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Doubly Perturbed S_3 neutrinos and the s_{13} mixing parameter

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We further study a predictive model for the masses and mixing matrix of three Majorana neutrinos. At zeroth order the model yielded degenerate neutrinos and a generalized “tribimaximal” mixing matrix. At first order the mass splitting was incorporated and the tribimaximal mixing matrix emerged with very small corrections but with a zero value for the parameter s_{13} . In the present paper a different, assumed weaker, perturbation is included which gives a non zero value for s_{13} and further corrections to other quantities. These corrections are worked out and their consequences discussed under the simplifying assumption that the conventional CP violation phase vanishes. It is shown that the existing measurements of the parameter s_{23} provide strong bounds on s_{13} in this model.

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I. INTRODUCTION

At present, the particle physics community is planning, as a follow-up to the enormously important experiments of the last decade [1]-[7], an extensive program with the goal of more accurately understanding the neutrino masses and mixings. There is really no accepted theory for an a priori prediction of these quantities. Hence it seems worthwhile to investigate in detail various theoretical models to develop plausible scenarios which might be tested.

Here we look more closely at a particular model presented in [8] and further studied in [9] and in [10]. That model assumed an initial permutation symmetry (S_3) which is motivated by the fact that the 3×3 matrix which transforms the defining representation to irreducible form is, up to a single parameter rotation, the same as the “tribimaximal” matrix, which is in, at least, rough agreement with the present experimental situation. The tribimaximal form is taken to be:

$$K_{TBM} = \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \equiv R. \quad (1)$$

The assumption was also made that, at zeroth order, the three neutrinos are degenerate. It may be seen from Table I of [8] that this is plausible for a large range of possible fits to the data. However such an assumption at first seems inconsistent with permutation symmetry which suggests two of the three neutrinos to be degenerate (however *not* the two “solar” neutrinos) and different in mass from the third. The proposed solution to this problem called for the introduction of a Majorana type phase, which does not affect the usual neutrino oscillations but does affect the rate for neutrinoless double beta decay. The complications involved in obtaining a suitable Higgs scheme for both the neutrino mass matrix and the charged lepton mass matrix (which can be arranged to be proportional to the unit matrix) in this approach are discussed in some detail in [8].

Of course, many interesting different models for neutrinos based on permutation symmetry have been discussed for a long time [11]- [17]. In addition, many interesting models with similar approaches to the tribimaximal mixing matrix have been vigorously pursued [18] -[30].

In the model under present discussion, the zeroth order piece of the prediagonal Majorana neutrino mass matrix has the well known S_3 invariant form:

$$M_\nu = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \equiv \alpha \mathbf{1} + \beta d. \quad (2)$$

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Here α and β are, in general, complex numbers while d is usually called the “democratic” matrix. As discussed in detail in [10] and [8], we take

$$\alpha = -i|\alpha|e^{-i\psi/2}, \quad (3)$$

where the physical phase ψ lies in the range:

$$0 < \psi \leq \pi. \quad (4)$$

For the assumed initial degeneracy, $|\alpha|$ is related to β , assumed real, by:

$$|\alpha| = \frac{3\beta}{2\sin(\psi/2)}. \quad (5)$$

The two zeroth order parameters are the degenerate neutrino masses, $|\alpha|$ and the phase ψ which contributes to the neutrinoless double beta decay amplitude.

The first order perturbation treated in [9] and [10] is

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & u \\ 0 & u & t \end{pmatrix} \quad (6)$$

where t and u are parameters. In general, t and u may be complex but they were assumed real for simplicity. This perturbation is well known as the “mu-tau” symmetry [31]-[33]. The assumed zeroth order degeneracy (which actually may be relaxed, if desired) forces us to use degenerate perturbation theory. Then Δ turns out (eg, section II of [10]) to be the only possible choice which forces the desired tribimaximal form (as opposed to the generalized tribimaximal form) of the first order mixing matrix.

Here we will choose for the second order perturbation, the matrix:

$$\Delta' = \begin{pmatrix} t' & u' & 0 \\ u' & t' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

For simplicity we again consider the parameters, t' and u' to be real.

Note that this second order perturbation preserves the S_2 subgroup which involves the 1-2 interchange. One might wonder about also including a perturbation, Δ'' which preserves the 1-3 S_2 subgroup. However, that is not expected to give anything new since the combination of Eqs.(2), (6) and (7) already has the same number of parameters as the most general symmetric matrix, M_ν .

