$$\begin{aligned} V^- &= \left\{(x,y,z) \in \mathbb{C}^3 : |xy| > \max\left\{R, 2a|z|, |x|^{3/2}, \frac{1}{\epsilon}|y|^{3/2}\right\}\right\}, \\ U^+ &= \cup_{n=0}^{\infty} H^{-n}(V^-), \qquad (4.2) \\ K^+ &= \mathbb{C}^3 \setminus U^+. \end{aligned}$$

In [CF], they showed that the orbits $H^n(w)$ of points in U^+ escape to infinity with super-exponential growth rate $(const)^{2^n}$. For the sake of completeness, we give a short proof of this fact for our simplified map.

Theorem 4.2.1. *There exists $\epsilon > 0$ and $R_0 > 1/\epsilon^{10}$ such that for all $R > R_0$, we have $H(V^-) \subset V^-$ and*

$$\begin{aligned} \frac{|xy|}{2} &< |x_1| < \frac{3|xy|}{2} \qquad (4.3) \\ (1 - |b|\epsilon^2)|x|^2 &< |y_1| < (1 + |b|\epsilon^2)|x|^2. \end{aligned}$$

Hence on U^+, $H^n(w)$ escape to infinity with the super-exponential growth $(const)^{2^n}$.

Proof. For the points in V^-, we have that $|2az| < |xy|$. Hence $|x_1 - xy| = |az| < |xy|/2$ which proves the first inequality above. On V^-, $|xy| > \frac{1}{\epsilon}|y|^{\frac{3}{2}}$ which implies that $|y| < \epsilon^2|x|^2$.

Thus

$$|y_1 - x^2| = |by| \le |b|\epsilon^2|x|^2$$

and this implies the second inequality above. We now prove that V^- is invariant under H. On V^-, $|xy| > |x|^{\frac{3}{2}}$ which implies that $|x| < |y|^2$. Since $\epsilon^2|x|^3 > |xy| > R > 1/\epsilon^{10}$,

$$|x| > \frac{1}{\epsilon^4} \text{ and } |y| > \frac{1}{\epsilon^2} \tag{4.4}$$

on V^-. Using this with the first inequality in (4.3), we obtain that

$$\begin{aligned} |x_1y_1| > |x|^3|y|\frac{1-|b|\epsilon^2}{2} &= 2|a||z_1||x|^2|y|\frac{1-|b|\epsilon^2}{4|a|} \\ &\ge 2|a||z_1|\frac{1-|b|\epsilon^2}{\epsilon^{10}4|a|} > 2|az_1|. \end{aligned}$$

(4.3) and (4.4) with the inequality $|y| < \epsilon^2|x|^2$ imply that

$$\begin{aligned} \max\left\{|x_1|^{\frac{3}{2}}, \frac{1}{\epsilon}|y_1|^{\frac{3}{2}}\right\} &\le \max\left\{\left(\frac{3|xy|}{2}\right)^{\frac{3}{2}}, \frac{1}{\epsilon}\left(1+|b|\epsilon^2\right)^{\frac{3}{2}}|x|^3\right\} \\ &\le \max\left\{\epsilon\left(\frac{3}{2}\right)^{3/2}|x|^{5/2}|y|, \frac{1}{\epsilon}(1+|b|\epsilon^2)^{3/2}|x|^3\right\} \\ &\le |x|^3|y|\max\left\{\epsilon^3\left(\frac{3}{2}\right)^{3/2}, \epsilon\left(1+|b|\epsilon^2\right)^{\frac{3}{2}}\right\} \le |x|^3|y|\frac{1-|b|\epsilon^2}{2} < |x_1y_1|. \end{aligned}$$

Thus $H(V^-) \subset V^-$. □

We now discuss the dynamics of H on K^+. Let

$$M_n = M_n(w) := \max\left\{R, 2|az_n|, |x_n|^{3/2}, \frac{1}{\epsilon}|y_n|^{3/2}\right\}. \tag{4.5}$$

Lemma 4.2.2. *On K^+, if R is sufficiently large, then we have*

(i) $|x_n| \leq \frac{3M_{n-1}}{2}$,

(ii) $M_n \leq \max\{(\frac{3M_{n-1}}{2})^{\frac{3}{2}}, \frac{1}{\epsilon}|y_n|^{\frac{3}{2}}\}$,

(iii) $M_n \leq C\max\{M_{n-1}^{\frac{3}{2}}, |x_{n-1}|^3\}$ *for some constant* C,

(iv) $M_n \leq \tilde{M}(w)^{(\sqrt{3})^n}$ *for some continuous function* $\tilde{M}(w)$.

Proof. (i) Note that on K^+, $|x_n y_n| \leq M_n$ for all $n \geq 0$. It follows from the definition of map H and the set M_n that

$$|x_n| \leq |x_{n-1}y_{n-1}| + |az_{n-1}| \leq M_{n-1} + \frac{M_{n-1}}{2} = \frac{3M_{n-1}}{2}.$$

(ii) The inequality $|z_n| = |x_{n-1}| \leq M_{n-1}^{\frac{2}{3}}$ with the estimate in (i) implies that

$$\begin{aligned} M_n &\leq \max\left\{M_{n-1}, 2|a|M_{n-1}^{\frac{2}{3}}, \left(\frac{3M_{n-1}}{2}\right)^{\frac{3}{2}}, \frac{1}{\epsilon}|y_n|^{\frac{3}{2}}\right\} \\ &= \max\left\{\left(\frac{3M_{n-1}}{2}\right)^{\frac{3}{2}}, \frac{1}{\epsilon}|y_n|^{\frac{3}{2}}\right\}. \end{aligned}$$

(iii) By definition of y_n

$$\begin{aligned} |y_n|^{\frac{3}{2}} &\leq (|x_{n-1}|^2 + |b||y_{n-1}|)^{\frac{3}{2}} \\ &\leq 2^{\frac{3}{2}}\max\left\{|x_{n-1}|^3, |b|^{\frac{3}{2}}|y_{n-1}|^{\frac{3}{2}}\right\}. \end{aligned}$$

Then

$$\frac{1}{\epsilon}|y_n|^{\frac{3}{2}} \leq 2^{\frac{3}{2}}\max\left\{\frac{1}{\epsilon}|x_{n-1}|^3, |b|^{\frac{3}{2}}|M_{n-1}|\right\}.$$

This estimate with (ii) proves (iii).

(iv) Since H is a degree two polynomial map, $M_{n-1} \leq CM_{n-2}^2$ for some constant C . So by (i) and (iii) we obtain that

$$M_n \leq C \max\left\{M_{n-2}^3, \left(\frac{3M_{n-2}}{2}\right)^3\right\} = \tilde{C}M_{n-2}^3.$$

Hence if n is even then $M_n \leq \tilde{C}M_{n-2}^3 \leq ... \leq (\tilde{C}M_0)^{(\sqrt{3})^n}$. If n is odd then $M_n \leq (\tilde{C}M_1)^{3^{\frac{n-1}{2}}} \leq ((\tilde{C}M_1)^2)^{(\sqrt{3})^n}$. Thus $M_n \leq \tilde{M}(w)^{(\sqrt{3})^n}$ where $\tilde{M}(w) = \max\{\tilde{C}M_0, (\tilde{C}M_1)^2\}$.

□

We now define the Green's function G^+ of H by

$$G^+(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ ||H^n(w)|| = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |x_n| = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |y_n|.$$

Theorem 4.2.3. *[CF] The above limits exist and are equal. G^+ is a continuous psh function in $\mathbb{C}^3$ and it is pluriharmonic on U^+. Moreover, $K^+ = \{G^+ = 0\}$ and $G^+ \circ H = 2G^+$.*

The Green's current of H is defined by $\mu^+ = dd^cG^+$. Then $H^*\mu^+ = 2\mu^+$ and supp $\mu^+ = \partial K^+$. Let us consider the induced map H on $\mathbb{P}^3$ and the Fubini-Study form ω on $\mathbb{P}^3$. Sibony ([S2, Theorem 1.6.1]) showed that $\frac{1}{2^n}(H^n)^*\omega$ converges to a closed positive current T_+ of bidegree $(1,1)$ which satisfies $H^*T_+ = 2T_+$ on $\mathbb{P}^3$. Moreover, by [S2, Theorem 1.8.1], T_+ does not charge the hyperplane at infinity and $T_+|_{\mathbb{C}^3} = \mu^+$ has mass one in $\mathbb{C}^3$.

Unlike the regular automorphisms (see [S2] for regular automorphisms), orbits of points in K^+ may escape to infinity. For example, $H(0, y, 0) = (0, b^n y, 0) \to [0 : 1 : 0 : 0]$ if $|b| > 1$. By Lemma 4.2.2 (iv), any such orbit may escape to infinity with a smaller super-exponential rate $(const)^{(\sqrt{3})^n}$. We will show that unbounded orbits of points in K^+ may accumulate only at two points, $P = [0 : 1 : 0 : 0]$ and $Q = [1 : 0 : 0 : 0]$. First we show that points in K^+ accumulate at infinity on the set $I^+_\infty = I^+ \cup \{t = y = 0\}$.

Theorem 4.2.4. $\overline{K^+} = K^+ \cup I_\infty$

Proof. Since

$$\overline{V^{-c}} = \{[x : y : z : t] \in \mathbb{P}^3 : |xy| \leq \max\{R|t^2|, 2a|zt|, |x|^{\frac{3}{2}}|t|^{\frac{1}{2}}, \frac{1}{\epsilon}|y|^{\frac{3}{2}}|t|^{\frac{1}{2}}\}\}$$

we have that

$$\overline{V^{-c}} = V^{-c} \cup \{x = t = 0\} \cup \{y = t = 0\} = V^{-c} \cup I_\infty.$$

Then

$$\overline{K^+} \subset \overline{V^{-c}} \subset V^{-c} \cup I^+_\infty$$

which implies that $\overline{K^+} \subset K^+ \cup I^+_\infty$.

Let $\tilde{H}$ be a homogeneous representation of the extension of H to $\mathbb{P}^3$ so that $||\tilde{H}(w)|| \leq ||w||^2$ and T_+ be the Green's current of H in $\mathbb{P}^3$. By [S2, Theorem 1.6.1], we have on $\mathbb{C}^4$,

$$\frac{1}{2^n} \log ||\tilde{H}^n(w)|| \searrow \tilde{G}(w) \text{ and } \pi^* T_+ = dd^c \tilde{G},$$

where $\pi : \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ is the canonical map. Note that for any $p \in I^+_\infty$, there exists a $\tilde{p} \in \mathbb{C}^4 \setminus \{0\}$ such that $\tilde{H}^2(\tilde{p}) = 0$ and $p = \pi(\tilde{p})$. Since $\tilde{G}(w) \leq \frac{1}{4} \log ||\tilde{H}^2(w)||$, by comparison theorem for Lelong numbers (see [D2]) we have that

$$\nu(\tilde{G},\tilde{p}) \geq \nu\left(\frac{1}{4}\log||\tilde{H}^2||,\tilde{p}\right) > 0.$$

Hence $\tilde{p} \in$ supp $\pi^*T_+ \subset \pi^{-1}(\text{supp } T_+)$, that is, $p \in$ supp T_+. [CF, Theorem 6.5] implies that supp $T_+ \subset \partial\overline{K^+}$, hence $I^+_\infty \subset \partial\overline{K^+}$. □

Theorem 4.2.5. *Unbounded orbits of points in K^+ under H can only cluster at $I^+_\infty \cap I^- = \{[1:0:0:0],[0:1:0:0]\}$.*

Proof. Since K^+ is invariant under H, Theorem 4.2.4 implies that an unbounded orbit can only cluster on I^+_∞. Let $w \in K^+$ and $w_{n_i} = H^{n_i}(w) \to w_0$. If $w_0 \notin I^-$, then $w_{n_i-1} = H^{n_i-1}(w) = H^{-1}(w_{n_i}) \to H^{-1}(w_0) = X^-$. Since H^{-1} is weakly regular $H^{n_i-1}(w)$ avoids a neighborhood of $I^- = I^-_\infty$. Thus $H^{-(n_i-1)}$ is well-defined. Since H^{-1} is weakly regular X^- is H^{-1} attracting. So $H^{-(n_i-1)}H^{n_i-1}(w) = w \to X^-$. This contradicts the fact that w is a fixed point in $\mathbb{C}^k$. Thus $w_0 \in I^+_\infty \cap I^- = \{[1:0:0:0],[0:1:0:0]\}$. □

Let $p \in I^+$. We denote by $B(p,\epsilon) \subset \mathbb{P}^n$ the ball with center p and radius ϵ with respect to the distance given by Fubini-Study metric. The blow-up $\mathcal{B}_p$ of H at p is defined by

$$\mathcal{B}_p = \cap_{\epsilon>0}\overline{H(B(p,\epsilon)\setminus I^+)}.$$

This is an analytic set consisting of all limit points of $H(w)$ as $I^+ \not\ni w \to p$. Since $B(p,\epsilon)\setminus I^+$ is connected for all $\epsilon > 0$, $\mathcal{B}_p$ is connected. The dimension of $\mathcal{B}_p$ is not 0, otherwise H would extend holomorphically at p and p would not be in I^+. We will write $\mathcal{B}^+_p$ for the blow-up of H at p and $\mathcal{B}^-_q$ for the blow-up of H^{-1} at q.

Proposition 4.2.6. *$\mathcal{B}_p^+ = I^-$ for any $p \in I^+ \setminus \{[0:0:1:0]\}$.*

Proof. Let $w_n \to p \in I^+ \subset \{t=0\}$ and $H(w_n) \to \tilde{p}$. If $\tilde{p} \notin \{t=0\}$ then $w_n = H^{-1}H(w_n) \to H^{-1}(\tilde{p}) \notin \{t=0\}$, a contradiction. Therefore we may assume that $\tilde{p} \in \{t=0\}$. Now we suppose that $\tilde{p} = [x:y:1:0] \notin I^-$. It follows that $w_n = H^{-1}H(w_n) \to X^- = [0:0:1:0] \neq p$, which gives a contradiction. Thus $\mathcal{B}_p^+ \subset I^-$. Since the dimension of $\mathcal{B}_p^+$ is non-zero, $\mathcal{B}_p^+ = I^-$. □

Proposition 4.2.7. *$\mathcal{B}_p^+ = \{t=0\}$ where $p = \{[0:0:1:0]\}$.*

Proof. First we show that $I^- = \{[x:y:0:0]\} \subset \mathcal{B}_p^+$. Consider the points $w_x = [x:x:1:x] \to p = [0:0:1:0]$ as $x \to 0$. We have $H(w_x) = [x^2+ax : x^2+bx^2 : x^2 : x^2] \to [1:0:0:0]$ as $x \to 0$. Hence $[1:0:0:0] \in \mathcal{B}_p^+$. For a fixed $\alpha \in \mathbb{C}$ we consider a sequence $w_{\alpha,x} = [x:x:1:\frac{(\alpha-1)x^2}{a}] \to p = [0:0:1:0]$ as $x \to 0$. Since $H(w_{\alpha,x}) = [\alpha x^2 : x^2 + \frac{b(\alpha-1)}{a}x^3 : \frac{\alpha-1}{a}x^3 : \frac{(\alpha-1)^2}{a}x^4] \to [\alpha:1:0:0]$ as $x \to 0$, it follows that $I^- \subset \mathcal{B}_p^+$ as $I^- = \{[x:y:0:0]\} = \{[1:0:0:0]\} \cup \{[\alpha:1:0:0] \mid \alpha \in \mathbb{C}\}$.