The combination of Eqs.(2), (6) and (7) was motivated by the group theory treatment of the strong interactions before QCD which led for example to the Gell-Mann Okubo mass formula [34]. In that case the initial term was flavor SU(3) invariant, the next term was invariant under the SU(2) isospin subgroup while the smallest last term was invariant under the SU(2) “U-spin” subgroup. In the present case the zeroth order term has the discrete group S_3 invariance and two different S_2 subgroups are left invariant by the two perturbations.

II. PERTURBATION ANALYSIS

In [10] we diagonalized the needed symmetric matrix:

$$R^T(\alpha\mathbf{1} + \beta d + \Delta)R = \alpha\mathbf{1} + \begin{pmatrix} t+u & \frac{\sqrt{2}}{3}(t+u) & 0 \\ \frac{\sqrt{2}}{3}(t+u) & 3\beta + \frac{2}{3}(t+u) & 0 \\ 0 & 0 & t-u \end{pmatrix}. \quad (8)$$

The diagonalization of this matrix gave the first order neutrino mixing matrix, $K^{(1)}$ as

$$K^{(1)} = RR_1, \quad (9)$$

where,

$$R_1 \approx \begin{pmatrix} 1 & \frac{\sqrt{2}}{9\beta}(t+u) & 0 \\ -\frac{\sqrt{2}}{9\beta}(t+u) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

This results in a diagonalization with complex eigenvalues. To make these real positive we multiplied $K^{(1)}$ on the right by a suitable diagonal matrix of phases.

To include the 2^{nd} -order perturbation, Eq. (7), we must diagonalize,

$$\begin{aligned} H &= R_1^T R^T (\alpha I + \beta d + \Delta + \Delta') R R_1 \\ &\equiv H^0 + H' \end{aligned} \quad (11)$$

where, after some computation and neglect of still higher order terms, we obtain:

$$H' = R_1^T R^T \Delta' R R_1 \approx \begin{pmatrix} \frac{5}{6}t' - \frac{2}{3}u' & -\frac{1}{3\sqrt{2}}(t'+u') & \frac{1}{2\sqrt{3}}(t'-2u') \\ -\frac{1}{3\sqrt{2}}(t'+u') & \frac{2}{3}(t'+u') & \frac{1}{\sqrt{6}}(t'+u') \\ \frac{1}{2\sqrt{3}}(t'-2u') & \frac{1}{\sqrt{6}}(t'+u') & \frac{1}{2}t' \end{pmatrix}. \quad (12)$$

We introduced the notation H^0 (Everything in Eq.(11) except for Δ') and H' to indicate that, rather than making an explicit diagonalization we will regard, the result to first order as a “zeroth order Hamiltonian”, the given second order term, Eq.(7) as a “first order perturbation” and use ordinary quantum mechanics perturbation theory to proceed. In that approach one has of course the corrections to the energies as:

$$E'_n = \langle \psi_n | H' | \psi_n \rangle, \quad (13)$$

while the corrections to the eigenvectors are,

$$\psi_m^{(1)} = \sum_{n \neq m} \frac{\langle \psi_n | H' | \psi_m \rangle}{E_m - E_n} \psi_n. \quad (14)$$

A more general perturbation approach, which gives the same results, is discussed in the Appendix. The lepton mixing matrix up to and including second order then reads:

$$K = R R_1 R_2 P = (\psi_1, \psi_2, \psi_3) P, \quad (15)$$

where the ψ_i are the columns of $R R_1 R_2$ and furthermore P is the phase matrix needed for the neutrino masses to be real positive; explicitly,

$$\begin{aligned} \psi_1 &= \frac{1}{\sqrt{6}} \begin{pmatrix} -2 - 2\frac{t+u}{9\beta} + 2\frac{t'+u'}{\beta} \\ 1 - 2\frac{t+u}{9\beta} - 3\frac{t'-2u'}{t-2u} + \frac{t'+u'}{9\beta} \\ 1 - 2\frac{t+u}{9\beta} + 3\frac{t'-2u'}{t-2u} + \frac{t'+u'}{9\beta} \end{pmatrix}, \\ \psi_2 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 - 2\frac{t+u}{9\beta} + \frac{t'+u'}{9\beta} \\ 1 + \frac{t+u}{9\beta} - \frac{t'+u'}{18\beta} + \frac{t'+u'}{6} \\ 1 + \frac{t+u}{9\beta} - \frac{t'+u'}{18\beta} - \frac{t'+u'}{6} \end{pmatrix}, \\ \psi_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{2}\frac{t'-2u'}{t-2u} - \frac{t'+u'}{9\beta} \\ 1 + \frac{1}{4}\frac{t'-2u'}{t-2u} - \frac{t'+u'}{9\beta} \\ -1 + \frac{1}{4}\frac{t'-2u'}{t-2u} - \frac{t'+u'}{9\beta} \end{pmatrix}, \end{aligned} \quad (16)$$

and the phase matrix has the form,

$$P = \begin{pmatrix} e^{-i\tau} & 0 & 0 \\ 0 & e^{-i\sigma} & 0 \\ 0 & 0 & e^{-i\rho} \end{pmatrix}, \quad (17)$$

wherein,

$$\begin{aligned} \tau &\approx \frac{\pi}{2} + \frac{1}{2} \tan^{-1} \left[\frac{\cot(\psi/2)}{1 - \frac{2(t+u)}{9\beta} - \frac{5t'}{9\beta} - \frac{4u'}{9\beta}} \right] \\ \sigma &\approx \pi - \frac{1}{2} \tan^{-1} \left[\frac{\cot(\psi/2)}{1 + \frac{4(t+u)}{9\beta} + \frac{4(t'+u')}{9\beta}} \right] \\ \rho &\approx \frac{\pi}{2} + \frac{1}{2} \tan^{-1} \left[\frac{\cot(\psi/2)}{1 - \frac{2(t+u)}{3\beta} - \frac{t'}{3\beta}} \right]. \end{aligned} \quad (18)$$

Note that we are free to subtract $(\tau + \sigma + \rho)/3$ from each of these three entries. Then the sum of the modified three entries will vanish in accordance with the requirement that there be only two independent Majorana phases. The real positive neutrino masses to second order are then:

$$\begin{aligned} m_1 &\approx \frac{3}{2} \beta \csc \frac{\psi}{2} \left[1 - \frac{2}{9\beta} (t + u + \frac{5}{2} t' - 2u') \sin^2 \frac{\psi}{2} \right], \\ m_2 &\approx \frac{3}{2} \beta \csc \frac{\psi}{2} \left[1 + \frac{4}{9\beta} (t + u + t' + u') \sin^2 \frac{\psi}{2} \right], \\ m_3 &\approx \frac{3}{2} \beta \csc \frac{\psi}{2} \left[1 - \frac{2}{3\beta} (t - u + \frac{1}{2} t') \sin^2 \frac{\psi}{2} \right]. \end{aligned} \quad (19)$$

Notice that the zeroth order masses have the characteristic strength, β while the first order masses are suppressed by $(t, u)/\beta$ and the second order masses are suppressed by $(t', u')/\beta$.

Also notice that the absolute values of the neutrino masses depend on the Majorana phase, ψ . However, the lepton number conserving neutrino oscillations can not depend on a Majorana phase [35]. As a check of this we see that the phase ψ cancels out when one considers the mass *differences*,

$$\begin{aligned} A &\equiv m_2^2 - m_1^2 \approx 3\beta(t + u) + \frac{9}{2}\beta t', \\ B &\equiv m_3^2 - m_2^2 \approx \beta(-5t + u) - \beta(\frac{7}{2}t' + 2u'), \\ C &\equiv m_3^2 - m_1^2 \approx 2\beta(-t + 2u) + \beta(t' - 2u'). \end{aligned} \quad (20)$$

Of course, A , B and C are not independent. There are two, presently unresolved, experimental possibilities:

$$\begin{aligned} \text{Type1} &: & m_3 > m_2 > m_1, \\ \text{Type2} &: & m_2 > m_1 > m_3. \end{aligned} \quad (21)$$

The corresponding relations are:

$$\begin{aligned} \text{Type1} &: & |C| = |B| + A, \\ \text{Type2} &: & |C| = |B| - A. \end{aligned} \quad (22)$$

These relations were obtained by using the known positive sign of A and that only the two possibilities $m_3^2 > m_2^2 > m_1^2$ and $m_2^2 > m_1^2 > m_3^2$ are allowed. In the literature some works specify A and $|B|$ while others specify A and $|C|$.