Let

$$\Gamma = \frac{\alpha + \sqrt{\alpha^2 - 4(\frac{-ab}{x} + b\beta)}}{2}$$

be a root of the equation

$$\Gamma^2 - \alpha\Gamma - \frac{ab}{x} + b\beta = 0 \tag{4.6}$$

where α and β are some constants in $\mathbb{C}$. We take

$$w_{\alpha,\beta,x} = \left[x : \frac{x(\alpha-\Gamma)}{b} : 1 : \frac{x}{\Gamma}\right] \to p = [0:0:1:0]$$

as $x \to 0$. It follows from (4.6) that

$$H(w_{\alpha,\beta,x}) = \left[\frac{-x^2\Gamma^2 + \alpha\Gamma x^2 + abx}{\Gamma b} : \frac{\alpha x^2}{\Gamma} : \frac{x^2}{\Gamma} : \frac{x^2}{\Gamma^2}\right] = \left[\beta : \alpha : 1 : \frac{1}{\Gamma}\right] \to [\beta : \alpha : 1 : 0],$$

as $x \to 0$. Hence $[\beta : \alpha : 1 : 0] \in \mathcal{B}_p^+$. Since $\{t = 0\} = I^- \cup \{[\beta : \alpha : 1 : 0] \mid \alpha, \beta \in \mathbb{C}\}$, $\{t = 0\} \subset \mathcal{B}_p^+$. □

4.3 Invariant Measures

For regular automorphisms of $\mathbb{C}^n$, Sibony ([S2, Theorem 2.5.2]) constructed an invariant probability measure $\mu = {T_+}^l \wedge {T_-}^{n-l}$ where dim $I^- = l - 1$. It is impossible to construct an invariant measure for H and H^{-1} by using the powers of the Green's currents since $\mu^+ \wedge \mu^+ = \mu^- \wedge \mu^- = 0$ in $\mathbb{C}^3$. Indeed,

$$\mu_\epsilon = dd^c \max\{G^+, \epsilon\} \wedge \mu^+ \to \mu^+ \wedge \mu^+$$

as $\epsilon \to 0$. We note that $K^+ = \{G^+ = 0\}$ and supp $\mu_\epsilon \subset \partial K^+$. Let $w \in \partial K^+$ and B be a neighborhood of w in $\mathbb{C}^3$ such that $G^+ < \epsilon$ in B. Hence $\max\{G^+, \epsilon\} = \epsilon$ and $\mu_\epsilon = 0$ on B. Thus $\mu_\epsilon = 0 = \mu^+ \wedge \mu^+$ in $\mathbb{C}^3$.

When f^{-1} is weakly regular and I^- is f-attracting, by [GS, Theorem 3.1], there is an invariant current σ_s of bidimension (s, s) where the dim $X^- = s-1$. Then the wedge product $\mu := \sigma_s \wedge T_-^s$ is an f-invariant measure. It is well-defined since G^- is locally bounded near $\overline{K^+}$.

In our case, H^{-1} is weakly regular. If $|b| > 1$, then I^- is $H-$attracting and the construction of invariant measure as in [GS] works for H. However, when $|b| \leq 1$, I^- is not $H-$attracting and their construction does not work.

Since H^{-1} is weakly regular, $X^- = [0 : 0 : 1 : 0]$ is an attracting point for H^{-1}. Using this fact with the estimates on K^- and some ideas from [GS], we construct an $H^{-1}-$ invariant current of bidimension $(1,1)$ which is supported on $\partial\overline{K^-}$.

Theorem 4.3.1. *There is a closed positive current σ of bidimension $(1,1)$ on $\mathbb{P}^3$ such that $(H^{-1})^*\sigma = 2\sigma$ and supp $\sigma \subset \partial\overline{K^-}$.*

Proof. Let ω be the standard Kähler form in $\mathbb{P}^3$ and $\omega|_{\mathbb{C}^3}$ be the restriction of ω to $\mathbb{C}^3$. We define

$$R_N = \frac{1}{N}\sum_{n=1}^{N}\frac{(H^{-n})^*\omega^2|_{\mathbb{C}^3}}{2^n}.$$

We still denote by R_N the trivial extension to $\mathbb{P}^3$. Then

$$\begin{aligned} ||R_N|| &= \int_{\mathbb{P}^3}\frac{1}{N}\sum_{n=1}^{N}\frac{(H^{-n})^*\omega^2}{2^n}\wedge\omega = \int_{\mathbb{C}^3}\frac{1}{N}\sum_{n=1}^{N}\frac{(H^{-n})^*\omega^2}{2^n}\wedge\omega \\ &= \int_{\mathbb{C}^3}\frac{1}{N}\sum_{n=1}^{N}\frac{(H^{n})^*\omega}{2^n}\wedge\omega^2 = \int_{\mathbb{P}^3}\frac{1}{N}\sum_{n=1}^{N}\frac{(H^{n})^*\omega}{2^n}\wedge\omega^2 = 1. \end{aligned}$$

Therefore there is a subsequence R_{N_j} which converges to a current σ in the sense of currents. Since $||R_{N_j}|| = 1$, σ has mass 1 in $\mathbb{P}^3$ and it is invariant under the pullback by $H|_{\mathbb{C}^3}^{-1}$. Indeed,

$$\begin{aligned} (H^{-1})^*R_{N_j} &= \frac{2}{N_j}\sum_{n=1}^{N_j}\frac{(H^{-(n+1)})^*\omega^2}{2^{n+1}} \\ &= \frac{2}{N_j}\left(N_jR_{N_j} - \frac{(H^{-1})^*\omega^2}{2} + \frac{(H^{-(N_j+1)})^*\omega^2}{2^{N_j+1}}\right) \to 2\sigma, \end{aligned}$$

as $||\frac{(H^{-(N_j+1)})^*\omega^2}{2^{N_j+1}}|| = 1$. On the other hand $(H^{-1})^*$ is continuous on currents in $\mathbb{C}^3$. Thus $(H^{-1})^*\sigma = 2\sigma$ on $\mathbb{C}^3$.

We first prove that supp $\sigma \subset \overline{K^-}$. By [GS, Theorem 2.2],

$$\overline{K^-} = K^- \cup I^-.$$

Let $X^- = \cap_{j=1}^{\infty} U_j$ where U_j's are decreasing open sets in $\mathbb{P}^3$ and $\epsilon > 0$. Since $\dim(X^-) = 0$ and T_+ is a current of bidimension $(2,2)$, $T_+ \wedge \omega^2(X^-) = 0$. Hence there is a U_J such that $T_+ \wedge \omega^2(\overline{U_J}) < \epsilon$. Let $B \subset \mathbb{P}^3 \setminus \overline{K^-}$ be a ball. Since H^{-1} is weakly regular, X^- is attracting, with basin U^- in $\mathbb{C}^3$. Thus there exists $M > 0$ such that $H^{-n}(B) \subset U_J$ for all $n \geq M$. By [S2, Theorem 1.6.1],

$$\frac{(H^n)^*(\omega)}{2^n} \to T_+ \text{ and } \frac{1}{N}\sum_{n=1}^{N}\frac{(H^n)^*(\omega)}{2^n} \to T_+.$$

Also the subsequence

$$\begin{aligned} T_{N_j} :&= \frac{1}{N_j}\sum_{n=M}^{N_j}\frac{(H^n)^*(\omega)}{2^n} \\ &= \frac{1}{N_j}\left(\sum_{n=1}^{N_j}\frac{(H^n)^*(\omega)}{2^n} - \sum_{n=1}^{M-1}\frac{(H^n)^*(\omega)}{2^n}\right) \to T_+ \end{aligned}$$

as $N_j \to \infty$. Hence $T_{N_j} \wedge \omega^2 \to T_+ \wedge \omega^2$ as measures and

$$\limsup_{N_j \to \infty} T_{N_j} \wedge \omega^2(\overline{U_J}) \leq T_+ \wedge \omega^2(\overline{U_J}) < \epsilon.$$