The following best fit values for the perturbation parameters βt and βu were given in the first order treatment [10]:

$$\begin{aligned}\beta t &\approx -4.13 \times 10^{-4} eV^2, \\ \beta u &\approx 4.39 \times 10^{-4} eV^2, \quad \text{Type1}\end{aligned}\tag{23}$$

$$\begin{aligned}\beta t &\approx 4.21 \times 10^{-4} eV^2, \\ \beta u &\approx -3.94 \times 10^{-4} eV^2 \quad \text{Type2.}\end{aligned}\tag{24}$$

III. ELEMENTS OF THE MIXING MATRIX

We employ the following parameterization [36] of the leptonic mixing matrix, K :

$$K = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\gamma} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\gamma} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\gamma} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\gamma} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\gamma} & c_{13}c_{23} \end{pmatrix} P,\tag{25}$$

where c_{12} is short for $\cos\theta_{12}$ for example. P is the diagonal matrix of Majorana type phases given in Eqs.(17) and (18) for the present model. For simplicity we are presently neglecting the conventional CP violation and thus setting $\gamma = 0$. To specify s_{12} , s_{13} and s_{23} , it is clearly sufficient to compare the (1-2), (1-3) and (2-3) matrix elements of K in Eq.(25) with those calculated in Eq.(16). This yields:

$$\begin{aligned}s_{12}c_{13} &= \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \frac{t+u}{9\beta} + \frac{1}{\sqrt{3}} \frac{t'+u'}{9\beta}, \\ s_{13} &= -\frac{1}{2\sqrt{2}} \frac{t'-2u'}{t-2u} - \frac{1}{\sqrt{2}} \frac{t'+u'}{9\beta}, \\ s_{23}c_{13} &= \frac{1}{\sqrt{2}} + \frac{1}{4\sqrt{2}} \frac{t'-2u'}{t-2u} - \frac{1}{\sqrt{2}} \frac{t'+u'}{9\beta}.\end{aligned}\tag{26}$$

For an initial orientation we see that at zeroth order, s_{13} vanishes and also K has the tribimaximal form. When the first order perturbation characterized by t and u is added, neither s_{13} nor s_{23} change. However s_{12} is somewhat modified as discussed previously in section IV of [10]. When the second order perturbation characterized by t' and u' is added, s_{13} finally becomes non-zero while both s_{12} and s_{23} suffer further corrections.

But something unusual is happening; there are terms for s_{13} and s_{23} which behave like t'/t and are manifestly of first order in strength. These arise from the energy difference denominator in Eq.(14). Since we had to use degenerate perturbation theory at first order this denominator is proportional to the first order ‘‘energy’’ corrections rather than the zeroth order energies. Keeping terms of actual first order in strength we find the interesting relation:

$$s_{13} \approx -2\delta s_{23},\tag{27}$$

where δs_{23} denotes the deviation of s_{23} from its tribimaximal value. Also the good approximation $c_{13} = 1$ was made.

IV. NUMERICAL ESTIMATES

Already, Fogli et. al. [37] and Schwetz et. al. [38] have pointed out that detailed analysis of existing neutrino oscillation experiments gives some hint for non zero s_{13} . Thus it seems interesting to see what predictions emerge from Eq.(27).

Expanding s_{23} around its ‘‘tribimaximal value’’ as $s_{23} = [s_{23}]_{TBM} + \delta s_{23}$, one gets:

$$(s_{23})^2 \approx \frac{1}{2} + \sqrt{2}\delta s_{23}.\tag{28}$$

Comparing with the results of a global analysis of the oscillation data given in Table A1 of [38] one then identifies, for respectively 1σ , 2σ and 3σ errors:

$$|\delta s_{23}| = 0.05, \quad 0.08, \quad 0.11.\tag{29}$$

Note that the three cases are associated with the experimental data relating to the 2-3 type neutrino oscillations. Using Eq.(27) then leads to the corresponding predictions,

$$|s_{13}| < 0.025, \quad 0.040, \quad 0.055. \quad (30)$$

It is amusing to note that these values range from about 1/4 to 1/2 of the “best fit” value $|s_{13}| = 0.11$, which is also presented in the first column of Table A1 in [38]. Of course, our estimates provide a test of the present theoretical model for neutrino parameters and have no connection with experimental data on $|s_{13}|$.

As discussed above, the theoretical estimate for $|s_{13}|$, is of characteristic first order strength, appearing as a ratio of a second order quantity divided by a first order quantity. Using Eq.(26) for s_{13} and neglecting the term of second order strength we can get an estimate of the relative second to first order effects:

$$\left| \frac{t' - 2u'}{t - 2u} \right| \approx 2\sqrt{2}|s_{13}| \approx 0.071, \quad 0.11, \quad 0.16, \quad (31)$$

wherein Eq.(30) was used. Evidently the second order effects seem to be suppressed by about 1/10 compared to the first order effects. On the other hand, as seen in Eq.(20), the quantities t' and u' enter in the true second order corrections for the neutrino mass differences. Thus those corrections are likely to be small— on the order of ten percent of the first order mass splittings.