This implies that $T_{N_j} \wedge \omega^2(\overline{U_J}) \leq \epsilon$ for all $N_j > N$ for some $N > 0$. Since R_{N_j} does not charge the hyperplane at infinity, we have that

$$\begin{aligned}
\int_B R_{N_j} \wedge \omega &= \int_{B \cap \mathbb{C}^3} \frac{1}{N_j} \sum_{n=1}^{N_j} \frac{(H^{-n})^* \omega^2}{2^n} \wedge \omega \\
&= \frac{1}{N_j} \sum_{n=1}^{N_j} \int_{H^{-n}(B \cap \mathbb{C}^3)} \frac{(H^n)^*(\omega)}{2^n} \wedge \omega^2 \\
&\leq \frac{1}{N_j} \left(\sum_{n=1}^{M-1} \int_{H^{-n}(B \cap \mathbb{C}^3)} \frac{(H^n)^*(\omega)}{2^n} \wedge \omega^2 + \sum_{n=M}^{N_j} \int_{U_J} \frac{(H^n)^*(\omega)}{2^n} \wedge \omega^2 \right) \\
&\leq \frac{M-1}{N_j} + \int_{\overline{U_J}} T_{N_j} \wedge \omega^2 \leq 2\epsilon,
\end{aligned}$$

if N_j is big enough. This shows that $\sigma \wedge \omega(B) = 0$.

Now we will show that σ has no mass in the interior of $\overline{K^-}$. Let $U \subset\subset \text{int}\, \overline{K^-}$. Since $\overline{K^-} = K^- \cup I_\infty^-$, U is actually contained in K^-. By [CF, Lemma 6.3], there is $C > 1$ such that $||H^{-n}(z)|| \leq C^n$ for all $z \in U$ and $n > 0$. In $\mathbb{C}^3$,

$$\begin{aligned}
R_{N_j} = \frac{1}{N_j} \sum_{n=1}^{N_j} \left(\frac{(H^{-n})^* \omega}{(\sqrt{2})^n} \right)^2 &\leq \left(\frac{1}{N_j^{\frac{1}{2}}} \sum_{n=1}^{N_j} \frac{(H^{-n})^* \omega}{(\sqrt{2})^n} \right)^2 \\
&= (dd^c G_{N_j})^2
\end{aligned}$$

where $G_{N_j}(z) := \frac{1}{N_j^{\frac{1}{2}}} \sum_{n=1}^{N_j} \frac{\log(1+||H^{-n}(z)||^2)^{\frac{1}{2}}}{(\sqrt{2})^n}$.

On U,

$$0 \leq G_{N_j} \leq \frac{1}{N_j^{1/2}} \sum_{n=1}^{N_j} \frac{n \log C}{(\sqrt{2})^n}.$$

Thus G_{N_j} converges to 0 locally uniformly on int K^- and hence

$$R_{N_j} \leq (dd^c G_{N_j})^2 \to 0,$$

which implies that σ has no mass on int K^-. Thus supp $\sigma \subset \partial \overline{K^-}$.

□

4.4 Possible Behavior on K^+

We know from Theorem 4.2.5 that an unbounded $H-$orbit of points in K^+ may accumulate only at two points, $[0:1:0:0]$ and $[1:0:0:0]$. We will first investigate the behavior of the orbits near the point $[1:0:0:0]$. We recall the set M_n is defined by

$$M_n = M_n(w) := \max\left\{R, 2|az_n|, |x_n|^{3/2}, \frac{1}{\epsilon}|y_n|^{3/2}\right\}. \tag{4.7}$$

If $H^n(x,y,z) = (x_n, y_n, z_n)$ is close to $[1:0:0:0]$ at infinity, then $M_n = |x_n|^{3/2}$ so $|x_n| > M_n^{1/2}$.

Lemma 4.4.1. *If $w \in K^+$ and $|x_{n-1}| > M_{n-1}^{\frac{1}{2}}$, then*

(a) $|x_n| \leq \epsilon^{\frac{-2}{3}} M_n^{\frac{1}{3}}$, and $|x_n| \leq \frac{1}{\epsilon}|y_n|^{\frac{1}{2}}$,

(b) $|y_{n-1}| \leq |a|\frac{|x_{n-2}|}{|x_{n-1}|} + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}$,

(c) $|x_{n-2}| < \epsilon^{\frac{-1}{2}} M_{n-2}^{\frac{1}{3}}$.

Proof. (a) Since $|x_{n-1}|^2 > M_{n-1} \geq |x_{n-1}y_{n-1}|$, we have $|y_{n-1}| < |x_{n-1}|$. It follows that

$$\frac{M_{n-1}}{2} \leq \frac{1}{2}|x_{n-1}|^2 \leq |y_n| = |x_{n-1}^2 + by_{n-1}| \leq \frac{3}{2}|x_{n-1}|^2,$$

since $|x_{n-1}| > M_{n-1}^{1/2} \geq R^{1/2}$ is large. Hence $|x_n| = |x_{n-1}y_{n-1} + az_{n-1}| \leq \frac{3M_{n-1}}{2} < 3|y_n|$. This implies that

$$|x_n y_n| \leq M_n = \max\{R, |2ax_{n-1}|, |x_n|^{\frac{3}{2}}, \frac{1}{\epsilon}|y_n|^{\frac{3}{2}}\} = \frac{1}{\epsilon}|y_n|^{\frac{3}{2}},$$

that is,

$$|x_n| \leq \frac{1}{\epsilon}|y_n|^{\frac{1}{2}} \leq \frac{1}{\epsilon}(\epsilon M_n)^{\frac{1}{3}} = \epsilon^{\frac{-2}{3}} M_n^{\frac{1}{3}}.$$

(b) As $|x_n| \leq \frac{1}{\epsilon}|y_n|^{\frac{1}{2}}$ we obtain

$$|x_{n-1}y_{n-1} + ax_{n-2}| < \frac{1}{\epsilon}\left(\frac{3}{2}|x_{n-1}|^2\right)^{\frac{1}{2}} = \frac{1}{\epsilon}\sqrt{\frac{3}{2}}|x_{n-1}|.$$

This gives $|y_{n-1} + a\frac{x_{n-2}}{x_{n-1}}| < \frac{1}{\epsilon}\sqrt{\frac{3}{2}}$.

(c) If $|x_{n-2}| < \epsilon^{\frac{-1}{2}} R^{\frac{1}{3}}$ we are done. Otherwise $|x_{n-2}| \geq \epsilon^{\frac{-1}{2}} R^{\frac{1}{3}}$ and by part (b) we get

$$|y_{n-1}| = |x_{n-2}^2 + by_{n-2}| \leq |a|\frac{|x_{n-2}|}{R^{\frac{1}{2}}} + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}.$$

This implies that

$$\left|1 + \frac{by_{n-2}}{x_{n-2}^2}\right| \leq \frac{|a|\epsilon^{\frac{1}{2}}}{R^{\frac{5}{6}}} + \frac{1}{R^{\frac{2}{3}}}\sqrt{\frac{3}{2}} < \frac{1}{2}.$$

So

$$\frac{1}{2}|x_{n-2}|^2 \leq |by_{n-2}| \leq \frac{3}{2}|x_{n-2}|^2$$

and

$$|x_{n-2}|^2 \leq 2|b||y_{n-2}| \leq 2|b|\epsilon^{\frac{2}{3}} M_{n-2}^{\frac{2}{3}} \leq M_{n-2}^{\frac{2}{3}}.$$

□

Remark 4.4.2. Lemma 4.4.1 (a) and (b) implies that if $w \in K^+$ and $|x_{n-1}| > M_{n-1}^{\frac{1}{2}}$, then $|x_n| < M_n^{\frac{1}{2}}$ and $|x_{n-2}| < M_{n-2}^{\frac{1}{2}}$.

Now we consider the orbits of points in K^+ for which every other iterate, $H^n(w)$ is near the point $[1 : 0 : 0 : 0]$.