V. SUMMARY AND DISCUSSION

In this work, we designated the zeroth order parameter as β , the first order parameters as t and u and the second order parameters as t' and u' . The first order corrections to the neutrino masses were suppressed by $(t, u)/\beta$ compared to zeroth order. For the mixing angles, the first order corrections had a previously obtained piece proportional to $(t, u)/\beta$ as well as a new piece proportional to $(t', u')/(t, u)$. The latter term arose because we are using degenerate perturbation theory and is clearly important for s_{13} to be non-zero and correlated to corrections of s_{23} .

Here, we have numerically neglected, for both masses and mixing angles terms proportional to $(t', u')/\beta$. In [10] we considered $(t, u)/\beta$ to be about 1/5. Here we found a characteristic strength of s_{13} to correspond to $(t', u')/(t, u)$ about 1/10. Both of these magnitudes are roughly similar.

Note that Eqs.(20) for the neutrino mass differences and Eqs.(26) for the mixing angles do contain pieces of actual second order strength. These should be interesting to study in the future when more precise data becomes available.

The first order corrected formula for the neutrinoless double beta decay factor is given in Eq.(51) of [10]. This was derived from Eq.(49) in which $(s_{13})^2$ was set to zero. Now s_{13} is not zero but its square contributes at a higher order. Furthermore, it is easy to see, using Eqs.(16), that the first two terms in Eq.(49) do not have any contributions of first order strength like $(t', u')/(t, u)$. Hence that formula for m_{ee} still holds to first order.

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Appendix A: Alternative perturbation method

We present here an alternative approach which leads to results in perturbation theory order by order. This can be applied to the case at hand or more generally when the mass matrix is invariant at zeroth order under a finite group G_0 and then we add perturbations of decreasing importance in the small parameter x such that for example the n^{th} perturbation is of order x^n and is invariant under a smaller group G_n . The mass matrix can then be written as an expansion in x ,

$$M(x) = M_0 + xM_1 + x^2M_2 + \dots \quad (A1)$$

where M_0 is invariant under G_0 , M_1 under G_1 and so on.

The eigenvalues (diagonal) and eigenvector matrices can also be expanded as,

$$\begin{aligned} M_d(x) &= M_{d0} + xM_{d1} + x^2M_{d2} + \dots \\ R(x) &= R_0 + xR_1 + x^2R_2 + \dots \end{aligned} \quad (\text{A2})$$

where,

$$R^T(x)M(x)R(x) = M_d(x) \quad (\text{A3})$$

is the eigenvalue equation.

If we differentiate Eq (A3) once we obtain:

$$R^T M' R + R^T M' R + R^T M R' = M'_d \quad (\text{A4})$$

which can be written as:

$$[M_d, R^T R'] + R^T M' R = M'_d \quad (\text{A5})$$

Here we used the orthonormality condition for the eigenvector matrix:

$$R^T R + R^T R' = 0 \quad (\text{A6})$$

Note that the matrix $R^T R$ which appears in what follows is antisymmetric (in each order of perturbation theory) and in consequence all of its derivatives will be antisymmetric.

The second derivative and third derivative equations will read:

$$\begin{aligned} [M'_d, R^T R'] + [M_d, (R^T R)'] + [R^T M' R, R^T R'] + R^T M'' R &= M''_d, \\ [M'_d, R^T R'] + 2[M'_d, (R^T R)'] + [M_d, (R^T R)'] + \\ [[R^T M' R, R^T R'], R^T R'] + 2[R^T M'' R, R^T R'] + [R^T M' R, (R^T R)'] + R^T M''' R &= M'''_d \end{aligned} \quad (\text{A7})$$

All commutators of diagonal matrices give zero on diagonal and in consequence the mass eigenvalues are obtained from the rest of the terms.

It is clear that by setting $x = 0$ one can associate the first derivative with the first order perturbation theory, second with second order and so on. The mass eigenvalues and the matrix $R^T R'$ can be extracted in each order from equations like Eq (A5) and Eq(A7).

Then one should use the orthonormality condition to obtain the eigenvector matrix according to:

$$R^T(x)R'(x) = R_0^T R_1 + x(R_1^T R_1 + 2R_0^T R_2) + \dots \quad (\text{A8})$$

Using this method and $G_0 = S_3$, $G_1 = S_{23}$ and $G_2 = S_{12}$ one retrieves the eigenvalues and eigenvectors in each order of perturbation theory. The results agree with those presented in the main text.

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