Lemma 4.4.3. *If $w \in K^+$, $|x_{n-1}| > M_{n-1}^{\frac{1}{2}}$, $|x_{n-3}| > M_{n-3}^{\frac{1}{2}}$ and $R > \frac{1}{\epsilon^9}$ then*

$$M_{n-2} = \frac{|y_{n-2}|^{\frac{3}{2}}}{\epsilon} \tag{4.8}$$

$$M_{n-1} \leq \frac{|x_{n-1}|^{\frac{3}{2}}}{\epsilon}. \tag{4.9}$$

Proof. On K^+, we have $|x_{n-3}y_{n-3}| \leq M_{n-3} < |x_{n-3}|^2$, which implies that

$$|y_{n-3}| \leq |x_{n-3}|. \tag{4.10}$$

It follows from (4.10) that

$$|y_{n-2}| = |x_{n-3}^2 + by_{n-3}| \geq \frac{|x_{n-3}|^2}{2} > |x_{n-3}| = |z_{n-2}|. \tag{4.11}$$

Since $|x_{n-3}| > M_{n-3}^{\frac{1}{2}}$, by Lemma 4.4.1 (a), we have $|x_{n-2}| \leq \frac{1}{\epsilon}|y_{n-2}|^{\frac{1}{2}}$. We also have $|y_{n-2}| = |x_{n-3}^2 + by_{n-3}| > \frac{|x_{n-3}|^2}{2} > \frac{R}{2}$. Hence

$$M_{n-2} = \max\left\{R, |2az_{n-2}|, |x_{n-2}|^{3/2}, \frac{|y_{n-2}|^{3/2}}{\epsilon}\right\} = \frac{|y_{n-2}|^{\frac{3}{2}}}{\epsilon}.$$

We will prove the second part now. Suppose that

$$|x_{n-2}| < (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}}. \tag{4.12}$$

Then by (4.8),

$$|x_{n-2}|^2 \leq \frac{|b|}{2}(\epsilon R)^{\frac{2}{3}} \leq \frac{|b|}{2}|y_{n-2}|. \tag{4.13}$$

Using 4.4.1 (b),

$$|y_{n-1}| < |a|\frac{|b|^{\frac{1}{2}}(\epsilon R)^{\frac{1}{3}}}{(2R)^{\frac{1}{2}}} + \frac{1}{\epsilon}\sqrt{\frac{3}{2}} < \frac{1}{\epsilon}R^{\frac{1}{4}} \leq \frac{1}{\epsilon}|x_{n-1}|^{\frac{1}{2}}. \tag{4.14}$$

It follows from (4.11),(4.12), (4.13) and (4.14) that

$$\begin{aligned}
\frac{|b|}{2}\epsilon^{\frac{2}{3}} M_{n-2}^{\frac{2}{3}} &= \frac{|b|}{2}|y_{n-2}| \leq |x_{n-2}^2 + by_{n-2}| = |y_{n-1}| \leq \frac{1}{\epsilon}|x_{n-1}|^{\frac{1}{2}} \\
&\leq \frac{1}{\epsilon}\left((\epsilon R)^{\frac{1}{3}}\left(\frac{|b|}{2}\right)^{\frac{1}{2}}|y_{n-2}| + |az_{n-2}|\right)^{\frac{1}{2}} \leq \frac{1}{\epsilon}|y_{n-2}|^{\frac{1}{2}}\left((\epsilon R)^{\frac{1}{3}}\left(\frac{|b|}{2}\right)^{\frac{1}{2}} + |a|\right)^{\frac{1}{2}} \\
&< \frac{\epsilon^{\frac{1}{3}} M_{n-2}^{\frac{1}{3}}}{\epsilon} R^{\frac{1}{6}}.
\end{aligned}$$

But this implies that $R^{\frac{1}{3}} \leq M_{n-2}^{\frac{1}{3}} \leq \frac{2R^{\frac{1}{6}}}{\epsilon^{\frac{4}{3}}|b|}$ so $R \leq \frac{2^6}{|b|^6\epsilon^8}$ which gives a contradiction since $R > \frac{1}{\epsilon^9}$. Therefore (4.12) does not hold. So

$$|x_{n-2}| \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}}. \tag{4.15}$$

It follows that

$$|x_{n-2}y_{n-2}| \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} |x_{n-3}^2 + by_{n-3}| > (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} \frac{|x_{n-3}|^2}{2},$$

since $|y_{n-3}| \leq |x_{n-3}|$. Then

$$|x_{n-1}| = |x_{n-2}y_{n-2} + ax_{n-3}| > (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} \frac{|x_{n-3}|^2}{4} > R. \tag{4.16}$$

Since $|x_{n-1}||y_{n-1}| \leq M_{n-1} \leq |x_{n-1}|^2$, we have

$$|y_{n-1}| \leq |x_{n-1}|. \tag{4.17}$$

We also have

$$|y_{n-2}| = |x_{n-3}^2 + by_{n-3}| \geq \frac{|x_{n-3}|^2}{2} = \frac{|z_{n-2}|^2}{2}. \tag{4.18}$$

It follows from (4.15) and (4.18) that

$$|x_{n-1}| \geq |x_{n-2}||y_{n-2}| - |az_{n-2}| \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} \frac{|y_{n-2}|}{2}.$$

By Lemma 4.4.1 (a) we have $|y_{n-2}| \geq \epsilon^2 |x_{n-2}|^2$. Hence we obtain

$$|x_{n-1}| \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} \frac{|y_{n-2}|}{2} \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} \frac{\epsilon^2 |x_{n-2}|^2}{2}.$$

Thus

$$|2az_{n-1}| = |2ax_{n-2}| \leq \frac{|x_{n-1}|^{3/2}}{\epsilon}. \tag{4.19}$$

Combining (4.16), (4.17) and (4.19), we obtain that $M_{n-1} \leq \frac{|x_{n-1}|^{\frac{3}{2}}}{\epsilon}$.

□

Using Lemma 4.4.3, we obtain the following relation between M_n and M_{n-2}.

Theorem 4.4.4. *If $w \in K^+$, $|x_{n-1}| > M_{n-1}^{\frac{1}{2}}$, $|x_{n-3}| > M_{n-3}^{\frac{1}{2}}$ and $R \geq \frac{1}{\epsilon^9}$ then $M_n \geq C(\epsilon)M_{n-2}^3$ where $C(\epsilon)$ is a constant depends on ϵ.*

Proof. Since $|y_{n-1}| \leq |x_{n-1}|$, we have

$$|y_n| \geq |x_{n-1}^2| - |by_{n-1}| \geq \frac{|x_{n-1}|^2}{2}.$$

This inequality with the second inequality of Lemma 4.4.3 implies that

$$M_n \geq \frac{|y_n|^{3/2}}{\epsilon} \geq \frac{|x_{n-1}|^3}{\epsilon 2^{3/2}} \geq \frac{\epsilon}{2^{\frac{3}{2}}} M_{n-1}^2. \tag{4.20}$$

By Lemma 4.4.1 (b),

$$|y_{n-1}| = |x_{n-2}^2 + by_{n-2}| \leq |a|\frac{|x_{n-2}|}{R^{\frac{1}{2}}} + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}. \tag{4.21}$$

This implies that

$$\left|1 + \frac{by_{n-2}}{x_{n-2}^2}\right| \leq \frac{|a|}{R^{\frac{3}{4}}} + \frac{1}{\epsilon R^{\frac{1}{2}}}\sqrt{\frac{3}{2}} < \frac{1}{2},$$

since $|x_{n-2}| \geq (\epsilon R)^{\frac{1}{3}} \left(\frac{|b|}{2}\right)^{\frac{1}{2}} > R^{\frac{1}{4}}$ by (4.15) and R is large. So

$$\frac{1}{2}|x_{n-2}|^2 \leq |by_{n-2}| \leq \frac{3}{2}|x_{n-2}|^2. \tag{4.22}$$

Then using (4.11) in the proof of Lemma 4.4.3 and (4.22), we get

$$|x_{n-1}| = |x_{n-2}y_{n-2} + az_{n-2}| \geq \left(\frac{|b|}{3}\right)^{\frac{1}{2}} |y_{n-2}|^{\frac{3}{2}} = \left(\frac{|b|}{3}\right)^{\frac{1}{2}} \epsilon M_{n-2}.$$

Hence

$$M_{n-1} \geq |x_{n-1}|^{\frac{3}{2}} \geq \left(\frac{|b|}{3}\right)^{\frac{3}{4}} \epsilon^{\frac{3}{2}} M_{n-2}^{\frac{3}{2}}. \tag{4.23}$$

Combining this with (4.20) we obtain that

$$M_n > \frac{\epsilon^4}{2^{\frac{3}{2}}} \left(\frac{|b|}{3}\right)^{\frac{3}{2}} M_{n-2}^3.$$

□

Remark 4.4.5. If we assume that $|x_{2n-1}|^2 > M_{2n-1}$ for all $n \geq 1$, then by Theorem 4.4.4,

$$M_{2n} \geq C(\epsilon) M_{2n-2}^3 \geq ... \geq C(\epsilon)^{1+3+...+3^{n-1}} M_0^{3^n} \geq [C(\epsilon)M_0]^{(\sqrt{3})^{2n}},$$

since $1 + 3 + ... + 3^{n-1} < 3^n$. By (4.23),

$$M_{2n-1} \geq C(\epsilon) M_{2n-2}^{\frac{3}{2}} \geq C(\epsilon)[(C(\epsilon)M_0)^{3^{n-1}}]^{\frac{3}{2}} \geq [(C(\epsilon)^2 M_0)^{\frac{3}{2}}]^{(\sqrt{3})^{2n-1}}.$$

Hence $M_n \geq M(w)^{(\sqrt{3})^n}$ for all $n \geq 0$ for some continuous function $M(w)$ of $w = (x, y, z) \in \mathbb{C}^3$. Combining this with Lemma 4.2.2 (iv), we obtain that $||H^n(w)|| \approx C(w)^{(\sqrt{3})^n}$. One can get the same result if $|x_{2n}|^2 > M_{2n}$ for all $n \geq 0$ is assumed.

In the following, we describe the behavior of $|y_n|$ for the points of K^+ satisfying the condition in Theorem 4.4.4.

Lemma 4.4.6. *If $w \in K^+$, $|x_{n-1}| > M_{n-1}^{\frac{1}{2}}$ and $|x_{n-3}| > M_{n-3}^{\frac{1}{2}}$ then $|y_{n-1}| \leq |a| + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}$.*

Proof. If $|x_{n-2}| < R^{\frac{1}{2}}$, then by Lemma 4.4.1 (b) $|y_{n-1}| \leq |a| + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}$. So we can assume that $|x_{n-2}| > R^{\frac{1}{2}}$. Since $|y_{n-2}| = |x_{n-3}^2 + by_{n-3}| \geq \frac{|x_{n-3}|^2}{2} > |x_{n-3}| = |z_{n-2}|$, we have

$$|x_{n-1}| \geq |x_{n-2}y_{n-2}| - |az_{n-2}| > \frac{R^{\frac{1}{2}}|y_{n-2}|}{2}. \tag{4.24}$$

By Lemma 4.4.1 (a) with (4.24) we have $|x_{n-1}| \geq \frac{R^{\frac{1}{2}}}{2}\epsilon^2|x_{n-2}|^2$. Thus

$$\frac{|x_{n-2}|}{|x_{n-1}|} \leq \frac{2|x_{n-2}|}{\epsilon^2 R^{\frac{1}{2}}|x_{n-2}|^2} = \frac{2}{\epsilon^2 R^{\frac{1}{2}}|x_{n-2}|} \leq \frac{2}{\epsilon^2 R} \leq 1$$

choosing $R > \frac{2}{\epsilon^2}$. Hence by Lemma 4.4.1 (b), $|y_{n-1}| \leq |a| + \frac{1}{\epsilon}\sqrt{\frac{3}{2}}$. □

Lemma 4.4.7. *On K^+, if $|x_{2n}|^2 > M_{2n}$ for all n, then for any $r > 0$, y_{2n} is contained in the ball $\mathbb{B}(\pm\sqrt{-b}, r)$ with center $\sqrt{-b}$ and radius r for all $n > N$ for some $N = N(r)$.*

Proof. Since $|x_{2n}y_{2n}| \leq M_{2n} \leq |x_{2n}|^2$, $|y_{2n}| \leq |x_{2n}|$ holds for all n. First we show that for any $\delta > 0$, $|z_{2n}| < \delta|x_{2n}|$ for n big enough. We have

$$|y_{2n-1}| = |x_{2n-2}^2 + by_{2n-2}| \geq \frac{|x_{2n-2}|^2}{2} > |x_{2n-2}| = |z_{2n-1}|.$$

If $|z_{2n}| < R^{\frac{1}{3}}$, then $|x_{2n}| > R^{\frac{1}{2}} > \frac{|z_{2n}|}{\delta}$ choosing $R > (\frac{1}{\delta})^6$ big enough. Thus we may assume that $|z_{2n}| = |x_{2n-1}| > R^{\frac{1}{3}}$. Using this with Lemma 4.4.1 (a), we obtain that

$$|x_{2n}| = |x_{2n-1}y_{2n-1} + az_{2n-1}| > \frac{R^{\frac{1}{3}}|y_{2n-1}|}{2} \geq \frac{R^{\frac{1}{3}}\epsilon^2|x_{2n-1}|^2}{2} = \frac{R^{\frac{1}{3}}\epsilon^2|z_{2n}|^2}{2}.$$

Then

$$\frac{|z_{2n}|}{|x_{2n}|} < \frac{2}{\epsilon^2 R^{1/3}|z_{2n}|} < \frac{2}{\epsilon^2 R^{\frac{2}{3}}} < \delta$$

choosing $R > (\frac{2}{\delta\epsilon^2})^{\frac{3}{2}}$. By Lemma 4.4.6,

$$|y_{2n+2}| = |x_{2n+1}^2 + by_{2n+1}| = |(x_{2n}y_{2n} + az_{2n})^2 + bx_{2n}^2 + b^2 y_{2n}| < A$$

for some constant A depending on ϵ. Since $|y_{2n}| < |x_{2n}|$ we have

$$\left| \left(y_{2n} + \frac{az_{2n}}{x_{2n}} \right)^2 + b \right| < \frac{\tilde{A}}{|x_{2n}|} < \delta$$

for R big enough and some constant $\tilde{A}$. That is, $y_{2n} + \frac{az_{2n}}{x_{2n}} \in \mathbb{B}(\pm\sqrt{-b}, \delta^{\frac{1}{2}})$. Since $\frac{|z_{2n}|}{|x_{2n}|} < \delta$, $y_{2n} \in \mathbb{B}(\pm\sqrt{-b}, \delta^{\frac{1}{2}} + |a|\delta)$. Lemma follows by choosing $\delta^{\frac{1}{2}} + |a|\delta < r$. □

4.5 More on the Behavior of H at Infinity

Let ϕ be an $\omega - psh$ function on $\mathbb{P}^n$. The Lelong number of ϕ at $x \in \mathbb{P}^n$ is defined by

$$\nu(\phi, x) = \sup\{\gamma > 0 : \phi(w) \leq \gamma \log \operatorname{dist}(w, x),\ w \text{ near } x\}.$$

The Lelong number of a $(1,1)$ current $T = \omega + dd^c\phi$ at a point x is defined by $\nu(T,x) := \nu(\phi, x)$.

Let $h : \mathbb{P}^2 \to \mathbb{P}^2$ be a weakly regular map of degree $\lambda > 1$ with indeterminacy set $|I| = 1$ and such that

$$\operatorname{dist}(h(w), I) \geq C \operatorname{dist}(w, I)^{\delta}, \tag{4.25}$$

for all $w \in \mathbb{P}^2 \setminus I$ and for some constants $C > 0$, $0 < \delta < \lambda$. For such maps, in [CG, Proposition 2.7], they show that the Lelong numbers $\nu(\lambda^{-n}(h^n)^*\omega, I)$ converge to $\nu(T, I)$ where ω is the Fubini-Study form of $\mathbb{P}^2$. This result can be used to calculate $\nu(T, I)$ for concrete maps of $\mathbb{P}^2$ satisfying the inequality (4.25). In order to obtain a similar convergence towards Lelong numbers of the Green current of our map H, we would need the estimate

$$\operatorname{dist}(H(w), I^+_\infty) \geq C \operatorname{dist}(w, I^+_\infty)^{\delta}, \tag{4.26}$$

for all $w \in \mathbb{P}^3 \setminus I^+$ and for some constants $C > 0$, $0 < \delta < 2$. However, this is impossible with $\delta < 2$. For example, let $w = [x : 1 : \frac{-x}{a} : 1]$ so $H(w) = [0 : x^2 + b : x : 1]$. If $|x|$ is large, then $\operatorname{dist}(w, I^+_\infty) = \operatorname{dist}(w, \{t = y = 0\}) \approx \frac{1}{|x|}$ and

$$dist(H(w), I^+_\infty) = dist(H(w), I^+) \approx \frac{1}{|x|^2} = dist(w, I^+_\infty)^2.$$

Hence if the inequality (4.26) holds for H, then δ must be at least 2. Below we show that this inequality holds for $\delta = 4$.

Proposition 4.5.1. $\operatorname{dist}(H(w), I_\infty^+) \geq C \operatorname{dist}(w, I_\infty^+)^4$ *for all* $w \in \mathbb{P}^3 \setminus I^+$ *and for some* $C > 0$.

Proof. Here we use the distance which is induced by Fubini-Study metric. It is equivalent to the Euclidean distance between two points in the same chart of $\mathbb{P}^3$. Let's recall that the indeterminacy set of H is $I^+ = \{t = x = 0\}$ and the extended indeterminacy set $I_\infty^+ = I^+ \cup I_2$ where $I_2 = \{t = y = 0\}$. We will consider several cases for w.

Case 1. We first assume that $w = [1 : y : z : 0] \in \{t = 0\} \setminus I^+$, $H(w) = [y : 1 : 0 : 0]$. If $y = 0$, then $\operatorname{dist}(w, I_\infty^+) = \operatorname{dist}(H(w), I_\infty^+) = 0$. Hence we can assume that $y \neq 0$. If $|z| > \max\{|y|, 1\}$, then

$$\operatorname{dist}(H(w), I_\infty^+) \approx \min\left\{\frac{1}{|y|}, |y|\right\} \geq \min\left\{\frac{1}{|z|}, \frac{|y|}{|z|}\right\} \approx \operatorname{dist}(w, I_\infty^+).$$

If $|z| \leq \max\{|y|, 1\}$, then $\operatorname{dist}(H(w), I_\infty^+\} \approx \operatorname{dist}(w, I_\infty^+) \approx \min\left\{\frac{1}{|y|}, |y|\right\}$.

Case 2. We consider now $w = [x : y : z : 1] \in \{t = 1\}$ and $|x| \geq \max\{|y|, |z|\}$. In this case, w is closer to the line I_2 than I^+ and

$$\operatorname{dist}(w, I_\infty^+) = \operatorname{dist}(w, I_2) \approx \max\left\{\frac{|y|}{|x|}, \frac{1}{|x|}\right\}.$$

We recall that $H([x : y : z : 1]) = [x_1 : y_1 : z_1 : 1] = [xy + az : x^2 + by : x : 1]$.

If $|y| > \max\{2|a|, 2|b|, 1\}$, then $|by| \leq |bx| \leq \frac{|x|^2}{2}$ and $|az| \leq |ax| \leq \frac{|xy|}{2}$. These imply that

$$\frac{|x|^2}{2} \leq |y_1| \leq \frac{3|x|^2}{2} \text{ and } \frac{|xy|}{2} \leq |x_1| \leq \frac{3|xy|}{2}.$$

Since $|y| > 1$,

$$\mathrm{dist}(H(w), I_\infty^+) \approx \max\left\{\frac{|x_1|}{|y_1|}, \frac{1}{|y_1|}\right\} \geq \max\left\{\frac{|y|}{3|x|}, \frac{2}{3|x|^2}\right\} = \frac{|y|}{3|x|} \approx \frac{1}{3}\,\mathrm{dist}(w, I_\infty^+).$$

If $|y| \leq \max\{2|a|, 2|b|, 1\}$, then

$$\mathrm{dist}(w, I_\infty^+) \lesssim \frac{1}{|x|},\ \frac{|x|^2}{2} \leq |y_1| \leq \frac{3|x|^2}{2} \text{ and } |x_1| \leq \frac{3|x|^2}{2}.$$

Hence

$$\mathrm{dist}(H(w), I_\infty^+) \approx \max\left\{\frac{|x_1|}{|y_1|}, \frac{1}{|y_1|}\right\} \geq \frac{2}{3|x|^2} \gtrsim \mathrm{dist}(w, I_\infty^+)^2.$$

Case 3. We now assume $w = [x : y : z : 1] \in \{t = 1\}$ and $|y| \geq \max\{|x|, |z|\}$. In this case, w is closer to the line I^+ than I_2 and

$$\mathrm{dist}(w, I_\infty^+) = \mathrm{dist}(w, I^+) \approx \max\left\{\frac{|x|}{|y|}, \frac{1}{|y|}\right\}.$$

If $|x| > \max\{1, 2|a|\}$ and $|by| \leq \frac{|x|^2}{2}$ then

$$\frac{|xy|}{2} \leq |x_1| \leq \frac{3|xy|}{2} \text{ and } \frac{|x|^2}{2} \leq |y_1| \leq \frac{3|x|^2}{2}.$$

We obtain that

$$\mathrm{dist}(H(w), I_\infty^+) \gtrsim \frac{|y_1|}{|x_1|} \geq \frac{|x|}{3|y|} \approx \mathrm{dist}(w, I_\infty^+).$$

If $|x| > \max\{1, 2|a|\}$ and $|by| > \frac{|x|^2}{2}$ then

$$\frac{|xy|}{2} \leq |x_1| \leq \frac{3|xy|}{2} \text{ and } |y_1| \leq 3|by|,$$

and hence

$$\mathrm{dist}(H(w), I_\infty^+) \gtrsim \frac{1}{|x_1|} \geq \frac{2}{3|xy|} \geq \frac{2}{3|2b|^{1/2}|y|^{3/2}}.$$

On the other hand,

$$\mathrm{dist}(w, I_\infty^+) \approx \frac{|x|}{|y|} \leq \frac{|2b|^{1/2}}{|y|^{1/2}}.$$

Therefore

$$\mathrm{dist}(H(w), I_\infty^+) \gtrsim \mathrm{dist}(w, I_\infty^+)^3.$$

If $|x| \leq \max\{1, 2|a|\}$ then

$$\mathrm{dist}(w, I_\infty^+) \lesssim \frac{1}{|y|},\ |y_1| \leq C\max\{|y|, 1\},\ |x_1| \leq C\max\{|y|, 1\},\ |z_1| = |x|,$$

for some constant $C > 0$. We have

$$\mathrm{dist}(H(w), I_\infty^+) \geq \mathrm{dist}(H(w), \{t = 0\}) \gtrsim \min\left\{\frac{1}{|y|}, \frac{1}{|x|}\right\} \gtrsim \frac{1}{|y|} \gtrsim \mathrm{dist}(w, I_\infty^+).$$

Case 4. We assume $w = [x : y : z : 1] \in \{t = 1\}$ and $|z| > |x| \geq |y|$. In this case w is closer to the line I_2 than I^+ and

$$\mathrm{dist}(w, I_\infty^+) = \mathrm{dist}(w, I_2) \approx \max\left\{\frac{|y|}{|z|}, \frac{1}{|z|}\right\}.$$

If $|x| \leq 2|b|$ then $|x_1| \leq C|z|$, $|y_1| \leq C|z|$ and $|z_1| \leq C|z|$ for some $C > 0$ since $|y| \leq |x|$. It follows that $\text{dist}(H(w), I^+_\infty) \gtrsim \frac{1}{|z|} \approx \text{dist}(w, I^+_\infty)$. Hence we can assume that $|x| \geq 2|b|$.

If $|y|^2 > 2|az|$, then $|xy| > |y|^2 > 2|az|$ and hence $\frac{|xy|}{2} \leq |x_1| \leq \frac{3|xy|}{2}$. We also have $|by| \leq |bx| \leq \frac{|x^2|}{2}$ which implies that $\frac{|x|^2}{2} \leq |y_1| \leq \frac{3|x|^2}{2}$. It follows that

$$\text{dist}(H(w), I^+_\infty) \approx \max\left\{\frac{|x_1|}{|y_1|}, \frac{1}{|y_1|}\right\} \geq \frac{|y|}{3|x|} \geq \frac{|y|}{3|z|} \approx \text{dist}(w, I^+_\infty).$$

If $|y|^2 < 2|az|$, then

$$\text{dist}(w, I^+_\infty) \lesssim \max\left\{\frac{|2a|^{1/2}}{|z|^{1/2}}, \frac{1}{|z|}\right\} \approx \frac{1}{|z|^{1/2}},$$

and

$$|x_1| \leq C|z|^{3/2},\ |y_1| \leq |z|^2,\ |z_1| \leq |z|.$$

Thus

$$\text{dist}(H(w), I^+_\infty) \geq \text{dist}(H(w), \{t = 0\}) \gtrsim \min\left\{\frac{1}{|x_1|}, \frac{1}{|y_1|}, \frac{1}{|z_1|}\right\} \geq \frac{1}{|z|^2} \gtrsim \text{dist}(w, I^+_\infty)^4.$$

Case 5. Lastly, we assume $w = [x : y : z : 1] \in \{t = 1\}$ and $|z| > |y| > |x|$. Then

$$\text{dist}(w, I^+_\infty) = \text{dist}(w, I^+) \approx \max\left\{\frac{|x|}{|z|}, \frac{1}{|z|}\right\}.$$

If $|x|^2 > \max\{|2a|, |2b|\}|z|$, then $|xy| > |x|^2 > 2|az|$ and hence $\frac{|xy|}{2} \leq |x_1| \leq \frac{3|xy|}{2}$. We also have $|by| \leq |bz| \leq \frac{|x^2|}{2}$ which implies that $\frac{|x|^2}{2} \leq |y_1| \leq \frac{3|x|^2}{2}$. It follows that

$$\mathrm{dist}(H(w), I_\infty^+) \approx \max\left\{\frac{|y_1|}{|x_1|}, \frac{1}{|x_1|}\right\} \geq \frac{|x|}{3|y|} \geq \frac{|x|}{3|z|} \gtrsim \mathrm{dist}(w, I_\infty^+).$$

If $|x|^2 < \max\{|2a|, |2b|\}|z|$, then

$$\mathrm{dist}(w, I_\infty^+) \lesssim \frac{1}{|z|^{1/2}},$$

and

$$|x_1| \leq C|z|^{\frac{3}{2}},\ |y_1| \leq C|z|,\ |z_1| \leq C|z|^{\frac{1}{2}},$$

for some $C > 0$. Thus

$$\mathrm{dist}(H(w), I_\infty^+) \geq \mathrm{dist}(H(w), \{t=0\}) \gtrsim \min\left\{\frac{1}{|x_1|}, \frac{1}{|y_1|}, \frac{1}{|z_1|}\right\} \geq C\frac{1}{|z|^{\frac{3}{2}}} \gtrsim \mathrm{dist}(w, I_\infty^+)^3.$$

□

4.6 A Two Dimensional Model

We will consider the case $a = 0$. Then H becomes a map of $\mathbb{C}^2$,

$$H(x, y) = (xy, x^2 + by).$$

We note that H is not an automorphism anymore. It determines a map $H : \mathbb{P}^2 \to \mathbb{P}^2$ by $H([x : y : t]) = [xy : x^2 + byt : t^2]$. The indeterminacy point is $I = [0 : 1 : 0]$. The sets K^+

and U^+ are defined as in (4.2) without the z coordinate. Then all the estimates for the map in $\mathbb{C}^3$ in Sections 4.2 and 4.4 hold for the reduced map in $\mathbb{C}^2$.

In the following examples, when $b^4 = 1$, we have some lines which are contained in K^+ and invariant under the second iterate H^2 of H. If w lies on these lines then $H^n(w) \approx C(w)^{(\sqrt{3})^n}$, so the maximal growth on K^+ given by the estimates from Section 4.2 does occur.

Example 4.6.1. The second iterate of H is

$$H^2(x, y) = H(xy, x^2 + by) = (xy(x^2 + by), x^2(y^2 + b) + b^2y),$$

so $x_2 = xy(x^2 + by)$ and $y_2 = x^2(y^2 + b) + b^2y$.

If $b^4 = 1$ and $y^2 = -b$ then $y_2^2 = b^4y^2 = -b^5 = -b$. We have four choices for b.

Case 1. If $b = 1$ then $H^2(x, y) = (x^3y + xy^2, x^2(y^2 + 1) + y)$. Hence $H^2(x, i) = (ix^3 - x, i)$ and $H^2(x, -i) = (-ix^3 - x, -i)$. So the lines $\{y = i\}$ and $\{y = -i\}$ are invariant under H^2.

Case 2. If $b = -1$ then $H^2(x, y) = (x^3y - xy^2, x^2(y^2 - 1) + y)$. Hence $H^2(x, 1) = (x^3 - x, 1)$ and $H^2(x, -1) = (-x^3 - x, -1)$. So the lines $\{y = 1\}$ and $\{y = -1\}$ are invariant under H^2.

Case 3. If $b = i$ then $H^2(x, y) = (x^3y + ixy^2, x^2(y^2 + i) - y)$. Hence $H^2(x, e^{\frac{3\pi i}{4}}) = (e^{\frac{3\pi i}{4}}x^3 + x, -e^{\frac{3\pi i}{4}})$ and $H^2(x, -e^{\frac{3\pi i}{4}}) = (-e^{\frac{3\pi i}{4}}x^3 + x, e^{\frac{3\pi i}{4}})$.

Case 4. If $b = -i$ then $H^2(x, y) = (x^3y - ixy^2, x^2(y^2 - i) - y)$. Hence $H^2(x, e^{\frac{\pi i}{4}}) = (e^{\frac{\pi i}{4}}x^3 + x, -e^{\frac{\pi i}{4}})$ and $H^2(x, -e^{\frac{\pi i}{4}}) = (-e^{\frac{\pi i}{4}}x^3 + x, e^{\frac{\pi i}{4}})$.

In all of the cases above, since $b^4 = 1$ and $y^2 = -b$, we have

$$H^2(x, \sqrt{-b}) = (x^3\sqrt{-b} - b^2x, b^2\sqrt{-b}).$$

Thus $|y_{2n}| = 1$, $|x_{2n}| \approx |x|^{3^n}$, $|x_{2n+1}| = |x_{2n}y_{2n}| \approx |x|^{3^n}$ and $|y_{2n+1}| = |x_{2n}^2 + by_{2n}| \approx |x|^{2.3^n}$ for all $n \geq 0$. Hence $H^n(w) \approx C(w)^{(\sqrt{3})^n}$ if w is contained in these lines.

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